The Anomalous Magnetic Moment of Muon

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Abstract. Known as one of the most hopeful fields to find new physics beyond the standard model, the anomalous magnetic moment of muon has gained much attention for a long time and become even more important after the Fermi National Accelerator Laboratory's result came out in 2018. This paper shows the general works and achievements in this exciting field. Those include experiments operated by the Brookhaven National Laboratory and Fermi National Accelerator Laboratory to measure the g-2 factor, the calculation based on the standard model, and a possible extension of the standard model that can explain the experimental results. This paper is an introduction for anyone interested in this field.

1. Introduction

It has been decades since the standard model (SM) was proposed. Through the years, it has overcome nearly every test, becoming the most successful theory in physics. However, for various reasons, the standard model cannot be the complete description of nature, e.g., the existence of dark matter and the failure to be combined with Einstein's theory. In that case, many theories have been created to replace the standard model, including supersymmetry and superstring[1]. However, since few experiment data directly contradict the SM, it turned out to be an obstacle to verifying the new theories and determining the details of the theory. The anomalous magnetic moment of the muon, known as one of the most accurate and significant results countering the SM, has raised more and more attention nowadays.

For a long time, people have known that measuring the anomalous magnetic moment of leptons could be a sensitive test to examine the SM, and that it serves as a tool to search for new physics. Among the three generations of leptons, the muon is the most promising one because its g value is more sensitive to SM extension by a factor typically of m_μ^2, and its lifetime is long enough to operate experiments to do any measurement[2]. Meanwhile, because of the charity violation of the decay of the muon, it provides an effective way to measure the g-factor[3]. In 2001, a breakthrough in this field was made by the Brookhaven National Laboratory, which showed a convincing conflict between the experiment and the calculation based on the SM (by a deviation of 2.7σ, with an accuracy of 1.3 ppm)[4]. So far, the most precious result is completed by the Fermi National Accelerator Laboratory (FNAL) in 2018. It has improved the accuracy to 0.46 ppm with conflicts with the theory of about 3.5σ[5]. In that case, it can be a powerful hint for people to find new physics.

This paper reviews the works in this field, including the experiments to measure the g factor and the SM way to calculate the result. Furthermore, this paper briefly reviews new models based on the new consequence of muon, basically based on supersymmetry. In the first part, this paper discusses the experiments done by the Brookhaven National Laboratory and the FNAL at a principal level. In the next part, the paper analyzes the calculation of the anomalous magnetic moment at the framework of the SM, including the QED, Electroweak and QCD. The final part focuses on several new models based on the experiment result. The author hopes that this paper can introduce this topic to anyone with a basic knowledge of quantum field theory.

2. The Principle of Measuring the g Factor

The principle of the experiment is similar among many different institutions, including the BNL and the FNAL. This method, which can be described as polarized muon beam storage experiments, has been well-explained in the work of Muon (g-2) Collaboration[2].

Generally, it determines the anomalous magnetic moment by measuring the precession of the muon in a storage ring with a highly uniform and precise known magnetic field. To understand this, one can imagine a muon living in a perfect storage ring with magnetic field B. When the muon orbits horizontally in the ring, its momentum processes at the cyclotron frequency $\omega_c = -eB/m\gamma$. For a relativistic muon polarized in the horizontal plane, the Larmor precession, combined with the Thomas precession, yields a total spin precession frequency of:

$$\omega_s = -g_\mu \frac{eB}{2m} - (1-\gamma)\frac{eB}{m\gamma} \tag{1}$$

The relative precession of the spin with respect to the momentum is:

$$\omega_a = \omega_s - \omega_c = -\left(\frac{g_\mu - 2}{2}\right)\frac{eB}{m} = -a_\mu \frac{eB}{m} \tag{2}$$

where $a_\mu = \left(\frac{g_\mu - 2}{2}\right)$ is the anomalous magnetic moment.

In that case, any experiment measuring the ω_a combined with precise knowledge of the storage ring can determine the a_μ through Equation (2).

However, before moving to the measurement of the a_μ, the crucial problem is to produce enough muon with the same polarization. One popular method to solve this problem is to produce the muon through the decay of the pion beam. Because of the parity violation of the weak decay, this method can produce a muon beam with a polarization of 95%, enough for further experiments[2].

When it comes to the measurement of the a_μ, parity violation also plays a central role. Given the parity violation of the decay of $\mu^+ \to e^+ + \bar{\nu}_\mu + \nu_e$, the energy of the positron in the rest frame of the muon is strongly correlated to the muon's spin direction. When the muon is transferred to the laboratory frame through a Lorentz boost, it turns out that the highest energy occurs when the spin and muon momentum directions are aligned, and that the lowest occurs when they are antialigned. As mentioned before, the spin direction respected to the momenta processes with an angular frequency of ω_a, and the direction in which the highest energy positron/electron occurs also varies with a frequency of ω_a.

In practice, it is undoubtedly hard to determine the direction where the positron with the highest energy comes. Instead, it is better to count the number of positrons from a specific direction whose energy is higher than a certain level denoted E_{th}. Its number calculated from this ideal model is

$$N(t) = N_0 e^{-\frac{t}{\gamma\tau_\mu}}\left(1 + A(E_{th})\cos(\omega_a t + \Phi)\right) \tag{3}$$

where $\gamma\tau_\mu$ is the lifetime of the muon in the laboratory frame.

If choosing the related parament wisely, one can gain a signal of the number of positrons with a clear fluctuation pattern, from which the ω_a and the a_μ, can be determined.

From the model, it is natural that the upper limit of the accuracy is determined by the strength of the magnetic field. The FNAL's achievements benefit more from the substantial improvements, including 2.5 times improved magnetic field intrinsic uniformity, detailed beam storage simulations, state-of-the-art tracking, calorimetry, and field metrology for measuring the beam properties, precession frequency, and magnetic field[3]. The results of the BNL and the FNAL experiments are listed below[4] [5]:

$$a_\mu(BNL) = 11659202(14)(6) \times 10^{-10}(1.3ppm)$$

$$a_\mu(FNAL) = 116592040(54) \times 10^{-11}(0.46ppm)$$

where the FNAL's uncertainty is about a quarter of that of the BNL.

3. Calculations of the Anomalous Magnetic Moment of the Muon Based on the SM

The difference between the theory and the experiment result of the anomalous magnetic moment of the electron once played a central role in the building of the QED and the SM. Nowadays, a similar situation occurs on muon, the electron's heavier partner, according to the SM. It is promising to discover new physics here.

Here, we briefly review the method to calculate the muon's magnetic moment by the SM to introduce what effects have been concluded to come to the results and what methods and effects are essential to this topic and serve as a hint for where to find new physics.

Generally, the a_μ arises from the virtual effect involving photons, leptons, hadrons, W, Z, and Higgs bosons. That is to say, the calculation exhausts all aspects of the SM, including the QED, the electroweak, and the QCD effect. Among them, the QED contributes the most and stands as the most certain part of the theory, while the QCD is the most uncertain one because of the confinement[6].

3.1. The Contribution of QED

From the QED, the interacting term of Lagrangian is:

$$\mathcal{L}_{int} = -ej^\mu A_\mu,\ where\ j^\mu = \bar{\psi}\gamma^\mu\psi \tag{4}$$

where j^μ is the electromagnetic current, and A_μ is a fixed classical potential.

Through that, the cross-section of the interaction can be calculated as[7]:

$$i\mathcal{M}(2\pi)\delta\left(p^{0'} - p^0\right) = -ie\bar{u}(p')\Gamma^\mu(p,p')u(p)A_\mu(p'-p) \tag{5}$$

where p, p', and q are the 4-momenta of the come-in muon, the come-out muon and the photon, respectively. This equation means that the higher-order Feynman Diagram corrects lepton's response to a given field, so the Lande g-factor is not strictly equal to 2. This effect is described by $\Gamma^\mu(p,p')$, which is equal to the γ matric when only the lowest-order Feynman Diagram is considered.

The $\Gamma^\mu(p,p')$ is defined as:

$$\Gamma^\mu(p,p') = \gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2m}F_2(q^2) \tag{6}$$

By calculating the cross-section of a magnetic dipole in a magnetic field, one can know the magnetic moment of muon is[7]:

$$<\mu> = \frac{e}{m}[F_1(0) + F_2(0)]\xi'^{\frac{\sigma}{2}}\xi \tag{7}$$

And in that case, the anomalous magnetic moment of muon is:

$$a_\mu = (g-2)/2 = F_2(0) \tag{8}$$

where the g is the Lande g-factor.

The lowest order of the Feynman diagram gives the result that the $a_\mu = \frac{\alpha}{2\pi},\ where\ \alpha$ is the fine-structure constant. As the order becomes higher, the mass of the muon matters. The final analytic result can be concluded in the form of:

$$a_\mu^{QED} = A_1 + A_2\left(\frac{m_\mu}{m_e}\right) + A_2\left(\frac{m_\mu}{m_\tau}\right) + A_3\left(\frac{m_\mu}{m_e}, \frac{m_\mu}{m_\tau}\right) \tag{9}$$

where the uncertainty main comes from the measurement of α, the mass of electron, muon, and tauon.

3.2. The Contribution of Electroweak

The contribution of Electroweak comes from the Feynman diagrams containing W, Z, and Higgs bosons. Because of the complexity of the theory itself, the calculation is much more complicated compared with the QED, but it can still gain high accuracy through perturbation.

The lowest-order contribution of the Electroweak can be written analytically as below:

$$a_\mu^{EW(1)} = \frac{G_F}{\sqrt{2}}\frac{m_\mu^2}{8\pi^2}\left[\frac{5}{3} + \frac{1}{3}(1-4s_w^2)^2\right] \propto \frac{\alpha}{4\pi^2 s_w^2}\frac{m_\mu^2}{M_W^2} \tag{10}$$

It is a general rule that the result can be insignificant because the mass of muon is the W, Z of Higgs bosons. The uncertainty of the electron weakens mainly from the uncertainty of Higgs's mass, and the process contains quarks.

3.3. The Contribution of QCD

The contribution of the QCD is the most difficult and uncertain one of this problem since it is impossible to complete the calculation through perturbation at the energy level. Several non-perturbative methods have been developed to solve this problem. The contribution of QCD is mainly from HVP (Hadronic Vacuum Polarization).

For a long time, the dispersion relation has been successfully used to calculate. The dispersion is a mathematical method to the related S-Matrix in this problem with some other cross sections that can be measured in the experiment. In that case, the calculation can be replaced by the experiment data. In practice, people often use the $e^{+}e^{-}$ annihilation's cross-section to do the calculation:

As an example, the lowest order can be written as:

$$a_{\mu}^{HVP,LO} = \frac{\alpha\text{^}2}{3\pi\text{^}2}\int_{M_{\pi}^{2}}^{\infty}\frac{K(s)}{s}R(s)ds \quad (11)$$

where K(s) is the kernel function, and the R(s) is the hadronic R-ratio defined as:

$$R(s) = \frac{\sigma^{0}(e^{+}e^{-}\to hadrons(+\gamma))}{\sigma_{pt}}, \sigma_{pt} = \frac{4\pi\alpha^{2}}{3s} \quad (12)$$

The uncertainty of this method is directly from the uncertainty of the measurement of the cross-section.

Another calculation method is called the Lattice QCD, which uses the Monte Carlo method to compute the Feynman path integral. This method calculates the contribution of the QCD directly from the first principles, so it is more competitive than the dispersion relation method. Lattice QCD has been successfully applied to this problem in the recent decade, thanks to the increasing computing power and the development in the calculation method.

In detail, to calculate the HVP contribution to the g-2 factor, one needs to calculate an integral[6]:

$$a_{\mu}^{HVP,LO} = \left(\frac{\alpha}{\pi}\right)^{2}\int_{0}^{+\infty} dQ^{2}f(Q^{2})\hat{\Pi}(Q^{2}) \quad (13)$$

where f is the kernel of the integral, and Π is the correlator of the electromagnetic currents in the momenta space. The Lattice QCD Method will be helpful in calculating this complex integral.

Several key features to improve the quality of the result from the Lattice QCD have been realized, including advanced noise reduction methods, corrections due to strong isospin breaking, and QED effects. Fortunately, it seems that the technology and methods to overcome the difficulties are all in place, so it is promising to see further improvement in this field.

3.4. Summary of These Contributions

Considering the three contributions mentioned above, nowadays, people can compute the magnetic moment with high accuracy and find its difference from the experiment done by Brookhaven National Laboratory of the Fermilab by about 3.4σ. The result is shown in the table below[6]:

Table 1. Calculation of the g-2 factor of the muon

Contribution	Value$\times 10^{11}$
QED	116,583,718.931(104)
Electroweak	153.6(1.0)
HVP	6,845(40)
HLBL	92(18)
Total SM Value	116,591,810(43)
Difference($\Delta a_{\mu} := a_{\mu}^{exp} - a_{\mu}^{SM}$)	279(76)

It is clearly shown that there is about an error of 3.7σ between the SM and the experiment, which cannot be treated as a casualty.

If we assume the calculation and the experiment don't make any mistakes, it is convincing evidence for the existence of physics beyond the SM and can help theorists to decide what is the new model for the new physics.

4. Some Research to Explain the G-2 Anomalies Based on the SUSY

Among many kinds of theories that extend beyond the SM, Supersymmetry (SUSY) serves as a promising candidate. By assuming particles' supersymmetry partners, some models based on the SUSY have successfully approached the results that meet the experiment. However, since the SUSY also has a crucial role in determining what dark matter is, one must consider cosmology when constructing models.

Early in the 1980s, theorists began constructing models to calculate the g-2 factor using the SUSY. Since the SUSY is a decoupling theory, the contribution of the theory decreases as the superpartner's mass increases. In that case, the g-2 factor contributes by the SUSY mainly from the loops containing sleptons (selectrons and smuons) and lighter chargino and/or neutralino. Moreover, as mentioned in the previous section, the mass of the particles makes sense in the question. So, the ratio of the two vacuum expectation values of the Higgs field, devoted as $tan\beta$, is a critical parament in the models[8]. Under particular conditions, the contribution of the SUSY is proportional to the $tan\beta$[9].

If we combine the SUSY with the experiment, the experiment result of the muon's g-2 factor implies: for $tan\beta = 35$, at least one superpartner has to be lighter than 450 GeV, and the heavier one must be no heavier than 900 Gev[8]. This is a promising result, showing that it is possible to discover the SUSY particles at accelerators if the theory is correct.

Except for the muon, a measurement of the electron fine structure constant results in a new SM prediction for the g-2 factor of the electron, which is 2.4σ higher than the value obtained by the measurement[10]:

$$\Delta a^e = a_e^{exp} - a_e^{SM} = -(8.8 \pm 3.6) \times 10^{-13} \quad (14)$$

In the previous section, it is made clear that the SM prediction of muon g-2 factor is 3.7σ below the experiment value. So, we know:

$$\frac{\Delta a_e}{\Delta a_\mu} \neq \left(\frac{m_e}{m_\mu}\right)^2 \quad (15)$$

This implies that the correction of the SUSY cannot be flavor-blind. Badziak and Sakurai showed that this feature could be explained via the Minimal Supersymmetric Standard Model (MSSM) by arranging the sizes of bino-slepton and chargino-sneutrino contributions differently between the electron and muon sectors[11].

In recent research on this topic, Li, Xiao and Yang noticed that if the related parament was chosen as what Badziak and Sakurai have done, it gave an over-abundance of the dark matter in our universe, as the bino-like lightest neutralino was assumed to be the dark matter candidate[12]. However, considering dark matter constraints and the LHC constraints, the author argued that this problem could be avoided if the assumption that dark matter was a super WIMP. This is an example of how experiments operate at the micro level will affect the universe's structure.

5. Conclusion

The experiments to measure the magnetic moment of the muon are reviewed in the first part of the paper. Based on the charity violation of the decay of the tau and muon, high-precision measurement can be operated, and the experiment results turn out to sharply contradict the prediction made by the SM, implying the existence of new physics.

In the second part, the paper analyzes the theory value calculated by the SM, and discusses the relative importance between three components of the calculation - QED, Electroweak, and QCD, respectively. The conclusion is that the QED contributes the most among the three but can be calculated most accurately by perturbation; the QCD is the most uncertain part of the theory. Hopefully, this condition can be eased by the development of the lattice QCD method.

Many models have been proposed to solve the contradiction between the theory and the experiment, and the SUSY seems a competitive candidate. Adjusting the related parament of the SUSY can give a semi-quantitative explanation of the experiment at the lowest level approximation. Meanwhile, the LHC and cosmology results restrict the choice of the parament.

It is worth mentioning that if the SUSY is correct, the restriction from the g-2 factor implies that it is likely to find supersymmetry particles on the accelerator.

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