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Integrable Systems and Lifshitz-invariant theories as the asymptotic dynamics of General Relativity on AdS_3

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Abstract

Applications to Condensed Matter and Integrable Systems are explored through the choice of boundary conditions for the gravitational field in three dimensions and negative cosmological constant (AdS_3 GR).

First, a map between two-dimensional Integrable Systems and General Relativity was established. We considered a family of boundary conditions depending on the integer N that reveal the equivalence between first-order Einstein equations and the Ablowitz–Kaup–Newell–Segur (AKNS) system. The latter encodes a broad family of integrable equations, e.g., KdV, MKdV, Sine-Gordon and Nonlinear Schrödinger. For $N = 1$ and odd values of N, and after some settings, we recovered the Brown-Henneaux boundary conditions and the KdV-type, respectively. The integrability of the AKNS system was mapped to an abelian infinite-dimensional asymptotic symmetry algebra of gravitational charges. We identified the conjugacy classes of the spatial holonomy, from where we conclude that particle sources and (extremal) black hole configurations are attainable.

Lastly, transport properties were studied in a two-dimensional holographic description of AdS_3 GR. This scalar theory, invariant under anisotropic scaling and known as the *anisotropic chiral boson*, is obtained after the choice of suitable boundary conditions that generalized the Brown-Henneaux case. Using bosonization techniques, we identified a fermionic current operator. In the context of linear response theory, we employed the Kubo formula to calculate a two-terminal conductance which, in its DC limit, reduces to the Ohm's law. An important feature of this result lies in the fact that the holographic DC conductivity depends explicitly on the dynamical exponent that controls the anisotropic scaling. The bulk realization of the linear response is related to a type of gravitational memory emerging in the context of near-horizon boundary conditions.

Keywords: Boundary conditions; Asymptotic Symmetries; Integrable Systems; AKNS hierarchy; Anisotropic scaling; Kubo formula.

Dedicatory

Vi Veri Universum Vivus Vici.

Fausto (?).

“Dormía y soñé que la vida era belleza; desperté y advertí que es deber”.

Immanuel Kant.

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Notations

In this thesis, we set $\hbar = 1$.

$K = \ell/4G$ is the Chern-Simons level, where ℓ is the AdS_3 radius and G the three-dimensional Newton constant.

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Chapter 1

Introduction

1.1 The holographic principle

Initially conceived by Thorne, 't Hooft, and Susskind [1–3], respectively, the holographic correspondence states that the dynamical content of a gravitational theory living on a D-dimensional volume is contained on a $(D-1)$-surface. Its first concrete realization was given in the context of string theory [4], known as the AdS/CFT conjecture. For a theory living on a D-dimensional asymptotically Anti-de Sitter (AdS) space, there is a dual $(D-1)$-dimensional conformally-invariant field theory. Specifically, the seminal correspondence was shown between a type IIB superstring theory on $\mathrm{AdS}_5 \times S^5$ with a $\mathcal{N}=4$ super Yang-Mills theory at the large N limit [4–7]. Since General Relativity, but seen as the low-energy regime of string theory, was mapped to a non-gravitational quantum field theory, the duality was also referred to as gauge/gravity correspondence.

Many works have continued AdS/CFT correspondences in its original context [8–11] (for classical reviews, see e.g., [12–15]), and it has been realized that this duality allowed to connect different branches of physics, for instances, quantum chromodynamics [16–22], particle physics [23], fluid dynamics [24–30], condensed matter theory (CMT) [31–44] and mathematics [45, 46].

A precursory result of the AdS/CFT correspondence, formulated almost ten years before Maldacena's article [4], is the one of J. D. Brown and Marc Henneaux [47], who proved that by describing the fall-off of the gravitational field on AdS_3, the theory exhibits asymptotically two independent copies, $\pm$, of the 2D conformal symmetry,

$$i\{\mathcal{L}_m^\pm, \mathcal{L}_n^\pm\} = (m-n)\mathcal{L}_{m+n}^\pm + \frac{c}{12} m^3 \delta_{m+n}\,,$$

where m, n are arbitrary integers and $c = 3\ell/2G$ stands for the *Brown-Henneaux central charge.*

This lower-dimensional result emphasizes the role of the asymptotic behavior of fields in order to elucidate holographic properties of gauge theories.

1.2 Asymptotic symmetries and holography beyond AdS

The imposition of suitable boundary conditions define asymptotic symmetries. These correspond to the set of gauge transformations that preserve the form of boundary conditions (a more detailed discussion regarding this topic can be found in article [48], reviews [49, 50] and thesis [51]). Concretely they can be defined as the following quotient group,

$$\text{Asymptotic symmetry group} = \frac{\text{Gauge transformations preserving boundary conditions}}{\text{Trivial gauge transformations}}\,.$$

Here, trivial gauge transformations are those carrying no global charges [52]. Thus, asymptotic symmetries extracts improper gauge transformations that alter the physical state of the system, and they allow to construct conservations laws associated to gauge symmetries, e.g., energy and angular momentum in General Relativity (GR) [53–55], electric charge in electrodynamics and in nonabelian gauge theories [56]; and from the holographic point of view, they correspond to global symmetries of the dual description [4].

In this context, let us briefly review how asymptotic symmetries played an important role in what is known as flat holography.

1.2.1 Flat holography

Bondi, van der Burg, Metzner and Sachs showed that the symmetry group of asymptotically flat spacetimes is an infinite-dimensional extension of Poincaré, labeled as the BMS group [57–59], which correspond to the semidirect product between the Lorentz group and the infinite-dimensional abelian group of spacetimes supertranslations, being the latter a generalization of spacetime translations.

In this regard, an infinite-dimensional extension of Lorentz transformations corresponds to local conformal transformations, known as superrotations. This allows to enhance the BMS group as the semidirect product between superrotations and supertranslations.

According to Barnich and Troessaert [60–62], this reconsideration corresponds to the compatible asymptotic symmetry group in asymptotically flat four-dimensional spacetimes. As a consequence, this result allows to employ the techniques of 2D conformal field theory at the null infinity as an holographic four-dimensional reduction of gravity, as suggested long time ago in [63–65]. The existence of this asymptotic symmetry group gives an indication that the holographic principle can be extended to its flat counterpart, known as *flat holography*.

Motivated by the aforementioned works, in recent years this asymptotic symmetry has gained a lot of interest due its higher-dimensional uplift [66, 67], its appearance in the near-horizon region of nonextremal black holes [68], the soft hair proposal to solve the information loss in black hole evaporation [69–79] (for details, see [80]), its connection between soft theorems in Quantum Field

Theory and memory effects [81] through the Strominger's triangle [82, 83] and the newly Celestial Holography program (for a review, see e.g., [84–87]).

1.3 Three-dimensional gravity

Without sources, three-dimensional General Relativity (3D GR) is a trivial theory from the bulk perspective [88–90], namely, since the Weyl tensor vanishes identically in three dimensions, the theory does not propagate *bulk* degrees of freedom, such as gravitational waves. This implies that the Riemann tensor may be fixed in terms of the Ricci tensor, so the geometry can be locally classified according to the value of the cosmological constant.

Although this property, boundaries let asymptotic degrees of freedom arise [91]. So the dynamical content is encoded in the choice of asymptotic boundary conditions. Besides, the theory can be written as a Chern-Simons action for all values of the cosmological constant [92, 93], allowing to perform the asymptotic analysis bypassing the metric formalism. Moreover, black hole solutions are admitted for negatively curved spacetimes [94, 95].

1.3.1 BTZ black hole

The Bañados-Teitelboim-Zanelli (BTZ) [94, 95] black hole is an exact stationary and axially-symmetric solution to 3D GR with negative cosmological constant. In Boyer-Lindquist-like coordinates (t, r, ϕ), the metric is

$$ds^2 = -N^2(r)dt^2 + \frac{dr^2}{N^2(r)} + r^2\left(d\phi + N^\phi(r)dt\right)^2 ,$$

where the lapse and shift functions are

$$N(r) = \left(-8MG + \frac{r^2}{\ell^2} + \frac{16G^2J^2}{r^2}\right)^{1/2} , \quad N^\phi = -\frac{4GJ}{r^2} ,$$

respectively, and the angular coordinate ranges from $0 < \phi \leq 2\pi$. Here M denotes the mass and J the angular momentum.

This black hole has Killing vectors ∂_t and ∂_ϕ, and possess two event horizons radii $r_\pm$ when the condition $N(r_\pm) = 0$ is fulfilled, with

$$r_\pm^2 = 4MG\ell^2\left(1 \pm \sqrt{1 - \left(\frac{J}{M\ell}\right)^2}\right) .$$

Positive roots are characterized when $|J| \leq M\ell$ with $M > 0$. Hence, extremal configurations are reached when the former inequality is saturated, i.e., $|J| = M\ell$.

The BTZ black hole has some similarities and differences with the Kerr black hole. Regarding the similarities, its causal properties are akin, it has an outer event horizon radius $r = r_+$, with Killing horizon given by the vector field $\xi = \partial_t + \Omega_h \partial_\phi$, where Ω_h is the angular velocity of the event horizon $\Omega_h = -\left(g_{tt}/g_{t\phi}\right)|_{r=r_+} = -N^\phi(r_+)$; an ergosphere region is present, fufills the Bekenstein-Hawking area law, possess a Hawking temperature, emerge when gravitational collapse occurs and exhibits mass inflation [96]. Regarding the differences, it is asymptotically AdS_3,

$$ds^2_{AdS_3} = -\left(1 + \frac{r^2}{\ell^2}\right) + \frac{dr^2}{\left(1 + \frac{r^2}{\ell^2}\right)} + d\phi^2 \,,$$

rather than asymptotically flat, and no curvature singularity appears at the origin.

Because the Ricci tensor of the BTZ is equal to the one of AdS_3, it reveals that this solution is locally AdS_3, just as we discussed previously.

From the holographic point of view, this solution has shown its versatility by being a fruitful toy model in the understanding of the semiclassical properties of black holes. An important question concerning the aforementioned object is to understand the microscopic origin of the Bekenstein-Hawking macroscopic quarter of the area law [97]. Motivated by successful results in string theory [98], and using the fact that GR on AdS_3 with the Brown-Henneaux boundary conditions is dual to a theory with conformal global symmetry, Strominger found [99] that the entropy of the BTZ black hole can be microscopically computed by means of the Cardy formula

$$S = 2\pi\left(\sqrt{\frac{c_L}{6}E_L} + \sqrt{\frac{c_R}{6}E_R}\right) \,,$$

where $c_{L,R}$, stands for the left and right copies of the Brown-Henneaux central charge, and $E_{L,R}$ the left and right energies that depend on M and J.

This features led the using of the BTZ black hole as a laboratory to test new ideas in holography.

1.3.2 Infinite-dimensional symmetries and 3D gravity

Choosing suitable boundary conditions allows us to explore aspects of the holographic nature of the gravitational field in 3D, which enjoys infinite-dimensional asymptotic symmetries. In this regard, it is worth mentioning successful examples, such as the ones on AdS_3, asymptotically flat spacetimes, and higher-spin gravity.

Symmetries on AdS_3 and its deformations

The Brown-Henneaux case [47] is the starting point to explore conformal holography on AdS_3. In the presence of minimally and nonminimally coupled scalar matter field, the usual central charge $c = 3\ell/2G$ was obtained in [100], while in higher-curvature gravity, a modification of the latter quantity was provided by scaling arguments in [101]. In the context of Topological Massive Gravity [102, 103], the latter was modified in [104], while specific fall-off for asymptotically AdS_3 solutions, compatible with conformal symmetry, were described in [105, 106]; and in New Massive Gravity [107, 108], in article [109].

Different infinite-dimensional asymptotic symmetry algebras were obtained by the use of novel asymptotic boundary conditions, e.g., warped conformal [110–112], centerless warped conformal [68], twisted warped conformal [113], Heisenberg [114], a generalization of conformal, Heisenberg and warped [115], $sl(2,\mathbb{R})$ current algebra [116], a generalization of the latter through [117], and a two centrally extended Virasoro plus a centrally extended (time-dependent) Weyl sector [118], nevertheless, what we will show along this thesis is that conformal symmetry (and its deformations) is not the only part of this "landscape" in three dimensions.

Symmetries on asymptotically flat spacetimes

An appealing feature of holography on AdS_3 is that many of their results and lessons can be pushed towards its flat limit after a suitable contraction [119–124]. In 3D asymptotically flat spacetimes, the three-dimensional version of the BMS group appears, labeled as BMS_3. The latter corresponds to the semidirect product of the diffeomorphism group on the circle with an abelian ideal of supertranslations [125], whose symmetry algebra admits a nontrivial central extension [126],

$$\begin{aligned}
i\{\mathcal{J}_m, \mathcal{J}_n\} &= (m-n)\mathcal{J}_{m+n}\,, \\
i\{\mathcal{J}_m, \mathcal{P}_n\} &= (m-n)\mathcal{P}_{m+n} + \frac{1}{4G} m^3 \delta_{m+n}\,, \\
i\{\mathcal{P}_m, \mathcal{P}_n\} &= 0\,,
\end{aligned}$$

where $\mathcal{P}_m$ and $\mathcal{J}_m$ are the superrotation and supertranslations generators, respectively. (Remarkably, the two-dimensional Virasoro algebra reduces to the Galilean conformal algebra (GCA) through the nonrelativistic contraction $c \to \infty$ [119, 127, 128], being the latter isomorphic to BMS_3.) The latter algebra appear in different physical contexts, e.g., string theory [129–136], higher spins [137–144] and massless Klein-Gordon fields [145], and in recent years has shown its importance by appearing in the near-horizon geometry of three-dimensional black holes [68, 146–148], applied in holographic entanglement entropy [149–164] and asymptotically counting of growing microstates by means of compatible Cardy formulas [165–167] of the flat-analogue version of the BTZ black hole, which is framed in the context of the so-called flat cosmological solutions; and it was generalized [168] to supersymmetry [169–171] and hypergravity [172, 173].

A comprehensive review in 3D flat holography can be found in [174], while selected thesis in this topic, in references [175–177].

Symmetries in higher-spin gravity

As we have seen, Virasoro symmetry plays a central role in holography. Nevertheless, it is possible to further generalize it when we study higher-spin theories.

Higher spin gravities are gauge theories containing a massless spin 2 particle that couples to higher-spin fields. These theories are interesting since they might have better quantum properties than Einstein's gravity, whose quantum behavior is not understood, along with several holographic applications, e.g., [178–184].

Although higher-spin fields suffer inconsistencies in their interactions [185, 186], these can be circumvented by means of Vasiliev equations [187, 188], which non-minimally couple them with an infinite tower of spin fields $s = 0, 1, 2, \ldots, \infty$, in the presence of cosmological constant. It is important to mention that the perspective provided by Vasiliev reconciles spacetime and gauge symmetries in a non-trivial form [189], thus allowing to avoid the no-go theorem of Coleman and Mandula [190] (for a review on no-go versus yes-go examples on higher-spin gravity, see [142]).

Since there is no action principle and only certain sectors of Vasiliev's theory are known, one possible route to gain insights is to reduce the theory to a toy model that captures the most relevant properties of higher-spin fields. In this regard, the three-dimensional case can be fruitful since interacting higher particles with gravity can be non perturbatively described by a Chern-Simons field theory [191–193] (for a review, see [194]).

The above can be further simplified since it is possible to consistently truncate the infinite tower of particles to the case of a finite number of fields with spin $s = 2, 3, \ldots, N$. Therefore, the simplest case to deal with corresponds to $N = 3$, which is gravity non-minimally coupled to a spin-three field with a negative cosmological constant. This allows to find exact black hole solutions, endowed with a non-trivial spin 3 hair, as reported in [195–198].

In the previously mentioned context, asymptotic symmetries were studied in [199–204], along with introduction of chemical potentials [197, 198, 205], unambiguously characterizing their global charges, which turns out to be the Zamolodchikov W_N algebra [206].

W algebras appeared originally in the context of 2D Conformal Field Theory [206–209] and they are closely related to Integrable Systems as a second Hamiltonian structure in KdV-like hierarchies such as the Boussinesq equation [210].

1.4 Integrable Systems and the asymptotic dynamics of 3D GR

Just as infinite-dimensional symmetries emerge naturally in 3D GR, they also appear in the context of Integrable Systems.

Several well-known nonlinear equations, such as the Korteweg-de Vries (KdV) [211, 212], Nonlinear Schrödinger (NLS) [213], Sine-Gordon (SG) [214], Toda lattice [215], and many other that appear in nature [216–218], including in GR [219], are known to be solvable. These equations belong to a more broad topic, known as Integrable Systems (classical textbooks are, e.g., [220–222]).

Different definitions of Integrable Systems can be found in the literature. For example, by Liouville's theorem, an integrable equation admits sufficiently conserved quantities in order to integrate the equations of motion [223]. Concretely, for a finite $2n$-dimensional symplectic Hamiltonian system, we say it is *completely integrable* if a maximal set of n functionally-independent conserved quantities are admitted, *partially integrable* if there are less than n, *superintegrable* if there are greater than n, and *maximally superintegrable* when an amount of $2n-1$ conserved quantities exists [224].

Although Liouville's theorem provides a definition of what finite-dimensional integrable systems are, there is no universally accepted definition for its infinite-dimensional analogue [222]. This is the reason why integrable systems manifests through the following broadly criteria [225]:

- The existence of "enough" conserved quantities, and
- the possibility to obtain explicit solutions,

A relevant type of solutions, supported by a solid experimental basis [226–230] are solitons. According to [221], these waves are localized within a spacetime region, they are of permanent form, scatter with other solitons and emerge from the collision unchanged, up to a phase shift. They appear due to a "fine process"; the nonlinearity is canceled with certain effects (such as dissipation in the KdV or dispersion in the Burgers equation). Examples are the kink and breathers (periodic solitons) of SG, n-soliton and cnoidal waves of KdV [231], peakons of Camassa-Holm [218], Peregrine [232] and Akhmediev [233] breathers of NLS, among others.

These waves are constructed by particular methods, for instances, the Inverse Scattering [234–237] and Bäcklund transformations [238–242]. The first is a nonlinear extension of Fourier analysis (for details, see [243]), while the second extends the superposition principle to the nonlinear case (for a introductory discussion, see [244]; for concrete examples, see [245–248]; and for a detailed treatise on the subject, see [249]). Other techniques are the Hirota's bilinear method [250], Painlevé criteria [251, 252], Lie groups [253], Consistent Tanh Expansion [254–259], nonlocal symmetries [260, 261] and Darboux transformations [262, 263].

The existence of conserved charges is a consequence of the invariance under particular transformations [264]. In this context, a large number of equations in field theory [265, 266] admit the following formulation,

$$\dot{u} = \mathcal{D}_1\left(\frac{\delta H_2}{\delta u}\right) = \mathcal{D}_2\left(\frac{\delta H_1}{\delta u}\right) , \tag{1.4.1}$$

where $u = u(t, \phi)$ stands for the dynamical field, $H_n = \int d\phi\ \mathcal{H}_n$, with $\mathcal{H}_n = \mathcal{H}_n[u]$ a density depending on u and its spatial derivatives, $\delta/\delta u$ denotes the functional derivative operator with respect to u and $\mathcal{D}_{1,2}$ are integro-differential operators. If $\mathcal{D}_1$ and $\mathcal{D}_2$ are compatible Hamiltonian operators, we say that Eq. (1.4.1) is a *bi-Hamiltonian system* [253]. If we are able to solve the recurrence relation

$$\frac{\delta H_{n+1}}{\delta u} = \mathcal{D}_1^{-1} \circ \mathcal{D}_2 \left(\frac{\delta H_n}{\delta u} \right) , \tag{1.4.2}$$

for all positive integers n (namely, we are able to find H_{n+1} in terms of H_n), the bi-Hamiltonian formulation with this property implies the existence of *infinite* Poisson-commuting quantities [267, 268],

$$\dot{H}_n = \{H_n, H_m\}_{1,2} = 0 \, , \quad n = 0, 1, 2, \dots \, , \tag{1.4.3}$$

where $\{,\}_{1,2}$ stands for the Poisson bracket associated to the Hamiltonian operators $\mathcal{D}_{1,2}$,

$$\{F[u], G[u]\}_{1,2} = \int d\phi \, \frac{\delta F}{\delta u} \mathcal{D}_{1,2} \left(\frac{\delta G}{\delta u} \right) , \tag{1.4.4}$$

respectively. The involution of charges H_n in (1.4.3) reveals then the integrability of the system [269–274]. This fact implies that these nonlinear equations possess a thoroughly symmetric structure.

It is in this context where Integrable Systems meet 3D GR: Through the imposition of suitable boundary conditions for the gravitational field, the involution of the infinite conserved charges of the treated integrable hierarchy translates to the infinite-dimensional abelian asymptotic symmetries of gravitational charges devoid of central extensions.

In holography, the above means that sectors of the asymptotic dynamics of 3D GR will be described by an integrable hierarchy, i.e., the Einstein equations of motion will be shown equivalent to the aforementioned system and the global symmetries of the putative dual field theory satisfies the classical algebra (1.4.3).

Recent works have shown indications about the holographic relationship between the asymptotic nature of 3D GR and Integrable Systems. On AdS_3, motivated by articles [197, 198], Pérez, Tempo, and Troncoso introduced suitable field-dependent asymptotic behavior of the Lagrange multipliers of the Chern-Simons 1-form gauge connection, so at the boundary, gravitational excitations were governed by the KdV hierarchy [275], from where, exploiting its anisotropic scaling and the duality between the high and low-temperature behavior, a compatible Cardy formula for the entropy was found. Moreover, its Hawking-Page phase transition was analyzed in [276]. Similarly, the holographic connection with the Benjamin-Ono [277], the KdV/MKdV [278] and, for the case of gravity coupled to spin-N fields, the Boussinesq hierarchy [279], was also established.

On the other hand, in the context of asymptotically flat spacetimes, the flat analog of the KdV hierarchy [275] was found in [280].

1.5 Nonrelativistic holography and General Relativity

Lifshitz symmetry appears in physical systems that are invariant under the following anistropic scaling

$$t \to \lambda^z t\,, \quad x^i \to \lambda x^i\,, \quad i = 1, 2, \dots\,, \tag{1.5.1}$$

where z is known as the dynamical exponent. When $z = 1$ and $z = 2$ it is possible to enhance the symmetry of the spacetime in order to include the Lorentz and Galilean group, respectively, while for others z, boost invariance is explicitly broken.

We have four motivations to discuss this type of holography. First, this symmetry allows us to explore holographic aspects beyond AdS [281–289] on account with its nontrivial presence in black hole physics [290–296]; secondly, due its intimate connection with integrable hierarchies in 3D GR, as shown in [275–277]; thirdly, its appearance in a variety of condensed matter systems, such as heavy fermions [297, 298], semiconductors [299] and quantum critical phenomena [300] (thus bringing the possibility to address holography in nonrelativistic set-ups); and lastly, it allows the computation of quantities of physical interest, e.g., (entanglement) entropy [301–304], transport properties [305–314] and quasinormal modes [315], in analogy to Virasoro and BMS.

It is in the context of 3D GR where we can concretize holographic relationships with 2D anisotropic scaling field theories, since Lifshitz behavior emerge as global symmetries of the putative dual theory [275, 277, 278, 280, 316, 317], which are succesfully generalized to higher-spin gravity [275, 318–325].

2D Lifshitz-invariant models are known to appear in 2D (anti) ferromagnet systems [326, 327] and in the chiral Potts model [328]. One appealing attribute of these lower-dimensional models are the duality that they exhibit between the high and low-temperature behavior [329]. If a gap energy separates both regimes, this duality allows an account for the asymptotic growth of states, leading to a compatible Cardy-like formula for the entropy that depends on the dynamical exponent z, where for $z = 1$, the entropy reduces to the standard one of conformal field theory.

The anisotropic version [329] not only tends to tie with integrable hierarchies, as said, but with number theory as well [277].

1.6 Aim of this thesis

According to the previous discussion, the general objective of this thesis is to expand the "landscape" of GR on AdS_3 through the choice of a new set of suitable asymptotic boundary conditions. By characterizing its asymptotic symmetries, we will address this through two examples.

Because in 3D the Chern-Simons equations of motion are a zero-curvature condition (see more in Eq. (2.2.1)), in article [330] we report on a family of boundary conditions labeled on a single integer, N, $\bar{N}$, that reveals the equivalence between first-order Einstein equations and the Ablowitz-Kaup-Newell-Segur (AKNS) system [234], which comprises a large number of well known integrable hierarchies, e.g. the KdV, MKdV, SG, and NLS. For example, for $N = 1$ and odd values of N, and after some settings, we obtain the Brown-Henneaux [47] and KdV-type boundary conditions [275], respectively.

In integrable models, a nontrivial task is to find its conserved functionals and prove its involution. One of the main advantages in the zero-curvature description of spacetime dynamics,

$$f_{t\phi} = \partial_t a_\phi - \partial_\phi a_t + [a_t, a_\phi] = 0\,, \tag{1.6.1}$$

lies in the ability to employ the trace formula [331] in order to explicitly reveal the aforementioned quantities. From the gravity side, it allows to relate boundary conditions, the gauge fields a that generates the zero-curvature equations of motion, and conserved charges in a single formula. Particularly, it is described in Eq. (2.1.20), and treated in detail in Appendix C.3.

Thus, the integrability of the AKNS system is mapped to an abelian infinite-dimensional asymptotic symmetry algebra of gravitational charges. By characterizing the conjugacy classes of the angular holonomy, particle sources and (extremal) black hole configurations were shown attainable from the asymptotic behavior. Hence, local excitations of black holes obey several integrable equations. Summarizing, integrable systems were provided with a gravitational account, and viceversa, in such a way that the properties of one flowed into the properties of the other.

In [332], we formulated the anisotropic chiral boson description [333] as a dual field theory of AdS_3 GR through suitable boundary conditions that induce Lifshitz symmetry. We show that this theory corresponds to local excitations of the gravitational field at the near-horizon sector of a Rindler spacetime. Due to the fractional statistics of this theory, with help of bosonization elements, we construct a fermionic quantum charge operator. By means of the Kubo formula [334, 335], we compute a two-terminal conductance at its DC limit. As a consequence of its symmetry, the latter depends explicitely on the dynamical exponent z. Additionally, we conclude that chiral bosons stores DC information of the boundary electric current, changing the charges of the system. These features can be recognized as the lower-dimensional analogue of 4D gravitational memory effect [82, 336].

1.7 Outline of this thesis

This thesis is divided in two parts. The connection between General Relativity on AdS_3 with Integrable Systems is explored in the first part, while the connection with Condensed Matter Theory and Lifshitz symmetry is treated in the second one.

In Chapter 2, we introduce the AKNS integrable model and we prove its integrability, according to [234, 267, 268].

In Chapter 3 we review the role of asymptotic boundary conditions in the Chern-Simons formulation of AdS_3 GR. As an example, we treat the Brown-Henneaux case in detail.

In Chapter 4, through the choice of consistent AKNS-type boundary conditions, we connect AdS_3 GR with the aforementioned integrable hierarchy.

This finish the first part of this thesis.

In Chapter 5 we review aspects of linear response theory and we obtain the Kubo formula.

In Chapter 6 we review aspects regarding the anisotropic chiral boson theory. We emphasize its $u(1)$ symmetry, its anyonic nature and we obtain that the associated $u(1)$ Noether charges can be interpreted as fermionic quantities.

In Chapter 7 we introduce novel asymptotic boundary conditions that induces Lifshitz symmetry. Through the Hamiltonian reduction of Chern-Simons theory, anisotropic chiral excitations (with an external source) effectively describe gravitational boundary degrees of freedom. We use the Kubo formula to obtain a two-terminal conductance that depends on the anisotropic dynamical exponent z. We geometrize this result and we understand how can be interpreted as 3D memory effect.

This finish the second part of this thesis.

In Chapter 7 we conclude this thesis work and we discuss future prospects.

Part I

Integrable Systems and General Relativity

Chapter 2

The AKNS system

As we have reviewed in the introduction, the robustness and diversity of methods for solving integrable nonlinear partial differential equations [238–242, 244–248, 250–263] are of vital significance for a wide variety of physics areas [243], and the AdS/CFT correspondence as well [337]. In particular, in this chapter we will focus our attention on the Ablowitz-Kaup-Newell-Segur (AKNS) system, reported originally in article [234]; a $1+1$ integrable model that comprises several famous differential equations, such as the Korteweg-de Vries (KdV), Modified Korteweg-de Vries (MKdV), Sine-Gordon (SG), and nonlinear Schrödinger (NLS) equations into one single formalism. Its integrability was manifested by means of the Inverse Scattering Transformation (IST) [234]. However, as is usually the case with several integrable hierarchies, the AKNS model admits a Lax representation and a zero curvature formulation. The latter will bring us the key relationship for its future geometrization in the context of 3D GR, as it shows an intimate connection with the $sl(2,\mathbb{R})$ Lie algebra.

2.1 Aspects of the system

2.1.1 The model

In this section, the AKNS system is postulated. Following recursive methods, we explicitly construct the first three hierarchies associated to this system.

With coordinates t and ϕ, where $0 \leq \phi \leq 2\pi$, the (Wick-rotated version of the) AKNS model is given by the following system of nonlinear partial differential equations

$$\dot{r} + \frac{1}{\ell}\left(C' - 2rA - 2\xi C\right) = 0\,, \tag{2.1.1a}$$

$$\dot{p} + \frac{1}{\ell}\left(B' + 2pA + 2\xi B\right) = 0\,, \tag{2.1.1b}$$

$$A' - pC + rB = 0\,, \tag{2.1.1c}$$

for dynamical functions $r = r(t,\phi)$ and $p = p(t,\phi)$, where $u' \equiv \partial_\phi u$, $\dot{u} \equiv \partial_t u$, ℓ is a constant with dimension of length (which in the GR side will correspond to the AdS_3 radius), ξ a constant without dimensions referred in the literature as *spectral parameter* and A, B and C are composite arbitrary functions of the fields $r(t,\phi)$ and $p(t,\phi)$ that has to be specified. In order to construct the AKNS conserved densities, we will assume the fields are identified in 0 and 2π, i.e., $r(t,0) = r(t,2\pi)$ and $p(t,0) = p(t,2\pi)$.

A simple method to find solutions of the previous system consist to perform a polynomial expansion in ξ for the functions A, B and C as following

$$A = \sum_{n=0}^{N} A_n \xi^{N-n}, \quad B = \sum_{n=0}^{N} B_n \xi^{N-n}, \quad C = \sum_{n=0}^{N-n} C_n \xi^{N-n}\,, \tag{2.1.2}$$

where N is an arbitrary positive integer. Replacing in (2.1.1a), (2.1.1b) and (2.1.1c), respectively, we obtain the following set of dynamical equations

$$\dot{r} = \frac{1}{\ell}\left(-C_N' + 2rA_N\right), \quad \dot{p} = \frac{1}{\ell}\left(-B_N' - 2pA_N\right)\,, \tag{2.1.3}$$

while every coefficients of the expansion (2.1.2) satisfy the following recursive relations

$$A_n' = pC_n - rB_n\,, \tag{2.1.4a}$$

$$B_{n+1} = -\frac{1}{2}B_n' - pA_n\,, \tag{2.1.4b}$$

$$C_{n+1} = \frac{1}{2}C_n' - rA_n\,, \tag{2.1.4c}$$

along with conditions $B_0 = C_0 = 0$. The deduction of the above relationships are performed in Appendix C.1.

It is possible to explicitly construct every term of the previous recursion. If we replace $n = 0$ in (2.1.4a), we readily find $A_0' = 0$. Integrating, we obtain that A_0 is an arbitrary constant. To simplify the discussion, let us assume for the moment that this constant is fixed to the unity. Then, readily $B_1 = -p$. We can perform the same analysis and find $C_1 = -r$. Therefore, we find that $A_1' = 0$, from where we obtain a different constant after integration. For the sake of simplicity, fixing the latter and subsequent integration constants to be zero, we can find B_2, C_2 and so on.

However, if we let all integration constants survive, the analysis will be similar (see Appendix C.2). Hence, for the forthcoming analysis, we will consider the aforementioned homogeneous choice.

We write here the first three coefficients

$$\begin{aligned} &A_0 = 1, \quad A_1 = 0\,, \quad A_2 = -\frac{1}{2}pr\,, \\ &B_0 = 0\,, \quad B_1 = -p\,, \quad B_2 = \frac{1}{2}p'\,, \\ &C_0 = 0\,, \quad C_1 = -r\,, \quad C_2 = -\frac{1}{2}r'\,. \end{aligned} \tag{2.1.5}$$

With them, we can explicitly write the first three dynamical equations according to (2.1.3) and recover the mentioned integrable hierarchies. For $N = 1$, we replace its associated coefficients, A_1, B_1 and C_1, and we obtain the chiral boson equations

$$\dot{p} = \frac{1}{\ell}p', \quad \dot{r} = \frac{1}{\ell}r'\,, \tag{2.1.6}$$

while for $N = 2$ we arrive to the following nonlinear differential equations

$$\dot{p} = \frac{1}{\ell}\left(p^2 r - \frac{1}{2}p''\right)\,, \quad \dot{r} = \frac{1}{\ell}\left(-pr^2 + \frac{1}{2}r''\right)\,; \tag{2.1.7}$$

and for $N = 3$ we get the following nonlinear system with third derivatives

$$\dot{p} = \frac{1}{\ell}\left(-\frac{3}{2}pp'r + \frac{1}{4}p'''\right)\,, \quad \dot{r} = \frac{1}{\ell}\left(-\frac{3}{2}prr' + \frac{1}{4}r'''\right)\,. \tag{2.1.8}$$

It is possible to compute subsequent equations for $N > 3$, however, let us recover different integrable equations at this point. Case $N = 1$ correspond to the chiral boson system, $N = 2$ to the Wick rotated nonlinear Schrödinger equation[1], and $N = 3$ recovers KdV and MKdV when $r = -1$ (or $p = -1$) and $p = -r$, respectively.

As we have said, the Sine-Gordon (SG) equation is also included in this formalism. To make it appear explicitly, let us redefine the fields as

$$p = -\frac{1}{2}u'\,, \quad r = \frac{1}{2}u'\,, \quad A = \frac{1}{4\xi}\cos u\,, \quad B = C = \frac{1}{4\xi}\sin u\,, \tag{2.1.11}$$

[1] The argument is the following. The nonlinear Schrödinger equation and its complex conjugate equation are given by

$$i\dot{p} = \frac{1}{\ell}\left(p^2 p^* - \frac{1}{2}p''\right)\,, \quad -i\dot{p}^* = \frac{1}{\ell}\left(p^{*2} p - \frac{1}{2}p^{*\prime\prime}\right)\,. \tag{2.1.9}$$

Performing a Wick rotation $t \to i\tau$, we obtain

$$\dot{p} = \frac{1}{\ell}\left(p^2 p^* - \frac{1}{2}p''\right)\,, \quad \dot{p}^* = \frac{1}{\ell}\left(-p^{*2} p + \frac{1}{2}p^{*\prime\prime}\right)\,. \tag{2.1.10}$$

Labeling $p^* = r$, we arrive to (2.1.7). We stress that the functions p and r must be real, so as a final step, we must impose $r \in \mathbb{R}$.

at the level of equations (2.1.1), yielding

$$\dot{u}' = \frac{1}{\ell} \sin u\,. \tag{2.1.12}$$

Aspects regarding the geometrization of this equation will be given in future works.

2.1.2 Conserved quantities as a recursive construction

As said in the introduction, the identification of conserved charges is crucial to prove the integrability of any system of nonlinear partial differential equations. Following [234], here we present a recursive deduction of these quantities.

From the equations of motion (2.1.1b) and (2.1.1a), it is possible to write B and C as

$$B = \frac{1}{2\xi}\left(-2pA - B' - \ell\dot{p}\right)\,, \quad C = \frac{1}{2\xi}\left(-2rA + C' + \ell\dot{r}\right)\,. \tag{2.1.13}$$

Using A' given in (2.1.1c), and after an integration by parts, we arrive to

$$A' = \frac{1}{2\xi}\frac{\partial}{\partial t}\left(\ell pr\right) + \frac{1}{2\xi}\frac{\partial}{\partial\phi}\left(pC + rB\right) - \frac{1}{2\xi}\left(p'C + r'B\right)\,. \tag{2.1.14}$$

Repeating the process, we have the following recursive series

$$\begin{aligned} A' = &\frac{\ell}{2\xi}\frac{\partial}{\partial t}\left(pr\right) + \frac{1}{2\xi}\frac{\partial}{\partial\phi}\left(pC + rB\right) - \frac{1}{\left(2\xi\right)^2}\frac{\partial}{\partial\phi}\left(p'C - r'B\right) \\ &- \frac{1}{\left(2\xi\right)^2}\frac{\partial}{\partial\phi}\left(2prA\right) - \frac{\ell}{\left(2\xi\right)^2}\frac{\partial}{\partial t}\left(p'r - pr'\right) + \frac{\ell}{\left(2\xi\right)^2}\left(\dot{p}'r + p\dot{r}'\right) \\ &- \frac{1}{\left(2\xi\right)^2}\left[\left(2p^2r - p''\right)C - \left(2pr^2 - r''\right)B\right]\,, \end{aligned} \tag{2.1.15}$$

which can be written in the following form,

$$A' = \frac{\ell}{2\xi}\frac{\partial}{\partial t}\left(pr\right) - \frac{\ell}{\left(2\xi\right)^2}\frac{\partial}{\partial t}\left(p'r - pr'\right) + \frac{\partial}{\partial\phi}\left(\ldots\right) + \ldots\,. \tag{2.1.16}$$

We encoded all terms with spatial derivatives in the first ellipsis, while lower powers in ξ in the second one. Integrating in ϕ between 0 and 2π, we obtain

$$A(2\pi) - A(0) = \frac{\ell}{2\xi}\frac{\partial}{\partial t}\int_0^{2\pi} d\phi\, pr - \frac{\ell}{\left(2\xi\right)^2}\frac{\partial}{\partial t}\int_0^{2\pi} d\phi\, \left(p'r - pr'\right)\,. \tag{2.1.17}$$

The total derivatives on the angular component vanishes because p and r are identified when $\phi = 0, 2\pi$, as said below Eq. (2.1.1). Besides, the latter identification implies that $A(2\pi) = A(0)$, and since ξ is arbitrary, we obtain a sequence of globally conserved quantities.

Thus, the first nontrivial densities are

$$\mathcal{H}_2 = -pr, \quad \mathcal{H}_3 = \frac{1}{4}\left(p'r - pr'\right), \quad \dots, \tag{2.1.18}$$

where $H_n = \int d\phi\, \mathcal{H}_n$. An explicit list of these quantities can be found in Appendix C.5.

2.1.3 Integrability

In this section we address the integrability of the AKNS system. An important step is to establish the relationship between the coefficients A_n, B_n and C_n with the found conserved quantities H_n and its functional derivatives, $\delta H_n/\delta p$ and $\delta H_n/\delta r$. This will lead us to easily write the AKNS equations of motion as a bi-Hamiltonian system.

In Appendix C.3, it is established the aforementioned relationship,

$$A_n = \frac{n-1}{2}\mathcal{H}_n\,, \quad B_n = \frac{\delta H_{n+1}}{\delta r} \equiv \mathcal{R}_{n+1}\,, \quad C_n = \frac{\delta H_{n+1}}{\delta p} \equiv \mathcal{P}_{n+1}\,, \tag{2.1.19}$$

for $n \geq 1$ by means of the trace formula [331]

$$\frac{\delta}{\delta u}\int \operatorname{tr}\left(a_t \frac{\partial a_\phi}{\partial \xi}\right) d\phi = \xi^{-\gamma}\frac{\partial}{\partial \xi}\xi^{\gamma} \operatorname{tr}\left(a_t \frac{\partial a_\phi}{\partial u}\right) . \tag{2.1.20}$$

Here, $u = \begin{pmatrix} p \\ r \end{pmatrix}$, tr stands for the matrix trace, γ an arbitrary constant to be determined and a are 1-form gauge fields (this quantities will make sense when we write the AKNS system in the zero-curvature formulation, in Section 2.2). The aforementioned is a general tool in the zero-curvature formulation of Integrable Systems that allow us to establish a relationship between the coefficients of the polynomial expansion (2.1.2) and functional conserved densities $\mathcal{H}_n$. Succesful applications of the trace formula are, e.g., the N-AKNS [331], Kaup-Newell [338] and Camassa-Holm [339] hierarchies. For details, see Appendix C.3.

Thus, naturally we can write the AKNS system (2.1.3) in a bi-Hamiltonian fashion

$$\begin{pmatrix} \dot{r} \\ \dot{p} \end{pmatrix} = \mathcal{D}_1 \begin{pmatrix} \mathcal{R}_{N+1} \\ \mathcal{P}_{N+1} \end{pmatrix} = \mathcal{D}_2 \begin{pmatrix} \mathcal{R}_{N+2} \\ \mathcal{P}_{N+2} \end{pmatrix}, \tag{2.1.21}$$

where $\mathcal{D}_1$ and $\mathcal{D}_2$ are the following compatible Hamiltonian operators [253]

$$\mathcal{D}_1 = \frac{1}{\ell}\begin{pmatrix} -2r\partial_\phi^{-1} r & -\partial_\phi + 2r\partial_\phi^{-1} p \\ -\partial_\phi + 2p\partial_\phi^{-1} r & -2p\partial_\phi^{-1} p \end{pmatrix}, \quad \mathcal{D}_2 = \frac{1}{\ell}\begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}. \tag{2.1.22}$$

The Poisson bracket of the first $\{,\}_1$ and second $\{,\}_2$ Hamiltonian structures are defined then as

$$\{F[r,p],G[r,p]\}_{1,2}=\int d\phi\,\begin{pmatrix}\frac{\delta F}{\delta r} & \frac{\delta F}{\delta p}\end{pmatrix}\mathcal{D}_{1,2}\begin{pmatrix}\frac{\delta G}{\delta r}\\ \frac{\delta G}{\delta p}\end{pmatrix}\,, \tag{2.1.23}$$

respectively.

The involution of charges is a consequence of the bi-Hamiltonian formulation and the compatibility of operators $\mathcal{D}_{1,2}$ [253, 267, 268] (see Section 1.4 for a detailed review). Define the transpose operator $\mathfrak{M}^{\mathrm{t}}=\mathcal{D}_1^{-1}\circ\mathcal{D}_2$, which satisfies the following recursion relation (recall Eq. (1.4.2))

$$\frac{\delta H_{n+1}}{\delta u}=\mathfrak{M}^{\mathrm{t}}\left(\frac{\delta H_{n+2}}{\delta u}\right)\,,\quad n\in\mathbb{N}_0\,, \tag{2.1.24}$$

then it is possible to prove that the conserved quantities H_n *Poisson-commute* with each other under the first and second brackets (2.1.23), i.e.,

$$\dot{H}_n=\{H_n,H_m\}_{1,2}=0\,,\quad n=1,2,\dots\,. \tag{2.1.25}$$

As showed in Appendix C.4, since all charges are in involution, we have thus proven the integrability of the AKNS system.

2.2 Zero-curvature formulation

Integrable systems can be written in different forms. One of them, and the AKNS is not the exception, is the zero-curvature formulation,

$$f_{t\phi}\equiv\partial_t a_\phi-\partial_\phi a_t+[a_t,a_\phi]=0\,, \tag{2.2.1}$$

where a_ϕ and a_t are some connections spanned in a particular Lie algebra. In the context of AdS_3 GR, the dynamics of the gravitational field is equivalent to (2.2.1), since Einstein equations can be written as two independent copies of the zero-curvature equation in the Chern-Simons formulation of gravity

$$F^{\pm}=d\mathcal{A}^{\pm}+\mathcal{A}^{\pm\,2}=0\,, \tag{2.2.2}$$

where $\mathcal{A}^{\pm}=\mathcal{A}^{\pm}_\mu(t,r,\phi)dx^\mu$, with $\mu=t,r,\phi$. After a suitable choice of function $b_\pm$ in the gauge transformation

$$a^{\pm}=b_\pm^{-1}\left(d+\mathcal{A}^{\pm}\right)b_\pm\,, \tag{2.2.3}$$

it is possible to gauge-away the radial dependence of the fields [91], so Eq. (2.2.2) turns equivalent to (2.2.1).

Regarding the deduction of Eq. (2.2.1, the zero-curvature equation of motion arise as a compatibility condition of a more fundamental auxiliary system. Consider the following linear problem

$$\partial_\phi U = U a_\phi , \quad \partial_t U = U a_t \,, \tag{2.2.4}$$

where U is a column vector with dependence on t, ϕ and ξ. Equation (2.2.1) can be obtained by the following procedure: Derive with respect to the time and the angle the first and second equation of (2.2.4), respectively,

$$\partial_t \partial_\phi U = \partial_t U a_\phi + U \partial_t a_\phi \,, \quad \partial_\phi \partial_t U = \partial_\phi U a_t + U \partial_\phi a_t \,. \tag{2.2.5}$$

Substracting them we find

$$U \left(\partial_t a_\phi - \partial_\phi a_t \right) + \partial_t U a_\phi - \partial_\phi U a_t = 0 \,. \tag{2.2.6}$$

Replacing (2.2.4), we finally obtain

$$U f_{t\phi} = 0 \,. \tag{2.2.7}$$

If $U \neq 0$, we arrive to (2.2.1).

Regarding to the AKNS system, it can be obtained if we consider the following gauge connections

$$a_\phi = -2\xi L_0 - p L_1 + r L_{-1} \,, \quad a_t = \frac{1}{\ell} \left(-2A L_0 + B L_1 - C L_{-1} \right) \,, \tag{2.2.8}$$

where $L_{0,\pm 1}$ are the generators of the $sl(2,\mathbb{R})$ Lie algebra,

$$[L_n, L_m] = (n - m) L_{n+m} \,, \tag{2.2.9}$$

for $n = 0, \pm 1$ (see Appendix A for further properties of the $sl(2,\mathbb{R})$ Lie algebra). As emphasized in the introduction, one of the advantages to write integrable equations as a zero-curvature formulation lies in the possibility to construct its conserved functionals by means of the trace formula (2.1.20).

As we will see in next chapters, the zero-curvature formulation of Integrable Systems finds a natural application on AdS_3 GR, since the Einstein equations can be written as a zero-curvature equation of motion (2.2.1), as we previously emphasized. From the holographic perspective, the choice of the 1-form gauge connection will correspond to boundary conditions for the gravitational side, from where it will be possible to obtain the AKNS system as the asymptotic dynamics of

AdS_3 GR . This map will be addressed in Chapter 4.

Chapter 3

The role of boundary conditions in three-dimensional General Relativity

3.1 Preliminary discussion

In three dimensions, GR is known is abscent of propagating degrees of freedom, such as bulk gravitational waves. Physically, this is due the vanishing of the Weyl tensor, leaving the theory as a topological field one. We can prove this argument by counting the numer of independent components that the Riemann tensor has, since it can be written in terms of the Weyl tensor as

$$R_{\mu\nu\lambda\rho} = W_{\mu\nu\lambda\rho} + \frac{2}{D-2}\left(g_{\lambda[\mu}R_{\nu]\rho} + R_{\lambda[\mu}g_{\nu]\rho}\right) - \frac{2}{(D-1)(D-2)}g_{\lambda[\mu}g_{\nu]\rho}R\,. \tag{3.1.1}$$

Due the antisymmetry properties and the Bianchi identities, the Riemann tensor has $D^2(D^2-1)/12$ independent components. Therefore, the Weyl tensor has

$$\frac{D^2\left(D^2-1\right)}{12} - \frac{D(D+1)}{2} = \frac{D(D+1)(D+2)(D-3)}{12}\,, \tag{3.1.2}$$

independent components, vanishing identically in $D=3$, proving that in this dimension there are no gravitational waves. Another consequence of this result lies in the ability to classify the geometry of each 3D solution in the vacuum with the value of the cosmological constant Λ.

Taking the trace of vacuum Einstein equations, we obtain that $R = 6\Lambda$. Replacing this value on the latter equations, we can fix the Ricci tensor to $R_{\mu\nu} = 2\Lambda g_{\mu\nu}$. Because the Weyl tensor vanishes, it is possible to prove that the Riemann tensor is proportional to the cosmological constant,

$$R_{\mu\nu\lambda\rho} = \Lambda\left(g_{\lambda\mu}g_{\rho\nu} - g_{\lambda\nu}g_{\rho\mu}\right)\,. \tag{3.1.3}$$

Therefore, we conclude that every solution of 3D Einstein equations can be classified as following,

- $\Lambda > 0$ correspond to solutions that are locally dS_3,
- $\Lambda = 0$ correspond to solutions that are locally Mink_3,
- $\Lambda < 0$ correspond to solutions that are locally AdS_3.

This is the reason why the discovery of the BTZ black hole on AdS_3 [95] was surprising, since it was believed that no black hole solution was allowed in this theory.

Although 3D GR is a trivial theory from the bulk perspective, characterizing asymptotic degrees of freedom will be given by fixing boundary conditions [91]. Thus, *all* of the dynamical content of the theory is captured in the election of suitable boundary conditions of the gravitational field. The criteria to choosing them, originally reported in [54], are:

(a) To ensure the differentiability of the action principle,

(b) generators of conserved charges must be finite,

(c) includes gravitational solutions of physical interest, such as black holes.

In this chapter we review the asymptotic aspects of three-dimensional General Relativity with negative cosmological constant by using the Chern-Simons formulation [92, 93]. The Brown–Henneaux example is treated in detail in order to illustrate the previous criteria, and to prepare the necessary concepts of Chapter 4.

3.2 Three-dimensional General Relativity

In the vacuum and with negative cosmological constant, General Relativity can be described in terms of two copies of the Chern-Simons action [92, 93]

$$I = I_{\mathrm{CS}}[\mathcal{A}^{+}] - I_{\mathrm{CS}}[\mathcal{A}^{-}]\,, \tag{3.2.1}$$

where the Chern-Simons action is defined as

$$I^{\pm}_{\mathrm{CS}}[\mathcal{A}^{\pm}] = \frac{K}{4\pi}\int_{\mathcal{M}}\left\langle \mathcal{A}^{\pm}d\mathcal{A}^{\pm} + \frac{2}{3}\mathcal{A}^{\pm 3}\right\rangle\,. \tag{3.2.2}$$

K is known as the *level* of the theory and is defined as $K = \ell/4G$, where ℓ is the AdS_3 radius and G the three-dimensional Newton constant. $\mathcal{M} = \Sigma\times\mathbb{R}$ is a manifold in three dimensions, with Σ a two-dimensional spacelike manifold with coordinates (r,ϕ), with $0 \le r < \infty$ and $0 \le \phi \le 2\pi$, and $\mathbb{R}$ correspond to time. Wedge product is understood and $\langle . \rangle$ stands for the invariant nondegenerate bilinear form.

For GR on AdS$_3$, the 1-form gauge connection $\mathcal{A}$ is spanned in the Lie algebra $\mathfrak{g} = \mathfrak{g}_+ + \mathfrak{g}_-$, where $\mathfrak{g}_\pm$ denotes the two independent copies of $sl(2,\mathbb{R})$. For Minkowski space, the gauge connection is spanned in the $iso(2,1)$ Lie algebra, while for dS$_3$ in $sl(2,\mathbb{C})$.

The connection splits as $\mathcal{A} = \mathcal{A}^+ + \mathcal{A}^-$, and it is related with the triad $e^a = e^a_\mu dx^\mu$ and the spin connection $\omega^a_\mu dx^\mu$ as following

$$\mathcal{A}^\pm = \omega \pm \frac{e}{l} \,. \tag{3.2.3}$$

The first-order Einstein equations reads as the zero-curvature condition

$$F^\pm \equiv d\mathcal{A}^\pm + \mathcal{A}^{\pm\,2} = 0 \,. \tag{3.2.4}$$

This is the same equation where we can write the AKNS system of Chapter 2, but now there are two (decoupled) $sl^\pm(2,\mathbb{R})$ copies.

3.2.1 Constraint analysis and boundary terms

Here we review the constraint structure of Chern–Simons theory, following [340, 341]. Further details can be found in Appendices D.1 and D.2.

Performing a $2+1$ splitting $\mathcal{A}^\pm_\mu = (\mathcal{A}^\pm_0, \mathcal{A}^\pm_i)$, in components, we can write the Chern–Simons action as

$$I^\pm_H = -\frac{K}{4\pi}\int_{\Sigma\times\mathbb{R}} dt d^2x \epsilon^{ij} \left\langle \mathcal{A}^\pm_i \dot{\mathcal{A}}^\pm_j - \mathcal{A}^\pm_0 F^\pm_{ij} \right\rangle + \mathcal{B}^\pm \,, \tag{3.2.5}$$

where $\mathcal{B}^\pm$ is a boundary term chosen such that it ensures the differentiability of the action $I^\pm_H$, $d^2x = drd\phi$ and $\epsilon^{ij} \equiv \epsilon^{0ij}$. We can see that $\mathcal{A}^\pm_0$ is a Lagrange multiplier and $F^\pm_{ij}$ a constraint of the theory. Performing an infinitesimal variation of (3.2.5), we obtain

$$\delta I^\pm_H = -\frac{K}{4\pi}\int dt d^2x \; \epsilon^{ij} \Big\langle 2F^\pm_{i0}\,\delta\mathcal{A}^\pm_j - \delta\mathcal{A}^\pm_0\, F^\pm_{ij} + \partial_t \left(\mathcal{A}^\pm_i\,\delta\mathcal{A}^\pm_j\right) - \partial_i\left(2\mathcal{A}^\pm_0\,\delta\mathcal{A}^\pm_j\right) \Big\rangle + \delta\mathcal{B}^\pm \;, \tag{3.2.6}$$

Vanishing the terms that contributes to the equations of motion, the action (3.2.6) is just surface integrals

$$\delta I^\pm_H = -\frac{K}{4\pi}\int dt d^2x \; \epsilon^{ij} \Big\langle \partial_t \left(\mathcal{A}^\pm_i\,\delta\mathcal{A}^\pm_j\right) - \partial_i\left(2\mathcal{A}^\pm_0\,\delta\mathcal{A}^\pm_j\right) \Big\rangle + \delta\mathcal{B}^\pm \;. \tag{3.2.7}$$

Assuming the fields vanishes for t_1 and t_2, as following

$$\delta\mathcal{A}^\pm_j\,(t_1, r, \phi) = 0 \,, \quad \text{and} \quad \delta\mathcal{A}^\pm_j\,(t_2, r, \phi) = 0 \,,$$

the surface integral reads

$$\delta I_H^{\pm} = -\frac{K}{2\pi}\int dt d^2x \left\langle \partial_r \left(\epsilon^{r\phi}\mathcal{A}_0^{\pm}\,\delta\mathcal{A}_{\phi}^{\pm}\right) + \partial_{\phi}\left(\epsilon^{\phi r}\mathcal{A}_0^{\pm}\,\delta\mathcal{A}_r^{\pm}\right)\right\rangle + \delta\mathcal{B}^{\pm}\ . \tag{3.2.8}$$

Choosing manifold orientation $\epsilon^{r\phi} = 1$, and angular periodicity in the fields, the angular surface integral also vanishes. Using Stokes' theorem, we arrive then to the following surface integral

$$\delta I_H^{\pm} = \frac{K}{2\pi}\int_{\partial\mathcal{M}} dt d\phi \left\langle \mathcal{A}_0^{\pm}\,\delta\mathcal{A}_{\phi}^{\pm}\right\rangle + \delta\mathcal{B}^{\pm}\ . \tag{3.2.9}$$

Because the action principle must be differentiable, we find that the infinitesimal variation of the boundary term is

$$\delta\mathcal{B}^{\pm} = -\frac{K}{2\pi}\int_{\partial\mathcal{M}} dt\, d\phi \left\langle \mathcal{A}_0^{\pm}\,\delta\mathcal{A}_{\phi}^{\pm}\right\rangle. \tag{3.2.10}$$

We can see from (3.2.10) that specific asymptotic behavior of the fields is necessary in order to integrate the boundary term $\delta\mathcal{B}^{\pm}$. The choice is subtle, since $\mathcal{A}_0$ is a Lagrange multiplier and is not fixed by the equations of motion due the presence of first-class constraints [52].

According to [91], if we perform a gauge transformation

$$a^{\pm} = b_{\pm}^{-1}\left(d + \mathcal{A}^{\pm}\right) b_{\pm}\,, \tag{3.2.11}$$

where $b_{\pm}(r)$ its a particular global gauge parameter, it will capture the radial dependence of the connection $\mathcal{A}^{\pm}$.

The gauge fields now are $a^{\pm} = a_t^{\pm}(t,\phi)dt + a_{\phi}^{\pm}(t,\phi)d\phi$, i.e., they do not depend on the radial coordinate r. Thus, the boundary term reads

$$\delta\mathcal{B}^{\pm} = -\frac{K}{2\pi}\int_{\partial\mathcal{M}} dt\, d\phi \left\langle a_t^{\pm}\delta a_{\phi}^{\pm}\right\rangle. \tag{3.2.12}$$

Suppose we have boundary conditions of the form $a_{\phi}^{\pm} = \mathcal{L}^{\pm}(t,\phi)L_0$ and $a_t^{\pm} = \frac{1}{\ell}\mu^{\pm}(t,\phi)L_0$, where $\mu^{\pm}$ is a composite function of $\mathcal{L}^{\pm}(t,\phi)$. With this election, we do not know how to integrate the associated surface integral (3.2.12), since it is unknown the explicit field-dependent form of $\mu^{\pm}$ [197, 198] and [275]. Following the latter reference, imposing $\mu^{\pm} = \frac{\delta H^{\pm}}{\delta a_{\phi}^{\pm}}$, the boundary term (3.2.12) readily integrates. Moreover, the equation of motion is $\dot{\mathcal{L}}^{\pm} = \pm\frac{1}{\ell}\partial_{\phi}\frac{\delta H^{\pm}}{\delta a_{\phi}^{\pm}}$. This is a Hamiltonian approach of the zero-curvature equation of motion. The latter election shows the connection between AdS_3 GR with Integrable Systems.

3.2.2 Generator of charges

Using the approach of Jackiw [342], the constraint analysis is performed in this section.

In the Chern-Simons formulation, gauge transformations can be seen as the following infinitesimal transformation (we discard the $\pm$-notation although the forthcoming results stands for both copies)

$$\delta\mathcal{A} = d\Lambda + [\mathcal{A}, \Lambda]\,. \tag{3.2.13}$$

According to [340], gauge transformations are generated by the constraint of the theory,

$$G\left[\Lambda\right] = \frac{K}{4\pi}\int_{\Sigma} d^2x\ \epsilon^{ij}\left\langle \Lambda F_{ij}\right\rangle + Q[\Lambda]\,, \tag{3.2.14}$$

with a boundary term chosed such that ensures the differentiability of the constraint. Regarding this boundary term, we will take care of it the next page.

It is necessary to compute the symplectic structure, namely, the Poisson bracket of the theory. For this purpose, consider the action (3.2.5)

$$I_H = -\frac{K}{4\pi}\int_{\Sigma\times\mathbb{R}} dtd^2x\ \epsilon^{ij} g_{ab}\left(\mathcal{A}_i^a\dot{\mathcal{A}}_j^b - \mathcal{A}_0^a F_{ij}^b\right)\,, \tag{3.2.15}$$

where $g_{ab} = \langle L_a, L_b\rangle$ is the Killing metric of $sl(2,\mathbb{R})$. Define

$$l_c^j(x) = \frac{K}{4\pi}\epsilon^{jk} g_{cd}\mathcal{A}_k^d(x). \tag{3.2.16}$$

Then, the 2-symplectic structure reads

$$\sigma_{ab}^{ij}(x,x') = \frac{\delta l_a^j(x)}{\delta\mathcal{A}_i^b(x')} - \frac{\delta l_b^i(x)}{\delta\mathcal{A}_j^a(x')} = -\frac{K}{2\pi}\epsilon^{ij} g_{ab}\delta^{(2)}(x-x') \tag{3.2.17}$$

where it is possible to explicitly compute its inverse, $J_{ij}^{ab}(x,x')$

$$J_{lj}^{db}(x,x') = -\frac{2\pi}{K} g^{bd}\epsilon_{jl}\delta^{(2)}(x-x')\,. \tag{3.2.18}$$

Therefore, the Poisson bracket of the theory is

$$\left\{\mathcal{A}_i^a(x), \mathcal{A}_j^b(x')\right\} \equiv J_{ij}^{ab}(x,x') = \frac{2\pi}{K}\epsilon_{ij} g^{ab}\delta^{(2)}(x-x')\,. \tag{3.2.19}$$

We can now compute the infinitesimal variation of the field $\mathcal{A}_i^a(x)$ as

$$\delta\mathcal{A}_i^a(x) = \frac{K}{4\pi}\int d^2x'\ \epsilon^{kl} g_{cd}\Lambda^c(x')\left\{\mathcal{A}_i^a(x), F_{kl}^d(x')\right\}\,. \tag{3.2.20}$$

By means of Poisson bracket (3.2.19) we find that indeed the constraint generates gauge transformations

$$\delta\mathcal{A}_i^a(x) = \{\mathcal{A}_i^a(x), G[\Lambda]\} = \partial_i\Lambda^a(x) + f^a{}_{bc}\mathcal{A}_i^b(x)\Lambda^c(x)\,. \tag{3.2.21}$$

Following [54], we will now take care of the boundary term $Q[\Lambda]$. By Noether's theorem, if $\delta\mathcal{A}_i^a$ is a symmetry of the theory, then a conserved charge G exists, and also conversely: If a conserved charge G exists, then an associated symmetry $\delta\mathcal{A}_i^a$ exists. If we compute the variation of the constraint, we see that it must be suplemented by a boundary term δQ in order to make it differentiable. We define then the improved generator $\delta\overline{G}$ as

$$\overline{G}[\Lambda] = G[\Lambda] + Q[\Lambda]\ , \tag{3.2.22}$$

where

$$\delta Q[\Lambda] = -\frac{K}{2\pi}\int_{\partial\mathcal{M}} d\phi\,\Big\langle \Lambda\,\delta a_\phi \Big\rangle\,. \tag{3.2.23}$$

Explicit calculations regarding the latter procedures can be found in Appendix D.2.2.

As said in the previous section, we must consider appropriate boundary conditions in order to make $\delta\mathcal{B}$ and now δQ integrable. This election is not trivial, since boundary conditions must also generate the asymptotic symmetry algebra.

3.3 Example: Brown–Henneaux boundary conditions

In this section we reconsider the previously discussed concepts by means of the celebrated Brown-Henneaux boundary conditions [47]. We will explicitly compute the boundary term (3.2.12), the charges (3.2.23), asymptotic symmetries and the algebra they fulfill, which will be two independent copies of Virasoro, with central extension $c = 3\ell/2G$.

The Brown-Henneaux asymptotic boundary conditions in the Chern–Simons formulation are

$$a_\phi^\pm = L_{\pm 1} - \frac{2\pi}{K}\mathcal{L}^\pm(t,\phi)L_{\mp 1}\,,\quad a_t^\pm = \pm\frac{1}{\ell}a_\phi^\pm\,. \tag{3.3.1}$$

The equations of motion are known as the chiral boson equation

$$\dot{\mathcal{L}}^\pm = \pm\frac{1}{\ell}\mathcal{L}^{\pm\prime}\,. \tag{3.3.2}$$

Any general solution of this equation will have the form $\mathcal{L}^\pm = \mathcal{L}^\pm(t-\phi)$.

In what follows we will find that these boundary conditions are suitable, namely they fulfill the criteria [54] of Section 3.1.

We computed the Hamiltonian action of 3D GR and we obtained the boundary term (3.2.12). With the Brown-Henneaux choice (3.3.1), the boundary term readily integrates as

$$\mathcal{B}^{\pm} = \mp \frac{K}{4\pi\ell} \int_{\partial\mathcal{M}} dt\, d\phi \, \left\langle a_\phi^{\pm\,2} \right\rangle \tag{3.3.3}$$

Therefore, we accomplished the first item of the criteria listed in Page 22. Now we must find the set of gauge transformations that preserve the form of the boundary conditions and calculate its symmetries at the boundary.

The 1-form gauge field $a^{\pm}$ is preserved if satisfies (3.2.13), where $\Lambda^{\pm}$ is an infinitesimal gauge parameter whose general form is

$$\Lambda^{\pm}\left[\alpha^{\pm}, \mu^{\pm}, \beta^{\pm}\right] = \alpha^{\pm} L_0 \pm \mu^{\pm} L_{\pm 1} \mp \beta^{\pm} L_{\mp 1}\,. \tag{3.3.4}$$

$\alpha^{\pm}$, $\mu^{\pm}$ and $\beta^{\pm}$ are arbitrary functions on the time and the angle.

The infinitesimal gauge transformation of the angular component is

$$\begin{aligned} \delta a_\phi^{\pm} = -\frac{2\pi}{K}\delta\mathcal{L}^{\pm} L_{\mp 1} = \left(-2\beta^{\pm} + \frac{4\pi}{K}\mu^{\pm}\mathcal{L}^{\pm} + \alpha^{\pm\prime}\right) L_0 \\ \pm \left(\frac{2\pi}{K}\alpha^{\pm}\mathcal{L}^{\pm} - \beta^{\pm\prime}\right) L_{\mp 1} \pm \left(\alpha^{\pm} + \mu^{\pm\prime}\right) L_{\pm 1}\,. \end{aligned} \tag{3.3.5}$$

The latter implies the following system of equations

$$\beta^{\pm} = \frac{1}{2}\alpha^{\pm\prime} + \frac{2\pi}{K}\mu^{\pm}\mathcal{L}^{\pm}\,, \quad \delta\mathcal{L}^{\pm} = \mp\left(\alpha^{\pm}\mathcal{L}^{\pm} - \frac{K}{2\pi}\beta^{\pm\prime}\right)\,, \quad \alpha^{\pm} = -\mu^{\pm\prime}\,. \tag{3.3.6}$$

We can replace the value of $\alpha^{\pm}$ and leave $\beta^{\pm}$ as a function of $\mu^{\pm}$, as promised,

$$\beta^{\pm} = -\frac{1}{2}\mu^{\pm\prime\prime} + \frac{2\pi}{K}\mu^{\pm}\mathcal{L}^{\pm}\,. \tag{3.3.7}$$

With this results, we can find that the infintesimal transformation of $\mathcal{L}^{\pm}$, which is given by

$$\delta\mathcal{L}^{\pm} = \pm\mathcal{D}^{\pm}\mu^{\pm}\,, \tag{3.3.8}$$

where $\mathcal{D}^{\pm}$ is the nontrivial operator

$$\mathcal{D}^{\pm} := \mathcal{L}^{\pm\prime} + 2\mathcal{L}^{\pm}\partial_\phi - \frac{K}{4\pi}\partial_\phi^3\,. \tag{3.3.9}$$

Transformation (3.3.8) is known as a *large* gauge transformation, or the *asymptotic symmetry* that preserve the form of the angular boundary condition. Therefore, $\Lambda^{\pm}$ reads

$$\Lambda^{\pm}\left[\mu^{\pm}\right] = -\mu^{\pm\prime} L_0 \pm \mu^{\pm} L_{\pm 1} \mp \left(-\frac{1}{2}\mu^{\pm\prime\prime} + \frac{2\pi}{K}\mu^{\pm}\mathcal{L}^{\pm}\right) L_{\mp 1}\,. \tag{3.3.10}$$

$\Lambda^{\pm}$ encodes the family of asymptotic permissible transformations that we found before.

On the other hand, invariance of the temporal boundary condition $a_t^{\pm}$ fulfills

$$\begin{aligned}\delta a_t^{\pm} = \mp\frac{2\pi}{K\ell}\delta\mathcal{L}^{\pm} L_{\mp 1} = \left(\pm\frac{1}{\ell}\mu^{\pm\prime\prime} - \dot{\mu}^{\pm\prime}\right) L_0 + \frac{2\pi}{K\ell}\Big[-\mu^{\pm\prime}\mathcal{L}^{\pm} \mp \ell\left(\mu^{\pm}\dot{\mathcal{L}}^{\pm} + \dot{\mu}^{\pm}\mathcal{L}^{\pm}\right) \\ \pm\frac{K\ell}{4\pi}\dot{\mu}^{\pm\prime\prime}\Big] L_{\mp 1} \mp \left(\pm\frac{1}{\ell}\mu^{\pm\prime} - \dot{\mu}^{\pm}\right) L_{\pm 1}\,.\end{aligned} \tag{3.3.11}$$

Therefore, we arrive to the following set of equations

$$\dot{\mu}^{\pm\prime} = \pm\frac{1}{\ell}\mu^{\pm\prime\prime}\,, \quad \dot{\mu}^{\pm} = \pm\frac{1}{\ell}\mu^{\pm\prime}\,, \quad \delta\mathcal{L}^{\pm} = \pm\mu^{\pm\prime}\mathcal{L}^{\pm} + \ell\left(\mu^{\pm}\dot{\mathcal{L}}^{\pm} + \dot{\mu}^{\pm}\mathcal{L}^{\pm}\right) - \frac{k\ell}{4\pi}\dot{\mu}^{\pm\prime\prime}\,. \tag{3.3.12}$$

The second and first equations are the chiral boson equations and its derivative, respectively. Equating the last equation with (3.3.8), and using the two latter obtained, we arrive to the following condition

$$\ell\,\mu^{\pm}\left(\dot{\mathcal{L}}^{\pm} \mp \frac{1}{\ell}\mathcal{L}^{\pm\prime}\right) = 0\,. \tag{3.3.13}$$

Because $\mu^{\pm}$ is arbitrary, the latter vanishes when we use the equations of motion (3.3.2). The function $\mu^{\pm}$ satisfies the chiral boson equation as showed, thereby it has the form $\mu^{\pm} = \mu^{\pm}(t-\phi)$. It is important to say that the function $\mu^{\pm}$ is not a *pure gauge* parameter; it has its own dynamics due the obtained symmetries at the boundary.

Thus, we have separated the gauge transformations from the ones that generate conserved charges and we have identified its dynamics. Now we aim to compute the asymptotic charges.

The variation of the generator of charges is given by Eq. (3.2.23), where $\Lambda^{\pm}$ is the permissible gauge parameter found in (3.3.10). Because $\mu^{\pm}$ only depends on the time and the angle since satisfies the chirality condition (so $\delta\mu^{\pm} = 0$), we can find $Q^{\pm}$ readily. It has the following expression

$$Q^{\pm}\left[\mu^{\pm}\right] = \mp\int_{\partial\mathcal{M}} d\phi\;\mu^{\pm}\mathcal{L}^{\pm}\,. \tag{3.3.14}$$

Now we aim to compute the asymptotic symmetry algebra. With (3.3.14), we identify the charge generator as a Fourier transform,

$$Q^{\pm}\left[\mu^{\pm} = e^{-im\phi}\right] = \mp\int d\phi\; e^{-im\phi}\mathcal{L}^{\pm} \equiv \mp\mathcal{L}_m^{\pm}\,, \tag{3.3.15}$$

where

$$\mathcal{L}_m^{\pm} = \int d\phi \, e^{-im\phi} \mathcal{L}^{\pm} \,. \tag{3.3.16}$$

In general, the algebra of charges can be computed as the variation with respect to $\mu_2^{\pm}$ of the charge generator, evaluated in $\mu_1^{\pm}$ [47],

$$\left\{ Q^{\pm}\left[\mu_1^{\pm}\right], Q^{\pm}\left[\mu_2^{\pm}\right] \right\} = \delta_{\mu_2^{\pm}} Q^{\pm}\left[\mu_1^{\pm}\right] \,. \tag{3.3.17}$$

This implies that $\left\{ Q^{\pm}\left[\mu_1^{\pm}\right], Q^{\pm}\left[\mu_2^{\pm}\right] \right\} = \pm\delta_{\mu_2^{\pm}} \int d\phi \; \mu_1^{\pm}\mathcal{L}^{\pm} = \pm \int d\phi \; \mu_1^{\pm}\delta_{\mu_2^{\pm}}\mathcal{L}^{\pm}$ [47]. After some calculations (see Appendix D.3), the asymptotic symmetry algebra is

$$i\{\mathcal{L}_m^{\pm}, \mathcal{L}_n^{\pm}\} = (m-n)\mathcal{L}_{m+n}^{\pm} + \frac{K}{2} m^3 \delta_{m+n,0} \,, \tag{3.3.18a}$$

with the known Brown–Henneaux central charge $c = 6K = 3\ell/2G$.

As we see, the choice of boundary conditions is not trivial, since they induce asymptotic symmetries that serve as spectrum of the holographic dual theory, as we reviewed in the introduction. This is not the only election that we can consider.

Let us briefly discuss what kind of spacetime generates the aforementioned election.

3.3.1 Holonomy

The holonomy is a geometrical quantity that measures if parallel transport is preserved or not along closed loops.

Spatial (angular) deffects are recognized by the angular holonomy $M^{\pm}$,

$$M^{\pm} = \mathrm{Tr}\left(\mathcal{P} \exp \oint d\phi \; a_{\phi}^{\pm} \right) \,, \tag{3.3.19}$$

where the angular gauge connection $a_{\phi}^{\pm}$ is given, in this case, by Eq. (3.3.1) and $\mathcal{P}$ is the path-ordered operator. We stress the fact that the trace in (3.3.19) ensures the gauge invariance of the angular holonomy.

For the AdS_3 set-up that we consider in this thesis, it is important to classify the $SL(2,\mathbb{R})$ conjugacy classes [343], characterized by the following values:

1. $M^{\pm} < 2$ correspond to a elliptic conjugacy class. This type of configurations corresponds to classical particle sources, inducing conical singularities.

2. If $M^{\pm} = 2$ we have parabolic conjugacy classes. They correspond to extremal black holes configurations.

3. Finally, if $M^{\pm} > 2$, it typifies hyperbolic conjugacy classes that characterize black hole solutions.

	$SL(2,\mathbb{R})$ holonomy conjugacy classes		
Holonomy value	$M^{\pm} > 2$	$M^{\pm} = 2$	$M^{\pm} < 2$
Conjugacy classes	Hyperbolic	Parabolic	Elliptic
Gravitational configurations	Black hole	Extremal black hole	Particle sources

Table 3.1: $SL(2,\mathbb{R})$ holonomy conjugacy classes and its characterization.

Thus, the holonomy allow to characterize spacetimes without the need to explicitly compute them [344], although the geometry of every solution in 3D GR with $\Lambda < 0$ will locally coincide with AdS_3.

For the Brown-Henneaux case, then the angular holonomy yields the value

$$M^{\pm} = 2\cosh\left(2\pi\sqrt{\frac{\mathcal{L}_0^{\pm}}{K}}\right) , \tag{3.3.20}$$

where a Fourier expansion was performed. From this result we can see that elliptic, parabolic and hyperbolic configurations are attainable, leading to conical singularities, extremal and black hole solutions.

3.3.2 The metric

The metric can be recovered from the asymptotic boundary conditions with the following relation

$$g_{\mu\nu} = \frac{\ell^2}{2}\left\langle\left(\mathcal{A}_\mu^+ - \mathcal{A}_\mu^-\right)\left(\mathcal{A}_\nu^+ - \mathcal{A}_\nu^-\right)\right\rangle , \tag{3.3.21}$$

where

$$\mathcal{A}^{\pm} = b_{\pm}^{-1}\left(d + a_{\pm}\right) b_{\pm} . \tag{3.3.22}$$

$b_{\pm}$ is an arbitrary gauge parameter that captures the radial dependence, as said before.

Consider the gauge parameter

$$b_{\pm}(r) = e^{\pm\frac{r}{\ell}L_0} . \tag{3.3.23}$$

In light-cone coordinates $x^{\pm} = t/\ell \pm \phi$, the aforementioned metric (that manifestly contains the

BTZ black hole solution) can be obtained from the boundary dynamics, yielding

$$ds^2 = \ell^2 \left[dr^2 + \frac{2\pi}{K}\left(\mathcal{L}^+ \left(dx^+\right)^2 + \mathcal{L}^- \left(dx^-\right)^2\right) - \left(e^{2r} + \frac{4\pi^2}{K^2}\mathcal{L}^+\mathcal{L}^-\right) dx^+ dx^- \right] . \qquad (3.3.24)$$

By construction it is a solution of vacuum Einstein equations, there are two arbitrary functions $\mathcal{L}^{\pm}(t, \phi)$. This is the most general metric that makes explicit the Virasoro invariance, known as *Bañados metric* [345].

Chapter 4

Geometrization of the AKNS system

In this chapter we describe a novel relation between the large family of integrable models given by the AKNS hierarchy and the spacetime dynamics of AdS_3 GR [330]. The diverse properties of the AKNS system form a large class of suitable boundary conditions that generalize [47] and [275].

We start the discussion based on the formalism developed in Chapters 2 and 3.

Consider a manifold with coordinates t and ϕ, with $0 \leq \phi \leq 2\pi$. Using the zero-curvature formulation of Integrable Systems, we postulate the AKNS boundary conditions of the Chern-Simons formulation of AdS_3 GR

$$\begin{aligned} a_\phi^\pm &= \mp 2\xi^\pm L_0 - p^\pm L_{\pm 1} + r^\pm L_{\mp 1} \,, \\ a_t^\pm &= \frac{1}{\ell}(-2A^\pm L_0 \pm B^\pm L_{\pm 1} \mp C^\pm L_{\mp 1}) \,, \end{aligned} \tag{4.0.1}$$

where ℓ is the AdS_3 radius, $p^\pm = p^\pm(t,\phi)$ and $r^\pm = r^\pm(t,\phi)$ are the fields carrying the boundary degrees of freedom of the theory, $A^\pm$, $B^\pm$ and $C^\pm$ are composite arbitrary functions of $r^\pm(t,\phi)$ and $p^\pm(t,\phi)$ that has to be specified. As we said in Eqs. (2.2.1) and (3.2.4), Einstein equations in the first-order formalism now reads as the vanishing 2-form curvature

$$f_{t\phi}^\pm = \partial_t a_\phi^\pm - \partial_\phi a_t^\pm + \left[a_t^\pm, a_\phi^\pm\right] = 0 \,.$$

Therefore, for the AKNS boundary conditions (4.0.1), the zero-curvature yields the following set of differential equations (2.1.1),

$$\dot{p}^\pm = \mp\frac{1}{\ell}\left(B^{\pm\prime} + 2p^\pm A^\pm + 2\xi^\pm B^\pm\right) \,, \tag{4.0.2a}$$

$$\dot{r}^\pm = \mp\frac{1}{\ell}\left(C^{\pm\prime} - 2r^\pm A^\pm - 2\xi^\pm C^\pm\right) \,, \tag{4.0.2b}$$

$$A^{\pm\prime} = p^\pm C^\pm - r^\pm B^\pm \,. \tag{4.0.2c}$$

Using the same polynomial ansatz of Eq. (2.1.2), we obtain at order ${\xi^{\pm}}^0$ the following dynamical equations

$$\dot{p}^{\pm} = \mp\frac{1}{\ell}\left(B_N^{\pm\prime} + 2p^{\pm}A_N^{\pm}\right) , \quad \dot{r}^{\pm} = \mp\frac{1}{\ell}\left(C_N^{\pm\prime} - 2r^{\pm}A_N^{\pm}\right) , \tag{4.0.3}$$

in complete analogy with (2.1.3); while for the remaining terms we obtain the recursion relations

$$A_n^{\pm\prime} = p^{\pm}C_n^{\pm} - r^{\pm}B_n^{\pm}, \tag{4.0.4a}$$

$$B_{n+1}^{\pm} = -\frac{1}{2}B_n^{\pm\prime} - p^{\pm}A_n^{\pm}, \tag{4.0.4b}$$

$$C_{n+1}^{\pm} = \frac{1}{2}C_n^{\pm\prime} - r^{\pm}A_n^{\pm}, \tag{4.0.4c}$$

subjected to conditions $B_0^{\pm} = C_0^{\pm} = 0$. As we see, every result of Chapter 2 works in this scenario.

In order to figure if the boundary conditions (4.0.1) are suitable, which is the general objective of this chapter, it is first necessary to briefly expose the main results of Chapter 2 but in terms of the chiral/antichiral sector.

Using the trace formula [331] (see Appendix C.3), one may cast $A_n^{\pm}$, $B_n^{\pm}$ and $C_n^{\pm}$ as

$$A_n^{\pm} = \frac{n-1}{2}\mathcal{H}_n^{\pm} , \quad B_n^{\pm} = \mathcal{R}_{n+1}^{\pm} , \quad C_n^{\pm} = \mathcal{P}_{n+1}^{\pm} , \tag{4.0.5}$$

for $n \geq 1$, where the quantities $\mathcal{R}_{n+1}^{\pm}$ and $\mathcal{P}_{n+1}^{\pm}$ are functional derivatives of the conserved charges $H_n^{\pm}$, defined as following

$$\mathcal{R}_{n+1}^{\pm} = \frac{\delta H_{n+1}^{\pm}}{\delta r^{\pm}} , \quad \mathcal{P}_{n+1}^{\pm} = \frac{\delta H_{n+1}^{\pm}}{\delta p^{\pm}} . \tag{4.0.6}$$

The latter allows to write the AKNS system as a bi-Hamiltonian equation

$$\begin{pmatrix} \dot{r}^{\pm} \\ \dot{p}^{\pm} \end{pmatrix} = \mp\mathcal{D}_1^{\pm}\begin{pmatrix} \mathcal{R}_{N+1}^{\pm} \\ \mathcal{P}_{N+1}^{\pm} \end{pmatrix} = \mp\mathcal{D}_2\begin{pmatrix} \mathcal{R}_{N+2}^{\pm} \\ \mathcal{P}_{N+2}^{\pm} \end{pmatrix} , \tag{4.0.7}$$

where the compatible Hamiltonian operators $\mathcal{D}_1^{\pm}$ and $\mathcal{D}_2^{\pm}$ are

$$\mathcal{D}_1^{\pm} = \frac{1}{\ell}\begin{pmatrix} -2r^{\pm}\partial_\phi^{-1}r^{\pm} & -\partial_\phi + 2r^{\pm}\partial_\phi^{-1}p^{\pm} \\ -\partial_\phi^{\pm} + 2p^{\pm}\partial_\phi^{-1}r^{\pm} & -2p^{\pm}\partial_\phi^{-1}p^{\pm} \end{pmatrix} , \quad \mathcal{D}_2 = \frac{1}{\ell}\begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix} , \tag{4.0.8}$$

respectively. These results will be the basis where we can establish the holographic relationship that lies between 3D GR and Integrable Systems. The connecting bridge for this formal relationship, as we have emphasized in the introduction, are asymptotic symmetries, generated by the choice of *suitable* boundary conditions [54].

In what follows, we will prove that boundary conditions (4.0.1) are suitable, fulfilling the criteria discussed in Section 3.1.

4.1 Integration of the boundary term

Performing the polynomial expansion (2.1.2), the variation of the boundary term (3.2.12) reads

$$\delta\mathcal{B}^{\pm} = \pm\frac{K}{2\pi}\int\frac{dt}{\ell}\sum_{n=0}^{N}\int d\phi\,\left(C_n^{\pm}\delta p^{\pm} + B_n^{\pm}\delta r^{\pm}\right)\xi^{\pm\,N-n}\,. \tag{4.1.1}$$

By means of (4.0.5), we arrive to

$$\delta\mathcal{B}^{\pm} = \pm\frac{K}{2\pi}\int\frac{dt}{\ell}\sum_{n=0}^{N}\int d\phi\,\left(\frac{\delta H_{n+1}^{\pm}}{\delta p^{\pm}}\delta p^{\pm} + \frac{\delta H_{n+1}^{\pm}}{\delta r^{\pm}}\delta r^{\pm}\right)\xi^{\pm\,N-n}\,. \tag{4.1.2}$$

Hence, using the AKNS boundary conditions (4.0.1), we readily have a well-defined action with the following boundary term

$$\mathcal{B}^{\pm} = \pm\frac{K}{2\pi}\int\frac{dt}{\ell}\sum_{n=0}^{N}H_{n+1}^{\pm}\xi^{\pm\,N-n}\,. \tag{4.1.3}$$

4.2 Asymptotic symmetries and asymptotic algebra

As we reviewed in Section 3.3, asymptotic symmetries are the set of gauge transformations that preserve the form of imposed asymptotic boundary conditions, which in this case are (4.0.1). Its infinitesimal form is

$$\delta a^{\pm} = d\Lambda^{\pm} + \left[a^{\pm},\Lambda^{\pm}\right]. \tag{4.2.1}$$

The exercise is analogue as the one performed for the Brown–Henneaux case of Section 3.3. We want to characterize asymptotic symmetries, i.e., the ones that generate conserved charges, namely, Noetherian charges. In order to achieve that, notice that the angular component of (4.2.1) is

$$\delta a_{\phi}^{\pm} - \partial_{\phi}\Lambda^{\pm} + \left[\Lambda^{\pm}, a_{\phi}^{\pm}\right] = 0\,, \tag{4.2.2}$$

which is similar to the zero-curvature condition (2.2.1), where we can "recognize" $\partial_t \leftrightarrow \delta$ and $a_t^{\pm} \leftrightarrow \Lambda^{\pm}$ [197]. Inspired by this analogy, we write $\Lambda^{\pm}$ in the following (reminiscent of $a_t^{\pm}$ in (4.0.1)) general form

$$\Lambda^{\pm} = -2\alpha^{\pm}L_0 \pm \beta^{\pm}L_{\pm1} \mp \gamma^{\pm}L_{\mp1}\,, \tag{4.2.3}$$

where $\alpha^{\pm}$, $\beta^{\pm}$ and $\gamma^{\pm}$ are field-dependent arbitrary functions that has to be specified. As a consequence, the angular component (4.2.2) reads as the following system of equations

$$\delta p^{\pm} = \mp\left(2\alpha^{\pm}p^{\pm} + 2\xi^{\pm}\beta^{\pm} + \beta^{\pm\prime}\right), \tag{4.2.4a}$$

$$\delta r^{\pm} = \mp\left(-2\alpha^{\pm}r^{\pm} - 2\xi^{\pm}\gamma^{\pm} + \gamma^{\pm\prime}\right), \tag{4.2.4b}$$

$$\alpha^{\pm\prime} = p^{\pm}\gamma^{\pm} - r^{\pm}\beta^{\pm}. \tag{4.2.4c}$$

Following the lessons of Chapter 2, if we specify $\alpha^{\pm}$, $\beta^{\pm}$ and $\gamma^{\pm}$ according to the polynomial expansion (2.1.2),

$$\alpha^{\pm} = \sum_{m=0}^{M}\alpha_m^{\pm}\xi^{\pm M-m}, \quad \beta^{\pm} = \sum_{m=0}^{M}\beta_m^{\pm}\xi^{\pm M-m}, \quad \gamma^{\pm} = \sum_{m=0}^{M}\gamma_m^{\pm}\xi^{\pm M-m}, \tag{4.2.5}$$

where M is a number that labels the infinite family of permissible gauge transformations (which is not necessarily equal to N), we readily obtain that the infinitesimal transformation of the fields $r^{\pm}$ and $p^{\pm}$ are just the analogue versions of the AKNS equations (2.1.3) and (2.1.4a),

$$\delta p^{\pm} = \mp\left(\beta_M^{\pm\,\prime} + 2p^{\pm}\alpha_M^{\pm}\right), \tag{4.2.6a}$$

$$\delta r^{\pm} = \pm\left(-\gamma_M^{\pm\,\prime} + 2\alpha_M^{\pm}r^{\pm}\right), \tag{4.2.6b}$$

$$\alpha_M^{\pm\,\prime} = p^{\pm}\gamma^{\pm} - r^{\pm}\beta^{\pm}, \tag{4.2.6c}$$

with conditions $\beta_0^{\pm} = \gamma_0^{\pm} = 0$ (analogue to conditions $B_0^{\pm} = C_0^{\pm} = 0$). These are the asymptotic infinitesimal transformations of the fields that preserve the form of boundary conditions (4.0.1).

It is clear by construction that $a_t^{\pm}$ belongs to the above family of permissible gauge parameters at

$$a_t^{\pm} = \pm\frac{1}{\ell}\Lambda^{\pm}|_{M=N}. \tag{4.2.7}$$

Because (4.2.1) yields an equation analogue to the zero-curvature condition, conservation along the temporal component $a_t^{\pm}$ reduces to combinations of the equations of motion. Thus, it does not imply any further condition on the gauge parameter (see Appendix E.1).

By virtue of relationships (4.0.5), we find that the coefficients of (4.2.5) satisfy the known identities

$$\alpha_m^{\pm} = \frac{m-1}{2}\mathcal{H}_m^{\pm}, \quad \beta_m^{\pm} = \mathcal{R}_{m+1}^{\pm}, \quad \gamma_m^{\pm} = \mathcal{P}_{m+1}^{\pm}. \tag{4.2.8}$$

Therefore, expansion (4.2.5) reads

$$\alpha^{\pm} = \sum_{m=0}^{M} \left(\frac{m-1}{2}\right) \mathcal{H}_{m}^{\pm}\, \xi^{\pm\, M-m}, \tag{4.2.9a}$$

$$\beta^{\pm} = \sum_{m=0}^{M} \mathcal{R}_{m+1}^{\pm}\, \xi^{\pm\, M-m}, \tag{4.2.9b}$$

$$\gamma^{\pm} = \sum_{m=0}^{M} \mathcal{P}_{m+1}^{\pm}\, \xi^{\pm\, M-m}. \tag{4.2.9c}$$

As we reviewed in the previous chapter, from the Hamiltonian point of view, gauge transformation (4.2.1) is generated by the boundary term $Q^{\pm}\left[\Lambda^{\pm}\right]$ that must be suplemented to the first class constraint in order to make it differentiable. Using the boundary conditions (4.0.1), the differentiated charge (3.2.23) reads

$$\delta Q^{\pm}\left[\Lambda^{\pm}\right] = -\frac{K}{2\pi}\int_{\partial\mathcal{M}} d\phi\ \left(\beta^{\pm}\delta r^{\pm} + \gamma^{\pm}\delta p^{\pm}\right) \tag{4.2.10}$$

In an analogue version when we integrated the boundary term in Section 4.1, we find then

$$\delta Q^{\pm}\left[\Lambda^{\pm}\right] = \pm\frac{K}{2\pi}\sum_{m=0}^{M}\int_{\partial\mathcal{M}} d\phi\ \left(\mathcal{R}_{m+1}^{\pm}\delta r^{\pm} + \mathcal{P}_{m+1}^{\pm}\delta p^{\pm}\right)\xi^{\pm\, M-m}\,, \tag{4.2.11}$$

yielding

$$Q^{\pm}\left[\Lambda^{\pm}\right] = \pm\frac{K}{2\pi}\sum_{m=0}^{M} H_{m+1}^{\pm}\, \xi^{\pm\, M-m}\,. \tag{4.2.12}$$

The charges are finite, and they are just the conserved quantities of the AKNS system.

We are now in position to compute its algebra. Recall that the algebra of charges between two arbitrary gauge parameters $\Lambda^{\pm}$ and $\bar{\Lambda}^{\pm}$ was computed in the Brown–Henneaux case, given by expression (3.3.17). Because, $\left\{Q^{\pm}\left[\Lambda^{\pm}\right], Q^{\pm}\left[\bar{\Lambda}^{\pm}\right]\right\} = \bar{\delta}Q^{\pm}\left[\Lambda^{\pm}\right]$, we readily obtain

$$\left\{Q^{\pm}\left[\Lambda^{\pm}\right], Q^{\pm}\left[\bar{\Lambda}^{\pm}\right]\right\} = \pm\frac{K}{2\pi}\int_{\partial\mathcal{M}} d\phi\ \left(\beta^{\pm}\bar{\delta}r^{\pm} + \gamma^{\pm}\bar{\delta}p^{\pm}\right)\,. \tag{4.2.13}$$

We can recast the latter expression as

$$\left\{Q^{\pm}\left[\Lambda^{\pm}\right], Q^{\pm}\left[\bar{\Lambda}^{\pm}\right]\right\} = \pm\frac{K}{2\pi}\int_{\partial\mathcal{M}} d\phi\ \begin{pmatrix}\beta^{\pm} & \gamma^{\pm}\end{pmatrix}\begin{pmatrix}\bar{\delta}r^{\pm}\\ \bar{\delta}p^{\pm}\end{pmatrix}\,, \tag{4.2.14}$$

which is the same as

$$\left\{Q^{\pm}\left[\Lambda^{\pm}\right],Q^{\pm}\left[\bar{\Lambda}^{\pm}\right]\right\}=\pm\frac{K}{2\pi}\int_{\partial\mathcal{M}}d\phi\,\begin{pmatrix}\mathcal{R}^{\pm}_{m+1} & \mathcal{P}^{\pm}_{m+1}\end{pmatrix}\mathcal{D}^{\pm}_{1}\begin{pmatrix}\mathcal{R}^{\pm}_{\bar{M}+1}\\ \mathcal{P}^{\pm}_{\bar{M}+1}\end{pmatrix}. \tag{4.2.15}$$

This the first Poisson bracket between H_{m+1} and $H_{\bar{M}+1}$, defined in (2.1.23). Because the conserved charges of the AKNS system are in involution (see Eq. (2.1.25)), the Poisson bracket of gravitational charges vanishes,

$$\left\{Q^{\pm}\left[\Lambda^{\pm}\right],Q^{\pm}\left[\bar{\Lambda}^{\pm}\right]\right\}=\pm\frac{K}{2\pi}\left\{H^{\pm}_{m+1},H^{\pm}_{\bar{M}+1}\right\}_{1}=0\,. \tag{4.2.16}$$

We computed the asymptotic symmetry algebra of charges, which realize an infinite-dimensional abelian one.

4.3 Holonomy and gravitational configurations

For the AKNS boundary conditions, its angular component is given by (4.0.1), from where the holonomy (3.3.19) reads

$$M^{\pm}=2\cosh\left(2\pi\sqrt{\left(\xi^{\pm}\right)^{2}+p^{\pm}_{0}r^{\pm}_{0}}\right). \tag{4.3.1}$$

$p^{\pm}_{0}$ and $r^{\pm}_{0}$ are the zero modes of the Fourier expansions. The remarkable fact of this result indicates that all gravitational configurations are attainable, namely, the AKNS boundary conditions includes particle sources, extremal and ordinary black hole solutions. This is because we can always work with zero modes and the spectral parameter such that the hyperbolic cosine turns complex and transforms into a cosine, allowing to reach values less than 2 (See Table 3.1).

Because the action functional is differentiable, the generator of charges are finite and black hole solutions lies in the conjugacy class of boundary conditions, the imposed AKNS asymptotic behavior are adequate.

4.4 Recovering specific boundary conditions

Here we recover specific boundary conditions from the asymptotic behavior. For details, see Appendix E.2.

It is desirable to make explicit how we particularly recover the Brown-Henneaux [47] and KdV-type [275] boundary conditions.

Recalling expansion (2.1.2), truncated at some integer number N. We have a large family of boundary conditions labeled by this number (e.g., see Eq. (E.2.5f)). Fixing $N = 1$, we have

$$a^{\pm}_{t,N=1} = \frac{1}{\ell}\left[-2L_0\xi^{\pm} \pm \left(-p^{\pm}L_{\pm 1} + r^{\pm}L_{\mp 1}\right)\right] . \tag{4.4.1}$$

If $\xi^{\pm} = 0$, we define $p^{\pm} = -1$ and $r^{\pm} = -\frac{2\pi}{K}\mathcal{L}^{\pm}(t,\phi)$, we obtain the Brown–Henneaux boundary conditions (3.3.1). On the other hand, for odd values of N, i.e.,

$$\begin{aligned}
a^{\pm}_{t,N=3} = \frac{1}{\ell}\Big\{&-2L_0\xi^{\pm\,3} \pm \left(-p^{\pm}L_{\pm 1} + r^{\pm}L_{\mp 1}\right)\xi^{\pm\,2} + \Big[-\frac{1}{2}\left(p^{\pm\prime}r^{\pm} - p^{\pm}r^{\pm\prime}\right)L_0 \\
&\pm\frac{1}{4}\left(2p^{\pm\,2}r^{\pm} - p^{\pm\prime\prime}\right)L_{\pm 1} \mp \frac{1}{4}\left(2p^{\pm}r^{\pm\,2} - r^{\pm\prime\prime}\right)L_{\mp 1}\Big]\xi^{\pm} \\
&-\frac{1}{4}\left(3p^{\pm\,2}r^{\pm\,2} + p^{\pm\prime}r^{\pm\prime} - p^{\pm\prime\prime}r^{\pm} - p^{\pm}r^{\pm\prime\prime}\right)L_0 \\
&\pm\frac{1}{8}\left(-6p^{\pm}p^{\pm\prime}r^{\pm} + p^{\pm\prime\prime\prime}\right)L_{\pm 1} \mp \frac{1}{8}\left(6p^{\pm}r^{\pm}r^{\pm\prime} - r^{\pm\prime\prime\prime}\right)L_{\mp 1}\Big\} ,
\end{aligned} \tag{4.4.2a}$$

$$\begin{aligned}
a^{\pm}_{t,N=5} = \frac{1}{\ell}\Big\{&-2L_0\xi^{\pm\,5} \pm \left(-p^{\pm}L_{\pm 1} + r^{\pm}L_{\mp 1}\right)\xi^{\pm\,4} + \Big[-\frac{1}{2}\left(p^{\pm\prime}r^{\pm} - p^{\pm}r^{\pm\prime}\right)L_0 \\
&\pm\frac{1}{4}\left(2p^{\pm\,2}r^{\pm} - p^{\pm\prime\prime}\right)L_{\pm 1} \mp \frac{1}{4}\left(2p^{\pm}r^{\pm\,2} - r^{\pm\prime\prime}\right)L_{\mp 1}\Big]\xi^{\pm\,3} \\
&+\Big[-\frac{1}{4}\left(3p^{\pm\,2}r^{\pm\,2} + p^{\pm\prime}r^{\pm\prime} - p^{\pm\prime\prime}r^{\pm} - p^{\pm}r^{\pm\prime\prime}\right)L_0 \\
&\pm\frac{1}{8}\left(-6p^{\pm}p^{\pm\prime}r^{\pm} + p^{\pm\prime\prime\prime}\right)L_{\pm 1} \mp \frac{1}{8}\left(6p^{\pm}r^{\pm}r^{\pm\prime} - r^{\pm\prime\prime\prime}\right)L_{\mp 1}\Big]\xi^{\pm\,2} \\
&+\Big[-\frac{1}{4}\left(3p^{\pm\,2}r^{\pm\,2} + p^{\pm\prime}r^{\pm\prime} - p^{\pm\prime\prime}r^{\pm} - p^{\pm}r^{\pm\prime\prime}\right)L_0 \\
&\pm\frac{1}{8}\left(-6p^{\pm}p^{\pm\prime}r^{\pm} + p^{\pm\prime\prime\prime}\right)L_{\pm 1} \mp \frac{1}{8}\left(6p^{\pm}r^{\pm}r^{\pm\prime} - r^{\pm\prime\prime\prime}\right)L_{\mp 1}\Big]\xi^{\pm} \\
&-\frac{1}{8}\Big[6p^{\pm\,2}r^{\pm}r^{\pm\prime} - p^{\pm\prime\prime}r^{\pm\prime} + p^{\pm\prime}r^{\pm\prime\prime} + p^{\pm\prime\prime\prime}r^{\pm} - p^{\pm}\left(6p^{\pm\prime}r^{\pm\,2} - r^{\pm\prime\prime\prime}\right)\Big]L_0 \\
&\pm\frac{1}{16}\Big[-6p^{\pm\,3}r^{\pm\,2} + 6p^{\pm\prime\,2}r^{\pm} + 4p^{\pm}\left(p^{\pm\prime}r^{\pm\prime} + 2p^{\pm\prime\prime}r^{\pm}\right) + 2p^{\pm\,2}r^{\pm\prime\prime} - p^{\pm\prime\prime\prime\prime}\Big]L_{\pm 1} \\
&\mp\frac{1}{16}\Big[-6p^{\pm\,2}r^{\pm\,3} + 4p^{\pm\prime}r^{\pm}r^{\pm\prime} + 2p^{\pm\prime\prime}r^{\pm\,2} + p^{\pm}\left(6r^{\pm\prime\,2} + 8r^{\pm}r^{\pm\,2}\right) - r^{\pm\prime\prime\prime\prime}\Big]L_{\mp 1}\Big\} , \\
&\vdots
\end{aligned} \tag{4.4.2b}$$

and letting $p^{\pm} = -1$ we obtain the KdV-type boundary conditions, reported in [275].

4.5 The metric

Just like for the Brown–Henneaux case of Subsection 3.3.2, the metric may be recovered by means of the following relation

$$g_{\mu\nu} = \frac{\ell^2}{2}\left\langle\left(\mathcal{A}^{+}_{\mu} - \mathcal{A}^{-}_{\mu}\right)\left(\mathcal{A}^{+}_{\nu} - \mathcal{A}^{-}_{\nu}\right)\right\rangle . \tag{4.5.1}$$

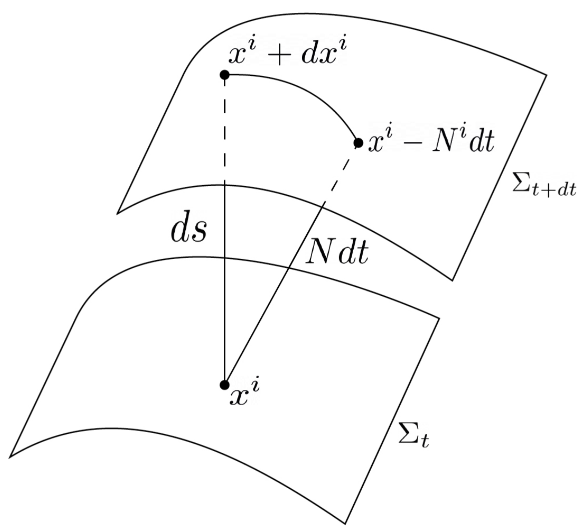

Figure 4.1: ADM decomposition.

We choose the explicit group element

$$b_\pm(r) = \exp\left[\pm \log\left(\frac{r}{\ell}\right) L_0\right] , \tag{4.5.2}$$

thus we arrive to the following metric components

$$g_{tt} = -\frac{B^+B^-r^2}{\ell^2} + \left(A^+ - A^-\right)^2 + B^+C^+ + B^-C^- - \frac{C^+C^-\ell^2}{r^2} , \tag{4.5.3a}$$

$$g_{tr} = \frac{\left(A^- - A^+\right)\ell}{r} , \tag{4.5.3b}$$

$$\begin{aligned} g_{t\phi} = \frac{\left(B^-p^+ - B^+p^-\right) r^2}{2\ell} &+ \frac{\ell}{2}\left[2\left(\xi^+ + \xi^-\right)\left(A^+ - A^-\right) - B^+r^+ + B^-r^- \right. \\ &\left. - C^+p^+ + C^-p^-\right] + \frac{\ell^3}{2r^2}\left(C^-r^+ - C^+r^-\right) , \end{aligned} \tag{4.5.3c}$$

$$g_{rr} = \frac{\ell^2}{r^2} , \tag{4.5.3d}$$

$$g_{r\phi} = -\frac{\ell^2}{r}\left(\xi^+ + \xi^-\right) , \tag{4.5.3e}$$

$$g_{\phi\phi} = p^+p^-r^2 + \left[p^+r^+ + p^-r^- + \left(\xi^+ + \xi^-\right)^2\right]\ell^2 + \frac{\ell^4}{r^2}r^+r^- . \tag{4.5.3f}$$

In ADM coordinates $ds^2 = -N^2dt^2 + (N^i dt + dx^i)(N^j dt + dx^j)\gamma_{ij}$, the metric can be seen in the following form. The lapse function is

$$N^2 = \frac{r^2}{4\ell^2}\frac{\left(\Omega^+\omega^- + \Omega^-\omega^+\right)^2}{\omega^-\omega^+} , \tag{4.5.4}$$

and the shift functions are

$$N^r = \frac{r}{\ell}\left(A^- - A^+ + \frac{1}{2}\left(\xi^+ + \xi^-\right)\left(\frac{\Omega^-}{\omega^-} - \frac{\Omega^+}{\omega^+}\right)\right), \tag{4.5.5a}$$

$$N^\phi = \frac{1}{2\ell}\left(\frac{\Omega^-}{\omega^-} - \frac{\Omega^+}{\omega^+}\right). \tag{4.5.5b}$$

Additionally, the spatial metric reads

$$\gamma_{ij} = \begin{pmatrix} \frac{\ell^2}{r^2} & -\frac{\ell^2}{r}\left(\xi^+ + \xi^-\right) \\ -\frac{\ell^2}{r}\left(\xi^+ + \xi^-\right) & \ell^2\left(\xi^+ + \xi^-\right)^2 + r^2\omega^-\omega^+ \end{pmatrix}. \tag{4.5.6}$$

The auxiliary functions $\Omega^\pm$ and $\omega^\pm$ are defined as

$$\Omega^\pm \equiv B^\pm - \frac{\ell^2}{r^2}C^\mp, \quad \omega^\pm \equiv p^\pm + \frac{\ell^2}{r^2}r^\mp. \tag{4.5.7}$$

The difference with the Brown–Henneaux case lies in the $\gamma_{r\phi}$ component and the form of the lapse and shift functions.

The boundary dynamics arises from the asymptotic behavior of the lapse function N and shift vectors N^i [197], which depends on the dynamical functions and consequently induces a non-trivial surface evolution at the boundary. This property has been used previously in [275] and [278] to connect the dynamics of AdS_3 GR with the KdV and Gardner integrable hierarchies, respectively.

We constructed the metric associated to the AKNS boundary conditions, which contains two sets of dimensionless functions $\{A^\pm, B^\pm, C^\pm, p^\pm, r^\pm\}$, labeled by the $\pm$ superscript, and chosen to depend only on the coordinates t and ϕ. On the other hand, the two quantities $\xi^\pm$ are constants without dimensions.

4.6 Remarks

Proving the consistency of boundary conditions (4.0.1) imply that the AKNS system is equivalent to the asymptotic dynamics of AdS_3 GR.

When we say that the boundary conditions are "suitable", we refer to the criteria of Section 3.1, namely, the AKNS boundary conditions integrate the boundary term in the Hamiltonian action, define finite asymptotic charges, the asymptotic symmetry algebra closes in an abelian infinite-dimensional form with devoid of central extensions and they includes all possible gravitational configurations by calculating the angular holonomy. We consistently recovered the boundary conditions [47, 275] and we understood that the dynamical fields $p^\pm$ and $r^\pm$ induce deformations on the spacetime through the Lapse function (4.5.4) and the Shift vectors (4.5.5).

It would be desirable to translate our results to the metric formalism. For example, to obtain the abelian asymptotic symmetry algebra from the group of isommetries at the boundary (namely, to solve the asymptotic Killing equation by impossing suitable boundary conditions). To recover the NLS, KdV, MKdV or SG metrics explicitly. Other avenue is to provide an interpretation of the mass and angular momentum in terms of the dynamical fields. Finally, a more general question involves the meaning of the Inverse Scattering Transformation and soliton construction under the light of curved spacetimes.

Part II

Condensed Matter Theory and General Relativity

Chapter 5

Linear response theory

Response theory aims to understand how a system reacts to a particular applied external perturbation [346], yielding a *response function*, or susceptibility $\chi(t, t')$ associated to the former. This is the main quantity that one tries to compute and hopefully, measure in a experiment.

Concrete examples are the application of an electric field or momentum transport in some liquids. This perturbations will give an associated conductivity and viscosity, respectively.

In general, it is hard to obtain an analytic expression of susceptibilities. Therefore one may approximate susceptibilities to its first order in the perturbation. This is what is known as *linear response theory*. In the second part of this thesis, the aim is to obtain the first-order susceptibility of a dual theory to 3D GR by means of the Kubo formula [334, 335].

Along with its natural application in Condensed Matter Theory [347], the Kubo formula find diverse applications in the lines of the holographic correspondence. Black holes [348, 349], Hall [350] and strongly correlated viscosities [351], anomalies [352, 353], higher-spin interactions [354] (with spin-3 possible experimental measurement [355]), two-dimensional [356] and three-dimensional [25, 357, 358] relativistic fluids, and electrical conductivity and charge susceptibility fixed by the value of the central charge [359] are part of them.

The objective of this chapter is to establish the Kubo formula. In Chapter 7, it will be used to compute a two-terminal holographic conductance associated to the boundary dynamics of AdS_3 GR with a specific asymptotic behavior. To deduce it, we will consider the Dirac interaction picture of quantum mechanics in presence of a time-dependent arbitrary perturbation. Then, by Taylor expanding the time-evolution operator, we will obtain the first-order approximation of the mean value of an arbitrary operator sensible to this perturbation. The first-order contribution of this approximation will correspond to the retarded Green function, from where we will be able to establish a relationship between the latter and the susceptibility of the theory.

5.1 Susceptibilities are retarded Green function

Consider a Hamiltonian perturbation

$$H = H_0 + V(t)\,, \tag{5.1.1}$$

where H_0 is the free part of the theory and $V(t)$ its perturbation.

Consider an arbitrary operator $\mathcal{O}(t,x)$. If the perturbation is small, then we write the first-order change in the expectation value of this operator

$$\delta \langle O(t,x)\rangle := \langle \mathcal{O}(t,x)\rangle - \langle \mathcal{O}(t,x)\rangle\,\big|_{V=0}\,, \tag{5.1.2}$$

as

$$\delta \langle \mathcal{O}(t,x)\rangle = \int dt'\,\chi(t,t')V(t')\,, \tag{5.1.3}$$

where $\chi(t,t')$ is the linear response function, or susceptibility associated to the perturbation V.

Consider now the following differential equation

$$\mathcal{D}\left[y(t)\right] = F(t)\,, \tag{5.1.4}$$

where $\mathcal{D}$ is some differential operator, $y = y(t)$ an arbitrary function and $F(t)$ a driving force. Formally, the solution of this differential equation is

$$y(t) = y_0 + \int dt'\,G(t,t')F(t')\,, \tag{5.1.5}$$

with y_0 the homogenous solution and $G(t,t')$ stands for the Green function or Kernel of the system. One thus recognize the susceptibility as the Green function of the system, however, we must consider causality conditions: The susceptibility follows the perturbation, rather than preceed it. So we may associate this condition of the response function, or susceptibility, with the *retarded* Green function, namely, the Green function that vanishes for negative time. Hence, we have

$$\chi(t,t') = G^R(t,t')\,, \tag{5.1.6}$$

where the index R labels the retarded prescription.

One more condition has to be specified. We will study systems that are invariant under time-translations, so the susceptibility depend only on the time difference $\chi(t-t')$. Thus,

$$\chi(t-t') = 0\,, \quad \text{for} \quad t-t' < 0\,. \tag{5.1.7}$$

5.2 Interaction picture

Here we review the interaction picture for a time-dependent perturbation $V = V(t)$. By means of the evolution operator $U(t, t_0)$, we will calculate the wavefunction $|\Psi\rangle$. Under the action of the perturbation we will find that Dyson equation governs the dynamics of the time-evolution operator. We can solve it recursively [360], whose solution will have a precise form on $V(t)$.

Again, consider the Hamiltonian of the form (5.1.1),

$$H = H_0 + V(t)\,,$$

In the Heisenberg picture, operators $\mathcal{O}(t, x)$ evolve with the full Hamiltonian in the form

$$\mathcal{O}(t, x) = e^{iH(t-t_0)}\mathcal{O}(0, x)e^{-iH(t-t_0)}\,. \tag{5.2.1}$$

Here we aim compute the expectation value of these operators. This can be achieved by means of the following formula

$$\langle \mathcal{O}(t, x)\rangle = \mathrm{Tr}\left[\mathcal{O}(t, x)\rho(t)\right], \tag{5.2.2}$$

where $\rho(t)$ is the density matrix of the system. Let us rewrite expression (5.2.2) in the following form

$$\langle \mathcal{O}(t, x)\rangle = \mathrm{Tr}\left[\mathcal{O}(t, x)e^{iH(t-t_0)}\rho(0, x)e^{-iH(t-t_0)}\right]. \tag{5.2.3}$$

Using the cyclic property of the trace, we obtain

$$\begin{aligned}\langle O(t, x)\rangle = \mathrm{Tr}\Big[&\left(e^{iH_0(t-t_0)}\mathcal{O}(t, x)e^{-iH_0(t-t_0)}\right)\times \\ &\times\left(e^{iH_0(t-t_0)}e^{iH(t-t_0)}\rho(0, x)e^{-iH(t-t_0)}e^{-iH_0(t-t_0)}\right)\Big],\end{aligned} \tag{5.2.4}$$

which allow us to consider the following definitions

$$\mathcal{O}_I(t, x) := e^{iH_0(t-t_0)}\mathcal{O}(t, x)e^{-iH_0(t-t_0)}\,, \tag{5.2.5a}$$

$$\rho_I(t, x) := e^{iH_0(t-t_0)}e^{iH(t-t_0)}\rho(0, x)e^{-iH(t-t_0)}e^{-iH_0(t-t_0)} \equiv e^{iH_0(t-t_0)}\rho(t, x)e^{-iH_0(t-t_0)}\,, \tag{5.2.5b}$$

where the subindex I is denoted for the interacting picture. Therefore, we obtain that the mean value of some arbitrary operator $\mathcal{O}(t, x)$ will be given in terms of its interacting operators

$$\langle \mathcal{O}_I(t, x)\rangle = \mathrm{Tr}\left[\mathcal{O}_I(t, x)\rho_I(t, x)\right]. \tag{5.2.6}$$

From the Heisenberg picture (5.2.1), and using definition (5.2.5a), we find that $\mathcal{O}(t,x)$ may be written as

$$\mathcal{O}(t,x) \equiv U(t,t_0)\mathcal{O}_I(t,x)U^\dagger(t,t_0)\,, \tag{5.2.7}$$

where

$$U(t,t_0) = e^{iH(t-t_0)}e^{-iH_0(t-t_0)}\,. \tag{5.2.8}$$

Because the wavefunction for $t > t_0$ evolves as

$$|\Psi\rangle = e^{iH(t-t_0)}\,|0\rangle\;, \tag{5.2.9}$$

in terms of $U(t,t_0)$, it reads as

$$|\Psi\rangle = e^{-iH_0(t-t_0)}U^\dagger(t,t_0)\,|0\rangle\;. \tag{5.2.10}$$

We aim now to determine the form of operator $U(t,t_0)$. Taking a temporal derivative of the previous equation, we obtain the Tomonaga-Schwinger equation for $U(t,t_0)$,

$$i\frac{\partial U(t,t_0)}{\partial t}\,|0\rangle = V(t,t_0)U(t,t_0)\,|0\rangle\;, \tag{5.2.11}$$

with initial condition $U(t,t) = 1$. This equation can be formally solved in terms of a Dyson-series [360]

$$U(t,t_0) = \mathcal{T}\exp\left(-i\int_{t_0}^{t} V(t',t_0)dt'\right)\,, \tag{5.2.12}$$

where $\mathcal{T}$ is the time-ordering operator.

We have the tools to compute the first-order susceptibility associated to an arbitrary perturbation. The relation that is going to allow us to compute it is known as the *Kubo formula* [334, 335]. In the next section we will obtain it explicitly.

5.3 The Kubo formula

At first order, the evolution operator $U(t,t_0)$ can be Taylor-expanded as

$$U(t,t_0) \approx 1 - i\int_{t_0}^{t} dt'\,V(t',t_0) + \dots\,, \tag{5.3.1}$$

where the ellipsis denote second-order contributions. Therefore, by virtue of relation (5.2.7), it is possible to prove that the expectation value of the arbitrary operator $\mathcal{O}(t,x)$ is (see Appendix F.1 for the explicit deduction)

$$\langle\mathcal{O}(t,x)\rangle \simeq \langle\mathcal{O}(t,x)\rangle\big|_{V=0} + i\int_{-\infty}^{t} dt'\ \langle[V(t',t_0),\mathcal{O}_I(t,x)]\rangle\ . \tag{5.3.2}$$

where the initial state was put in the far past $t_0 \to -\infty$. According to definition (5.1.2), and inserting a Heaviside theta function $\Theta(t-t')$ to extend the range of the time integration, the first order contribution reads

$$\delta\langle O(t,x)\rangle = i\int_{-\infty}^{\infty} dt'\,\Theta(t-t')\,\langle[V_I(t'),\mathcal{O}_I(t,x)]\rangle\ . \tag{5.3.3}$$

Let us suppose now a perturbation of the form

$$V(t,x) = \lambda\int dx\,\mu(x)B(x)\,, \tag{5.3.4}$$

with $\mu(x)$ a source, and $B(x)$ an observable operator. As we will see in the next chapters, the introduction of this chemical potential on the gravity side will induce an holographic difussion of chiral bosons that leaves an electrostatic current voltage after its passage, just like in the Quantum Hall Effect (see Chapter 7, in particular Eq. (7.2.16)).

Then the perturbed Hamiltonian (5.1.1) reads

$$H = H_0 + \lambda\int dx\,\mu(x)B(x)\,. \tag{5.3.5}$$

Hence, (5.3.3) is

$$\delta\langle\mathcal{O}(t,x)\rangle = i\int_{-\infty}^{\infty} dt'\ \int dx'\ \mu(t',x')\Theta(t-t')\,\langle[B_I(t',x'),\mathcal{O}_I(t,x)]\rangle\ , \tag{5.3.6}$$

where λ was fixed to the unity since we are considering only linear contributions. We can recognize the integrand expression as the susceptibility of the system. We arrive then to what is known as the *Kubo formula*,

$$\chi_{\mathcal{O}B}(t-t';x,x') = -i\Theta(t-t')\,\langle[\mathcal{O}_I(t,x),B_I(t',x')]\rangle\ , \tag{5.3.7}$$

where the linear response of its associated operator is

$$\delta\langle\mathcal{O}(t,x)\rangle = \int dt'\,dx'\,\chi_{\mathcal{O}B}(t-t';x,x')\mu(t',x')\,. \tag{5.3.8}$$

We obtained then the first-order susceptibility and mean value of an arbitrary operator $\mathcal{O}$.

From the holographic point of view, it is important to mention that the susceptibility is local in the frequency space. This implies that subsequent calculations will enjoy this property, meaning that no global geometric issues, such as global holonomies in black hole backgrounds will be considered. This implies that we can stand on an event horizon background and perform local calculations on that region of the spacetime (see Appendix F.2).

Chapter 6

Anisotropic chiral bosons and bosonization

Lifshitz holography [283] allows to explore holographic properties in non-relativistic frameworks. In this context, a simple theory that exhibits anisotropic scaling in $1+1$ dimensions is the *anisotropic chiral boson theory* [333], which naturally generalizes the chiral boson case [91, 361, 362]. Its action principle is given by

$$I^{\pm}\left[\varphi_{\pm}\right]=\frac{K}{8\pi}\int dt\,dx\left[\dot{\varphi}_{\pm}\varphi_{\pm}'\mp v\left(\partial_{\phi}^{\frac{z+1}{2}}\varphi_{\pm}\right)^{2}\right], \tag{6.0.1}$$

where dots stands for time derivatives while primes for spatial derivatives. The number z is odd integer and corresponds to the dynamical exponent of anisotropic scaling, while v the velocity of chiral excitations. Its equations of motion are

$$\partial_x\left(\dot{\varphi}_{\pm}\pm v\left(-1\right)^{\frac{z+1}{2}}\partial_x^{z}\varphi_{\pm}\right)=0\,, \tag{6.0.2}$$

which reveals the existence of gauge symmetries $\varphi^{\pm}\to\varphi^{\pm}+f^{\pm}(t)$. By means of a Fourier expansion, it is possible to find that the dispersion relation $\omega_k^{\pm}$ of the theory is,

$$\omega_k^{\pm}=\pm vk^{z}\,. \tag{6.0.3}$$

As we see, different chiralities propagates in differents directions. Observe that the parity $\mathcal{P}$ and time-reversal operators $\mathcal{T}$, defined as the action $x\to-x$ and $t\to-t$, respectively, swaps the chirality, since

$$\mathcal{P}\left[I^{\pm}\right] = -I^{\mp}\,, \tag{6.0.4}$$

$$\mathcal{T}\left[I^{\pm}\right] = -I^{\mp}\,. \tag{6.0.5}$$

Thus, the theory is $\mathcal{PT}$-invariant. It is worth mentioning that (6.0.1) recovers the Floreanini-Jackiw case for $z = 1$ [361], and 2D conformal algebra is obtained from nonlocal infinitesimal symmetries [333]. Besides, the infinitesimal transformation

$$\delta\varphi_{\pm} = \eta_k^{\pm}\,, \quad \eta_k^{\pm} = e^{i(kx \pm \omega_k t)}\,, \tag{6.0.6}$$

generates a $u(1)$-current algebra, whose Noether charge is

$$\mathcal{J}_{\pm} = \pm\frac{K}{4\pi}\partial_x\varphi_{\pm}\,, \tag{6.0.7}$$

and its conjugate conserved charge, defined as

$$\mathcal{I}_{\pm} = \mp\frac{K}{4\pi}\partial_t\varphi_{\pm}\,, \tag{6.0.8}$$

fulfills the continuity equation

$$\partial_t\mathcal{J}_{\pm} + \partial_x\mathcal{I}_{\pm} = 0\,. \tag{6.0.9}$$

Details can be found in Appendix G.1.

From the quantum of point of view, the theory was consistently cuantized by means of Dirac [363], Jackiw symplectic [220, 342] and path integral method [52, 364, 365]. In particular, by means of the second method (see Appendix G.2), the Dirac bracket of the fields reads

$$\{\varphi_{\pm}(x), \partial_{x'}\varphi_{\pm}(x')\} = \pm\frac{4\pi}{K}\delta(x - x')\,, \tag{6.0.10}$$

where the passage to quantum mechanics follows the the prescription $\{,\} \to i\,[,]$.

Because the theory exhibits conformal behavior, it is possible to calculate the entropy of a gas of non-interacting anisotropic chiral bosons by means of a compatible extension of Cardy formula [275, 277, 329, 333], whose leading term agrees with the asymptotic sum of power-partitions from number theory [366].

6.1 Fractional statistics

One of the cornerstones of quantum physics is the exchange statistics that emerges when the many-particle wavefunction transforms under interchange of indistinguishable particles. In this regard, anyons are quasiparticles that occur only in two-dimensional systems, whose statistic is *fractional*; it is neither bosonic nor fermionic [367–369].

Anyons had two versions: Abelian and non-abelian. When we interchange abelian anyons, the wavefunction acquires a phase factor $e^{i\theta}$, while in the non-abelian case, the wavefunction not only acquires the latter, but instead, it can change to a fundamentally different quantum state. They lie at the root of many physical phenomena, e.g., topological quantum computation [370, 371], Fractional Quantum Hall Effect [372–377][1], exclusion statistics (generalized Pauli exclusion principle for anyons) [379–383] and Majorana fermions in solid state systems [384]. Remarkably, the abelian flavor was recently detected by two experiments [385, 386].

Here we show that anisotropic chiral bosons fulfill an abelian fractional statistics where the operator $\mathcal{J}_\pm$, defined as

$$\mathcal{J}_\pm = \pm\frac{K}{4\pi}\varphi'_\pm\,, \tag{6.1.1}$$

is the associated Noether current of symmetry (6.0.6), and corresponds to the fermionic number operator under bosonization. Details of the forthcoming calculations can be seen in detail in Appendix G.3.

Following [361], we define the fermionic annihilation $c_\pm$ and creation $c_\pm^\dagger$ operators

$$c_\pm(x) =: e^{-i\sqrt{\frac{K}{2}}\varphi_\pm(x)} :\,, \quad c_\pm^\dagger(x) =: e^{i\sqrt{\frac{K}{2}}\varphi_\pm(x)} :\,, \tag{6.1.2}$$

respectively, where $:\mathcal{O}:$ is the normal-ordering of the operator $\mathcal{O}$. The bosonic fields $\varphi_\pm$ can be expanded as a superposition of creation and annihilation modes

$$\varphi_\pm(x) = \theta_\pm(x) + \theta_\pm^\dagger(x)\,, \tag{6.1.3}$$

where $\theta_+(x)$ correspond to the creation operator and $\theta_+^\dagger(x)$ to the annihilation one, while $\theta_-(x)$ to the annihilation operator and $\theta_-^\dagger(x)$ to the creation one. Performing the passage to quantum mechanics, the Dirac bracket (6.0.10), reads as the following commutation rule

$$[\varphi_\pm(x), \varphi_\pm(x')] = \mp\frac{2i\pi}{K}\text{sign}(x - x')\,. \tag{6.1.4}$$

[1] In fact, the quantum Hall state at filling fraction $\nu = 5/2$ may provide the first experimental evidence of a non-abelian phase [378].

This allow us to find the bosonic algebra of creation and annihilation operators, given by

$$\left[\theta_\pm(x), \theta_\pm^\dagger(x')\right] = \pm\frac{2}{K}\log\left[-2i\pi\left(x - x' + i\eta\right)\right] , \tag{6.1.5}$$

where $\eta \to 0^+$ is a regulator. With the bosonic algebra of creation and annihilation operators obtained, we can prove that operators (6.1.2) satisfy the fermionic anticommutation rule

$$\{c_\pm(x), c_\pm(x')\} = \left\{c_\pm^\dagger(x), c_\pm^\dagger(x')\right\} = \left\{c_\pm(x), c_\pm^\dagger(x')\right\} = 0\,, \quad x \neq x'\,. \tag{6.1.6}$$

Performing a Taylor expansion of the operator $\mathcal{J}_\pm$, and by means of commutation rule (6.1.5) , it is possible to find

$$: c_-^\dagger(x) c_-(x) := -\frac{1}{2\pi}\sqrt{\frac{K}{2}}\partial_x\varphi_-(x)\,, \tag{6.1.7}$$

and

$$: c_+(x) c_+^\dagger(x) := \frac{1}{2\pi}\sqrt{\frac{K}{2}}\partial_x\varphi_+(x)\,. \tag{6.1.8}$$

Therefore, for the chiral/antichiral sector, we arrive to

$$\mathcal{J}_+(x) = \sqrt{\frac{K}{2}} : c_+(x) c_+^\dagger(x) :, \quad \mathcal{J}_-(x) = \sqrt{\frac{K}{2}} : c_-^\dagger(x) c_-(x) : \,. \tag{6.1.9}$$

This justifies calling $\mathcal{J}_\pm$ as the electric charge density. Besides, by virtue of the conservation law (6.0.9), we say that $\mathcal{I}_\pm$ is its conjugate electric charge operator.

Chapter 7

Holographic two-terminal conductance and memory effect

In this chapter, we perform the Hamiltonian reduction of Chern–Simons theory by imposing suitable boundary conditions. The asymptotic degrees of freedom of the theory are going to be captured by (two independent copies) of the anisotropic chiral boson theory, characterized by the action principle (6.0.1). Using the tools developed in Chapter 5, particularly by means of the Kubo formula, we will compute a holographic two-terminal conductance in the DC limit. Finally, we aim to interpret this result as the dynamics of the gravitational field within a spacetime.

Along this chapter, we follow [91, 316, 387].

7.1 Anisotropic chiral movers as gravitational boundary excitations

7.1.1 From Chern–Simons to Wess-Zumino-Novikov-Witten (WZNW)

In Chapter 3.3, in particular in Eq. (3.2.5), we found that the Chern–Simons action can be written as

$$I^{\pm} = I_H^{\pm} + \mathcal{B}^{\pm}\,, \tag{7.1.1}$$

where

$$I_H^{\pm} = -\frac{K}{4\pi}\int dt\, d^2x\, \epsilon^{ij}\left\langle \mathcal{A}_i^{\pm}\dot{\mathcal{A}}_j^{\pm} - \mathcal{A}_0^{\pm}F_{ij}^{\pm}\right\rangle\,, \tag{7.1.2}$$

is the bulk action, and $\mathcal{B}^{\pm}$ a boundary term, whose variation is

$$\delta\mathcal{B}^{\pm} = -\frac{K}{2\pi}\int_{\partial\mathcal{M}} dt\, d\phi \left\langle \mathcal{A}_0^{\pm}\delta\mathcal{A}_\phi^{\pm}\right\rangle . \tag{7.1.3}$$

As said in Chapter 3, the bulk action has a constraint (3.2.14). Without holonomies, a solution of the latter is

$$\mathcal{A}_i^{\pm} = G_{\pm}^{-1}\partial_i G_{\pm} , \tag{7.1.4}$$

where $i = r, \phi$ and $G_{\pm}(t, r, \phi) \in \mathrm{SL}^{\pm}(2, \mathbb{R})$. Replacing in the bulk action, we obtain

$$I_H^{\pm} = -\frac{K}{4\pi}\int dt\, d^2x\, \epsilon^{ij} \left\langle G_{\pm}^{-1}\partial_i G_{\pm}\dot{G}_{\pm}^{-1}\partial_j G_{\pm} + G_{\pm}^{-1}\partial_i G_{\pm} G_{\pm}^{-1}\partial_j \dot{G}_{\pm}\right\rangle . \tag{7.1.5}$$

The second term is a total derivative,

$$\epsilon^{ij}\left\langle G_{\pm}^{-1}\partial_i G_{\pm} G_{\pm}^{-1}\partial_j \dot{G}_{\pm}\right\rangle = -\epsilon^{ij}\left\langle \partial_i G_{\pm}^{-1}\partial_j \dot{G}_{\pm}\right\rangle = -\epsilon^{ij}\partial_i \left\langle G_{\pm}^{-1}\partial_j \dot{G}_{\pm}\right\rangle , \tag{7.1.6}$$

while the first term may be readily written as

$$\epsilon^{ij}\left\langle G_{\pm}^{-1}\partial_i G_{\pm}\dot{G}_{\pm}^{-1}\partial_j G_{\pm}\right\rangle = -\epsilon^{ij}\left\langle G_{\pm}^{-1}\partial_i G_{\pm} G_{\pm}^{-1}\dot{G}_{\pm} G_{\pm}^{-1}\partial_j G_{\pm}\right\rangle . \tag{7.1.7}$$

Choosing manifold orientation $\epsilon^{r\phi} = 1$, the action may be splitted into

$$I_H^{\pm} = I_{\mathrm{WZ}}^{\pm} + I_{\mathrm{Nlsm}}^{\pm} , \tag{7.1.8}$$

where $I_{\mathrm{WZ}}^{\pm}$ is known as the Wess-Zumino term and $I_{\mathrm{Nlsm}}^{\pm}$ is the nonlinear sigma model contribution, defined as

$$I_{\mathrm{WZ}}^{\pm} = \frac{K}{4\pi}\int dt\, d^2x\, \epsilon^{ij}\left\langle G_{\pm}^{-1}\partial_i G_{\pm} G_{\pm}^{-1}\dot{G}_{\pm} G_{\pm}^{-1}\partial_j G_{\pm}\right\rangle , \tag{7.1.9a}$$

$$I_{\mathrm{Nlsm}}^{\pm} = \frac{K}{4\pi}\int_{\partial\mathcal{M}} dt\, d\phi \left\langle G_{\pm}^{-1}\dot{G}_{\pm}'\right\rangle , \tag{7.1.9b}$$

respectively. Primes denote derivatives with respect to ϕ. For latter purposes, it is important to observe that the integrand of (7.1.9a) can be written as

$$\left\langle \left(G_{\pm}^{-1}dG_{\pm}\right)^3\right\rangle = -3\epsilon^{ij}\left\langle G_{\pm}^{-1}\partial_i G_{\pm} G_{\pm}^{-1}\dot{G}_{\pm} G_{\pm}^{-1}\partial_j G_{\pm}\right\rangle . \tag{7.1.10}$$

Let us establish now a bulk/boundary relationship between the solutions of the constraint equation. The constraint solution (7.1.4) holds at the bulk, but at the boundary we can write

$$a_i^{\pm} = g_{\pm}^{-1}\partial_i g_{\pm}\,, \tag{7.1.11}$$

with $G_{\pm}$ the bulk extension of $g_{\pm}$. Because a radial gauge transformation reads

$$\mathcal{A}^{\pm} = b_{\pm}^{-1}\left(d + a^{\pm}\right)b_{\pm}\,, \tag{7.1.12}$$

we have the following equality

$$G_{\pm}^{-1}dG_{\pm} = b_{\pm}^{-1}db_{\pm} + b_{\pm}^{-1}g_{\pm}^{-1}dg_{\pm}b_{\pm} = b_{\pm}^{-1}g_{\pm}^{-1}d\left(g_{\pm}b_{\pm}\right)\,. \tag{7.1.13}$$

This implies that

$$G_{\pm}(t,r,\phi) = g_{\pm}(t,\phi)b_{\pm}(r)\,, \tag{7.1.14}$$

which allows to relate $G_{\pm}$ with $g_{\pm}$ and the gauge parameter $b_{\pm}$, given by the following election

$$b_{\pm}(r) = e^{\pm\log(r/\ell)L_0}\,. \tag{7.1.15}$$

Using the cyclic property of the trace, $I^{\pm}_{\text{Nlsm}}$ reduces to

$$I^{\pm}_{\text{Nlsm}} = \frac{K}{4\pi}\int_{\partial\mathcal{M}} dt\,d\phi\,\left\langle b_{\pm}^{-1}g_{\pm}^{-1}\partial_t\left(g_{\pm}b_{\pm}\right)'\right\rangle = \frac{K}{4\pi}\int_{\partial\mathcal{M}} dt\,d\phi\,\left\langle g_{\pm}^{-1}\dot{g}_{\pm}'\right\rangle\,. \tag{7.1.16}$$

The addition $I^{\pm}_{\text{WZ}}+I^{\pm}_{\text{Nlsm}}$ is known as the Wess-Zumino-Novikov-Witten (WZNW) model [388–391].

We solved the constraint of the theory and we obtained, as a dual theory of AdS$_3$ GR, the aforementioned model.

7.1.2 From WZNW to anisotropic chiral bosons

Here we address the reduction from the WZNW model to the anisotropic chiral boson theory. This reduction is going to be addressed through the Gauss decomposition for $G_{\pm}$ and $g_{\pm}$,

$$G_{\pm}(t,r,\phi) = e^{X_{\pm}(t,r,\phi)L_{\pm1}}e^{\pm\Phi_{\pm}(t,r,\phi)L_0}e^{Y_{\pm}(t,r,\phi)L_{\mp1}}\,, \tag{7.1.17a}$$

$$g_{\pm} = e^{x_{\pm}(t,\phi)L_{\pm1}}e^{\pm\varphi_{\pm}(t,\phi)L_0}e^{y_{\pm}(t,\phi)L_{\mp1}}\,. \tag{7.1.17b}$$

respectively.

For the chiral sector, the Gauss decomposition (7.1.17a) of G_+ acquires the following representation

$$G_+ = \begin{pmatrix} 1 & -X_+ \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{-\Phi_+/2} & 0 \\ 0 & e^{\Phi_+/2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ Y_+ & 1 \end{pmatrix} , \tag{7.1.18}$$

while for the antichiral one, the representation of G_- is

$$G_- = \begin{pmatrix} 1 & 0 \\ X_- & 1 \end{pmatrix} \begin{pmatrix} e^{\Phi_-/2} & 0 \\ 0 & e^{-\Phi_-/2} \end{pmatrix} \begin{pmatrix} 1 & -Y_- \\ 0 & 1 \end{pmatrix} . \tag{7.1.19}$$

Replacing the latter in (7.1.10), and after direct calculations, the integrand of the WZ action reads

$$\left\langle \left(G_\pm^{-1} dG_\pm\right)^3 \right\rangle = -3dr\, dt\, d\phi\, \epsilon^{\alpha\beta\gamma} \partial_\alpha \left(e^{\Phi_\pm} \partial_\beta X_\pm \partial_\gamma Y_\pm\right) , \tag{7.1.20}$$

yielding

$$I_{\mathrm{WZ}}^{\pm} = \frac{K}{4\pi} \int dr\, dt\, d\phi\, e^{\Phi_\pm} \partial_r \left(\epsilon^{tr\phi} \dot{X}_\pm Y'_\pm + \epsilon^{t\phi t} X'_\pm \dot{Y}_\pm\right) ; \tag{7.1.21a}$$

where periodicity on the fields was assumed. With manifold orientation $\epsilon^{tr\phi} = 1$, we arrive then to the following reduction

$$I_{\mathrm{WZ}}^{\pm} = \frac{K}{4\pi} \int_{\partial\mathcal{M}} dt d\phi\, e^{\Phi_\pm} \left(\dot{X}_\pm Y'_\pm - X'_\pm \dot{Y}_\pm\right) . \tag{7.1.22}$$

Now we aim to perform a similar procedure to the nonlinear sigma term (7.1.16). Integrating by parts the angular derivative in (7.1.16), we readily obtain

$$I_{\mathrm{Nlsm}}^{\pm} = \frac{K}{4\pi} \int_{\partial\mathcal{M}} dt d\phi \left[\frac{1}{2} \dot{\varphi}_\pm \varphi'_\pm - e^{\varphi_\pm} \left(\dot{x}_\pm y'_\pm + x'_\pm \dot{y}_\pm\right)\right] . \tag{7.1.23}$$

If we add the WZ–term plus the Nlsm, namely the WZNW–model, we arrive to the boundary description of AdS$_3$ GR, up to the boundary term (7.1.3), which will be treated at the end.

We need a relation between the bulk functions $\{X_\pm, \Phi_\pm, Y_\pm\}$ and the boundary $\{x_\pm, \varphi_\pm, y_\pm\}$ ones, because they are a different set of functions. The path will be given by means of Eq. (7.1.14), with gauge parameter $b_\pm = e^{\pm \log(r/\ell) L_0}$. Hence, for the chiral sector we have

$$\begin{pmatrix} e^{-\Phi_+/2}\left(1 - e^{\Phi_+} X_+ Y_+\right) & -e^{\Phi_+/2} X_+ \\ e^{\Phi_+/2} Y_+ & e^{\Phi_+/2} \end{pmatrix} = \begin{pmatrix} e^{-\varphi_+/2} \sqrt{\frac{\ell}{r}} \left(1 - e^{\varphi_+} x_+ y_+\right) & -e^{\varphi_+/2} \sqrt{\frac{r}{\ell}} x_+ \\ e^{\varphi_+/2} \sqrt{\frac{\ell}{r}} y_+ & e^{\varphi_+/2} \sqrt{\frac{r}{\ell}} \end{pmatrix} .$$

From elements 22, 12 and 21 we obtain the consistency conditions $e^{\Phi_+} = \frac{r}{\ell} e^{\varphi_+}$, $X_+ = x_+$ and $Y_+ = \frac{\ell}{r} y_+$, respectively. On the other hand, for the antichiral sector, we have the following matrix equality

$$\begin{pmatrix} e^{\Phi_-/2} & -e^{\Phi_-/2}Y_- \\ e^{\Phi_-/2}X_- & e^{-\Phi_-/2}\left(1-e^{\Phi_-}X_-Y_-\right) \end{pmatrix} = \begin{pmatrix} e^{\varphi_-/2}\sqrt{\frac{r}{\ell}} & -e^{\varphi_-/2}y_-\sqrt{\frac{\ell}{r}} \\ e^{\varphi_-/2}x_-\sqrt{\frac{r}{\ell}} & e^{-\varphi_-/2}\sqrt{\frac{\ell}{r}}\left(1-e^{\varphi_-}x_-y_-\right) \end{pmatrix} .$$

Thus, from elements 11, 12 and 21 we get conditions $e^{\Phi_-} = \frac{r}{\ell}e^{\varphi_-}$, $Y_- = \frac{\ell}{r}y_-$ and $X_- = x_-$, respectively. In sum, we have the forthcoming consistency conditions for both sectors

$$e^{\Phi_\pm} = \frac{r}{\ell}e^{\varphi_\pm} , \quad X_\pm = x_\pm , \quad Y_\pm = \frac{\ell}{r}y_\pm , \tag{7.1.24}$$

With them, $I_H^\pm = I_{\text{WZNW}}^\pm = I_{\text{WZ}}^\pm + I_{\text{Nlsm}}^\pm$ reduces to the 2-dimensional surface integral

$$I_{\text{WZNW}}^\pm = \frac{K}{4\pi}\int_{\partial\mathcal{M}} dt\, d\phi \left(\frac{1}{2}\dot{\varphi}_\pm\varphi'_\pm - 2e^{\varphi_\pm}x'_\pm\dot{y}_\pm\right) . \tag{7.1.25}$$

At this stage, the procedure was general, however, we aim to obtain a further reduction of the previous action by imposing suitable boundary conditions. These are given by

$$a_\phi^\pm = \pm\frac{4\pi}{K}\mathcal{J}_\pm(t,\phi)L_0 , \tag{7.1.26}$$

$$a_t^\pm = \pm v\,(-1)^{\frac{z-1}{2}}\,\partial_\phi^{z-1}a_\phi^\pm + \mu(t,\phi)L_0 , \tag{7.1.27}$$

where $v > 0$ is a coupling constant with dimensions of $[\text{length}]^{z-1}$ and, as we will see next, μ to an external perturbation, and z to the odd integer dynamical exponent of Lifshitz scaling. It is important to say that for $z = 1$ we obtain the Brown-Henneaux case in the Chern-Simons formulation.

Recalling solution (7.1.11), we can equate the latter with the angular boundary condition (7.1.26), as following

$$g_\pm^{-1}\partial_\phi g_\pm = \pm\frac{4\pi}{K}\mathcal{J}_\pm(t,\phi)L_0 . \tag{7.1.28}$$

By virtue of the Gauss decomposition (7.1.17b), it implies the following relationships

$$e^{\varphi_\pm}x'_\pm y_\pm + \frac{1}{2}\varphi'_\pm = \pm\frac{2\pi}{K}\mathcal{J}_\pm , \quad e^{\varphi_\pm}x'_\pm = 0 , \quad e^{\varphi_\pm}x'_\pm y_\pm^2 + y'_\pm + y_\pm\varphi'_\pm = 0 . \tag{7.1.29}$$

Replacing the middle equation into the first one, we arrive to the following crucial relation,

$$\mathcal{J}_\pm = \pm\frac{K}{4\pi}\partial_\phi\varphi_\pm , \tag{7.1.30}$$

allowing us to connect the angular (spatial) derivative of the field $\varphi_\pm$ with the asymptotic function

$\mathcal{J}_{\pm}(t,\phi)$. The exponential of the WZNW action (7.1.25) vanishes by virtue of the second equation of (7.1.29), so the WZNW reduces to the kinetic integral functional

$$I^{\pm}_{\text{kinetic}} = \frac{K}{8\pi}\int dt\, d\phi\, \dot{\varphi}_{\pm}\varphi'_{\pm}\,. \tag{7.1.31}$$

7.1.3 Reduction of the boundary term

Now we must take care of the boundary term (7.1.3). After a gauge transformation, it reads

$$\delta\mathcal{B}^{\pm} = -\frac{K}{2\pi}\int_{\partial\mathcal{M}} dt\, d\phi \left\langle a^{\pm}_{t}\delta a^{\pm}_{\phi}\right\rangle\,,$$

where the lowercase fields $a^{\pm}$ do not have radial dependence.

Replacing the angular boundary condition (7.1.26) and the temporal one (7.1.27) in the variation of the boundary term, we obtain

$$\delta\mathcal{B}^{\pm} = -\frac{K}{2\pi}\int_{\partial\mathcal{M}} dt\, d\phi \left\langle \pm v\,(-1)^{\frac{z-1}{2}}\,\partial_{\phi}^{z-1}a^{\pm}_{\phi}\delta a^{\pm}_{\phi}\right\rangle - \int_{\partial\mathcal{M}} dt\, d\phi\, \mu\delta\mathcal{J}_{\pm}\,. \tag{7.1.32}$$

Then, the boundary term integrates as following,

$$\mathcal{B}^{\pm} = \mp\frac{Kv}{4\pi}\int_{\partial\mathcal{M}} dt\, d\phi \left\langle \left(\partial_{\phi}^{\frac{z-1}{2}}a^{\pm}_{\phi}\right)^{2}\right\rangle - \int_{\partial\mathcal{M}} dt\, d\phi\, \mu\mathcal{J}_{\pm}\,, \tag{7.1.33}$$

rendering a well-defined action. According to the Gauss decomposition (7.1.17b) and the second consistency condition of (7.1.29), the latter boundary contribution– that deforms the WZNW model–, yield a term proportional to $\sim\left(\partial_{\phi}^{\frac{z+1}{2}}\varphi_{\pm}\right)^{2}$, namely

$$\mathcal{B}^{\pm}_{(z)} = \mp\frac{Kv}{8\pi}\int_{\partial\mathcal{M}} dt\, d\phi \left(\partial_{\phi}^{\frac{z+1}{2}}\varphi_{\pm}\right)^{2} \mp\frac{K}{4\pi}\int_{\partial\mathcal{M}} dt\, d\phi\, \mu\varphi'_{\pm}\,. \tag{7.1.34}$$

We have all the ingredients to obtain the anisotropic chiral boson theory, with a perturbation μ, as an effective description that captures the boundary degrees of freedom of the gravitational field. Replacing the previous result and the kinetic term (7.1.31) in the action (7.1.1), we obtain

$$I^{\pm}\left[\varphi_{\pm}\right] = \frac{K}{8\pi}\int dt\, d\phi \left[\dot{\varphi}_{\pm}\varphi'_{\pm} \mp v\left(\partial_{\phi}^{\frac{z+1}{2}}\varphi_{\pm}\right)^{2} \mp 2\mu\varphi'_{\pm}\right]\,, \tag{7.1.35}$$

where the gravitational description is given by the substraction of the two anisotropic chiral boson actions (6.0.1), with the source μ coupled to $\mathcal{J}_{\pm}$.

In summary, by solving the constraint equation $F^{\pm}_{ij}=0$ and by means of Gauss decomposition (7.1.17), we performed the Hamiltonian reduction of AdS_3 GR . We obtained at the boundary a substraction

$$I\left[\varphi_+,\varphi_-\right]=I^+\left[\varphi_+\right]-I^-\left[\varphi_-\right]\,, \tag{7.1.36}$$

of two anisotropic chiral boson action (7.1.35) [333].

7.2 Boundary two-terminal conductance

In the previous section, we proved that the anisotropic chiral boson theory (with the external source μ coupled to $\mathcal{J}_\pm$), is a holographic description of AdS_3 GR through suitable boundary conditions. Using the $u(1)$ operator $\mathcal{J}_\pm$ and its conjugate current $\mathcal{I}_\pm$, given in (6.0.7) and (6.0.8),

$$\mathcal{J}_\pm(t,x)=\pm\frac{K}{4\pi}\partial_x\varphi_\pm(t,x)\,,\quad \mathcal{I}_\pm(t,x)=\mp\frac{K}{4\pi}\partial_t\varphi_\pm\,, \tag{7.2.1}$$

respectively, we aim to obtain its associated susceptibility through the Kubo formula (5.3.7)

$$\tilde{\chi}^\pm_{\mathcal{I},\mathcal{J}}(\omega;x,x')=-i\int_{-\infty}^{\infty}dt\,\Theta(t)e^{i\omega t}\;\langle[\mathcal{I}_\pm(x,t),\mathcal{J}_\pm(x',0)]\rangle\;. \tag{7.2.2}$$

As reviewed in the previous chapter and Appendix G.3.2, anisotropic chiral excitations exhibits anyonic nature, allowing us to interpret $\mathcal{J}_\pm$ and $\mathcal{I}_\pm$ as electric charge and density fermionic operators, respectively.

7.2.1 Anisotropic chiral susceptibility

Here we obtain the holographic susceptibility associated to the anisotropic chiral bosons. Consider the $u(1)$ operators defined in (6.0.7) and (6.0.8). We aim to compute the linear response associated to these operators, which are given by the following expression

$$\delta\left\langle\tilde{\mathcal{I}}_\pm(\omega,x)\right\rangle=\int_{\infty}^{\infty}dx'\,\tilde{\mu}(x',\omega)\;\tilde{\chi}^\pm_{\mathcal{I},\mathcal{J}}\left(\omega;x,x'\right), \tag{7.2.3}$$

where $\tilde{\mu}(x',\omega)$ is the Fourier frequency-space source and $\tilde{\chi}^\pm_{\mathcal{I},\mathcal{J}}(\omega;x,x')$ the susceptibility. According to the bosonization performed, $\mathcal{J}_\pm$ and $\mathcal{I}_\pm$ are related with electric charge transport, so we expect that the quantity

$$\delta\left\langle\tilde{\mathcal{I}}_{\text{tot}}\right\rangle\equiv\delta\left\langle\tilde{\mathcal{I}}_+\right\rangle+\delta\left\langle\tilde{\mathcal{I}}_-\right\rangle\;, \tag{7.2.4}$$

gives the total expected current intensity carried by chiral bosons with AC frequency ω. Nevertheless, in order to find the latter quantity it is necessary to calculate the susceptibility of the theory.

Recall that this can be found by means of the Kubo formula

$$\tilde{\chi}^{\pm}_{\mathcal{I},\mathcal{J}}(\omega;x,x') = -i\int_{-\infty}^{\infty} dt\,\Theta(t)e^{i\omega t}\ \langle[\mathcal{I}_\pm(x,t),\mathcal{J}_\pm(x',0)]\rangle\ . \tag{7.2.5}$$

We start explaining how to compute the different-time commutator. Because the fields $\varphi_\pm(t,x)$ satisfies the superposition principle, they can be expanded in Fourier series

$$\varphi_\pm(x,t) = \int_{-\infty}^{\infty}\frac{dk}{k}\,e^{i(kx\pm\omega_k t)}b_{\pm,k}\,, \tag{7.2.6}$$

where $b_{\pm,k}$ correspond to the k-th wavenumber annihilation operator, $b^\dagger_{\pm,k}$ the k-th wavenumber creation operator, and ω_k its dispersion relation (6.0.3).

Before calculating the commutator of $\mathcal{I}_\pm$ with $\mathcal{J}_\pm$, it is necessary first to obtain the commutation algebra of annihilation operators $b_{\pm,k}$. Recall that the anisotropic chiral bosons satisfy the commutation rule (6.1.4),

$$[\varphi_\pm(x),\partial_{x'}\varphi_\pm(x')] = \pm\frac{4i\pi}{K}\delta(x-x')\,. \tag{7.2.7}$$

If we invert the Fourier transform, we can find the commutation algebra of annihilation operators

$$[b_{\pm,k},b_{\pm,k'}] = \pm\frac{2}{K}k\delta(k+k')\,. \tag{7.2.8}$$

Details can be found in Appendix H.1.

Now we are in position to compute the different-time commutator. Because the operators $\mathcal{J}_\pm$ and $\mathcal{I}_\pm$ are proportional to the space and time derivative of the fields, respectively, we have to take derivatives of the fields in order to build these operators. Then, using (7.2.8), we readily obtain

$$[\mathcal{I}_\pm(x,t),\mathcal{J}_\pm(x',0)] = \left[\mp\frac{K}{4\pi}\partial_t\varphi_\pm(x,t),\pm\frac{K}{4\pi}\partial_{x'}(x',0)\right] = \frac{K}{8\pi^2}\int_{-\infty}^{\infty}dk\,\omega_k e^{ik(x-x')}e^{\pm i\omega_k t}\,. \tag{7.2.9}$$

Thus, we obtained the different-time commutator associated to the operators $\mathcal{I}_\pm$ and $\mathcal{J}_\pm$. Replacing this result in the Kubo–formula (7.2.5), we arrive to the following integral expression

$$\tilde{\chi}^{\pm}_{\mathcal{I},\mathcal{J}}(\omega;x,x') = -\frac{iK}{8\pi^2}\int_{-\infty}^{\infty}dk\,\omega_k e^{ik(x-x')}\int_{-\infty}^{\infty}dt\,\Theta(t)e^{i(\omega\pm\omega_k)t}\,. \tag{7.2.10}$$

The time-integral is just the Fourier transform of the theta Heaviside function, given by

$$\int_{-\infty}^{\infty}\Theta(t)e^{i(\omega\pm\omega_k)t} = \frac{i}{\omega\pm\omega_k+i\epsilon}\,,$$

where ϵ is a positive regulator that appears as a consequence of the susceptibility causality. As proved in Appendix H.2, the susceptibility acquires then the following form

$$\tilde{\chi}^{\pm}_{\mathcal{I},\mathcal{J}}(\omega; x, x') \equiv \frac{K}{4\pi}\left[\pm\delta(x-x') \mp (\omega + i\epsilon)\, F^{\pm}\,(x - x'; \omega)\right] , \tag{7.2.11}$$

where the function $F^{\pm}(x - x'; \omega)$ is the integral expression

$$F^{\pm}(x - x'; \omega) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{e^{ik(x-x')}}{\omega \pm \omega_k + i\epsilon} , \tag{7.2.12}$$

which has $p = 0, 1, 2, \ldots, z-1$ simple poles. Using the residue theorem, we obtain

$$\omega F^{\pm}(y; \omega) = -i\Theta(y) \sum_{k_p^{\pm} \in \mathrm{Im}_{>}} \frac{\omega_{k_p^{\pm}} e^{ik_p^{\pm} y}}{\omega'_{k_p^{\pm}}} + i\Theta(-y) \sum_{k_p^{\pm} \in \mathrm{Im}_{<}} \frac{\omega_{k_p^{\pm}} e^{ik_p^{\pm} y}}{\omega'_{k_p^{\pm}}} , \tag{7.2.13}$$

where $y = x - x'$ and $\gtrless$ denotes the upper/lower complex plane sector, respectively.

The latter result is general, since we never used the dispersion relation $\omega_k = vk^z$ of the anisotropic chiral boson, so in principle, any dispersion relation is valid at this point.

If we now specialize to the anisotropic chiral boson case, the numerator and denominator of the latter expression get simplified and the dependence on the dynamical exponent $z^{\pm}$ emerge through functions $\Delta^{\pm}_{\gtrless}$, yielding

$$\omega F^{\pm}(y; \omega) = -\Theta(y)\partial_y \Delta^{\pm}_{>}(y; \omega) + \Theta(-y)\partial_y \Delta^{\pm}_{<}(y; \omega) , \tag{7.2.14}$$

where $\Delta^{\pm}_{\gtrless}$ is defined as

$$\Delta^{\pm}_{\gtrless}(y; \omega) = \sum_{k_p^{\pm} \in \mathrm{Im}_{\gtrless}} \frac{e^{ik_p^{\pm} y}}{z^{\pm}} . \tag{7.2.15}$$

For details in the explicit contour integration of function (7.2.12), see Appendix H.3.

7.2.2 Total current intensity and conductance

We computed the associated susceptibility to the operators $\mathcal{I}_{\pm}$ and $\mathcal{J}_{\pm}$. We have to further compute the associated total electric current, given by the linear response (7.2.3).

We must now specify a particular chemical potential. Turning on an external source that corresponds to an electrostatic potential that produces a fermionic difussion, as seen in Fig. 7.1,

$$\tilde{\mu}(x', \omega) = V_L \Theta(x_L - x') + V_R \Theta(x' - x_R) . \tag{7.2.16}$$

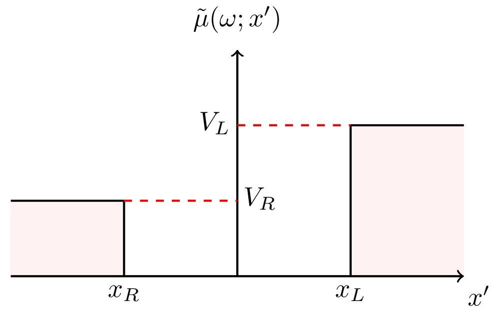

Figure 7.1: Chemical potential $\tilde{\mu}$ in frequency space, where we supposed (only for drawing purposes) $V_L > V_R$.

and replacing the chemical potential (7.2.16) in (7.2.3) we arrive to

$$\delta \left\langle \tilde{\mathcal{I}}_\pm(\omega, x) \right\rangle = \mp \frac{K}{4\pi} \int_{-\infty}^{\infty} dy \, \left[V_L \Theta(x_L - x + y) + V_R \Theta(x - y - x_R) \right] \times . \\ \times \left[-\delta(y) - \Theta(y) \partial_y \Delta^\pm_>(y; \omega) + \Theta(-y) \partial_y \Delta^\pm_<(y; \omega) \right] . \tag{7.2.17}$$

After direct manipulations (see Appendix H.4), the linear response of the mean current intensity reads

$$\begin{aligned} \delta \left\langle \tilde{\mathcal{I}}_\pm(\omega, x) \right\rangle = \mp \frac{K}{4\pi} \{ & - \left[V_L \Theta(x_L - x) + V_R \Theta(x - x_R) \right] \\ & + V_L \left[\Theta(x - x_L) \Delta^\pm_>(x - x_L; \omega) - \Theta(x_L - x) \Delta^\pm_<(x - x_L; \omega) \right] \\ & - V_R \left[\Theta(x - x_R) \Delta^\pm_>(x - x_R; \omega) - \Theta(x_R - x) \Delta^\pm_<(x - x_R; \omega) \right] \\ & + \left[V_L \Theta(x_L - x) + V_R \Theta(x - x_R) \right] \left(\Delta^\pm_>(0; \omega) + \Delta^\pm_<(0; \omega) \right) \} . \end{aligned} \tag{7.2.18}$$

This expression is valid in the AC regime. Nevertheless, we aim to recover an Ohm's law in the DC limit $\omega \to 0$. This means that it is necessary to calculate expression $\Delta^\pm_\gtrless(0)$, given in definition (7.2.15). It turns out that $\Delta^\pm_\gtrless(0)$ is a constant that counts the number of poles. On the upper and lower complex plane, the $i\epsilon$–prescription implies that we have $\frac{z^\pm \mp 1}{2}$ poles on the upper plane and $\frac{z^\pm \pm 1}{2}$ poles on the lower one. In sum,

$$\Delta^\pm_>(0) = \frac{z^\pm \mp 1}{2z^\pm}, \quad \Delta^\pm_<(0) = \frac{z^\pm \pm 1}{2z^\pm} . \tag{7.2.19}$$

Thus, $\Delta^\pm_>(0) + \Delta^\pm_<(0) = 1$. This implies that the first and second terms cancel with the seventh and eigth ones of (7.2.18), from where we obtain

$$\delta \left\langle \tilde{\mathcal{I}}_\pm(\omega, x) \right\rangle |_{\omega \to 0} = \mp \frac{K}{4\pi} \left\{ V_L \left[\Theta(x - x_L) - \frac{z^\pm \pm 1}{2z^\pm} \right] - V_R \left[\Theta(x - x_R) - \frac{z^\pm \pm 1}{2z^\pm} \right] \right\} . \tag{7.2.20}$$

We obtained the mean current intensity in a first-order approximation of the chiral/antichiral sectors. Then, the total mean current intensity reads

$$\delta\left\langle\tilde{\mathcal{I}}_{\text{tot}}\right\rangle \equiv \delta\left\langle\tilde{\mathcal{I}}_{+}\right\rangle + \delta\left\langle\tilde{\mathcal{I}}_{-}\right\rangle = \frac{K}{8\pi}\Delta V\left(\frac{1}{z^{+}}+\frac{1}{z^{-}}\right), \tag{7.2.21}$$

where we defined the electric potential difference $\Delta V = V_L - V_R$. We can recognize the two-terminal conductance σ given by the following expression

$$\sigma = \frac{\delta\left\langle\tilde{\mathcal{I}}_{\text{tot}}\right\rangle}{\Delta V} = \frac{K}{8\pi}\left(\frac{1}{z^{+}}+\frac{1}{z^{-}}\right). \tag{7.2.22}$$

This is the two-terminal conductance associated to anisotropic chiral excitations as an effective description of AdS_3 GR at the radial infinity, subjected to the perturbation μ given in Eq. (7.2.16).

7.3 The gravitational side

Through establishing the bulk/boundary correspondence, here we aim to provide a gravitational perspective of operators $\mathcal{J}_{\pm}$ and $\mathcal{I}_{\pm}$.

7.3.1 The bulk correspondence

Consider the Chern-Simons action coupled to a covariantly conserved current source $J^{\mu}_{\pm}$,

$$I^{\pm}_{\text{CS}}\left[\mathcal{A}^{\pm}; J_{\pm}\right] = I^{\pm}_{\text{CS}}\left[\mathcal{A}^{\pm}\right] - \int d^3x \left\langle \mathcal{A}^{\pm}_{\mu} J^{\mu}_{\pm}\right\rangle. \tag{7.3.1}$$

First-order Einstein equations in this formalism reads as

$$\frac{K}{2\pi}\epsilon^{\alpha\mu\nu}F^{\pm}_{\mu\nu} = J^{\alpha}_{\pm}. \tag{7.3.2}$$

This allow to identify a physical process in the bulk where an external source modifies the Chern-Simons field $\mathcal{A}^{\pm}$. Choosing orientation $\epsilon^{rxt} = 1$, we obtain

$$\frac{K}{2\pi}F^{\pm}_{tx} = J^{r}_{\pm}. \tag{7.3.3}$$

With a gauge transformation, where $b_{\pm}(r)$ is the gauge parameter that captures the radial dependence of the fields, the gauge connection will not depend on the radius. With this transformation, Einstein equations (7.3.3) simplifies, and acquires the following form

$$b^{-1}_{\pm} f^{\pm}_{tx} b_{\pm} = \frac{2\pi}{K} J^{r}_{\pm}, \tag{7.3.4}$$

with $f^{\pm}_{tx} = \partial_t a^{\pm}_x - \partial_x a^{\pm}_t + \left[a^{\pm}_t, a^{\pm}_x\right]$. The source is defined through a current pointing in the radial direction

$$J^r_{\pm} = -\frac{K}{2\pi}\partial_x \mu b^{-1}_{\pm} L_0 b_{\pm}\,, \qquad J^t_{\pm} = J^x_{\pm} = 0\,, \tag{7.3.5}$$

where its precise dependence on the group element $b_{\pm}$ ensures the conservation law $\partial_\alpha J^\alpha_{\pm} + \left[\mathcal{A}^{\pm}_\alpha, J^\alpha_{\pm}\right] = 0$. The functional dependence of μ will be chosen so that (7.3.4) describes the boundary perturbation of the previous section.

Because (7.3.4) reduces to $f^{\pm}_{tx} = \partial_x \mu L_0$ and with the following boundary conditions

$$a^{\pm}_x = \pm\frac{4\pi}{K}\mathcal{J}_{\pm} L_0\,, \quad a^{\pm}_t = \mp\frac{4\pi}{K}\mathcal{I}_{\pm} L_0\,. \tag{7.3.6}$$

we can write (7.3.4) as

$$\partial_t \mathcal{J}_{\pm} + \partial_x \mathcal{I}_{\pm} = \pm\frac{K}{4\pi}\partial_x \mu\,. \tag{7.3.7}$$

Now, from boundary conditions (7.1.27), we directly obtain that $\mathcal{I}_{\pm}$ is related with $\mathcal{J}_{\pm}$ as

$$\mathcal{I}_{\pm} = \mathcal{I}^{\text{in}}_{\pm} \pm v\,(-1)^{\frac{z+1}{2}}\,\partial^{z-1}_x \mathcal{J}_{\pm}\,, \tag{7.3.8}$$

or

$$\delta\mathcal{I}_{\pm} = \pm v\,(-1)^{\frac{z+1}{2}}\,\partial^{z-1}_x \mathcal{J}_{\pm}\,, \tag{7.3.9}$$

where $\mathcal{I}^{\text{in}}_{\pm}$ is an initial condition to be fixed. Importantly, this constant will be interpreted in light of a particular initial gravitational configuration where the system consequently evolves. Therefore, by virtue of relationship (7.3.8), equation (7.3.7) transforms into the anisotropic chiral boson equation

$$\partial_t \mathcal{J}_{\pm} \pm v\,(-1)^{\frac{z+1}{2}}\,\partial^z_x \mathcal{J}_{\pm} = \pm\frac{K}{4\pi}\partial_x \mu\,. \tag{7.3.10}$$

The formal solution of the latter is

$$\mathcal{J}_{\pm}(t,x) = \mathcal{J}^{\text{in}}_{\pm} \pm \frac{K}{4\pi}\int dt'\,dx'\,G^{\pm}_{\text{R}}(x-x', t-t')\,\partial_{x'}\mu(t',x')\,, \tag{7.3.11}$$

where R dubs the retarded Green function. The reason to consider the retarded prescription lies in causality, in the sense that we measure after the physical process has been taken place, and not before. Mathematically, this can be accomplished if we impose that the ordinary Green function vanishes for negative time, i.e.,

$$G^{\pm}(x-x', t-t' < 0) = 0\,.$$

Because the retarded propagator emerge after the perturbation has been placed, it is a sensible quantity that, according to linear response theory (see Chapter 5), can be recognized as the susceptibility of the theory $\chi(x-x', t-t')$. Thus, $\mathcal{J}_{\pm} - \mathcal{J}_{\pm}^{\text{in}} \equiv \delta\left\langle \mathcal{J}_{\pm} \right\rangle$ stands for the expectation value of the electric charge operator; while analogue analysis stands for $\delta\left\langle \mathcal{I}_{\pm} \right\rangle$, the expectation value of the intensity current of both sectors.

Expanding in Fourier modes, the retarded Green function reads

$$G_{\text{R}}^{\pm}(x-x', t-t') = \frac{1}{(2\pi)^2}\int dk\, d\omega\, e^{ik(x-x')} e^{-i\omega(t-t')} \tilde{G}_{\text{R}}^{\pm}(k,\omega)\,, \tag{7.3.12}$$

where it is possible to obtain the inverse of the differential operator $\partial_t \pm v\,(-1)^{\frac{z+1}{2}}\,\partial_x^z$,

$$\tilde{G}_{\text{R}}^{\pm}(k,\omega) = \frac{i}{\omega \pm \omega_k + i\epsilon}\,, \tag{7.3.13}$$

with $\omega_k = vk^z$. As proved in Appendix H.5, replacing in (7.3.11), we obtain

$$\mathcal{J}_{\pm}(t,x) = \mathcal{J}_{\pm}^{\text{in}} \pm \frac{iK}{8\pi}\int dt'\, dx'\, d\omega\, F^{\pm}\left(x-x',\omega\right) \partial_{x'}\mu(t',x')\,, \tag{7.3.14}$$

where the integral function $F^{\pm}$ is the same as the definition given in (7.2.12),

$$F^{\pm}(x-x',\omega) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{e^{ik(x-x')}}{\omega \pm \omega_k + i\epsilon}\,.$$

Because $\mathcal{I}_{\pm}$ is related with $\mathcal{J}_{\pm}$ through Eq. (7.3.8), then with result (7.3.14) it is possible to obtain

$$\tilde{\mathcal{I}}_{\pm} = \tilde{\mathcal{I}}_{\pm}^{\text{in}} \mp \frac{K}{4\pi}\omega \int dx'\, F^{\pm}(x-x',\omega)\,\mu(x-x',\omega)\,. \tag{7.3.15}$$

We have to solve the integral expression. In order to make contact with the holographic result of the previous section, let us now specify the source in the following form

$$\tilde{\mu}(x',t) = \delta(t)\left[V_L\delta(x_L - x') + V_R\delta(x'-x_R)\right]\,, \tag{7.3.16}$$

drawn in Fig. 7.1. The temporal localization of the external source models a process where the initial and final configurations corresponds to solutions of vacuum Einstein's equation. This election yield the same result as the one obtained from the putative dual theory (7.2.22) at the DC limit, but we recovered it now from the bulk perspective.

Particular questions must be addressed at this point:

(I) To answer how we recover an associated spacetime where anisotropic chiral bosons evolves within,

(II) to elucidate the geometric interpretation of operators $\mathcal{I}_{\pm}$ and $\mathcal{J}_{\pm}$,

(III) answer if the computation of σ, obtained in (7.2.22) is characterized by an adiabatic process, and

(IV) to understand what gravitational phenomena can be associated to these boundary excitations.

7.3.2 Linear response for near-horizon boundary conditions

In order to answer points *(I)* and *(II)*, let us construct a spacetime solution. In terms of Chern–Simons gauge fields, the aforementioned can be obtained from the following relationship

$$g_{\mu\nu} = \frac{\ell^2}{2}\left\langle \left(\mathcal{A}_\mu^+ - \mathcal{A}_\mu^-\right)\left(\mathcal{A}_\nu^+ - \mathcal{A}_\nu^-\right)\right\rangle . \tag{7.3.17}$$

To construct those solutions, consider a particular radial dependence of the group element $b_{\pm}$ [114, 147],

$$b_{\pm}(r) = \exp\left[\pm\frac{r}{2\ell}\left(L_1 - L_{-1}\right)\right] . \tag{7.3.18}$$

Because the spatial component of the gauge field (7.1.26) lies on the hyperbolic conjugacy class (see table 3.1), the resulting gravitational configurations are typified by black hole solutions [343]. Hence, closer to the Rindler horizon $r = 0$, it is possible to find the following spacetime metric

$$ds^2 = \left(\frac{2\pi\ell^2}{K}\right)^2\left(\mathcal{I}_{\text{tot}}^2 dt^2 - 2\mathcal{I}_{\text{tot}}\rho_{\text{tot}} dt dx + \rho_{\text{tot}}^2 dx^2\right) + dr^2 + \mathcal{O}\big(r^2\big) , \tag{7.3.19}$$

where we defined $\mathcal{I}_{\text{tot}} = \mathcal{I}_+ + \mathcal{I}_-$ and $\rho_{\text{tot}} = \mathcal{J}_+ + \mathcal{J}_-$. The operators $\mathcal{J}_{\pm}$ and $\mathcal{I}_{\pm}$ corresponds to the ones specified in (6.0.7) and (6.0.8), respectively. The $\mathcal{O}\big(r^2\big)$ terms stands for subleading components.

From the geometry, we can conclude the following. Operators $\mathcal{I}_{\text{tot}}$ and ρ_{tot} induces a rotation to the black hole, while the horizon area is measured by an observer in a rotating frame close to $r = 0$, whose operator is defined as an integral of the total charge density in Fourier space

$$A = \int dx\, \tilde{\rho}_{\text{tot}} . \tag{7.3.20}$$

Because $\tilde{\rho}_{\text{tot}}$ appears in the g_{xx} component, let us study its behavior in the DC limit. After a

Fourier transform, (7.3.7) may be readily written as

$$i\omega\delta\tilde{\rho}_{\rm tot} - \partial_x \delta\tilde{\mathcal{I}}_{\rm tot} = 0\,. \tag{7.3.21}$$

Recall the integral expression (7.2.12) for function $F^{\pm}$ when we computed the retarded Green function (or susceptibility) associated to $\mathcal{I}_{\pm}$ and $\mathcal{J}_{\pm}$,

$$F^{\pm}(x-x';\omega) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{e^{ik(x-x')}}{\omega \pm \omega_k + i\epsilon}\,.$$

In its denominator we have the polynomial equation $Q(k) = \omega \pm vk^z + i\epsilon$. When we use residue theorem, we will have a sum of z poles such that $Q(k_p^{\pm}) = 0$, where $k_p^{\pm} \propto \omega^{1/z}$. Recall now $\Delta_{\gtrless}(0)$, defined in (7.2.15), which is a sum of this poles at the DC limit (see comment before Eq. (7.2.19)). It has maximum order of $\mathcal{O}\big(\omega^{1/z}\big)$. Recalling that the integration and algebraic manipulation of $\Delta_{\gtrless}(x-x';\omega)$ yields finally the total current intensity, its maximum order at the DC limit then is

$$\delta\tilde{\mathcal{I}}_{\rm tot}|_{\omega\to 0} = \sigma\Delta V + \mathcal{O}\Big(\omega^{1/z}\Big)\,. \tag{7.3.22}$$

By means of Eq. (7.3.21), then the order of $\delta\tilde{\rho}_{\rm tot}$ is

$$\delta\tilde{\rho}_{\rm tot} \propto \mathcal{O}\Big(\omega^{1/z-1}\Big)\,. \tag{7.3.23}$$

Because in the DC limit this quantity diverges, the metric component g_{xx} will have the same behavior at this regime.

Regarding question *(III)*, the perturbed process we consider when we computed the two-terminal conductance (7.2.22) is adiabatic, since, from Eq. (7.3.21), we obtain

$$\delta\mathcal{A} = 0\,. \tag{7.3.24}$$

If the black hole fulfills the Bekenstein-Hawking area law [97], this result guarantees that this law holds in the entire process. Thus, we have chiral bosons in a localized near-horizon sector that do not alter the entropy of the black hole. Moreover, these excitations diffuse anisotropically according to Ohm's law.

7.3.3 Memory effect

The strategy to answer question (IV) of Page 66 is to turn off and on the external source μ and map some of the previous results into the language of chiral bosons. The reason is because chiral bosons corresponds to improper diffeomorphisms of AdS$_3$: They change the physical state of the system since they capture the boundary degrees of freedom of the theory.

Without the source μ, the scalar fields $\varphi_\pm$ parametrize the global cover of AdS_3. Recalling definitions (6.0.7) and (6.0.8), we obtain

$$d\varphi_\pm = \pm\frac{K}{4\pi}\left(\mathcal{J}_\pm dx - \mathcal{I}_\pm dt\right) . \tag{7.3.25}$$

Thus, we can parametrize the AdS_3 metric as

$$ds^2 = -\frac{l^2}{4}\sinh^2\left(\frac{r}{l}\right)(d\varphi_+ + d\varphi_-)^2 + dr^2 + \frac{l^2}{4}\cosh^2\left(\frac{r}{l}\right)(d\varphi_+ - d\varphi_-)^2 . \tag{7.3.26}$$

The diffeomorphism (7.3.25) makes evident that the degrees of freedom yielding $\mathcal{J}_\pm$ and $\mathcal{I}_\pm$ belong to the set of large gauge transformations.

Since we are always standing locally on AdS_3, we want to see how the metric evolves from a certain initial setting $\varphi_\pm^{\text{in}}$. We start with configuration

$$\varphi_\pm^{\text{in}}(t,x) = \frac{4\pi}{K}(t\pm x)\,\mathcal{J}_\pm^{\text{in}} . \tag{7.3.27}$$

The latter corresponds to the BTZ black hole with inner and outer horizons $\frac{2\pi\ell^2}{K}\left(\mathcal{J}_+^{\text{in}} + \mathcal{J}_-^{\text{in}}\right)$, respectively [392]. In order to make this choice consistent with (7.3.8), we set the initial value of the electric current to be $\mathcal{I}_\pm^{\text{in}} = \mp\mathcal{J}_\pm^{\text{in}}$.

If we now turn on the source μ, the dynamic is dictated by Eq. (7.3.7), which, after replacing Eq. (7.3.8), transforms into the anisotropic chiral equation for $\mathcal{J}_\pm$, (7.3.10). The solution of this equation is given by (7.3.14).

After the choice of the chemical potential (7.2.16), we can write $\mathcal{J}_\pm$ as,

$$\mathcal{J}_\pm = \mathcal{J}_\pm^{\text{in}} \mp \frac{K}{4\pi(vzt)^{1/z}}\Theta(t)\left\{V_L \text{Ai}_z\left[\pm\frac{x-x_L}{(vzt)^{1/z}}\right] - V_R \text{Ai}_z\left[\pm\frac{x-x_R}{(vzt)^{1/z}}\right]\right\} , \tag{7.3.28}$$

where $\text{Ai}_z(x)$ corresponds to the higher-order Airy function of the first kind, defined in Eq. (B.0.3) of Appendix H.5 as

$$\text{Ai}_z(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{i\left(\frac{k^z}{z}+kx\right)} . \tag{7.3.29}$$

There are two crucial steps in order to write (7.3.28) in terms of anisotropic chiral bosons. Physically, we perturbed the system with a temporally localized source, so when the perturbation is turned off, the solution (7.3.28) still holds for $t > 0$. However, according to (7.3.7), without the source we can express the following output functions

$$\mathcal{J}_\pm^{\text{out}} = \pm\frac{K}{4\pi}\partial_x\varphi_\pm^{\text{out}} , \quad \mathcal{I}_\pm^{\text{out}} = \mp\frac{K}{4\pi}\partial_t\varphi_\pm^{\text{out}} . \tag{7.3.30}$$

We can write the higher-order Airy function as a derivative on x. According to (B.3.2), if we define $\psi_z(x)$ as

$$\psi_z(x) = \frac{1}{2i\pi}\text{p.v.}\int_{-\infty}^{\infty} dk\, \frac{e^{i\left(\frac{k^z}{z}+kx\right)}}{k}\,, \tag{7.3.31}$$

where $\psi_z(x)$ is the antiderivative of the z-order Airy function by virtue of Eq. (B.3.1), namely,

$$\frac{\partial}{\partial x}\psi_z(x) = \text{Ai}_z(x)\,, \tag{7.3.32}$$

then (7.3.28) reads as

$$\varphi_{\pm}^{\text{out}}(t,x) = \varphi_{\pm}^{\text{in}}(x,t) \mp V_L\,\psi_z\left(\pm\frac{x-x_L}{(vz^{\pm}t)^{1/z^{\pm}}}\right) \pm V_R\,\psi_z\left(\pm\frac{x-x_R}{(vz^{\pm}t)^{1/z^{\pm}}}\right)\,. \tag{7.3.33}$$

In the final evolution of the process, we end again with the AdS_3 geometry (7.3.26). The difference lies in the form of the function $\varphi_{\pm}$, which changed nontrivially, storing sensible physical information of the process that has occured, since it is possible to obtain an electrical two-terminal conductance from this result.

Consider the total electric current defined as $\delta\langle\mathcal{I}_{\text{tot}}\rangle \equiv \delta\langle\mathcal{I}_{+}\rangle + \delta\langle\mathcal{I}_{+}\rangle$. Then the conductivity is defined as the time-average of the total electric current over voltage, as following

$$\sigma = \frac{1}{\Delta V}\int_{-\infty}^{\infty} dt\,\delta\langle\mathcal{I}_{\text{tot}}\rangle = -\frac{K}{4\pi\Delta V}\int_{-\infty}^{\infty} dt\,\partial_t(\delta\varphi_{+} - \delta\varphi_{-})\,, \tag{7.3.34}$$

where $\delta\varphi_{\pm} \equiv \varphi_{\pm}^{\text{out}}(x,t) - \varphi_{\pm}^{\text{in}}(x,t)$. After the integration, we arrive to chiral bosons evaluated in $t = \pm\infty$. Due the causality condition, the chiral bosons at minus infinity vanishes. Using the fact that $\psi_z^{\pm}(0) = 1/2z^{\pm}$ (see Eq. (B.3.3)), we can write

$$\delta\varphi_{\pm}(t=\infty, x) = \mp\frac{\Delta V}{2z^{\pm}}\,. \tag{7.3.35}$$

yielding the same result (7.2.22). This outcome puts in evidence two facts. The first is that chiral bosons belong to the set of improper diffeomorphisms, as said, since they capture the boundary degrees of freedom of the theory by storing DC sensible information of the boundary electric current. This means that, from the canonical point of view, they change the charges of the system, despite we are always standing on AdS_3.

Although the external source is turned off, there are permanent imprints in the solution space of anisotropic chiral bosons, as showed in (7.3.33). This linear response effect, where a susceptibility (or retarded Green function) arise from a holographic perturbation, can be mapped as a three-dimensional version of the 4D gravitational memory effect, where the passage of a gravitational wave induces a permanent displacement of the detectors [82, 336].

7.4 Remarks

Through the choice of suitable boundary conditions, we formulated a 2D scalar theory - the anisotropic chiral boson [333]- as a putative dual description of AdS_3 GR. Through the Kubo formula, we obtained an Ohm's law in the DC limit $\omega \to 0$. An important feature of this result lies in the fact that the holographic conductivity depends explicitly on the dynamical exponent that controls the anisotropic scaling.

A black hole interpretation was given, since we proved that anisotropic chiral bosons corresponds to local excitations of the gravitational field at the near-horizon region of the BTZ black hole. Because chiral bosons evolves in a particular form, they change the physical state of the system by storing boundary DC information. Thus, through the identification of large gauge transformations, we identified a 3D analogue gravitational memory effect.

Conclusions

Due the lack of bulk gravitational propagating degrees of freedom, we argued along this thesis, that the holographic character of 3D GR is given by the choice of suitable boundary conditions. We emphasized this by working on AdS_3 geometries.

On one side, we established a dictionary between Integrable Systems, in particular the Ablowitz-Kaup-Newell-Segur (AKNS) system, and AdS_3 GR, through boundary conditions and asymptotic symmetries.

On the other hand, the introduction of an external source in Lifshitz-type boundary conditions allowed us to geometrize quantities of a perturbed holographic description of AdS_3 GR, and connect it with 3D memory effect.

Here we conclude this thesis by exposing different possible avenues that can expand what has been treated along these pages.

Discussion of Part I

All of the results obtained in Chapter 4 are given in the Chern-Simons approach of GR on AdS_3. Although this framework enjoys some advantages than the metric formalism, it is desirable to express results in terms of the metric field $g_{\mu\nu}$, e.g., to explicitly write the spacetimes where NLS, KdV or MKdV equations evolves within; the form of their asymptotic Killing vectors and how the mass and angular momentum reads in terms of the dynamical fields $p^{\pm}$ and $r^{\pm}$.

Another path to follow is performing the Hamiltonian reduction of AdS_3 GR by solving its first-class constraint, with [393] and without global holonomies; and obtain at the boundary an action princple whose Euler-Lagrange equations yields the AKNS system or modifications of it. In this regard, it is desirable to study this action functional by canonical analysis. It would be expected to recover the action reported in [316].

More general prospects involves gravitational or black hole interpretations of different methods used in the Integrable Systems literature regarding the construction of N-soliton solutions, e.g., the Inverse Scattering Transformation or Bäcklund ones. Better understanding of this map involves to construct solitons in the Chern-Simons approach and map these results to the metric formalism. For example, we expect that Bäcklund transformations would allow us to find general Miura-like maps at the level of Einstein equations, or to find specific (proper or improper) change of coordinates where a 1-soliton solution maps to a self-interacting one through Bianchi nonlinear superposition principle.

Hamiltonian methods are not the only ones that have been explored in the construction of gravitational charges. Covariant methods, such as Balasubramanian-Kraus [394] or Noether-Wald [395, 396] have been of profound importance for nowadays theoretical gravitational physics. Since general AKNS black holes are not stationary, it is necessary to work with fixed N.

In the BTZ black hole, the entropy and the mass are generated by the Killing vector $\xi = \partial_t - N^\phi \partial_\phi$, while angular momentum by ∂_ϕ, so it is expected that this method works for particular AKNS black holes.

Other avenue regards higher-spin gravity. We would search to generalize boundary conditions (4.0.1) for the gravitational field nonminimally-coupled to a spin-3 particle by spanning the gauge connection in two independent copies of the $sl^{\pm}(3,\mathbb{R})$ Lie algebra,

$$\begin{aligned} &[H_i, H_j] = 0, \quad \left[H_i, E_j^{\pm}\right] = \pm K_{ji} E_j^{\pm}, \quad \left[E_i^{+}, E_j^{-}\right] = \delta_{ij} H_j, \\ &\left[E_i^{\pm}, E_3^{\pm}\right] = 0, \quad \left[E_3^{+}, E_3^{-}\right] = H_1 + H_2, \quad \left[H_i, E_3^{+}\right] = \pm E_3^{\pm}, \\ &\left[E_1^{\pm}, E_2^{\pm}\right] = \pm E_3^{\pm}, \quad \left[E_1^{\pm}, E_3^{\mp}\right] = \mp E_3^{\mp}, \quad \left[E_2^{\pm}, E_3^{\mp}\right] = \mp E_1^{\mp}, \end{aligned} \tag{7.4.1}$$

where $H_1, H_2, E_1^{\pm}, E_2^{\pm}, E_3^{\pm}$ are the generators of the algebra, and K_{ij} its Cartan matrix. Unlike the $sl(2,\mathbb{R})$ case where two dynamical equations of motion (4.0.3) and three recurrence relationships (4.0.4) appears, here it is expected to obtain six dynamical equations of motion with eighteen recursion relationships, thus generalizying the AKNS system to the self-interacting spin-3 case. Besides, an abelian realization through a nonlinear redefinition of the Zamolodchikov W_3 generators in terms of the new dynamical fields should be accomplished. After this warm-up, a spin-N generalization is desirable.

It is known that the Poincaré algebra $iso(2,1)$ can be recovered from the Anti-de Sitter Lie algebra $so(2,2)$

$$\begin{aligned} [J_a, J_b] &= \epsilon_{abc} J^c \,, \\ [J_a, P_b] &= \epsilon_{abc} P^c \,, \\ [P_a, P_b] &= -\Lambda \epsilon_{abc} J^c \,, \end{aligned} \tag{7.4.2}$$

where $\Lambda = -1/l^2 < 0$ is the cosmological constant, by the Innönü–Wigner contraction $\ell \to \infty$. Future prospects for results of Chapter 4, are to recover its flat limit by the aforementioned contraction. However, the reduction is not straightforward, since we must redefine the dynamical functions in order to take this limit safely. We expect that some dynamical terms present in the $so(2,2)$ case dissapear after taking this limit. Additionally, it is not clear if the conditions $B_0^{\pm} = C_0^{\pm} = 0$ holds in the flat case, in the sense that, in order to achieve the usual recursion relations, a double power expansion to N and M in the spectral parameters must be taken, so the initial conditions that one obtained in the $sl(2,\mathbb{R})$ case, maps into initial conditions to row and columns of $B_{n,m}$ and $C_{n,m}$, whose indices will stand in different foots.

Regarding the $\Lambda > 0$ (de Sitter) case, the gauge connection, along with a particular reality condition, is spanned in the $sl(2,\mathbb{C})$ Lie algebra. In this case, the possible AKNS configuration will evolve on cosmological set-ups.

Discussion of Part II

Although the anisotropic chiral boson theory is defined for odd z, a possible prospect to continue with the research line exposed in Part II of this thesis is to consider the even extension of the aforementioned theory. That is, to find suitable boundary conditions where the Hamiltonian reduction leads to an even version field description. In this context, it is proposed to perform the same analysis of Chapter 6 and 7, i.e., to perturb the theory with a certain chemical potential and, since we also expect anyons excitations, to write the possible $u(1)$ operators in their fermionic formulation, so as find an associated conductivity in the DC limit.

In order to consider an eternal black hole (and not a local one as we treated in [332]), we should consider two asymptotically AdS_3 boundaries [393]. When we performed the Hamiltonian reduction in Chapter 7, we solved the constraint without holonomies, however, if we aim to consider them, it is necessary to include them in the solution of the zero-curvature equation of motion. This change the holographic dual theory in the sense that the zero modes couple the dynamical fields on the two different boundaries. Under the light of linear response theory, it would allow us to find transport coefficients that interpolate both boundaries.

A natural avenue to pursue is to extend the construction of Chapter 7 but when the gravitational field nonminimally coupled a higher spin field. In this regard, the simplest case to consider is a spin-3 field. As mentioned above, in this case the Chern-Simons connection spans into two independent copies of the $sl^{\pm}(3,\mathbb{R})$ algebra.

If we generalize the Lifshitz-like boundary conditions (7.1.26) and (7.1.27), but now for $sl^{\pm}(3,\mathbb{R})$, then we expect to obtain the self-interacting higher version of the anisotropic chiral boson after a Hamiltonian reduction (with no global holonomies). In this sense, it would be desirable to compute transport coefficients using the reviewed methods of Chapter 5.

Another appealing line to explore is to study the fluctuation-dissipation theorem [397, 398] from the metric formalism. From linear response theory, it is known that the real part of the susceptibility, labeled χ' is associated to the reactive part of the theory. It is a function that says to which particular frequency ω the reaction is concentrated.

On the other hand, the imaginary part of the susceptibility, χ'', is associated to dissipative effects. This functions says us where the system naturally vibrates at some frequency, i.e., the regime where the system is able to absorb energy.

In statistical mechanics, the two-point correlation function $S(t)$ is associated to the variance or fluctuation of the system. Using the Kubo formula (5.3.7) [334, 335], and for translational invariance systems, the theorem reads

$$S(\omega) = -2\left[n_{\mathrm{B}}(\omega) + 1\right]\chi''(\omega)\,,$$

where $n_{\mathrm{B}}(\omega) = (e^{\beta\omega} - 1)^{-1}$ is the Bose–Einstein distribution function. We see that n_{B} correspond to thermal effects while the "-1" term represent quantum fluctuations. Observe that the classical limit is recovered at high temperatures $\beta\omega \ll 1$, where n_{B} approximates to $k_B T/\omega$, with k_B the Boltzmann constant. Thus, we obtain $S(\omega) \approx -2\,(kT/\omega)\,\chi''(\omega)$.

Future prospects involve a possible geometrization of the fluctuation-dissipation theorem. We would like to find the frequencies where the system absorbs energy and understand it metric counterparts, in the sense to what kind of gravitational phenomena we can associate this absorption process. Besides, if the Hamiltonian reduction in the eternal black hole set-up is plausible, an appealing question to ask is if the transport coefficient that interpolates the two boundaries is dissipative or not, and figure at what kind of frequency naturally vibrates.

Strominger [399] revealed the triangular relation between unrelated different physics topics associated to the infrared dynamics of physical massless particles theories. There are three corners: Soft theorems, memory effects and asymptotic symmetries.

From memory effect, according to [82], the Heaviside step function can be interpreted as a domain wall that connects to inequivalent vaccuas, related by an asymptotic symmetry. We can move from asymptotic symmetries to a soft theorem since every symmetry has its own Ward identity.

Ward identities are identities of the scattering amplitudes from where physical information can be obtained, e.g., to constrain the tensor structure of vacuum polarization or to say that longitudinal polarization of the photon is unphysical in a particular gauge. But soft theorems are relations of scattering amplitudes with and without soft particles, so we ended at the beginning.

We conjecture that the final part of the triangle can be addressed if we calculate the scattering amplitude of anisotropic chiral bosons and then finding its Ward identity, allowing us to translate the other two corners of the triangle in the language of particle physics and quantum field theory.

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Appendices

Appendix A

$sl(2,\mathbb{R})$ matrix representations and identities

In this appendix we present the $sl(2,\mathbb{R})$ algebra and its 2×2 matrix representation, along with the trace identities that the generators fulfill.

The $sl(2,\mathbb{R})$ algebra (2.2.9) is

$$[L_n, L_m] = (n-m)L_{n+m}\,, \tag{A.0.1}$$

where L_0, L_{-1} and L_1 are its generators that admits the following 2×2 matrix representation

$$L_{-1} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\,, \quad L_0 = \begin{pmatrix} -1/2 & 0 \\ 0 & 1/2 \end{pmatrix}\,, \quad L_1 = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}\,. \tag{A.0.2}$$

The generators satisfy the following trace identities

$$\langle L_{\pm 1}, L_0 \rangle = 0\,, \quad \langle L_1, L_{-1} \rangle = -1\,, \quad \langle L_0, L_0 \rangle = 1/2\,, \tag{A.0.3}$$

from where the Killing metric $g_{ab} = \langle L_a, L_b \rangle$ reads

$$g_{ab} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1/2 & 0 \\ -1 & 0 & 0 \end{pmatrix} \tag{A.0.4}$$

Appendix B

Higher-order Airy functions of the first kind

Following [400], in this appendix we show that the z-dependent plane-wave like Fourier integral

$$f_z(t,x) = \int_{-\infty}^{\infty} dk\, \exp\left[i(ky \pm \omega_k t)\right]\,, \qquad z \in \mathbb{N}\,, \tag{B.0.1}$$

where $\omega_k = vk^z$, can be written in terms of the higher-order Airy functions of the special kind Ai_{z1},

$$f_z(t,x) = \frac{2\pi}{(vzt)^{1/z}} \mathrm{Ai}_z \left[\pm \frac{x}{(vzt)^{1/z}}\right]\,, \tag{B.0.2}$$

defined as

$$\mathrm{Ai}_z(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk\, \exp\left[i\left(\frac{k^z}{z} + kx\right)\right]\,. \tag{B.0.3}$$

B.1 Ordinary Airy functions

Airy functions $\mathrm{Ai}(x)$ are special functions defined as the integral expression

$$\mathrm{Ai}(x) = \frac{1}{2\pi} \int_{0}^{\infty} dk \cos\left(\frac{k^3}{3} + kx\right) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk\, e^{i\left(\frac{k^3}{3} + kx\right)}\,. \tag{B.1.1}$$

Along this appendix, we will use the exponential representation of the Airy function. As a particular objective, we will show that the latter function fulfills the Airy equation

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2} \mathrm{Ai}(x) = x \mathrm{Ai}(x)\,. \tag{B.1.2}$$

To prove that, let us take a second derivative on u,

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\mathrm{Ai}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, \frac{\mathrm{d}^2}{\mathrm{d}x^2} e^{i\left(\frac{k^3}{3}+kx\right)} \tag{B.1.3a}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, (ik)^2 e^{i\left(\frac{k^3}{3}+kx\right)} \tag{B.1.3b}$$

$$= -\frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, k^2 e^{i\left(\frac{k^3}{3}+kx\right)}\,. \tag{B.1.3c}$$

If we add a zero in the integrand in order to conveneniently appear ordinary Airy functions, we obtain

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\mathrm{Ai}(x) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, \left(k^2 + x - x\right) e^{i\left(\frac{k^3}{3}+kx\right)} \tag{B.1.3d}$$

$$= -\frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, \left(k^2 + x\right) e^{i\left(\frac{k^3}{3}+kx\right)} + \frac{x}{2\pi}\int_{-\infty}^{\infty} dk\, e^{i\left(\frac{k^3}{3}+kx\right)} \tag{B.1.3e}$$

$$\equiv -\frac{1}{2i\pi}\int_{-\infty}^{\infty} dk \frac{\mathrm{d}}{\mathrm{d}k}\left[e^{i\left(\frac{k^3}{3}+kx\right)}\right] + x\mathrm{Ai}(x)\,. \tag{B.1.3f}$$

However, the integral vanishes since

$$\lim_{|k|\to\infty} e^{i\left(\frac{k^3}{3}+kx\right)} = 0\,, \tag{B.1.4}$$

when $\arg k = n\pi$, for $n \in \mathbb{Z}$. Hence (B.1.1) fulfills the Airy equation

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\mathrm{Ai}(x) = x\mathrm{Ai}(x)\,. \tag{B.1.5}$$

B.2 Higher-order Airy functions of the first kind

We define the higher-order Airy function of the first kind $\mathrm{Ai}_z(x)$ as

$$\mathrm{Ai}_z(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{i\left(\frac{k^z}{z}+kx\right)}\,, \qquad z \geq 1\,. \tag{B.2.1}$$

This generalized version of the Airy function reduces to particular known functions for lower values of z, e.g., for $z = 1$, it reduces to a delta distribution

$$\mathrm{Ai}_1(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{ik(x+1)} = \delta(x+1)\,. \tag{B.2.2}$$

For $z = 2$, we arrive to a Gaussian integral

$$\mathrm{Ai}_2(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{i\left(\frac{k^2}{2}+kx\right)} = \frac{1}{2\sqrt{\pi}}\,(1+i)\, e^{-\frac{i}{2}x^2}\,, \tag{B.2.3}$$

when $x \in \mathbb{R}$; while for $z = 3$, we readily recover the ordinary Airy function $\mathrm{Ai}(x)$,

$$\mathrm{Ai}_3(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{i\left(\frac{k^3}{3}+kx\right)} \equiv \mathrm{Ai}(x)\,. \tag{B.2.4}$$

Another feature of this generalized special function is that fulfills an extension of the Airy equation (B.1.5),

$$\frac{\partial^z}{\partial x^z}\mathrm{Ai}_z(x) = (-1)^{\frac{z}{2}+1} x \mathrm{Ai}_z(x)\,. \tag{B.2.5}$$

The proof is analogue to the one of ordinary Airy equation. Let us take z derivatives of the higher-order Airy function (B.2.1), as following

$$\frac{\partial^z}{\partial x^z}\mathrm{Ai}_z(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, \frac{\partial^z}{\partial x^z} e^{i\left(\frac{k^z}{z}+kx\right)} \tag{B.2.6a}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, (ik)^z e^{i\left(\frac{k^z}{z}+kx\right)} \tag{B.2.6b}$$

$$= \frac{(-1)^{z/2}}{2\pi}\int_{-\infty}^{\infty} dk\, k^z e^{i\left(\frac{k^z}{z}+kx\right)}\,. \tag{B.2.6c}$$

In the same spirit as Eq. (B.1.3d), we add now a zero to make appear higher-order Airy functions

$$\frac{\partial^z}{\partial x^z}\mathrm{Ai}_z(x) = \frac{(-1)^{z/2}}{2\pi}\int_{-\infty}^{\infty} dk\, (k^z + x - x) e^{i\left(\frac{k^z}{z}+kx\right)} \tag{B.2.6d}$$

$$= \frac{(-1)^{z/2}}{2\pi}\int_{-\infty}^{\infty} dk\, (k^z + x) e^{i\left(\frac{k^z}{z}+kx\right)} + \frac{(-1)^{\frac{z}{2}+1}x}{2\pi}\int_{-\infty}^{\infty} dk\, e^{i\left(\frac{k^z}{z}+kx\right)} \tag{B.2.6e}$$

$$\equiv \frac{(-1)^{z/2}}{2\pi}\int_{-\infty}^{\infty} dk\, (k^z + x) e^{i\left(\frac{k^z}{z}+kx\right)} + (-1)^{\frac{z}{2}+1} x \mathrm{Ai}_z(x)\,. \tag{B.2.6f}$$

The latter integral is a total derivative. Because

$$\lim_{|k|\to\infty} e^{i\left(\frac{k^{z+1}}{z+1}+kx\right)} = 0\,, \tag{B.2.7}$$

when $\arg k = 2n\pi/(z+1)$, with $n \in \mathbb{Z}$, the integral vanishes, yielding (B.2.5).

B.3 Antiderivative of the higher-order Airy functions

Let us define the antiderivative of higher-order Airy functions of the special kind; i.e., to find a function $\psi_z(x)$ such that fulfills

$$\frac{\partial}{\partial x}\psi_z(x) = \mathrm{Ai}_z(x)\,. \tag{B.3.1}$$

The function is readily

$$\psi_z(x) = \frac{1}{2i\pi}\,\text{p.v.}\int_{-\infty}^{\infty} dk\,\frac{e^{i\left(\frac{k^z}{z}+kx\right)}}{k}\,, \tag{B.3.2}$$

where p.v. stands for the Cauchy principal value.

B.3.1 Zero value

An important result that has direct application in the calculation of anisotropic chiral bosons DC current (7.3.35) occurs when $x = 0$, namely,

$$\psi_z(0) = \frac{1}{2i\pi}\text{p.v.}\int_{-\infty}^{\infty} dk\,\frac{e^{i\frac{k^z}{z}}}{k} = \frac{1}{2z}\,. \tag{B.3.3}$$

This result can be proved as following. Using De Moivre's formula, let us write

$$\psi_z(0) = \frac{1}{2i\pi}\text{p.v.}\int_{-\infty}^{\infty} \frac{dk}{k}\left[\cos\left(\frac{k^z}{z}\right) + i\sin\left(\frac{k^z}{z}\right)\right]\,. \tag{B.3.4a}$$

Because the odd cosine integral vanishes, we only need to compute the sine part

$$\psi_z(0) = \frac{1}{2\pi}\text{p.v.}\int_{-\infty}^{\infty} \frac{dk}{k}\sin\left(\frac{k^z}{z}\right)\,. \tag{B.3.4b}$$

With the change of variables $\frac{k^z}{z} = y$, then $k^{z-1}dk = dy$, yielding

$$\psi_z(0) = \frac{1}{2\pi}\text{p.v.}\int_{-\infty}^{\infty} dy\frac{\sin y}{x^n} \equiv \frac{1}{2z\pi}\text{p.v.}\int_{-\infty}^{\infty} dy\frac{\sin y}{y}\,. \tag{B.3.4c}$$

By standard complex analysis, it is possible to prove

$$\text{p.v.}\int_{-\infty}^{\infty} dy\frac{\sin y}{y} = \pi\,.$$

Finally, we arrive then to result (B.3.3),

$$\psi_z(0) = \frac{1}{2z}\,.$$

B.4 Fourier-type integral

Recalling the z-order Airy function defined in (B.2.1) as

$$\text{Ai}_z\left[\pm\frac{x}{\sqrt[z]{vzt}}\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\,\exp\left[i\left(\frac{k^z}{z} \pm \frac{kx}{\sqrt[z]{vzt}}\right)\right]\,, \tag{B.4.1}$$

with the change of variables $\kappa \sqrt[z]{vzt} = k$, we can directly write $d\kappa \sqrt[z]{vzt} = dk$, allowing to establish a relationship between the higher-order Airy functions and plane wave-like Fourier integral. Then,

$$\mathrm{Ai}_z\left[\pm \frac{x}{\sqrt[z]{vzt}}\right] = \frac{\sqrt[z]{vzt}}{2\pi} \int_{-\infty}^{\infty} d\kappa \, \exp\left[i\left(v\kappa^z t \pm \kappa x\right)\right] \, . \tag{B.4.2}$$

Defining $\kappa \to \pm\kappa$ (and then $\kappa \to k$), we obtain a propagating wave-like Fourier integral

$$\mathrm{Ai}_z\left[\pm \frac{x}{\sqrt[z]{vzt}}\right] = \frac{\sqrt[z]{vzt}}{2\pi} \int_{-\infty}^{\infty} dk \, \exp\left[i\left(kx \pm \omega_k t\right)\right] \, , \tag{B.4.3}$$

where $\omega_k = vk^z$. Finally, if we define the Fourier integral

$$f_z(x,t) = \int_{-\infty}^{\infty} dk \, \exp\left[i(kx \pm \omega_k t)\right] , \tag{B.4.4}$$

we are able to establish the following relationship with the higher-order Airy function

$$f_z(t,x) = \frac{2\pi}{\sqrt[z]{vzt}} \mathrm{Ai}_z\left[\pm \frac{x}{\sqrt[z]{vzt}}\right] \, , \tag{B.4.5}$$

arriving to (B.0.2).

Appendix C

Appendices of Chapter 2

C.1 Deduction of the AKNS recursive equations of motion

In this appendix we aim to obtain the recursive equations of motion (2.1.3)

$$\dot{r} = \frac{1}{\ell}\left(-C'_N + 2rA_N\right), \quad \dot{p} = \frac{1}{\ell}\left(-B'_N - 2pA_N\right), \tag{C.1.1}$$

where the coefficients A_n, B_n and C_n satisfy the relations (2.1.4a), (2.1.4b) and (2.1.4c),

$$A'_n = pC_n - rB_n\,, \tag{C.1.2a}$$

$$B_{n+1} = -\frac{1}{2}B'_n - pA_n\,, \tag{C.1.2b}$$

$$C_{n+1} = \frac{1}{2}C'_n - rA_n\,, \tag{C.1.2c}$$

$$B_0 = C_0 = 0\,. \tag{C.1.2d}$$

The deduction is the following. Consider the AKNS system,

$$\dot{r} + \frac{1}{\ell}\left(C' - 2rA - 2\xi C\right) = 0\,, \tag{C.1.3a}$$

$$\dot{p} + \frac{1}{\ell}\left(B' + 2pA + 2\xi B\right) = 0\,, \tag{C.1.3b}$$

$$A' - pC + rB = 0\,. \tag{C.1.3c}$$

Perform a polynomial ansatz in ξ,

$$A = \sum_{n=0}^{N-n} A_n \xi^{N-n}, \quad B = \sum_{n=0}^{N-n} B_n \xi^{N-n}, \quad C = \sum_{n=0}^{N-n} C_n \xi^{N-n}\,, \tag{C.1.4}$$

with N an arbitrary positive integer. Replacing the latter expansion in (C.1.3a), we have

$$\dot{r}+\frac{1}{\ell}\left[\sum_{n=0}^{N}\left(C_n'-2rA_n\right)\xi^{N-n}-2\sum_{n=0}^{N}C_n\xi^{N-(n-1)}\right]=0\,. \tag{C.1.5}$$

After relabeling indices of the last sum, and gathering terms with the same order in ξ, we obtain

$$\left[\dot{r}+\frac{1}{\ell}\left(C_N'-2rA_N\right)\right]+\frac{1}{\ell}\sum_{n=0}^{N-1}\left(C_n'-2rA_n+2C_{n+1}\right)\xi^{N-n}+\frac{1}{\ell}2C_0\xi^{N+1}=0\,. \tag{C.1.6}$$

Because ξ is arbitrary, we arrive to the equations

$$\dot{r}+\frac{1}{\ell}\left(C_N'-2rA_N\right)=0\,, \tag{C.1.7a}$$

$$C_{n+1}=\frac{1}{2}C_n'-rA_n\,, \tag{C.1.7b}$$

$$C_0=0\,. \tag{C.1.7c}$$

Thus, we obtained in particular the first equation of (C.1.1), (C.1.2b) and the second equality of (C.1.2d). Performing the same procedure by using the polynomial ansatz we obtain the rest of equations. This completes the procedure.

C.2 AKNS recursive construction with integration constants

Here we explicitly construct the recursive form of the coefficients A_n, B_n and C_n from expansion (2.1.2) with all of the integrations constants considered.

Recall from the equations of motion that we obtained $B_0 = C_0 = 0$. Replacing the latter in (2.1.4a), we readily get $A_0 = c_0$, where c_0 is an integration constant. Consider now $n = 0$ in (2.1.4b) and (2.1.4c): We arrive to $B_1 = -c_0 p$, $C_1 = -c_0 r$, respectively. Using the latter results, we arrive to $A_1 = c_1$. Thus, the equations of motion (2.1.3) for $N = 1$ are

$$\dot{p} = \frac{1}{\ell}\left(c_0 p' - 2c_1 p\right) , \quad \dot{r} = \frac{1}{\ell}\left(c_0 r' + 2c_1 r\right) . \tag{C.2.1}$$

We can continue and obtain subsequent coefficients A_2, B_2 and C_2 in order to obtain the associated equations of motion for $N = 2$, and so on.

We index in the following the first five inhomogeneous coefficients

$$\begin{aligned}
A_0 &= c_0\,,\\
A_1 &= c_1\,,\\
A_2 &= -\frac{1}{2}c_0 pr + c_2\,,\\
A_3 &= \frac{1}{4}c_0\left(p'r - pr'\right) - \frac{1}{2}c_1 pr + c_3\,,\\
A_4 &= \frac{1}{8}c_0\left(p'r' - p''r - pr'' + 3p^2r^2\right) + \frac{1}{4}c_1\left(p'r - pr'\right) - \frac{1}{2}c_2 pr + c_4\,,\\
B_0 &= 0\,,\\
B_1 &= -c_0 p\,,\\
B_2 &= \frac{1}{2}c_0 p' - c_1 p\,,\\
B_3 &= \frac{1}{2}c_0\left(-\frac{1}{2}p'' + p^2 r\right) + \frac{1}{2}c_1 p' - c_2 p,\\
B_4 &= -\frac{1}{4}c_0\left(-\frac{1}{2}p''' + 3pp'r\right) + \frac{1}{2}c_1\left(p^2 r - \frac{1}{2}p''\right) + \frac{1}{2}c_2 p' - c_3 p\,,\\
C_0 &= 0\,,\\
C_1 &= -c_0 r\,,\\
C_2 &= -\frac{1}{2}c_0 r' - c_1 r\,,\\
C_3 &= -\frac{1}{2}c_0\left(\frac{1}{2}r'' - pr^2\right) - \frac{1}{2}c_1 r' - c_2 r\,,\\
C_4 &= -\frac{1}{4}c_0\left(\frac{1}{2}r''' - 3prr'\right) + \frac{1}{2}c_1\left(pr^2 - \frac{1}{2}r''\right) - \frac{1}{2}c_2 r' - c_3 r\,.
\end{aligned}$$

The first five equations of motion are

$$N=0: \qquad \dot{p} = -\frac{2}{\ell} c_0 p \,,$$
$$\dot{r} = \frac{2}{\ell} c_0 r \,,$$

$$N=1: \qquad \dot{p} = \frac{1}{\ell}\left(c_0 p' - 2c_1 p\right) \,,$$
$$\dot{r} = \frac{1}{\ell}\left(c_0 r' + 2c_1 r\right) \,,$$

$$N=2: \qquad \dot{p} = \frac{1}{\ell}\left[c_0\left(-\frac{1}{2}p'' + p^2 r\right) + c_1 p' - 2c_2 p\right] \,,$$
$$\dot{r} = \frac{1}{\ell}\left[c_0\left(\frac{1}{2}r'' - p r^2\right) + c_1 r' + 2c_2 r\right] \,,$$

$$N=3: \qquad \dot{p} = \frac{1}{\ell}\left[-\frac{1}{4}c_0\left(6pp'r - p'''\right) - c_1\left(\frac{1}{2}p'' - p^2 r\right) + c_2 p' - 2c_3 p\right] \,,$$
$$\dot{r} = \frac{1}{\ell}\left[-\frac{1}{4}c_0\left(6prr' - r'''\right) - c_1\left(-\frac{1}{2}r'' + pr^2\right) + c_2 r' + 2c_3 r\right] \,,$$

$$N=4: \qquad \dot{p} = \frac{1}{\ell}\left[\frac{1}{8}c_0\left(-6p^3r^2 + 6p'^2 r + 4pp'r' + 6pp''r + 2p^2 r'' + 2prp'' - p''''\right)\right.$$
$$\left. + \frac{1}{4}c_1\left(-6pp'r + p'''\right) + c_2\left(p^2 r - \frac{1}{2}p''\right) + c_3 p' - 2c_4 p\right] \,,$$
$$\dot{r} = \frac{1}{\ell}\left[\frac{1}{8}c_0\left(6p^2r^3 - 4p'rr' - 6pr'^2 - 8prr'' - 2r^2 p'' + r''''\right)\right.$$
$$\left. + \frac{1}{4}c_1\left(-6prr' + r'''\right) + c_2\left(-pr^2 + \frac{1}{2}r''\right) + c_3 r' + 2c_4 r\right] \,.$$

Let us focus now in the equations of motion for the dynamical function p for case $N=0$ and $N=1$. We can readily see that the right-hand side equation associated to $N=0$ is written in the $N=1$ case, but the integration constant incremented in one order, namely c_1. This structure will be repeated for the following equations in the hierarchy, leading to the previous equations coupled to the next one. In order to simplify the analysis and decouple them, it is convenient to perform the forthcoming reduction. If we let $c_0 = 1$ and $c_i = 0$, for $i \neq 1$, the coefficients A_n, B_n and C_n

simplify as

$$
\begin{aligned}
A_0 &= 1\,, \\
A_1 &= 0\,, \\
A_2 &= -\frac{1}{2}pr\,, \\
A_3 &= \frac{1}{4}\left(p'r - pr'\right)\,, \\
A_4 &= \frac{1}{8}\left(p'r' - p''r - pr'' + 3p^2r^2\right)\,,
\end{aligned}
\tag{C.2.2}
$$

$$
\begin{aligned}
B_0 &= 0\,, \\
B_1 &= -p\,, \\
B_2 &= \frac{1}{2}p'\,, \\
B_3 &= \frac{1}{2}\left(-\frac{1}{2}p'' + p^2r\right), \\
B_4 &= -\frac{1}{4}\left(-\frac{1}{2}p''' + 3pp'r\right)\,,
\end{aligned}
\tag{C.2.3}
$$

$$
\begin{aligned}
C_0 &= 0\,, \\
C_1 &= -r\,, \\
C_2 &= -\frac{1}{2}r'\,, \\
C_3 &= -\frac{1}{2}\left(\frac{1}{2}r'' - pr^2\right)\,, \\
C_4 &= -\frac{1}{4}\left(\frac{1}{2}r''' - 3prr'\right)\,.
\end{aligned}
\tag{C.2.4}
$$

Thus, we obtain the next equations of motion,

$$N=0: \qquad \dot{p}=-\frac{2}{\ell}p\,, \qquad \dot{r}=\frac{2}{\ell}r\,,$$

$$N=1: \qquad \dot{p}=\frac{1}{\ell}p'\,, \qquad \dot{r}=\frac{1}{\ell}r'\,,$$

$$N=2: \qquad \dot{p}=\frac{1}{\ell}\left(-\frac{1}{2}p''+p^2r\right)\,, \qquad \dot{r}=\frac{1}{\ell}\left(\frac{1}{2}r''-pr^2\right)\,,$$

$$N=3: \qquad \dot{p}=\frac{1}{\ell}\left(-\frac{3}{2}pp'r+\frac{1}{4}p'''\right)\,, \qquad \dot{r}=\frac{1}{\ell}\left(-\frac{3}{2}prr'+\frac{1}{4}r'''\right)\,,$$

$$N=4: \qquad \dot{p}=\frac{1}{\ell}\left(-\frac{3}{4}p^3r^2+\frac{3}{4}p'^2r+\frac{1}{2}pp'r'+\frac{3}{4}pp''r+\frac{1}{4}p^2r''+\frac{1}{4}prr''-\frac{1}{8}p''''\right)\,,$$
$$\dot{r}=\frac{1}{\ell}\left(\frac{3}{4}p^2r^3-\frac{1}{2}p'rr'-\frac{3}{4}pr'^2-prr''-\frac{1}{4}r^2r''+\frac{1}{8}r''''\right)\,.$$

$N=1$ correspond to the chiral boson system, $N=2$ recovers the (Wick rotated) nonlinear Schrödinger equation, while $N=3$ recovers KdV and MKdV when $r=-1$ (or $p=-1$) and $p=-r$, respectively. The case $N=4$ is written for completitude.

C.3 Trace formula for the AKNS hierarchy

The trace formula [331] is a powerful tool that allows to establish a relationship between recursive quantities that appear when we consider the zero-curvature formulation, with conserved functionals. In particular, it has direct implications in the computation of the asymptotic symmetry algebra of gravitational charges in 3D GR and can be applied for all kind of boundary conditions in the Chern-Simons formulation.

It is given by the following expression

$$\frac{\delta}{\delta u}\int \operatorname{tr}\left(a_t\frac{\partial a_\phi}{\partial \xi}\right)d\phi = \xi^{-\gamma}\frac{\partial}{\partial \xi}\xi^{\gamma}\operatorname{tr}\left(a_t\frac{\partial a_\phi}{\partial u}\right)\,, \tag{C.3.1}$$

where tr denotes the matrix trace and γ is an arbitrary constant to be determined.

We are interested in the AKNS hierarchy. In this case, the function u takes the form $u = \begin{pmatrix} p \\ r \end{pmatrix}$, where $p = p(t,\phi)$ and $r = r(t,\phi)$. The left hand side of the trace formula is

$$\operatorname{tr}\left(a_t\frac{\partial a_\phi}{\partial \xi}\right) = \frac{2A}{\ell}\,, \tag{C.3.2}$$

while for the right hand side we obtain

$$\operatorname{tr}\left(a_t\frac{\partial a_\phi}{\partial p}\right) = -\frac{C}{\ell},\quad \operatorname{tr}\left(a_t\frac{\partial a_\phi}{\partial r}\right) = -\frac{B}{\ell}\,. \tag{C.3.3}$$

Expanding as it was performed in (C.1.4), the trace formula reads

$$\begin{aligned}
\begin{pmatrix} \delta/\delta p \\ \delta/\delta r \end{pmatrix}\int\sum_{n=0}^{N}2A_n\xi^{N-n}d\phi &= -\xi^{-\gamma}\frac{\partial}{\partial\xi}\sum_{n=0}^{N}\begin{pmatrix} C_n \\ B_n \end{pmatrix}\xi^{N+\gamma-n} \\
&= -\sum_{n=0}^{N}\begin{pmatrix} C_n \\ B_n \end{pmatrix}(N+\gamma-n)\,\xi^{N-n-1} \\
&= -\sum_{n=1}^{N+1}(N+\gamma-n+1)\begin{pmatrix} C_{n-1} \\ B_{n-1} \end{pmatrix}\xi^{N-n}\,.
\end{aligned}$$

Because $A_0 = 1$ (which implies that the first functional functional derivative of the left-hand side summation will vanish), the left hand side summation thus begins from $n = 1$. Additionally, on the right-hand side, when $n = N+1$, we will have expression $\gamma\begin{pmatrix} C_N \\ B_N \end{pmatrix}\xi^{-1}$. In contrast, at the left-hand side we will never reach negative powers of the spectral parameter expansion, hence, we will impose that the latter expression vanishes, yielding

$$\begin{pmatrix} \delta/\delta p \\ \delta/\delta r \end{pmatrix} \int \sum_{n=1}^{N} 2A_n \xi^{N-n} d\phi = -\sum_{n=1}^{N} (N+1+\gamma-n) \begin{pmatrix} C_{n-1} \\ B_{n-1} \end{pmatrix} \xi^{N-n} ,$$

and since ξ is arbitrary we can math the coefficient expansions for $1 \leq n \leq N$,

$$\begin{pmatrix} \delta/\delta p \\ \delta/\delta r \end{pmatrix} \int A_n \, d\phi = -\frac{1}{2} (N+1+\gamma-n) \begin{pmatrix} C_{n-1} \\ B_{n-1} \end{pmatrix} , \tag{C.3.4}$$

In principle, this is the relationship between the functional derivatives of A_n with B_{n-1} and C_{n-1}, however, in order to unambiguously characterize this relationship, we must now fix the value of γ. To do that, consider $n = 2$, which implies

$$\begin{pmatrix} \delta/\delta p \\ \delta/\delta r \end{pmatrix} \int pr \, d\phi = (-N+1-\gamma) \begin{pmatrix} r \\ p \end{pmatrix} .$$

Therefore, we arrive to the system $r = (-N+1-\gamma)\, r$ and $p = (-N+1-\gamma)\, p$. Thus, $\gamma = -N$, from where we obtain the important relationship

$$\begin{pmatrix} \delta/\delta p \\ \delta/\delta r \end{pmatrix} \int A_n \, d\phi = \frac{n-1}{2} \begin{pmatrix} C_{n-1} \\ B_{n-1} \end{pmatrix} . \tag{C.3.5}$$

If we recognize

$$B_{n-1} = \frac{\delta H_n}{\delta r} \equiv \mathcal{R}_n \, , \quad C_{n-1} = \frac{\delta H_n}{\delta p} \equiv \mathcal{P}_n \, , \tag{C.3.6}$$

the trace formula can be read as

$$\begin{pmatrix} \delta/\delta p \\ \delta/\delta r \end{pmatrix} \int A_n \, d\phi = \frac{n-1}{2} \begin{pmatrix} \delta/\delta p \\ \delta/\delta r \end{pmatrix} H_n \, ,$$

allowing to establish the following important relationship between the coefficient A_n and conserved densities, given in Eq. (2.1.19),

$$A_n = \frac{n-1}{2} \mathcal{H}_n \, , \quad n \geq 1 \, . \tag{C.3.7}$$

C.4 Involution of charges

In this appendix, we aim to prove the integrability of the AKNS system, namely, that Eq. (2.1.25)

$$\dot{H}_n = \{H_n, H_m\}_{1,2} = 0\,, \quad n \in \mathbb{N}\,, \tag{C.4.1}$$

holds. In order to prove this statement, it is convenient to recall two definitions. First, from Eq. (2.1.23) the Poisson bracket of the first $\{,\}_1$ and second $\{,\}_2$ Hamiltonian structures

$$\{F[r,p], G[r,p]\}_1 = \int d\phi \left(\frac{\delta F}{\delta u}\right)^t \mathcal{D}_1 \left(\frac{\delta G}{\delta u}\right)\,, \tag{C.4.2a}$$

$$\{F[r,p], G[r,p]\}_1 = \int d\phi \left(\frac{\delta F}{\delta u}\right)^t \mathcal{D}_2 \left(\frac{\delta G}{\delta u}\right)\,, \tag{C.4.2b}$$

where $u = \begin{pmatrix} p \\ r \end{pmatrix}$, whose operators $\mathcal{D}_1$ and $\mathcal{D}_2$ are defined as

$$\mathcal{D}_1 = \frac{1}{\ell}\begin{pmatrix} -2r\partial_\phi^{-1}r & -\partial_\phi + 2r\partial_\phi^{-1}p \\ -\partial_\phi + 2p\partial_\phi^{-1}r & -2p\partial_\phi^{-1}p \end{pmatrix}\,, \quad \mathcal{D}_2 = \frac{1}{\ell}\begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}\,, \tag{C.4.3}$$

respectively. And lastly, the recursion formula between conserved functionals (2.1.24),

$$\mathcal{D}_1\left(\frac{\delta H_{n+1}}{\delta u}\right) = \mathcal{D}_2\left(\frac{\delta H_{n+2}}{\delta u}\right)\,, \quad n \in \mathbb{N}_0\,. \tag{C.4.4}$$

Consider then the following Poisson bracket

$$\{H_n, H_m\}_1 = \int d\phi \left(\frac{\delta H_n}{\delta u}\right)^t \mathcal{D}_1 \left(\frac{\delta H_m}{\delta u}\right)\,. \tag{C.4.5a}$$

Applying Eq. (C.4.4), we can write the latter as

$$\{H_n, H_m\}_1 = \int d\phi \left(\frac{\delta H_n}{\delta u}\right)^t \mathcal{D}_2 \left(\frac{\delta H_{m+1}}{\delta u}\right) \tag{C.4.5b}$$

$$\equiv \{H_n, H_{m+1}\}_2\,, \tag{C.4.5c}$$

or

$$\{H_n, H_m\}_1 = -\{H_{m+1}, H_n\}_2\,. \tag{C.4.5d}$$

Again, according to (C.4.4), the latter expresion can be written as

$$\{H_n, H_m\}_1 = -\int d\phi \left(\frac{\delta H_{m+1}}{\delta u}\right)^t \mathcal{D}_2 \left(\frac{\delta H_n}{\delta u}\right) \tag{C.4.5e}$$

$$= -\int d\phi \left(\frac{\delta H_{m+1}}{\delta u}\right)^t \mathcal{D}_1 \left(\frac{\delta H_{n-1}}{\delta u}\right) \tag{C.4.5f}$$

$$\equiv -\{H_{m+1}, H_{n-1}\}, \tag{C.4.5g}$$

or

$$\{H_n, H_m\}_1 = \{H_{n-1}, H_{m+1}\}_1 . \tag{C.4.5h}$$

Performing the latter procedure $n-m$ times, we obtain

$$\begin{aligned}\{H_n, H_m\}_1 = \{H_{n-1}, H_{m+1}\}_1 = \{H_{n-2}, H_{m+2}\}_1 = \ldots &= \left\{H_{n-(n-m)}, H_{m+n-m}\right\}_1 \\ &\equiv \{H_m, H_n\}_1 .\end{aligned} \tag{C.4.6}$$

Thus, $\{H_n, H_m\}_1 = 0$. In the same fashion, the proof for the second Poisson bracket $\{,\}_2$ is analogue

$$\{H_n, H_m\}_2 = \{H_n, H_m\}_1 = \int d\phi \left(\frac{\delta H_n}{\delta u}\right)^t \mathcal{D}_2 \left(\frac{\delta H_m}{\delta u}\right) \tag{C.4.7a}$$

$$= \int d\phi \left(\frac{\delta H_n}{\delta u}\right)^t \mathcal{D}_1 \left(\frac{\delta H_{m-1}}{\delta u}\right) \tag{C.4.7b}$$

$$\equiv \{H_n, H_{m-1}\}_1 \tag{C.4.7c}$$

$$= -\{H_{m-1}, H_n\}_1 \tag{C.4.7d}$$

$$= -\int d\phi \left(\frac{\delta H_{m-1}}{\delta u}\right)^t \mathcal{D}_1 \left(\frac{\delta H_n}{\delta u}\right) \tag{C.4.7e}$$

$$= -\int d\phi \left(\frac{\delta H_{m-1}}{\delta u}\right)^t \mathcal{D}_2 \left(\frac{\delta H_{n+1}}{\delta u}\right) \tag{C.4.7f}$$

$$\equiv -\{H_{m-1}, H_{n+1}\}_2 \tag{C.4.7g}$$

$$= \{H_{n+1}, H_{m-1}\}_2 , \tag{C.4.7h}$$

where Eq. (C.4.4) was applied when passing to (C.4.7b) and (C.4.7f). Performing the latter procedure $m-n$ times, we obtain

$$\begin{aligned}\{H_n, H_m\}_2 = \{H_{n+1}, H_{m-1}\}_2 = \{H_{n+2}, H_{m-2}\} = \ldots &= \left\{H_{n+(m-n)}, H_{m-(m-n)}\right\}_2 \\ &= \{H_m, H_n\}_2 ,\end{aligned} \tag{C.4.7i}$$

arriving to $\{H_m, H_n\}_2 = 0$. Because charges are in involution for the two Poisson brackets $\{,\}_1$ and $\{,\}_2$, we thus have proved (C.4.1).

C.5 List of AKNS conserved densities

As showed in the previous appendix, the Hamiltonian densities are related with coefficients A_n according to Eq. (C.3.7)

$$A_n = \frac{n-1}{2}\mathcal{H}_n\,, \quad n \geq 1\,,$$

where the coefficients A_n were constructed in Eq. (C.2.2). Here we appendix a list of the first Hamiltonian densities $\mathcal{H}_n$, such that its functionals are

$$H_n = \int d\phi\, \mathcal{H}_n\,.$$

In particular,

$$\begin{aligned}
\mathcal{H}_1 &= 0\,,\\
\mathcal{H}_2 &= -pr\,,\\
\mathcal{H}_3 &= \frac{1}{4}\left(p'r - pr'\right)\,,\\
\mathcal{H}_4 &= \frac{1}{12}\left(p'r' - p''r - pr'' + 3p^2r^2\right)\,,\\
\mathcal{H}_5 &= \frac{1}{32}\left(-6pp'r^2 + 6p^2rr' - p''r' + p'r'' + p'''r - pr'''\right)\,,\\
\mathcal{H}_6 &= \frac{1}{80}\left(-10p^3r^3 + 5p'^2r^2 + 5p^2r'^2 + 10pp''r^2 + 10p^2rr''\right.\\
&\qquad\qquad \left. -p''r'' + p'''r' + p'r''' - p''''r - pr''''\right)\,,\\
\mathcal{H}_7 &= \frac{1}{192}\Big(30p^2p'r^3 - 30p^3r^2r' - 10p'rr' + 10pp'r'^2 - 20p'p''r^2 + 10pp''rr'\\
&\qquad - 10pp'rr'' + 20p^2r'r'' - 10pp'''r^2 + p'''r'' + 10p^2rr'''\\
&\qquad - p''r'''' - p''''r' + p'r'''' + p^{(5)}r - pr^{(5)}\Big)\,,\\
&\ \vdots
\end{aligned} \tag{C.5.1}$$

where $p^{(n)}$, $r^{(n)}$ stands for the nth angular derivative of functions p and r, respectively.

Appendix D

Appendices of Chapter 3

D.1 Dirac analysis of Chern-Simons theory

In this appendix, we explicitly show the Dirac constraint analysis [363] of $2+1$ Chern-Simons theory.

D.1.1 Primary Hamiltonian and Dirac's algorithm

In components, the Chern-Simons action reads

$$I_{\text{CS}} = \frac{K}{4\pi}\int_{\mathcal{M}} d^3x\, \epsilon^{\mu\nu\rho}\left\langle \mathcal{A}_\mu \partial_\nu \mathcal{A}_\rho + \frac{2}{3}\mathcal{A}_\mu \mathcal{A}_\nu \mathcal{A}_\rho \right\rangle , \tag{D.1.1}$$

where $\mathcal{M}$ is foliated with coordinates (t, r, ϕ), with $0 \leq r < \infty$ and $0 \leq \phi \leq 2\pi$. Consider the $2+1$ splitting $\mathcal{A}_\mu = (\mathcal{A}_0, \mathcal{A}_i)$. The previous action decompose as

$$\begin{aligned} I_{\text{CS}} = \frac{K}{4\pi}\int dt d^2x \Big[& \epsilon^{0ij}\left\langle \mathcal{A}_0 \partial_i \mathcal{A}_j + \frac{2}{3}\mathcal{A}_0\mathcal{A}_i\mathcal{A}_j \right\rangle \\ & +\epsilon^{i0j}\left\langle \mathcal{A}_i \partial_0 \mathcal{A}_j + \frac{2}{3}\mathcal{A}_i\mathcal{A}_0\mathcal{A}_j \right\rangle + \epsilon^{ij0}\left\langle \mathcal{A}_i \partial_j \mathcal{A}_0 + \frac{2}{3}\mathcal{A}_i\mathcal{A}_j\mathcal{A}_0 \right\rangle \Big] \end{aligned} \tag{D.1.2a}$$

$$\begin{aligned} = \frac{K}{4\pi}\int dt d^2x\, \epsilon^{ij} \Big\langle & -\mathcal{A}_i \dot{\mathcal{A}}_j + \mathcal{A}_0 \partial_i \mathcal{A}_j - \partial_j \mathcal{A}_i \mathcal{A}_0 + \partial_j\left(\mathcal{A}_i \mathcal{A}_0\right) \\ & + \frac{2}{3}\left(\mathcal{A}_0\mathcal{A}_i\mathcal{A}_j - \mathcal{A}_i\mathcal{A}_0\mathcal{A}_j + \mathcal{A}_i\mathcal{A}_j\mathcal{A}_0\right) \Big\rangle, \end{aligned} \tag{D.1.2b}$$

where $d^2x = drd\phi$, $\epsilon^{0ij} \equiv \epsilon^{ij}$, and we integrated by parts the third term. Regarding boundary terms, we will care about them in Section D.2.1. The last term can be rewritten as

$$\frac{2}{3}\epsilon^{ij}\left\langle \mathcal{A}_0\mathcal{A}_i\mathcal{A}_j - \mathcal{A}_i\mathcal{A}_0\mathcal{A}_j + \mathcal{A}_i\mathcal{A}_j\mathcal{A}_0 \right\rangle = \epsilon^{ij}\left\langle \mathcal{A}_0[\mathcal{A}_i, \mathcal{A}_j] \right\rangle . \tag{D.1.3}$$

Thus, we arrive to action

$$I_H\left[\mathcal{A}_0,\mathcal{A}_i\right]=-\frac{K}{4\pi}\int_{\Sigma\times\mathbb{R}}dt\,d^2x\,\epsilon^{ij}\left\langle\mathcal{A}_i\dot{\mathcal{A}}_j-\mathcal{A}_0F_{ij}\right\rangle+\mathcal{B}\,, \tag{D.1.4}$$

with F_{ij} defined as

$$F_{ij}=\partial_i\mathcal{A}_j-\partial_j\mathcal{A}_i+\left[\mathcal{A}_i,\mathcal{A}_j\right], \tag{D.1.5}$$

and a boundary term that ensures its differentiability. Recalling the Killing metric definition of Appendix A, we can write the Lagrangian density as

$$\mathcal{L}=-\frac{K}{4\pi}\epsilon^{ij}g_{ab}\left(\mathcal{A}_i^a\dot{\mathcal{A}}_j^b-\mathcal{A}_0^aF_{ij}^b\right), \tag{D.1.6}$$

where $F_{ij}^b=\partial_i\mathcal{A}_j^b-\partial_j\mathcal{A}_i^b+f^b{}_{cd}\mathcal{A}_i^c\mathcal{A}_j^d$.

We want to compute the Hamiltonian of the theory. Because we are working with a gauge theory, the constraint nature of the latter will soon appear explicitly. The canonical momentum can be read from (D.1.4), which is

$$\Pi_c^k=\frac{\partial\mathcal{L}}{\partial\dot{A}_k^c}=-\frac{K}{4\pi}\epsilon^{jk}g_{bc}\mathcal{A}_j^b\,. \tag{D.1.7}$$

For next purposes, it is convenient to invert the latter relationship by contracting with $\epsilon_{lk}g^{cd}$ as

$$\Pi_c^k\epsilon_{lk}g^{cd}=-\frac{K}{4\pi}\epsilon^{jk}\epsilon_{lk}g_{bc}g^{cd}\mathcal{A}_j^b=-\frac{K}{4\pi}\mathcal{A}_l^d\,.$$

Then, we can write the gauge field $\mathcal{A}$ in terms of the canonical momentum Π according to

$$\mathcal{A}_l^d=-\frac{4\pi}{K}\Pi_c^k\epsilon_{lk}g^{cd}\,. \tag{D.1.8}$$

From (D.1.7), we can identify two primary constraints

$$\left(\Phi^1\right)_c=\Pi_c^0\approx0\,, \tag{D.1.9a}$$

$$\left(\Phi^2\right)_c^k=\Pi_c^k+\frac{K}{4\pi}\epsilon^{jk}g_{bc}\mathcal{A}_j^b\approx0\,, \tag{D.1.9b}$$

where $\approx$ stands for the usual weakly-equal definition [52]. If Γ is the naive $2N$-dimensional phase space, two quantities F and G will be weakly-equal iff $F=G$ on the constrained surface Γ_c.

According to Dirac [363], let us compute now the canonical Hamiltonian H_c, defined as the following Legendre transformation

$$
\begin{aligned}
H_{\mathrm{c}} &= \int d^2x \left(\Pi^{\mu}_{a}\dot{\mathcal{A}}^{a}_{\mu} - \mathcal{L}\right) && \text{(D.1.10a)}\\
&= \int d^2x \left[\Pi^{i}_{a}\dot{\mathcal{A}}^{a}_{i} + \frac{K}{4\pi}\epsilon^{ij}g_{ab}\left(\mathcal{A}^{a}_{i}\dot{\mathcal{A}}^{b}_{j} - \mathcal{A}^{a}_{0}F^{b}_{ij}\right)\right] && \text{(D.1.10b)}\\
&= \int d^2x \left\{\Pi^{i}_{a}\left(-\frac{4\pi}{K}\dot{\Pi}^{j}_{b}\epsilon_{ij}g^{ba}\right) + \frac{K}{4\pi}\epsilon^{ij}g_{ab}\left[\left(-\frac{4\pi}{K}\Pi^{k}_{c}\epsilon_{ij}g^{ca}\right)\left(-\frac{4\pi}{K}\dot{\Pi}^{l}_{d}\epsilon_{jl}g^{db}\right)\right.\right.\\
&\qquad\left.\left. - \mathcal{A}^{a}_{0}F^{b}_{ij}\right]\right\} && \text{(D.1.10c)}\\
&= -\frac{K}{4\pi}\int d^2x\, \epsilon^{ij}g_{ab}\mathcal{A}^{a}_{0}F^{b}_{ij}\,, && \text{(D.1.10d)}
\end{aligned}
$$

or

$$H_{\mathrm{c}} = -\frac{K}{4\pi}\int d^2x\, \epsilon^{ij}\left\langle \mathcal{A}_0 F_{ij}\right\rangle\,. \tag{D.1.10e}$$

Then, the primary Hamiltonian H_{p}, i.e., the canonical Hamiltonian H_{c} added with the primary constraints, reads

$$H_{\mathrm{p}} = \int d^2x \left[-\frac{K}{4\pi}\epsilon^{ij}g_{ab}\mathcal{A}^{a}_{0}F^{b}_{ij} + (\lambda^1)^{a}_{i}\left(\Pi^{i}_{a} + \frac{K}{4\pi}\epsilon^{ji}g_{ba}\mathcal{A}^{b}_{j}\right) + (\lambda^2)^{a}\Pi^{0}_{a}\right]. \tag{D.1.11}$$

This quantity generates time evolution of any dynamical variable F. In this regard, it is important to emphasize the fact that every constraint of the theory must be preserved along time-evolution. Define the following Poisson bracket

$$\left\{A^{a}_{\mu}(x), \Pi^{\nu}_{b}(x')\right\} = \delta^{(2)}(x - x')\,, \tag{D.1.12}$$

where $\delta^{(2)}(x-x')$ is the 2-dimensional Dirac delta along spatial coordinates (r,ϕ). Before study the consistency of contraints (D.1.9), it is convenient to consider the following Poisson bracket

$$
\begin{aligned}
\left\{\Pi^{k}_{c}(x), F^{b}_{ij}(x')\right\} &= \left\{\Pi^{k}_{c}(x), \partial'_{i}\mathcal{A}^{b}_{j}(x') - \partial'_{j}\mathcal{A}^{b}_{i}(x') + f^{b}{}_{ad}\mathcal{A}^{a}_{i}(x')\mathcal{A}^{d}_{j}(x')\right\}\\
&= -\left[\delta^{b}_{c}\left(\delta^{k}_{j}\partial'_{i} - \delta^{k}_{i}\partial'_{j}\right) + f^{b}{}_{cd}\left(\delta^{k}_{i}\mathcal{A}^{d}_{j}(x') - \delta^{k}_{j}\mathcal{A}^{d}_{i}(x')\right)\right]\delta^{(2)}(x-x')\,.
\end{aligned} \tag{D.1.13}
$$

Let us begin with Φ^1, defined in (D.1.9a). Its time-evolution is given by

$$\left(\dot{\Phi}^1\right)_c = \left\{\Phi^1_c(x), H_{\rm p}\right\} \tag{D.1.14a}$$

$$\begin{aligned} &= \int d^2x' \left\{\Pi^0_c(x), -\frac{K}{4\pi}\epsilon^{ij} g_{ab}\mathcal{A}^a_0(x')F^b_{ij}(x') \right. \\ &\qquad \left. + (\lambda^1)^a_i(x')\left(\Pi^i_a(x') + \frac{K}{4\pi}\epsilon^{ji} g_{ba}\mathcal{A}^b_j(x')\right) + (\lambda^2)^a(x')\Pi^0_a(x')\right\} \end{aligned} \tag{D.1.14b}$$

$$= -\frac{K}{4\pi}\int d^2x'\, \epsilon^{ij} g_{ab}\left\{\Pi^0_c(x), \mathcal{A}^a_0(x')\right\}F^b_{ij}(x') \tag{D.1.14c}$$

$$= \frac{K}{4\pi}\epsilon^{ij} g_{cb}F^b_{ij} \approx 0\,. \tag{D.1.14d}$$

As we see, this constraint leads to a secondary one, namely

$$(\Phi^3)_c = \frac{K}{4\pi}\epsilon^{ij} g_{cb}F^b_{ij} \approx 0\,. \tag{D.1.15}$$

The time-evolution of Φ^2, defined in Eq. (D.1.9b), is

$$\left(\dot{\Phi}^2\right)^k_c = \left\{(\Phi^2)^k_c, H_{\rm p}\right\} \tag{D.1.16a}$$

$$\begin{aligned} &= \int d^2x' \left\{\Pi^k_c(x) + \frac{K}{4\pi}\epsilon^{lk} g_{dc}\mathcal{A}^d_l(x), -\frac{K}{4\pi}\epsilon^{ij} g_{ab}\mathcal{A}^a_0(x')F^b_{ij}(x') \right. \\ &\qquad \left. + (\lambda^1)^a_i(x')\left(\Pi^i_a(x') + \frac{K}{4\pi}\epsilon^{ji} g_{ba}\mathcal{A}^b_j(x')\right) + (\lambda^2)^a(x')\Pi^0_a(x')\right\} \end{aligned} \tag{D.1.16b}$$

$$\begin{aligned} &= \frac{K}{4\pi}\int d^2x \Big[-\epsilon^{ij} g_{ab}\mathcal{A}^a_0(x')\left\{\Pi^k_c(x), F^b_{ij}(x')\right\} \\ &\qquad + \epsilon^{ji} g_{ba}(\lambda^1)^a_i(x')\left\{\Pi^k_c(x), \mathcal{A}^b_j(x')\right\} + \epsilon^{lk} g_{dc}(\lambda^1)^a_i(x')\left\{\mathcal{A}^d_l(x), \Pi^i_a(x')\right\}\Big]\,. \end{aligned} \tag{D.1.16c}$$

By means of (D.1.13), we obtain

$$\left(\dot{\Phi}^2\right)^k_c = \frac{K}{2\pi}\int d^2x'\, \epsilon^{kj} g_{ac}\left[-\mathcal{A}^a_0(x')\partial'_j - f^a{}_{bd}\mathcal{A}^b_0(x')\mathcal{A}^d_j(x') - \left(\lambda^1\right)^a_j(x')\right]\delta^{(2)}(x-x') \tag{D.1.16d}$$

$$= \frac{K}{2\pi}\epsilon^{kj} g_{ac}\left[\partial_j\mathcal{A}^a_0 - f^a{}_{bd}\mathcal{A}^b_0\mathcal{A}^d_j - \left(\lambda^1\right)^a_j\right] \approx 0\,. \tag{D.1.16e}$$

As we see, the time-evolution of Φ^2 does not generate any new constraint, but a condition that fix the Lagrange multiplier λ^1. Regarding the consistency of the third constraint Φ^3, it can be directly computed using (D.1.13))

$$
\begin{aligned}
(\dot{\Phi}^3)_c &= \left\{(\Phi^3)_c(x), H_{\mathrm{p}}\right\} && \text{(D.1.17a)}\\
&= \int d^2x' \left\{\frac{K}{4\pi}\epsilon^{kl} g_{cd} F^d_{kl}(x), -\frac{K}{4\pi}\epsilon^{ij} g_{ab}\mathcal{A}^a_0(x') F^b_{ij}(x') \right.\\
&\qquad \left. + (\lambda^1)^a_i(x')\left(\Pi^i_a(x') + \frac{K}{4\pi}\epsilon^{ji} g_{ba}\mathcal{A}^b_j(x')\right) + (\lambda^2)^a(x')\Pi^0_a(x')\right\} && \text{(D.1.17b)}\\
&= \frac{K}{4\pi}\int d^2x'\, \epsilon^{kl} g_{cd}(\lambda^1)^a_i(x')\left\{F^d_{kl}(x), \Pi^i_a(x')\right\} && \text{(D.1.17c)}\\
&= \frac{K}{2\pi}\epsilon^{kj} g_{ac}\left[\partial_k \left(\lambda^1\right)^a_j + f^a{}_{de}\left(\lambda^1\right)^d_j \mathcal{A}^e_k\right] \approx 0\,. && \text{(D.1.17d)}
\end{aligned}
$$

Again, this is another condition for the Lagrange multiplier λ^1, so we end Dirac's algorithm here, since no new constraints emerged. Hence, in sum we have the following constraints

$$
\begin{aligned}
\left(\Phi^1\right)_c(x) &= \Pi^0_c(x) \approx 0\,,\\
\left(\Phi^2\right)^k_c(x) &= \Pi^k_c(x) + \frac{K}{4\pi}\epsilon^{jk} g_{bc}\mathcal{A}^b_j(x) \approx 0\,,\\
\left(\Phi^3\right)_c(x) &= \frac{K}{4\pi}\epsilon^{ij} g_{cb} F^b_{ij} \approx 0\,.
\end{aligned}
$$

Table D.1: Constraints of Chern-Simons theory

D.1.2 Classification of constraints and Dirac bracket

With the constraints identified and summarized in Table D.1, we aim here to construct the compatible Poisson bracket of the theory, i.e., the Dirac bracket $\{,\}_{\mathrm{DB}}$ defined as

$$
\begin{aligned}
\{F(x), G(x')\}_{\mathrm{DB}} &= \{F(x), G(x')\}\\
&\qquad - \int d^2y\, d^2y' \,\{F(x), \Phi^A(y)\} C^{-1}_{AB}(y, y')\left\{\Phi^B(y'), G(x')\right\}\,,
\end{aligned} \tag{D.1.18}
$$

where $C_{AB} = \left\{\Phi^A, \Phi^B\right\}$ and

$$
C^{AD} C^{-1}_{DB} = \delta^A_B = C^{-1}_{BD} C^{DA}\,. \tag{D.1.19}
$$

So, in order to compute the Dirac bracket of Chern-Simons theory, first it is necessary to obtain the form of matrix C^{AB}, which allow us to classify the constraints of the theory. According to [363], if the Poisson bracket between the function $F = F(\mathcal{A}, \Pi)$ and a constraint Φ^i, with $i = 1, 2, 3$, vanishes weakly, i.e.,

$$\left\{F, \Phi^i\right\} \approx 0\,, \quad \forall i\,, \tag{D.1.20}$$

then we can say that F is a first-class constraint. On the other hand, F will be a second-class one if there is at least one constraint such that its Poisson bracket with F does not vanishes weakly, i.e., $\exists i \left(\left\{F, \Phi^i\right\} \not\approx 0\right)$. Regarding Chern-Simons theory, it is direct to prove that Φ^1 is a first-class constraint, while Φ^2 and Φ^3 are second-class, since the only nontrivial Poisson brackets are

$$\begin{aligned}
\left\{\left(\Phi^2\right)^k_c(x), \left(\Phi^2\right)^l_d(x')\right\} &= \left\{\Pi^k_c(x) + \frac{K}{4\pi}\epsilon^{jk} g_{bc}\mathcal{A}^b_j(x), \Pi^l_d(x') + \frac{K}{4\pi}\epsilon^{il} g_{ed}\mathcal{A}^e_i(x')\right\} \\
&= \frac{K}{4\pi}\left(\epsilon^{jl} g_{bd}\left\{\Pi^k_c(x), \mathcal{A}^b_j(x')\right\} + \epsilon^{jk} g_{bc}\left\{\mathcal{A}^b_j(x), \Pi^l_d(x')\right\}\right) \\
&= -\frac{K}{2\pi}\epsilon^{kl} g_{cd}\delta^{(2)}(x-x')\,,
\end{aligned} \tag{D.1.21a}$$

$$\begin{aligned}
\left\{\left(\Phi^2\right)^k_c(x), \left(\Phi^3\right)_d(x')\right\} &= \left\{\Pi^k_c(x) + \frac{K}{4\pi}\epsilon^{jk} g_{bc}\mathcal{A}^b_j(x), \frac{K}{4\pi}\epsilon^{lm} g_{de}F^e_{lm}(x')\right\} \\
&= \frac{K}{4\pi}\epsilon^{ij} g_{db}\left\{\Pi^k_c(x), F^b_{ij}(x')\right\} \\
&= \frac{K}{2\pi}\epsilon^{kj}\left(g_{dc}\partial'_j + f_{dbc}\mathcal{A}^b_j(x')\right)\delta^{(2)}(x-x')\,,
\end{aligned} \tag{D.1.21b}$$

where Eq. (D.1.13) was used. We can summarize the latter results in Table D.2

$$\begin{aligned}
&\left\{\left(\Phi^1\right)_c(x), \left(\Phi^1\right)_d(x')\right\} \approx 0\,, \\
&\left\{\left(\Phi^1\right)_c(x), \left(\Phi^2\right)^k_d(x')\right\} \approx 0\,, \\
&\left\{\left(\Phi^1\right)_c(x), \left(\Phi^3\right)_d(x')\right\} \approx 0\,, \\
\\
&\left\{\left(\Phi^2\right)^k_c(x), \left(\Phi^1\right)_d(x')\right\} \approx 0\,, \\
&\left\{\left(\Phi^2\right)^k_c(x), \left(\Phi^2\right)^l_d(x')\right\} = -\frac{K}{2\pi}\epsilon^{kl} g_{cd}\delta^{(2)}(x-x')\,, \\
&\left\{\left(\Phi^2\right)^k_c(x), \left(\Phi^3\right)_d(x')\right\} = \frac{K}{2\pi}\epsilon^{kj}\left(g_{dc}\partial'_j + f_{dbc}\mathcal{A}^b_j(x')\right)\delta^{(2)}(x-x')\,, \\
\\
&\left\{\left(\Phi^3\right)_c(x), \left(\Phi^1\right)_d(x')\right\} \approx 0\,, \\
&\left\{\left(\Phi^3\right)_c(x), \left(\Phi^2\right)^k_d(x')\right\} = -\frac{K}{2\pi}\epsilon^{kj}\left(g_{dc}\partial_j - f_{dbc}\mathcal{A}^b_j(x)\right)\delta^{(2)}(x-x')\,, \\
&\left\{\left(\Phi^3\right)_c(x), \left(\Phi^3\right)_d(x')\right\} \approx 0\,.
\end{aligned}$$

Table D.2: Poisson brackets of constraints. We readily see that Φ^1 is first-class, while the remaining, Φ^2 and Φ^3, are second-class constraints

Although Φ^3 can be identified with a second-class constraint, it can be linearly combined with Φ^2 as

$$(\bar{\Phi}^3)_c(x) = (\Phi^3)_c(x) - \left[\partial_i(\Phi^2)^i_c(x) + f^a{}_{bc}\mathcal{A}^b_i(x)(\Phi^2)^i_a(x)\right] , \tag{D.1.22}$$

in order to convert it into a first-class one, since every Poisson bracket of the latter improved constraint vanishes with the others [401]. Because only Φ^2 remains as a second-class constraint of the theory, the Dirac bracket of Chern-Simons theory is defined as

$$\begin{aligned}\{F(x), G(x')\}_{\mathrm{DB}} &= \{F(x), G(x')\} \\ &\quad - \int d^2y\, d^2y' \left\{F(x), \left(\Phi^2\right)^k_c(y)\right\} \left(C^{-1}\right)^{cd}_{kl}(y, y')\left\{\left(\Phi^2\right)^l_d(y'), G(x')\right\},\end{aligned} \tag{D.1.23}$$

where

$$C^{kl}_{cd}(x, x') = \left\{\left(\Phi^2\right)^k_c(x), \left(\Phi^2\right)^l_d(x')\right\} = -\frac{K}{2\pi}\epsilon^{kl}g_{cd}\delta^{(2)}(x - x') . \tag{D.1.24}$$

Because the matrix C fulfills

$$\int dw\, C^{ki}_{ce}(y, w) \left(C^{-1}\right)^{ed}_{il}(w, y') = \delta^k_l\delta^d_c\delta^{(2)}(y - y') , \tag{D.1.25}$$

it is possible to find its inverse, which is given by

$$\left(C^{-1}\right)^{cd}_{kl}(x, x') = \frac{2\pi}{K}\epsilon_{kl}g^{cd}\delta^{(2)}(x - x') . \tag{D.1.26}$$

So now we have all the ingredients to compute the Dirac bracket (D.1.23) of Chern-Simons theory. Particularly, the bracket between $\mathcal{A}^a_i(x)$ with $\mathcal{A}^b_j(x')$ is

$$\begin{aligned}\left\{\mathcal{A}^a_i(x), \mathcal{A}^b_j(x')\right\}_{\mathrm{DB}} &= -\int d^2y\, d^2y' \left\{\mathcal{A}^a_i(x), \Pi^k_c(y)\right\} \left(\frac{2\pi}{K}\epsilon_{kl}g^{cd}\delta^{(2)}(y - y')\right) \times \\ &\qquad \times \left\{\Pi^l_d(y'), \mathcal{A}^b_j(x')\right\} \end{aligned} \tag{D.1.27a}$$

$$= \frac{2\pi}{K}\epsilon_{ij}g^{ab}\delta^{(2)}(x - x') , \tag{D.1.27b}$$

in agreement with result (3.2.19), obtained by means of Jackiw symplectic method. On the other hand, the Dirac bracket between $\mathcal{A}_i^a(x)$ with $\Pi_b^j(x')$ is

$$\begin{aligned}\left\{\mathcal{A}_i^a(x),\Pi_b^j(x')\right\}_{\text{DB}} &= \left\{\mathcal{A}_i^a(x),\Pi_b^j(x')\right\} - \int d^2y\,d^2y'\,\left\{\mathcal{A}_i^a(x),\left(\Phi^2\right)_c^k(y)\right\}\times \\ &\qquad\qquad \times\left(\frac{2\pi}{K}\epsilon_{kl}g^{cd}\delta^{(2)}(y-y')\right)\left\{\left(\Phi^2\right)_d^l(y'),\Pi_b^j(x')\right\} &\text{(D.1.28a)}\\ &= \delta_b^a\delta_i^j\delta^{(2)}(x-x') - \frac{1}{2}\int d^2y\,d^2y'\,\delta_c^a\delta_i^k\delta^{(2)}(x-y)\times \\ &\qquad\qquad \times\,\epsilon_{kl}g^{cd}\delta^{(2)}(y-y')\epsilon^{ml}g_{ed}\delta_b^e\delta_m^j\delta^{(2)}(y'-x') &\text{(D.1.28b)}\\ &= \frac{1}{2}\delta_b^a\delta_i^j\delta^{(2)}(x-x')\,. &\text{(D.1.28c)}\end{aligned}$$

Finally, the Dirac bracket between the canonical momentum turns out to be

$$\begin{aligned}\left\{\Pi_a^i(x),\Pi_b^j(x')\right\}_{\text{DB}} &= -\int d^2y\,d^2y'\,\left\{\Pi_a^i(x),\left(\Phi^2\right)_c^k(y)\right\}\left(\frac{2\pi}{K}\epsilon_{kl}g^{cd}\delta^{(2)}(y-y')\right)\times \\ &\qquad\qquad \times\left\{\left(\Phi^2\right)_d^l(y'),\Pi_b^j(x')\right\} &\text{(D.1.29a)}\\ &= -\int d^2y\,d^2y'\,\left\{\Pi_a^i(x),\frac{K}{4\pi}\epsilon^{mk}g_{ec}\mathcal{A}_m^e(y)\right\}\times \\ &\qquad \times\left(\frac{2\pi}{K}\epsilon_{kl}g^{cd}\delta^{(2)}(y-y')\right)\left\{\frac{K}{4\pi}\epsilon^{nl}g_{hd}\mathcal{A}_n^h(y'),\Pi_b^j(x')\right\} &\text{(D.1.29b)}\\ &= -\frac{1}{2}\int d^2y\,d^2y'\,\epsilon^{mk}g_{ec}\delta_m^i\delta_c^e\delta^{(2)}(x-y)\epsilon_{kl}g^{cd}\delta^{(2)}(y-y')\times \\ &\qquad\qquad \times\frac{K}{4\pi}\epsilon^{nl}g_{hd}\delta_d^h\delta_n^j\delta^{(2)}(y'-x') &\text{(D.1.29c)}\\ &= -\frac{K}{8\pi}\epsilon^{ik}g_{ac}\epsilon_{kl}g^{cd}\epsilon^{jl}g_{bd}\delta^{(2)}(x-x') &\text{(D.1.29d)}\\ &= -\frac{K}{8\pi}\epsilon^{ij}g_{ab}\delta^{(2)}(x-x')\,. &\text{(D.1.29e)}\end{aligned}$$

In sum, we have the following results, according to Table D.3,

$$\begin{aligned}\left\{\mathcal{A}_i^a(x),\mathcal{A}_j^b(x')\right\}_{\text{DB}} &= \frac{2\pi}{K}\epsilon_{ij}g^{ab}\delta^{(2)}(x-x')\,,\\ \left\{\mathcal{A}_i^a(x),\Pi_b^j(x')\right\}_{\text{DB}} &= \frac{1}{2}\delta_b^a\delta_i^j\delta^{(2)}(x-x')\,,\\ \left\{\Pi_a^i(x),\Pi_b^j(x')\right\}_{\text{DB}} &= -\frac{K}{8\pi}\epsilon^{ij}g_{ab}\delta^{(2)}(x-x')\,.\end{aligned}$$

Table D.3: Dirac brackets of Chern-Simons theory

Now we can impose the constraints, so the primary Hamiltonian of the theory (D.1.11) turns into the Chern-Simons Hamiltonian

$$H = -\frac{K}{4\pi}\int d^2x\,\epsilon^{ij}\left\langle\mathcal{A}_0F_{ij}\right\rangle\,, \tag{D.1.30}$$

allowing us to say that (D.1.4)

$$I_H\left[\mathcal{A}_0,\mathcal{A}_i\right] = -\frac{K}{4\pi}\int_{\Sigma\times\mathbb{R}} dt\, d^2x\,\left\langle \mathcal{A}_i\dot{\mathcal{A}}_j - \mathcal{A}_0 F_{ij}\right\rangle\,,$$

corresponds to the Hamiltonian action of Chern-Simons theory, where the canonical momentum is

$$\Pi^k_c(x) = -\frac{K}{4\pi}\epsilon^{jk} g_{bc}\mathcal{A}^b_j(x)\,.$$

D.2 Regge-Teitelboim analysis of Chern-Simons theory

In this appendix we explicitly show the details regarding the constraint analysis of Chern–Simons theory à la Regge-Teitelboim.

D.2.1 Boundary term

Let us begin with the Hamiltonian action of Chern-Simons theory

$$I_H\left[\mathcal{A}_0,\mathcal{A}_i\right] = -\frac{K}{4\pi}\int_{\Sigma\times\mathbb{R}} dt\, d^2x\,\epsilon^{ij}\left\langle \mathcal{A}_i\dot{\mathcal{A}}_j - \mathcal{A}_0 F_{ij}\right\rangle + \mathcal{B}\,, \tag{D.2.1}$$

with F_{ij} defined as

$$F_{ij} = \partial_i\mathcal{A}_j - \partial_k\mathcal{A}_i + \left[\mathcal{A}_i,\mathcal{A}_j\right], \tag{D.2.2}$$

where $F_{ij} = \partial_i\mathcal{A}_j - \partial_j\mathcal{A}_i + \left[\mathcal{A}_i,\mathcal{A}_j\right]$ is a constraint of the theory, $\mathcal{A}_0$ a Lagrange multiplier, and $\mathcal{B}$ a boundary term that must ensure that the action functional I_H is differentiable.

The equations of motion are obtained when we perform an infinitesimal variation on the dynamical fields, as following

$$
\begin{aligned}
\delta I_H = &-\frac{K}{4\pi}\int dtd^2x\ \epsilon^{ij}\Big\langle \delta\mathcal{A}_i\,\dot{\mathcal{A}}_j + \mathcal{A}_i\,\delta\dot{\mathcal{A}}_j - \delta\mathcal{A}_0\,F_{ij} \\
&\qquad - \mathcal{A}_0\left(\partial_i\,\delta\mathcal{A}_j - \partial_j\,\delta\mathcal{A}_j + \delta[\mathcal{A}_i,\mathcal{A}_j]\right)\Big\rangle + \delta\mathcal{B} \qquad \text{(D.2.3a)}\\
= &-\frac{K}{4\pi}\int dtd^2x\ \epsilon^{ij}\Big\langle \delta\mathcal{A}_i\,\dot{A}_j - \dot{A}_i\,\delta A_j + \partial_t\left(\mathcal{A}_i\,\delta\mathcal{A}_j\right) - \delta\mathcal{A}_0\,F_{ij} \\
&\qquad - \mathcal{A}_0\left(2\partial_i\,\delta\mathcal{A}_j + \delta\left(\mathcal{A}_i\mathcal{A}_j - \mathcal{A}_j\mathcal{A}_i\right)\right)\Big\rangle + \delta\mathcal{B} \qquad \text{(D.2.3b)}\\
= &-\frac{K}{4\pi}\int dtd^2x\ \epsilon^{ij}\Big\langle -2\dot{\mathcal{A}}_i\,\delta\mathcal{A}_j + \partial_t\left(\mathcal{A}_i\,\delta A_j\right) - \delta\mathcal{A}_0\,F_{ij} \\
&\qquad - \mathcal{A}_0\left(2\partial_i\,\delta\mathcal{A}_j + 2\delta\left(\mathcal{A}_i\mathcal{A}_j\right)\right)\Big\rangle + \delta\mathcal{B} \qquad \text{(D.2.3c)}\\
= &-\frac{K}{4\pi}\int dtd^2x\ \epsilon^{ij}\Big\langle -2\dot{\mathcal{A}}_i\,\delta\mathcal{A}_j + \partial_t\left(\mathcal{A}_i\,\delta A_j\right) - \delta\mathcal{A}_0\,F_{ij} \\
&\qquad + 2\partial_i\mathcal{A}_0\,\delta\mathcal{A}_j - \partial_i\left(2\mathcal{A}_0\,\delta\mathcal{A}_j\right) \qquad \text{(D.2.3d)}\\
&\qquad - 2\left(\mathcal{A}_0\,\delta\mathcal{A}_i\,\mathcal{A}_j + \mathcal{A}_0\mathcal{A}_i\,\delta\mathcal{A}_j\right)\Big\rangle + \delta\mathcal{B} \\
= &-\frac{K}{4\pi}\int dtd^2x\ \epsilon^{ij}\Big\langle -2\dot{\mathcal{A}}_i\,\delta\mathcal{A}_j + \partial_t\left(\mathcal{A}_i\,\delta A_j\right) - \delta\mathcal{A}_0\,F_{ij} \\
&\qquad + 2\partial_i\mathcal{A}_0\,\delta\mathcal{A}_j - \partial_i\left(2\mathcal{A}_0\,\delta\mathcal{A}_j\right) - 2[\mathcal{A}_0,\mathcal{A}_i]\,\delta\mathcal{A}_j\Big\rangle + \delta\mathcal{B}\,, \qquad \text{(D.2.3e)}
\end{aligned}
$$

where, in order to create the commutator, we used the cyclic property of the traces and changed indices using the properties of the Levi-Civita symbol. Finally we can write the variation of the Hamiltonian action as (in agreement with Eq. (3.2.6))

$$
\begin{aligned}
\delta I_H = -\frac{K}{4\pi}\int dtd^2x\ \epsilon^{ij}\Big\langle 2\left(-\dot{\mathcal{A}}_i + \partial_i\mathcal{A}_0 - [\mathcal{A}_0,\mathcal{A}_i]\right)\delta\mathcal{A}_j - \delta\mathcal{A}_0\,F_{ij} \\
+ \partial_t\left(\mathcal{A}_i\,\delta A_j\right) - \partial_i\left(2\mathcal{A}_0\,\delta\mathcal{A}_j\right)\Big\rangle + \delta\mathcal{B}\,. \qquad \text{(D.2.3f)}
\end{aligned}
$$

Hence, we have the following equations of motion

$$
\frac{\delta I_H}{\delta\mathcal{A}_j} = \partial_i\mathcal{A}_0 - \partial_0\mathcal{A}_i + [\mathcal{A}_i,\mathcal{A}_0] \equiv F_{i0} = 0 \quad , \quad \frac{\delta I_H}{\delta\mathcal{A}_0} = F_{ij} = 0\,. \qquad \text{(D.2.4)}
$$

Vanishing the terms proportional to the latter terms, the action (D.2.3f) reads

$$
\delta I_H = -\frac{K}{4\pi}\int dtd^2x\ \epsilon^{ij}\Big\langle \partial_t\left(\mathcal{A}_i\,\delta A_j\right) - \partial_i\left(2\mathcal{A}_0\,\delta\mathcal{A}_j\right)\Big\rangle + \delta\mathcal{B}\,. \qquad \text{(D.2.5)}
$$

Assuming the fields behave as

$$
\delta\mathcal{A}_j\left(t_1,r,\phi\right) = 0\,, \quad \text{and} \quad \delta\mathcal{A}_j\left(t_2,r,\phi\right) = 0\,,
$$

the surface integral reads

$$\delta I_H = \frac{K}{4\pi}\int dtd^2x\, \epsilon^{ij}\Big\langle \partial_i \left(2\mathcal{A}_0\, \delta\mathcal{A}_j\right) \Big\rangle + \delta\mathcal{B} \tag{D.2.6a}$$

$$= \frac{K}{2\pi}\int dtd^2x\, \Big\langle \partial_r \left(\epsilon^{r\phi}\mathcal{A}_0\, \delta\mathcal{A}_\phi\right) + \partial_\phi \left(\epsilon^{\phi r}\mathcal{A}_0\, \delta\mathcal{A}_r\right) \Big\rangle + \delta\mathcal{B}\ . \tag{D.2.6b}$$

Choosing manifold orientation $\epsilon^{r\phi} = 1$ and periodicity of the fields between 0 and 2π, the angular surface term vanishes, obtaining

$$\delta I_H = \frac{K}{2\pi}\int dtdrd\phi\, \Big\langle \partial_r \left(\mathcal{A}_0\, \delta\mathcal{A}_\phi\right) \Big\rangle + \delta\mathcal{B} = \frac{K}{2\pi}\int_{\partial\mathcal{M}} dtd\phi\, \Big\langle \mathcal{A}_0\, \delta\mathcal{A}_\phi \Big\rangle + \delta\mathcal{B}\ , \tag{D.2.6c}$$

where Stokes' theorem was used. Because the action must reach a minimum, the boundary term finally reads

$$\delta\mathcal{B} = -\frac{K}{2\pi}\int_{\partial\mathcal{M}} dtd\phi\, \Big\langle \mathcal{A}_0\, \delta\mathcal{A}_\phi \Big\rangle\,, \tag{D.2.7}$$

in agreement with (3.2.10).

D.2.2 Charge generator

In this section, we aim to prove that the constraint (3.2.14),

$$G\left[\Lambda\right] = \frac{K}{4\pi}\int_\Sigma d^2x\, \epsilon^{ij}\Big\langle \Lambda F_{ij} \Big\rangle = \frac{K}{4\pi}\int_\Sigma d^2x \epsilon^{ij} g_{ab}\Lambda^a F^b_{ij}\,, \tag{D.2.8}$$

generates infinitesimal gauge transformations (3.2.13),

$$\delta\mathcal{A} = d\Lambda + \left[\mathcal{A}, \Lambda\right], \tag{D.2.9}$$

which, in components, reads as

$$\delta\mathcal{A}^a_i(x) = \partial_i\Lambda^a(x) + f^a{}_{bc}\mathcal{A}^b_i(x)\Lambda^c(x)\,, \tag{D.2.10}$$

where $f^a{}_{bc}$ is the structure constant of the algebra.

From Jackiw symplectic approach of Eq. (3.2.19) and the Dirac brackets summarized in Table D.3, the bracket between the gauge fields reads

$$\left\{\mathcal{A}^a_i(x), \mathcal{A}^b_j(x'\right\} \equiv J^{ab}_{ij}(x,x') = \frac{2\pi}{K}\epsilon_{ij} g^{ab}\delta^{(2}(x-x')\,. \tag{D.2.11}$$

Thus, an infinitesimal variation of the field $\mathcal{A}_i^a(x)$ is

$$\delta\mathcal{A}_i^a(x) = \{\mathcal{A}_i^a(x), G[\Lambda]\} \tag{D.2.12a}$$

$$= \frac{K}{4\pi}\int d^2x'\ \epsilon^{kl} g_{cd}\Lambda^c(x')\big\{\mathcal{A}_i^a(x), F_{kl}^d(x')\big\}\,, \tag{D.2.12b}$$

where

$$\begin{aligned}
\epsilon^{kl}\big\{\mathcal{A}_i^a(x), F_{kl}^d(x')\big\} &= \epsilon^{kl}\Big\{\mathcal{A}_i^a(x), \partial_k\mathcal{A}_l^d(x') - \partial_l\mathcal{A}_k^d(x') + f^d{}_{ef}\mathcal{A}_k^e(x')\mathcal{A}_l^f(x')\Big\} \\
&= 2\epsilon^{kl}\left(\partial'_k\{\mathcal{A}_i^a(x), \mathcal{A}_l^d(x')\} + f^d{}_{ef}\{\mathcal{A}_i^a(x), \mathcal{A}_k^e(x')\}\mathcal{A}_l^f(x')\right) \\
&= \frac{4\pi}{K}\epsilon^{kl}\left(\epsilon_{il}g^{ad}\partial'_k\delta^{(2}(x-x') + f^d{}_{ef}\epsilon_{ik}g^{ae}\delta^{(2}(x-x')\mathcal{A}_l^f(x')\right) \\
&= \frac{4\pi}{K}\left(g^{ad}\partial'_i\delta^{(2}(x-x') - f^d{}_{ef}g^{ae}\delta^{(2}(x-x')\mathcal{A}_i^f(x')\right)\,.
\end{aligned}$$

Therefore, the variation is

$$\delta\mathcal{A}_i^a(x) = \int d^2x'\ \Lambda^c(x')g_{cd}\left(g^{ad}\partial'_i\delta^{(2}(x-x') - f^d{}_{ef}g^{ae}\delta^{(2}(x-x')\mathcal{A}_i^f(x')\right) \tag{D.2.12c}$$

$$= \int d^2x'\ \left(\Lambda^a(x')\partial'_i\delta^{(2}(x-x') - g^{ae}f_{cef}\Lambda^c(x')\mathcal{A}_i^f(x')\delta^{(2}(x-x')\right) \tag{D.2.12d}$$

$$= \int d^2x'\ \left(-\partial'_i\Lambda^a(x') - g^{ae}f_{cef}\Lambda^c(x')\mathcal{A}_i^f(x')\right)\delta^{(2}(x-x') \tag{D.2.12e}$$

$$= -\partial_i\Lambda^a(x) + f^a{}_{bc}\Lambda^b(x)\mathcal{A}_i^c(x) \tag{D.2.12f}$$

$$= -\left[\partial_i\Lambda^a(x) + f^a{}_{bc}\mathcal{A}_i^b(x)\Lambda^c(x)\right]\,, \tag{D.2.12g}$$

proving that the constraint (D.2.8) generate gauge transformations (D.2.10), allowing us to classify it as a first-class constraint [52].

An important aspect of the constraint is that it must be differentiable. To prove this statement, let us first recall the definition of the Poisson bracket of Chern-Simons theory. If F and G depend on $\mathcal{A}$, then the Poisson bracket is defined as

$$\{F, G\} = \int d^2x\, d^2x'\,\frac{\delta F}{\mathcal{A}_i^a(x)}J_{ij}^{ab}(x,x')\frac{\delta G}{\mathcal{A}_j^b(x')} \tag{D.2.13a}$$

$$= \frac{2\pi}{K}\int d^2x\, d^2x'\,\frac{\delta F}{\mathcal{A}_i^a(x)}\epsilon_{ij}g^{ab}\delta^{(2}(x-x')\frac{\delta G}{\mathcal{A}_j^b(x')} \tag{D.2.13b}$$

$$= \frac{2\pi}{K}\int d^2x'\,\frac{\delta F}{\mathcal{A}_i^a}\epsilon_{ij}g^{ab}\frac{\delta G}{\mathcal{A}_j^b}\,. \tag{D.2.13c}$$

Then, $\delta A_i^a(x)$ can be calculated as

$$\delta A_i^a(x) = \{\mathcal{A}_i^a(x), G\,[\Lambda]\} \tag{D.2.14a}$$

$$= \frac{2\pi}{K}\int d^2x' \frac{\delta\mathcal{A}_i^a}{\mathcal{A}_k^c}\epsilon_{kl}g^{cd}\frac{\delta G[\Lambda]}{\mathcal{A}_l^d} \tag{D.2.14b}$$

$$= \frac{2\pi}{K}\int d^2x'\, \delta_c^a\delta_i^k\epsilon_{kl}g^{cd}\delta^{(2}(x-x')\frac{\delta G[\Lambda]}{\mathcal{A}_l^d} \tag{D.2.14c}$$

$$= \frac{2\pi}{K}g^{ad}\epsilon_{il}\frac{\delta G[\Lambda]}{\mathcal{A}_l^d}\,. \tag{D.2.14d}$$

We must make sense of the functional derivative of $G\,[\Lambda]$ with respect to $\mathcal{A}_l^d$. Let us consider its definition, given in (D.2.8) and calculate its variation as,

$$\delta G\,[\Lambda] = \frac{K}{4\pi}\int d^2x\,\epsilon^{ij}\Lambda^b\delta F_{ij}^a \tag{D.2.15a}$$

$$= \frac{K}{2\pi}\int d^2x'\epsilon^{ij}\Lambda_b\left(\partial_i\delta\mathcal{A}_j^b + f^b{}_{cd}\mathcal{A}_i^c\delta\mathcal{A}_j^d\right)\,. \tag{D.2.15b}$$

Defining the covariant derivative $D_i := \partial_i + [A_i,\cdot]$, we can rewrite the latter as

$$\delta G\,[\Lambda] = \frac{K}{2\pi}\int d^2x'\epsilon^{ij}\Lambda_b D_i\delta\mathcal{A}_j^b\,. \tag{D.2.15c}$$

Intergrating by parts, we arrive to

$$\delta G\,[\Lambda] = \frac{K}{2\pi}\int d^2x'\,\epsilon^{ij}D_i\left(\Lambda_b\delta\mathcal{A}_j^b\right) - \frac{K}{2\pi}\int d^2x'\,\epsilon^{ij}D_i\Lambda_b\delta\mathcal{A}_j^b\,. \tag{D.2.15d}$$

Let us focus on the right-hand side. As we can see, the first term is not differentiable, since we cannot calculate its functional derivative with respect to $\mathcal{A}$ (it is a boundary term); while regarding the second term, its functional derivative will be proportional to the desired infinitesimal gauge transformation. This is the reason why we define the variation of the boundary term as

$$\delta Q\,[\Lambda] = -\frac{K}{2\pi}\int d^2x'\,\epsilon^{ij}D_i\left(\Lambda_b\delta\mathcal{A}_j^b\right)\,, \tag{D.2.16}$$

in such a way that the constraint is improved, redefining itself as well as

$$\overline{G}[\Lambda] = G[\Lambda] + Q[\Lambda]\,, \tag{D.2.17}$$

hence, its variation is

$$\delta\overline{G}[\Lambda] = \delta G[\Lambda] + \delta Q[\Lambda] \tag{D.2.18}$$

$$\equiv \delta G[\Lambda] - \frac{K}{2\pi}\int d^2x'\,\epsilon^{ij}D_i\left(\Lambda_b\delta\mathcal{A}_j^b\right)\,, \tag{D.2.19}$$

where

$$\delta\overline{G}[\Lambda] = -\frac{K}{2\pi}\int d^2x'\,\epsilon^{ij}D_i\Lambda_b\delta\mathcal{A}_j^b\,. \tag{D.2.20}$$

Therefore, by construction

$$\frac{\delta\overline{G}[\Lambda]}{\delta\mathcal{A}_j^b} = -\frac{K}{2\pi}\epsilon^{ij}D_i\Lambda_b\,. \tag{D.2.21}$$

Replacing the improved constraint in the place where the original constraint was in (D.2.14d), we see that

$$\delta A_i^a(x) = \frac{2\pi}{K}g^{ad}\epsilon_{il}\frac{\delta\overline{G}[\Lambda]}{\mathcal{A}_l^d} = \frac{2\pi}{K}g^{ad}\epsilon_{il}\left(-\frac{K}{2\pi}\epsilon^{kl}D_k\Lambda_d\right) = -D_i\Lambda^a\,, \tag{D.2.22}$$

being the right result for an infinitesimal gauge transformation with parameter Λ. Therefore, it is the improved constraint (D.2.17) that indeed generates gauge transformations, and not (D.2.8). Finally, let us compute the boundary term δQ. The covariant derivative acts as an ordinary derivative for a scalar of the inner group, we obtain

$$\delta Q\,[\Lambda] = -\frac{K}{2\pi}\int d^2x'\,\epsilon^{ij}\partial_i\left(\Lambda_b\delta\mathcal{A}_j^b\right) \tag{D.2.23a}$$

$$= -\frac{K}{2\pi}\int_{\partial\mathcal{M}} d^2x'\,\epsilon^{ij}\Lambda_b\delta\mathcal{A}_j^b \tag{D.2.23b}$$

$$= -\frac{K}{2\pi}\int_{\partial\mathcal{M}} d^2x'\left[\epsilon^{r\phi}\partial_r\left(\Lambda_b\delta\mathcal{A}_\phi^b\right) + \epsilon^{\phi r}\partial_\phi\left(\Lambda_b\delta\mathcal{A}_r^b\right)\right]\,. \tag{D.2.23c}$$

With manifold orientation $\epsilon^{r\phi} = 1$, and assuming periodicity on the fields, we finally arrive to Eq. (3.2.23),

$$\delta Q\,[\Lambda] = -\frac{K}{2\pi}\int_{\partial\mathcal{M}} d\phi\left\langle\Lambda\,\delta\mathcal{A}_\phi\right\rangle. \tag{D.2.23d}$$

D.3 Virasoro asymptotic algebra

We aim to compute the asymptotic symmetry algebra associated to Brown-Henneaux boundary conditions in the Chern-Simons formulation, given in Eq. (3.3.1)

$$a_\phi^\pm = L_{\pm 1} - \frac{2\pi}{K}\mathcal{L}^\pm(t,\phi)L_{\mp 1}\,, \quad a_t^\pm = \pm\frac{1}{\ell}a_\phi^\pm\,. \tag{D.3.1}$$

By means of the procedure developed in Section 3.3, the charge is given by Eq. (3.3.14)

$$Q^\pm\left[\mu^\pm\right] = \mp\int_{\partial\mathcal{M}} d\phi\;\mu^\pm\mathcal{L}^\pm\,. \tag{D.3.2}$$

Then, with $\mu^\pm$ as a Fourier mode, we can identify the charge generator as the following Fourier transform,

$$Q^\pm\left[\mu^\pm = e^{-im\phi}\right] = \mp\int d\phi\; e^{-im\phi}\mathcal{L}^\pm \equiv \mp\mathcal{L}_m^\pm\,, \tag{D.3.3}$$

where

$$\mathcal{L}_m^\pm = \int d\phi\; e^{-im\phi}\mathcal{L}^\pm\,. \tag{D.3.4}$$

According to Ref. [47], the algebra of charges can be computed as following

$$\left\{Q^\pm\left[\mu_1^\pm\right], Q^\pm\left[\mu_2^\pm\right]\right\} = \delta_{\mu_2^\pm}Q^\pm\left[\mu_1^\pm\right] \equiv \mp\delta_{\mu_2^\pm}\int d\phi\;\mu_1^\pm\mathcal{L}^\pm = \mp\int d\phi\;\mu_1^\pm\delta_{\mu_2^\pm}\mathcal{L}^\pm\,, \tag{D.3.5}$$

where $\delta_{\mu_2^\pm}\mathcal{L}^\pm = \pm\mathcal{D}^\pm\mu_2^\pm$, in line with (3.3.8), with operator $\mathcal{D}^\pm$ defined as

$$\mathcal{D}^\pm := \mathcal{L}^{\pm\prime} + 2\mathcal{L}^\pm\partial_\phi - \frac{K}{4\pi}\partial_\phi^3\,.$$

By means of (D.3.5), the asymptotic symmetry algebra of Fourier modes $\mathcal{L}_m^\pm$ is

$$i\{\mathcal{L}_m^\pm, \mathcal{L}_n^\pm\} = \mp i\int d\phi\; e^{-im\phi}\left(\pm\mathcal{D}^\pm e^{-in\phi}\right) \tag{D.3.6a}$$

$$= -i\int d\phi\; e^{-im\phi}\left(\mathcal{L}^{\pm\prime} + 2\mathcal{L}^\pm(-in) - \frac{K}{4\pi}(-in)^3\right)e^{-in\phi} \tag{D.3.6b}$$

$$= -i\int d\phi\;\left(-\mathcal{L}^\pm(-i)(m+n) - 2in\mathcal{L}^\pm - \frac{iK}{4\pi}n^3\right)e^{-i(m+n)} \tag{D.3.6c}$$

$$= -i\int d\phi\;\left(i(m-n)\mathcal{L}^\pm - \frac{iK}{4\pi}n^3\right)e^{-i(m+n)} \tag{D.3.6d}$$

$$= (m-n)\int d\phi\;\mathcal{L}^\pm e^{-i(m+n)} - \frac{K}{4\pi}n^3\cdot 2\pi\delta_{m+n,0} \tag{D.3.6e}$$

$$= (m-n)\mathcal{L}_{m+n}^\pm + \frac{K}{2}m^3\delta_{m+n,0}\,, \tag{D.3.6f}$$

with central charge $c = 6K = 3\ell/2G$. Thus, two independent copies of the 2D Virasoro algebra were obtained.

Appendix E

Appendices of Chapter 4

E.1 Conservation of gauge transformations along the temporal component

In this appendix, we show that gauge transformations (4.2.1)

$$\delta a = d\Lambda + [a, \Lambda] , \tag{E.1.1}$$

$$\bar{\delta} a = d\bar{\Lambda} + \left[a, \bar{\Lambda}\right] , \tag{E.1.2}$$

along the temporal component of AKNS boundary conditions reduces to combinations of the equations of motion.

From the canonical point of view, an infinitesimal gauge transformation with parameter Λ and $\bar{\Lambda}$, respectively, are given by [47]

$$\delta a_\phi = \{a_\phi, Q[\Lambda]\} , \tag{E.1.3}$$

$$\bar{\delta} a_\phi = \left\{a_\phi, Q\left[\bar{\Lambda}\right]\right\} . \tag{E.1.4}$$

Thus, the action of two infinitesimal gauge transformations reads

$$\delta\bar{\delta} a_\phi = \left\{\bar{\delta} a_\phi, Q[\Lambda]\right\} = \left\{\left\{a_\phi, Q\left[\bar{\Lambda}\right]\right\}, Q\right\} , \tag{E.1.5}$$

$$\bar{\delta}\delta a_\phi = \left\{\delta a_\phi, Q\left[\bar{\Lambda}\right]\right\} = \left\{\{a_\phi, Q[\Lambda]\}, Q\left[\bar{\Lambda}\right]\right\} . \tag{E.1.6}$$

Hence, the commutator reads

$$\left[\delta, \bar{\delta}\right] a_\phi = \left\{\left\{a_\phi, Q\left[\bar{\Lambda}\right]\right\}, Q\left[\Lambda\right]\right\} - \left\{\left\{a_\phi, Q\left[\Lambda\right]\right\}, Q\left[\bar{\Lambda}\right]\right\} . \tag{E.1.7}$$

Recalling the Jacobi identity

$$\big\{\big\{a_\phi, Q\left[\bar{\Lambda}\right]\big\}, Q\left[\Lambda\right]\big\} + \big\{\{Q\left[\Lambda\right], a_\phi\}, Q\left[\bar{\Lambda}\right]\big\} + \big\{\big\{Q\left[\bar{\Lambda}\right], Q\left[\Lambda\right]\big\}, a_\phi\big\} = 0\,,$$

we can readily write $\big\{\big\{a_\phi, Q\left[\bar{\Lambda}\right]\big\}, Q\left[\Lambda\right]\big\} - \big\{\{a_\phi, Q\left[\Lambda\right]\}, Q\left[\bar{\Lambda}\right]\big\} = \big\{\big\{Q\left[\bar{\Lambda}\right], Q\left[\Lambda\right]\big\}, a_\phi\big\}$. Therefore, the commutator vanishes

$$\big[\delta, \bar{\delta}\big] a_\phi = \big\{\big\{Q\left[\bar{\Lambda}\right], Q\left[\Lambda\right]\big\}, a_\phi\big\} = 0\,, \tag{E.1.8}$$

since charges are in involution, according to Eq. (4.2.16). On the other hand, the application of two successive infinitesimal gauge transformations reads

$$\delta\bar{\delta} a_\phi = \delta\left(\partial_\phi \bar{\Lambda} + \big[a_\phi, \bar{\Lambda}\big]\right) \tag{E.1.9a}$$

$$= \partial_\phi\left(\partial_\phi \bar{\Lambda} + \big[a_\phi, \bar{\Lambda}\big]\right) + \big[\partial_\phi \bar{\Lambda} + \big[a_\phi, \bar{\Lambda}\big], \Lambda\big] \tag{E.1.9b}$$

$$= \partial_\phi^2 \bar{\Lambda} + \partial_\phi\big[a_\phi, \bar{\Lambda}\big] + \big[\partial_\phi \bar{\Lambda}, \Lambda\big] + \big[\big[a_\phi, \bar{\Lambda}\big], \Lambda\big]\,, \tag{E.1.9c}$$

while

$$\bar{\delta}\delta a_\phi = \bar{\delta}\left(\partial_\phi \Lambda + [a_\phi, \Lambda]\right) \tag{E.1.10a}$$

$$= \partial_\phi\left(\partial_\phi \Lambda + [a_\phi, \Lambda]\right) + \big[\partial_\phi \Lambda + [a_\phi, \Lambda], \bar{\Lambda}\big] \tag{E.1.10b}$$

$$= \partial_\phi^2 \Lambda + \partial_\phi[a_\phi, \Lambda] + \big[\partial_\phi \Lambda, \bar{\Lambda}\big] + \big[[a_\phi, \Lambda], \bar{\Lambda}\big]\,. \tag{E.1.10c}$$

Therefore, the commutator is

$$\begin{aligned}\big[\delta, \bar{\delta}\big] a_\phi = \partial_\phi^2 \bar{\Lambda} + \partial_\phi\big[a_\phi, \bar{\Lambda}\big] + \big[\partial_\phi \bar{\Lambda}, \Lambda\big] + \big[\big[a_\phi, \bar{\Lambda}\big], \Lambda\big]& \\ - \partial_\phi^2 \Lambda - \partial_\phi[a_\phi, \Lambda] - \big[\partial_\phi \Lambda, \bar{\Lambda}\big] - \big[[a_\phi, \Lambda], \bar{\Lambda}\big]&\end{aligned} \tag{E.1.11a}$$

$$= \partial_\phi^2\left(\bar{\Lambda} - \Lambda\right) + \partial_\phi\big[\bar{\Lambda}, \Lambda\big] + \partial_\phi\big[a_\phi, \bar{\Lambda} - \Lambda\big] + \big[\big[a_\phi, \bar{\Lambda}\big], \Lambda\big] - \big[[a_\phi, \Lambda], \bar{\Lambda}\big]\,. \tag{E.1.11b}$$

Recalling the Jacobi identity

$$\big[\big[a_\phi, \bar{\Lambda}\big], \Lambda\big] + \big[[\Lambda, a_\phi], \bar{\Lambda}\big] + \big[\big[\bar{\Lambda}, \Lambda\big], a_\phi\big] = 0\,,$$

we are able to write the last two terms of Eq. (E.1.11b) as

$$\big[\big[a_\phi, \bar{\Lambda}\big], \Lambda\big] - \big[[a_\phi, \Lambda], \bar{\Lambda}\big] = \big[a_\phi, \big[\bar{\Lambda}, \Lambda\big]\big]\,.$$

Then, (E.1.11b) is

$$\left[\delta,\bar{\delta}\right]a_\phi = \partial_\phi^2\left(\bar{\Lambda}-\Lambda\right)+\partial_\phi\left[\bar{\Lambda},\Lambda\right]+\partial_\phi\left[a_\phi,\bar{\Lambda}-\Lambda\right]+\left[a_\phi,\left[\bar{\Lambda},\Lambda\right]\right] \tag{E.1.11c}$$

$$= \partial_\phi\left\{\partial_\phi\left(\bar{\Lambda}-\Lambda\right)+\left[\bar{\Lambda},\Lambda\right]\right\}+\left[a_\phi,\partial_\phi\left(\bar{\Lambda}-\Lambda\right)+\left[\bar{\Lambda},\Lambda\right]\right]. \tag{E.1.11d}$$

Let

$$\bar{\bar{\Lambda}} := \delta\bar{\Lambda}-\bar{\delta}\Lambda+\left[\Lambda,\bar{\Lambda}\right] \tag{E.1.12a}$$

$$= \partial_\phi\bar{\Lambda}+\left[\bar{\Lambda},\Lambda\right]-\partial_\phi\Lambda-\left[\Lambda,\bar{\Lambda}\right]+\left[\Lambda,\bar{\Lambda}\right] \tag{E.1.12b}$$

$$= \partial_\phi\left(\bar{\Lambda}-\Lambda\right)+\left[\bar{\Lambda},\Lambda\right]. \tag{E.1.12c}$$

Therefore, the commutator of gauge transformations (E.1.11d) closes as following

$$\left[\delta,\bar{\delta}\right]a_\phi = \partial_\phi\bar{\bar{\Lambda}}+\left[a_\phi,\bar{\bar{\Lambda}}\right]\equiv\bar{\bar{\delta}}a_\phi\,. \tag{E.1.13}$$

Now we can equate the latter with the canonical result (E.1.8), which allows to find the particular solution $\bar{\bar{\Lambda}}=0$, or

$$\bar{\delta}\Lambda = \delta\bar{\Lambda}+\left[\Lambda,\bar{\Lambda}\right]. \tag{E.1.14}$$

However, as it was said below Eq. (4.2.2), we recognized $\partial_t \leftrightarrow \delta$ and $a_t^\pm \leftrightarrow \Lambda^\pm$ [197]. Thus, the previous condition is analogue to

$$\bar{\delta}a_t = \partial_t\bar{\Lambda}+\left[a_t,\bar{\Lambda}\right]. \tag{E.1.15}$$

As we see, we obtained gauge transformations along temporal components from the angular one, provided the involution of conserved charges, inner feature of integrable systems. Thus, we do not obtain further conditions for the gauge parameter Λ.

E.2 Recovering specific boundary conditions

Here we explicitly show how the AKNS boundary conditions (4.0.1)

$$a_\phi^\pm = \mp 2\xi^\pm L_0 - p^\pm L_{\pm 1} + r^\pm L_{\mp 1}\,, \tag{E.2.1a}$$

$$a_t^\pm = \frac{1}{\ell}(-2A^\pm L_0 \pm B^\pm L_{\pm 1} \mp C^\pm L_{\mp 1})\,, \tag{E.2.1b}$$

encodes a family of boundary conditions as coefficients of powers of the spectral parameter $\xi^\pm$. Let us focus on the temporal component, written as

$$a_{t,N}^\pm = \sum_{n=0}^{N} a_{t,n}^\pm \xi^{\pm\, N-n}\,, \tag{E.2.2}$$

where

$$a_{t,n}^\pm = \frac{1}{\ell}\left(-2A_n^\pm L_0 \pm B_n^\pm L_{\pm 1} \mp C_n^\pm L_{\mp 1}\right)\,. \tag{E.2.3}$$

The coefficients $A_n^\pm$, $B_n^\pm$ and $C_n^\pm$ are given by expressions (C.2.2), (C.2.3) and (C.2.4), respectively. The first six coefficients of the latter expansion are

$$a_{t,n=0}^\pm = -\frac{2}{\ell} L_0\,, \tag{E.2.4a}$$

$$a_{t,n=1}^\pm = \pm\frac{1}{\ell}\left(-p^\pm L_{\pm 1} + r^\pm L_{\mp 1}\right)\,, \tag{E.2.4b}$$

$$a_{t,n=2}^\pm = \frac{1}{\ell}\left(p^\pm r^\pm L_0 \pm \frac{1}{2} p^{\pm\prime} L_{\pm 1} \pm \frac{1}{2} r^{\pm\prime} L_{\mp 1}\right)\,, \tag{E.2.4c}$$

$$a_{t,n=3}^\pm = \frac{1}{\ell}\Bigg[-\frac{1}{2}\left(p^{\pm\prime} r^\pm - p^\pm r^{\pm\prime}\right) L_0 \pm \frac{1}{4}\left(2p^{\pm\,2} r^\pm - p^{\pm\prime\prime}\right) L_{\pm 1} \mp \frac{1}{4}\left(2p^\pm r^{\pm\,2} - r^{\pm\prime\prime}\right) L_{\mp 1}\Bigg]\,, \tag{E.2.4d}$$

$$a_{t,n=4}^\pm = \frac{1}{\ell}\Bigg[-\frac{1}{4}\left(3p^{\pm\,2} r^{\pm\,2} + p^{\pm\prime} r^{\pm\prime} - p^{\pm\prime\prime} r^\pm - p^\pm r^{\pm\prime\prime}\right) L_0 \pm \frac{1}{8}\left(-6p^\pm p^{\pm\prime} r^\pm + p^{\pm\prime\prime\prime}\right) L_{\pm 1} \mp \frac{1}{8}\left(6p^\pm r^\pm r^{\pm\prime} - r^{\pm\prime\prime\prime}\right) L_{\mp 1}\Bigg]\,, \tag{E.2.4e}$$

$$\begin{aligned} a_{t,n=5}^\pm = \frac{1}{\ell}\Bigg\{ &-\frac{1}{8}\big[6p^{\pm\,2} r^\pm r^{\pm\prime} - p^{\pm\prime\prime} r^{\pm\prime} + p^{\pm\prime} r^{\pm\prime\prime} + p^{\pm\prime\prime\prime} r^\pm - p^\pm\left(6p^{\pm\prime} r^{\pm\,2} - r^{\pm\prime\prime\prime}\right)\big] L_0 \\ &\pm \frac{1}{16}\big[-6p^{\pm\,3} r^{\pm\,2} + 6p^{\pm\prime\,2} r^\pm + 4p^\pm\left(p^{\pm\prime} r^{\pm\prime} + 2p^{\pm\prime\prime} r^\pm\right) + 2p^{\pm\,2} r^{\pm\prime\prime} - p^{\pm\prime\prime\prime\prime}\big] L_{\pm 1} \\ &\mp \frac{1}{16}\big[-6p^{\pm\,2} r^{\pm\,3} + 4p^{\pm\prime} r^\pm r^{\pm\prime} + 2p^{\pm\prime\prime} r^{\pm\,2} + p^\pm\left(6r^{\pm\prime\,2} + 8r^\pm r^{\pm\,2}\right) - r^{\pm\prime\prime\prime\prime}\big] L_{\mp 1}\Bigg\}\,. \end{aligned} \tag{E.2.4f}$$

Thus, the first six temporal boundary conditions $a_{t,N}^{\pm}$ reads

$$a_{t,N=0}^{\pm} = -\frac{2}{\ell} L_0 \,, \tag{E.2.5a}$$

$$a_{t,N=1}^{\pm} = \frac{1}{\ell} \left[-2L_0 \xi^{\pm} \pm \left(-p^{\pm} L_{\pm 1} + r^{\pm} L_{\mp 1} \right) \right] \,, \tag{E.2.5b}$$

$$\begin{aligned} a_{t,N=2}^{\pm} = \frac{1}{\ell} \Bigg[& -2L_0 \xi^{\pm\, 2} \pm \left(-p^{\pm} L_{\pm 1} + r^{\pm} L_{\mp 1} \right) \xi^{\pm\, 1} - \frac{1}{2} \left(p^{\pm\prime} r^{\pm} - p^{\pm} r^{\pm\prime} \right) L_0 \\ & \pm \frac{1}{4} \left(2p^{\pm\, 2} r^{\pm} - p^{\pm\prime\prime} \right) L_{\pm 1} \mp \frac{1}{4} \left(2p^{\pm} r^{\pm\, 2} - r^{\pm\prime\prime} \right) L_{\mp 1} \Bigg] \,, \end{aligned} \tag{E.2.5c}$$

$$\begin{aligned} a_{t,N=3}^{\pm} = \frac{1}{\ell} \Bigg\{ & -2L_0 \xi^{\pm\, 3} \pm \left(-p^{\pm} L_{\pm 1} + r^{\pm} L_{\mp 1} \right) \xi^{\pm\, 2} + \Bigg[-\frac{1}{2} \left(p^{\pm\prime} r^{\pm} - p^{\pm} r^{\pm\prime} \right) L_0 \\ & \pm \frac{1}{4} \left(2p^{\pm\, 2} r^{\pm} - p^{\pm\prime\prime} \right) L_{\pm 1} \mp \frac{1}{4} \left(2p^{\pm} r^{\pm\, 2} - r^{\pm\prime\prime} \right) L_{\mp 1} \Bigg] \xi^{\pm} \\ & - \frac{1}{4} \left(3p^{\pm\, 2} r^{\pm\, 2} + p^{\pm\prime} r^{\pm\prime} - p^{\pm\prime\prime} r^{\pm} - p^{\pm} r^{\pm\prime\prime} \right) L_0 \\ & \pm \frac{1}{8} \left(-6p^{\pm} p^{\pm\prime} r^{\pm} + p^{\pm\prime\prime\prime} \right) L_{\pm 1} \mp \frac{1}{8} \left(6p^{\pm} r^{\pm} r^{\pm\prime} - r^{\pm\prime\prime\prime} \right) L_{\mp 1} \Bigg\} \,, \end{aligned} \tag{E.2.5d}$$

$$\begin{aligned} a_{t,N=4}^{\pm} = \frac{1}{\ell} \Bigg\{ & -2L_0 \xi^{\pm\, 4} \pm \left(-p^{\pm} L_{\pm 1} + r^{\pm} L_{\mp 1} \right) \xi^{\pm\, 3} + \Bigg[-\frac{1}{2} \left(p^{\pm\prime} r^{\pm} - p^{\pm} r^{\pm\prime} \right) L_0 \\ & \pm \frac{1}{4} \left(2p^{\pm\, 2} r^{\pm} - p^{\pm\prime\prime} \right) L_{\pm 1} \mp \frac{1}{4} \left(2p^{\pm} r^{\pm\, 2} - r^{\pm\prime\prime} \right) L_{\mp 1} \Bigg] \xi^{\pm\, 2} \\ & + \Bigg[-\frac{1}{4} \left(3p^{\pm\, 2} r^{\pm\, 2} + p^{\pm\prime} r^{\pm\prime} - p^{\pm\prime\prime} r^{\pm} - p^{\pm} r^{\pm\prime\prime} \right) L_0 \\ & \pm \frac{1}{8} \left(-6p^{\pm} p^{\pm\prime} r^{\pm} + p^{\pm\prime\prime\prime} \right) L_{\pm 1} \mp \frac{1}{8} \left(6p^{\pm} r^{\pm} r^{\pm\prime} - r^{\pm\prime\prime\prime} \right) L_{\mp 1} \Bigg] \xi^{\pm} \\ & - \frac{1}{4} \left(3p^{\pm\, 2} r^{\pm\, 2} + p^{\pm\prime} r^{\pm\prime} - p^{\pm\prime\prime} r^{\pm} - p^{\pm} r^{\pm\prime\prime} \right) L_0 \\ & \pm \frac{1}{8} \left(-6p^{\pm} p^{\pm\prime} r^{\pm} + p^{\pm\prime\prime\prime} \right) L_{\pm 1} \mp \frac{1}{8} \left(6p^{\pm} r^{\pm} r^{\pm\prime} - r^{\pm\prime\prime\prime} \right) L_{\mp 1} \Bigg\} \,, \end{aligned} \tag{E.2.5e}$$

$$
\begin{aligned}
a^{\pm}_{t,N=5} = \frac{1}{\ell}\Bigg\{ & -2L_0\xi^{\pm\,5} \pm \left(-p^{\pm}L_{\pm 1} + r^{\pm}L_{\mp 1}\right)\xi^{\pm\,4} + \Bigg[-\frac{1}{2}\left(p^{\pm\prime}r^{\pm} - p^{\pm}r^{\pm\prime}\right)L_0 \\
& \pm\frac{1}{4}\left(2p^{\pm\,2}r^{\pm} - p^{\pm\prime\prime}\right)L_{\pm 1} \mp \frac{1}{4}\left(2p^{\pm}r^{\pm\,2} - r^{\pm\prime\prime}\right)L_{\mp 1}\Bigg]\xi^{\pm\,3} \\
& + \Bigg[-\frac{1}{4}\left(3p^{\pm\,2}r^{\pm\,2} + p^{\pm\prime}r^{\pm\prime} - p^{\pm\prime\prime}r^{\pm} - p^{\pm}r^{\pm\prime\prime}\right)L_0 \\
& \pm\frac{1}{8}\left(-6p^{\pm}p^{\pm\prime}r^{\pm} + p^{\pm\prime\prime\prime}\right)L_{\pm 1} \mp \frac{1}{8}\left(6p^{\pm}r^{\pm}r^{\pm\prime} - r^{\pm\prime\prime\prime}\right)L_{\mp 1}\Bigg]\xi^{\pm\,2} \\
& + \Bigg[-\frac{1}{4}\left(3p^{\pm\,2}r^{\pm\,2} + p^{\pm\prime}r^{\pm\prime} - p^{\pm\prime\prime}r^{\pm} - p^{\pm}r^{\pm\prime\prime}\right)L_0 \\
& \pm\frac{1}{8}\left(-6p^{\pm}p^{\pm\prime}r^{\pm} + p^{\pm\prime\prime\prime}\right)L_{\pm 1} \mp \frac{1}{8}\left(6p^{\pm}r^{\pm}r^{\pm\prime} - r^{\pm\prime\prime\prime}\right)L_{\mp 1}\Bigg]\xi^{\pm} \\
& - \frac{1}{8}\left[6p^{\pm\,2}r^{\pm}r^{\pm\prime} - p^{\pm\prime\prime}r^{\pm\prime} + p^{\pm\prime}r^{\pm\prime\prime} + p^{\pm\prime\prime\prime}r^{\pm} - p^{\pm}\left(6p^{\pm\prime}r^{\pm\,2} - r^{\pm\prime\prime\prime}\right)\right]L_0 \\
& \pm\frac{1}{16}\left[-6p^{\pm\,3}r^{\pm\,2} + 6p^{\pm\prime\,2}r^{\pm} + 4p^{\pm}\left(p^{\pm\prime}r^{\pm\prime} + 2p^{\pm\prime\prime}r^{\pm}\right) + 2p^{\pm\,2}r^{\pm\prime\prime} - p^{\pm\prime\prime\prime\prime}\right]L_{\pm 1} \\
& \mp\frac{1}{16}\left[-6p^{\pm\,2}r^{\pm\,3} + 4p^{\pm\prime}r^{\pm}r^{\pm\prime} + 2p^{\pm\prime\prime}r^{\pm\,2} + p^{\pm}\left(6r^{\pm\prime\,2} + 8r^{\pm}r^{\pm\,2}\right) - r^{\pm\prime\prime\prime\prime}\right]L_{\mp 1}\Bigg\}.
\end{aligned}
\tag{E.2.5f}
$$

Appendix F

Appendices of Chapter 5

F.1 Kubo formula deduction

The explicit deduction of the Kubo formula [334, 335] is presented in this appendix.

Let us start from (5.2.12)

$$U(t,t_0) = \mathcal{T}\exp\left(-i\int_{t_0}^{t} V(t',t_0)dt'\right)\,, \tag{F.1.1}$$

where $\mathcal{T}$ stands for the time-ordering operator. We can Taylor expand the latter as

$$U(t,t_0) \approx 1 - i\int_{t_0}^{t} dt'\,V(t',t_0) + (-i)^2\int_{t_0}^{t} dt'\int_{t_0}^{t_1} dt'' V(t')V(t'') + \dots\,, \tag{F.1.2}$$

where the ellipsis stands for higher-order terms in the expansion. In the interaction picture, as it was obtained in (5.2.7), every operator can be decomposed in the following form

$$\mathcal{O}(t,x) \equiv U(t,t_0)\mathcal{O}_I(t,x)U^\dagger(t,t_0)\,. \tag{F.1.3}$$

Thus, the expectation value of an arbitrary operator is

$$\langle\mathcal{O}(t,x)\rangle = \mathrm{Tr}\left[\rho(t,x)\mathcal{O}(t,x)\right] \tag{F.1.4a}$$

$$= \mathrm{Tr}\left[\rho_I(t,x)U^\dagger(t,t_0)\mathcal{O}(t,x)U(t,t_0)\right]\,, \tag{F.1.4b}$$

where the cyclic property of the trace was used in the last line. Keeping only first-order terms in the forthcoming expansion, $\langle \mathcal{O}(t,x)\rangle$ can be approximated as

$$\begin{aligned}\langle \mathcal{O}(t,x)\rangle \approx \operatorname{Tr}\Bigg[\rho_I(t,x)\left(1+i\int_{t_0}^{t} dt'\, V(t',t_0)+\dots\right)\mathcal{O}(t,x)\times \\ \times\left(1-i\int_{t_0}^{t} dt'\, V(t',t_0)+\dots\right)\Bigg]\end{aligned} \tag{F.1.4c}$$

$$= \operatorname{Tr}\left[\rho_I(t,x)\left(\mathcal{O}(t,x)+i\int_{t_0}^{t} dt'\,[V(t',t_0),\mathcal{O}(t,x)]+\dots\right)\right] \tag{F.1.4d}$$

$$= \operatorname{Tr}[\rho_I(t,x)\mathcal{O}(t,x)] + i\operatorname{Tr}\left[\rho_I(t,x)\int_{t_0}^{t} dt'\,[V(t',t_0),\mathcal{O}(t,x)]\right]. \tag{F.1.4e}$$

Defining

$$\langle \mathcal{O}(t,x)\rangle\big|_{V=0} := \operatorname{Tr}[\rho_I(t,x)\mathcal{O}(t,x)]\,, \tag{F.1.5}$$

and according to definition (5.1.2), $\delta\langle O(t,x)\rangle := \langle \mathcal{O}(t,x)\rangle - \langle \mathcal{O}(t,x)\rangle\big|_{V=0}$, we arrive to

$$\delta\langle O(t,x)\rangle = i\int_{t_0}^{t} dt'\ \langle [V_I(t',t_0),\mathcal{O}_I(t,x)]\rangle\ . \tag{F.1.6}$$

Placing the initial state in the far past $t_0\to-\infty$, and plugging a Heaviside theta function $\Theta(t-t')$ to extend the range of integration, defined as

$$\Theta(t-t') = \begin{cases} 1\,, & t>t' \\ 0\,, & t\le t'\,, \end{cases} \tag{F.1.7}$$

we arrive to (5.3.3),

$$\delta\langle O(t,x)\rangle = i\int_{-\infty}^{\infty} dt'\,\Theta(t-t')\,\langle [V_I(t'),\mathcal{O}_I(t,x)]\rangle\ . \tag{F.1.8}$$

F.2 Localness of the susceptibility

Here we prove that the susceptibility is local in the frequency space ω.

We will perform a Fourier transform (FT) to Eq. (5.3.8),

$$\delta \langle \mathcal{O}(t,x)\rangle = \int dt'\, dx'\, \chi_{\mathcal{O}B}(t-t';x,x')\mu(t',x')\,. \tag{F.2.1}$$

For simplicity, consider a FT to the arbitrary operator $O(t)$

$$\mathcal{O}(\omega) = \int dt\, e^{i\omega t}\mathcal{O}(t)\,. \tag{F.2.2}$$

Hence, (5.3.8) reads

$$\delta \langle \mathcal{O}(\omega)\rangle = \int dt\, dt'\, dx' \chi_{\mathcal{O}B}(t-t';x,x')\mu(t',x')e^{i\omega t} \tag{F.2.3a}$$

$$= \int dx'\, dt'\, dt\, \left[e^{i\omega(t-t')}\chi_{\mathcal{O}B}(t-t';x,x')\right]\, e^{i\omega t'}\mu(t',x') \tag{F.2.3b}$$

$$\equiv \int dx'\, \chi_{\mathcal{O}B}(\omega;x,x')\,\mu(\omega,x')\,. \tag{F.2.3c}$$

Thus, the susceptibility is local in the frequency ω: If the perturbation acts at frequency ω, it will responds at the same frequency.

Appendix G

Appendices of Chapter 6

G.1 $u(1)$-current symmetry

In this appendix we show that the action principle (6.0.1)

$$I^{\pm}\left[\varphi_{\pm}\right]=\frac{K}{8\pi}\int dt\,dx\,\left[\dot{\varphi}_{\pm}\varphi_{\pm}^{\prime}\mp v\left(\partial_{x}^{\frac{z+1}{2}}\varphi_{\pm}\right)^{2}\right]\,, \tag{G.1.1}$$

(where dots stands for temporal derivatives and primes for spatial derivatives) exhibits $u(1)$ symmetry,

$$\delta\varphi_{\pm}=\delta\eta_{k}^{\pm}\,,\quad \eta_{k}^{\pm}=e^{i(kx\pm\omega_{k}t)}\,, \tag{G.1.2}$$

where $\omega_k = vk^z$; and determine its corresponding Noether charge.

Under arbitrary infinitesimal transformations, the action transforms as

$$\delta I^{\pm}=\frac{K}{8\pi}\int dt\,dx\,\left[\delta\dot{\varphi}_{\pm}\varphi_{\pm}^{\prime}+\dot{\varphi}_{\pm}\delta\varphi_{\pm}^{\prime}\mp 2v\left(\partial_{x}^{\frac{z+1}{2}}\varphi_{\pm}\right)\partial_{x}^{\frac{z+1}{2}}\delta\varphi_{\pm}\right]\,. \tag{G.1.3}$$

Replacing $\delta\varphi_{\pm}=\eta_{k}^{\pm}$, we have

$$\delta I^{\pm}=\frac{K}{8\pi}\int dt\,dx\,\left[\dot{\eta}_{k}^{\pm}\varphi_{\pm}^{\prime}+\dot{\varphi}_{\pm}\eta_{k}^{\pm\prime}\mp 2v\left(\partial_{x}^{\frac{z+1}{2}}\varphi_{\pm}\right)\partial_{x}^{\frac{z+1}{2}}\eta_{k}^{\pm}\right] \tag{G.1.4a}$$

$$=\frac{K}{8\pi}\int dt\,dx\,\left[-2\dot{\eta}_{k}^{\pm\prime}\varphi_{\pm}\mp 2v\left(-1\right)^{\frac{z+1}{2}}\varphi_{\pm}\partial_{x}^{z+1}\eta_{k}^{\pm}+\partial_{t}\left(\eta_{k}^{\pm\prime}\varphi_{\pm}\right)\right]\,, \tag{G.1.4b}$$

where we integrated by parts and assumed periodic boundary conditions on the fields (thus eliminating spatial boundary terms).

We can rewrite the above equation as

$$\delta I^{\pm} = -\frac{K}{4\pi}\int dt\, dx\, \varphi_{\pm}\partial_x \left[\partial_t \pm v(-1)^{\frac{z\pm1}{2}}\partial_x^z\right]\eta_k^{\pm} + \frac{K}{8\pi}\int dt\, dx\, \partial_t\left(\varphi_{\pm}\eta_k^{\pm}\right) . \tag{G.1.4c}$$

If $\eta_k^{\pm}$ satisfies the equations of motion,

$$\left[\partial_t \pm v(-1)^{\frac{z\pm1}{2}}\partial_x^z\right]\eta_k^{\pm} = 0\,,$$

then the form of $\eta_k^{\pm}$ is determined as

$$\eta_k^{\pm} = e^{i(kx\pm\omega_k t)}\,. \tag{G.1.5}$$

Hence, the variation of the action is

$$\delta I^{\pm} = \frac{K}{8\pi}\int dt\, dx\, \partial_t\left(\varphi_{\pm}\eta_k^{\pm\prime}\right) . \tag{G.1.6}$$

On the other hand, starting from (G.1.3), if we integrate by parts and cancel the terms that contribute to the equation of motion, then we have that

$$\delta I^{\pm} = \frac{K}{8\pi}\int dt\, dx\, \partial_t\left(\delta\varphi_{\pm}\varphi_{\pm}'\right) . \tag{G.1.7a}$$

If we again integrate by parts, and substitute the corresponding infinitesimal transformation $\delta\varphi_{\pm} = \eta_k^{\pm}$, we obtain

$$\delta I^{\pm} = -\frac{K}{8\pi}\int dt\, dx\, \partial_t\left(\varphi_{\pm}\eta_k^{\pm\prime}\right) \tag{G.1.7b}$$

Equating the last equation with (G.1.6), we arrive to the following Noether charge

$$J_{\pm}[\eta_k^{\pm}] = \int dx\, \eta_k^{\pm}\mathcal{J}_{\pm}\,, \tag{G.1.8}$$

where

$$\mathcal{J}_{\pm} = \frac{K}{4\pi}\partial_x\varphi_{\pm}\,. \tag{G.1.9}$$

Additionally, we see that $\mathcal{J}_{\pm}$ satisfies the continuity equation

$$\partial_t\mathcal{J}_{\pm} + \partial_x\mathcal{I}_{\pm} = 0\,, \tag{G.1.10}$$

so its conjugate quantity $\mathcal{I}_\pm$ turns out to be

$$\mathcal{I}_\pm = -\frac{K}{4\pi}\partial_t\varphi_\pm\,. \tag{G.1.11}$$

As showed in (7.1.36), at the boundary, the asymptotic degrees of freedom will be captured by the action principle

$$I\left[\varphi_+,\varphi_-\right] = I^+[\varphi_+] - I^-\left[\varphi_-\right]\,, \tag{G.1.12}$$

so the negative copy will have a global negative sign. This will affect the symmetries, so in this case, the Noether charge and its conjugate quantity will be readily

$$\mathcal{J}_\pm = \pm\frac{K}{4\pi}\partial_x\varphi_\pm\,,\qquad \mathcal{I}_\pm = \mp\frac{K}{4\pi}\partial_t\varphi_\pm\,. \tag{G.1.13}$$

G.2 Jackiw quantization of anisotropic chiral bosons

Following [342, 402] it is possible to read the Dirac bracket of the theory by analizying the theory in its symplectic form.

Define the generalized set of coordinates $z^a = (q^i, p_j)^{\mathrm{t}}$ with $a = 1, 2, \ldots, 2N$. In this formalism, the action of every theory is

$$\int dt \, \left(l_a \dot{z}^a - H \right) , \tag{G.2.1}$$

with $l_a = (p_i, 0)$. An infinitesimal variation of the action reads

$$\int dt \, \left[\left(\partial_a l_b - \partial_b l_a \right) \dot{z}^b - \partial_a H \right] \delta z^a \, . \tag{G.2.2}$$

If we define the 2-form symplectic matrix

$$\omega_{ab} = \partial_a l_b - \partial_b l_a \, , \tag{G.2.3}$$

the equations of motion can be written as

$$\dot{z}^a = \omega^{ab} \partial_b H \, . \tag{G.2.4}$$

These equations can be obtained from the Poisson bracket

$$\{ f, g \} = \partial_a f \omega^{ab} \partial_b g \, , \tag{G.2.5}$$

where $\left\{ z^a, z^b \right\} = \omega^{ab}$. Now we are going to apply the symplectic formalism to the anisotropic chiral boson in order to obtain the Dirac bracket.

The Lagrangian density of the anisotropic chiral boson theory is[1]

$$\mathcal{L}^{\pm} = \pm \frac{K}{8\pi} \left[\dot{\varphi}_{\pm} \varphi'_{\pm} \mp v \left(\partial_{\phi}^{\frac{z+1}{2}} \varphi_{\pm} \right)^2 \right] , \tag{G.2.6}$$

so it is possible to read

$$l_{\pm}(x) = \mp \frac{K}{8\pi} \partial_x \varphi_{\pm}(x) \, . \tag{G.2.7}$$

[1]Note that we put a global $\pm$ sign in front of the Lagrangian density (G.2.6). We did this since in Chapter 7, it will be proved that with suitable boundary conditions, AdS_3 GR will be reduced to an action principle whose Lagrangian density will have the form $\mathcal{L}[\varphi_+, \varphi_-] = \mathcal{L}[\varphi_+] - \mathcal{L}[\varphi_-]$.

Therefore, the symplectic matrix $\omega_\pm$ is

$$\omega_\pm(x, x') = \mp\frac{K}{4\pi}\partial_x\delta(x - x')\,. \tag{G.2.8}$$

Because its inverse satisfies,

$$\int dy\,\omega_\pm(x, y)J_\pm(y, x') = \delta(x - x')\,. \tag{G.2.9}$$

we can readily obtain the Dirac bracket of the theory,

$$J_\pm(x, x') \equiv \{\varphi_\pm(x), \varphi_\pm(x')\} = \pm\frac{4\pi}{K}\partial_{x'}^{-1}\delta(x - x')\,. \tag{G.2.10}$$

Taking a derivative in x' to both sides, we finally obtain

$$\{\varphi_\pm(x), \partial_{x'}\varphi_\pm(x')\} = \pm\frac{4\pi}{K}\delta(x - x')\,. \tag{G.2.11}$$

With the symplectic structure obtained, the quantization of (G.2.11) proceeds with the prescription $\{,\} \to i\,[,]$.

G.3 Bosonization

In this appendix we perform the explicit computations regarding the fermionic interpretation of the operators $\mathcal{J}_\pm$ and $\mathcal{I}_\pm$.

Define the operators [361]

$$c_\pm(x) =: e^{-i\sqrt{\frac{K}{2}}\varphi_\pm(x)} :, \quad c^\dagger_\pm(x) =: e^{i\sqrt{\frac{K}{2}}\varphi_\pm(x)} :, \tag{G.3.1}$$

where $:\mathcal{O}:$ denotes the normal-ordering of the operator $\mathcal{O}$.

We calculated the bosonic equal-time Dirac bracket in Eq. (G.2.10). Performing the passage to quantum mechanics, the Dirac bracket transforms to a commutator, and the functions $\varphi_\pm$ can be interpreted as anisotropic chiral boson operators that satisfies

$$[\varphi_\pm(x), \varphi_\pm(x')] = \mp\frac{4i\pi}{K}\partial_x^{-1}\delta(x-x') \,. \tag{G.3.2}$$

Recalling that $\partial_x \mathrm{sign}(x-x') = 2\delta(x-x')$, where

$$\mathrm{sign}(x-x') = \begin{cases} -1\,, & x-x'<0\,,\\ 0\,, & x-x'=0\,,\\ 1\,, & x-x'>0\,, \end{cases} \tag{G.3.3}$$

the previous commutator can be read as

$$[\varphi_\pm(x), \varphi_\pm(x')] = \mp\frac{2i\pi}{K}\mathrm{sign}(x-x') \,. \tag{G.3.4}$$

G.3.1 Bosonic creation and annihilation commutation algebra

Because anisotropic chiral excitations can be decomposed in positive and negative frequency modes, we can write its excitations as a superposition of creation and annihilation operators, as following

$$\varphi_\pm(x) = \theta_\pm(x) + \theta^\dagger_\pm(x) \,. \tag{G.3.5}$$

It is important to say that for the chiral sector, $\theta_+(x)$ correspond to a creation operator and $\theta^\dagger_+(x)$ to an annihilation one, while for the antichiral sector, $\theta_-(x)$ to a annihilation mode and $\theta^\dagger_-(x)$ to a creation one. The antichiral sector obeys the usual convention, while the chiral one an inverted one (see Table G.1)

We aim to compute the algebra of the creation and annihilation modes. In order to do that,

	Chiral sector	Antichiral sector
Creation operator	$\theta_+(x)$	$\theta_-^\dagger(x)$
Annihilation operator	$\theta_+^\dagger(x)$	$\theta_-(x)$

Table G.1: Bosonic creation and annihilation modes of each sector.

consider the next useful representation of the Dirac delta

$$2i\pi\delta(x-x') = \frac{1}{x-x'-i\eta} + \frac{1}{x'-x-i\eta}\,, \tag{G.3.6}$$

where $\eta \to 0^+$ is a positive regulator. Now, let us calculate the commutation algebra $[\partial_x\varphi_\pm(x), \varphi_\pm(x')]$ by replacing the creations and annihilation operators,

$$[\partial_x\varphi_\pm(x), \varphi_\pm(x')] = \left[\partial_x\theta_\pm(x), \theta_\pm^\dagger(x')\right] + \left[\partial_x\theta_\pm^\dagger(x), \theta_\pm(x')\right], \tag{G.3.7}$$

where we assummed that

$$[\theta_\pm(x), \theta_\pm(x')] = 0\,, \quad \left[\theta_\pm^\dagger(x), \theta_\pm^\dagger(x')\right] = 0\,. \tag{G.3.8}$$

Recalling that the commutator fulfills (G.2.11), and using the Delta representation (G.3.6), we can obtain from (G.3.7) the following relationship

$$\left[\partial_x\theta_\pm(x), \theta_\pm^\dagger(x')\right] + \left[\partial_x\theta_\pm^\dagger(x), \theta_\pm(x')\right] = \mp\frac{2}{K}\left(\frac{1}{x-x'-i\eta} + \frac{1}{x'-x-i\eta}\right). \tag{G.3.9}$$

Equating left and right-handed terms, we arrive to the differential equations

$$\left[\partial_x\theta_\pm(x), \theta_\pm^\dagger(x')\right] = \mp\frac{2}{K}\left(\frac{1}{x'-x-i\eta}\right), \tag{G.3.10a}$$

$$\left[\partial_x\theta_\pm^\dagger(x), \theta_\pm(x')\right] = \mp\frac{2}{K}\left(\frac{1}{x-x'-i\eta}\right). \tag{G.3.10b}$$

In order to obtain the creation and annihilation algebra, we can integrate the first commutation relation with respect to x, which leads to

$$\left[\theta \pm (x), \theta_\pm^\dagger(x')\right] = \pm\frac{2}{K}\log\left(x'-x-i\eta\right) + c_1\,,$$

where c_1 is an integration constant. If we integrate the second one, we arrive to

$$\left[\theta_\pm^\dagger(x), \theta_\pm(x')\right] = \mp\frac{2}{K}\log\left(x-x'-i\eta\right) + c_2\,.$$

If we flip the commutator and relabel $x \to x'$, we can recognize $c_1 = c_2$. A consistent value for the integration constant is given by $c_1 = \pm\frac{2}{K}\log(2i\pi)$, so the algebra of creation and annihilation operators reads

$$\left[\theta_\pm(x), \theta_\pm^\dagger(x')\right] = \pm\frac{2}{K}\log\left[-2i\pi\left(x - x' + i\eta\right)\right] . \tag{G.3.11}$$

We obtained then the bosonic algebra of creation and annihilation operators.

G.3.2 Majorana fermions

Here we show that the bosonic anisotropic particles possess a non-abelian statistics that allows to interpret them as Majorana fermiones

Fermionic creation and annihilation operators

We aim to compute the algebra of operators $c_\pm$ and $c_\pm^\dagger$, given in (G.3.1). We are going to prove that they can be interpreted as fermionic creation and annihilation operators.

Let us compute the anticommutator $\{c_\pm(x), c_\pm(x')\}$. Consider the combination

$$c_\pm(x)c_\pm(x') =: e^{-i\sqrt{\frac{K}{2}}\varphi_\pm(x)} :: e^{-i\sqrt{\frac{K}{2}}\varphi_\pm(x')} : . \tag{G.3.12}$$

In particular, let us focus on the negative copy first.

Recalling the creation and annihilation expansion (G.3.5), we make sense of the normal ordering as following

$$c_-(x)c_-(x') = e^{-i\sqrt{\frac{K}{2}}\theta_-^\dagger(x)} e^{-i\sqrt{\frac{K}{2}}\theta_-(x)} e^{-i\sqrt{\frac{K}{2}}\theta_-^\dagger(x')} e^{-i\sqrt{\frac{K}{2}}\theta_-(x')} . \tag{G.3.13}$$

Using the Baker–Campbell–Haussdorff identity $e^A e^B = e^{[A,B]} e^B e^A$, we have

$$\begin{aligned} c_-(x)c_-(x') &= e^{-i\sqrt{\frac{K}{2}}\theta_-^\dagger(x)} \left(e^{-\frac{K}{2}\left[\theta_-(x), \theta_-^\dagger(x')\right]} e^{-i\sqrt{\frac{K}{2}}\theta_-^\dagger(x')} e^{-i\sqrt{\frac{K}{2}}\theta_-(x)}\right) e^{-i\sqrt{\frac{K}{2}}\theta_-(x')} && \text{(G.3.14a)} \\ &= e^{-i\sqrt{\frac{K}{2}}\theta_-^\dagger(x)} \left(e^{\log\left[-2i\pi\left(x - x' + i\eta\right)\right]} e^{-i\sqrt{\frac{K}{2}}\theta_-^\dagger(x')} e^{-i\sqrt{\frac{K}{2}}\theta_-(x)}\right) e^{-i\sqrt{\frac{K}{2}}\theta_-(x')} && \text{(G.3.14b)} \\ &= \left[-2i\pi\left(x - x' + i\eta\right)\right] e^{-i\sqrt{\frac{K}{2}}\left(\theta_-^\dagger(x) + \theta_-^\dagger(x')\right)} e^{-i\sqrt{\frac{K}{2}}\left(\theta_-(x) + \theta_-(x')\right)} , && \text{(G.3.14c)} \end{aligned}$$

where we used, from the first to the second line, the commutation relation (G.3.11). On the other hand, if we relabel $x \to x'$, we can write $c_-(x')c_-(x)$, yielding

$$c_-(x')c_-(x) = \left[2i\pi\left(x - x'\right)\right] e^{-i\sqrt{\frac{K}{2}}\left(\theta_-^\dagger(x) + \theta_-^\dagger(x')\right)} e^{-i\sqrt{\frac{K}{2}}\left(\theta_-(x) + \theta_-(x')\right)} , \tag{G.3.15}$$

where the limit $\eta \to 0^+$ was taken. After adding the last two terms, we directly obtain $\{c_-(x), c_-(x')\} = 0$.

Let us perform now the analogue procedure for the chiral sector,

$$c_+(x)c_+(x') =: e^{-i\sqrt{\frac{K}{2}}\varphi_+(x)} :: e^{i\sqrt{\frac{K}{2}}\varphi_+(x')} : \tag{G.3.16a}$$

$$= e^{-i\sqrt{\frac{K}{2}}\theta_+(x)}e^{-i\sqrt{\frac{K}{2}}\theta_+^\dagger(x)}e^{-i\sqrt{\frac{K}{2}}\theta_+(x')}e^{-i\sqrt{\frac{K}{2}}\theta_+^\dagger(x')} \tag{G.3.16b}$$

$$= e^{-i\sqrt{\frac{K}{2}}\theta_+(x)}\left(e^{-\frac{K}{2}\left[\theta_+^\dagger(x),\theta_+(x')\right]}e^{-i\sqrt{\frac{K}{2}}\theta_+(x')}e^{-i\sqrt{\frac{K}{2}}\theta_+^\dagger(x)}\right)e^{-i\sqrt{\frac{K}{2}}\theta_+^\dagger(x')} \tag{G.3.16c}$$

$$= e^{-i\sqrt{\frac{K}{2}}\theta_+(x)}\left(e^{\log\left[-2i\pi\left(x-x'+i\eta\right)\right]}e^{-i\sqrt{\frac{K}{2}}\theta_+(x')}e^{-i\sqrt{\frac{K}{2}}\theta_+^\dagger(x)}\right)e^{-i\sqrt{\frac{K}{2}}\theta_+^\dagger(x')} \tag{G.3.16d}$$

$$= \left[-2i\pi\left(x-x'\right)\right]e^{-i\sqrt{\frac{K}{2}}\left(\theta_+(x)+\theta_+(x')\right)}e^{-i\sqrt{\frac{K}{2}}\left(\theta_+^\dagger(x)+\theta_+^\dagger(x')\right)}\,. \tag{G.3.16e}$$

Again, $c_+(x')c_+(x)$ can be calculated from the latter if $x \to x'$. So $\{c_+(x), c_+(x')\} = 0$, and therefore the result holds for the two copies. A direct consequence is that $\left\{c_\pm^\dagger(x), c_\pm^\dagger(x')\right\} = 0$ fulfills.

We only need to compute the anticommutator of $c_\pm(x)$ and $c_\pm^\dagger(x')$. Let us start with the negative copy. The combination $c_-(x)c_-^\dagger(x')$ is

$$c_-(x)c_-^\dagger(x') =: e^{-i\sqrt{\frac{K}{2}}\varphi_-(x)} :: e^{i\sqrt{\frac{K}{2}}\varphi_-(x')} : \tag{G.3.17a}$$

$$= e^{-i\sqrt{\frac{K}{2}}\theta_-^\dagger(x)}e^{-i\sqrt{\frac{K}{2}}\theta_-(x)}e^{i\sqrt{\frac{K}{2}}\theta_-^\dagger(x')}e^{i\sqrt{\frac{K}{2}}\theta_-(x')} \tag{G.3.17b}$$

$$= e^{-i\sqrt{\frac{K}{2}}\theta_-^\dagger(x)}\left(e^{\frac{K}{2}\left[\theta_-(x),\theta_-^\dagger(x')\right]}e^{i\sqrt{\frac{K}{2}}\theta_-^\dagger(x')}e^{-i\sqrt{\frac{K}{2}}\theta_-(x)}\right)e^{i\sqrt{\frac{K}{2}}\theta_-(x')} \tag{G.3.17c}$$

$$= e^{-i\sqrt{\frac{K}{2}}\theta_-^\dagger(x)}\left(e^{-\log\left[-2i\pi\left(x-x'+i\eta\right)\right]}e^{i\sqrt{\frac{K}{2}}\theta_-^\dagger(x')}e^{-i\sqrt{\frac{K}{2}}\theta_-(x)}\right)e^{i\sqrt{\frac{K}{2}}\theta_-(x')} \tag{G.3.17d}$$

$$= \frac{e^{-i\sqrt{\frac{K}{2}}\left(\theta_-^\dagger(x)-\theta_-^\dagger(x')\right)}e^{-i\sqrt{\frac{K}{2}}\left(\theta_-(x)-\theta_+(x')\right)}}{-2i\pi\left(x-x'+i\eta\right)}\,, \tag{G.3.17e}$$

while $c_-^\dagger(x')c_-(x)$ reads

$$c_-^\dagger(x')c_-(x) =: e^{i\sqrt{\frac{K}{2}}\varphi_-(x')} :: e^{-i\sqrt{\frac{K}{2}}\varphi_-(x)} : \tag{G.3.18a}$$

$$= e^{i\sqrt{\frac{K}{2}}\theta_-^\dagger(x')}e^{i\sqrt{\frac{K}{2}}\theta_-(x')}e^{-i\sqrt{\frac{K}{2}}\theta_-^\dagger(x)}e^{-i\sqrt{\frac{K}{2}}\theta_-(x)} \tag{G.3.18b}$$

$$= e^{i\sqrt{\frac{K}{2}}\theta_-^\dagger(x')}\left(e^{\frac{K}{2}\left[\theta_-(x'),\theta_-^\dagger(x)\right]}e^{-i\sqrt{\frac{K}{2}}\theta_-^\dagger(x)}e^{i\sqrt{\frac{K}{2}}\theta_-(x')}\right)e^{-i\sqrt{\frac{K}{2}}\theta_-(x)} \tag{G.3.18c}$$

$$= e^{i\sqrt{\frac{K}{2}}\theta_-^\dagger(x')}\left(e^{-\log\left[-2i\pi(x'-x+i\eta)\right]}e^{-i\sqrt{\frac{K}{2}}\theta_-^\dagger(x)}e^{i\sqrt{\frac{K}{2}}\theta_-(x')}\right)e^{-i\sqrt{\frac{K}{2}}\theta_-(x)} \tag{G.3.18d}$$

$$= \frac{e^{-i\sqrt{\frac{K}{2}}\left(\theta_-^\dagger(x)-\theta_-^\dagger(x')\right)}e^{-i\sqrt{\frac{K}{2}}\left(\theta_-(x)-\theta_-(x')\right)}}{-2i\pi\left(x'-x+i\eta\right)}\,. \tag{G.3.18e}$$

Taking the limit $\eta \to 0^+$, and adding the latter two results, we readily obtain that $\left\{c_-(x), c_-^\dagger(x')\right\} = 0$.

Now let us compute $\left\{c_+(x), c_+^\dagger(x')\right\}$ for the chiral sector. The combination $c_+(x)c_+^\dagger(x')$ is

$$
\begin{aligned}
c_+(x)c_+^\dagger(x') &=: e^{-i\sqrt{\frac{K}{2}}\varphi_+(x)} :: e^{i\sqrt{\frac{K}{2}}\varphi_+(x')} : && \text{(G.3.19a)}\\
&= e^{-i\sqrt{\frac{K}{2}}\theta_+(x)} e^{-i\sqrt{\frac{K}{2}}\theta_+^\dagger(x)} e^{i\sqrt{\frac{K}{2}}\theta_+(x')} e^{i\sqrt{\frac{K}{2}}\theta_+^\dagger(x')} && \text{(G.3.19b)}\\
&= e^{-i\sqrt{\frac{K}{2}}\theta_+(x)} \left(e^{\frac{K}{2}\left[\theta_+^\dagger(x),\theta_+(x')\right]} e^{i\sqrt{\frac{K}{2}}\theta_+(x')} e^{-i\sqrt{\frac{K}{2}}\theta_+^\dagger)x)} \right) e^{i\sqrt{\frac{K}{2}}\theta_+^\dagger(x')} && \text{(G.3.19c)}\\
&= e^{-i\sqrt{\frac{K}{2}}\theta_+(x)} \left(e^{-\log\left[-2i\pi\left(x-x'+i\eta\right)\right]} e^{i\sqrt{\frac{K}{2}}\theta_+(x')} e^{-i\sqrt{\frac{K}{2}}\theta_+^\dagger(x)} \right) e^{i\sqrt{\frac{K}{2}}\theta_+^\dagger(x')} && \text{(G.3.19d)}\\
&= \frac{e^{-i\sqrt{\frac{K}{2}}\left(\theta_+(x)-\theta_+(x')\right)} e^{-i\sqrt{\frac{K}{2}}\left(\theta_+^\dagger(x)-\theta_+^\dagger(x')\right)}}{-2i\pi\left(x-x'+i\eta\right)}\,. && \text{(G.3.19e)}
\end{aligned}
$$

On the other hand, $c_+^\dagger(x')c_+(x)$ is

$$
\begin{aligned}
c_+^\dagger(x')c_+(x) &=: e^{i\sqrt{\frac{K}{2}}\varphi_+(x')} :: e^{-i\sqrt{\frac{K}{2}}\varphi_+(x)} : && \text{(G.3.20a)}\\
&= e^{i\sqrt{\frac{K}{2}}\theta_+(x')} e^{i\sqrt{\frac{K}{2}}\theta_+^\dagger(x')} e^{-i\sqrt{\frac{K}{2}}\theta_+(x)} e^{-i\sqrt{\frac{K}{2}}\theta_+^\dagger(x)} && \text{(G.3.20b)}\\
&= e^{i\sqrt{\frac{K}{2}}\theta_+(x')} \left(e^{\frac{K}{2}\left[\theta_+^\dagger(x'),\theta_+(x)\right]} e^{-i\sqrt{\frac{K}{2}}\theta_+(x)} e^{i\sqrt{\frac{K}{2}}\theta_+^\dagger(x')} \right) e^{-i\sqrt{\frac{K}{2}}\theta_+^\dagger(x)} && \text{(G.3.20c)}\\
&= e^{i\sqrt{\frac{K}{2}}\theta_+(x')} \left(e^{-\log\left[-2i\pi\left(x'-x+i\eta\right)\right]} e^{-i\sqrt{\frac{K}{2}}\theta_+(x)} e^{i\sqrt{\frac{K}{2}}\theta_+^\dagger(x')} \right) e^{-i\sqrt{\frac{K}{2}}\theta_+^\dagger(x)} && \text{(G.3.20d)}\\
&= \frac{e^{-i\sqrt{\frac{K}{2}}\left(\theta_+(x)-\theta_+(x')\right)} e^{-i\sqrt{\frac{K}{2}}\left(\theta_+^\dagger(x)-\theta_+^\dagger(x')\right)}}{-2i\pi\left(x'-x+i\eta\right)}\,. && \text{(G.3.20e)}
\end{aligned}
$$

Again, taking the limit $\eta \to 0$, we readily obtain that $\left\{c_+(x), c_+^\dagger(x')\right\} = 0$, proving this result for the chiral/antichiral sector. Therefore, we proved that the following anticommutation algebra fulfills

$$
\left\{c_\pm(x), c_\pm(x')\right\} = \left\{c_\pm^\dagger(x), c_\pm^\dagger(x')\right\} = \left\{c_\pm(x), c_\pm^\dagger(x')\right\} = 0\,, \quad x \neq x'\,. \qquad \text{(G.3.21)}
$$

Operator $c_-(x)$ stands for the fermionic annihilation operator, $c_+(x)$ to the creation one, and for the chiral sector, $c_+(x)$ stands for the fermionic creation operator and $c_+^\dagger(x)$ to the annihilation one (see Table G.2).

	Chiral sector	Antichiral sector
Creation operator	$c_+(x)$	$c_-^\dagger(x)$
Annihilation operator	$c_+^\dagger(x)$	$c_-(x)$

Table G.2: Fermionic creation and annihilation operators of each sector.

Majorana fermions

Majorana fermions are particles that are their own antiparticle [403], and appear in different contexts, such as neutrino physics [404, 405], Quantum Hall Effect [384], and superconductors [406].

If we define

$$c_{\pm}(t,x) = \frac{\psi_i(t,x) + i\psi_j(t,x)}{\sqrt{2}}\,, \quad c^{\dagger}_{\pm}(t,x) = \frac{\psi_i(t,x) - i\psi_j(t,x)}{\sqrt{2}}\,, \tag{G.3.22}$$

where $\psi^{\pm}_i(t,x)$ denotes a collection of $i = 1, \ldots, n$ fermionic particles, we have

$$\psi^{\pm}_i(t,x) = \frac{c_{\pm}(t,x) + c^{\dagger}_{\pm}(t,x)}{\sqrt{2}}\,, \quad \psi^{\pm}_j(t,x) = \frac{c_{\pm}(t,x) - c^{\dagger}_{\pm}(t,x)}{\sqrt{2}i} \tag{G.3.23}$$

From (G.3.21), we readily obtain that $\psi^{\pm}_i$ is a fermion, because satisfies

$$\left\{\psi^{\pm}_i(x), \psi^{\pm}_j(x')\right\} = 0 \quad x \neq x'\,. \tag{G.3.24}$$

Furthermore, from definition (G.3.23), it is possible to prove that $\psi^{\pm}_{i,j} = \left(\psi^{\pm}_{i,j}\right)^{\dagger}$. This condition allows us to interpret $\psi^{\pm}_{i,j}$ as Majorana fermions.

We can write the number operators $n_-(x) = c^{\dagger}_-(x)c_-(x)$, $n_+(x) = c_+(x)c^{\dagger}_+(x)$ in terms of these Majorana fermions as following

$$n_-(x) = c^{\dagger}_-(x)c_-(x) = 1 + i\psi^-_i(x)\psi^-_j(x)\,, \tag{G.3.25a}$$

$$n_+(x) = c_+(x)c^{\dagger}_+(x) = 1 - i\psi^+_i(x)\psi^+_j(x)\,. \tag{G.3.25b}$$

The fermionic number (charge density) operator of the two sectors are $n_{\pm}(x) \propto \mp i\psi^{\pm}_i(x)\psi^{\pm}_j(x)$. This operator is the one that is going to provide the fermionic interpretation of $\mathcal{J}_{\pm}$.

Fermionic charge operator

Here we aim to compute the fermionic number operator in terms of anisotropic chiral fields $\varphi_{\pm}(x)$.

For the antichiral sector, let us calculate $: c^{\dagger}_-(x)c_-(x') :$, just like we did in Section G.3.2,

$$: c^{\dagger}_-(x)c_-(x') : =: e^{i\sqrt{\frac{K}{2}}\varphi_-(x)} :: e^{-i\sqrt{\frac{K}{2}}\varphi_-(x)} : \tag{G.3.26a}$$

$$= e^{i\sqrt{\frac{K}{2}}\theta^{\dagger}_-(x)} e^{i\sqrt{\frac{K}{2}}\theta_-(x)} e^{-i\sqrt{\frac{K}{2}}\theta^{\dagger}_-(x')} e^{-i\sqrt{\frac{K}{2}}\theta_-(x')} \tag{G.3.26b}$$

$$= e^{i\sqrt{\frac{K}{2}}\theta^{\dagger}_-(x)} \left(e^{\frac{K}{2}\left[\theta_-(x),\theta^{\dagger}_-(x')\right]} e^{-i\sqrt{\frac{K}{2}}\theta^{\dagger}_-(x')} e^{i\sqrt{\frac{K}{2}}\theta_-(x)}\right) e^{-i\sqrt{\frac{K}{2}}\theta_-(x')} \tag{G.3.26c}$$

$$= e^{i\sqrt{\frac{K}{2}}\theta^{\dagger}_-(x)} \left(e^{-\log\left[-2i\pi\left(x-x'+i\eta\right)\right]} e^{-i\sqrt{\frac{K}{2}}\theta^{\dagger}_-(x')} e^{i\sqrt{\frac{K}{2}}\theta_-(x)}\right) e^{-i\sqrt{\frac{K}{2}}\theta_-(x')} \tag{G.3.26d}$$

$$= \frac{e^{i\sqrt{\frac{K}{2}}\left(\theta^{\dagger}_-(x)-\theta^{\dagger}_-(x')\right)} e^{i\sqrt{\frac{K}{2}}\left(\theta_-(x)-\theta_-(x')\right)}}{-2i\pi(x-x'+i\eta)}\,. \tag{G.3.26e}$$

If we define $\Delta x = x - x'$, the Taylor expansions of $\Delta\theta_-(x) \equiv \theta_-(x) - \theta_-(x')$ and $\Delta\theta^{\dagger}_-(x) \equiv$

$\theta_{-}^{\dagger}(x) - \theta_{-}^{\dagger}(x')$ are

$$\Delta\theta_{-}(x) \equiv \theta_{-}(x) - \theta_{-}(x') \approx \Delta x \partial_x \theta_{-}(x) - \frac{1}{2}(\Delta x)^2 \partial_x^2 \theta_{-}(x)\,,$$
$$\Delta\theta_{-}^{\dagger}(x) \equiv \theta_{-}^{\dagger}(x) - \theta_{-}^{\dagger}(x') \approx \Delta x \partial_x \theta_{-}^{\dagger}(x) - \frac{1}{2}(\Delta x)^2 \partial_x^2 \theta_{-}^{\dagger}(x)\,,$$

respectively. Therefore, the Taylor expansion of the number operator is

$$\begin{aligned} : c_{-}^{\dagger}(x) c_{-}(x') : \approx \frac{1}{-2i\pi\,(x - x' + i\eta)} \Bigg\{ &\left[1 + i\sqrt{\frac{K}{2}} \Delta\theta_{-}^{\dagger}(x) - \frac{K}{4}\left(\Delta\theta_{-}^{\dagger}(x)\right)^2\right] \times \\ &\times \left[1 + i\sqrt{\frac{K}{2}} \Delta\theta_{-}(x) - \frac{K}{4}\left(\Delta\theta_{-}(x)\right)^2\right] \Bigg\}\,. \end{aligned} \tag{G.3.27}$$

At second order, the latter expansion is

$$\begin{aligned} : c_{-}^{\dagger}(x) c_{-}(x') : = \frac{i}{2\pi\,(\Delta x + i\eta)} \Bigg\{ &1 + i\sqrt{\frac{K}{2}} \Delta x \partial_x \left(\theta_{-}^{\dagger}(x) + \theta_{-}(x)\right) \\ &- \frac{(\Delta x)^2}{2} \Bigg[i\sqrt{\frac{K}{2}} \partial_x^2 \left(\theta_{-}^{\dagger}(x) + \theta_{-}(x)\right) \\ &+ \frac{K}{2}\left(\left(\partial_x \theta_{-}^{\dagger}(x)\right)^2 + \left(\partial_x \theta_{-}(x)\right)^2 + 2\partial_x \theta_{-}^{\dagger}(x) \partial_x \theta_{-}(x)\right) \Bigg] \Bigg\}\,. \end{aligned} \tag{G.3.28}$$

We can recognize the bosonic field $\varphi_{-}(x)$ according to (G.3.5), and taking the limit $\eta \to 0^{+}$, we can write

$$\begin{aligned} : c_{-}^{\dagger}(x) c_{-}(x') : = \frac{i}{2\pi\Delta x} \Bigg\{ &1 + i\sqrt{\frac{K}{2}} \Delta x \partial_x \varphi_{-}(x) \\ &- \frac{(\Delta x)^2}{2} \left[i\sqrt{\frac{K}{2}} \partial_x^2 \varphi_{-}(x) + \frac{K}{2} : \left(\partial_x \varphi_{-}(x)\right)^2 : \right] \Bigg\} \end{aligned} \tag{G.3.29a}$$

$$\begin{aligned} = \frac{i}{2\pi\Delta x} - \frac{1}{2\pi}\sqrt{\frac{K}{2}} \partial_x \varphi_{-}(x) & \\ + \frac{\Delta x}{4\pi} \left[\sqrt{\frac{K}{2}} \partial_x^2 \varphi_{-}(x) - \frac{iK}{2} : \left(\partial_x \varphi_{-}(x)\right)^2 : \right]. & \end{aligned} \tag{G.3.29b}$$

The first term correspond to the vacuum term. If we take the limit $\Delta x \to 0$, we find that the number operator is

$$: c_{-}^{\dagger}(x) c_{-}(x) := -\frac{1}{2\pi}\sqrt{\frac{K}{2}} \partial_x \varphi_{-}(x)\,. \tag{G.3.30}$$

This is operator $\mathcal{J}_-(x)$, defined in (7.1.30). Hence we can establish the following relationship

$$\mathcal{J}_-(x) = \sqrt{\frac{K}{2}} : c_-^\dagger(x) c_-(x) : \, . \tag{G.3.31}$$

An analogue procedure can be performed for the chiral sector, yielding

$$\mathcal{J}_+(x) = \sqrt{\frac{K}{2}} : c_+(x) c_+^\dagger(x) : \, . \tag{G.3.32}$$

Therefore, we can say that the $u(1)$ Noether charge $\mathcal{J}_\pm$ is the electric charge density of the anisotropic chiral boson theory.

Appendix H

Appendices of Chapter 7

In this appendix we explicitly show the calculations mentioned in Chapter 7.

H.1 $u(1)$ creation and annihilation algebra

Consider the Fourier-expanded chiral field

$$\varphi_{\pm}(x,t) = \int_{-\infty}^{\infty} \frac{dk}{k}\, e^{i(kx \pm \omega_k t)} b_{\pm,k}\,, \tag{H.1.1}$$

where $b_{\pm,k}$ correspond to the k-th wavenumber annihilation operator and $b^{\dagger}_{\pm,k}$ the k-th wavenumber creation operator. We want to invert relation (H.1.1) and expand the annihilation operator in Fourier modes to compute the commutator of $b_{\pm}(k)$ with $b_{\pm}(k')$. For this, we multiply by $e^{-i(k'x \pm \omega_{k'} t)}$,

$$\varphi_{\pm}(t,x) e^{-i(k'x \pm \omega_{k'} t)} = \int_{-\infty}^{\infty} \frac{dk}{k} e^{i(k-k')x} e^{\pm i(\omega_k - \omega_{k'})t} b_{\pm}(\omega_k, k)\,. \tag{H.1.2}$$

Integrating on x, we obtain

$$\begin{aligned} \int_{\infty}^{\infty} dx\, \varphi_{\pm}(t,x) e^{-i(k'x \pm \omega_{k'} t)} &= \int_{-\infty}^{\infty} \frac{dk}{k} \left(\int_{-\infty}^{\infty} dx\, e^{i(k-k')x} \right) e^{\pm i(\omega_k - \omega_{k'})t} b_{\pm}(\omega_k, k) \\ &= \int_{-\infty}^{\infty} \frac{dk}{k} \left(2\pi\delta(k-k')\right) e^{\pm i(\omega_k - \omega_{k'})t} b_{\pm}(\omega_k, k)\,. \end{aligned} \tag{H.1.3}$$

Thus,

$$b_{\pm}(\omega_k, k) = \frac{k}{2\pi} \int_{\infty}^{\infty} dx\, \varphi_{\pm}(t,x) e^{-i(kx \pm \omega_k t)}\,. \tag{H.1.4}$$

Then the commutator of operators $b_\pm(k)$ is

$$[b_\pm(k), b_\pm(k')] = \frac{kk'}{(2\pi)^2}\int_{-\infty}^{\infty} dx\, dx'\ [\varphi_\pm(x), \varphi_\pm(x')] e^{-i(kx+k'x')} e^{\mp i(\omega_k+\omega_{k'})t}\,. \tag{H.1.5}$$

Observe that $k' e^{-i'kx'} = i\partial_{x'} e^{-ik'x'}$, hence

$$[b_\pm(k), b_\pm(k')] = \frac{ik}{(2\pi)^2}\int_{-\infty}^{\infty} dx\, dx'\ [\varphi_\pm(x), \varphi_\pm(x')] e^{-i(kx\pm\omega_k t)} \partial_{x'} e^{-i(k'x'\pm\omega_{k'} t)}\,. \tag{H.1.6}$$

We can integrate by parts, suppose the fields vanishes sufficiently fast in the boundaries and using the algebra of chiral fields,

$$[\varphi_\pm(x), \partial_{x'}\varphi_\pm(x')] = \pm\frac{4i\pi}{K}\delta(x-x')\,, \tag{H.1.7}$$

we obtain

$$[b_\pm(k), b_\pm(k')] = -\frac{ik}{(2\pi)^2}\int_{-\infty}^{\infty} dx\, dx'\ [\varphi_\pm(x), \partial_{x'}\varphi_\pm(x')] e^{-i(kx\pm\omega_k t)} e^{-i(k'x'\pm\omega_{k'} t)}. \tag{H.1.8a}$$

$$= -\frac{ik}{(2\pi)^2}\int_{-\infty}^{\infty} dx\, dx'\ \left(\pm\frac{4i\pi}{K}\delta(x-x')\right) e^{-i(kx\pm\omega_k t)} \times \times e^{-i(k'x'\pm\omega_{k'} t)}. \tag{H.1.8b}$$

$$= \pm\frac{k}{\pi K}\int_{-\infty}^{\infty} dx\ e^{-i(k+k')x} e^{\mp i(\omega_k+\omega_{k'})t} \tag{H.1.8c}$$

$$= \pm\frac{2}{K} k\delta(k+k')\,, \tag{H.1.8d}$$

in agreement with (7.2.8).

H.2 Susceptibility expression

We aim to obtain Eq. (7.2.11) for the susceptibility. As we have seen in Section 7.2, recall that

$$\tilde{\chi}^{\pm}_{\mathcal{I},\mathcal{J}}(\omega;x,x') = -i\int_{-\infty}^{\infty} dt\,\Theta(t)e^{i\omega t}\ \langle[\mathcal{I}_{\pm}(x,t),\mathcal{J}_{\pm}(x',0)]\rangle \tag{H.2.1a}$$

$$= -\frac{iK}{8\pi^2}\int_{-\infty}^{\infty} dt\,\Theta(t)e^{i\omega t}\int_{-\infty}^{\infty} dk\,\omega_k e^{ik(x-x')}e^{\pm i\omega_k t} \tag{H.2.1b}$$

$$= -\frac{iK}{8\pi^2}\int_{-\infty}^{\infty} dk\,\omega_k e^{ik(x-x')}\int_{-\infty}^{\infty} dt\,\Theta(t)e^{i(\omega\pm\omega_k)t}. \tag{H.2.1c}$$

As said in Eq. (7.2.9), we consider the Fourier transform of the theta Heaviside function

$$\int_{-\infty}^{\infty} dt\,\Theta(t)e^{i(\omega\pm\omega_k)t} = \frac{i}{\omega\pm\omega_k+i\epsilon},$$

where ϵ is a regulator that appears as a consequence of the susceptibility causality, i.e., the susceptibility vanishes for $t<0$. Therefore,

$$\tilde{\chi}^{\pm}_{\mathcal{I},\mathcal{J}}(\omega;x,x') = \frac{K}{8\pi^2}\int_{-\infty}^{\infty} dk\,\frac{\omega_k}{\omega\pm\omega_k+i\epsilon}e^{ik(x-x')} \tag{H.2.20d}$$

$$= \frac{K}{8\pi^2}\int_{-\infty}^{\infty} dk\,\frac{\omega_k\pm(\omega+i\epsilon)\mp(\omega+i\epsilon)}{\omega\pm\omega_k+i\epsilon}e^{ik(x-x')} \tag{H.2.20e}$$

$$\equiv \frac{K}{8\pi^2}\int_{-\infty}^{\infty} dk\,\frac{\pm(\omega\pm\omega_k+i\epsilon)\mp(\omega+i\epsilon)}{\omega\pm\omega_k+i\epsilon}e^{ik(x-x')} \tag{H.2.20f}$$

$$= \frac{K}{8\pi^2}\int_{-\infty}^{\infty} dk\left(\pm1\mp\frac{\omega+i\epsilon}{\omega\pm\omega_k+i\epsilon}\right)e^{ik(x-x')} \tag{H.2.20g}$$

$$= \frac{K}{4\pi}\left(\pm\delta(x-x')\mp\int_{-\infty}^{\infty}\frac{dk}{2\pi}\,\frac{\omega+i\epsilon}{\omega\pm\omega_k+i\epsilon}e^{ik(x-x')}\right) \tag{H.2.20h}$$

$$\equiv \frac{K}{4\pi}\left[\pm\delta(x-x')\mp(\omega+i\epsilon)\,F^{\pm}\left(x-x';\omega\right)\right]\ , \tag{H.2.20i}$$

where the function $F^{\pm}(x-x';\omega)$ is defined as

$$F^{\pm}(x-x';\omega) = \int_{-\infty}^{\infty}\frac{dk}{2\pi}\,\frac{e^{ik(x-x')}}{\omega\pm\omega_k+i\epsilon}\,. \tag{H.2.21}$$

Thus, we arrived to Eq. (7.2.11).

H.3 Complex integral

We start from Eq. (7.2.12),

$$F^{\pm}(x-x';\omega) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{e^{ik(x-x')}}{\omega \pm \omega_k + i\epsilon}\,. \tag{H.3.1}$$

Using standard complex calculus, we aim to solve this integral and arrive to Eq. (7.2.14).

Define $y = x - x'$. The function $F^{\pm}$ now reads as

$$F^{\pm}(y;\omega) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{e^{iky}}{\omega \pm \omega_k + i\epsilon}.$$

Extending the integral to the complex plane, we see it has a simple pole $k_p^{\pm}$, with $p = 0, 1, 2, \ldots, z^{\pm} - 1$ when $\omega_{k_p^{\pm}} = \mp(\omega + i\epsilon)$. In order to fix ideas, let us work with the dispersion relation of anisotropic chiral bosobserve that for this case, the simple pole $k_p^{\pm}$ satisfies the equation $\omega \pm v\left(k_p^{\pm}\right)^{z^{\pm}} + i\epsilon = 0$, which yields

$$k_p^{\pm} = \mp e^{2p\pi/z^{\pm}}\left(\omega + i\epsilon\right)^{1/z^{\pm}} \equiv \mp e^{2p\pi/z^{\pm}} \omega^{1/z^{\pm}} \left(1 + \frac{i\epsilon}{\omega}\right)^{1/z^{\pm}} \approx \mp \omega^{1/z^{\pm}} e^{i(2p\pi+\epsilon)/z^{\pm}}.$$

Hence, every pole will have $\omega^{1/z^{\pm}}$ complex module and $\theta = (2p\pi + \epsilon)/z^{\pm}$ angle. As an example, consider case $z^{\pm} = 1$. Then we have an unique pole given by expression

$$k_0^{\pm} = \mp\omega e^{i\epsilon} = \mp\omega\left(\cos\epsilon + i\sin\epsilon\right) \approx \mp\omega\left(1 + i\epsilon\right).$$

On the other hand, for $z^{\pm} = 3$, we will have the following poles

$$k_p^{\pm} = \mp\omega^{1/3} e^{i(2p\pi+\epsilon)/3} = \mp\omega^{1/3} \begin{cases} e^{i\epsilon} \approx 1 + i\epsilon, & p = 0, \\ e^{i(2\pi+\epsilon)/3} \approx -\frac{1}{2} + \frac{i\sqrt{3}}{2}, & p = 1, \\ e^{i(4\pi+\epsilon)} \approx -\frac{1}{2} - \frac{i\sqrt{3}}{2}, & p = 2. \end{cases}$$

The two aforementioned cases can be seen pictorically in figure (H.1).

Now that we settled the idea, we may solve $F^{\pm}$. In general, it may be extended to the complex plane by considering the following contour integral

$$\oint_{C_{\gtrless}} \frac{dz}{2\pi} \frac{e^{izy}}{\omega \pm \omega_z + i\epsilon} = \oint_{\Gamma_{\gtrless}} \frac{dz}{2\pi} \frac{e^{izy}}{\omega \pm \omega_z + i\epsilon} + \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{e^{iky}}{\omega \pm \omega_k + i\epsilon},$$

where $C_{\gtrless}$ is a general contour that considers the arc $\Gamma_{\gtrless}$ and the real line. For $y > 0$, we will close

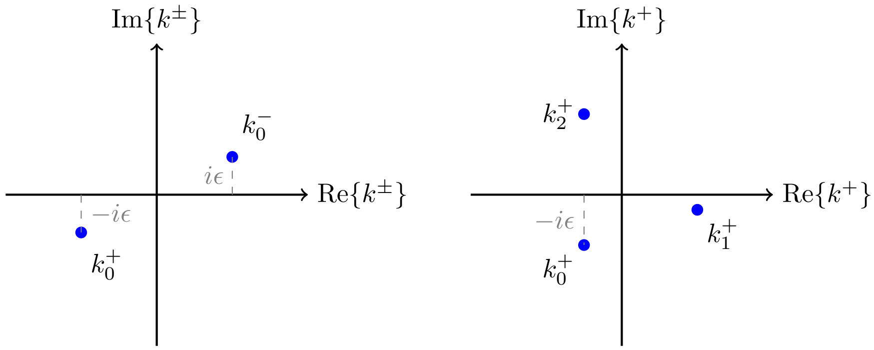

Figure H.1: Left picture: Poles for case $z^{\pm} = 1$. Right picture: Poles for case $z^+ = 3$. For the $\pm$ sector, the pole $k_0^{\pm}$ is always shifted to the lower and upper complex plane, respectively.

the contour from above and in anticlockwise manner, yielding

$$\oint_{C_>} \frac{dz}{2\pi} \frac{e^{izy}}{\omega \pm \omega_z + i\epsilon} = \oint_{\Gamma_>} \frac{dz}{2\pi} \frac{e^{izy}}{\omega \pm \omega_z + i\epsilon} + \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{e^{iky}}{\omega \pm \omega_k + i\epsilon}.$$

By Jordan's lemma, the $\Gamma_>$ contour integral goes to zero. Thus, the complex integral is the real one and it can be solved by using the residue theorem

$$\oint_{C_>} \frac{dz}{2\pi} \frac{e^{izy}}{\omega \pm \omega_z + i\epsilon} = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{e^{iky}}{\omega \pm \omega_k + i\epsilon} = i\Theta(y)\,\mathrm{Res}\left[\frac{e^{iky}}{\omega \pm \omega_k + i\epsilon}, \omega_k = +(\omega + i\epsilon)\right] \quad \text{(H.3.2a)}$$

$$= i\Theta(y) \sum_{k_p^{\pm} \in \mathrm{Im}_>} \frac{e^{ik_p^{\pm} y}}{\pm\omega'_{k_p^{\pm}}}. \quad \text{(H.3.2b)}$$

On the other hand, for $y < 0$, we will close the contour from below and in clockwise manner, and by Jordan's lemma, we obtain

$$\int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{e^{iky}}{\omega \pm \omega_k + i\epsilon} = -i\Theta(-y)\,\mathrm{Res}\left[\frac{e^{iky}}{\omega \pm \omega_k + i\epsilon}, \omega_k = -(\omega + i\epsilon)\right] \quad \text{(H.3.3a)}$$

$$= -i\Theta(-y) \sum_{k_p^{\pm} \in \mathrm{Im}_<} \frac{e^{ik_p^{\pm} y}}{\pm\omega'_{k_p^{\pm}}}. \quad \text{(H.3.3b)}$$

Therefore, $F^{\pm}(y;\omega)$ is

$$F^{\pm}(y;\omega) = \pm i\Theta(y) \sum_{k_p^{\pm} \in \mathrm{Im}_>} \frac{e^{ik_p^{\pm} y}}{\omega'_{k_p^{\pm}}} \mp i\Theta(-y) \sum_{k_p^{\pm} \in \mathrm{Im}_<} \frac{e^{ik_p^{\pm} y}}{\omega'_{k_p^{\pm}}}. \quad \text{(H.3.4)}$$

Thus, if we take the limit $\epsilon \to 0$, $(\omega + i\epsilon)F^{\pm}(y;\omega)$ reads

$$\omega F^{\pm}(y;\omega) = \pm i\Theta(y) \sum_{k_p^{\pm} \in \mathrm{Im}_>} \frac{\left(\mp\omega_{k_p^{\pm}}\right) e^{ik_p^{\pm} y}}{\omega'_{k_p^{\pm}}} \mp i\Theta(-y) \sum_{k_p^{\pm} \in \mathrm{Im}_<} \frac{\left(\mp\omega_{k_p^{\pm}}\right) e^{ik_p^{\pm} y}}{\omega'_{k_p^{\pm}}} \tag{H.3.5a}$$

$$= -i\Theta(y) \sum_{k_p^{\pm} \in \mathrm{Im}_>} \frac{\omega_{k_p^{\pm}} e^{ik_p^{\pm} y}}{\omega'_{k_p^{\pm}}} + i\Theta(-y) \sum_{k_p^{\pm} \in \mathrm{Im}_<} \frac{\omega_{k_p^{\pm}} e^{ik_p^{\pm} y}}{\omega'_{k_p^{\pm}}}. \tag{H.3.5b}$$

The latter result is general, since we did not replace the dispersion relation of the anisotropic chiral boson on any calculation. On the other hand, If we now specialize to the anisotropic chiral boson case, we obtain

$$\omega F^{\pm}(y;\omega) = -i\Theta(y) \sum_{k_p^{\pm} \in \mathrm{Im}_>} \frac{k_p^{\pm} e^{ik_p^{\pm} y}}{z^{\pm}} + i\Theta(-y) \sum_{k_p^{\pm} \in \mathrm{Im}_<} \frac{k_p^{\pm} e^{ik_p^{\pm} y}}{z^{\pm}} \tag{H.3.25c}$$

$$= -\Theta(y)\partial_y \sum_{k_p^{\pm} \in \mathrm{Im}_>} \frac{e^{ik_p^{\pm} y}}{z^{\pm}} + \Theta(-y)\partial_y \sum_{k_p^{\pm} \in \mathrm{Im}_<} \frac{e^{ik_p^{\pm} y}}{z^{\pm}} \tag{H.3.25d}$$

$$\equiv -\Theta(y)\partial_y \Delta_>^{\pm}(y;\omega) + \Theta(-y)\partial_y \Delta_<^{\pm}(y;\omega) \tag{H.3.25e}$$

where $\Delta_{\gtrless}^{\pm}$ is defined as

$$\Delta_{\gtrless}^{\pm}(y;\omega) = \sum_{k_p^{\pm} \in \mathrm{Im}_{\gtrless}} \frac{e^{ik_p^{\pm} y}}{z^{\pm}}\,. \tag{H.3.26}$$

Hence, we arrived to Eq. (7.2.14).

H.4 Linear response

In this Appendix we show the explicit calculations in order to arrive to Eq. (7.2.18).

We start from Eq. (7.2.17)

$$\begin{aligned}\delta\left\langle\tilde{\mathcal{I}}_\pm(\omega,x)\right\rangle &= \mp\frac{K}{4\pi}\int_{-\infty}^{\infty}dy\,\left[V_L\Theta(x_L-x+y)+V_R\Theta(x-y-x_R)\right]\times \\ &\qquad\times\left[-\delta(y)-\Theta(y)\partial_y\Delta_>^\pm(y;\omega)+\Theta(-y)\partial_y\Delta_<^\pm(y;\omega)\right] \qquad \text{(H.4.1a)}\\ &\equiv \mp\frac{K}{4\pi}\left(I_0+I_1+I_2\right)\,, \qquad \text{(H.4.1b)}\end{aligned}$$

where I_0 is

$$\begin{aligned}I_0 &= -\int_{-\infty}^{\infty}dy\,\left[V_L\Theta(x_L-x+y)+V_R\Theta(x-y-x_R)\right]\delta(y) \qquad \text{(H.4.2a)}\\ &= -\left[V_L\Theta(x_L-x)+V_R\Theta(x-x_R)\right]\,. \qquad \text{(H.4.2b)}\end{aligned}$$

On the other hand, I_1 is

$$\begin{aligned}I_1 &= -\int_{-\infty}^{\infty}dy\,\left[V_L\Theta(x_L-x+y)+V_R\Theta(x-y-x_R)\right]\Theta(y)\partial_y\Delta_>^\pm(y;\omega) \qquad \text{(H.4.3a)}\\ &= \int_{-\infty}^{\infty}dy\,\partial_y\left\{\left[V_L\Theta(x_L-x+y)+V_R\Theta(x-y-x_R)\right]\Theta(y)\right\}\Delta_>^\pm(y;\omega) \qquad \text{(H.4.3b)}\\ &= \int_{-\infty}^{\infty}dy\,\left\{\left[V_L\delta(x_L-x+y)+V_R\delta(x-y-x_R)\right]\Theta(y)\right.\\ &\qquad\left.\left[V_L\Theta(x_L-x+y)+V_R\Theta(x-y-x_R)\right]\delta(y)\right\}\Delta_>^\pm(y;\omega) \qquad \text{(H.4.3c)}\\ &= V_L\Theta(x-x_L)\Delta_>^\pm(x-x_L;\omega)-V_R\Theta(x-x_R)\Delta_>^\pm(x-x_R;\omega)\\ &\qquad+\left[V_L\Theta(x_L-x)+V_R\Theta(x-x_R)\right]\Delta_>^\pm(0;\omega)\,. \qquad \text{(H.4.3d)}\end{aligned}$$

Finally, I_2 is

$$\begin{aligned}I_2 &= \int_{-\infty}^{\infty}dy\,\left[V_L\Theta(x_L-x+y)+V_R\Theta(x-y-x_R)\right]\Theta(-y)\partial_y\Delta_<^\pm(y;\omega) \qquad \text{(H.4.4a)}\\ &= -\int_{-\infty}^{\infty}dy\,\partial_y\left\{\left[V_L\Theta(x_L-x+y)+V_R\Theta(x-y-x_R)\right]\Theta(-y)\right\}\Delta_<^\pm(y;\omega) \qquad \text{(H.4.4b)}\\ &= -\int_{-\infty}^{\infty}dy\,\left\{\left[V_L\delta(x_L-x+y)-V_R\delta(x-y-x_R)\right]\Theta(-y)\right.\\ &\qquad\left.-\left[V_L\Theta(x_L-x+y)+V_R\Theta(x-y-x_R)\right]\delta(-y)\right\}\Delta_<^\pm(y;\omega) \qquad \text{(H.4.4c)}\\ &= -V_L\Theta(x_L-x)\Delta_<^\pm(x-x_L;\omega)+V_R\Theta(x_R-x)\Delta_<^\pm(x-x_R;\omega)\\ &\qquad+\left[V_L\Theta(x_L-x)+V_R\Theta(x-x_R)\right]\Delta_<^\pm(0;\omega)\,. \qquad \text{(H.4.4d)}\end{aligned}$$

Therefore, (H.4.1b) is

$$\begin{aligned}\delta\left\langle\tilde{\mathcal{I}}_{\pm}(\omega,x)\right\rangle = \mp\frac{K}{4\pi}\Big\{ & -\left[V_L\Theta(x_L-x)+V_R\Theta(x-x_R)\right] \\ & + V_L\left[\Theta(x-x_L)\Delta^{\pm}_{>}(x-x_L;\omega)-\Theta(x_L-x)\Delta^{\pm}_{<}(x-x_L;\omega)\right] \\ & - V_R\left[\Theta(x-x_R)\Delta^{\pm}_{>}(x-x_R;\omega)-\Theta(x_R-x)\Delta^{\pm}_{<}(x-x_R;\omega)\right] \\ & + \left[V_L\Theta(x_L-x)+V_R\Theta(x-x_R)\right]\left(\Delta^{\pm}_{>}(0;\omega)+\Delta^{\pm}_{<}(0;\omega)\right)\Big\}\ ,\end{aligned} \tag{H.4.5}$$

in agreement with (7.2.18).

H.5 Bulk two-terminal conductance

Here we show how the holographic result exposed in (7.2.22) can be obtained from the bulk perspective.

Consider the anisotropic chiral boson equation given in (7.3.10),

$$\left[\partial_t + v(-1)^{\frac{z^\pm+1}{2}}\partial_x^{z^\pm}\right]\mathcal{J}_\pm = \pm\frac{K}{4\pi}\partial_x\mu\,., \tag{H.5.1}$$

with odd $z^\pm$. As said in Eq. (7.3.11), the formal solution to this differential equation comes in terms of the retarded Green function $G_\mathrm{R}^\pm$,

$$\mathcal{J}_\pm(t,x) = \mathcal{J}_\pm^\mathrm{in} \pm \frac{K}{4\pi}\int dt'\,dx'\,G_\mathrm{R}^\pm(x-x',t-t')\partial_{x'}\mu\,, \tag{H.5.2}$$

where $\mathcal{J}_\pm^\mathrm{in}$ is the homogenous solution (i.e., the solution without the chemical potential source) and $G_\mathrm{R}^\pm$ fulfills the Green equation

$$\left[\partial_t + v(-1)^{\frac{z^\pm+1}{2}}\partial_x^{z^\pm}\right]G_\mathrm{R}^\pm(t-t',x-x') = \delta(t-t')\delta(x-x')\,. \tag{H.5.3}$$

In order to find $G_\mathrm{R}^\pm$, it is convenient to consider the next Fourier expansion

$$G_\mathrm{R}^\pm(t-t',x-x') = \int\frac{dk\,d\omega}{(2\pi)^2}e^{i\left[k(x-x')-\omega(t-t')\right]}\tilde{G}_\mathrm{R}^\pm(\omega,k)\,. \tag{H.5.4}$$

Replacing in (H.5.3), and using the integral representation of Dirac deltas, we arrive to the following equation in Fourier space

$$\int\frac{dk\,d\omega}{(2\pi)^2}\left[-i\omega \pm v(-1)^{\frac{z^\pm+1}{2}}i^{z^\pm}k^{z^\pm}\right]\tilde{G}_\mathrm{R}^\pm(\omega,k)e^{i\left[k(x-x')-\omega(t-t')\right]} = \int\frac{dk\,d\omega}{(2\pi)^2}e^{i\left[k(x-x')-\omega(t-t')\right]}\,. \tag{H.5.5}$$

Hence $\tilde{G}_\mathrm{R}^\pm(\omega,k)$ is the inverse of the differential operator as following

$$\tilde{G}_\mathrm{R}^\pm(k,\omega) = \frac{1}{-i\omega \pm v(-1)^{\frac{z^\pm+1}{2}}i^{z^\pm}k^{z^\pm}} = \frac{1}{-i\omega \mp ivk^{z^\pm}} = \frac{i}{\omega\pm\omega_k}\,. \tag{H.5.6}$$

As discussed in Appendix H.3, the retarded prescription will be guaranteed by means of the regulator $\epsilon\to$ in form,

$$\tilde{G}_\mathrm{R}^\pm(k,\omega) = \frac{i}{\omega\pm\omega_k + i\epsilon}\,. \tag{H.5.7}$$

Replacing the latter in (H.5.4), and then in (H.5.2), we obtain

$$\mathcal{J}_\pm(t,x) = \mathcal{J}_\pm^{\text{in}} \pm \frac{K}{4\pi}\int dt'\,dx'\int dk\,d\omega\frac{e^{ik(x-x')}e^{-i\omega(t-t')}}{\omega\pm\omega_k+i\epsilon}\partial_{x'}\mu(t',x')\,. \tag{H.5.8}$$

This expression is not new, since it is possible to recognize the function (7.2.12),

$$F^\pm(x-x',\omega) = \int_{-\infty}^{\infty}\frac{dk}{2\pi}\frac{e^{ik(x-x')}}{\omega\pm\omega_k+i\epsilon}\,, \tag{H.5.9}$$

allowing us to rewrite Eq. (H.5.8) as

$$\mathcal{J}_\pm(t,x) = \mathcal{J}_\pm^{\text{in}} \pm \frac{iK}{8\pi^2}\int dt'\,dx'\int dk\,F^\pm(x-x',\omega)e^{-i\omega(t-t')}\partial_{x'}\mu(t',x')\,, \tag{H.5.10}$$

in agreement with (7.3.14). This expression will be useful in memory effect calculations.

With the charge density $\mathcal{J}_\pm$ obtained, let us proceed to compute the current intensity $\mathcal{I}_\pm$, being the latter related with the former by means of equation (7.3.8).

$$\mathcal{I}_\pm(t,x) = \mathcal{I}_\pm^{\text{in}} \pm v\,(-1)^{\frac{z^\pm+1}{2}}\,\partial_x^{z^\pm-1}\mathcal{J}_\pm\,. \tag{H.5.11}$$

Replacing (H.5.8) in the previous equation, we arrive to the following expression (in frequency space)

$$\mathcal{I}_\pm(x,\omega) = \mathcal{I}_\pm + \frac{iK}{4\pi}v(-1)^{\frac{z+1}{2}}\partial_x^{z^\pm-1}\int dx'\left(\int\frac{dk}{2\pi}\frac{e^{ik(x-x')}}{\omega\pm\omega_k+i\epsilon}\right)\partial_{x'}\mu(x,x';\omega)\,. \tag{H.5.12}$$

Integrating by parts the chemical potential, we obtain

$$\mathcal{I}_\pm(x,\omega) = \mathcal{I}_\pm - \frac{iK}{4\pi}v(-1)^{\frac{z^\pm+1}{2}}\partial_x^{z^\pm-1}\int dx'\left(\partial_{x'}\int\frac{dk}{2\pi}\frac{e^{ik(x-x')}}{\omega\pm\omega_k+i\epsilon}\right)\mu(x,x';\omega) \tag{H.5.13a}$$

$$\equiv \mathcal{I}_\pm^{\text{in}} - \frac{iK}{4\pi}v(-1)^{\frac{z^\pm+1}{2}}\partial_x^{z^\pm-1}\int dx'\,\partial_{x'}F^\pm(x-x',\omega)\mu(x,x';\omega)\,. \tag{H.5.13b}$$

If $y = x-x'$, then we can rewrite the above expression as

$$\mathcal{I}_\pm(x,\omega) = \mathcal{I}_\pm^{\text{in}} - \frac{iK}{4\pi}v(-1)^{\frac{z^\pm+1}{2}}\int dy\,\partial_y^{z^\pm}F^\pm(y,\omega)\mu(x,x-y;\omega)\,. \tag{H.5.13c}$$

Explicitly

$$\mathcal{I}_\pm(x,\omega) = \mathcal{I}_\pm^{\text{in}} - \frac{iK}{4\pi}v(-1)^{\frac{z+1}{2}}\int\frac{dy\,dk}{2\pi}\frac{i^{z^\pm}k^{z^\pm}e^{iky}}{\omega\pm\omega_k+i\epsilon}\mu(x,x-y;\omega)\,. \tag{H.5.13d}$$

Noting that $i^{z^\pm} = -i(-1)^{\frac{z^\pm+1}{2}}$, we can write

$$\mathcal{I}_\pm(x,\omega) = \mathcal{I}_\pm^{\text{in}} + \frac{K}{4\pi}\int \frac{dy\,dk}{2\pi}\frac{\omega_k e^{iky}}{\omega \pm \omega_k + i\epsilon}\mu(x, x-y;\omega)\,, \tag{H.5.13e}$$

or

$$\mathcal{I}_\pm(\Delta x,\omega) = \mathcal{I}_\pm^{\text{in}} + \frac{K}{4\pi}\int \frac{dy\,dk}{2\pi}\left(\frac{\omega_k \pm (\omega+i\epsilon) \mp (\omega+i\epsilon)}{\omega \pm \omega_k + i\epsilon}\right) e^{iky}\mu(x, x-y;\omega) \tag{H.5.13f}$$

$$= \mathcal{I}_\pm^{\text{in}} + \frac{K}{4\pi}\int \frac{dy\,dk}{2\pi}\left(\pm 1 \mp \frac{\omega+i\epsilon}{\omega \pm \omega_k + i\epsilon}\right) e^{iky}\mu(x, x-y;\omega) \tag{H.5.13g}$$

$$= \mathcal{I}_\pm^{\text{in}} \pm \frac{K}{8\pi^2}\mu(x,x;\omega) \mp \frac{K}{4\pi}(\omega+i\epsilon)\int dy\, F^\pm(y,\omega)\mu(x,x-y;\omega)\,, \tag{H.5.13h}$$

which is reminiscent of Eq. (7.2.11). If we choose μ according to (7.2.16),

$$\tilde{\mu}(x',\omega) = V_L\Theta(x_L - x') + V_R\Theta(x' - x_R)\,, \tag{H.5.14}$$

(see Fig. 7.1) then the vacuum term proportional to $\mu(x,x;\omega)$ vanishes and we recover the same boundary result at the DC limit, but now from the bulk perspective.

H.6 Chiral bosons two-terminal conductance

In this appendix, we show the details in the obtention of Eq. (7.3.28). In particular, we use results from Appendix B.

Starting from Eq. (H.5.8),

$$\mathcal{J}_\pm(t,x) = \mathcal{J}_\pm^{\text{in}} \pm \frac{K}{4\pi}\int dt'\,dx' \int dk\,d\omega \frac{e^{ik\Delta x}e^{-i\omega\Delta t}}{\omega \pm \omega_k + i\epsilon}\partial_{x'}\mu(t',x')\,,$$

and introducing the chemical potential given in (7.2.16), the latter equation reads as

$$\begin{aligned}\mathcal{J}_\pm(t,x) &= \mathcal{J}_\pm^{\text{in}} \pm \frac{K}{4\pi}\int dt'\,dx' \int dk\,d\omega \frac{e^{i\left[k(x-x')-\omega(t-t')\right]}}{\omega \pm \omega_k + i\epsilon}\times \\ &\qquad\qquad \times\,\delta(t')\left[-V_L\delta(x'-x_L)+V_R\delta(x'-x_R)\right] &\text{(H.6.1a)}\\ &= \mathcal{J}_\pm^{\text{in}} \pm \frac{K}{4\pi}\int dx' \int dk\,d\omega \frac{e^{ik(x-x')}e^{-i\omega t}}{\omega \pm \omega_k + i\epsilon}\left[-V_L\delta(x'-x_L)+V_R\delta(x'-x_R)\right] &\text{(H.6.1b)}\end{aligned}$$

Using the following Fourier integral

$$\int d\omega \frac{e^{-i\omega\Delta t}}{\omega \pm \omega_k + i\epsilon} = -2i\pi\Theta(t)e^{\pm i\omega_k t}\,, \tag{H.6.2}$$

then $\mathcal{J}_\pm$ reads

$$\begin{aligned}\mathcal{J}_\pm(t,x) &= \mathcal{J}_\pm^{\text{in}} \mp \frac{K}{8\pi^2}\Theta(t)\int dx' \int dk\, e^{ik(x-x')}e^{\pm i\omega_k t}\left[V_L\delta(x'-x_L)-V_R\delta(x'-x_R)\right] &\text{(H.6.3a)}\\ &= \mathcal{J}_\pm^{\text{in}} \mp \frac{K}{8\pi^2}\Theta(t)\int dk\left(V_L e^{ik(x-x_L)} - V_R e^{ik(x-x_R)}\right)e^{\pm i\omega_k t}\,. &\text{(H.6.3b)}\end{aligned}$$

According to the integral definition (B.0.1) of Appendix B, we can recognize the function

$$f_z(t,x-x') = \int_{-\infty}^{\infty} dk\, e^{i\left[k(x-x')\pm\omega_k t\right]}\,, \tag{H.6.4}$$

which, by virtue of Eq. (B.0.2) admits a relationship with the higher-order Airy function of the first kind, Ai_z, as

$$f_z(t,x-x') = \frac{2\pi}{(vzt)^{1/z}}\text{Ai}_z\left[\pm\frac{x-x'}{(vzt)^{1/z}}\right]\,. \tag{H.6.5}$$

Thus, we can write

$$\mathcal{J}_{\pm}(t,x) = \mathcal{J}_{\pm}^{\text{in}} \mp \frac{K}{8\pi^2}\Theta(t)\left[V_L f_z^{\pm}(t,x-x_L) - V_R f_z^{\pm}(t,x-x_R)\right] \tag{H.6.6a}$$

$$= \mathcal{J}_{\pm}^{\text{in}} \mp \frac{K}{4\pi(vz^{\pm}t)^{1/z^{\pm}}}\Theta(t)\left\{V_L \text{Ai}_{z^{\pm}}\left[\pm\frac{x-x_L}{(vz^{\pm}t)^{1/z^{\pm}}}\right] - V_R \text{Ai}_{z^{\pm}}\left[\pm\frac{x-x_R}{(vz^{\pm}t)^{1/z^{\pm}}}\right]\right\}, \tag{H.6.6b}$$

in agreement with (7.3.28).

Now, for $t > 0$, and by considering Eqs. (7.3.30)

$$\mathcal{J}_{\pm}^{\text{out}} = \pm\frac{K}{4\pi}\partial_x\varphi_{\pm}^{\text{out}}, \quad \mathcal{I}_{\pm}^{\text{out}} = \mp\frac{K}{4\pi}\partial_t\varphi_{\pm}^{\text{out}}, \tag{H.6.7}$$

and also (B.3.1),

$$\frac{\partial}{\partial x}\psi_z(x) = \text{Ai}_z(x),$$

we can write (H.6.6b) as a sum of total partial derivatives

$$\pm\frac{K}{4\pi}\partial_x\varphi_{\pm}^{\text{out}} = \pm\frac{K}{4\pi}\partial_x\varphi_{\pm}^{\text{in}} - \frac{K}{4\pi}\partial_x\left[V_L\psi_z^{\pm}\left(\pm\frac{x-x_L}{(vz^{\pm}t)^{1/z^{\pm}}}\right) - V_R\psi_{z^{\pm}}\left(\pm\frac{x-x_R}{(vz^{\pm}t)^{1/z^{\pm}}}\right)\right] \tag{H.6.8a}$$

$$\therefore \varphi_{\pm}^{\text{out}} = \varphi_{\pm}^{\text{in}} \mp V_L\psi_z^{\pm}\left(\pm\frac{x-x_L}{(vz^{\pm}t)^{1/z^{\pm}}}\right) \pm V_R\psi_z^{\pm}\left(\pm\frac{x-x_R}{(vz^{\pm}t)^{1/z^{\pm}}}\right), \tag{H.6.8b}$$

in concordance with (7.3.33). Therefore,

$$\delta\varphi_{\pm} = \mp V_L\psi_z^{\pm}\left(\pm\frac{x-x_L}{(vz^{\pm}t)^{1/z^{\pm}}}\right) \pm V_R\psi_z^{\pm}\left(\pm\frac{x-x_R}{(vz^{\pm}t)^{1/z^{\pm}}}\right). \tag{H.6.9}$$

Now, the two-terminal conductance σ is postulated in Eq. (7.3.34) as

$$\sigma = \frac{1}{\Delta V}\int_{-\infty}^{\infty} dt\,(\delta\langle\mathcal{I}_+\rangle + \delta\langle\mathcal{I}_-\rangle), \tag{H.6.10a}$$

thus, with the operator $\mathcal{I}_{\pm}$ defined in (H.6.7), we obtain

$$= \frac{1}{\Delta V}\int_{-\infty}^{\infty} dt\left(-\frac{K}{4\pi}\partial_t\delta\varphi_+ + \frac{K}{4\pi}\partial_t\varphi_-\right) \tag{H.6.10b}$$

$$= -\frac{K}{4\pi\Delta V}\int_{-\infty}^{\infty} dt\,\partial_t(\delta\varphi_+ - \delta\varphi_-) \tag{H.6.10c}$$

$$= -\frac{K}{4\pi\Delta V}\left\{[\delta\varphi_+(t=\infty) - \delta\varphi_-(t=\infty)] - [\delta\varphi_+(t=-\infty) - \delta\varphi_-(t=-\infty)]\right\}. \tag{H.6.10d}$$

Because chiral bosons vanishes in $t \to -\infty$, only the first two term survives, allowing us to write

$$\sigma = -\frac{K}{4\pi\Delta V}\left[\delta\varphi_+(t=\infty) - \delta\varphi_-(t=\infty)\right] , \tag{H.6.10e}$$

with,

$$\delta\varphi_\pm(t=\infty,x) = \mp V_L \lim_{t\to\infty} \psi_z^\pm\left(\pm\frac{x-x_L}{(vz^\pm t)^{1/z^\pm}}\right) \pm V_R \lim_{t\to\infty} \psi_z^\pm\left(\pm\frac{x-x_R}{(vz^\pm t)^{1/z^\pm}}\right) \tag{H.6.11a}$$

$$= \mp V_L\psi_z^\pm(0) \pm V_R\psi_z^\pm(0) \tag{H.6.11b}$$

$$= \mp\frac{V_L}{2z^\pm} \pm \frac{V_R}{2z^\pm} = \mp\frac{\Delta V}{2z^\pm} , \tag{H.6.11c}$$

in agreement with (7.3.35).