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The method LPLDE applied to high orders

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Abstract. This notes expose results of applying the LPLDE method to some nonlinear periodic problems as the Duffing equation, the sextic and octic oscillators and the Van der Pol equation.

1. Introduction

Relevance of new approximation methods to find the solutions to nonlinear problems comes along with the role this kind of phenomena plays in nature. The LPLDE method, proposed by Amore and Aranda in [1], works by combining the ideas of two methods, the Lindsted-Poincare method (LP) and the Linear Delta Expansion method (LDE).

The first rescales the time and expresses the solution and the (unknown) frequency as series in a small parameter ε. After substituting such expansions in the problem and collecting power like terms of ε, is necessary to solve recursively the resulting set of equations. Corrections of the frecuency are chosen in such manner that secular terms are avoided when each equation is solved.

The second is a nonperturbative method, whose idea is to transform the insoluble original equation into a soluble one dependent on two arbitrary parameters: λ and δ, in such a way that for $\delta = 1$ the transformed equation becomes again the original. Dependence on λ is minimized by applying the Principle of Minimal Sensibility (PMS).

2. Duffing equation

We first consider the motion equation of a particle with unitary mass in a potential of the form

$$V(x) = \frac{x^2}{2} + \mu \frac{x^{2N}}{2N}. \tag{1}$$

For $N = 3$ and $N = 4$ such equations correspond to the sextic and octic oscillators, for $N = 2$ we get the Duffing equation

$$\frac{d^2x}{dt^2}(t) + x(t) = -\mu x^3(t). \tag{2}$$

According to the LPLDE method we rearrange the original equation by rescaling time, $\tau = \Omega t$, and introducing λ and δ as follows

$$\Omega^2 \frac{d^2x}{dt^2}(\tau) + (1+\lambda^2)x(\tau) = \delta \left[-\mu x^3(\tau) + \lambda^2 x(\tau) \right] \tag{3}$$

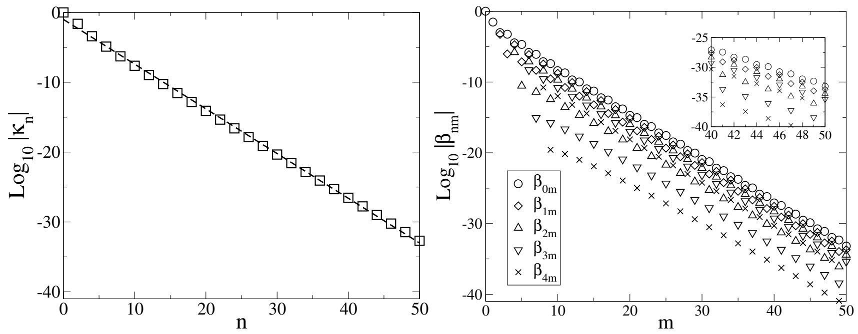

Figure 1. Numerical coefficients κ_n and β_n in Equation (9) and Equation (10) decay exponentially with the order of expansion. The line corresponds to the fitting $\kappa_n = 0.0663e^{-1.46225n}$.

Notice that if $\delta = 1$ we get back Equation Equation (2). Now if we substitute the expressions

$$\Omega^2 = \sum_{n=0}^{N_{max}} \delta^n \alpha_n, \qquad x(t) = \sum_{n=0}^{N_{max}} \delta^n x_n(\tau), \tag{4}$$

by collecting terms proportional to δ^k we obtain the equation of order k, $(k = 0 \ldots N_{max})$, and by imposing the resonant contributions to order n to vanish we obtain the coefficients α_n.

By proceeding in this manner, once the expressions for the solution and frequency to the order of approximation required N_{max} have been obtained, we calculate the optimal value of λ by applying the PMS condition $\partial\Omega^2/\partial\lambda = 0$ and solving it for λ. Although this last equation could not be solved analytically, the optimal value of λ (denoted λ_{PMS}) to the third order

$$\lambda_{PMS} = \sqrt{\frac{3\mu A^2}{4}}, \tag{5}$$

approximates very well the optimal values of λ to any other order, this allows to obtain fully analytical expressions.

Since the solutions $x_n(\tau)$ have the form

$$x_n(\tau) = \sum_{m=0}^{n} \bar{c}_{nm} \cos(2m+1)\tau \tag{6}$$

we can write the expression in Equation (4) equivalently as

$$x_{approx}(t) = \sum_{n=0}^{N_{max}} c_n^{(approx)} \cos[(2n+1)\Omega_{(approx)}t] \tag{7}$$

where

$$c_n^{(approx)} = \sum_{m=0}^{N_{max}} \bar{c}_{nm}. \tag{8}$$

The coefficients α_n, as well as the approximate Fourier coefficients obtained with the LPLDE method, follow a pattern. The α_n can be written as

$$\alpha_0 = 1 + \frac{3}{4}A^2\mu, \quad \alpha_{2n} = -\frac{\kappa_{2n}(A^2\mu)^{2n}}{(1+\frac{3}{4}A^2\mu)^{2n-1}}, \quad \alpha_{2n+1} = 0 \tag{9}$$

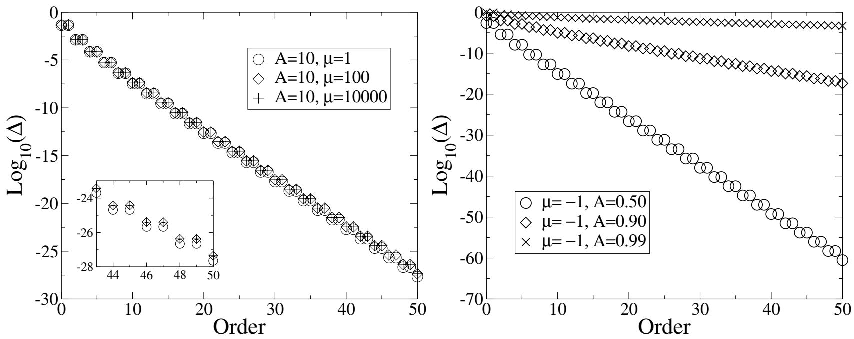

Figure 2. Logarithm of the error defined in Equation (11). Left: $A = 10$ and positive μ. Right: Negative μ and A=0.5,0.9 and 0.99.

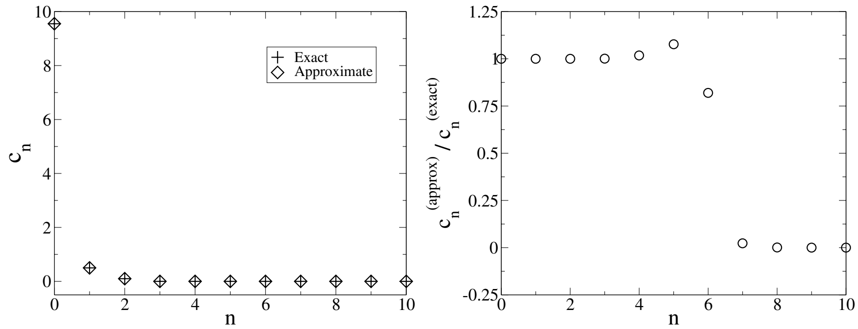

Figure 3. Comparison of the Fourier coefficients for the Duffing case with $\mu = 10^4$ and $A = 10$. Right: Ratio of the exact and the approximate coefficients.

whereas the $\bar{c}_{nm}$ coefficients can be written as

$$\bar{c}_{nm} = \frac{\beta_{nm} A (A^2\mu)^n}{(1 + \frac{3A^2\mu}{4})^n}. \tag{10}$$

Both, κ_{2n} and β_{nm}, are purely numerical coefficients decaying exponentially with the order of the expansion (Figure 1).

We now define the error

$$\Delta = \left| \frac{\Omega^2_{approx} - \Omega^2_{exact}}{\Omega^2_{exact}} \right| \times 100. \tag{11}$$

For the case of positive μ, corresponding to a single well potential, the error is practically unaffected with the size of μ. This could have been expected since the coefficients α_n go to 0 faster than κ_n for $\mu > 0$ (Figure 2, left).

Coefficients $c_n^{(approx)}$ of the approximate solution Equation (7) are compared with the Fourier coefficients of the exact numerical solution (Figure 3)

$$x_{exact}(t) = \sum_{n=0}^{\infty} c_n^{(exact)} \cos[(2n+1)\Omega_{(exact)} t]. \tag{12}$$

It can be noticed that the first can be reproduced with great accuracy. The coefficients with higher frequency, although poorly approximated, do not influence too much in the approximation because of their small contributions.

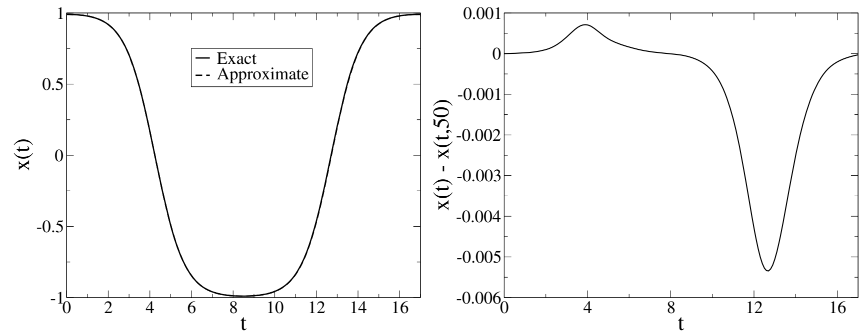

Figure 4. Left: Exact (numerical) and approximate solutions computed up to order 50, with $\mu = -1$ and $A = 0.99$. Right: The difference of exact and approximate solutions $x(t) - x(t, 50)$.

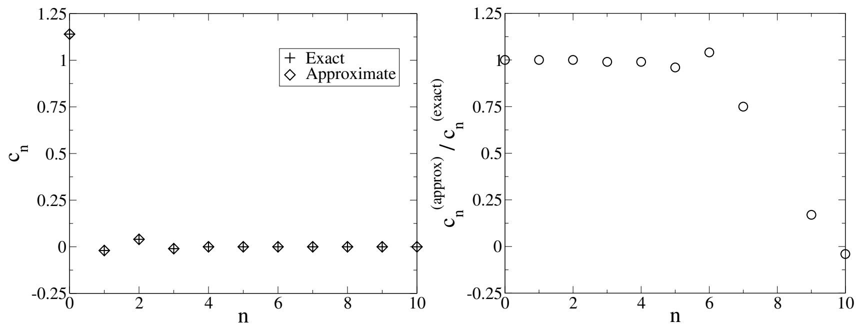

Figure 5. Comparison of the Fourier coefficients for the Duffing case with $\mu = -1$ and $A = 0.99$. Right: Ratio of the exact and the approximate coefficients.

The case of negative μ corresponds to a double wall potential, with maxima at amplitudes ± 1, the points of unstable equilibrium, and oscillatory behaviour exist only for amplitudes between them.

Although in this case the error decaying rate is more dependent to the amplitude (Figure 2-right), the approximation is good as well (Figure 4). This dependent behaviour could have been expected, because the size of the denominator in equation is smaller and changing with the order.

3. Sextic and Octic Oscillators

An analogous analysis for sextic and octic oscillators shows that the LPLDE method works very well also for these anharmonic potentials, for both positive and negative values of μ. Computations are carried out similarly to those of the Duffing equation.

Figure 6 shows the decaying behaviour of the error for the octic oscillator and a comparison of Fourier coefficients; sextic oscillator analogous plots are alike to these but not displayed here.

As for the Duffing case, the optimal values of λ to any order, once the PMS condition has been applied, are well approximated with the values of λ to the third order of expansion

$$\lambda_{sextic} = \sqrt{\frac{211 A^4 \mu}{312}}, \qquad \lambda_{octic} = \sqrt{\frac{10885 A^6 \mu}{16896}} \tag{13}$$

obtaining fully analytical expressions when they are used, like is for the Duffing equation.

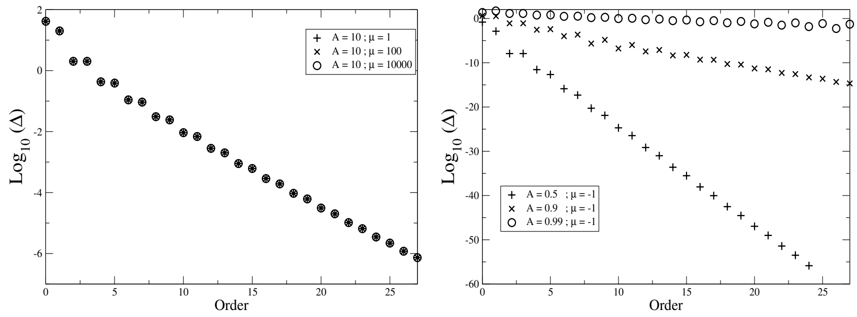

Figure 6. Logarithm of the error Equation (11) for the octic oscillator. Left: For A=10 and positive values of μ. Right: For $\mu = -1$ and $A = 0.5, 0.9$ and 0.99.

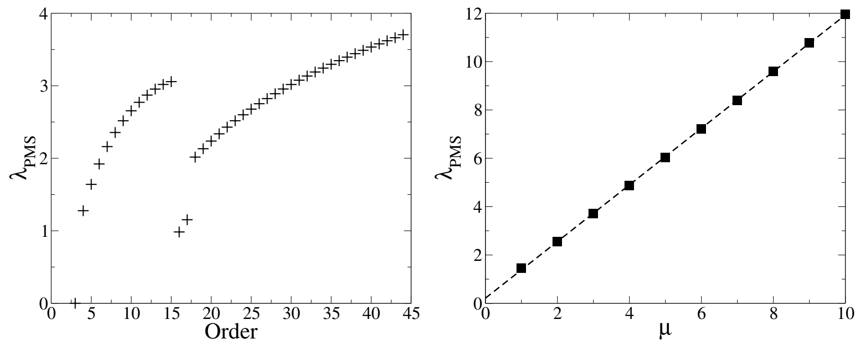

Figure 7. Left: For fixed $\mu = 3$ the optimal λ is strongly order dependent. Right: Optimal value of λ as function μ, to order 44. The dashed line is the fit $\lambda = 0.21599 + 1.17166\mu$

4. Van der Pol equation

Now we apply the LPLDE method to the Van der Pol equation which corresponds to a nonconservative system: the term on the right either diminishes or enhances the oscillations depending upon the size of μ.

$$\ddot{x} + x = \mu(1 - x^2)\dot{x} \tag{14}$$

In order to tackle this problem we change the original equation by rescaling time and by introducing arbitrary λ and δ,

$$\Omega^2 \ddot{x}(\tau) + (1 + \lambda)x(\tau) = \delta \left[\mu\Omega(1 - x^2)\dot{x} + \lambda^2 x(\tau)\right], \tag{15}$$

where derivatives are now with respect to τ. As in the previous cases we assume the expansions

$$\Omega = \sum_{n=0}^{N_{max}} \delta^n \gamma_n, \qquad x(\tau) = \sum_{n=0}^{N_{max}} \delta^n x_n(\tau) \tag{16}$$

and substitute them into Equation (15).

Then we make use of δ as order-selecting parameter to get a set of linear equations which, contrary to previously treated cases, contain not only cosine terms, but also sine terms. As before, coefficients γ_n are fixed by imposing that resonant terms at a given order to vanish.

Unlike the Duffing and the oscillators cases, where we estimate the optimal value of λ to any order with the optimal to the third order, for a given μ, the optimal value of λ is strongly order

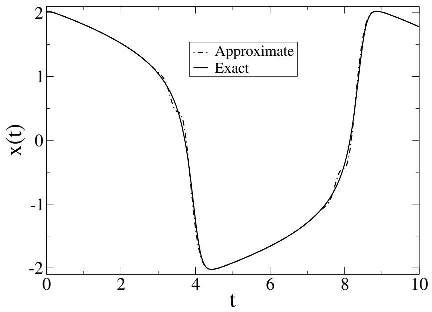

Figure 8. Approximate and exact solutions of the Van der Pol equation computed to order 44 for $\mu = 3$.

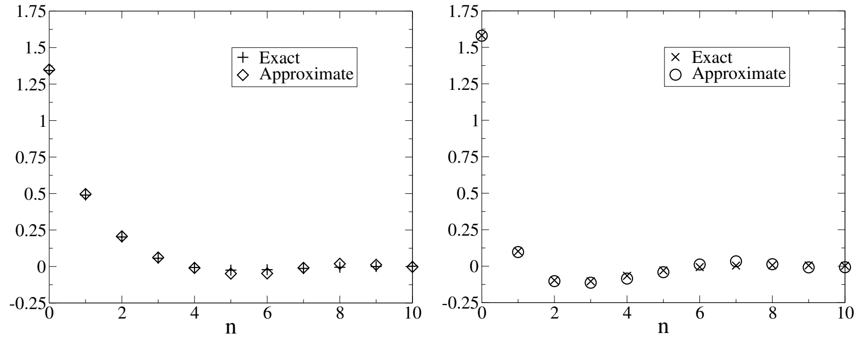

Figure 9. Comparison of Fourier coefficients to the order 44 for $\mu = 3$. Left: Cosine coefficients. Right: Sine coefficients.

dependent, as it can be observed in the left plot of Figure 7. However, as shown on the right plot, at a fixed order of expansion (44 in this case) the λ_{PMS} is linearly μ dependent. This allows to obtain expressions for both the period and the solution with the value of λ as the only numerical estimated term.

We get a maximum error of 12% with $\mu = 10$ for periods obtained with the LPLDE method compared with exact numerical results from [7]. Smaller errors are obtained with smaller μ's.

Exact (numerical) and approximated solutions are plotted in Figure 8 for $\mu = 3$. We can observe there that the LPLDE method applied to order 44 gives a quite good approximation of the solution which can be improved by taking the expansion to higher orders. Notice that for such values of μ the LP method is not applicable. Finally, Figure 9 shows a comparison of Fourier coefficients of the approximate and exact solutions.

The nonperturbative nature of LDE, in the cases presented, allows the LPLDE method to reach nonperturbative regimes and deal with large nonlinearities, obtaining errors even smaller than those obtained with the LP only.

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