Published by IOP Publishing for SISSA

Received: *May 26, 2009*
Accepted: *July 30, 2009*
Published: *August 18, 2009*

# Pressure to order $g^8 \log g$ of massless $\phi^4$ theory at weak coupling

**Jens O. Andersen,**[a,1] **Lars T. Kyllingstad**[b] **and Lars E. Leganger**[b]

[a] *Niels Bohr International Academy, Niels Bohr Institute, Blegdamsvej 17, DK-2100 Copenhagen, Denmark*

[b] *Department of Physics, Norwegian University of Science and Technology, Høgskoleringen 5, N-7491 Trondheim, Norway*

*E-mail:* andersen@tf.phys.ntnu.no, lars.kyllingstad@ntnu.no, lars.leganger@ntnu.no

Abstract: We calculate the pressure of massless $\phi^4$-theory to order $g^8 \log(g)$ at weak coupling. The contributions to the pressure arise from the hard momentum scale of order $T$ and the soft momentum scale of order $gT$. Effective field theory methods and dimensional reduction are used to separate the contributions from the two momentum scales: The hard contribution can be calculated as a power series in $g^2$ using naive perturbation theory with bare propagators. The soft contribution can be calculated using an effective theory in three dimensions, whose coefficients are power series in $g^2$. This contribution is a power series in $g$ starting at order $g^3$. The calculation of the hard part to order $g^6$ involves a complicated four-loop sum-integral that was recently calculated by Gynther, Laine, Schröder, Torrero, and Vuorinen. The calculation of the soft part requires calculating the mass parameter in the effective theory to order $g^6$ and the evaluation of five-loop vacuum diagrams in three dimensions. This gives the free energy correct up to order $g^7$. The coefficients of the effective theory satisfy a set of renormalization group equations that can be used to sum up leading and subleading logarithms of $T/gT$. We use the solutions to these equations to obtain a result for the free energy which is correct to order $g^8 \log(g)$. Finally, we investigate the convergence of the perturbative series.

Keywords: NLO Computations, Thermal Field Theory, Renormalization Group

ArXiv ePrint: 0903.4596

[1] On leave from: Department of Physics, Norwegian University of Science and Technology, N-7491 Trondheim, Norway



doi:10.1088/1126-6708/2009/08/066

JHEP08(2009)066

## Contents



JHEP08(2009)066

## 1 Introduction

In recent years there has been significant progress in our understanding of thermal field theories in equilibrium [1–4]. Part of the progress is based on the developement of the calculational technology necessary to perform loop calculations beyond the first correction. The motivation to carry out such difficult higher-order calculations of e.g. the pressure in thermal QCD is its relevance to heavy-ion collisions and the early universe. The pressure in nonabelian gauge theories has been calculated perturbatively through order $g^4$ in ref. [5, 6], to order $g^5$ in refs. [7, 8], and to order $g^6 \log(g)$ in ref. [9]. There are three momentum scales that contribute to the pressure in thermal QCD - hard momenta of order $T$, soft momenta

of order $gT$, and supersoft momenta of order $g^2T$. The next order — order $g^6$ — is the first order at which all three momentum scales contribute to the pressure and it is also the order at which perturbation theory breaks down due to infrared divergences [10, 11]. The pressure contains a nonperturbative contribution from the supersoft scale that can be estimated numerically [12–14]. It also contains a presently unknown contribution from the hard scale. This contribution can be calculated by evaluating highly nontrivial four-loop vacuum diagrams with unresummed propagators. As a step in this direction, Gynther, Laine, Schröder, Torrero, and Vuorinen considered the simpler problem of $\phi^4$-theory at finite temperature and calculated the free energy to order $g^6$ [15]. A difficult part of the calculation was to evaluate the four-loop triangle sum-integral, using the techniques developed by Arnold and Zhai in refs. [5, 6].

In hot field theories at weak coupling, the momentum scales in the plasma are well separated and it is advantageous to use effective field theory methods to organize the calculations of the pressure into separate contributions from the hard, soft and supersoft scales. The basic idea is that the mass of the nonzero Matsubara modes are of order $T$ and heavy. Since these modes are heavy, they decouple from the light modes, i.e. the static Matsubara modes. In particular, all fermionic modes decouple since their masses are always of order $T$. The contributions from the nonzero Matsubara modes to thermodynamic quantities can be calculated using bare propagators and are encoded in the parameters of the effective theory. Integrating out the hard scale $T$, i.e. integrating out the nonzero Matsubara frequencies, leaves us with an effective dimensionally reduced theory for the scales $gT$ and $g^2T$ [8]. In the case of QCD, the effective theory is an SU($N$) gauge theory coupled to an adjoint Higgs. The process is known as dimensional reduction [16–20]. The next step is to construct a second effective theory for the scale $g^2T$ by integrating out the scale $gT$ from the problem [8]. It amounts to integrating out the adjoint Higgs and this step can also be made in perturbation theory. This effective theory is a nonabelian gauge theory in three dimensions, which is confining with a nonperturbative mass gap of order $g^2T$ [11]. This theory must be treated nonperturbatively and gives the nonperturbative contribution to the pressure mentioned above.

In the present paper we consider the thermodynamics of massless $\phi^4$-theory and calculate the pressure through order $g^8\log(g)$ in a weak-coupling expansion using effective field theory. Calculations in scalar field theory are simplified by the fact that the supersoft scale $g^2T$ does not appear and so we only need to construct a single effective theory for the soft scale $gT$. This theory is infrared safe to all orders in perturbation theory due to the generation of a thermal mass of order $gT$. Compared to the $g^6$-calculations of ref. [15], the next order requires the matching of the mass parameter to three loops and the evaluation of some five-loop vacuum diagrams in the effective theory. The matching involves a nontrivial three-loop sum-integral that was calculated recently in ref. [21].

The paper is organized as follows. In section II, we briefly discuss effective field theory and determine the coefficients of the dimensionally reduced theory. In section III, we use the effective theory and calculate the soft contributions to the pressure. In section IV, we present and discuss our final results for the pressure. In section V, we summarize. In appendix A and B, we list the necessary sum-integrals and integrals. In appendix C, we



JHEP08(2009)066

calculate explicitly some of the new three-dimensional integrals that we need.

## 2 Effective field theory

In this section, we briefly discuss the three-dimensional effective field theory and the matching procedure used to determine its coefficients. For a detailed discussion, see e.g. refs. [19, 20].

The Euclidean Lagrangian density for a massless scalar field with a $\Phi^4$-interaction is

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \Phi)^2 + \frac{g^2}{24}\Phi^4 + \Delta\mathcal{L} \,, \tag{2.1}$$

where $g$ is the coupling constant and $\Delta\mathcal{L}$ includes counterterms. This term reads

$$\Delta\mathcal{L} = \frac{1}{2}\Delta Z_\Phi (\partial_\mu \Phi)^2 + \frac{1}{24}\Delta g^2 \Phi^4 \,. \tag{2.2}$$

In the present case we need the counterterm $\Delta g^2$ to next-to-leading order i $g^2$. It is given by

$$\Delta g^2 = \left[\frac{3}{2\epsilon}\alpha + \left(\frac{9}{4\epsilon^2} - \frac{17}{12\epsilon}\right)\alpha^2\right] g^2 \,, \tag{2.3}$$

where $\alpha = g^2/(4\pi)^2$. We denote by $\phi(x)$ the field in the effective theory. It can be approximately, i.e. up to field redefinitions, be identified with zero-frequency mode of the field $\Phi$ in the original theory. The Lagrangian of the effective theory can be then be written as

$$\mathcal{L}_{\text{eff}} = \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{g_3^2}{24}\phi^4 + \cdots \,, \tag{2.4}$$

where $m$ is the mass of the theory and $g_3^2$ is the quartic coupling. The dots indicate an infinite series of higher-order operators consistent with the symmetries, such as rotational invariance and the discrete symmetry $\phi \to -\phi$. In eq. (2.4), we have omitted a coefficient $f$ of the unit operator. Its interpretation is that it gives the contribution to the free energy from the hard scale $T$.

For the calculation of the pressure to order $g^8\log(g)$, we need to know $f$ and the mass parameter $m^2$ to order $g^6$ and the coupling constant $g_3^2$ to order $g^4$, i.e. we consider $\phi^4$-theory in three spatial dimensions.[1] This theory is superrenormalizable and only the mass needs renormalization [22]. The parameters in the effective Lagrangian (2.4) are determined by calculating static correlation functions in the two theories at long distances $R$, i.e. $R \gg 1/T$, and demanding that they be the same [19]. In the matching calculations, we are employing *strict perturbation theory* [19]. This amounts to doing perturbative calculations in power series in $g^2$ in which we treat the mass parameter as a perturbation in the effective theory. The Lagrangian is therefore split into a free and an interacting part according to

$$\mathcal{L}_{\text{eff}}^{\text{free}} = \frac{1}{2}(\nabla\phi)^2 \,, \tag{2.5}$$

$$\mathcal{L}_{\text{eff}}^{\text{int}} = \frac{1}{2}m^2\phi^2 + \frac{g_3^2}{24}\phi^4 + \cdots \,. \tag{2.6}$$

[1] Power counting tells one that the operator $(\phi\nabla\phi)^2$ contributes to the free energy first at order $g^8$.

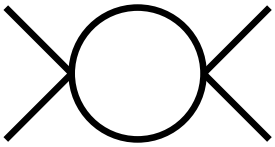

**Figure 1**. One-loop Feynman graph that contributes to the coupling $g_3^2$ in the effective theory.

Strict perturbation theory gives rise to infrared divergences in the calculation that physically are cut off by the generation of a thermal mass $m$. The same infrared divergences appear in the loops in the full theory and so they cancel in the matching calculations. The incorrect treatment of the infrared divergences and the physics on the scale $gT$ is not problematic since this will be taken care of by calculations in the effective theory. The matching calculations treat the physics on the hard scale correctly and the physics on that scale is encoded in the parameters of the three-dimensional effective Lagrangian.

However, the matching calculations of the parameters in $\mathcal{L}_{\text{eff}}$ are complicated by ultraviolet divergences. Those divergences that are associated with the full four-dimensional theory are removed by renormalization of the coupling constant $g$. The remaining divergences are cancelled by the extra counterterms that are determined by the ultraviolet divergences in the effective theory. These divergences are regulated by introducing a cutoff $\Lambda$. The cutoff $\Lambda$ can be thought of as an arbitrary factorization scale that separates the scale $T$ from the scale $gT$ (or smaller) which can be treated in the effective theory [19]. The parameters in the effective theory therefore depend on the cutoff $\Lambda$ in order to cancel the $\Lambda$-dependence of the loop integrals in the effective theory.

### 2.1 Coupling constant

To leading order in the coupling $g^2$, we can simply read off the coupling $g_3^2$ from the Lagrangian of the full theory. Making the replacement $\Phi \to \sqrt{T}\phi$ in the Lagrangian (2.1) and comparing $\int_0^\beta d\tau\, \mathcal{L}$ with $\mathcal{L}_{\text{eff}}$, we conclude that $g_3^2 = g^2 T$. The one-loop graph needed for the matching of the coupling $g_3^2$ to next-to-leading order in $g^2$ is shown in figure 1. Since the loop correction vanishes in the effective theory due to the fact that we are using massless propagators, the matching equation reduces to

$$g_3^2 = g^2 T - \frac{3}{2} g^4 T \sum\!\!\!\!\!\!\int_P \frac{1}{P^4} + \Delta_1 g^2 T \,, \tag{2.7}$$

where $\Delta_1 g^2$ is the order-$g^4$ coupling constant counterterm in eq. (2.3). After renormalization, we find

$$g_3^2(\Lambda) = g^2(\mu) T \left[ 1 - \frac{3g^2}{(4\pi)^2} \left( \log \frac{\mu}{4\pi T} + \gamma_E \right) - \right. \\ \left. - \frac{3g^2}{(4\pi)^2} \left( \log^2 \frac{\mu}{4\pi T} + 2\gamma_E \log \frac{\mu}{4\pi T} + \frac{\pi^2}{8} - 2\gamma_1 \right) \epsilon \right] \,, \tag{2.8}$$

where $g^2 = g^2(\mu)$ is the coupling constant at the scale $\mu$ in the $\overline{\text{MS}}$ scheme and we have kept the order-$\epsilon$ terms in $g_3^2$ for later use. We have used the renormalization group equation

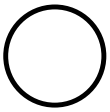

**Figure 2**. One-loop vacuum diagram.

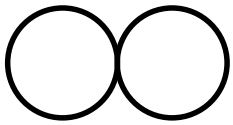

**Figure 3**. Two-loop vacuum diagram.

for the running coupling constant $g^2$,

$$\mu\frac{\partial}{\partial\mu}\alpha = 3\alpha^2 - \frac{17}{3}\alpha^3 \,, \tag{2.9}$$

to change the scale from $\Lambda$ to $\mu$. The right-hand side of eq. (2.8) is independent of $\Lambda$. In fact, since the coupling $g_3^2$ does not require renormalization in three dimensions, it satisfies the renormalization group equation

$$\Lambda\frac{\partial}{\partial\Lambda}g_3 = 0 \,. \tag{2.10}$$

### 2.2 Coefficient of unit operator

The partition function in the full theory is given by the path integral

$$Z = \int \mathcal{D}\Phi\, e^{-\int_0^\beta d\tau \int d^3x \mathcal{L}} \,, \tag{2.11}$$

and the pressure is then given by $\mathcal{P} = T\log\mathcal{Z}/V$, where $V$ is the volume of the system. In terms of the effective theory, the partition function can be written as

$$\mathcal{Z} = e^{-fV}\int \mathcal{D}\phi\, e^{-\int d^3x \mathcal{L}_{\rm eff}} \,. \tag{2.12}$$

The matching then yields

$$\log\mathcal{Z} = -fV + \log\mathcal{Z}_{\rm eff} \,, \tag{2.13}$$

where $\mathcal{Z}_{\rm eff}$ is the partitition function of the three-dimensional theory. Equivalently, we can write $\mathcal{F} = \mathcal{F}_{\rm hard} + \mathcal{F}_{\rm soft}$, where $\mathcal{F}_{\rm hard} = fT$ and $\mathcal{F}_{\rm soft} = -T\log\mathcal{Z}_{\rm eff}/V$. Now since calculations in strict perturbation theory in the effective theory is carried out using bare propagators, there is no scale in the vacuum graphs. This implies that they vanish in dimensional regularization and that $\log\mathcal{Z}_{\rm eff} = 0$. Eq. (2.13) then tells us that $f$ is given by a strict loop expansion in four dimensions.

The vacuum diagrams through four loops are shown in figures 2–5.



JHEP08(2009)066

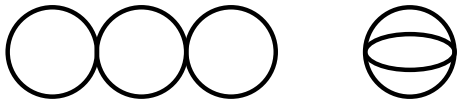


**Figure 4**. Three-loop vacuum diagrams.

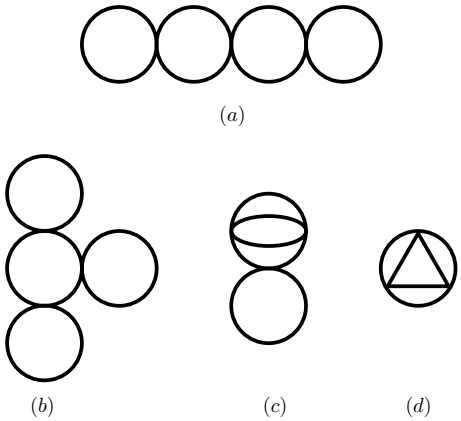


**Figure 5**. Four-loop vacuum diagrams.

We can then write

$$\mathcal{F}_{\rm hard} = \mathcal{F}_0^{(h)} + \mathcal{F}_1^{(h)} + \mathcal{F}_{\rm 2a}^{(h)} + \mathcal{F}_{\rm 2b}^{(h)} + \mathcal{F}_{\rm 3a}^{(h)} + \mathcal{F}_{\rm 3b}^{(h)} + \mathcal{F}_{\rm 3c}^{(h)} + \mathcal{F}_{\rm 3d}^{(h)} + \\ + \frac{\mathcal{F}_1^{(h)}}{g^2}(\Delta_1 g^2 + \Delta_2 g^2) + 2\left(\frac{\mathcal{F}_{\rm 2a}^{(h)}}{g^2} + \frac{\mathcal{F}_{\rm 2b}^{(h)}}{g^2}\right)\Delta_1 g^2 \,, \tag{2.14}$$

where $\Delta_1 g^2$ and $\Delta_2 g^2$ are the order-$g^4$ and order-$g^6$ coupling constant counterterms, respectively, given in eq. (2.3). The superscript $h$ indicates that the expression gives the hard contribution to the free energy. The expressions for the diagrams are

$$\mathcal{F}_0^{(h)} = \frac{1}{2}\sum\!\!\!\!\!\!\int_P \log P^2 \,, \tag{2.15}$$

$$\mathcal{F}_1^{(h)} = \frac{1}{8}g^2\left(\sum\!\!\!\!\!\!\int_P \frac{1}{P^2}\right)^2 \,, \tag{2.16}$$

$$\mathcal{F}_{\rm 2a}^{(h)} = -\frac{1}{16}g^4\left(\sum\!\!\!\!\!\!\int_P \frac{1}{P^2}\right)^2 \sum\!\!\!\!\!\!\int_Q \frac{1}{Q^4} \,, \tag{2.17}$$

$$\mathcal{F}_{\rm 2b}^{(h)} = -\frac{1}{48}g^4 \sum\!\!\!\!\!\!\int_{PQR} \frac{1}{P^2Q^2R^2(P+Q+R)^2} \,, \tag{2.18}$$

$$\mathcal{F}_{\rm 3a}^{(h)} = \frac{1}{32}g^6\left(\sum\!\!\!\!\!\!\int_P \frac{1}{P^2}\right)^2\left(\sum\!\!\!\!\!\!\int_Q \frac{1}{Q^4}\right)^2 \,, \tag{2.19}$$

$$\mathcal{F}_{\rm 3b}^{(h)} = \frac{1}{48}g^6 \sum\!\!\!\!\!\!\int_P \frac{1}{Q^6}\left(\sum\!\!\!\!\!\!\int_P \frac{1}{P^2}\right)^3 \,, \tag{2.20}$$

$$\mathcal{F}_{\rm 3c}^{(c)} = \frac{1}{24} g^6 \sum\!\!\!\!\!\!\int_P \frac{1}{P^2} \sum\!\!\!\!\!\!\int_{KQR} \frac{1}{K^4 Q^2 R^2 (K+Q+R)^2} \,, \tag{2.21}$$

$$\mathcal{F}_{\rm 3d}^{(h)} = \frac{1}{48} g^6 \sum\!\!\!\!\!\!\int_P [\Pi(P)]^3 \,, \tag{2.22}$$

where the symbol $\sum\!\!\!\!\!\!\int$ is defined in eq. (A.1) and the self-energy $\Pi(P)$ is defined in eq. (A.13). The expressions for the sum-integrals are listed in appendix A. After renormalization, the final expression is [15]

$$\begin{aligned}
\mathcal{F}_{\rm hard}(\Lambda) = &-\frac{\pi^2 T^4}{90} \times \\
&\times \Bigg\{ 1 - \frac{5}{4}\alpha + \frac{15}{4}\alpha^2 \left[ \log\frac{\mu}{4\pi T} + \frac{1}{3}\gamma_E + \frac{31}{45} + \frac{4}{3}\frac{\zeta'(-1)}{\zeta(-1)} - \frac{2}{3}\frac{\zeta'(-3)}{\zeta(-3)} \right] + \\
&\quad + \frac{15}{16}\alpha^3 \times \\
&\quad \times \Bigg[ \frac{\pi^2}{\epsilon} - 12\log^2\frac{\mu}{4\pi T} - \left( \frac{1084}{45} + 8\gamma_E + 32\frac{\zeta'(-1)}{\zeta(-1)} - 16\frac{\zeta'(-3)}{\zeta(-3)} \right) \times \\
&\qquad \times \log\frac{\mu}{4\pi T} + 8\pi^2 \log\frac{\Lambda}{4\pi T} - \frac{134}{9} - \frac{25}{3}\gamma_E^2 - \frac{1}{27}\zeta(3) + \frac{31}{15}\gamma_E - \\
&\qquad - \frac{\pi^2}{2} + 4\gamma_E\pi^2 - \frac{206}{9}\frac{\zeta'(-1)}{\zeta(-1)} - \frac{16}{3}\gamma_1 + 8\gamma_E\frac{\zeta'(-3)}{\zeta(-3)} + \\
&\qquad + \frac{4}{3}\gamma_E\frac{\zeta'(-1)}{\zeta(-1)} - 8\left(\frac{\zeta'(-1)}{\zeta(-1)}\right)^2 - \frac{20}{3}\frac{\zeta''(-1)}{\zeta(-1)} - \\
&\qquad - \frac{2}{3}C'_{\rm ball} + 2C^a_{\rm triangle} + \pi^2 C^b_{\rm triangle} \Bigg] + \mathcal{O}(\epsilon) \Bigg\} \,,
\end{aligned} \tag{2.23}$$

where $\alpha = \alpha(\mu)$, $C'_{\rm ball} = 48.7976$, $C^a_{\rm triangle} = -25.7055$, and $C^b_{\rm triangle} = 28.9250$. We have used the renormalization group equation for $g^2$ to change the renormalization scale from $\Lambda$ to $\mu$. Note that the final results contains a pole in $\epsilon$. We cancel it by adding a counterterm $T\delta f$ [8]. The term $\delta f$ can be determined by calulating the ultraviolet divergences in the effective theory. The triangle diagram in three dimensions has a logarithmic ultraviolet divergence and the counterterm needed to cancel this divergence is given by

$$\delta f = \frac{g_3^6 \pi^2}{1536(4\pi)^4 \epsilon} \,. \tag{2.24}$$

If we express the counterterm in terms of the coupling $g$ of the full theory, we must take into account that $g_3^6$ multiplies a pole in $\epsilon$ and it therefore picks up finite terms. These terms will be of order $g^8$ and can be neglected in the present calculation.[2] The coefficient $f$ satisfies the evolution equation

$$\Lambda \frac{\partial}{\partial \Lambda} f = -\frac{\pi^2}{192(4\pi)^4} g_3^6 \,. \tag{2.25}$$

This follows from the scale dependence of the triangle diagram in three dimensions and the fact that the $\Lambda$-dependence of $f$ must cancel the scale dependence in the effective theory.

[2] Note that minimal subtraction in the full theory and in the effective theory are not equivalent. The difference is the finite terms mentioned above [8].

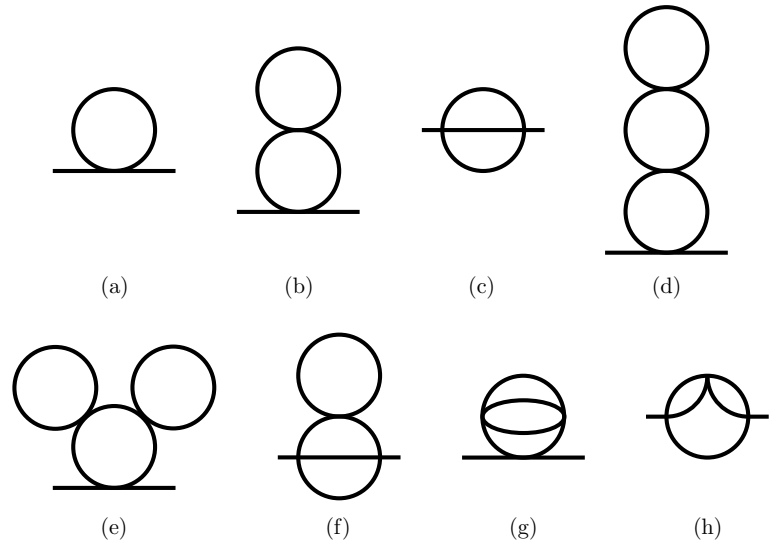


**Figure 6**. Feynman graphs that contribute to the self-energy through three loops.

### 2.3 Mass parameter

The simplest way of determining the mass parameter $m^2$ is by matching the Debye or screening mass $m_D$ in the full theory and in the effective theory [19]. The Debye mass $m_D$ is given by the pole of static propagator, i.e. by

$$p^2 + \tilde{\Pi}(p_0 = 0, p) = 0 \,, \qquad p^2 = -m_D^2 \,, \tag{2.26}$$

where $\tilde{\Pi}(p_0, p)$ denotes the self-energy function. In the effective theory, the equation is

$$p^2 + m^2 + \Pi_{\text{eff}}(p) = 0 \,, \qquad p^2 = -m_D^2 \,, \tag{2.27}$$

where $\Pi_{\text{eff}}(p)$ is the self-energy in the effective theory. Since the self-energy in the full theory is expanded around $p = 0$, we should do to the same in the effective theory (see discussion below). The loop integrals are therefore evaluated at zero external momentum and since the matching is carried out using massless propagators there is no scale in the loop integrals. They therefore vanish in in dimensional regularization, i.e. $\Pi_{\text{eff}}(0) = \Pi'_{\text{eff}}(0) = \cdots = 0$. Using this fact and equating (2.26) and (2.27), we obtain $m^2 \approx m_D^2$ [3]

$$m_D^2 = \tilde{\Pi}(p_0 = 0, p = im_D) \,. \tag{2.28}$$

The diagrams that contribute to the self-energy $\tilde{\Pi}(P)$ through three loops are shown in figure 6. The self-energy $\tilde{\Pi}(P)$ is given by

$$\begin{aligned} \tilde{\Pi}(P) = {} & \tilde{\Pi}_1^{(h)}(P) + \tilde{\Pi}_2^{(h)}(P) + \tilde{\Pi}_3^{(h)}(P) + \\ & + \frac{\tilde{\Pi}_1^{(h)}(P)}{g^2} \left(\Delta_1 g^2 + \Delta_2 g^2\right) + 2\frac{\tilde{\Pi}_2^{(h)}(P)}{g^2} \Delta_1 g^2 \,. \end{aligned} \tag{2.29}$$

[3] Note that we use the symbol "$\approx$" to emphasize that the the mass parameter $m^2$ is equal to the Debye mass $m_D^2$ only in strict perturbation theory. The interpretation is that $m$ gives the contribution to the Debye mass from the hard scale $T$.



JHEP08(2009)066

The expression for the various terms in the self-energy are given by

$$\tilde{\Pi}_1^{(h)}(P) = \frac{1}{2}g^2 \sum\!\!\!\!\!\!\int_Q \frac{1}{Q^2} \,, \tag{2.30}$$

$$\tilde{\Pi}_{2a}^{(h)}(P) = -\frac{1}{4}g^4 \sum\!\!\!\!\!\!\int_{QR} \frac{1}{Q^4R^2} \,, \tag{2.31}$$

$$\tilde{\Pi}_{2b}^{(h)}(P) = -\frac{1}{6}g^4 \sum\!\!\!\!\!\!\int_{QR} \frac{1}{Q^2R^2(P+Q+R)^2} \,, \tag{2.32}$$

$$\tilde{\Pi}_{3a}^{(h)}(P) = \frac{1}{8}g^6 \sum\!\!\!\!\!\!\int_Q \frac{1}{Q^2} \left( \sum\!\!\!\!\!\!\int_R \frac{1}{R^4} \right)^2 \,, \tag{2.33}$$

$$\tilde{\Pi}_{3b}^{(h)}(P) = \frac{1}{8}g^6 \sum\!\!\!\!\!\!\int_Q \frac{1}{Q^6} \left( \sum\!\!\!\!\!\!\int_R \frac{1}{R^2} \right)^2 \tag{2.34}$$

$$\tilde{\Pi}_{3c}^{(h)}(P) = \frac{1}{4}g^6 \sum\!\!\!\!\!\!\int_K \frac{1}{K^2} \sum\!\!\!\!\!\!\int_{QR} \frac{1}{Q^4R^2(P+Q+R)^2} \,, \tag{2.35}$$

$$\tilde{\Pi}_{3d}^{(h)}(P) = \frac{1}{12}g^6 \sum\!\!\!\!\!\!\int_{KQR} \frac{1}{K^4Q^2R^2(K+Q+R)^2} \,, \tag{2.36}$$

$$\tilde{\Pi}_{3e}^{(h)}(P) = \frac{1}{4}g^6 \sum\!\!\!\!\!\!\int_Q \frac{1}{(P+Q)^2} [\Pi(Q)]^2 \,. \tag{2.37}$$

Since the leading-order solution to eq. (2.28) gives a value of $p$ that is of the order $gT$, it is justified to expand the loop diagrams in a Taylor series around $p = 0$. We can then write eq. (2.28) as

$$m_D^2 = \tilde{\Pi}_1^{(h)}(0) + \tilde{\Pi}_2^{(h)}(0) + \tilde{\Pi}_2^{(h)\prime}(0)p^2 + \tilde{\Pi}_3(0) + \cdots \,, \quad p^2 = -m_D^2 \,, \tag{2.38}$$

or $m_D^2 = \tilde{\Pi}_1^{(h)}(0) + \tilde{\Pi}_2^{(h)}(0) + \tilde{\Pi}_3^{(h)}(0) - \tilde{\Pi}_1(0)\tilde{\Pi}_2'(0)$ . We then need the two-loop self-energy diagram $\tilde{\Pi}_{2\mathrm{b}}(P)$ to order $p^2$, while the three-loop self-energy diagrams $\tilde{\Pi}_{3\mathrm{c}}^{(h)}(P)$ and $\tilde{\Pi}_{3\mathrm{e}}^{(h)}(P)$ can be evaluated at $p = 0$. This yields

$$\tilde{\Pi}_{2\mathrm{b}}^{(h)}(P) = -\frac{1}{6}g^4 \sum\!\!\!\!\!\!\int_{QR} \frac{1}{Q^2R^2(Q+R)^2} - \frac{1}{6}g^4p^2 \sum\!\!\!\!\!\!\int_{QR} \frac{(4/d)q^2 - Q^2}{Q^6R^2(Q+R)^2} + \mathcal{O}(p^4) \,, \tag{2.39}$$

$$\tilde{\Pi}_{3c}^{(h)}(0) = \frac{1}{4}g^6 \sum\!\!\!\!\!\!\int_K \frac{1}{K^2} \sum\!\!\!\!\!\!\int_{QR} \frac{1}{Q^4R^2(Q+R)^2} \,, \tag{2.40}$$

$$\tilde{\Pi}_{3e}^{(h)}(0) = \frac{1}{4}g^6 \sum\!\!\!\!\!\!\int_Q \frac{1}{Q^2} [\Pi(Q)]^2 \,. \tag{2.41}$$

The sum-integrals needed are listed in appendix A. After renormalization, we obtain

$$\begin{aligned} m^2(\Lambda) = {} & \frac{1}{24}g^2(\Lambda)T^2 \times \\ & \times \Bigg\{ 1 + \frac{g^2}{(4\pi)^2} \left[ \frac{1}{\epsilon} + \log\frac{\Lambda}{4\pi T} + 2 - \gamma_E + 2\frac{\zeta'(-1)}{\zeta(-1)} \right] - \frac{6g^4}{(4\pi)^4} \times \\ & \times \left[ \frac{1}{\epsilon} \left( \log\frac{\Lambda}{4\pi T} + \gamma_E \right) + \frac{7}{2}\log^2\frac{\Lambda}{4\pi T} + \left( \frac{19}{18} + 5\gamma_E + 2\frac{\zeta'(-1)}{\zeta(-1)} \right) \log\frac{\Lambda}{4\pi T} + \right. \\ & \left. + \frac{2851}{864} - \frac{95}{48}\gamma_E^2 - \frac{119}{144}\gamma_E - \frac{1}{144}\zeta(3) - 9\gamma_1 + \frac{\zeta'(-1)}{\zeta(-1)} \left( \frac{113}{72} + \frac{17}{12}\gamma_E \right) - \right. \end{aligned}$$



JHEP08(2009)066

$$-\frac{1}{4}\frac{\zeta''(-1)}{\zeta(-1)}+\frac{29}{32}\pi^2-2\gamma_E\log(2\pi)+2\log^2(2\pi)-\frac{1}{24}C'_{\rm ball}+\frac{1}{4}C_I\Bigg]+$$
$$+\mathcal{O}(\epsilon)\Bigg\}\,, \tag{2.42}$$

where $g=g(\Lambda)$ and $C_I=-38.4672$. The mass parameter through order $g^4$ is known to order $\epsilon$ [9], but we only need it to order $\epsilon^0$. We notice that the mass parameter contains uncancelled poles in $\epsilon$. It is advantageous to write the mass term as a sum of a finite piece $\tilde{m}^2$ and a counterterm $\Delta m^2$, where

$$\begin{aligned}
\tilde{m}^2(\Lambda) &= \frac{1}{24}g^2(\mu)T^2\times\\
&\quad\times\Bigg\{1+\frac{g^2}{(4\pi)^2}\left[4\log\frac{\Lambda}{4\pi T}-3\log\frac{\mu}{4\pi T}+2-\gamma_E+2\frac{\zeta'(-1)}{\zeta(-1)}\right]-\\
&\qquad-\frac{6g^4}{(4\pi)^4}\left[4\log^2\frac{\Lambda}{4\pi T}-\frac{3}{2}\log^2\frac{\mu}{4\pi T}+\left(\frac{19}{18}-\gamma_E+2\frac{\zeta'(-1)}{\zeta(-1)}\right)\log\frac{\mu}{4\pi T}+\right.\\
&\qquad\quad+4\gamma_E\log\frac{\Lambda}{4\pi T}+\frac{2851}{864}-\frac{95}{48}\gamma_E^2-\frac{119}{144}\gamma_E-\frac{1}{144}\zeta(3)-7\gamma_1+\\
&\qquad\quad+\frac{\zeta'(-1)}{\zeta(-1)}\left(\frac{113}{72}+\frac{17}{12}\gamma_E\right)-\frac{1}{4}\frac{\zeta''(-1)}{\zeta(-1)}+\frac{25}{32}\pi^2-2\gamma_E\log(2\pi)+\\
&\qquad\quad\left.+2\log^2(2\pi)-\frac{1}{24}C'_{\rm ball}+\frac{1}{4}C_I\right]+\mathcal{O}(\epsilon)\Bigg\}\,,
\end{aligned} \tag{2.43}$$

$$\begin{aligned}
\Delta m^2(\Lambda) &= \frac{g^4T^2}{24(4\pi)^2\epsilon}\left[1-\frac{6g^2}{(4\pi)^2}\left(\log\frac{\mu}{4\pi T}+\gamma_E\right)-\right.\\
&\qquad\left.-\frac{6g^2}{(4\pi)^2}\left(\log^2\frac{\mu}{4\pi T}+2\gamma_E\log\frac{\mu}{4\pi T}+\frac{\pi^2}{8}-2\gamma_1\right)\epsilon\right]\,,\\
&= \frac{g_3^4(\Lambda)}{24(4\pi)^2\epsilon}\,,
\end{aligned} \tag{2.44}$$

where $g=g(\mu)$ and we have used eq. (2.9) to change the renormalization scale from $\Lambda$ to $\mu$. The term $\Delta m^2$ acts as a counterterm in the effective theory. In fact, the sunset diagram in three dimensions that contribute to the self-energy is logarithmically divergent, whose divergence exactly is given by the right-hand side of eq. (2.44) [22]. The mass parameter $\tilde{m}$ in three dimensions therefore satisfies the evolution equation

$$\Lambda\frac{\partial}{\partial\Lambda}\tilde{m}^2=\frac{1}{6}\frac{g_3^4}{(4\pi)^2}\,. \tag{2.45}$$

In the remainder of the paper, we will use $m$ instead of $\tilde{m}$ for covenience.

## 3 Soft contributions

In this section, we calculate the soft contributions $\mathcal{P}_{\rm soft}$ to the pressure. This requires the calculations of vacuum diagrams in the effective theory (2.4) through five loops. In order to take into account the soft scale $gT$, we now include the mass term $m^2$ in the free part of the Lagrangian and only the quartic term in eq. (2.4) is treated as an interaction. The

inclusion of the mass term in the propagators cuts off the infrared divergences that plagues naive perturbation theory in the full theory.

The one-loop vacuum diagram is shown in figure 2. Its contribution to the free energy is given by

$$\mathcal{F}_0^{(s)} = \frac{1}{2} T \int_p \log\left(p^2 + m^2\right) \,, \tag{3.1}$$

where the superscript $(s)$ indicates that the expression gives the soft contribution to the free energy. Using the expression in the appendix B, we obtain

$$\mathcal{F}_0^{(s)} = -\frac{m^3 T}{12\pi} \,. \tag{3.2}$$

The two-loop vacuum diagram is shown in figure 3. Its contribution to the free energy is given by

$$\mathcal{F}_1^{(s)} = \frac{1}{8} g_3^2 T \left(\int_p \frac{1}{p^2 + m^2}\right)^2 \,. \tag{3.3}$$

Using the expression in the appendix B, we obtain

$$\mathcal{F}_1^{(s)} = \frac{g_3^2 m^2 T}{8(4\pi)^2} \,. \tag{3.4}$$

The three-loop vacuum diagrams are shown in figure 4. The contribution to the free energy is given by

$$\mathcal{F}_2^{(s)} = \mathcal{F}_{2\mathrm{a}}^{(s)} + \mathcal{F}_{2\mathrm{b}}^{(s)} + \frac{\partial \mathcal{F}_0^{(s)}}{\partial m^2} \Delta m^2 \,, \tag{3.5}$$

where $\Delta m^2$ is the mass counterterm (2.44) in the effective theory and

$$\mathcal{F}_{2\mathrm{a}}^{(s)} = -\frac{1}{16} g_3^4 T \left(\int_p \frac{1}{p^2 + m^2}\right)^2 \int_q \frac{1}{(q^2 + m^2)^2} \,, \tag{3.6}$$

$$\mathcal{F}_{2\mathrm{b}}^{(s)} = -\frac{1}{48} g_3^4 T \int_{pqr} \frac{1}{p^2 + m^2} \frac{1}{q^2 + m^2} \frac{1}{r^2 + m^2} \frac{1}{(\mathbf{p} + \mathbf{q} + \mathbf{r})^2 + m^2} \,. \tag{3.7}$$

Using the expression in the appendix B, we obtain

$$\mathcal{F}_2^{(s)} = \frac{g_3^4 m T}{96(4\pi)^3} \left[8 \log \frac{\Lambda}{2m} + 9 - 8 \log 2\right] \,. \tag{3.8}$$

We note that all poles in $\epsilon$ cancel as they must since there are no divergences from the hard part proportional to $g_3^4 m$.

The four-loop vacuum diagrams are shown in figure 5. The contribution to the free energy is given by

$$\mathcal{F}_3^{(s)} = \mathcal{F}_{3\mathrm{a}}^{(s)} + \mathcal{F}_{3\mathrm{b}}^{(s)} + \mathcal{F}_{3\mathrm{c}}^{(s)} + \mathcal{F}_{3\mathrm{d}}^{(s)} + \frac{\partial \mathcal{F}_1^{(s)}}{\partial m^2} \Delta m^2 \,, \tag{3.9}$$



JHEP08(2009)066

where the expressions for the diagrams are

$$\mathcal{F}_{3a}^{(s)} = \frac{1}{32} g_3^6 T \left( \int_p \frac{1}{p^2+m^2} \right)^2 \left( \int_q \frac{1}{(q^2+m^2)^2} \right)^2 , \tag{3.10}$$

$$\mathcal{F}_{3b}^{(s)} = \frac{1}{48} g_3^6 T \left( \int_p \frac{1}{p^2+m^2} \right)^3 \int_q \frac{1}{(q^2+m^2)^3} , \tag{3.11}$$

$$\mathcal{F}_{3c}^{(s)} = \frac{1}{24} g_3^6 T \int_{pqr} \frac{1}{(p^2+m^2)^2} \frac{1}{q^2+m^2} \frac{1}{r^2+m^2} \frac{1}{(\mathbf{p}+\mathbf{q}+\mathbf{r})^2+m^2} \times \\ \times \int_s \frac{1}{s^2+m^2} , \tag{3.12}$$

$$\mathcal{F}_{3d}^{(s)} = \frac{1}{48} g_3^6 T \int_{pqrs} \frac{1}{q^2+m^2} \frac{1}{(\mathbf{p}+\mathbf{q})^2+m^2} \frac{1}{r^2+m^2} \frac{1}{(\mathbf{p}+\mathbf{r})^2+m^2} \times \\ \times \frac{1}{s^2+m^2} \frac{1}{(\mathbf{p}+\mathbf{s})^2+m^2} . \tag{3.13}$$

Using the expressions in the appendix B, we obtain

$$\mathcal{F}_3^{(s)} = \frac{g_3^6 T}{768(4\pi)^4} \left[ -4\left(4-\pi^2\right) \log \frac{\Lambda}{2m} - 4 + 16\log 2 - 42\zeta(3) + \pi^2(1+2\log 2) \right] + \\ + \frac{g_3^6 T \pi^2}{1536(4\pi)^4 \epsilon} . \tag{3.14}$$

The pole in $\epsilon$ in eq. (3.14) arises from the triangle diagram in eq. (3.13). This pole is cancelled by the counterterm in eq. (2.24).

The five-loop vacuum diagrams are shown in figure 7. The contributions to the free energy are given by

$$\mathcal{F}_4^{(s)} = \mathcal{F}_{4a}^{(s)} + \mathcal{F}_{4b}^{(s)} + \mathcal{F}_{4c}^{(s)} + \mathcal{F}_{4d}^{(s)} + \mathcal{F}_{4e}^{(s)} + \mathcal{F}_{4f}^{(s)} + \mathcal{F}_{4g}^{(s)} + \mathcal{F}_{4h}^{(s)} + \mathcal{F}_{4i}^{(s)} + \mathcal{F}_{4j}^{(s)} + \\ + \frac{\partial \mathcal{F}_2^{(s)}}{\partial m^2} \Delta m^2 + \frac{1}{2} \frac{\partial^2 \mathcal{F}_0^{(s)}}{(\partial m^2)^2} (\Delta m^2)^2 . \tag{3.15}$$

where the expressions for the diagrams are

$$\mathcal{F}_{4a}^{(s)} = -\frac{1}{64} g_3^8 T \left( \int_p \frac{1}{p^2+m^2} \right)^2 \left( \int_q \frac{1}{(q^2+m^2)^2} \right)^3 , \tag{3.16}$$

$$\mathcal{F}_{4b}^{(s)} = -\frac{1}{32} g_3^8 T \left( \int_p \frac{1}{p^2+m^2} \right)^3 \int_q \frac{1}{(q^2+m^2)^2} \int_r \frac{1}{(r^2+m^2)^3} , \tag{3.17}$$

$$\mathcal{F}_{4c}^{(s)} = -\frac{1}{128} g_3^8 T \left( \int_p \frac{1}{p^2+m^2} \right)^4 \int_q \frac{1}{(q^2+m^2)^4} , \tag{3.18}$$

$$\mathcal{F}_{4d}^{(s)} = -\frac{1}{16} g_3^8 T \int_{pqrs} \frac{1}{(q^2+m^2)^2} \frac{1}{(\mathbf{p}+\mathbf{q})^2+m^2} \frac{1}{r^2+m^2} \frac{1}{(\mathbf{p}+\mathbf{r})^2+m^2} \times \\ \times \frac{1}{s^2+m^2} \frac{1}{(\mathbf{p}+\mathbf{s})^2+m^2} \int_t \frac{1}{t^2+m^2} , \tag{3.19}$$

$$\mathcal{F}_{4e}^{(s)} = -\frac{1}{48} g_3^8 T \int_{pqr} \frac{1}{(p^2+m^2)^3} \frac{1}{q^2+m^2} \frac{1}{r^2+m^2} \frac{1}{(\mathbf{p}+\mathbf{q}+\mathbf{r})^2+m^2} \times \\ \times \left( \int_s \frac{1}{s^2+m^2} \right)^2 , \tag{3.20}$$

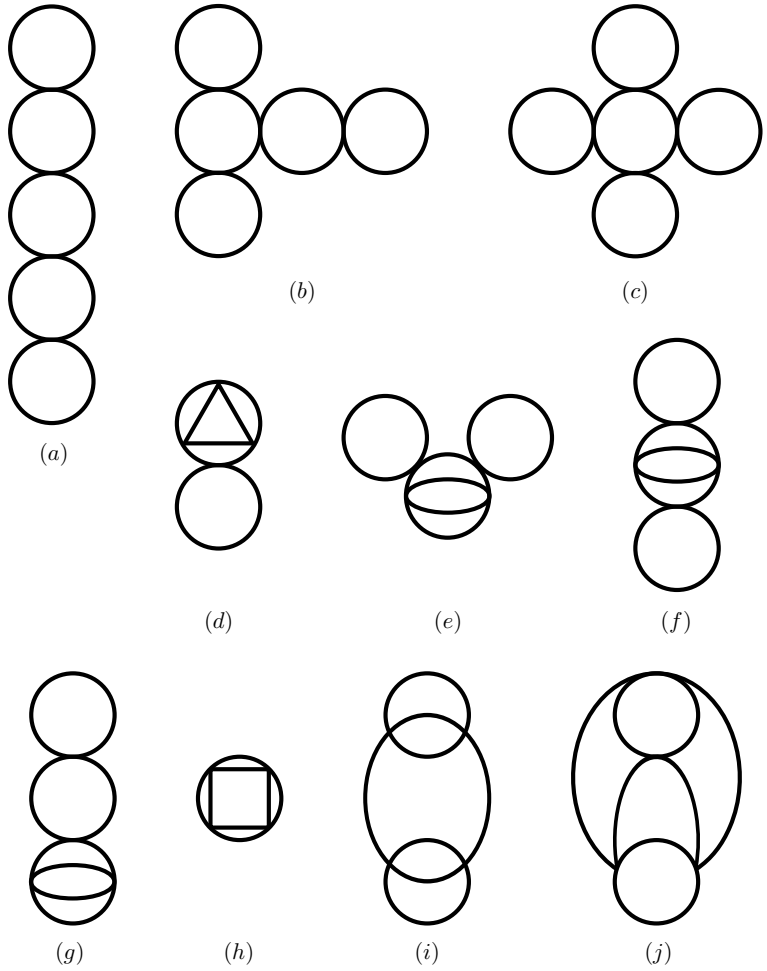


**Figure 7**. Five-loop vacuum diagrams that contribute to the soft part of the free energy.

$$\mathcal{F}_{4\mathrm{f}}^{(s)} = -\frac{1}{32} g_3^8 T \int_{pqr} \frac{1}{(p^2+m^2)^2} \frac{1}{(q^2+m^2)^2} \frac{1}{r^2+m^2} \frac{1}{(\mathbf{p}+\mathbf{q}+\mathbf{r})^2+m^2} \times \\ \times \left( \int_s \frac{1}{s^2+m^2} \right)^2 , \tag{3.21}$$

$$\mathcal{F}_{4\mathrm{g}}^{(s)} = -\frac{1}{48} g_3^8 T \int_{pqr} \frac{1}{(p^2+m^2)^2} \frac{1}{q^2+m^2} \frac{1}{r^2+m^2} \frac{1}{(\mathbf{p}+\mathbf{q}+\mathbf{r})^2+m^2} \times \\ \times \int_s \frac{1}{s^2+m^2} \int_t \frac{1}{(t^2+m^2)^2} , \tag{3.22}$$

$$\mathcal{F}_{4\mathrm{h}}^{(s)} = -\frac{1}{128} g_3^8 T \int_{pqrst} \frac{1}{q^2+m^2} \frac{1}{(\mathbf{p}+\mathbf{q})^2+m^2} \frac{1}{r^2+m^2} \frac{1}{(\mathbf{p}+\mathbf{r})^2+m^2} \times \\ \times \frac{1}{s^2+m^2} \frac{1}{(\mathbf{p}+\mathbf{s})^2+m^2} \frac{1}{t^2+m^2} \frac{1}{(\mathbf{p}+\mathbf{t})^2+m^2} , \tag{3.23}$$

$$\mathcal{F}_{4\mathrm{i}}^{(s)} = -\frac{1}{144} g_3^8 T \int_p \frac{1}{(p^2+m^2)^2} \int_{qr} \frac{1}{q^2+m^2} \frac{1}{r^2+m^2} \frac{1}{(\mathbf{p}+\mathbf{q}+\mathbf{r})^2+m^2} \times$$



JHEP08(2009)066

$$\times \int_{st} \frac{1}{s^2+m^2}\frac{1}{t^2+m^2}\frac{1}{(\mathbf{p}+\mathbf{s}+\mathbf{t})^2+m^2}\,, \tag{3.24}$$

$$\mathcal{F}_{4\mathrm{j}}^{(s)} = -\frac{1}{32}g_3^8 T\int_{pqrst}\frac{1}{q^2+m^2}\frac{1}{(\mathbf{p}+\mathbf{q})^2+m^2}\frac{1}{(\mathbf{p}+\mathbf{r})^2+m^2}\frac{1}{(\mathbf{t}+\mathbf{r})^2+m^2}\frac{1}{r^2+m^2}\times$$
$$\times\frac{1}{(\mathbf{p}+\mathbf{s})^2+m^2}\frac{1}{(\mathbf{s}+\mathbf{t})^2+m^2}\frac{1}{s^2+m^2}\,. \tag{3.25}$$

Using the expressions in the appendix B, we obtain

$$\mathcal{F}_4^{(s)} = -\frac{g_3^8 T}{288m(4\pi)^5}\times$$
$$\times\left[\log^2\frac{\Lambda}{2m}+\frac{1}{4}(1-8\log 2)\log\frac{\Lambda}{2m}-\frac{15}{64}-\frac{3}{8}\pi^2+\frac{9}{8}\pi^2\log 2+\right.$$
$$\left.+\frac{23}{4}\log 2+6\log^2 2-6\log 3-\frac{81}{16}\zeta(3)+5\mathrm{Li}_2(\tfrac{1}{4})+9\mathrm{C}_{4\mathrm{j}}\right]\,, \tag{3.26}$$

where $C_{4j} = 0.443166$. We note that all poles in $\epsilon$ cancel as they must since there are no divergences from the hard part proportional to $g_3^8/m$. Adding eqs. (3.2), (3.3), (3.8), (3.14), and (3.26) as well as the counterterm eq. (2.24), we obtain the soft contribution to the free energy through five loops

$$\mathcal{F}_{0+1+2+3+4}^{(s)} = -\frac{m^3T}{12\pi}+\frac{g_3^2m^2T}{8(4\pi)^2}+\frac{g_3^4mT}{96(4\pi)^3}\left[8\log\frac{\Lambda}{2m}+9-8\log 2\right]+$$
$$+\frac{g_3^6T}{768(4\pi)^4}\times$$
$$\times\left[-4(4-\pi^2)\log\frac{\Lambda}{2m}-4+16\log 2-42\zeta(3)+\pi^2(1+2\log 2)\right]-$$
$$-\frac{g_3^8T}{288m(4\pi)^5}\times$$
$$\times\left[\log^2\frac{\Lambda}{2m}+\frac{1}{4}(1-8\log 2)\log\frac{\Lambda}{2m}-\frac{15}{64}-\frac{3}{8}\pi^2+\frac{9}{8}\pi^2\log 2+\right.$$
$$\left.+\frac{23}{4}\log 2+6\log^2 2-6\log 3-\frac{81}{16}\zeta(3)+5\mathrm{Li}_2(\tfrac{1}{4})+9\mathrm{C}_{4\mathrm{j}}\right]\,. \tag{3.27}$$

Using the evolution equations for $g_3^2$ and $m^2$, it easy to check that the free energy, eq. (2.23) plus eq. (3.27) is independent of the factorization scale $\Lambda$.

By expanding the coupling $g_3^2$ (2.8) and the mass parameter $m^2$ (2.43) to the appropriate orders in the various terms in (3.27), we obtain the soft contribution through order $g^7$. This yields

$$\mathcal{F}_{\mathrm{soft}} = -\frac{\pi^2T^4}{90}\times$$
$$\times\left\{\frac{5\sqrt{6}}{3}\alpha^{3/2}-\frac{15}{2}\alpha^2-\frac{15\sqrt{6}}{2}\alpha^{5/2}\left[\log\frac{\mu}{4\pi T}-\frac{2}{3}\log\alpha+C_5\right]-\right.$$
$$\left.-\frac{15}{16}\alpha^3\times\right.$$

$$
\begin{aligned}
&\times \left[ -48 \log \frac{\mu}{4\pi T} + 16 \frac{\zeta'(-1)}{\zeta(-1)} - 32\gamma_E - 84\zeta(3) + 8 + 16 \log \frac{2}{3} + \right. \\
&\qquad \left. + 16 \log \alpha + \pi^2 \left( 2 + 12 \log 2 - 4 \log \frac{2}{3} - 4 \log \alpha + 8 \log \frac{\Lambda}{4\pi T} \right) \right] + \\
&+ \frac{225\sqrt{6}}{8} \alpha^{7/2} \times \\
&\times \left\{ \left[ \log^2 \frac{\mu}{4\pi T} + \left( \frac{221}{135} + \frac{2}{3}\gamma_E - \frac{4}{3} \log \frac{2}{3} - \frac{4}{3} \frac{\zeta'(-1)}{\zeta(-1)} - \frac{4}{3} \log \alpha \right) \log \frac{\mu}{4\pi T} + \right. \right. \\
&\qquad + \left( \frac{2}{15} + \frac{8}{45} \frac{\zeta'(-1)}{\zeta(-1)} - \frac{52}{45}\gamma_E + \frac{8}{45} \log \frac{2}{3} \right) \log \alpha + \frac{4}{45} \log^2 \alpha + \\
&\qquad \left. \left. + C_7 \right] \right\} ,
\end{aligned}
\tag{3.28}
$$

where the constants $C_5$ and $C_7$ are defined below.

## 4 Results and discussion

The full pressure is given by minus the sum of eq. (2.23) and eq. (3.27). The strict weak-coupling result for the pressure through order $g^7$ is minus the sum of eq. (2.23) and eq. (3.28). This yields

$$
\begin{aligned}
\mathcal{P} = \mathcal{P}_{\rm ideal} &\times \\
&\times \left\{ 1 - \frac{5}{4}\alpha + \frac{5\sqrt{6}}{3}\alpha^{3/2} + \frac{15}{4}\alpha^2 \left[ \log \frac{\mu}{4\pi T} + C_4 \right] - \right. \\
&\quad - \frac{15\sqrt{6}}{2}\alpha^{5/2} \left[ \log \frac{\mu}{4\pi T} - \frac{2}{3} \log \alpha + C_5 \right] - \\
&\quad - \frac{45}{4}\alpha^3 \left[ \log^2 \frac{\mu}{4\pi T} - \frac{1}{3} \left( \frac{269}{45} - 2\gamma_E - 8 \frac{\zeta'(-1)}{\zeta(-1)} + 4 \frac{\zeta'(-3)}{\zeta(-3)} \right) \log \frac{\mu}{4\pi T} + \right. \\
&\qquad \left. + \frac{1}{3}(4 - \pi^2) \log \alpha + C_6 \right] + \\
&\quad + \frac{225\sqrt{6}}{8}\alpha^{7/2} \times \\
&\quad \times \left[ \log^2 \frac{\mu}{4\pi T} + \left( \frac{221}{135} + \frac{2}{3}\gamma_E - \frac{4}{3} \log \frac{2}{3} - \frac{4}{3} \frac{\zeta'(-1)}{\zeta(-1)} - \frac{4}{3} \log \alpha \right) \log \frac{\mu}{4\pi T} + \right. \\
&\qquad \left. \left. + \left( \frac{2}{15} + \frac{8}{45} \frac{\zeta'(-1)}{\zeta(-1)} - \frac{52}{45}\gamma_E - \frac{8}{45} \log \frac{2}{3} \right) \log \alpha + \frac{4}{45} \log^2 \alpha + C_7 \right] \right\},
\end{aligned}
\tag{4.1}
$$

where $\mathcal{P}_{\rm ideal} = \pi^2 T^4/90$ and where the constants $C_4 - C_7$ are

$$
C_4 \equiv -\frac{59}{45} + \frac{1}{3}\gamma_E + \frac{4}{3}\frac{\zeta'(-1)}{\zeta(-1)} - \frac{2}{3}\frac{\zeta'(-3)}{\zeta(-3)} , \tag{4.2}
$$

$$
C_5 \equiv \frac{5}{6} + \frac{1}{3}\gamma_E - \frac{2}{3} \log \frac{2}{3} - \frac{2}{3}\frac{\zeta'(-1)}{\zeta(-1)} , \tag{4.3}
$$

$$
C_6 \equiv \frac{1}{3}(4 - \pi^2) \log \frac{2}{3} + \frac{103}{54} + \frac{1}{18} C'_{\rm ball} - \frac{1}{6} C^a_{\rm triangle} - \frac{\pi^2}{12} C^b_{\rm triangle} + \frac{4}{9}\gamma_1 - \frac{511}{180}\gamma_E +
$$

$$+\frac{25}{36}\gamma_E^2+\frac{5\pi^2}{24}-\frac{\pi^2}{3}\gamma_E+\pi^2\log 2+\left(\frac{175}{54}-\frac{1}{9}\gamma_E\right)\frac{\zeta'(-1)}{\zeta(-1)}+\frac{2}{3}\left(\frac{\zeta'(-1)}{\zeta(-1)}\right)^2+$$
$$+\frac{5}{9}\frac{\zeta''(-1)}{\zeta(-1)}-\frac{2}{3}\gamma_E\frac{\zeta'(-3)}{\zeta(-3)}-\frac{2267}{324}\zeta(3)\,, \tag{4.4}$$

$$C_7=-\frac{1457}{810}+\frac{1}{45}C'_{\rm ball}-\frac{2}{15}C_I+\frac{749}{270}\gamma_E+\frac{56}{15}\gamma_1-\frac{11}{20}\pi^2+\frac{2}{15}\frac{\zeta''(-1)}{\zeta(-1)}+$$
$$+\frac{16}{15}\gamma_E\log(2\pi)-\frac{16}{15}\log^2(2\pi)-\frac{52}{45}\gamma_E\log\frac{2}{3}-\frac{19}{27}\frac{\zeta'(-1)}{\zeta(-1)}-\frac{38}{45}\gamma_E\frac{\zeta'(-1)}{\zeta(-1)}+$$
$$+\frac{4}{45}\left(\frac{\zeta'(-1)}{\zeta(-1)}\right)^2+\frac{34}{15}\log\frac{2}{3}+\frac{2}{5}\pi^2\log 2+\frac{4}{45}\log^2 3+\frac{28}{15}\log^2 2-$$
$$-\frac{8}{45}\log 2\log 3+\frac{8}{45}\frac{\zeta'(-1)}{\zeta(-1)}\log\frac{2}{3}-\frac{97}{54}\zeta(3)+\frac{16}{9}{\rm Li}_2(\tfrac{1}{4})+\frac{97}{90}\gamma_E^2+\frac{16}{5}C_{4\rm j}\,, \tag{4.5}$$

where $C^a_{\rm triangle}=-25.7055$ and $C^b_{\rm triangle}=28.9250$. The numerical values of $C_4-C_7$ are

$$C_4 = 1.09775\,, \tag{4.6}$$
$$C_5 = -0.0273205\,, \tag{4.7}$$
$$C_6 = -6.5936\,, \tag{4.8}$$
$$C_7 = -0.862\,. \tag{4.9}$$

Note that the $\Lambda$-dependence cancels in the result (4.1). Using eq. (2.9) for the running of $\alpha$, it is straightforward to check that the final result eq. (4.1) is RG invariant up to higher-order corrections. The order-$g^4$ result was obtained by Frenkel, Saa, and Taylor [23], the order-$g^5$ result by Parwani and Singh [24], the order-$g^6\log(g)$ result by Braaten and Nieto [19], and the order-$g^6$ result by Gynther et al. [15]. The latter was later reproduced in ref. [21] using screened perturbation theory [25–27] by taking the weak-coupling limit for the mass parameter, $m=gT/\sqrt{24}$.

An expansion of the pressure in powers of $g$ is given in eq. (4.1). It is accurate up to corrections of order $g^8\log(g)$. A more accurate expression can be obtained by using the fact that our short-distance coefficients satisfy a set of evolution equations. The solutions to the evolution equations are

$$g_3^2(\Lambda) = g_3^2(2\pi T)\,, \tag{4.10}$$
$$f(\Lambda) = f(2\pi T)-\frac{\pi^2 g_3^6(2\pi T)}{192(4\pi)^4}\log\frac{\Lambda}{2\pi T}\,, \tag{4.11}$$
$$m^2(\Lambda) = m^2(2\pi T)+\frac{g_3^4(2\pi T)}{6(4\pi)^2}\log\frac{\Lambda}{2\pi T}\,. \tag{4.12}$$

If we substitute the short-distance coefficients (4.10) and (4.12) into eq. (3.27) and add the short-distance contribution (4.11), setting $\Lambda=gT/\sqrt{24}$ everwhere, and expand the resulting expression in powers of $g$, we obtain the complete result for the pressure, which is correct up to order $g^8\log(g)$. The contributions to the free energy $\mathcal{F}$ of order $g^8\log(g)$ come from (4.11) and from using (4.12) to expand the $g_3^2m^2T$ term in (3.27). This yields

$$\mathcal{F}_{\rm g^8\,log(g)} = \frac{3g^8T^4}{64(4\pi)^6}\,(\log 2-\gamma_E)\,(4-\pi^2)\log(g)\,. \tag{4.13}$$



JHEP08(2009)066

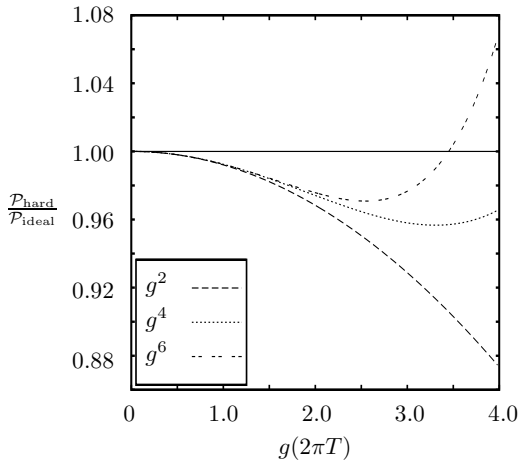


**Figure 8**. Hard contributions $\mathcal{P}_{\rm hard}$ to the pressure $\mathcal{P}$ normalized to $\mathcal{P}_{\rm ideal}$ to order $g^2$, $g^4$, and $g^6$.

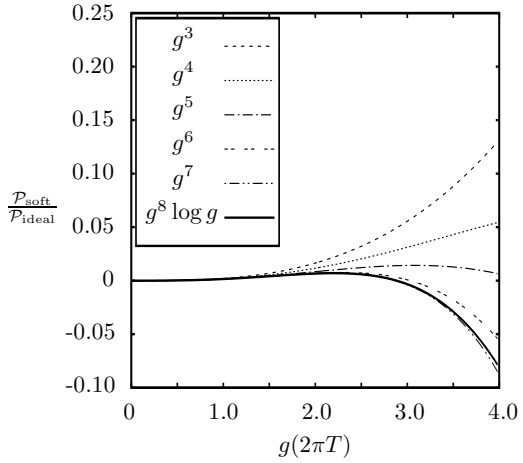


**Figure 9**. Soft contributions $\mathcal{P}_{\rm soft}$ to the pressure $\mathcal{P}$ normalized to $\mathcal{P}_{\rm ideal}$ to order $g^3$, $g^4$, $g^5$, $g^6$, $g^7$, and $g^8 \log(g)$.

Moreover, using the solutions to the flow equations, we are summing up leading logarithms of the form $g^{2n+3}\log^n(g)$ and e.g. subleading logarithms of the form $g^{2n+5}\log^n(g)$, where $n = 2, 3, \ldots$. These terms are obtained by expanding out the $m^3T$ and $g_3^4 mT$ terms in (3.27), respectively.

In figure 8, we show the various loop orders of $\mathcal{P}_{\rm hard}$ normalized to $\mathcal{P}_{\rm ideal}$ to orders $g^2$, $g^4$, and $g^6$, where $\mathcal{P}_{\rm hard}$ is given by minus eq. (2.23).[4] We have chosen $\mu = 2\pi T$ and $\Lambda = 2\pi T$. We notice that the successive approximations are larger than the previous one. In figure 9, we show the weak-coupling expansion of $\mathcal{P}_{\rm soft}$ normalized to $\mathcal{P}_{\rm ideal}$ to orders $g^3$, $g^4$, $g^5$ $g^6$, $g^7$, and $g^8\log(g)$, where $\mathcal{P}_{\rm soft}$ is given by minus the sum of eqs. (3.28) and (4.13).

In figure 10, we show the weak-coupling expansion of the pressure $\mathcal{P}$ given by (4.1) minus (4.13) normalized to $\mathcal{P}_{\rm ideal}$ to orders $g^2$, $g^3$ $g^4$, $g^5$ $g^6$, $g^7$, and $g^8\log(g)$ . The convergence properties of the successive approximations of the sum $\mathcal{P} = \mathcal{P}_{\rm hard} + \mathcal{P}_{\rm soft}$

[4]Note that we omit the pole in $\epsilon$ in eq. (2.23) in the plots of the hard part.



JHEP08(2009)066

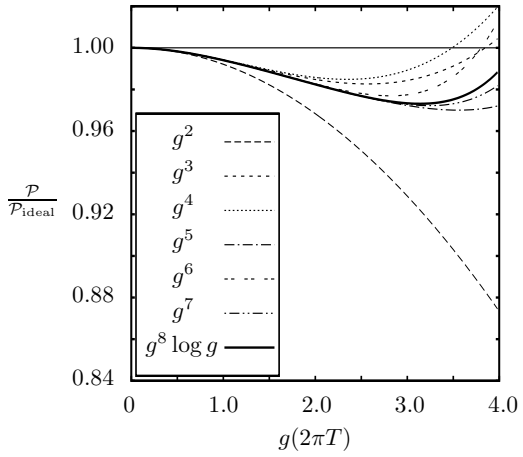


**Figure 10**. Weak-coupling expansion of the pressure $\mathcal{P}$ normalized to $\mathcal{P}_{\rm ideal}$ to order $g^2$, $g^3$, $g^4$, $g^5$, $g^6$, $g^7$, and $g^8 \log(g)$.

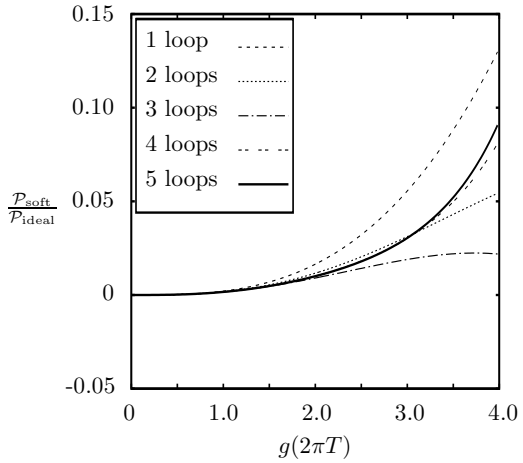


**Figure 11**. Soft contributions $\mathcal{P}_{\rm soft}$ to the pressure $\mathcal{P}$ normalized to $\mathcal{P}_{\rm ideal}$ at one through five loops.

clearly is better than the convergence properties of the successive approximations to $\mathcal{P}_{\rm hard}$ and $\mathcal{P}_{\rm soft}$ separately.

In figure 11, we plot the successive loop orders of minus eq. (3.27) normalized to $\mathcal{P}_{\rm ideal}$. In the one- and two-loop approximations, we use the leading-order results for $g_3^2$ and for $m^2$. At three and four loops, we use the leading-order result for $g_3^2$ and next-to-leading order result for $m^2$. Finally, at five loops, we use the solutions to the evolution equations for $g_3^2$, $f$, and $m^2$. The renormalization scale is $\mu = 2\pi T$ and the factorization scale is $\Lambda = gT/\sqrt{24}$. These approximations represent a selective resummation of higher-order terms. Clearly, the convergence is better than the strict perturbative expansion. In particular, the three-, four-, and five-loop approximations are very close.

In figure 12, we plot the successive loop orders of the the pressure which is given by the sum of minus eq. (2.23) minus eq. (3.27), and minus (4.13), normalized to $\mathcal{P}_{\rm ideal}$, starting



JHEP08(2009)066

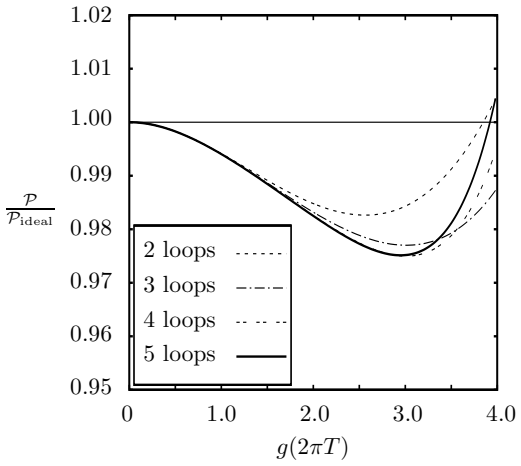


**Figure 12**. Successive approximations to the pressure $\mathcal{P}$ normalized to $\mathcal{P}_{\rm ideal}$ at two through five loops.,

at two loops. We are using the same approximations for $g_3^2$ and $m^2$ as in the previous plot. Again we notice that the convergence of $P$ is better than $P_{\rm hard}$ and $P_{\rm soft}$ separately. In fact the convergence is very good as the 3-loop through 5-loop approximations are very close. It is not surprising that a selective resummation improves the convergence of the series. This was also notice in screened perturbation theory [21, 25–27].

## 5 Summary

In the present paper, we have calculated the pressure to order $g^8\log(g)$ in massless $\phi^4$-theory at weak coupling. The first step is the determination of the coefficients in the dimensionally reduced effective field theory. This calculation encodes the physics of the hard scale $T$. The mass parameter was needed to order $g^6$ and involves a nontrivial three-loop sum-integral that was recently calculated in ref. [21]. The second step consists of using the effective theory to calculate the vacuum diagrams through five loops. All loop diagrams in the effective theory but one could be calculated analytically with dimensional regularization. This way of organizing the calculations is more economical and efficient than resummed perturbation theory.

The parameters of the effective theory, $g_3^2$, $f$, and $m^2$, satisfy a set of evolution equations. The solutions of these equations show that the parameters depend explicitly on the renormalization scale. This dependence is necessary to cancel the dependence on the scale in the effective theory [19]. The fact the our final result for the pressure is independent of the renormalization scale is a nontrivial check of the calculations. Furthermore, by choosing $\Lambda = gT/\sqrt{24}$ and using the solutions to the evolution equations, we were able to sum up leading logarithms of the form $g^{2n+3}\log^n(g)$ and e.g. subleading logarithms of the form $g^{2n+5}\log^n(g)$, where $n = 2, 3, \ldots$. as well as obtaining the coefficient of the $g^8\log(g)$ term.

As pointed out in ref. [15], it would be advantageous to develop the machinery of calculating complicated multiloop sum-integrals in an automated fashion as has been done for

Feynman diagrams at zero temperature. Perhaps such techniques could provide analytical expressions for the constants that today are known only numerically. This is necessary if one wants to tackle the formidable problem of calculating the hard part of the $g^6$-contribution to the free energy of QCD.


## Acknowledgments

The authors would like to thank T. Brauner, B. Kastening, and M. Laine for useful discussions. The authors would like to thank the Niels Bohr International Academy for kind hospitality.




## A Sum-integrals

In the imaginary-time formalism for thermal field theory, the 4-momentum $P = (P_0, \mathbf{p})$ is Euclidean with $P^2 = P_0^2 + \mathbf{p}^2$. The Euclidean energy $p_0$ has discrete values: $P_0 = 2n\pi T$ for bosons, where $n$ is an integer. Loop diagrams involve sums over $P_0$ and integrals over $\mathbf{p}$. With dimensional regularization, the integral is generalized to $d = 3-2\epsilon$ spatial dimensions. We define the dimensionally regularized sum-integral by

$$\sum\!\!\!\!\!\!\int_P \equiv \left(\frac{e^\gamma \mu^2}{4\pi}\right)^\epsilon T \sum_{P_0 = 2n\pi T} \int \frac{d^{3-2\epsilon}p}{(2\pi)^{3-2\epsilon}} , \tag{A.1}$$

where $3-2\epsilon$ is the dimension of space and $\mu$ is an arbitrary momentum scale. The factor $(e^\gamma/4\pi)^\epsilon$ is introduced so that, after minimal subtraction of the poles in $\epsilon$ due to ultraviolet divergences, $\mu$ coincides with the renormalization scale of the $\overline{\rm MS}$ renormalization scheme.

### A.1 One-loop sum-integrals

The massless one-loop sum-integral is given by

$$\begin{aligned} \mathcal{I}_n &\equiv \sum\!\!\!\!\!\!\int_P \frac{1}{P^{2n}} \\ &= (e^{\gamma_E}\mu^2)^\epsilon \frac{\zeta(2n-3+2\epsilon)}{8\pi^2} \frac{\Gamma(n-\frac{3}{2}+\epsilon)}{\Gamma(\frac{1}{2})\Gamma(n)} (2\pi T)^{4-2n-2\epsilon} , \end{aligned} \tag{A.2}$$

where $\zeta(x)$ is Riemann's zeta function. Specifically, we need the sum-integrals

$$\begin{aligned} \mathcal{I}'_0 &\equiv \sum\!\!\!\!\!\!\int_P \log P^2 \\ &= -\frac{\pi^2 T^4}{45}\left[1 + \mathcal{O}(\epsilon)\right] , \end{aligned} \tag{A.3}$$

$$\begin{aligned} \mathcal{I}_1 = \frac{T^2}{12}\left(\frac{\mu}{4\pi T}\right)^{2\epsilon} \Bigg[1 + \left(2 + 2\frac{\zeta'(-1)}{\zeta(-1)}\right)\epsilon + \\ + \left(4 + \frac{\pi^2}{4} + 4\frac{\zeta'(-1)}{\zeta(-1)} + 2\frac{\zeta''(-1)}{\zeta(-1)}\right)\epsilon^2 + \mathcal{O}(\epsilon^3)\Bigg] , \end{aligned} \tag{A.4}$$

$$\mathcal{I}_2 = \frac{1}{(4\pi)^2}\left(\frac{\mu}{4\pi T}\right)^{2\epsilon}\left[\frac{1}{\epsilon} + 2\gamma_E + \left(\frac{\pi^2}{4} - 4\gamma_1\right)\epsilon + \mathcal{O}(\epsilon^2)\right] , \tag{A.5}$$

$$\mathcal{I}_3 = \frac{1}{(4\pi)^4 T^2}\left[2\zeta(3) + \mathcal{O}(\epsilon)\right] . \tag{A.6}$$

### A.2 Two-loop sum-integrals

We need three two-loop sum-integral that are listed below:

$$\begin{aligned}
\mathcal{I}_{\rm sun} &= \sum\!\!\!\!\!\!\int_{PQ} \frac{1}{P^2Q^2(P+Q)^2} \\
&= \mathcal{O}(\epsilon) , \qquad\qquad (A.7)\\
\sum\!\!\!\!\!\!\int_{PQ} \frac{P^2+(2/d)p^2}{P^6Q^2(P+Q)^2} &= \frac{3}{4(4\pi)^4}\left(\frac{\mu}{4\pi T}\right)^{4\epsilon}\left[\frac{1}{\epsilon^2}+\left(\frac{5}{6}+4\gamma_E\right)\frac{1}{\epsilon}+\frac{89}{36}+\frac{\pi}{2}+\right. \\
&\qquad \left.+\frac{10}{3}\gamma_E+4\gamma_E^2-8\gamma_1+\mathcal{O}(\epsilon)\right] , \qquad (A.8)\\
\sum\!\!\!\!\!\!\int_{PQ} \frac{P^2-(4/d)p^2}{P^6Q^2(P+Q)^2} &= \frac{1}{4(4\pi)^4}\left(\frac{\mu}{4\pi T}\right)^{4\epsilon}\left[\frac{1}{\epsilon}+\frac{19}{6}+4\gamma_E+\mathcal{O}(\epsilon)\right] . \qquad (A.9)
\end{aligned}$$

The setting-sun sum-integral was first calculated by Arnold and Zhai in refs. [5, 6]. The remaining two-loop sum-integrals were calculated by Braaten and Petitgirard [28, 29] using the techniques developed in [5, 6].

### A.3 Three-loop sum-integrals

We need the following three-loop sum-integrals:

$$\begin{aligned}
\mathcal{I}_{\rm ball} &= \sum\!\!\!\!\!\!\int_{PQR} \frac{1}{P^2Q^2R^2(P+Q+R)^2} \\
&= \frac{T^4}{24(4\pi)^2}\left(\frac{\mu}{4\pi T}\right)^{6\epsilon}\left[\frac{1}{\epsilon}+\frac{91}{15}+8\frac{\zeta'(-1)}{\zeta(-1)}-2\frac{\zeta'(-3)}{\zeta(-3)}+\mathcal{O}(\epsilon)\right] , \qquad (A.10)\\
\mathcal{I}'_{\rm ball} &= \sum\!\!\!\!\!\!\int_{PQR} \frac{1}{P^4Q^2R^2(P+Q+R)^2} \\
&= \frac{T^2}{8(4\pi)^4}\left(\frac{\mu}{4\pi T}\right)^{6\epsilon}\left[\frac{1}{\epsilon^2}+\left(\frac{17}{6}+4\gamma_E+2\frac{\zeta'(-1)}{\zeta(-1)}\right)\frac{1}{\epsilon}+\right. \\
&\quad \left.+\frac{1}{2}\gamma_E\left(17+15\gamma_E+12\frac{\zeta'(-1)}{\zeta(-1)}\right)+C'_{\rm ball}+\mathcal{O}(\epsilon)\right] , \qquad (A.11)
\end{aligned}$$

and

$$\begin{aligned}
&\sum\!\!\!\!\!\!\int_P \frac{1}{P^2}\left\{[\Pi(P)]^2-\frac{2}{(4\pi)^2\epsilon}\Pi(P)\right\} = \\
&\quad = -\frac{T^2}{4(4\pi)^4}\left(\frac{\mu}{4\pi T}\right)^{6\epsilon}\times \\
&\qquad \times\left\{\frac{1}{\epsilon^2}+\frac{1}{\epsilon}\left[\frac{4}{3}+2\frac{\zeta'(-1)}{\zeta(-1)}+4\gamma_E\right]+\right. \\
&\qquad\quad +\frac{1}{3}\left[46-8\gamma_E-16\gamma_E^2-104\gamma_1-24\gamma_E\log(2\pi)+24\log^2(2\pi)+\frac{45\pi^2}{4}+\right. \\
&\qquad\qquad \left.\left.+24\frac{\zeta'(-1)}{\zeta(-1)}+2\frac{\zeta''(-1)}{\zeta(-1)}+16\gamma_E\frac{\zeta'(-1)}{\zeta(-1)}\right]+C_I+\mathcal{O}(\epsilon)\right\} , \qquad (A.12)
\end{aligned}$$



JHEP08(2009)066

where the self-energy $\Pi(P)$ is defined as

$$\Pi(P) = \sum\!\!\!\!\!\!\int_Q \frac{1}{Q^2(P+Q)^2} \,, \tag{A.13}$$

and $C'_{\rm ball} = 48.7976$ and $C_I = -38.4672$. The massless basketball sum-integral was first calculated by Arnold and Zhai in refs. [5, 6]. The sum-integral eq. (A.11) was calculated by Gynther *et al.* in ref. [15]. The expression for the sum-integral eq. (A.12) was calculated in ref. [21].

### A.4 Four-loop sum-integrals

We also need a single four-loop sum-integral which was calculated in ref. [15]:

$$\begin{aligned}
&\sum\!\!\!\!\!\!\int_P \left\{ [\Pi(P)]^3 - \frac{3}{(4\pi)^2\epsilon}[\Pi(P)]^2 \right\} = \\
&\quad = -\frac{T^4}{16(4\pi)^4} \times \\
&\qquad \times \left[ \frac{1}{\epsilon^2} + \left( 4\log\frac{\mu}{4\pi T} + \frac{10}{3} + 4\frac{\zeta'(-1)}{\zeta(-1)} \right) \frac{1}{\epsilon} + (2\log\frac{\mu}{4\pi T} + \gamma_E)^2 + \right. \\
&\qquad\quad \left. + \left( \frac{6}{5} - 2\gamma_E + 4\frac{\zeta'(-3)}{\zeta(-3)} \right) (2\log\frac{\mu}{4\pi T} + \gamma_E) + C^a_{\rm triangle} \right] - \\
&\qquad - \frac{T^4}{512(4\pi)^2} \left[ \frac{1}{\epsilon} + 8\log\frac{\mu}{4\pi T} + 4\gamma_E + C^b_{\rm triangle} \right] + \mathcal{O}(\epsilon) \,,
\end{aligned} \tag{A.14}$$

where $C^a_{\rm triangle} = -25.7055$ and $C^b_{\rm triangle} = 28.9250$.

## B Three-dimensional integrals

Dimensional regularization can be used to regularize both the ultraviolet divergences and infrared divergences in 3-dimensional integrals over momenta. The spatial dimension is generalized to $d = 3 - 2\epsilon$ dimensions. Integrals are evaluated at a value of $d$ for which they converge and then analytically continued to $d = 3$. We use the integration measure

$$\int_p \equiv \left( \frac{e^\gamma \mu^2}{4\pi} \right)^\epsilon \int \frac{d^{3-2\epsilon}p}{(2\pi)^{3-2\epsilon}} \,. \tag{B.1}$$

### B.1 One-loop integrals

The one-loop integral is given by

$$\begin{aligned}
I_n &\equiv \int_p \frac{1}{(p^2+m^2)^n} \\
&= \frac{1}{8\pi} (e^{\gamma_E}\mu^2)^\epsilon \frac{\Gamma(n - \frac{3}{2} + \epsilon)}{\Gamma(\frac{1}{2})\Gamma(n)} m^{3-2n-2\epsilon} \,.
\end{aligned} \tag{B.2}$$

Specifically, we need

$$\begin{aligned}
I'_0 &\equiv \int_p \log(p^2+m^2) \\
&= -\frac{m^3}{6\pi} \left( \frac{\mu}{2m} \right)^{2\epsilon} \left[ 1 + \frac{8}{3}\epsilon + \left( \frac{52}{9} + \frac{\pi^2}{4} \right) \epsilon^2 + \mathcal{O}\left(\epsilon^3\right) \right] \,,
\end{aligned} \tag{B.3}$$



JHEP08(2009)066

$$I_1 = -\frac{m}{4\pi}\left(\frac{\mu}{2m}\right)^{2\epsilon}\left[1+2\epsilon+\left(4+\frac{\pi^2}{4}\right)\epsilon^2+\mathcal{O}\left(\epsilon^3\right)\right], \tag{B.4}$$

$$I_2 = \frac{1}{8\pi m}\left(\frac{\mu}{2m}\right)^{2\epsilon}\left[1+\frac{\pi^2}{4}\epsilon^2+\mathcal{O}\left(\epsilon^3\right)\right], \tag{B.5}$$

$$I_3 = \frac{1}{32\pi m^3}\left(\frac{\mu}{2m}\right)^{2\epsilon}\left[1+2\epsilon+\frac{\pi^2}{4}\epsilon^2+\mathcal{O}\left(\epsilon^3\right)\right], \tag{B.6}$$

$$I_4 = \frac{1}{64\pi m^5}\left(\frac{\mu}{2m}\right)^{2\epsilon}\left[1+\frac{8}{3}\epsilon+\left(\frac{4}{3}+\frac{\pi^2}{4}\right)\epsilon^2+\mathcal{O}\left(\epsilon^3\right)\right]. \tag{B.7}$$

### B.2 Two-loop integrals

We need the following two-loop integral

$$\begin{aligned} I_{\rm sun}(p=im) &= \int_{qr}\frac{1}{q^2+m^2}\frac{1}{r^2+m^2}\frac{1}{(\mathbf{p}+\mathbf{q}+\mathbf{r})^2+m^2}\bigg|_{p=im} \\ &= \frac{1}{4(4\pi)^2}\left(\frac{\mu}{2m}\right)^{4\epsilon}\times \\ &\quad\times\left[\frac{1}{\epsilon}+6-8\log 2+\left(36-\frac{\pi^2}{6}-48\log 2+8\log^2 2\right)\epsilon+\mathcal{O}(\epsilon^2)\right]. \end{aligned} \tag{B.8}$$

This integral was calculated to order $\epsilon^0$ in ref. [19] and to order $\epsilon$ in refs. [28, 29].

### B.3 Three-loop integrals

We need the following three-loop integrals:

$$\begin{aligned} I_{\rm ball} &= \int_{pqr}\frac{1}{p^2+m^2}\frac{1}{q^2+m^2}\frac{1}{r^2+m^2}\frac{1}{(\mathbf{p}+\mathbf{q}+\mathbf{r})^2+m^2} \\ &= -\frac{m}{(4\pi)^3}\left(\frac{\mu}{2m}\right)^{6\epsilon}\times \\ &\quad\times\left[\frac{1}{\epsilon}+8-4\log 2+4\left(13+\frac{17}{48}\pi^2-8\log 2+\log^2 2\right)\epsilon+\mathcal{O}\left(\epsilon^2\right)\right], \end{aligned} \tag{B.9}$$

$$\begin{aligned} I'_{\rm ball} &= \int_{pqr}\frac{1}{(p^2+m^2)^2}\frac{1}{q^2+m^2}\frac{1}{r^2+m^2}\frac{1}{(\mathbf{p}+\mathbf{q}+\mathbf{r})^2+m^2} \\ &= \frac{1}{8m(4\pi)^3}\left(\frac{\mu}{2m}\right)^{6\epsilon}\times \\ &\quad\times\left[\frac{1}{\epsilon}+2-4\log 2+4\left(1+\frac{17}{48}\pi^2-2\log 2+\log^2 2\right)\epsilon+\mathcal{O}\left(\epsilon^2\right)\right], \end{aligned} \tag{B.10}$$

$$\begin{aligned} J &= \int_{pqr}\frac{1}{(q^2+m^2)^2}\frac{1}{(\mathbf{p}+\mathbf{q})^2+m^2}\frac{1}{(r^2+m^2)^2}\frac{1}{(\mathbf{p}+\mathbf{r})^2+m^2} \\ &= \frac{1}{16m^3(4\pi)^3}\left(\frac{\mu}{2m}\right)^{6\epsilon}\left[1+\mathcal{O}\left(\epsilon\right)\right], \end{aligned} \tag{B.11}$$

$$\begin{aligned} K &= \int_{pqr}\frac{1}{(q^2+m^2)^3}\frac{1}{(\mathbf{p}+\mathbf{q})^2+m^2}\frac{1}{r^2+m^2}\frac{1}{(\mathbf{p}+\mathbf{r})^2+m^2} \\ &= \frac{1}{32m^3(4\pi)^3}\left(\frac{\mu}{2m}\right)^{6\epsilon}\left[\frac{1}{\epsilon}+5-4\log 2+\mathcal{O}\left(\epsilon\right)\right]. \end{aligned} \tag{B.12}$$

The massive basketball was calculated in ref. [19] to order $\epsilon^0$, and to order $\epsilon$ in ref. [9]. $I'_{\rm ball}$ can be obtained by differentiation of $I_{\rm ball}$ with respect to $m$. The 3-loop integrals $J$ and $K$ are calculated in appendix C.

### B.4 Four-loop integrals

We need the following two four-loop integrals

$$\begin{aligned} I_{\rm triangle} &= \int_{pqrs} \frac{1}{q^2+m^2}\frac{1}{(\mathbf{p}+\mathbf{q})^2+m^2}\frac{1}{r^2+m^2}\frac{1}{(\mathbf{p}+\mathbf{r})^2+m^2}\frac{1}{s^2+m^2}\frac{1}{(\mathbf{p}+\mathbf{s})^2+m^2} \\ &= \frac{\pi^2}{32(4\pi)^4}\left(\frac{\mu}{2m}\right)^{8\epsilon}\left[\frac{1}{\epsilon}+2+4\log 2-\frac{84}{\pi^2}\zeta(3)+\mathcal{O}(\epsilon)\right], \qquad \text{(B.13)} \\ I'_{\rm triangle} &= \int_{pqrs} \frac{1}{(q^2+m^2)^2}\frac{1}{(\mathbf{p}+\mathbf{q})^2+m^2}\frac{1}{r^2+m^2}\frac{1}{(\mathbf{p}+\mathbf{r})^2+m^2}\frac{1}{s^2+m^2}\frac{1}{(\mathbf{p}+\mathbf{s})^2+m^2} \\ &= \frac{\pi^2}{48m^2(4\pi)^4}\left(\frac{\mu}{2m}\right)^{8\epsilon}\left[1+\mathcal{O}(\epsilon)\right]. \qquad \text{(B.14)} \end{aligned}$$

The triangle diagram was calculated in ref. [30]. The diagram $I'_{\rm triangle}$ follows from the triangle diagram upon differentiation with respect to $m^2$.

### B.5 Five-loop integrals

$$\begin{aligned} I_{\rm rung} &= \int_{pqrst} \frac{1}{q^2+m^2}\frac{1}{(\mathbf{p}+\mathbf{q})^2+m^2}\frac{1}{r^2+m^2}\frac{1}{(\mathbf{p}+\mathbf{r})^2+m^2}\frac{1}{s^2+m^2}\frac{1}{(\mathbf{p}+\mathbf{s})^2+m^2}\times \\ &\quad\times\frac{1}{t^2+m^2}\frac{1}{(\mathbf{p}+\mathbf{t})^2+m^2} \\ &= \frac{1}{2m(4\pi)^5}\left(\frac{\mu}{2m}\right)^{10\epsilon}\left[\pi^2\log 2-\frac{9}{2}\zeta(3)+\mathcal{O}(\epsilon)\right], \qquad \text{(B.15)} \\ I_{\rm doublesun} &= \int_{pqrst} \frac{1}{(p^2+m^2)^2}\frac{1}{q^2+m^2}\frac{1}{r^2+m^2}\frac{1}{(\mathbf{p}+\mathbf{q}+\mathbf{r})^2+m^2}\times \\ &\quad\times\frac{1}{s^2+m^2}\frac{1}{t^2+m^2}\frac{1}{(\mathbf{p}+\mathbf{s}+\mathbf{t})^2+m^2} \\ &= \frac{1}{32m(4\pi)^5}\left(\frac{\mu}{2m}\right)^{10\epsilon}\times \\ &\quad\times\left[\frac{1}{\epsilon^2}+(4-8\log 2)\frac{1}{\epsilon}-4+\frac{31}{12}\pi^2-96\log 3+64\log 2+104\log^2 2\right. \\ &\quad\left.+80\mathrm{Li}_2(\tfrac{1}{4})+\mathcal{O}(\epsilon)\right], \qquad \text{(B.16)} \\ I_{\rm 4j} &= \int_{pqrst} \frac{1}{q^2+m^2}\frac{1}{(\mathbf{p}+\mathbf{q})^2+m^2}\frac{1}{(\mathbf{p}+\mathbf{r})^2+m^2}\frac{1}{(\mathbf{t}+\mathbf{r})^2+m^2}\frac{1}{r^2+m^2}\times \\ &\quad\times\frac{1}{(\mathbf{p}+\mathbf{s})^2+m^2}\frac{1}{(\mathbf{t}+\mathbf{s})^2+m^2}\frac{1}{s^2+m^2} \\ &= \frac{1}{m(4\pi)^5}\left(\frac{\mu}{2m}\right)^{10\epsilon}\left[C_{\rm 4j}+\mathcal{O}(\epsilon)\right], \qquad \text{(B.17)} \end{aligned}$$

where $C_{\rm 4j}=0.443166$. The integrals are calculated in appendix C.

## C Explicit calculations

In this appendix, we calculate explicitly some of the multi-loop vacuum diagrams in three dimensions.

The three-loop integral $J$ in eq. (B.11) can be written as

$$J = \int_p \left[I'_{\rm bubble}(p)\right]^2 , \tag{C.1}$$

where

$$I'_{\rm bubble}(p) = \int_q \frac{1}{(q^2+m^2)^2}\frac{1}{(\mathbf{p}+\mathbf{q})^2+m^2} . \tag{C.2}$$

By power counting it is easy to see that both $J$ and $I'_{\rm bubble}$ are finite in three spatial dimension. The latter then reduces to

$$I'_{\rm bubble}(p) = \frac{1}{8\pi m}\frac{1}{p^2+4m^2} . \tag{C.3}$$

Inserting eq. (C.3) into eq. (C.1) and using eq. (B.5) with $\epsilon = 0$ and a mass of $2m$. we obtain eq. (B.11).

The integral $K$ can be calculated by noting the relation

$$I''_{\rm ball} = -2K - 3J . \tag{C.4}$$

The integral $I_{\rm rung}$ in (B.15) can be written as

$$I_{\rm rung} = \int_p I^4_{\rm bubble}(p) , \tag{C.5}$$

where

$$I_{\rm bubble}(p) = \int_q \frac{1}{q^2+m^2}\frac{1}{(\mathbf{p}+\mathbf{q})^2+m^2} . \tag{C.6}$$

The integrals $I_{\rm rung}$ and $I_{\rm bubble}(p)$ are convergent in three dimensions. The latter then reduces to

$$I_{\rm bubble}(p) = \frac{1}{4\pi p}\arctan\frac{p}{2m} . \tag{C.7}$$

$I_{\rm rung}$ can now be easily found and the result is given by eq. (B.15).

The diagram appearing in $\mathcal{F}_{5i}$ can be written as

$$I_{\rm doublesun} = \int_p \frac{1}{(p^2+m^2)^2} I^2_{\rm sun}(p) , \tag{C.8}$$

where $I_{\rm sun}(p)$ is

$$I_{\rm sun}(p) = \int_{qr} \frac{1}{q^2+m^2}\frac{1}{r^2+m^2}\frac{1}{(\mathbf{p}+\mathbf{q}+\mathbf{r})^2+m^2} . \tag{C.9}$$

In order to isolate the divergences in (C.8), we add and subtract $I_{\rm sun}(p=im)$, and rewrite it as

$$I_{\rm doublesun} = \int_p \frac{1}{(p^2+m^2)^2} \times \\ \times \Big\{ [I_{\rm sun}(p) - I_{\rm sun}(p=im)]^2 + 2 I_{\rm sun}(p) I_{\rm sun}(p=im) - \\ - I^2_{\rm sun}(p=im) \Big\} \ . \tag{C.10}$$

We denote the three terms above by $I_{\rm ds1}$, $I_{\rm ds2}$, and $I_{\rm ds3}$. We first consider $I_{\rm ds1}$. The difference $I_{\rm sun}(p) - I_{\rm sun}(p=im)$ is finite and can be calculated directly in three dimensions. We obtain

$$I_{\rm sun}(p) - I_{\rm sun}(p=im) = \\ = -\frac{1}{(4\pi)^2} \left(\frac{\mu}{2m}\right)^{4\epsilon} \left[ \frac{3m}{p} \arctan \frac{p}{3m} + \frac{1}{2} \ln \frac{p^2+9m^2}{64 m^2} + \mathcal{O}(\epsilon) \right] \ . \tag{C.11}$$

The first term $I_{\rm ds1}$ is finite in three dimensions. Using eq. (C.11), we obtain

$$I_{\rm ds1} = \frac{1}{2m(4\pi)^5} \left(\frac{\mu}{2m}\right)^{10\epsilon} \left[ 6 \log^2 2 - 6 \log 3 + 4 \log 2 + 5 {\rm Li}_2(\tfrac{1}{4}) + \mathcal{O}(\epsilon) \right] \ . \tag{C.12}$$

The second term $I_{\rm ds2}$ can be written as

$$I_{\rm ds2} = 2 I_{\rm sun}(p=im) \int_{pqr} \frac{1}{(p^2+m^2)^2} \frac{1}{q^2+m^2} \frac{1}{r^2+m^2} \frac{1}{(\mathbf{p}+\mathbf{q}+\mathbf{r})^2+m^2} \ . \tag{C.13}$$

Using eqs. (B.8) and (B.10), we obtain

$$I_{\rm ds2} = \frac{1}{16 m (4\pi)^5} \left(\frac{\mu}{2m}\right)^{10\epsilon} \times \\ \times \left[ \frac{1}{\epsilon^2} + (8 - 12 \log 2) \frac{1}{\epsilon} + 52 + \frac{5}{4} \pi^2 - 96 \log 2 + 44 \log^2 2 + \mathcal{O}(\epsilon) \right] \ . \tag{C.14}$$

Similarly, $I_{\rm ds3}$ can be written as

$$I_{\rm ds3} = -I^2_{\rm sun}(p=im) I_2 \\ = -\frac{1}{32 m (4\pi)^5} \left(\frac{\mu}{2m}\right)^{10\epsilon} \times \\ \times \left[ \frac{1}{\epsilon^2} + (12 - 16 \log 2) \frac{1}{\epsilon} + 108 - \frac{\pi^2}{12} - 192 \log 2 + 80 \log^2 2 + \mathcal{O}(\epsilon) \right] \ . \tag{C.15}$$

Adding eqs. (C.12), (C.14), and (C.15), we obtain

$$I_{\rm doublesun} = \frac{1}{32 m (4\pi)^5} \left(\frac{\mu}{2m}\right)^{10\epsilon} \times \\ \times \left[ \frac{1}{\epsilon^2} + (4 - 8 \log 2) \frac{1}{\epsilon} - 4 + \frac{31}{12} \pi^2 - 96 \log 3 + 64 \log 2 + 104 \log^2 2 + \right. \\ \left. + 80 {\rm Li}_2(\tfrac{1}{4}) + \mathcal{O}(\epsilon) \right] \ . \tag{C.16}$$

Let us finally discuss the five-loop integral appearing in eq. (B.17). It can be written as

$$I_{4\mathrm{j}} = \int_{pq} I_{\mathrm{bubble}}(p)\,[\Pi_{\mathrm{tri}}(p,q)]^2 \ , \tag{C.17}$$

where

$$\Pi_{\mathrm{tri}}(p,q) = \int_r \frac{1}{r^2+m^2}\frac{1}{(\mathbf{p}+\mathbf{r})^2+m^2}\frac{1}{(\mathbf{q}+\mathbf{r})^2+m^2} \ . \tag{C.18}$$

The diagram (C.18) is finite in three dimensions and can be written as [31, 32]

$$\Pi_{\mathrm{tri}}(p,q) = \frac{\arctan(\sqrt{D}/C)}{8\pi\sqrt{D}} \ , \tag{C.19}$$

where

$$C = \frac{p^2+q^2+\mathbf{p}\cdot\mathbf{q}+4m^2}{m^2} \ , \tag{C.20}$$

$$D = \frac{p^2q^2(\mathbf{p}-\mathbf{q})^2+4m^2[p^2q^2-(\mathbf{p}\cdot\mathbf{q})^2]}{4m^6} \ . \tag{C.21}$$

The integral (C.17) can now be evaluated numerically by first averaging over angles and then integrating over $p$ and $q$. This yields

$$I_{4\mathrm{j}} = \frac{1}{m(4\pi)^5}\left(\frac{\mu}{2m}\right)^{10\epsilon}[0.443166] \ . \tag{C.22}$$

## References


[1] J.-P. Blaizot, E. Iancu and A. Rebhan, *Thermodynamics of the high-temperature quark gluon plasma*, hep-ph/0303185 [SPIRES].

[2] D.H. Rischke, *The quark-gluon plasma in equilibrium*, *Prog. Part. Nucl. Phys.* **52** (2004) 197 [nucl-th/0305030] [SPIRES].

[3] U. Kraemmer and A. Rebhan, *Advances in perturbative thermal field theory*, *Rept. Prog. Phys.* **67** (2004) 351 [hep-ph/0310337] [SPIRES].

[4] J.O. Andersen and M. Strickland, *Resummation in hot field theories*, *Ann. Phys.* **317** (2005) 281 [hep-ph/0404164] [SPIRES].

[5] P. Arnold and C.-X. Zhai, *The three loop free energy for pure gauge QCD*, *Phys. Rev.* **D 50** (1994) 7603 [hep-ph/9408276] [SPIRES].

[6] P. Arnold and C.-X. Zhai, *The three loop free energy for high temperature QED and QCD with fermions*, *Phys. Rev.* **D 51** (1995) 1906 [hep-ph/9410360] [SPIRES].

[7] C.-X. Zhai and B.M. Kastening, *The Free energy of hot gauge theories with fermions through* $g^5$, *Phys. Rev.* **D 52** (1995) 7232 [hep-ph/9507380] [SPIRES].

[8] E. Braaten and A. Nieto, *Renormalization effects in a dilute Bose gas*, *Phys. Rev.* **B 55** (1997) 8090 [hep-th/9609047] [SPIRES].

[9] K. Kajantie, M. Laine, K. Rummukainen and Y. Schroder, *Four-loop vacuum energy density of the* SU($N_c$) *+ adjoint Higgs theory*, *JHEP* **04** (2003) 036 [hep-ph/0304048] [SPIRES].




JHEP08(2009)066

[10] A.D. Linde, *Infrared problem in thermodynamics of the Yang-Mills gas*, *Phys. Lett.* **B 96** (1980) 289 [SPIRES].

[11] D.J. Gross, R.D. Pisarski and L.G. Yaffe, *QCD and instantons at finite temperature*, *Rev. Mod. Phys.* **53** (1981) 43 [SPIRES].

[12] A. Hietanen, K. Kajantie, M. Laine, K. Rummukainen and Y. Schroder, *Plaquette expectation value and gluon condensate in three dimensions*, *JHEP* **01** (2005) 013 [hep-lat/0412008] [SPIRES].

[13] A. Hietanen and A. Kurkela, *Plaquette expectation value and lattice free energy of three-dimensional* SU($N$) *gauge theory*, *JHEP* **11** (2006) 060 [hep-lat/0609015] [SPIRES].

[14] F. Di Renzo, M. Laine, V. Miccio, Y. Schroder and C. Torrero, *The leading non-perturbative coefficient in the weak-coupling expansion of hot QCD pressure*, *JHEP* **07** (2006) 026 [hep-ph/0605042] [SPIRES].

[15] A. Gynther, M. Laine, Y. Schroder, C. Torrero and A. Vuorinen, *Four-loop pressure of massless O(N) scalar field theory*, *JHEP* **04** (2007) 094 [hep-ph/0703307] [SPIRES].

[16] P.H. Ginsparg, *First order and second order phase transitions in gauge theories at finite temperature*, *Nucl. Phys.* **B 170** (1980) 388 [SPIRES].

[17] T. Appelquist and R.D. Pisarski, *High-temperature Yang-Mills theories and three-dimensional quantum chromodynamics*, *Phys. Rev.* **D 23** (1981) 2305 [SPIRES].

[18] N.P. Landsman, *Limitations to dimensional reduction at high temperature*, *Nucl. Phys.* **B 322** (1989) 498 [SPIRES].

[19] E. Braaten and A. Nieto, *Effective field theory approach to high temperature thermodynamics*, *Phys. Rev.* **D 51** (1995) 6990 [hep-ph/9501375] [SPIRES].

[20] K. Kajantie, M. Laine, K. Rummukainen and M.E. Shaposhnikov, *Generic rules for high temperature dimensional reduction and their application to the standard model*, *Nucl. Phys.* **B 458** (1996) 90 [hep-ph/9508379] [SPIRES].

[21] J.O. Andersen and L. Kyllingstad, *Four-loop screened perturbation theory*, *Phys. Rev.* **D 78** (2008) 076008 [arXiv:0805.4478] [SPIRES].

[22] K. Farakos, K. Kajantie, K. Rummukainen and M.E. Shaposhnikov, 3$D$ *physics and the electroweak phase transition: perturbation theory*, *Nucl. Phys.* **B 425** (1994) 67 [hep-ph/9404201] [SPIRES].

[23] J. Frenkel, A.V. Saa and J.C. Taylor, *The pressure in thermal scalar field theory to three loop order*, *Phys. Rev.* **D 46** (1992) 3670 [SPIRES].

[24] R. Parwani and H. Singh, *The pressure of hot* ($g^2\phi^4$) *theory at order* $g^5$, *Phys. Rev.* **D 51** (1995) 4518 [hep-th/9411065] [SPIRES].

[25] F. Karsch, A. Patkos and P. Petreczky, *Screened perturbation theory*, *Phys. Lett.* **B 401** (1997) 69 [hep-ph/9702376] [SPIRES].

[26] J.O. Andersen, E. Braaten and M. Strickland, *Screened perturbation theory to three loops*, *Phys. Rev.* **D 63** (2001) 105008 [hep-ph/0007159] [SPIRES].

[27] J.O. Andersen and M. Strickland, *Mass expansions of screened perturbation theory*, *Phys. Rev.* **D 64** (2001) 105012 [hep-ph/0105214] [SPIRES].

JHEP08(2009)066

[28] E. Braaten and E. Petitgirard, *Solution to the 3-loop $\phi$-derivable approximation for scalar thermodynamics*, *Phys. Rev.* **D 65** (2002) 041701 [hep-ph/0106045] [SPIRES].

[29] E. Braaten and E. Petitgirard, *Solution to the 3-loop $\phi$-derivable approximation for massless scalar thermodynamics*, *Phys. Rev.* **D 65** (2002) 085039 [hep-ph/0107118] [SPIRES].

[30] A. Vuorinen, *The pressure of QCD at finite temperature and quark number density*, hep-ph/0402242 [SPIRES].

[31] B.G. Nickel, *Evaluation of simple Feynman graphs*, *J. Math. Phys.* **19** (1978) 542 [SPIRES].

[32] B. Kastening, private communication.

JHEP08(2009)066