Kundt waves in modified gravitational models

A Baykal[†1], **T Dereli**[‡2]

[1]Niğde Ömer Halisdemir University, Faculty of Arts and Sciences, Department of Physics, Bor yolu üzeri, Merkez Yerleşke, 51240 Niğde, TURKEY

[2] Department of Physics, College of Sciences, Koç University, 34450 Sarıyer, İstanbul, TURKEY

E-mail: [†]abaykal@ohu.edu.tr [‡]tdereli@ku.edu.tr

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Abstract. Kundt's class of gravitational wave metrics belonging to Petrov type-N are constructed for generic $f(R)$ model with minimally coupled electromagnetic field, a Gauss-Bonnet extended gravitational model for vacuum in four spacetime dimensions, and RF^2-type nonminimally coupled Einstein-Maxwell models including a cosmological constant.

1. Introduction

The Kundt class of metrics is a well known class of solutions containing several distinct Petrov types of algebraically special metrics. Non-twisting Petrov type-N solutions with a cosmological constant and an expanding null geodesic congruence belong to the Robinson-Trautman class, whereas those with a non-expanding null geodesic congruence belong to Kundt's class. These classes of solutions have been classified and interpreted as the gravitational waves propagating in constant curvature spacetimes by Bičák, and Podolský [1] relatively recently despite the fact that Kundt's class of metrics was introduced by Kundt [2] long time ago.

More recently, the invariant classification of the Kundt's class of solutions for vacuum are given by McNutt et al.. [3] by using Cartan-Karlhede algorithm in four spacetime dimensions. Bičák and Pravda [4] also discussed curvature invariants of all type-N metrics and they showed that the curvature invariants belonging to the Kundt class of metrics are trivial (See, also Ref. [5]).

In a more general geometrical context, the plane-fronted gravitational wave metrics have also been investigated in metric-affine and in Riemann-Cartan geometries by a number of researchers [10–19]. The generalization of this class of metrics to higher spacetime dimensions was presented by Obukhov [14]. Garcia and Plebanski [20], and Salazar et al. [21] studied the nontwisting type-N solutions to Einstein-Maxwell theory including a cosmological constant, whereas the Kundt's class of metrics belonging to Petrov type-III with a cosmological constant have been studied by Griffiths et al.. [8]. Petrov type-N and type-II Kundt class of metrics in type-D and type-O backgrounds are studied by Podolsky and Ortaggio [9], who also showed that the Einstein-Maxwell theory does not admit a conformally-flat solution with a cosmological constant.

In higher dimensions, using a higher dimensional version of the Newman-Penrose formalism [25, 26] Ortaggio et al.. [29] constructed Kundt's class of solutions to Einstein vacuum field equations with a cosmological constant belonging to type-III and type-N solutions according to the classification scheme of the Weyl tensor in higher dimensions [26–30] that is based on alignment properties of Weyl tensor.

A higher dimensional classification scheme for the Kundt's class of metrics is also introduced by Podolský and Švarc [6]. Moreover, Podolskỳ and Žofka surveyed the constraints on the matter fields admitted by the Kundt class of metrics in higher dimensions [7]. Furthermore, by considering the higher order Euler-Poincare forms complementing the Einstein-Hilbert Lagrangian, Gleiser and Dotti constructed wave solutions [22] using this class of metrics. Macías and Lozano [23] constructed plane-fronted wave solutions to a four dimensional model that is obtained from a five dimensional Chern-Simons gravity in five spacetime dimensions by means of a Klauza-Klein reduction on a circle.

By making use of a Kerr-Schild type metric ansatz, Debney [31] constructed Kundt class of solutions to Einstein-Maxwell equations. More recently, Gürses, et al.. [32] have shown that Kerr-Schild type ansatz of the form $g = g_{backgr.} + V\lambda \otimes \lambda$ which involves a shear-free, twist-free, and nonexpanding null congruence $\lambda = \lambda_\alpha dx^\alpha$ on a given maximally symmetric background metric $g_{backgr.}$, are universal metrics [33–37]. The universal metrics solve the field equations of a given metric theory governed by a gravitational Lagrangian of the general form $f(g, R, \nabla R \ldots)$ involving Riemann tensor (symbolically denoted by R here for simplicity), and also its covariant derivatives.

In three spacetime dimensions, Chow et al.. [38] studied the Petrov-Segre classification of the solutions to the topologically massive gravity [39] including a cosmological constant and they reduce all the solutions in the literature to three distinct cases: the biaxial timelike-squashed case and the spacelike-squashed AdS^3 case and the AdS pp-waves case, belonging to Petrov-Segre types D_t, D_s and N, respectively. In a subsequent paper [40], Chow et al. constructed new solutions belonging to the Kundt class of metrics having Petrov-Segre types D, N and III, and they also showed that such solutions have constant scalar curvature polynomial invariants.

The present work can be considered as extensions of the previous works by Mohseni [41] who previously discussed the plane fronted-wave solutions to some modified gravitational models, and the works of Dereli and Sert [42], and Gürses and Halil [43] that introduce pp-wave solutions to Prasanna's and the Horndeski's nonminimally coupled Einstein-Maxwell models, respectively. In the same vein, Macías and Lozano [23] previously studied Kundt class solutions to the models involving the nonminimal coupling of the Faraday tensor to gravitational theories obtained by kaluza-Klein reduction of the Chern-Simons curvature invariants.

The paper is organized as follows. We use a null tetrad formulation [44] and the exterior algebra of differential forms on pseudo-Riemannian manifolds. The metric ansatz and a null coframe that will be used in all subsequent sections are introduced in the following section. A technique, due to Osváth et al.. [45], to obtain particular solution to equation for the profile function is also discussed in this section. Sec. II also serves to set the geometrical notation and conventions as well. In Appendix B, the original of Newman-Penrose spin coefficients and curvature scalars are identified in terms of the geometrical quantities introduced.

The metric ansatz is used to construct wave solutions to generic $f(R)$ model [46] in Sec. III, Gauss-Bonnet (GB) extended vacuum Einstein-Maxwell model in four spacetime dimensions in Sec. IV, the nonminimally coupled Einstein Maxwell model [49] introduced by Prasanna in Sec. V and to the Horndeski's generalized Einstein-Maxwell model [50] in Sec. VI. We show that the equations for the wave profile for nonminimal couplings considered reduce to that of an effective Einstein-Maxwell equations for maximally symmetric background with a cosmological constant. For the Horndeski model, an explicit expression is derived for the particular solution.

A derivation of the Horndeski's unique model using a Kaluza-Klein ansatz to reduction of the $(n+1)$-dimensional Euler-Poincarè invariant to n spacetime dimensions is presented. The field equations that follow from the reduced Lagrangian are derived by using an action principle in the Appendix A. The paper concludes with brief comments on the general properties of the solutions.

2. The metric ansatz

The class of metrics we study belongs to the general Kundt's class of metrics [2] which was introduced and studied by Osváth, et al.. [45]. In terms of some local null coordinates $\{x^\alpha\} = \{u, v, \zeta, \bar{\zeta}\}$, for $\alpha = 0, 1, 2, 3$, the metric ansatz [17] can be expressed in the form

$$g = -2\left(\frac{Q}{P}\right)^2 du(Sdu + dv) + \frac{2}{P^2}d\zeta d\bar{\zeta} \tag{1}$$

where the functions in the ansatz are defined as

$$P \equiv 1 + \frac{\lambda}{6}|\zeta|^2, \qquad Q \equiv 1 - \frac{\lambda}{6}|\zeta|^2, \qquad S \equiv -\frac{\lambda}{6}v^2 + \frac{Q}{2P}H(u,\zeta,\bar{\zeta}), \tag{2}$$

whereas the real profile function $H = H(u, \zeta, \bar{\zeta})$ and the constant parameter λ are to be determined by the gravitational field equations.

A simplified form of the ansatz (1) originally considered by Osvath et al.. [45] is adopted in the discussion below for a first orientations to Kundt wave type solutions in modified gravitational models (especially, for the nonminimally coupled RF^2 models) which do have fairly complicated field equations compared to the case with minimally coupled matter fields.

We make use of exterior algebra of differential forms in our calculations, also adopting a null coframe, however there are other choices of rigid frames as well. For example a semi-null coframe adopted in [14, 17] to study the same class of metrics.

We find it convenient to adopt the following Newman-Penrose (NP) null coframe

$$k = k_\alpha dx^\alpha = du, \quad l = l_\alpha dx^\alpha = \left(\frac{Q}{P}\right)^2 (Sdu + dv), \quad m = m_\alpha dx^\alpha = \frac{d\zeta}{P}, \tag{3}$$

for the metric ansatz given in eq. (1). In terms of the coframe 1-forms, the metric can be written in the form

$$g = \eta_{ab}\theta^a \otimes \theta^b \tag{4}$$

where the nonvanishing components of the flat metric are $-\eta_{01} = -\eta_{10} = \eta_{23} = \eta_{32} = +1$. A tensorial index belonging to the NP frame are raised/lowered by the flat metric η^{ab} and η_{ab} with $\{\theta^a\} = k, l, m, \bar{m}$ for $a = 0, 1, 2, 3$, respectively.

The coordinate components of tensorial quantities are labeled by the letters in the Greek alphabet, whereas the indices referring to an NP frame are labeled by the lowercase letters in the Latin alphabet. A numerical tensorial indice below refers exclusively to the null coframe defined in (3).

In terms of the NP formalism introduced, for example, in Refs. [51, 52], the frame fields associated to the coframe (3) can be written as

$$D = -(\partial_u - S\partial_v), \qquad \Delta = -\left(\frac{P}{Q}\right)^2 \partial_v, \qquad \delta = P\partial_{\bar{\zeta}}, \qquad \bar{\delta} = P\partial_\zeta. \tag{5}$$

The set of basis frame fields $\{e_a\}$ associated to (3) are explicitly given by $e_0 \equiv -\Delta = -l^\nu\partial_\mu$, $e_1 \equiv -D = -k^\mu\partial_\mu$, $e_2 \equiv \bar{\delta} = \bar{m}^\mu\partial_\mu$, and $e_3 \equiv \delta = m^\mu\partial_\mu$ with $k^\mu = g^{\mu\nu}k_\nu$, etc.. In terms of the frame fields (5), the exterior derivative acting on scalars can be written in the form

$$d = -k\Delta - lD + m\bar{\delta} + \bar{m}\delta \tag{6}$$

that is of practical value in calculations.

Because we make exclusive use of the differential forms, the symbols for the basis frame fields do not appear explicitly in the presentation below, and particularly the basis frame field $D = -e_1$ is not to be confused with the covariant exterior derivative, almost universally denoted also by D as well.

In terms of the NP null basis coframe, the invariant volume 4-form with a particular orientation is defined as $*1 = ik \wedge l \wedge m \wedge \bar{m}$ ($i^2 = -1$), and consequently, the Hodge dual of a p-form can be calculated by introducing completely antisymmetric permutation symbol $\epsilon_{0123} = +i$ relative to the null coframe basis defined in eq. (4).

With the indices referring to the NP null coframe, the connection 1-forms ${\omega^0}_3$ can be obtained by solving Cartan's first structure equations

$$d\theta^a + {\omega^a}_b \wedge \theta^b = 0 \tag{7}$$

for the metric (1), and they are explicitly given by

$${\omega^0}_3 = -\frac{\lambda}{3Q}\zeta k, \qquad {\omega^1}_2 = \frac{Q^2}{P}S_\zeta k - \frac{\lambda}{3Q}\bar{\zeta}l, \tag{8}$$

$${\omega^0}_0 - {\omega^3}_3 = \frac{\lambda}{3}vk - \frac{\lambda}{3Q}\left(1 + \frac{Q}{2}\right)\bar{\zeta}m - \frac{\lambda P}{6Q}\zeta\bar{m}, \tag{9}$$

where a coordinate subscript to a function denotes a partial derivative with respect to the coordinate.

The corresponding curvature 2-forms, which are defined in terms of the Riemann tensor as ${\Omega^a}_b = \frac{1}{2}{R^a}_{bcd}\theta^c \wedge \theta^d$, can be expressed in terms of the connection 1-forms by the Cartan's second structure equation in the form

$${\Omega^0}_3 = d{\omega^0}_3 + \left({\omega^0}_0 - {\omega^3}_3\right) \wedge {\omega^0}_3, \tag{10}$$

$${\Omega^1}_2 = d{\omega^1}_2 - \left({\omega^0}_0 - {\omega^3}_3\right) \wedge {\omega^1}_2, \tag{11}$$

$${\Omega^0}_0 - {\Omega^3}_3 = d\left({\omega^0}_0 - {\omega^3}_3\right) + 2{\omega^0}_3 \wedge {\omega^1}_2. \tag{12}$$

For the connection 1-forms given explicitly in eqs. (8) and (9), the corresponding curvature 2-forms take the form

$${\Omega^0}_3 = \frac{\lambda}{3}k \wedge m, \qquad {\Omega^0}_0 - {\Omega^3}_3 = -\frac{\lambda}{3}(k \wedge l - m \wedge \bar{m}), \tag{13}$$

$${\Omega^1}_2 = -\frac{PQ}{2}\left(H_{\zeta\zeta} + \frac{\lambda\bar{\zeta}}{3P}H_\zeta\right)k \wedge m - \frac{PQ}{2}\left(H_{\zeta\bar{\zeta}} + \frac{\lambda}{3P^2}H\right)k \wedge \bar{m} + \frac{\lambda}{3}l \wedge \bar{m}. \tag{14}$$

From the curvature expressions above, one can readily verify that $R = 6\lambda$. Consequently, one can show that the Einstein forms corresponding to the metric ansatz (1) have the form

$$*G^0 = \lambda * k, \qquad *G^2 = \lambda * m, \qquad *G^3 = \lambda * \bar{m}, \tag{15}$$

$$*G^1 = -PQ\left(H_{\zeta\bar{\zeta}} + \frac{\lambda}{3P^2}H\right) * k + \lambda * l, \tag{16}$$

by making use of the useful contraction formula $*G^a = -\frac{1}{2}\Omega_{bc} \wedge *\left(\theta^a \wedge \theta^b \wedge \theta^c\right)$.

Consequently, for the metric ansatz (1), the Einstein field equations are satisfied with a cosmological constant by construction, and the only off-diagonal Ricci component relative to the null coframe (3) turns out to be

$$R_{00} = PQ\left(H_{\zeta\bar{\zeta}} + \frac{\lambda}{3P^2}H\right), \tag{17}$$

and that this particular Ricci component corresponds to the real Ricci spinor Φ_{22} in the NP notation (See the definitions provided in Appendix B).

Note that the numerical indices refer to the null coframe, and therefore the cosmological vacuum equations reduces to $l^\alpha l^\beta R_{\alpha\beta} = R_{00} = 2\Phi_{22} = 0$. This is a *linear* partial differential equation of the form

$$\left(\boldsymbol{\Delta} + \frac{2\lambda}{3}\right) H = 0, \tag{18}$$

for the profile function H in terms of the Laplacian operator $\boldsymbol{\Delta} \equiv 2P^2\partial_\zeta\partial_{\bar\zeta}$ on the wave-fronts defined by the transverse metric $2P^{-2}d\zeta d\bar\zeta$.

The general solution to the homogeneous equation (18) can be written in the form

$$H_h(u,\zeta,\bar\zeta) = h_\zeta + \bar h_{\bar\zeta} - \frac{\lambda}{3P}\left(\zeta\bar h + \bar\zeta h\right) \tag{19}$$

where $h = h(u,\zeta)$ is an analytic function of the complex coordinate variable ζ and u-dependence remains determined by the metric field equations. The only non-vanishing component of the Weyl tensor relative to a NP null coframe (3) is $\Psi_4 = l^\alpha \bar m^\beta l^\mu \bar m^\nu C_{\alpha\beta\mu\nu}$ representing a transverse component propagating in the $k^\alpha\partial_\alpha$ direction [52]. With the help of the definitions given in Appendix B and the curvature 2-form expression (14), one concludes that the only non-vanishing Weyl curvature component has the expression

$$\Psi_4 = -\frac{PQ}{2}\left(H_{\zeta\zeta} + \frac{\lambda\bar\zeta}{3P^2}H_\zeta\right). \tag{20}$$

The result in eq. (20) evidently implies that for a non-vanishing expression on the right-hand side, the metric is of Petrov type-N. The vector field $k^\alpha\partial_\alpha$ is along the four-fold principle null direction and for the conformally-flat solutions can be obtained by requiring $\Psi_4 = 0$ in addition to the metric field equation $\Phi_{22} = 0$.

For the vacuum, the expression for the Weyl component can be simplified by feeding the general solution (19) into the expression in eq. (20). Thus, by making use of the general solution to the vacuum field equations, one finds that

$$\Psi_4 = -\frac{PQ}{2}h_{\zeta\zeta\zeta}. \tag{21}$$

The Weyl curvature expression (21) implies that the metric ansatz (1) is conformally-flat in the case where $h(u,\zeta)$ is at most a quadratic polynomial in ζ.

For $h = \alpha(u)\zeta$ in the general homogeneous solution given in eq. (19), H_h takes the form

$$H_h(u,\zeta,\bar\zeta) = (\alpha(u) + \bar\alpha(u))\frac{Q}{P} \tag{22}$$

where α is an arbitrary complex function of the coordinate u. On the other hand, for the profile function of the form (22), the Weyl component Ψ_4 also vanishes identically so that the solution (22) is conformally-flat (See, for example, [17], for a discussion on this particular solution).

Sec. 5 and Sec. 6 are devoted to study of nonminimally coupled Einstein-Maxwell models, and therefore the definitions and ansatz involving electromagnetic field variables associated to the metric ansatz (1) for the GR case are now in order.

Because the only non-vanishing Ricci tensor is Φ_{22}, the metric ansatz in eq. (1) admits a null electromagnetic field with $\Phi_2 = F_{\alpha\beta}\bar m^\alpha l^\beta$ component only. The Faraday 2-form is closed $dF = 0$, and therefore it is locally exact. Such a Faraday 2-form can be derived from the gauge potential of the form $A = a(u, \zeta, \bar{\zeta})du$ defined in terms of a real function $a = a(u, \zeta, \bar{\zeta})$. It has the explicit form

$$F = -a_\zeta du \wedge d\zeta - a_{\bar{\zeta}} du \wedge d\bar{\zeta}. \tag{23}$$

F defined in eq. (23) satisfies the relations $k \wedge F = 0 = k^\alpha i_\alpha F$ in terms of the null coframe 1-form k, and also it defines a principle null direction for the electromagnetic field. The source-free Maxwell's equations reduce to

$$d * F = -2P^2 a_{\zeta\bar{\zeta}} * k = 0. \tag{24}$$

Consequently, the Maxwell's equations are satisfied identically provided that one has $a_{\zeta\bar{\zeta}} = 0$ outside a four-current source in the geometry defined by the metric ansatz.

The canonical energy momentum form that follows from the Lagrangian

$$L_F[F, g] = -\frac{1}{2} F \wedge *F \tag{25}$$

by a coframe variation can be written in the form

$$* T_a[F] = i_a F \wedge *F - \frac{1}{2} i_a (F \wedge *F) \tag{26}$$

where $i_a \equiv i_{e_a}$ stands for the contraction operator with respect to the basis frame field e_a.

Note that for the null electromagnetic field defined in eq. (23) with $F \wedge *F = 0 = F \wedge F$, the energy momentum form simplifies further, leaving only the first term on the right-hand side in eq. (26). Thus, for the Faraday 2-form defined in eq. (23), there is only one non-vanishing component of energy-momentum form, and it is explicitly given by

$$* T^1[F] = -P^2 a_\zeta a_{\bar{\zeta}} * k. \tag{27}$$

Consequently, Einstein-Maxwell equations for vacuum with a cosmological constant yields the inhomogeneous linear equation

$$H_{\zeta\bar{\zeta}} + \frac{\lambda}{3P^2} H = \frac{2\kappa P}{Q} a_\zeta a_{\bar{\zeta}}, \tag{28}$$

for the profile function H.

The particular solution to the inhomogeneous equation (28) is given by Osváth et al.. [45]. Explicitly, one can construct the particular solution of (28) by assuming that it has the form

$$H_p \equiv \mu_\zeta + \bar{\mu}_{\bar{\zeta}} - \frac{\lambda}{3P} \left(\zeta\bar{\mu} + \bar{\zeta}\mu \right), \tag{29}$$

which is formally akin to the general homogeneous solution expression given in (19), expressed in terms of a complex function $\mu = \mu(u, \zeta, \bar{\zeta})$. The function $H^{(1)}$ defined by

$$H^{(1)} \equiv \mu_\zeta - \frac{\lambda\bar{\zeta}}{3P} \mu \tag{30}$$

satisfies the equation

$$H^{(1)}_{\zeta\bar{\zeta}} + \frac{\lambda}{3P^2} H^{(1)} = \frac{\kappa P}{Q} a_\zeta a_{\bar{\zeta}}. \tag{31}$$

Consequently, eq. (31) can be written in terms of the function μ by feeding the expression in eq. (30) into eq. (31) as

$$\left[P^2 \left(\frac{\mu_{\bar{\zeta}}}{P^2} \right)_\zeta \right]_\zeta = \kappa \frac{P}{Q} a_\zeta a_{\bar{\zeta}}. \tag{32}$$

Finally, the resulting equation can directly be integrated to have

$$\mu(u,\zeta,\bar{\zeta}) = \kappa \int^{\bar{\zeta}} d\bar{\zeta}' P^2 \int^{\zeta} \frac{d\zeta'}{P^2} \int^{\zeta'} \frac{d\zeta'' P}{Q} a_{\zeta''} a_{\bar{\zeta}'}. \tag{33}$$

The result is obtained by carrying out integrations indicated in (33) by treating ζ and $\bar{\zeta}$ as independent complex variables. Subsequently one uses them to construct the general solution of the equation (28) in the form $H = H_h + H_p$ as a linear superposition of the homogeneous solution given in eq. (19). The particular solution H_p can be expressed in terms of the complex function μ with the help of eq. (31,) and the definition $H_p = H^{(1)} + \bar{H}^{(1)}$.

Although the particular solution (33) has an expression formally similar to the the general homogeneous solution (19), the form of the particular solution H_p does not lead to a simplified expression for the Weyl component as in (21) in terms of the function μ. In this particular case, one finds the general expression

$$\Psi_4 = -\frac{PQ}{2} h_{\zeta\zeta\zeta} - \frac{Q}{2P} \left(P^2 (H_p)_\zeta\right)_\zeta \tag{34}$$

for the Weyl spinor.

Note that the construction of the general form the particular solution, and consequently the expressions for Ψ_4 for the modified gravitational models discussed below proceed in the same way as in the vacuum case. Therefore, the expression for the Weyl component in eq. (34) remains valid for the modified gravitational models studied below with the corresponding particular solutions. On the other hand, because of the term corresponding to the particular solution H_p, the $\times$ and $+$ polarization modes persist even for a conformally-flat homogeneous part of the exact solution. In this regard, the form of a particular solution for a given modified gravitational model with electromagnetic coupling allows one to compare the properties of the gravitational wave solutions for such models.

It is worthwhile to investigate whether the metric ansatz (1) can be used to construct gravitational wave solution to a given modified gravitational models, in particular the models that alter the minimal coupling between the gravitational and electromagnetic fields. An example of such models arises in the context of gravitational models modified by terms originating from quantum field theoretical considerations on a curved background [65]. For such models involving nonminimal RF^2 couplings discussed below, it can be shown that H_p part of the profile function for the ansatz (1) gets modified. Consequently, the method of obtaining a particular solution introduced above can also be applied to such models.

Moreover, in addition to such nonminimally coupled models, we also discuss some modified gravitational models popular in the context of cosmology admitting maximally-symmetric vacuum solutions.

3. The Gravitational waves in $f(R)$ models

In order to facilitate a direct comparison of the plane-fronted wave solutions for a given $f(R)$ model with the well-known solutions, we find it convenient to work with the original form of the $f(R)$ field equations [46–48, 53], although it is possible to work with Brans-Dicke-type scalar-tensor equivalents with a particular potential term for the scalar field. The gravitational wave solutions in the linearised approximation to generic $f(R)$ model have previously been studied by Capozziello et al.. [54] in the flat background.

3.1. The metric field equations

It is convenient to start from a Lagrangian 4-form of the form

$$L = \frac{1}{2\kappa} \left(f(R) * 1 + 2\Lambda * 1\right) - \frac{1}{2} F \wedge *F \tag{35}$$

where f is assumed to be a sufficiently smooth algebraic function of scalar curvature R, and κ and Λ are the gravitational coupling and cosmological constants, respectively.

The metric field equations that follow from the total Lagrangian (35) can be obtained by a suitable coframe variation $\delta L/\delta\theta_a \equiv *E^a = 0$ with $E^a \equiv E^a{}_b\theta^b$. With respect to an orthonormal or a null NP coframe, they have the same form with $*E^a$ explicitly given by

$$* E^a = -f' * R^a + \frac{1}{2} f * \theta^a + D * (df' \wedge \theta^a) + \Lambda * \theta^a + \kappa * T^a[F], \tag{36}$$

where a prime denotes a derivative with respect to scalar curvature, $f' \equiv \frac{df}{dR}$, and $*T^a[F] \equiv \delta L_m/\delta\theta_a$ stands for the canonical energy-momentum form for electromagnetic field with the explicitly expression given in eq. (26) [47]. The terms that are third and fourth order in the derivatives of the metric components can conveniently be expressed in the form

$$D * (df' \wedge \theta^a) = f'' \left[d * (dR \wedge \theta^a) + \omega^a{}_b \wedge *(dR \wedge \theta^b) \right] + f''' dR \wedge *(dR \wedge \theta^a) \tag{37}$$

where $df' = f''dR$.

It is straightforward to see that, even in the case where $\Lambda = 0$, the vacuum field equations (36) admit a maximally symmetric solutions in addition to the flat Minkowski solution, provided that the function f satisfies the constraint $f(0) = 0$. Assuming that the scalar curvature in this case is equal to a non-vanishing constant $R = R_0$, the field equations then reduce to form

$$f'(R_0)R^a = \left(\tfrac{1}{2} f(R_0) + \Lambda\right) \theta^a \tag{38}$$

by making use of the fact that $df' = f''(R_0)dR_0 \equiv 0$ in this case, and consequently, the higher order terms all of which are displayed in (37) vanish identically. In this case, the Ricci 1-forms R^a and, consequently by calculating the contraction of the Ricci 1-forms (38), one obtains an algebraic constraint equation satisfied by the curvature scalar as

$$R_0 = \frac{2\,(f(R_0) + 2\Lambda)}{f'(R_0)}. \tag{39}$$

We shall assume that the algebraic function f has a form such that the constraint (39) yields at least a non-vanishing root, such that $f(R_0) \neq 0$. On the other hand, depending on the explicit form of the function $f(R)$, eq. (39) admits several distinct roots, or else, multiple root(s) as well. Also note that for the case with $f(R) = R^2$, the constraint equation (39) equation implies that $\Lambda = 0$. Moreover, for the case $f(R) = R^2$, the vacuum field equations with $\Lambda = 0$ are identically satisfied for the pp-wave metric ansatz.

For $\lambda \mapsto 0$ in the metric ansatz, the metric reduces to the pp-wave metric for which $R = 0$ identically. in this case, for the pp-wave ansatz to be a valid ansatz for generic $f(R)$ gravity, $f(0) = 0$ is to be satisfied, and the vacuum equations then yield the same equation for the profile function since the Einstein tensor is multiplied by the constant $f'(0)$. Such gravitational wave solutions in flat background have been discussed previously by Mohseni [41]. On the other hand, the vacuum field equations (36) for a generic $f(R)$ model allow gravitational wave solutions on a curved background.

3.2. The solutions for the electro-vacuum case

By comparing the diagonal components of the field equations $*E^a = 0$ with the Einstein forms given explicitly in eq. (16), it is straightforward to obtain the algebraic equation satisfied by the constant parameter λ in the metric ansatz (1). One finds

$$\Lambda + \frac{1}{2}\left(f(R_0) - R_0 f'(R_0)\right) - f'(R_0)\lambda = 0. \tag{40}$$

Assuming that there exists a real root R_0 satisfying the constraint in eq. (40) for the scalar curvature, one can construct the solutions to the equations $E_{00} = 0$ for the profile function as in the GR case. The corresponding solution to generic $f(R)$ model equations with the Lagrangian 4-form given in eq. (35) can explicitly be written in the form given in eq. (1) with the profile function given by

$$H(u, \zeta, \bar{\zeta}) = h_\zeta + \bar{h}_{\bar{\zeta}} - \frac{\tilde{\lambda}}{3P^2}\left(\zeta\bar{h} + \bar{\zeta}h\right) \tag{41}$$

with the constant parameter $\tilde{\lambda}$ is given by

$$\tilde{\lambda} = \frac{\Lambda}{f'(R_0)} + \frac{f(R_0) - R_0 f'(R_0)}{2f'(R_0)}. \tag{42}$$

Note that, the constant parameter $\tilde{\lambda}$ appearing in the equation for the profile function is now determined by an algebraic equation determining the background scalar curvature R_0. Because generic $f(R)$ models admit maximally symmetric vacuum solutions as briefly discussed above, $\tilde{\lambda}$ survives even if one assumes that $\Lambda = 0$. For $\Lambda = \lambda = 0$, and also assuming $f(0) = 0$ (with the consistency condition $f'(0) \neq 0$), one recovers the general pp-wave solutions presented in [41] (for example, the Aichelburg-Sexl type solutions correspond to $h(u, \zeta) \sim \delta(u) \log \zeta$ which require an appropriate null particle source) by specializing the complex function $h(u, \zeta)$ in (41).

One can immediately find the equations for the profile function for $f(R)$ gravity coupled to electromagnetic field with the Lagrangian of the form given in (35). In this case, one readily finds

$$H_{\zeta\bar{\zeta}} + \frac{\tilde{\lambda}}{3P^2}H = \frac{2\kappa}{f'(R_0)}\frac{Q}{P}a_\zeta a_{\bar{\zeta}}. \tag{43}$$

The general solution to eq. in (43) can be written as a superposition of the homogeneous solution of the form in (19) and the particular solution of the form (29) with a corresponding μ defined in (33) by replacing the constants κ and λ with $\kappa/f'(R_0)$ and $\tilde{\lambda}$, respectively.

It is also possible to consider the models based on a slightly more general Lagrangian density of the form $f(R, T, R_{ab}T^{ab}) * 1$, involving nonminimal matter momentum tensor T_{ab} coupled to Ricci tensor R_{ab} as well as trace of the matter energy momentum $T \equiv T^a{}_a$ coupling to curvature scalar [55–57]. Any gravitational model based on these interaction lead to fairly involved field equations. In particular, note that for the electromagnetic energy-momentum tensor of the form given in (26), the nonminimal $R_{ab}T^{ab}[F]$ coupling can be expressed as a superposition of RF^2-type nonminimal couplings as

$$R_{ab}T^{ab}[F] * 1 = R^a \wedge *T^a[F] = -F_a \wedge R^a \wedge *F - \frac{1}{2}RF \wedge *F \tag{44}$$

involving Ricci tensor R_{ab} and scalar curvature R.

The nonminimal coupling of type $RF\wedge*F$ is not particularly interesting for the metric ansatz (1) and in Sections V and VI, the gravitational wave solutions based on the ansatz (1) for the some simpler models with other RF^2-type nonminimal couplings involving Riemann and Ricci tensors will be discussed.

4. Gravitational waves in a Gauss-Bonnet extended Einstein gravity

In four spacetime dimensions, the Euler-Poincaré invariant, more often referred to as Gauss-Bonnet scalar $\mathcal{G}$, can conveniently be defined in terms of the particular contraction of the curvature 2-forms of the form

$$\mathcal{G} * 1 = \frac{1}{4}\Omega_{ab} \wedge \Omega_{bc} * \theta^{abcd} = (R^{abcd}R_{abcd} - 4R_{ab}R^{ab} + R^2) * 1. \tag{45}$$

One can show that, relative to an orthonormal or a null coframe, $\mathcal{G} * 1$ is an exact form given by

$$\mathcal{G} * 1 = d\left[\frac{1}{4}\left(\omega_{ab} \wedge d\omega_{cd} + \tfrac{2}{3}\omega_{ab} \wedge \omega_{ce} \wedge \omega^{e}{}_{d}\right)\epsilon^{abcd}\right], \tag{46}$$

and therefore such a term in gravitational Lagrangian 4-form does not contribute to the metric field equations. In spacetime dimensions greater than four, the term of the form (45) yields metric field equations that are of second order in the partial derivatives of the metric components [58].

A way to incorporate the GB term in (45) into the gravitational field equations in four spacetime dimensions, is to couple it with a dynamic scalar field. On the other hand, one can also consider modified GB term of the algebraic form $f(\mathcal{G})$. We follow the latter prescription, and obtain the ensuing field equations in some generality.

4.1. The metric field equations

In four spacetime dimensions, the general gravitational Lagrangian 4-form involving a second order GB term can be written of the form [59, 60]

$$L = \frac{1}{2\kappa}F(R,\mathcal{G}) * 1 + L_m[g,\psi] \tag{47}$$

where F is now assumed to be sufficiently smooth algebraic function of the GB scalar $\mathcal{G}$ and the scalar curvature R and L_m is a Lagrangian 4-form for a matter field ψ minimally coupled to metric field g. The metric field equations that follow from the Lagrangian 4-form (47) by a coframe variational derivative can be written in the form [41, 53, 61]

$$\frac{1}{2}F_R\Omega_{bc} \wedge *\theta^{abc} + \frac{1}{2}(F - RF_R - \mathcal{G}F_{\mathcal{G}}) * \theta^a + D\left[*(dF_R \wedge \theta^a) - \Omega_{bc} * (dF_{\mathcal{G}} \wedge \theta^{abc})\right] + \kappa * T^a[\psi] = 0 \tag{48}$$

where we introduced the short hand notations for the partial derivatives:

$$F_R \equiv \frac{\partial F}{\partial R}, \qquad F_{\mathcal{G}} \equiv \frac{\partial F}{\partial \mathcal{G}}. \tag{49}$$

Note that for $F(R,\mathcal{G}) = f(R) + \mathcal{G}$, the metric field equations (48) reduce to those of $f(R)$ equations given in (36) above as the consistency of the equations for the models studied above requires.

Because no additional insight is to be gained by increasing the complexity of the algebraic equation satisfied by the parameter λ, we find it convenient to discuss the vacuum field equations following from a particular form of the Lagrangian density of the form $F(R,\mathcal{G}) = \frac{1}{2}R + f(\mathcal{G})$. In this case, the general field metric equations in eq. (48) reduce to a simplified form as

$$- *G^a + (f - \mathcal{G}f_{\mathcal{G}}) * \theta^a - \Omega_{bc} \wedge D * (df_{\mathcal{G}} \wedge \theta^{abc}) = 0, \tag{50}$$

where the fourth-order derivatives of the metric are contained in the third term on the left-hand side of eq. (50) with $df_{\mathcal{G}} = \frac{df}{d\mathcal{G}}d\mathcal{G}$ and

$$D * (df_{\mathcal{G}} \wedge \theta^{abc}) = -\epsilon^{abcd}D(i_d df_{\mathcal{G}}). \tag{51}$$

It is straightforward to deduce that the reduced vacuum equations given in eq. (50) admit a maximally symmetric solution, for which case $\mathcal{G}$ and consequently $f(\mathcal{G})$ become constants, similar to the case of generic $f(R)$ model discussed in the previous section. For the metric ansatz (1), the Gauss-Bonnet scalar $\mathcal{G}$ can be calculated to have the value $\mathcal{G} = \lambda^2/9$ with the help of the expression (45).

4.2. The gravitational wave solutions

For a constant curvature spacetime to be a vacuum solution in the GB modified GR model above, the constant parameter λ has to satisfy the algebraic relation

$$f\left(\frac{\lambda^2}{9}\right) - \frac{\lambda^2}{9} f'\left(\frac{\lambda^2}{9}\right) - \lambda = 0, \tag{52}$$

for consistency, and the function $f(\mathcal{G})$ is such that eq. (52) has at least a real root. The equation for the profile function then is same as in eq. (18), and consequently, the solution also has the same form as the homogeneous solution given in eq. (19), provided that λ_0 is a root of the constraint equation (52). Finally, compared to the GR solution, only the curvature parameter of the background spacetime is replaced by λ_0 for the metric ansatz (1) in accordance with the algebraic constraint in eq. (52).

5. Prasanna's nonminimally coupled model

One can consider the coupling the Faraday 2-form to the curvature tensor in a mathematically consistent manner [62], as for example, RF^2-type couplings listed in [42] by Dereli and Sert (See, also, Ref. [63]). Without a cosmological term, Dereli and Sert studied *pp*-wave solutions to the nonminimally coupled Einstein-Maxwell model that follow from the Lagrangian (53).

Although such nonminimal coupling terms are not compatible with the strong principle of equivalence [62,64], Drummond and Hathrell showed in a remarkable paper [65] that the effective action for the photon involves nonminimal RF^2 interaction terms in a curved background arising from the one-loop polarization calculations, and such nonminimal curvature interactions allow superluminal propagation for low energy photons.

RF^2-type nonminimal coupling terms also naturally arise from the Kaluza-Klein reduction of quadratic-curvature terms, and in particular, the quadratic Gauss-Bonnet term studied previously by Dereli and Üçoluk [66].

The electro-vacuum pp-waves with a nonminimal coupling of type $RF \wedge *F$, or more generally the nonminimal interaction of type $f(R)F \wedge *F$, have previously been considered by Mohseni, concluding that they are identical to the corresponding solutions in GR [41].

It is also interesting to note that the differential equation for the profile function for pp-wave metric in Horndeski's generalized Einstein-Maxwell model [50] with the nonminimal coupling terms of the form $F \wedge F_{ab} * \tilde{\Omega}^{ab}$ where $\tilde{\Omega}^{ab}$ stands for the double-dual curvature 2-form involving the coupling terms $RF \wedge *F$, $R_a \wedge F^a \wedge *F$ and $F \wedge F_{ab} * \tilde{\Omega}_{ab}$ presented in [43] is identical to the one in a nonminimally coupled theory with the nonminimal coupling term $F \wedge F_{ab} * \Omega^{ab}$ [42].

We first consider the simplest possible nonmiminal RF^2 coupling defined by $F \wedge F_{ab} * \Omega^{ab}$ in order to gain some insight for the gravitational wave solutions in a curved background for such models.

5.1. The Lagrangian and the field equations

The nonminimally coupled Einstein-Maxwell model with a cosmological constant λ we want to study is governed by the Lagrangian 4-form

$$L = \frac{1}{2\kappa}(R + 2\lambda) * 1 - \frac{1}{2} F \wedge *F + \frac{\gamma}{2} F \wedge F_{ab} * \Omega^{ab} \tag{53}$$

where γ is a constant multiplying the nonminimal coupling term, namely the term

$$F \wedge F_{ab} * \Omega^{ab} = \tfrac{1}{2} F_{ab} R^{ab}{}_{cd} F^{cd} * 1, \tag{54}$$

first considered by Prasanna [49]. The field equations for the Faraday 2-form F that follow from the nonminimally coupled Einstein-Maxwell Lagrangian (53) are

$$dF = 0, \qquad d * (F - \gamma F_{ab} \Omega^{ab}) = 0. \tag{55}$$

One can readily verify that the ansatz (23) for the Faraday 2-form and for the metric ansatz (1), the modified Maxwell's equations (55) are satisfied.

The metric field equations $*E^a = 0$ which follow from a coframe variation of the Lagrangian density (53), $*E_a \equiv \delta L/\delta\theta^a$ can be written explicitly in the convenient form as [42]

$$* E^a = - * G^a + \lambda * \theta^a + \kappa * T^a[F] + \kappa\gamma(\tfrac{1}{2}F^a{}_c F_b \wedge *\Omega^{cb} - *T^a[F, F_{bc}\Omega^{bc}] + D\lambda^a) = 0 \quad (56)$$

where the Lagrange multiplier 2-form λ^a is explicitly given by

$$\lambda^a = 2i_b D(F^{ba} * F) + \tfrac{1}{2}\theta^a \wedge i_b i_c D(F^{bc} * F). \quad (57)$$

Consequently, the coframe field equations containing $D\lambda^a$ term involve, in general, the third order derivatives of the gauge potential for the model.

The energy-momentum 3-forms $*T_a[F, F_{cd}\Omega^{cd}] = T_{ab}[F, F_{cd}\Omega^{cd}] * \theta^b$ that arises from the nonminimal interaction term is explicitly given by

$$* T_a[F, F_{cd}\Omega^{cd}] \equiv \tfrac{1}{4}(F_a \wedge F_{bc} * \Omega^{bc} + F_{bc}(i_a\Omega^{bc}) \wedge *F - F \wedge F_{bc} i_a * \Omega^{bc} - F_{bc}\Omega^{bc} \wedge i_a * F). \quad (58)$$

The expression in eq. (58) for the 1-form $*T_a[F, F_{cd}\Omega^{cd}]$ has an explicit form similar to that of the canonical energy-momentum form $*T^a[F]$ defined in eq. (26). By definition, $*T_a[F, F_{cd}\Omega^{cd}]$ has a vanishing trace, and it is symmetrical under the interchange $F \leftrightarrow F_{ab}\Omega^{ab}$: $*T_a[F, F_{cd}\Omega^{cd}] = *T_a[F_{cd}\Omega^{cd}, F]$. It is also symmetric with respect to its indices: $\theta^a \wedge *T^b[F, F_{cd}\Omega^{cd}] = \theta^b \wedge *T^a[F, F_{cd}\Omega^{cd}]$. In the expression for the energy-momentum forms, the short-hand notations $F_a \equiv i_a F$ and $F_{ab} \equiv i_b i_a F$ are used for convenience.

5.2. The Gravitational wave solutions

Before presenting the differential equation satisfied by the profile function obtained by inserting the metric ansatz (1) into the field equations (56), we first note the following relation

$$F_{ab}\Omega^{ab} = \frac{2\lambda}{3}F \quad (59)$$

which is of some practical importance in simplifying the metric field equations as well. Consequently, the energy-momentum 3-forms defined in (58) reduce to

$$* T_a[F, F_{bc}\Omega^{bc}] = *T_a\left[F, \frac{2\lambda}{3}F\right] = \frac{2\lambda}{3} * T_a[F] \quad (60)$$

for the metric ansatz (1) and for the Faraday 2-form of the form (23). Moreover, by making use of the gravitational Bianchi identity satisfied by the curvature 2-form, $\Omega^a{}_b \wedge \theta^b = 0$, and the relation (59), one can show that

$$F_{ac}F_b \wedge *\Omega^{cb} = -\frac{1}{2}F_a \wedge F_{bc} * \Omega^{bc} = -\frac{\lambda}{2}F_a \wedge *F. \quad (61)$$

By making use of the expression in equation (57), it is straightforward to verify that the only non-vanishing component of the Lagrange multiplier 2-form is λ^1, and it is given by

$$\lambda^1 = -(i_a DF^{1a}) * F + DF^{1a} \wedge *(F \wedge \theta_a) \quad (62)$$

More conveniently, the non-vanishing component of the Lagrange multiplier 2-form λ^1 reduces to the form

$$\lambda^1 = i\mathcal{A}k \wedge m - i\bar{\mathcal{A}}k \wedge \bar{m}, \quad (63)$$

where the complex function $\mathcal{A}$ introduced in the equation for convenience has the explicitly expression

$$\mathcal{A} \equiv P^3 a_{\zeta\zeta} a_{\bar{\zeta}} + \frac{\lambda P^2(Q+1)}{3Q} \bar{\zeta} a_{\bar{\zeta}} a_{\zeta} + \frac{\lambda P^2}{3Q} \zeta a_{\zeta} a_{\zeta}. \tag{64}$$

Thus, the components of the exterior derivative expression for $D\lambda^1$ turn out to contain at most second derivatives of the gauge potential for the ansatz (1). In addition to a shift in the constant multiplying the electromagnetic energy momentum form, the effect of the nonminimal interaction comes from the Lagrange multiplier term (63). Consequently, the contribution of the Lagrange multiplier term $D\lambda^a$ to the metric equations originates from $D\lambda^1$ term which can be calculated by making use of the simplifications $\lambda^2 = 0 = \lambda^3$. Thus, the expression for the covariant exterior derivative reduces to

$$D\lambda^1 = d\lambda^1 - \omega^0{}_0 \wedge \lambda^1. \tag{65}$$

Consequently, with the help of the explicit expressions for the connection 1-forms, this formula takes the explicit form

$$D\lambda^1 = 2Re\left(P\mathcal{A}_{\bar{\zeta}} + \frac{\lambda\zeta}{2Q}\mathcal{A}\right) * k. \tag{66}$$

Eventually, in parallel to the GR case, $l^\alpha l^\beta E_{\alpha\beta} = E_{00} = 0$ gives the inhomogeneous equation of the form

$$H_{\zeta\bar{\zeta}} + \frac{\lambda}{3P^2}H = \kappa\left(2 - \frac{\gamma\lambda}{3}\right)\frac{P}{Q}a_{\zeta}a_{\bar{\zeta}} - \frac{2\kappa\gamma}{PQ}Re\left(P\mathcal{A}_{\bar{\zeta}} + \frac{\lambda\zeta}{2Q}\mathcal{A}\right) \tag{67}$$

for the profile function H. Here $Re(\cdot)$ denotes a real part of an expression inside the bracket.

For both of the nonminimal coupling Einstein-Maxwell models we considered, the ansatz (1) leads to an inhomogeneous equation of type similar to (67). The particular solution of an equation of this type can be constructed as a contour integral [45] in the same way as the particular solution H_p is constructed as in the Einstein-Maxwell case in terms of the complex function μ in eq. (33) and with the general expression in eq. (34).

6. Horndeski's nonminimally coupled model

Gürses and Halil [43] previously studied pp-wave solutions to the Horndeski model. Horndeski [67] himself studied Petrov type III and Petrov type-N solutions investigating the extension of the Mariot-Robinson and Kundt-Trumper theorems to the generalized Einstein-Maxwell model [50].

We construct type-N gravitational waves by taking a cosmological constant into account for this model.

6.1. The Horndeski's Lagrangian and the field equations

In terms of differential forms, the original Lagrangian introduced by Horndeski [50] can be written out as a Lagrangian 4-form of the form

$$L = \tfrac{1}{2\kappa}(R + 2\lambda) * 1 - \tfrac{1}{2}F \wedge *F + \tfrac{\gamma}{2}F \wedge F_{ab} * \tilde{\Omega}^{ab}, \tag{68}$$

where $\tilde{\Omega}^{ab}$ is the double dual curvature 2-form defined in terms of double dual Riemann tensor ${}^*R^{*a}{}_{bcd}$ as

$$\tilde{\Omega}^a{}_b = \tfrac{1}{2}\, {}^*R^{*a}{}_{bcd}\theta^c \wedge \theta^d, \tag{69}$$

and that the double-dual Riemann tensor has components expressed in terms of the permutation symbols as

$${}^*R^{*ab}{}_{cd} \equiv \tfrac{1}{4}\epsilon^{ab}{}_{mn}R^{mn}{}_{pr}\epsilon^{pr}{}_{cd}. \tag{70}$$

Note that the particular nonminimal coupling in (68) is a special linear combination of the nonminimal couplings

$$F \wedge F_{ab} * \tilde{\Omega}^{ab} = -F \wedge *(RF + 2F_a \wedge R^a + F_{ab}\Omega^{ab}) \tag{71}$$

where $R_a \equiv R_{ab}\theta^b = i_b\Omega^b{}_a$ is the Ricci 1-form and R is the scalar curvature. The individual terms on the right-hand side in (71) can be expressed in the component form as

$$RF \wedge *F = \tfrac{1}{2}F^{ab}F_{ab}R * 1, \tag{72}$$

$$F_a \wedge R^a \wedge *F = F_{ac}F^c{}_b R^{ab} * 1, \tag{73}$$

$$\Omega_{ab}F^{ab} \wedge *F = \tfrac{1}{2}F_{ab}F^{cd}R^{ab}{}_{cd} * 1, \tag{74}$$

in terms of the scalar curvature, Ricci and Riemann tensors, respectively. Note also that the nonminimal coupling term in the Horndeski Lagrangian (68) can also be written in the following equivalent forms:

$$F \wedge F_{ab} * \tilde{\Omega}^{ab} = -F \wedge \tilde{F}_{ab}\Omega^{ab} = -F \wedge \Omega_{ab} * (F \wedge \theta^a \wedge \theta^b). \tag{75}$$

The field equation that follow from the Lagrangian (68) by a coframe variation by using the first order formalism is relegated to Appendix A.

As in the previous models discussed above, the metric field equations that follow from the Horndeski Lagrangian (68) can be written as $*E^a = \delta L/\delta\theta_a = 0$ with

$$* E^a = - * G^a + \lambda * \theta^a + \kappa * T^a[F] - \tfrac{\gamma\kappa}{2}F^a \wedge F_{bc} * \tilde{\Omega}^{bc} + \gamma\kappa D(F_b \wedge D\tilde{F}^{ab}) \tag{76}$$

and $\tilde{F}^{ab} \equiv i^b i^a * F$ stand for the components of the dual Faraday 2-form.

The source-free field equation for the Faraday 2-form $F = dA$ takes the form

$$dF = 0, \qquad d * (F - kF_{ab}\tilde{\Omega}^{ab}) = 0. \tag{77}$$

In contrast to the Prasanna's model studied in the previous section, for the Horndeski's model with the set of dynamic variables as $\{g, A\}$, the system of field equations (76) and (77) are all of second order. In this regard, it is the unique RF^2-type coupling among all the other couplings mentioned.

Note that the modified electromagnetic field equations in eq. (77) contain only the second order derivatives of metric components as a consequence of the second Bianchi identity, namely the identity $D * \tilde{\Omega}^{ab} = -\frac{1}{2}\epsilon^{ab}{}_{cd}D\Omega^{cd} \equiv 0$.

On the other hand, one can verify, from the modified electromagnetic equations (55) and (77) with a source term of the form $*J$ with a four current 1-form $J = J_\alpha dx^\alpha$, that the local charge conservation is satisfied for both of the models as a consequence of the operator identity $d^2 \equiv 0$. Therefore, the local conservation of charge is consistent with both the Horndeski's second order model and the Prasanna's third order RF^2 models.

6.2. The gravitational wave solutions

For the Faraday 2-form of the form (23), and the metric ansatz (1), the nonminimal coupling term reduces to

$$F_{ab}\tilde{\Omega}^{ab} = -\tfrac{\lambda}{3}F. \tag{78}$$

Consequently, the modified Maxwell's equations (77) are satisfied identically, in parallel to the case in the Prasanna model. Such a simplifying relation between the curvature and the Faraday 2-forms in both of the nonmimimal coupling cases is peculiar to the maximally symmetric background, and hence to the metric ansatz (1).

As another consequence of the relation in given eq. (78), one has

$$F_a \wedge F_{bc} * \tilde{\Omega}^{cb} = -\tfrac{\lambda}{3} * T_a[F] \tag{79}$$

as in the non-mimimal case studied above.

The key relation in eq. (78) also helps reduce the metric field equations (76) to a more manageable form. For the metric ansatz (1), the only non-vanishing component of the Lagrange multiplier 2-form, as before, is given by

$$\lambda^1 = Pa_{\bar{\zeta}}\left(P^2 a_\zeta\right)_\zeta ik \wedge m - Pa_\zeta \left(P^2 a_{\bar{\zeta}}\right)_{\bar{\zeta}} ik \wedge \bar{m} \tag{80}$$

for this case as well. Consequently, by making use of the expression in (65), the equation for the profile function takes the form

$$H_{\zeta\bar{\zeta}} + \frac{\lambda}{3P^2}H = \kappa\left(2 + \frac{\lambda\gamma}{3}\right)\frac{Q}{P}a_\zeta a_{\bar{\zeta}} + \frac{\gamma\kappa}{P}\left\{\left[\frac{P^2 a_{\bar{\zeta}}}{Q}\left(P^2 a_\zeta\right)_\zeta\right]_{\bar{\zeta}} + \left[\frac{P^2 a_\zeta}{Q}\left(P^2 a_{\bar{\zeta}}\right)_{\bar{\zeta}}\right]_\zeta\right\}. \tag{81}$$

In the limit $\lambda \mapsto 0$, the background becomes flat and the metric (1) takes the form of a pp-wave metric. Consequently, one recovers the result obtained by Gürses and Halil [43], and Dereli and Sert [42]. In this limit, eq.(81) becomes

$$H_{\zeta\bar{\zeta}} = \kappa a_\zeta a_{\bar{\zeta}} + \kappa\gamma a_{\zeta\zeta} a_{\bar{\zeta}\bar{\zeta}}. \tag{82}$$

One can readily verify that the general solution to this equation can be expressed in terms of functions $h(u,\zeta)$ and $f(u,\zeta)$ which are analytic in the variable ζ in the form

$$H(u,\zeta,\bar{\zeta}) = h_\zeta + h_{\bar{\zeta}} + \kappa(f\bar{f} + \gamma f_\zeta \bar{f}_{\bar{\zeta}}) \tag{83}$$

with $a(u,\zeta,\bar{\zeta}) \equiv f(u,\zeta) + \bar{f}(u,\bar{\zeta})$.

The particular solution to (81) cannot be expressed in a simple form as in eq. (83). Moreover, the solution to eq. (81) does not allow one to interpret the resulting metric as describing a partially massive spin-2 graviton and partially massless spin-2 photon with the assumption that $f(u,\zeta) = \alpha(u)\zeta + \beta(u)\zeta^2$ in the flat background [42]. However, in parallel to the GR case, the particular solution to eq. (81) can be constructed by using an appropriate function μ which is defined in eq. (29). In this case, for a given admissible gauge potential, $\mu(u,\zeta,\bar{\zeta})$ can be expressed in the form

$$\mu = \kappa \int^{\bar{\zeta}} d\bar{\zeta}' P^2 \int^{\zeta} \frac{d\zeta'}{P^2} \int^{\zeta'} d\zeta'' \left\{\left(1 + \frac{\lambda\gamma}{6}\right)\frac{Q}{P}a_{\zeta''}a_{\bar{\zeta}'} + Re\left(\frac{\gamma\kappa}{P^2 Q}\left[\frac{P^2 a_{\bar{\zeta}'}}{Q}\left(P^2 a_{\zeta''}\right)_{\zeta''}\right]_{\bar{\zeta}'}\right)\right\}. \tag{84}$$

The general expression for the complex function $\mu(\zeta,\bar{\zeta})$ in eq. (84) is valid for an electromagnetic potential which satisfies eq. (77).

7. Concluding remarks

For a given profile function H as a solution, one can calculate the corresponding Weyl spinor Ψ_4 with the explicit expression given in eq. (20). Thus, the way the plane-fronted gravitational wave solutions presented above for each model allows a direct comparison of the models discussed in a unified manner.

In a generic metric theory of gravity, for which the metric is assumed to be determined by some field equations, the possible polarization modes gravitational waves in the far zone are attributed to some particular components of the Riemann tensor, namely R_{0i0j} components of the Riemann curvature tensor. Thus, for i, j standing for the spatial components, there are at most six distinct polarization modes that are independent of the field equations that determine the metric. In a previous study of these polarization modes, and their invariance under null frame rotations in [68, 69], these polarization modes are identified with the real and imaginary parts of the NP scalars $\Psi_4(u), \Psi_2(u), \Psi_3(u)$ and $\Phi_{22}(u)$, as a function of the retarded real null coordinate u. In the recent works [70–73], the properties of gravitational waves are studied for some particular metric theories. These polarization modes are also relevant to all the models discussed in the current work.

The technical approach adopted here can also be used to introduce a more refined linearised approximation scheme relative to a null coframe that allows some additional gravitational wave features to be added to the metric perturbations in a convenient manner.

The geodesic deviation equation for test particles following neighbouring geodesic curves can be decomposed into, for example, the effects arising from the curved background, and the effects arising from a type-N gravitational waves [1]. In all the modified gravitational models discussed above, both of these effects are present. In particular, $\times$ and $+$ polarization modes registered by the test particles correspond to the real and the imaginary parts of the Weyl spinor Ψ_4, respectively. Consequently, the part of Ψ_4 which is determined by the particular solution H_p has an effect on the transverse polarizations modes, and the corresponding amplitude.

For both of the nonminimally coupled electromagnetic cases discussed above, the field equations lead to an effective T_{00} component for the nonminimally coupled electromagnetic field. Consequently, the $\Phi_{22} = R_{00}$ curvature components corresponding to transverse breathing modes [68, 69] are modified in contrast to the GR case.

Finally, note that the gravitational wave solutions constructed in this work follow the pattern introduced in [32], and therefore they imply that the universal metrics of Kerr-Schild-Kundt type of metrics can be extended to the cases that include the general RF^2-type nonminimal couplings as well.

Appendix A: A derivation of the Horndeski's generalized Lagrangian and the field equations

It follows from the Lovelock's theorem [58] that the field equations for the Einstein-Hilbert Lagrangian supplemented by a GB term (45)in five dimensions contain only the second order derivatives of the five dimensional pseudo-Riemannian metric. By introducing a Kaluza-Klein ansatz into the five dimensional metric, the dimensional reduction of the GB term (45) on a circle is bound to yield RF^2- and F^4-type gauge field couplings in four spacetime dimensions [66] that are second order in all the field variables. In this manner, the Horndeski Lagrangian given in eq. (68) was rederived by Buchdahl [74] a few years after Horndeski introduced it [50] following a completely different approach.

Following Buchdahl, it is sufficient to introduce a simple $(n+1)$-dimensional Kaluza-Klein metric ansatz of the form

$$g^{(n+1)} = \eta_{AB}\mathbf{\Theta}^A \otimes \mathbf{\Theta}^B = g^{(n)} + \mathbf{\Theta}^n \otimes \mathbf{\Theta}^n \qquad (85)$$

adapted to the local product form $U^n \times S^1$ of the total manifold. Here, $\mathbf{\Theta}^n = dy + A(x)$ in terms of the local coordinates $\{x^\mu, y\}$ on some chart $U^n \subset M^n$, and y denotes the compactified coordinate on S^1. The indices have the range $A, B, C, \ldots = 0, 1, 2, \ldots (n-1), n$, and one can introduce a suitable local orthonormal coframe θ^a on some coordinate neighbourhood on $M^{(n)}$ of the form

$$g^{(n)} = \eta_{ab}\theta^a \otimes \theta^b$$

with $\mathbf{\Theta}^a(x, y) = \theta^a(x)$. The lowercase Latin indices refer to the pseudo-Riemannian manifold M^n and run over the range $a, b, c, \ldots = 0, 1, 2, \ldots, (n-1)$.

The Levi-Civita connection 1-forms on $M^{(n+1)}$ satisfying the metricity condition, $\mathbf{\Gamma}_{AB} + \mathbf{\Gamma}_{BA} = 0$, and the structure equations with vanishing torsion

$$d\mathbf{\Theta}^A + \mathbf{\Gamma}^A{}_B \wedge \mathbf{\Theta}^B = 0$$

can be expressed in terms of the Levi-Civita connection 1-forms ω_{ab} on base manifold $M^{(n)}$ and the field strength $F = dA$ as

$$\mathbf{\Gamma}^a{}_b = \omega^a{}_b - \tfrac{1}{2}F^a{}_b\mathbf{\Theta}^n, \qquad \mathbf{\Gamma}^a{}_n = -\tfrac{1}{2}F^a. \tag{86}$$

Consequently, the curvature 2-forms $\mathbf{\Omega}^{AB} = d\mathbf{\Gamma}^{AB} + \mathbf{\Gamma}^A{}_C \wedge \mathbf{\Gamma}^{CB}$ can be expressed in the form

$$\mathbf{\Omega}^{ab} = \pi^{ab} + \tau^{ab} \wedge \mathbf{\Theta}^n, \qquad \mathbf{\Omega}^{an} = -\rho^a - \sigma^a \wedge \mathbf{\Theta}^n, \tag{87}$$

in terms of convenient tensor-valued forms

$$\pi^{ab} = \Omega^{ab} - \tfrac{1}{2}F^{ab}F - \tfrac{1}{4}F^a \wedge F^b, \qquad \tau^{ab} = -\tfrac{1}{2}DF^{ab},$$
$$\rho^a = \tfrac{1}{2}DF^a, \qquad \sigma^a = -\tfrac{1}{4}F^a{}_bF^b$$

that were introduced in [66] previously. In $n+1 \geq 5$ spacetime dimensions, the metric field equations that follow from the dimensionally-continued Euler-Poincare (EP) form explicitly read

$$L^{(n+1)} = \tfrac{1}{2}\mathbf{\Omega}_{AB} \wedge \star\mathbf{\Theta}^{AB} + \tfrac{1}{4}\mathbf{\Omega}_{AB} \wedge \mathbf{\Omega}_{CD} \wedge \star\mathbf{\Theta}^{ABCD}. \tag{88}$$

The metric field equations that follow from the Lagrangian in (88) containing the quadratic curvature term, can be obtained straightforwardly by a coframe variational derivative, $\delta L^{(n+1)}/\delta\mathbf{\Theta}_A = 0$. They explicitly read

$$\tfrac{1}{2}\mathbf{\Omega}_{BC} \wedge \star\mathbf{\Theta}^{ABC} + \tfrac{1}{4}\mathbf{\Omega}_{BC} \wedge \mathbf{\Omega}_{DE} \wedge \star\mathbf{\Theta}^{ABCDE} = 0. \tag{89}$$

In the equations above $\star$ denotes the Hodge dual operator defined on $M^{(n+1)}$ which determines the volume element on the base manifold by $\star 1 = *1 \wedge \mathbf{\Theta}^n$. Moreover, by feeding the Kaluza-Klein metric ansatz (85) into the field equations (89), one ends up with the equations involving only the second order derivatives of the set of variables $\{g^{(n)}, A\}$ of the ansatz (85) that parametrises the $g^{(n+1)}$ metric components, as a consequence of the fact that the metric filed equations in eq. (89) are manifestly second order in the derivatives.

Equivalently, one can also reduce the action (88), and subsequently derive the field equations therefrom. More precisely, by inserting the ansatz (85) into the Lagrangian density (88), one finds that the reduced Lagrangian n-form $L^{(n)}_{red.}$ defined by

$$L^{(n+1)} = L^{(n)}_{red.} \wedge dy$$

has the explicit expression

$$\begin{aligned} L^{(n)}_{red.} &= \tfrac{1}{2}\pi_{ab} \wedge *\theta^{ab} - \sigma_a \wedge *\theta^a + \tfrac{1}{4}\pi_{ab} \wedge \pi_{cd} \wedge *\theta^{abcd} - (\pi_{ab} \wedge \sigma_c + \tau_{ab} \wedge \rho_c) \wedge *\theta^{abc} \\ &= \tfrac{1}{2}\Omega_{ab} \wedge *\theta^{ab} - \tfrac{1}{2}F \wedge *F + \tfrac{1}{4}\Omega_{ab} \wedge \Omega_{cd} \wedge *\theta^{abcd} - \tfrac{1}{4}\Omega_{ab} \wedge F \wedge *(\theta^{ab} \wedge F) \\ &\quad + \tfrac{3}{32}F \wedge F \wedge *(F \wedge F) \end{aligned}$$

up to a disregarded exact n-form. Each term in $L^{(n)}_{red.}$ separately leads to second order field equations in the set of metric field variables of $M^{(n+1)}$, namely, $g^{(n)}_{\mu\nu}$, and the gauge potential A.

In four spacetime dimensions, the GB term is an exact form, and therefore it can be disregarded from the total Lagrangian. Consequently, up to redefinitions of the coupling constants, the reduced Lagrangian $L^{(n)}_{red.}$ corresponds to the Lagrangian density of the Horndeski's model after discarding the F^4 self-interaction terms for $n = 4$. In four spacetime dimensions, one can verify that the resulting RF^2 coupling can be written in terms of the double-dual curvature 2-form $\tilde{\Omega}^{ab}$, or else the linear combination of nonlinear couplings to the curvature components as

$$\Omega_{ab} \wedge F \wedge *(\theta^{ab} \wedge F) = -F \wedge F_{ab} * \tilde{\Omega}^{ab} = F \wedge F_{ab} * \Omega^{ab} - 2F \wedge *(R_a \wedge F^a) + RF \wedge *F. \quad (90)$$

The coupling to the double-dual curvature on the left-hand side can be considered as a generalization of the Horndeski-type RF^2 nonminimal interaction term to higher dimensions. The Kaluza-Klein reduction mechanism can be used to derive more general nonminimal couplings [75] that leads to second order equations for the field variables.

Returning back to the Horndeski's Lagrangian, the second order field equations that follow from the Lagrangian given in eqn. (68) can explicitly be found by a constrained variational derivative as follows.

In deriving the field equation by making use of the first order formalism (see, for example, Ref. [42] in the context of nonminimal coupling considered above), the connection and the coframe 1-forms are assumed to be independent gravitational variables, and the Riemannian case is recovered by introducing vanishing torsion ($\Theta^a = d\theta^a + \omega^a{}_b \wedge \theta^b = 0$), and vanishing non-metricity ($D\eta_{ab} = -\eta_{ac}\omega^c{}_b - \eta_{bc}\omega^c{}_a = 0$) constraints for the independent connection 1-forms.

For the metricity constraint, we assume that $\omega_{ab} = -\omega_{ba}$, and the zero-torsion constraint $\Theta^a = 0$ is imposed on the connection by introducing a Lagrange multiplier terms of the form $\lambda_a \wedge (d\theta^a + \omega^a{}_b \wedge \theta^b)$ to the original Lagrangian 4-form. Eventually, the Lagrange multiplier 2-form λ^a is to be eliminated in favour of the remaining variables to obtain the metric field equations.

In the first order formalism, only the first order exterior derivatives of the independent variables appear. For example, the Lagrangian 4-form is assumed to depend the gravitational variables θ^a and $\omega^a{}_b$ and their derivatives $d\theta^a$ and $d\omega_{ab}$, which are to be replaced by more convenient counterparts Θ^a and $\Omega^a{}_b$, respectively. Note that all the field equations studied above for the gravitational models cobsidered above can conveniently be discussed using this approach. In particular, some other RF^2 couplings the field equations were derived using the first order formalism [47, 76–79] previously. The derivation of the field equations for the Horndeski's model, which involves subtle technical points, is presented in a streamlined manner as follows.

To this end, one starts with the original Lagrangian 4-form extended by the constraint terms of the form

$$L'[\theta^a, \omega^a{}_b, F, \lambda^a, \mu] = \tfrac{1}{2\kappa}(R + 2\lambda) * 1 - \tfrac{1}{2}F \wedge *F - \tfrac{\gamma}{2}F \wedge F_{ab} * \tilde{\Omega}^{ab} + \lambda_a \wedge \Theta^a + \mu \wedge dF. \quad (91)$$

The variational derivatives commute with exterior derivative and also with an integral, thus for simplicity of notation, we suppress the integral symbol before the Lagrangian 4-form, and that the variational derivative is applied to an action integral $I = \int_U L'$ over a compact region U. The Lagrange multiplier 2-form μ is introduced to render the Faraday 2-form closed: $dF = 0$.

After a straightforward exterior algebra calculation, one can show that the total variational derivative of the Lagrangian 4-form with respect to the independent variables takes the form

$$\begin{aligned}\delta L' =& \delta\theta_a \wedge (-\tfrac{1}{\kappa} * G^a + \lambda * \theta^a + *T^a[F] + \gamma D\lambda^a - \tfrac{\gamma}{2}F^a \wedge F_{bc} * \tilde{\Omega}^{bc}) \\ &+ \delta\omega_{ab} \wedge \frac{1}{2}\left[D\left(\tfrac{1}{\kappa} * (\theta^a \wedge \theta^b) - \gamma\tilde{F}^{ab}F\right) - \theta^a \wedge \lambda^b + \theta^b \wedge \lambda^a\right] \\ &+ \delta F \wedge (-d\mu - *F + \gamma F_{ab} * \tilde{\Omega}^{ab}) + \delta\mu \wedge dF + \delta\lambda_a \wedge \Theta^a, \end{aligned} \quad (92)$$

up to a discarded exact form. Therefore, subject to the constraints $\delta L'/\delta\lambda_a = \Theta^a = 0$ and $\delta L'/\delta\mu = dF = 0$, the equations for the connection 1-forms, $\delta L'/\delta\omega_{ab} = 0$, can uniquely be solved for the Lagrange multiplier 2-form as

$$\lambda^a = F_b \wedge D\tilde{F}^{ab} + \theta^a \wedge d * (F \wedge F). \tag{93}$$

In the calculations above, note that it is essential to have a closed Faraday 2-form because the terms of the form $i_a D\tilde{F}^a = i_a \nabla^a \tilde{F} = -d^\dagger * \tilde{F} = *dF = 0$ show up in the derivation of the expression for the Lagrange multiplier 2-form (93). Consequently, the metric field equations (76) follow from (93) and (92).

Finally, it is worth to emphasize that among the F^2R-type nonminimal couplings given in eqs. (72)-(74), only for the particular case of the Horndeski's Lagrangian (68), Lagrange multiplier terms, namely $D\lambda^a$ terms, contain at most second order derivatives of the field variables $\{g, A\}$ as a consequence of vanishing torsion $\Theta^a = D\theta^a = 0$, and the identities $d^2 \equiv 0$ and

$$D^2\tilde{F}^{ab} = \Omega^a{}_c\tilde{F}^{cb} + \Omega^b{}_c\tilde{F}^{ac} \tag{94}$$

for the exterior derivative and the covariant exterior derivatives, respectively. In this regard, the expression for the Lagrange multiplier 2-form in equation (93) is to be compared to the one, for example, given in equation (57) for the Prasanna's model.

Appendix B: Newman-Penrose scalars

In the text, the NP quantities are expressed as contractions of the associated tensors with the null frame/coframe components with labels referring to a coordinate basis [52]. More conveniently, and perhaps in a manner that is of considerable practical use, the NP quantities can be identified as components of differential forms belonging to a null coframe [51].

Note however that, in contrast to the presentation of the NP formalism by making use of a null frame fields, for example in Refs. [51] and [52], the discussion above makes exclusive use of the null coframe fields $k, l, m, \bar{m}$ and refers an associated null frame fields to facilitate the comparison. The use of the coframe fields lead to a slightly differences in the expressions for the associated the basis frame fields and vice versa.

A technical advantage of our approach is that it allows us to obtain the metric field equations with the tensorial indices identified in terms of the NP scalars at once with the help of the definitions provided in this Appendix.

In general, the field equations derived from modified gravitational Lagrangian do not simply determine the anti-self dual part of the curvature 2-form interms of matter fields but they lead to higher order equations involving all the curvature components. The approach here is well-adapted for the application of the NP formalism to modified gravitational models as well. The expressions referring to a coordinate coframe, or an associated frame fields are not necessarily required along the calculations, except for an explicit expression for the exterior derivative in eq. (6). The NP scalars are expressed in terms of contractions only for convenience of the reader accustomed to the calculations using a coordinate coframe.

The spin coefficients defined originally by Newman and Penrose can be expressed as the components of the complex connection 1-forms, $\omega_{ab} = \eta_{ac}\omega^c{}_b = -\omega_{ba}$, introduced in the text by the 1-form equations

$$\omega^0{}_3 = -\tau k - \kappa l + \rho m + \sigma\bar{m}, \tag{95}$$

$$\tfrac{1}{2}\left(\omega^0{}_0 - \omega^3{}_3\right) = \gamma k + \epsilon l - \alpha m - \beta\bar{m}, \tag{96}$$

$$\omega^1{}_2 = \nu k + \pi l - \lambda m - \mu\bar{m}, \tag{97}$$

The curvature scalars Φ_{ik} with $i, k = 0, 1, 2$ and Ψ_k with $k = 0, 1, 2, 3, 4$ can be defined by the expressions for the curvature 2-forms as

$$\begin{aligned}
\Omega^0{}_3 &= -\Phi_{00} l \wedge m - \Phi_{01}(k \wedge l + m \wedge \bar{m}) + \Phi_{02} k \wedge \bar{m} + \tfrac{1}{12} R k \wedge m \\
&\quad - \Psi_0 l \wedge \bar{m} - \Psi_1 (k \wedge l - m \wedge \bar{m}) + \Psi_2 k \wedge m, \qquad (98)\\
\tfrac{1}{2}\left(\Omega^0{}_0 - \Omega^3{}_3\right) &= \Phi_{10} l \wedge m + \Phi_{11}(k \wedge l + m \wedge \bar{m}) - \Phi_{12} k \wedge \bar{m} - \tfrac{1}{24} R (k \wedge l - m \wedge \bar{m}) \\
&\quad + \Psi_1 l \wedge \bar{m} + \Psi_2 (k \wedge l - m \wedge \bar{m}) - \Psi_3 k \wedge m, \qquad (99)\\
\Omega^1{}_2 &= \Phi_{20} l \wedge m + \Phi_{21}(k \wedge l + m \wedge \bar{m}) - \Phi_{22} k \wedge \bar{m} + \tfrac{1}{12} R l \wedge \bar{m} \\
&\quad + \Psi_2 l \wedge \bar{m} + \Psi_3 (k \wedge l - m \wedge \bar{m}) - \Psi_4 k \wedge m. \qquad (100)
\end{aligned}$$

where self-dual components are identified in terms of Ψ_{ik} and R, whereas anti-self-dual components are identified in terms of Φ_{ik}.

By using the curvature 2-form definitions (98)-(100), one can verify that the Weyl curvature spinors can be expressed in the form as contraction of associated frame fields with the Weyl curvature tensor in the form

$$\Psi_0 = k^\mu m^\nu k^\alpha m^\beta C_{\mu\nu\alpha\beta}, \qquad (101)$$
$$\Psi_1 = k^\mu l^\nu k^\alpha m^\beta C_{\mu\nu\alpha\beta}, \qquad (102)$$
$$\Psi_2 = -k^\mu m^\nu l^\alpha \bar{m}^\beta C_{\mu\nu\alpha\beta}, \qquad (103)$$
$$\Psi_3 = l^\mu k^\nu l^\alpha \bar{m}^\beta C_{\mu\nu\alpha\beta}, \qquad (104)$$
$$\Psi_4 = l^\mu \bar{m}^\nu l^\alpha \bar{m}^\beta C_{\mu\nu\alpha\beta}, \qquad (105)$$

that are in accordance with the definitions given in Refs. [51, 52].

With the help of the Cartan's second structure equations in (10)-(12), the above curvature definitions in (98)-(100), and the convenient expression for exterior derivative (6), the complex scalar field equations originally derived by Newman and Penrose can be reproduced after some straightforward exterior algebra computations.

Only the Ricci spinors Φ_{ik} are determined by the Einstein field equations. For example, for the Einstein-Maxwell theory, they are determined in terms of Maxwell spinors using the present formalism as follows. The self-dual Faraday 2-form $\mathcal{F} = \frac{1}{2}(F - i * F)$ can be expanded into the basis self-dual 2-forms in terms of the Maxwell spinors as

$$\mathcal{F} = -\Phi_0 l \wedge \bar{m} - \Phi_2 (k \wedge l - m \wedge \bar{m}) + \Phi_2 k \wedge m. \qquad (106)$$

Using the expansion in eq (106), the Maxwell scalars Φ_k can be expressed in terms of the contractions as

$$\Phi_0 = F_{\mu\nu} k^\mu m^\nu, \qquad \Phi_2 = F_{\mu\nu} \bar{m}^\mu l^\nu, \qquad \Phi_1 = \tfrac{1}{2} F_{\mu\nu}(k^\mu l^\nu + \bar{m}^\mu m^\nu). \qquad (107)$$

In terms of the self-dual 2-form $\mathcal{F} =$, the source-free Maxwell's equations take the form $d\mathcal{F} = 0$, and the vanishing of the individual 3-form components yield the NP form of the Maxwell's equations as a set of scalar field equations for Φ_k with $k = 0, 1, 2$.

The electromagnetic field ansatz (23) discussed above has only Φ_2 component corresponding to a null component that satisfies $\mathcal{F} \wedge \mathcal{F} = 2(F \wedge F - iF \wedge *F) = 0$. Consequently, the canonical energy momentum 1-form defined in (26) can also be written in the form

$$* T_a[F] = -2i i_a \mathcal{F} \wedge \bar{\mathcal{F}}, \qquad (108)$$

and by making use of these curvature expressions (98)-(100) in the Einstein forms, some straightforward exterior algebra manipulations reproduce the Einstein-Maxwell field equations,

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namely the equations $*G^a = \kappa * T^a[F]$, in the scalar form as $\Phi_{ik} = \kappa\Phi_i\bar{\Phi}_k$. (Here $\kappa = 8\pi G/c^4$ denotes the gravitational coupling constant). In the same manner, the above definitions for the NP scalars in terms of tensor-valued forms allow one to construct other algebraically special solutions to the field equations discussed above using the first order formalism in terms of the original NP quantities.

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