On cylindrically symmetric gravitational waves in expanding spacetimes

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Abstract

We show a global existence theorem for the vacuum Einstein equations with cylindrical symmetry. Also Kasner-like solutions near initial singularity are constructed. These results support the validity of the strong cosmic censorship.

One of the most important problems in general relativity is the understanding of the global behavior of solutions to the Einstein equations. It is known that the Einstein equations lead to the formation of singularities. Typical examples are given by the Friedman-Robertson-Walker (spatially homogeneous and isotropic) spacetime, which evolves from the "big bang" singularity, and the Schwarzschild (stationary and spherically symmetric) spacetime, which has the central singularity. Then, the question of whether singularities generally occur in physical spacetimes or not, has been an important question for year. Penrose and Hawking have given some answers by their *singularity theorems* [4]. Rephrased in the words of the initial value problem, the question is that of the timelike and null geodesic completeness of the maximal future Cauchy development. The singularity theorems give us negative answers, i.e. generic spacetimes are timelike or null geodesically incomplete.

If the singularities can be seen, this means violation of predictability, because we cannot put appropriate boundary conditions on the singularities. Since predictability is a fundamental requirement of classical physics, it seems reasonable to require it to be valid whole spacetime. This implies that physical spacetimes should be globally hyperbolic in Leray's sense. Motivated by these considerations, Penrose proposed the strong cosmic censorship conjecture:

Conjecture 1 (Klainerman [6]) *Generic Cauchy data sets have maximal Cauchy developments which are locally inextendible as Lorentzian manifolds.*

This is one of the most important and unsolved questions in classical general relativity. We need two steps to prove the validity of the conjecture: (1) show global existence theorems for solutions to the Einstein(-matter) equations in an appropriate time coordinate, (2) analyze asymptotic behavior of the solutions and show inextendiblity of spacetime manifold. Thus, an important aspect of this strong cosmic censorship conjecture is relation to the global Cauchy problem for the Einstein(-matter) equations.

Unfortunately, the problem of proving global existence theorems for the full Einstein-matter equations is beyond the reach of the mathematics presently available. To make some progress, it is necessary to concentrate on simplified models. The most common simplifications are to look at solutions with various types of symmetry and solutions for small data.

Recently, new spacetimes which describe cylindrical gravitational waves in expanding universe are proposed [3]. The metric of the (generalized) spacetimes is given by

$$g = -e^{2(\eta-U)}dt^2 + e^{2(\eta-U)}dr^2 + e^{2U}(dx + Ady)^2 + e^{-2U}R^2dy^2, \tag{1}$$

where $\partial/\partial x$ and $\partial/\partial y$ are Killing vector fields generating the $U(1)\times R$ group action, and η, U, A and R are functions of $t \in (0,\infty)$ and $r \in (0,\infty)$. These new spacetimes would model localized inhomogeneities in Big Bang cosmology. Now we put a gauge condition, $R = rt$ [3]. The system of the evolution part of the Einstein equations is equivalent with the following wave maps $u : (M^{2+1}, G) \mapsto (N^2, h)$:

$$S_{\mathrm{WM}} = \int dtdr\sqrt{-G}G^{\alpha\beta}h_{AB}\partial_\alpha u^A\partial_\beta u^B, \tag{2}$$

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where

$$G = -dt^2 + dr^2 + t^2 r^2 d\psi^2, \qquad h = dU^2 + \frac{e^{4U}}{4r^2t^2} dA^2.$$

One of our main results is to show existence of global solutions for the above system and by using this we can prove the following theorem:

Theorem 1 *Let $(\mathcal{M}, g)$ be the maximal Cauchy development of C_0^∞ initial data for the cylindrically symmetric system. Then, $\mathcal{M}$ can be covered by Cauchy surfaces of constant time t with each value in the range $(0, \infty)$. Moreover, this maximal Cauchy development is timelike future geodesically complete, hence inextendible into the future direction.*

The method of the proof is the standard energy estimate (so-called *light cone estimate*). Theorems of Christodoulou-Tahvildar-Zadeh are also used [1, 2].

Another result is to construct Kasner-like (asymptotically velocity-terms dominated (AVTD)) solutions as $t \to 0$. To do this we will apply the Fuchsian algorithm developed by Kichenassamy and Rendall [5] to our system. This algorithm consists of the following steps:

- Decompose the unknown into a prescribed singular part and a regular part $\mathcal{U}$.
- If the system can be written as a Fuchsian system of the form

$$t\partial_t \mathcal{U} + N(x)\mathcal{U} = t^\alpha f(t, x, \mathcal{U}, \partial_x \mathcal{U}), \qquad \alpha > 0, \tag{3}$$

one can use the following theorem:

Theorem 2 *(Kichenassamy-Rendall [5]) Assume that N is an analytic matrix near $x = 0$ such that there is a constant C with $\| \sigma^N \| \leq C$ for $0 < \sigma < 1$, where σ^N is the matrix exponential of $N \ln \sigma$. Also, suppose that f is a locally Lipschitz function of $\mathcal{U}$ and $\partial_x \mathcal{U}$ which preserves analyticity in x and continuity in t. Then, the Fuchsian system (3) has a unique solution in a neighborhood of $x = 0$ and $t = 0$ which is analytic in x and continuous in t, and tends to zero as $t \to 0$.*

Remark 1 *The sufficient condition for N is non-negativity of eigenvalues of N.*

Now, the Geroch-Ernst potential, given by

$$\dot{A} = -Re^{-4U} w', \qquad A' = -Re^{-4U} \dot{w},$$

will be used for the convenience of computation. From this and replacing U by $z/2$, the evolution part of the Einstein equations become

$$D^2 z - t^2 \Delta z = -e^{-2z} \left((Dw)^2 - t^2 (\nabla w)^2 \right), \tag{4}$$

$$D^2 w - t^2 \Delta w = 2 \left(Dz Dw - t^2 \nabla z \nabla w \right), \tag{5}$$

where

$$D := t\partial_t, \qquad \Delta := \partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\phi^2, \qquad \nabla f \nabla g := \partial_r f \partial_r g + \frac{1}{r^2} \partial_\phi f \partial_\phi g.$$

To avoid a coordinate singularity at $r = 0$, the Cartesian coordinate will be used:

$$x = r\cos\phi, \qquad y = r\sin\phi, \qquad \Delta := \partial_x^2 + \partial_y^2, \qquad \nabla f \nabla g := \partial_x f \partial_x g + \partial_y f \partial_y g.$$

Note that the form of the evolution equations does not change. Dropping the spatial derivative parts from equations (4) and (5) and solving the equations, we have the following formal solutions:

$$z(t, x, y) = k(x, y) \ln t + \phi(x, y) + t^\epsilon u(t, x, y), \tag{6}$$

$$w(t, x, y) = w_0(x, y) + t^{2k(x,y)} \left(\psi(x, y) + v(t, x, y) \right), \tag{7}$$

where $\epsilon > 0$ is a small constant. Next, the Fuchsian system (3) for u and v will be reduced with the following conditions: $\alpha = 1$, $f = f(t, x, y, \mathcal{U}_i)$ is a regular function and

$$\begin{aligned} \mathcal{U} &= (\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3, \mathcal{U}_4, \mathcal{U}_5, \mathcal{U}_6, \mathcal{U}_7, \mathcal{U}_8) \\ &:= (u, Du, t\partial_x u, t\partial_y u, v, Dv, t\partial_x v, t\partial_y v) \end{aligned} \tag{8}$$

$$N = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \epsilon^2 & 2\epsilon & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2k & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad 0 < \epsilon < \min\{2k, 2-2k\}. \tag{9}$$

By using the theorem 2, we have

Theorem 3 *Suppose that k, ϕ, w_0 and ψ are real analytic functions of r and $0 < \epsilon < \min\{2k, 2-2k\}$. Then, there is a unique solution of the Einstein equations (4) and (5) of the form (6) and (7) in a neighborhood of $t = 0$ such that u and v tend to zero as $t \to 0$.*

Thus, solutions become AVTD (Kasner-like) ones near singularities at $t = 0$. In this case, the Kretschmann invariant $R_{\mu\nu\lambda\delta}R^{\mu\nu\lambda\delta}$ blows up as $t \to 0$, thus our spacetime is inextendible into the past direction if the solution (6) and (7) is generic. Combining our two theorems, it has been verified the validity of the strong cosmic censorship conjecture for our spacetimes.

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