GENERALIZED GELL-MANN AND LOW GROUP TRANSFORMATIONS: SCALING VARIABLES AND CROSS-OVER EFFECTS

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In dealing with critical phenomena the reduction schemes commonly used in many-body theory are useless. In fact infinitely many degrees of freedom are strongly correlated among themselves, owing to the fluctuations which extend over a coherence distance ξ and a characteristic time τ_c, both diverging at the critical point.

However, at least for the static case where only one infinity of degrees of freedom related to the length scale is involved, the introduction of a new reduction scheme is, at least in principle, quite simple.

This is related to the existence of "classes of universality", each comprising systems with equal symmetry, whose critical behaviour is found to be same, i.e. independent of the details of each given hamiltonian. Therefore a transformation on the parameters characterizing each system, leading to the asymptotic elimination of the "irrelevant" variables, was needed.

On the contrary the identification of those parameters on which the critical indices actually depend, led on the one hand to their perturbative ($\frac{1}{n}$ or ε -expansion) evaluation and on the other to the possibility to switch from one class of universality to another through the cross-over mechanism.

After Kadanoff's[1] formulation of scaling, to achieve these goals the Gell-Mann and Low[2] renormalization group transformation was proposed,[3] while Wilson[4] constructed his own group.

By means of the ε -expansion the same results were obtained with the two groups at least in the first few orders in ε, as first indicated in[5] up to the second order and then in[6] up to the fourth order by means of the Callan and Symanzik equation.

It was then felt[7] that no special significance should be assigned to the transformation one is considering but rather to the mechanism by which the asymptotic scale invariance in the physical variables is obtained. Actually it has been quite generally stated[8] that an infinity of equivalent transformations can be generated by means of an arbitrary diffeomorphism.

The choice of the transformation depends mainly on the simplicity of the calculations for the particular problem considered. When more than one relevant fields or "coupling constants" are present simultaneously, as for instance, in the tricritical phenomena, standard field theoretic methods become very complicated, because the renormalization functions depend non-linearly on the variables. In the ordinary Gell-Mann and Low group for instance even for the isotropic critical point the parameters of the transformation depend on the mass and the renormalized coupling constant. In fact already for this simple case a new realization of the group transformation à la Gell-Mann and Low was introduced in(9). This group is linear in the field φ and in the bare mass $\tau \sim T-T_c$ and therefore the transformation parameters depend only on the renormalized coupling constant as in the Callan and Symanzik equation, and its differential equation does not have the inhomogeneous term as the Gell-Mann and Low group.

On the other hand recently groups non linear in φ have been successfully applied to lattice systems(10) in one or two dimensions, where the results of the ε-expansion cannot be extended. The properties of a particular non linear group have also been studied(11) for a continuous model. We shall introduce generalized group transformations(8), which include all those previously mentioned.

Generalized Group Transformations

According to universality we can try to discuss critical phenomena by means of an "effective hamiltonian" with short range interactions

$$H = \int d^d x \left\{ \frac{1}{2} (\nabla\varphi)^2 + \sum_i \frac{1}{i!} \mu_i(x) \varphi^i(x) \right\} \tag{1}$$

with $(\nabla\varphi)^2 = \sum_{\sigma=1}^{n} (\nabla\varphi_\sigma)^2$, $\mu_1\varphi_1 = \sum_{\sigma=1}^{n} h_\sigma \varphi_\sigma$, $\varphi^2 = \sum_{\sigma=1}^{n} \varphi_\sigma^2$, $\varphi^4 = (\varphi^2)^2, \dots$

$\mu_1(x)$ is the external field and $\mu_2 = \tau \sim T-T_c/T_c$ is the deviation from the critical temperature. The relation of this field-theory model with the standard statistical models such as the Ising, x-y or Heisemberg models has been discussed in (12). Long range forces of the type $\frac{1}{r^{d+\sigma}}$, anisotropic exchange effects, cubic anisotropy can be included by modifying the gradient term, the quadratic and quartic interaction, respectively, where as the φ^6 term will be associated to the tricritical behaviour. These features lead to cross-ever effects from one critical behaviour to another as reviewed by Fisher.(14)

The logarithm of the partition function F is a functional of the $\{\mu_i(x)\}$. By successive Legendre transformations we can eliminate these "fields" μ_i's in favour of the corresponding conjugate "density" variables ω_i's, generating a set of thermodynamic potentials which are functionals of combinations of fields and densities $\{\bar{\nu}_i\} \equiv \{\mu_\ell, \omega_j\}$, $\ell \neq j$. In each case successive functional derivatives generate all the physically relevant quantities.

In particular the n-th derivative with respect to one variable generates the comulant of n-th order of the conjugate variable . For instance the first Legendre transformation of F gives rise to the functional $\Gamma(\varphi, \mu_2, \cdots)$, which is in turn the generator of the one-particle irreducible n-point vertex function $\Gamma^{(n)}(x_1, \ldots, x_n)$

By means of a differentiable transformation labelled by a parameter λ having the dimensions of the inverse of a length we operate on the dimensionless variables $\{\nu_i = \frac{\nu_i}{\lambda^{x_i^0}}\}$, where x_i^0 is the bare dimensionality of ν_i in terms of the inverse of a length ($x_\varphi^0 = 1 - \varepsilon/2$, $x_\tau^0 = 2$ and $x_{\mu_4}^0 = \varepsilon$ where $\varepsilon = 4 - d$).

The transformation is determined by a set of differential equations

$$\lambda' \frac{\partial \nu_i'}{\partial \lambda'} = \Psi_i (\{ \nu_j' \} ; \lambda') \tag{2}$$

where the Ψ_i functions are to be specified.

We define as generalized group transformations(8) those transformations, on the variables $\{\nu_i\}$ which leave the partition function unchanged except for an irrelevant multiplicative constant. This means that the thermodynamic potentials are the invariants of the system (2) of differential equations of the transformation and thus satisfy the linear differential equation

$$\left[\sum_i \{ \Psi_i (\{ \nu_j' \} ; \lambda') \frac{\delta}{\delta \nu_i'} + \lambda' \frac{\partial}{\partial \lambda'} \right] \Gamma = 0 \tag{3}$$

<u>Scaling variables</u>

We make the further assumption that on the critical surface of the system there exists a fixed point of the trasformation i.e. a point in the parameter space where if we iterate the transformation further the $\{\nu_i'\}$ are left unchanged. From eqs.(2) it follows that a fixed point is a solution of the equations

$$\Psi_i (\{ \nu_j^* \} ; \lambda^*) = 0 \tag{4}$$

If we linearize eqs.(2) around the fixed point, we see immediately that the eigenfunctions of the matrix $\sigma_{ij} = - \frac{\delta \Psi_i}{\delta \nu_j}$, when evaluated at the fixed point, are the asymptotic scaling variables and its eigenvalues are the corresponding critical exponents. The thermodynamic potential becomes a homogeneous function of the same variables.

As originally stressed in (4), (12) and (13) all the requirements of the universality hypothesis are now satisfied. Different classes of universality correspond to different fixed points with different symmetry properties. Moreover the critical exponents are connected to the derivatives of the transformation functions evaluated at the fixed point which are non-singular. Perturbation theory can now be applied and this is at the origin of the $1/n$ and the ε-expansions.

As stressed by Jona-Lasinio(8) two transformations where the matrices σ_{ij}

are related by a similarity transformation lead to the same critical indices i.e. are equivalent. This is the case if the two transformations are differentiably conjugate.

Non-linear terms allow to discuss corrections to the asymptotic power-law behaviour and possibly the cross over from one type of critical behaviour to another i.e. from one fixed to point to another.[14]

In general for each set $\{x_i\}$ of critical exponents related to a given fixed point we could look for the scaling variables[13] $g_i'(\{\nu_j'\})$ which under the group transformation scale exactly as $g_i' = (\lambda'/\lambda)^{-x_i} g_i$ and therefore satisfy the equations

$$\lambda' \frac{\partial g_i'}{\partial \lambda'} = -x_i g_i' \quad ; \qquad \Big[-\sum_i x_i \{ g_i' \frac{\delta}{\delta g_i'} + \lambda' \frac{\partial}{\partial \lambda'} \Big] \tilde{\Gamma}(\{g_i'\}) = 0 \tag{5}$$

The scaling variables are not unique since they depend on which group transformation is considered. More generally we can now say that equivalent group transformations lead to various sets of scaling variables with the same critical exponents.

On the contrary two sets of scaling variables $\{g_i'\}$ and $\{g_i'^{o}\}$ corresponding to two different sets of critical exponents $\{x_i\}$ and $\{x_i^o\}$ cannot be equivalent i.e. the cross-over from one critical behaviour to another is controlled by a non-differentiable transformation.

In each particular case an explicit determination of the scaling variables would completely solve the problem of critical phenomena. This was done in the large n limit by Ma[15] and is not feasible in general.

A cross-over model

If the transformation is linear in φ, a linear relationship in the correlation functions is induced, which is local in the Gell-Mann and Low case[3][5][9] and non-local in the Wilson case.[12] This is the main reason why recently[16] it was possible to give a formal evidence that the two groups are asymptotically equivalent in the sense specified above, whereas the comparison with non linear groups is much more difficult.

As in (9) we now consider the simplest model Hamiltonian with only $u\varphi^4$ interaction. The original set of variables is simply $\{\nu_i\} \equiv \{\varphi, \tau, u\}$. The standard renormalization procedure is to express the transformation in terms of correlation functions by the choice of a normalization point. In the Gell-Mann and Low group this is possible owing to the presence of the multiplicative factor in the linear relationship for the correlation functions, by choosing

$$\frac{\partial\Gamma^{(2)}}{\partial k'^2}\bigg|_{N.P.} = 1 \quad , \quad \frac{\partial\Gamma^{(2)}}{\partial\tau'}\bigg|_{N.P.} = 1 \quad , \quad \frac{\Gamma^{(4)}}{u'}\bigg|_{N.P.} = 1 , \tag{6}$$

provided the energy correlation functions also transform linearly.

Since τ' and φ' are the two obviously relevant fields, we parametrize the transformation in terms of u' only as specified in (9). Successive functional derivatives of the group equation (3) for Γ generate the corresponding equations for $\frac{\partial\Gamma^{(2)}}{\partial k'^2}$, $\frac{\partial\Gamma^{(2)}}{\partial\tau'}$ and $\Gamma^{(4)}$, which once specified at the N.P. determine the quantities $\psi_\varphi(u')$, $\psi_\tau(u')$ and $\psi_u(u')$.

Normalization conditions (6) correspond to the subtractions necessary in diagrammatic technique to perform wave function, mass and vertex renormalization.

If the transformation is non linear, differentiations with respect to φ' and τ' of eq.(3), lead to coupled equations for different correlation functions, and the choice of a normalization point may become meaningless.

Our linear transformation is now completely specified and we can study[9] the fixed-point solution at d = 4- ε dimension. We obtain a trivial fixed point $u^* = 0$ with classical exponents $\{x_i^o\}$, which is unstable for $d < 4$ and a non-trivial fixed point $u^* \sim O(\varepsilon)$ with non-classical exponents $\{x_i\}$, which is stable for $d < 4$. There is an additional relevant field u in the trivial case and therefore this fixed point can be associated with tricritical behaviour.[17] u becames the cross-over field from critical to tricritical behaviour at the lowest order, with cross over exponent ϕ given by $x_u = \phi/\nu$. Since $u^* = 0$, the φ^6 term becomes essential in the description of tricritical phenomena. This model can be studied around d = 3 in the same way as the φ^4 model around d = 4. At d = 3 the indices are classical with logarithmic corrections as found in (17) by Wilson's method.

If we expand the group equation for τ' and u' at second order we obtain

$$\lambda' \frac{\partial\tau'}{\partial\lambda'} = \psi_\tau(u') \simeq -x_\tau^o \tau' - (x_\tau - x_\tau^o) \frac{u'}{u^*} \tau' \tag{7}$$

$$\lambda \frac{\partial u'}{\partial\lambda'} = \psi_u(u') \simeq -x_u^o \, u' \left(1 - \frac{u'}{u^*}\right) \tag{8}$$

The transformation becomes completely parametrized by the critical exponents $\{x_i^o\}$ and $\{x_i\}$ evaluated at the lowest order.

If we now assume that $\{x_i^o\}$ and $\{x_i\}$ are two sets of indices obtained by other sources and consider eqs.(7) and (8) as exact, this model coincides with the one hypotized by Riedel and Wegner[18] to discuss cross-over effects in various cases such as tricritical systems, the Askin and Teller model and Fisher exponent renormalization.

The two sets of scaling variables are now easily determined

$$g'_{\tau} = \tau'\left(\frac{u'}{u^*}\right)^x \qquad\qquad \mathring{g}'_{\tau} = \tau'\left(1-\frac{u'}{u^*}\right)^x$$
$$g'_u = \frac{1-u'/u^*}{u'/u^*} \qquad\qquad \mathring{g}'_u = \frac{u'/u^*}{1-u'/u^*} \tag{9}$$

with $x = x_{\tau} - x^{\circ}_{\tau}/x^{\circ}_u$

They, asymptotically near to each one of the fixed points, reduce to τ', $u'-u^*$ or τ' and u' respectively. They are related by the non-equivalent transformation

$$g'_{\tau}(\lambda') = \mathring{g}'_{\tau}(\lambda')\,[\mathring{g}'_u(\lambda')]^x \quad , \qquad g'_u(\lambda') = [\mathring{g}'_u(\lambda')]^{-1} \tag{10}$$

which allows for the discussion of the whole cross-over region.[18]

As previously indicated in the discussion of the model hamiltonian, when a more involved symmetry property is taken into account a new cross-over field is present, which leads to a further renormalization condition of type (6). The related critical exponent coincides with the new cross-over index. The explicit evaluation of critical indices by means of the ϵ-expansion in all the cases previously indicated is described by Yamazaki in the paper presented at this conference[18] to which we also direct the reader for the original references.

In this scheme it becomes natural to consider any transport coefficient also as an "additional field" to be renormalized. The corresponding renormalization function leads to a further anomalous dimension, which is the dynamical critical index associated with the transport coefficient under consideration. Of course, in order to introduce bare transport coefficients, one is compelled to start with an equation of motion of "the time-dependent Landau-Ginzburg" type as it was actually done in (20) and (21).

The important problem still under consideration is "if and how" is it possible to derive scaling starting from the microscopic interactions instead of performing the renormalization on the equations of motion. Quite heavy work of coarse - graining is necessary to derive these equations of motion in the very few cases where it is possible.[22] We feel that this process of introducing "hydrodynamics" into the theory is a necessary step to obtain dynamic scaling, if dynamic scaling hold at all.

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This talk was originally given at the Congressino on "The Renormalization Group and its applications in statistical phisics" - Trieste July 1974.