SLAC-TN-05-028

LCLS-TN-02-6
October 2002

Undulator Gravitational Deflection

Gordon B. Bowden
Stanford Linear Accelerator Center, Stanford University, Stanford, CA 94309

Abstract

This note estimates distortions imposed by gravity on LCLS undulator strongbacks. Because of the strongback's asymmetric cross section, gravitational forces cause both torsion as well as simple bending. The superposition of these two effects yields a 4.4 μm maximum deflection and a 0.16 milli radian rotation of the undulator axis. The choice of titanium is compared to aluminum.

1 Undulator Strongback Dimensions

LCLS strongbacks will be 0.8 ton titanium cylinders 3.4 meters long by .305 meter diameter. Rectangular channels are cut into the sides to house the permanent magnet undulator structure. Dimensions are shown below in Fig. 1. Table 1 contains strongback parameters.

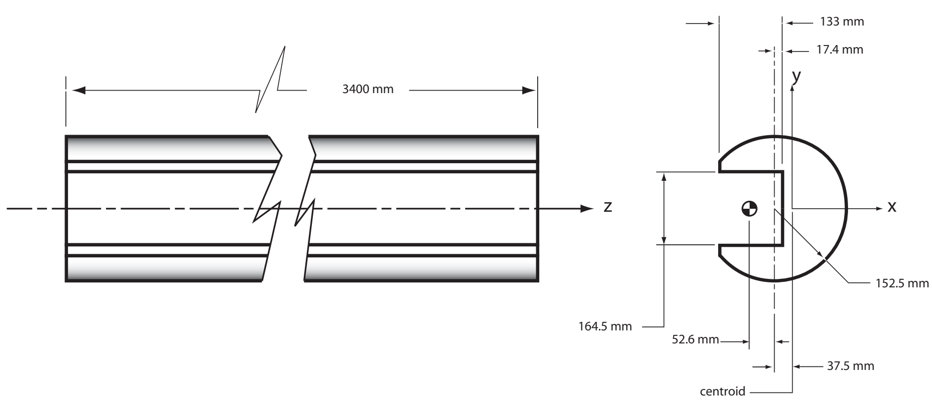

Figure 1: Undulator strongback, approx. dimensions measured from LCLS CDR.

Work supported in part by the Department of Energy contract DE-AC02-76SF00515
Stanford Linear Accelerator Center, Stanford University, Stanford, California, 94309

Table 1 **Strongback Parameters**			
section area	$A_{section}$	4.62×10^4	mm^2
section inertia	I_{xx}	3.65×10^8	mm^4
centroid location*	x_c	37.53	mm
undulator axis*	x_u	-52.60	mm
shear ctr.*	x_s	115.66	mm
strongback weight/length	w	0.208	kg/mm
PM weight/length	w_{pm}	0.030	kg/mm
torsion stiffness	D	5.95×10^{11}	kg*mm^2
Ti Young's modulus	E	11.977×10^3	kg/mm^2
Ti shear modulus	G	4.368×10^3	kg/mm^2
Ti Poisson's ratio	ϵ	0.33	unitless
Ti density	ρ	4.47×10^{-6}	kg/mm^3
* x dimen to section circle ctr			

2 Deflection of non-symmetrical sections

Gravitational forces act throughout the strongback's volume. To calculate gravitational distortion without approximation would require integrating the effect of these distributed body forces over the entire volume of the strongback. In this note, the strongback will be treated instead as a simple beam with weight distributed uniformly along the axis of its centroid at x_c. Beams with fully symmetric cross sections suffer only simple bending if forces act at the symmetry axis. However even if the beam cross section is not symmetric, there is still a point in the cross section called the center of flexure or shear center x_s where forces can be applied causing only simple bending, free of any torsion. This shear center usually lies within the beam cross section but for thin walled structural sections, x_s may even fall outside the section. (For fully symmetric sections, $x_s = x_c$ and is the center of symmetry.) The bending problem for a beam with asymmetric cross section can be decomposed into the superposition of 2 simpler problems:

bending: Deflection of the shear center axis by simple bending of the beam under the influence of distributed weight w (kg/mm) and reaction forces from the supports.

torsion: When mass ctr is not coincident with the shear center, gravity loads the beam with axial torsion moments dM/dz (mm*kg/mm) distributed along its length. Support reaction forces counter balance these moments (otherwise the beam will roll over). The resulting torsion α (radians/mm) vertically displaces the undulator axis by rotation around the shear center x_s.

3 Bending: optimal support location

Optimal location for support of a uniform beam loaded by gravity is defined by that pair of support positions where the overhanging ends of the beam droop the same amount as the sag at midpoint between the two supports. Any other support locations have larger deflection. This pair of support points are called the Airy points after Astronomer Royal, Sir George Airy. Derivation of their location follows.
Consider one half of the symmetric beam length $l = L/2$. The free end of the beam and the midpoint of the beam are required to have the same deflection (figure 2). This deflection is the superposition of the downward droop of the half beam under uniform load w (kg/mm) and the upward bending of the half beam by reaction force wl (kg) at location a to be determined. Location a is adjusted until these 2 components of the total deflection just cancel at the free ends of the beam. The sum of these two contributions to the end point deflection is set to zero and solved for a:

$$\frac{wl}{6EI}(a^3 - 3l^2a + 2l^3) - \frac{wl^4}{8EI} = 0 \tag{1}$$

This yields a cubic equation for the reaction force location $\alpha \equiv a/l$,

$$\alpha^3 - 3\alpha + \frac{5}{4} = 0. \tag{2}$$

This equation has 3 real unequal roots. The Handbook of Chemistry and Physics lists the roots in parametric form based on angle ϕ defined by $\cos(\phi) \equiv -\frac{1}{2}(5/4), \quad \phi = 128.68°$:

$$\alpha = 2\cos(\frac{\phi}{3}), \quad 2\cos(\frac{\phi}{3} + 120°), \quad 2\cos(\frac{\phi}{3} + 240°) \tag{3}$$

The only relevant root $(0 < \alpha < 1)$ is the last: $\alpha = 2\cos(42.894° + 240°) = .4463$. This means that optimal support locations are $(.4463/2)L = .22315L$ in from the ends of the full length beam. The maximum amplitude of the deflection curve occurs slightly outboard of

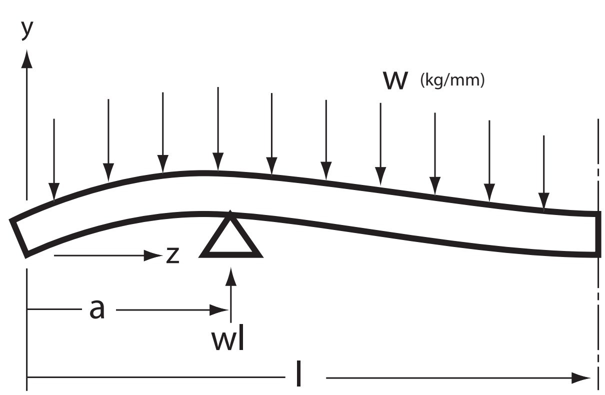

Figure 2: Optimal support location for minimal deflection under uniform load

the support location a. The coordinates of the top of the deflection curve measured from the free end of the beam are:

$$\begin{aligned} z_{max} &= \sqrt[3]{l^3 - 3l(l-a)^2} \\ y_{max} &= y_A + \theta z_{max} + \frac{w}{24EI}(4l^3 z_{max} - z_{max}^4 - 3l^4) \\ y_A &= \frac{wl}{6EI}(2l^3 - 3al^2 + a^3) \\ \theta &= -\frac{wl}{2EI}(l-a)^2 \end{aligned} \qquad (4)$$

Evaluated for the LCLS undulator, $y_{max} = .0017$ mm at $z_{max} = 732.25$ mm. This 1.7 micron deflection is compared with the deflection of the same beam supported at its end points which would be 49 times larger:

$$\delta = \frac{5}{384}\frac{wL^4}{EI} = .08279 \text{ mm} \qquad (5)$$

4 Torsion constants x_s and D

Calculation of torsion on a non-symmetrical beam requires two constants: (1) shear center location $(x_s - x_c)$ which is the separation between the twisting axis and the section's centroid and (2) torsional stiffness D, the angular twist per unit length generated by unit axial torque. For a few simple cross sections, $(x_s - x_c)$ and D can be looked up in structural engineering handbooks, but usually these properties of the beam cross section must be calculated from elastic theory. See for example Sokolnikoff[1]. Torsion stiffness D and shear center location x_s both require solution of 2D partial differential equations over the cross section. *Matlab toolbox* **PDE** was used to compute D and x_s. Figure 3 shows the solution for Sokolnikoff's function Ψ[1] for D. Figure 4 shows the ϕ_2 solution for $(x_s - x_c)$. Equations are numbered following Sokolnikoff but symbols for shear modulus G and poisson's ratio ϵ have been changed to follow notation of Roark[3] and Timoshenko[4]. Formula (53.4) for the shear center $(x_s - x_c)$ assumes that the coordinate system origin is located at the section's centroid x_c. Torsion function ϕ_2 has von Neumann boundary conditions where ν is the boundary's unit normal vector. Two material properties enter into these equations. For the torsion stiffness D, eq.(34.10), shear modulus $G = 4.368 \times 10^3$ kg/mm^2 for titanium. For shear center $(x_s - x_c)$, eq.(53.4), titanium's poisson ratio $\epsilon = .33$. Numerical values for x_s and D are calculated after solutions of eq. 35.7 and eq. 34.5 and are included in Table 1.

5 Torsion curve

Gravity acting on the strongback's center of mass w at x_c and permanent magnets w_{pm} at x_u generates a moment/unit length around the shear center axis $dM/dz = (x_s - x_c)w + (x_s - x_u)w_{pm}$. Each support carries half the total weight W. At the 1st support, rotation ϕ is locked to zero. At the 2nd support, reaction forces generate an additional moment.

Calculation of torsion stiffness D

$$(35.7) \quad \nabla^2 \Psi = -2 \quad \text{in section region } R$$

$$(34.6) \quad \Psi = 0 \quad \text{on section boundary } C$$

$$(34.10) \quad D = 2G \iint_R \Psi \, dx \, dy$$

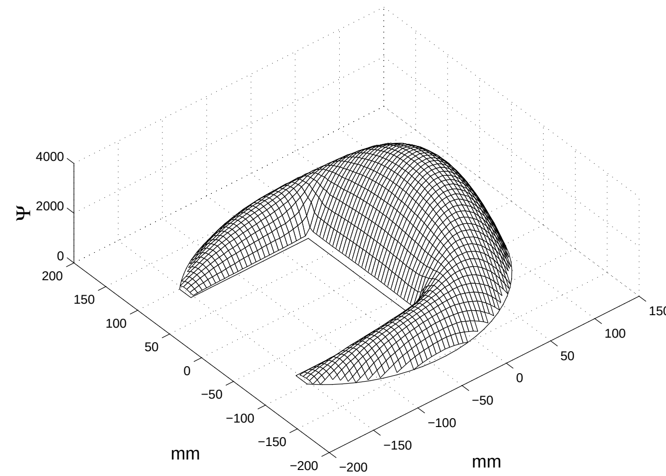

Figure 3: Stress Function Ψ (mm^2)

Calculation of shear center x_s

$$(34.5) \quad \nabla^2 \phi_2 = 0 \quad \text{in section region } R$$

$$(52.16) \quad \frac{d\phi_2}{d\nu} = [(1+\epsilon)y^2 - \epsilon x^2] \cos(y, \nu) \quad \text{normal derivative on boundary } C$$

$$(53.4) \quad (x_s - x_c) = \frac{1}{2(1+\epsilon)I_{xx}} \iint_R \left[x \frac{\partial \phi_2}{\partial y} - y \frac{\partial \phi_2}{\partial x} - (1+\epsilon)xy^2 + \epsilon x^3 \right] dx \, dy$$

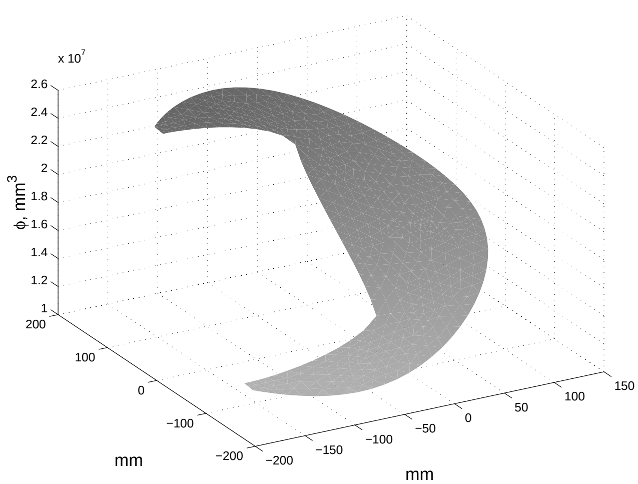

Figure 4: Torsion Function ϕ_2 (mm^3)

Geometry of 2nd support can be chosen by design to generate a torsion cancelling the twist which has accumulated along the axis between the two support locations. Twisting angle ϕ will then reach a minimal maximum at midspan. Support geometry is illustrated in Figure 5 with 2nd support reaction force at x_r. (illustration exaggerates the offset of the roller support circle from the strongback ctr.) Twisting moments which develop along the span between the two support points consist of two components:

(1) moment $dM/dz = (x_s - x_c)w + (x_s - x_u)w_{pm}$ due to strongback and magnet weight times their offsets from the shear center. This moment accumulates along the length of the strongback reaching a maximum at the 1st support where the rotation ϕ is fixed to zero.

(2) a constant axial moment along the strongback between the supports due to the offset $(x_s - x_r)$ between the 2nd support reaction force $W/2$ and the shear center at x_s.

Along the length of the strongback beyond the fixed support 1, the sum of these moments can be integrated to find twist angle ϕ. If the total weight $W = (w + w_{pm})L$ and the torsionally free length of the strongback $l' = (1 - .2231)L$, then

$$\begin{aligned} D\frac{d\phi}{dz} = M(z) &= \frac{dM}{dz}(l' - z) - \frac{W}{2}(x_s - x_r) \\ D\phi &= \frac{dM}{dz}(l'z - \frac{z^2}{2}) - \frac{W}{2}(x_s - x_r)z \end{aligned} \quad (6)$$

The offset $(x_s - x_r)$ of the 2nd support reaction force which cancels ϕ at $z = l = (1 - 2 * .2231)L$ is:

$$(x_s - x_r) = \frac{L}{W}\frac{dM}{dz} = 89.5 \text{ mm.} \quad (7)$$

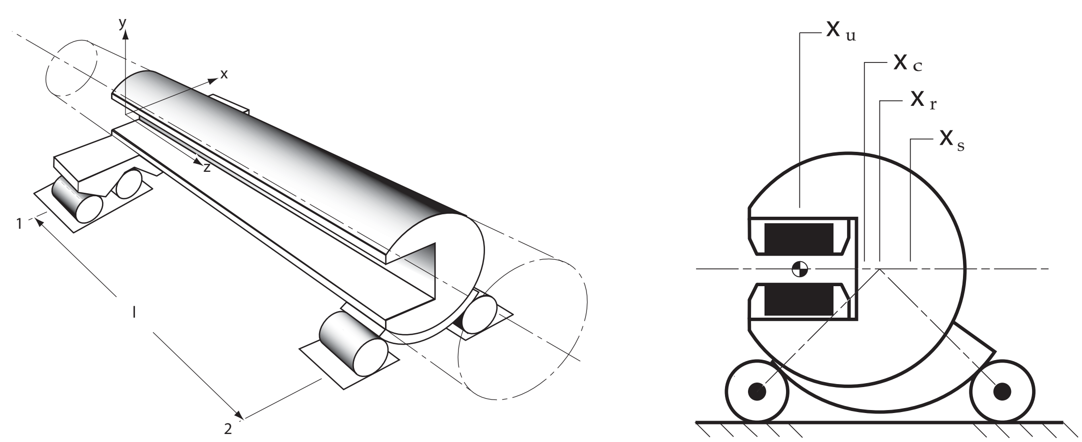

Figure 5: Support geometry for cancellation of torsion

The remaining maximum twist occurs at mid span at $z = l/2$ where $\phi_{max} = 1.59 \times 10^{-5}$ radian. Since the undulator axis x_u is offset 168.25 mm from the strongback shear center x_s, this rotation causes an additional $(1.59 \times 10^{-5})(168.25 \text{ mm}) = 2.7$ micron deflection of the undulator axis.

6 Conclusion: Titanium vs Aluminum

Gravity loads on Ti strongbacks cause a 1.7 μm beam deflection plus a 2.7 μm torsional deflection totaling to 4.4 μm at mid-span. These numbers are challenging to measure over a strongback length of 3.4 meters. They are negligible compared to practical machining tolerances. But this small calculated deflection depends on an optimal support geometry. If the strongback were supported from its extreme ends without cancellation of gravitational torsion moment, distortion could be orders of magnitude larger.
Mechanical/thermal properties of titanium and aluminum are compared in Table 2. There is almost no distinction between titanium and aluminum from the the standpoint of gravitational deflection. Gravitational deflection depends on the ratio of elastic modulus to density. For Ti, $E/\rho = 2.679 \times 10^6$ meters while for Al, $E/\rho = 2.59 \times 10^6$ meters. Deflections in a Ti strongback would be only 4 percent less than in aluminum.

Table 2 **Ti vs Al**

property		Ti	Al	
Young's modulus	E	11.977×10^3	7.045×10^3	kg/mm^2
shear modulus	G	4.368×10^3	2.642×10^3	kg/mm^2
Poisson's ratio	ϵ	0.33	0.30	unitless
density	ρ	4.47×10^{-6}	2.72×10^{-6}	kg/mm^3
thermal expansion	α	8.4×10^{-6}	23.9×10^{-6}	°C^{-1}
thermal conductivity	λ	0.171	2.37	watts/cm/°C

Comparison of titanium with aluminum from the standpoint of thermal distortion depends on both expansion coefficient and thermal conductivity [2]. Transverse thermal gradients cause the strongback to bend toward the warmer side. Long bars of width h and length L with transverse thermal gradients deflect $\delta/\Delta T = \alpha L^2/8h$ mm/°C. For the LCLS strongback made from titanium, $\delta/\Delta T = 39.8$ μm/°C. Made from aluminum, $\delta/\Delta T = 113.2$ μm/°C. It might appear that titanium would be less affected by thermal disturbances than aluminum because of its lower expansion coefficient. But thermal distortions are driven by heat fluxes. Surprisingly small transverse fluxes lead to significant distortion. The ratio of thermal expansion α to thermal conductivity λ determines the strongback's sensitivity to heat flux. Deflection/transverse heat flux $\delta/q = \frac{L^2}{8}\frac{\alpha}{\lambda}$. Made from titanium, $\delta/q = .89$ μm/watt/m^2. Made from aluminum, $\delta/q = .18$ μm/watt/m^2. Aluminum's high thermal conductivity more than compensates for its higher thermal expansion. Thermal gradients and the distortion they cause will be lower in aluminum than in titanium. Ti's poor thermal conductivity also effects precision machining where localized cutting heat leads to distortion during machining operations.

References

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