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Gravitational form factors and decoupling in $2D$

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ABSTRACT

We calculate and analyse non-local gravitational form factors induced by quantum matter fields in curved two-dimensional space. The calculations are performed for scalars, spinors and massive vectors by means of the covariant heat kernel method up to the second order in the curvature and confirmed using Feynman diagrams. The analysis of the ultraviolet (UV) limit reveals a generalized "running" form of the Polyakov action for a nonminimal scalar field and the usual Polyakov action in the conformally invariant cases. In the infrared (IR) we establish the gravitational decoupling theorem, which can be seen directly from the form factors or from the physical beta function for fields of any spin.

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1. Introduction

The decoupling of quantum massive field at the energies which are much smaller than the mass of the field is the cornerstone of the effective approach to quantum field theory. The application of this idea to gravity is a subject which attracts a special interest (see, e.g., [1] for a recent review). Practical calculations of decoupling of massive matter fields in four dimensional ($4D$) spacetime have been done by means of Feynman diagrams and also within the heat kernel methods [2–4].

There is an extensive literature about the use of effective field theory methods in quantum gravity. Starting from the pioneer publications [5,6], there were a lot of works (see, e.g., [7] for reviews and further references). Along with the rich theoretical contents, these works pave the way for interesting applications in astrophysics. An important point in the application of effective methods to quantum gravity is the "belief" in the universality of quantum general relativity as an IR limit of quantum gravity of any sort, at least for the local versions of the theory. The calculation which could resolve this issue has been proposed long ago by one of the present authors [8,9]. However, a practical realization of this program meets serious obstacles, especially because of the technical difficulties of making calculations in higher derivative gravity within a mass-dependent renormalization scheme. In this situation it is tempting to start from the simplest model of quantum gravity, and in this respect the two dimensional ($2D$) quantum gravity is a very useful model to test general concepts.

In the present work we start the investigation of decoupling of massive degrees of freedom in $2D$. In order to explore the mechanisms and special features of two dimensional gravity, we consider the loops of matter fields and thus extend the results which were obtained in $4D$. The nice feature of the gravitational contributions of matter fields in two dimensions is that the ones of the massless conformal fields are pretty well known from the integration of the trace anomaly [10,11], being covariant non-local Polyakov action [12]. The integration of non-conformal and in particular massive fields gives a new perspective to these classical results about anomaly [13] and in this sense it complements the recent derivation of the Polyakov action via the functional renormalization group methods [14], which effectively treat all fields as massive due to the use of the regulator term.

In what follows we summarize the derivation of non-local form factors in the vacuum gravitational sector of massive matter fields in $2D$. Regardless of its relative simplicity compared to the $4D$ vacuum calculations, we shall observe that the $2D$ results may be interesting, since they enable one to explore general issues as the relation between massless limit and the anomaly-induced effective action in the UV from one side and the decoupling of massive degrees of freedom at the IR end of the energy scale.

The paper is organized as follows. In Sec. 2 we present a brief review of the necessary results from the heat kernel methods, based on the works [15,16], [17] and [4]. Some useful technical information is separated into Appendix A, and an explanation of how

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the surface term can be dealt with using Feynman diagrams into Appendix B. Sec. 3 gives general well-known expressions for the vacuum effective action of a theory which included sets of massive scalars, fermions and vectors. Sec. 4 describes the derivation of form factors of a massive scalar for the gravitational terms in $2D$. Furthermore, in the two subsections we describe the generalization of the Polyakov action to the case of a nonminimal massless scalar and the high-energy (UV) and low-energy (IR) limits in the physical beta functions derived within the momentum-subtraction renormalization scheme. Technical discussion of the result for the non-conformal case can be found in Appendix C. Sec. 5 shows the derivation and discuss the structure of the form factor and beta functions for the Dirac fermion. In Sec. 6 we show the same for the Proca field and also discuss the discontinuity which takes place in the UV, in which one finds a discontinuity with the expression for the massless gauge field. Finally, in Sec. 7 we draw our conclusions and outline perspectives.

2. Effective action in $2D$: general considerations

In $2D$ Riemann and Ricci tensors can be expressed via the Ricci scalar, as

$$R_{\mu\nu\alpha\beta} = \frac{1}{2} R\big(g_{\mu\alpha} g_{\nu\beta} - g_{\nu\alpha} g_{\mu\beta}\big), \qquad R_{\mu\nu} = \frac{1}{2} R\, g_{\mu\nu}. \tag{1}$$

This means, in particular, that the Einstein tensor is identically zero such that the part of Einstein's classical equations describing the metric tensor is null. Furthermore, the power counting in $2D$ shows much less divergences. Both features make the $2D$ quantum gravity models much easier to explore compared than their $4D$ counterparts [18,19], and produce results which are compatible with the ones of conformal field theory [20,21]. Specifically, all these works were devoted to the quantization of a nonlocal Polyakov action or its metric-scalar equivalents.

In a $2D$ semi-classical approach, in which only matter fields are quantized, the effective action for the metric may be induced by quantum fields. In the UV limit such an effective action can be calculated through the integration of the conformal anomaly [22]. However, in order to describe the phenomena at various energy scales, one has to take the masses of quantum fields into account. Consider a free massive matter field Φ characterized by the quadratic action

$$S[\Phi] = \frac{1}{2} \int \mathrm{d}^D x \sqrt{g}\, \Phi \left(\mathcal{O} + m^2\right) \Phi, \tag{2}$$

in which $\mathcal{O}$ is a second-order covariant differential operator also known as a Laplace type operator.

We can use the heat kernel expansion to compute the one-loop effective action of vacuum by appropriately integrating over the heat kernel time in the Euclidean signature,

$$\Gamma[g] = -\frac{1}{2} \mathrm{Tr} \int_0^\infty \frac{\mathrm{d}s}{s}\, \mathrm{e}^{-sm^2}\, \mathcal{H}(s), \tag{3}$$

in which we introduced the heat kernel solution $\mathcal{H}(s)$, which formally is a bi-scalar $\mathcal{H}(s) = \mathcal{H}(s; x, x')$ and whose explicit form will be shown in Appendix A following [4,16]. Generically, the functional trace includes taking coincidence limit $x' \to x$ and a covariant integration over $\int \mathrm{d}^D x \sqrt{g}$. Additionally, since the parameter s is dual to an energy the s-integration ranges from the UV at $s = 0$ to the IR at $s = \infty$. While the IR limit converges thanks to the presence of the mass $m^2 > 0$, the UV might require regularization and could induce a running in the gravitational couplings for the effective action $\Gamma[g]$ through renormalization.

In fact, along with a finite non-local part, the effective action includes divergences. In dimensional regularization these appear through integrals of the form [23]

$$-\frac{1}{2} \int_0^\infty \frac{\mathrm{d}s}{s^{n+1}}\, \mathrm{e}^{-sm^2} \sim \frac{1}{2n} + \text{finite}\,. \tag{4}$$

From the form of the heat kernel expansion and considering only the small-s asymptotics we observe that for $D = 2-\epsilon$ the potential divergences can appear as poles $1/\epsilon$ up to the first order in the curvature expansion when $n = D/2 - 1$. If instead $D = 4 - \epsilon$ the divergences could appear up to the second order in the curvature expansion when $n = D/2 - 2$.

In order to cancel the divergences one has to introduce counterterms $\Delta S[g; \mu]$, then the renormalized one-loop effective action becomes

$$\Gamma_{\text{ren}}[g] = \Gamma[g] + \Delta S[g; \mu]\,, \tag{5}$$

in which μ is the mass scale introduced for dimensional reasons and at which the subtraction of the divergences occurs. As long as we are concerned with free fields, the counter terms depend only on the metric and hence only the parameters of the vacuum action may be running with scale.

In the minimal subtraction ($\overline{\text{MS}}$) scheme of renormalization the counter terms simply remove the poles $1/\epsilon$ (modulo some local finite part). In this scheme the subtraction is performed at the arbitrarily high scale μ and the effects of masses on the running of effective parameters is lost. If otherwise the subtraction is performed at a physical energy scale, say q^2, then the effects that masses have in the IR become visible in the running of the vacuum parameters. The two extreme regimes of interest are the UV limit $q^2/m^2 \gg 1$ and the IR limit $q^2/m^2 \ll 1$, in which the fluctuations freeze-out below a threshold defined by the mass of the quantum field.

Explicit calculations in $4D$ [2,3] have shown that the physical running agrees with the expectations based on the Appelquist–Carazzone theorem [24], namely the beta function β of any vacuum parameter displays the two limits

$$\begin{aligned} \beta_{\text{UV}} &= \beta_{\overline{\text{MS}}} && \text{for } q^2/m^2 \gg 1 \\ \beta_{\text{IR}} &\propto \frac{q^2}{m^2} && \text{for } q^2/m^2 \ll 1\,. \end{aligned} \tag{6}$$

In the next sections we explore the status of the theorem in the $2D$ case.

3. Vacuum sector of a general theory at one loop

It is always interesting to keep in mind a general model of interacting matter fields. At one loop the vacuum form factor does not depend on the interactions and is given by the algebraic sum of the contributions of the fields with spins 0, 1/2, 1 with different masses. Let us consider the vacuum form factors in the theory with n_{s} minimally or nonminimally coupled scalar fields, n_{f} minimally coupled Dirac spinors, and n_{p} minimally coupled Proca fields on a general $2D$ background with metric $g_{\mu\nu}$. Instead of massive Proca fields we could consider massless vectors, and later on we shall see the difference between this case and the massless limit of the Proca model.

For the sake of simplicity, let us assume that all matter fields of each spin have the same masses. Then the effective action is given by the expression

$$\Gamma[g] = \frac{n_s}{2}\,\mathrm{Tr}_s \ln\left(-\Delta_g + \xi R + m_s^2\right) - n_f\,\mathrm{Tr}_f \ln\left(\not{D} + m_f\right) + \frac{n_p}{2}\,\mathrm{Tr}_v \ln\left(-\Delta_g + m_v^2\right). \tag{7}$$

In this formula we denoted $\Delta_g = \nabla^2$ independently of the type of bundle (scalar, spinor or vector), and we assume that the functional traces must be taken accordingly. The first term includes the scalar contributions and, for our current purposes, it does not need further manipulation, let us instead consider the other two terms.

The Dirac operator $\not{D}$ could be squared inside the trace to obtain

$$\mathrm{Tr}_f \ln\left(\not{D} + m_f\right) = \frac{1}{2}\,\mathrm{Tr}_f \ln\Big(-\Delta_g + \frac{R}{4} + m_f^2\Big) \tag{8}$$

in which we used an explicit form for the spin connection. The latter is also used to evaluate the commutator of covariant derivatives over the spin bundle

$$[\nabla_\mu, \nabla_\nu] = \Omega_{\mu\nu} = -\frac{1}{4}\gamma^\alpha\gamma^\beta R_{\alpha\beta\mu\nu}. \tag{9}$$

The trace constituting the contribution of the Proca fields can be rewritten as the difference between a vector trace and a scalar trace. We refer to [23] and [25] for two detailed and different derivations of this property in curved space. Since these derivations do not depend explicitly on the spacetime dimension we shall not present the details here, but just give the final result

$$\frac{1}{2}\,\mathrm{Tr}_v \ln\left(-\Delta_g + m_v^2\right) = \frac{1}{2}\,\mathrm{Tr}\ln\left(-g_{\mu\nu}\Delta_g + R_{\mu\nu} + g_{\mu\nu}m_v^2\right) - \frac{1}{2}\,\mathrm{Tr}\ln\left(-\Delta_g + m_v^2\right) \tag{10}$$

The second term is equivalent to a minimally coupled scalar field with $\xi = 0$.

The final result of all the above manipulations is

$$\Gamma[g] = \frac{n_s}{2}\,\mathrm{Tr}_s \log\left(-\Delta_g + \xi R + m_s^2\right) - \frac{n_f}{2}\,\mathrm{Tr}_f \log\left(-\Delta_g + \frac{R}{4} - m_f^2\right) + \frac{n_p}{2}\,\mathrm{Tr}_v \log\left(-\Delta_g + \mathrm{Ric} + m_v^2\right) - \frac{n_p}{2}\,\mathrm{Tr}_s \log\left(-\Delta_g + m_v^2\right). \tag{11}$$

In the next sections we derive the above contributions following the methods anticipated in Sec. 2 with the appropriate replacements of the masses of the fields. We anticipate the notation β_G^s, β_G^f, β_G^p and β_G^g for the contributions to the running of the inverse Newton constant induced by scalar, fermionic, Proca and massless vector gauge degrees of freedom, respectively [2].

4. Massive scalar field in 2*D* gravity

We begin our computation of the induced vacuum effective action by considering the effects induced by a massive non-minimally coupled scalar field throughout the entirety this section. We consider this to be the most essential example and therefore we use it to flesh out all the details of our computation. Consider a scalar field φ with the classical action

$$S[\varphi] = \frac{1}{2}\int \mathrm{d}^2x\sqrt{g}\,\varphi\left(-\Delta_g + \xi R + m^2\right)\varphi. \tag{12}$$

In general this action is not conformally invariant, but it is well-known that classically the invariance can be recovered in the simultaneous limit of $m^2 \to 0$ and $\xi \to 0$. More generally, in this limit one could evaluate the effective action $\Gamma[g]$ by integrating the conformal anomaly, because no conformal invariant structures can be expected in $2D$. The result of this procedure is the Polyakov term which is a non-local action that is quadratic in the Ricci scalar. Outside the protection of the conformal limit one can instead expect all powers of Ricci scalar and more complicated structure of non-localities. However, for the sake of practical calculations, we will consider only terms up to the second order in the curvature of the non-local effective action.

On the top of the above considerations we have to introduce the action of vacuum, which should be local, covariant and sufficient to renormalize all possible divergences. In $2D$ such an Euclidean action includes only Einstein–Hilbert and cosmological terms,

$$S_{\mathrm{vac}}[g_{\mu\nu}] = \frac{1}{16\pi G_N}\int \mathrm{d}^2x\sqrt{g}\,(2\Lambda - R). \tag{13}$$

Indeed, the above vacuum action is sufficient for all types of matter fields, so the renormalization of vacuum reduce to the renormalization of the Newton constant G_N and the cosmological constant Λ.

4.1. Derivation of effective action and β-function

The regularized effective action is defined as modified version of the general expression (3),

$$\Gamma[g] = -\frac{1}{2}\left(4\pi\mu^2\right)^{\epsilon/2}\,\mathrm{Tr}\int\frac{\mathrm{d}s}{s}\,\mathrm{e}^{-sm^2}\,\mathcal{H}(s). \tag{14}$$

Here μ is the renormalization parameter, which is used to preserve the dimension in $D = 2 - \epsilon$ dimension.

We now have to evaluate the heat kernel $\mathcal{H}(s)$ using the methods of Appendix A. Since we are interested in the limit $\epsilon \to 0$, we use the relations (1) for the evaluation of the effective action. In this case the trace of the heat kernel simplifies considerably and can be written as

$$\begin{aligned}\mathrm{Tr}\,\mathcal{H}(s) &= \frac{1}{(4\pi s)^{D/2}}\int \mathrm{d}^2x\sqrt{g}\Big\{1 + s\Big[G_R(-s\Delta_g) + \xi G_E(-s\Delta_g)\Big]R \\ &\quad + s^2R\Big[F_R(-s\Delta_g) + \frac{1}{2}F_{Ric}(-s\Delta_g) + \xi F_{RE}(-s\Delta_g) \\ &\quad + \xi^2F_E(-s\Delta_g)\Big]R\Big\} + \mathcal{O}\left(\mathcal{R}^3\right),\end{aligned} \tag{15}$$

in which the functions G_R, G_E, F_R, F_{Ric}, F_{RE} and F_E are given in Appendix A. Let us stress once more that this formula is essentially simpler than its $4D$ counterpart thanks to the use of the $2D$ relations (1) and thanks to the fact that the commutator of covariant derivatives $\Omega_{\mu\nu} = 0$ when acting on scalars. At the same time the general dimension D appears in the intermediate formula (15) indicating the use of dimensional regularization.

It is convenient to introduce a condensed notation which simplifies both divergent and finite parts of the effective action. We define

$$\frac{1}{\bar{\epsilon}} = \frac{2}{\epsilon} + \ln\Big(\frac{4\pi\mu^2}{m^2}\Big) - \gamma,$$
$$z = -\frac{\Delta_g}{m^2}, \quad a = \sqrt{\frac{4z}{4+z}}, \quad Y = 1 - \frac{1}{a}\log\left|\frac{1+a/2}{1-a/2}\right|, \tag{16}$$

in which γ is the Euler–Mascheroni constant. Taking the integral over the proper time variable s, after some manipulations we obtain the result for the effective action up to the second order in the scalar curvature

$$\Gamma_{\rm s}[g] = \frac{1}{4\pi}\int {\rm d}^2x\sqrt{g}\,\Big\{\frac{m^2}{2\,\bar\epsilon} + \frac{m^2}{2} + \frac{1}{2\,\bar\epsilon}\Big(\xi - \frac{1}{6}\Big)R + \frac{B(z)}{2}R + \frac{1}{24}R\,\frac{C(z)}{\Delta_g}\,R\Big\}\,, \tag{17}$$

in which we defined the functions

$$B(z) = \frac{1}{18} + 2\Big(\xi - \frac{1}{4}\Big)Y + \frac{2Y}{3a^2},$$
$$C(z) = -\frac{1}{2} - \frac{6Y}{a^2} + 3(1-4\xi)Y + \frac{3}{8}a^2(1-4\xi)^2(1-Y)\,. \tag{18}$$

Finally, subtracting the divergence at the scale defined by Euclidean momentum q, we can obtain the beta function for the inverse Newton constant. The physical beta function can be computed by acting with the derivative ${\rm d}/{\rm d}\ln(q/q_0)$ on the coefficient of the Ricci scalar in the finite part of the effective action. Following the strategy described in [2,3], one can perform the computation directly in coordinate space by trading $q^2 \leftrightarrow -\Delta_g$ in the final expression. In this way we obtain

$$\beta_G^s = \frac{1}{4\pi}z\,B'(z) = \frac{1}{8\pi}\Big[-\frac{1}{6} + (1-2\xi)Y - \frac{2Y}{a^2} + \frac{1}{8}(1-4\xi)a^2(1-Y)\Big]. \tag{19}$$

One observation is in order. Differently than the $4D$ case explored in [2,3], here we could establish the beta-function for the inverse Newton constant. The reason for this difference is *not* the change of dimension, but rather the fact that we used the heat kernel surface terms such as $G_R(-s\Delta_g)R$. More details on how to obtain the surface terms can be found in [4,17], while non-trivial applications appear in [26]. In the next subsections we present a brief discussion of the properties of the main results (17) and (19).

4.2. Recovering the Polyakov action in the conformal limit

We are now interested in the behavior of the effective action (17) under the conformal limit $\xi \to 0$ and $m^2 \to 0$. The term quadratic in the curvatures of (17) can be understood as a generalization of the Polyakov action [12], and therefore it is possible to see it as a toy model of what one can expect in the more realistic $4D$ case for the non-conformal scalar.

Let us start from the simplest case with $\xi = 0$ and nonzero mass. It is easy to see that the Polyakov action must be recovered in the limit $z \to \infty$. This is the UV regime since z diverges in the limit $m^2 \to 0$ at fixed momenta q^2, or alternatively $q^2 \to \infty$ at fixed m^2. The function $C(z)$ was normalized so that it interpolates with the central charge

$$C(z) \to 1\,, \qquad \text{for} \qquad z \to \infty\,. \tag{20}$$

As expected from local conformal invariance, in this limit the quadratic term actually becomes Polyakov action

$$\Gamma_{\rm s}[g] \longrightarrow \frac{1}{96\pi}\int {\rm d}^2x\sqrt{g}\,R\frac{1}{\Delta_g}R = \Gamma_{\rm P}[g]\,, \tag{21}$$

in which we exclusively displayed the limit of the part of $\Gamma_{\rm s}[g]$ that is quadratic in R.

In general, for a nonzero mass there are also infinitely many other terms which display higher powers of R. The correspondence with the Polyakov action (21) requires that all these terms vanish in the conformal limit, but the proof of this fact is beyond the scope of the present article which deals only with the second order form factors. Let us note, however, that the proof could constitute a relevant step, particularly because of the interesting discussion of the role of expansion in curvature tensor components in $4D$ that appeared in [27,28].

Even more interesting is to consider the theory with an arbitrary value of ξ. Taking the UV limit in the general expression (17), we can only expand about a small but nonzero mass m^2 and we arrive at a logarithmically corrected Polyakov action

$$\Gamma_{\rm s}[g] \longrightarrow \Gamma_{\rm P}[g] - \frac{\xi}{8\pi}\int {\rm d}^2x\sqrt{g}\,R\Big[\frac{1}{\Delta_g} - \xi\,\frac{\ln\big(-\Delta_g/m^2\big)}{\Delta_g}\Big]R\,. \tag{22}$$

One can see that the massless limit of the second term inside the integral is singular for $\xi \neq 0$. Therefore the presence of a non-conformal value for ξ forbids the massless limit. At the same time in the conformal case $\xi = 0$ the result trivially interpolates the Polyakov action as expected from (21). A more detailed discussion of the IR singularity for $\xi \neq 0$ can be found in Appendix C.

4.3. Two extreme regimes for β_G^s

The beta function β_G^s is a general expression which is valid at all energy scales. Let us consider the UV regime with $z \gg 1$, or equivalently $q^2 \gg m^2$; and the IR one with $z \ll 1$, or $q^2 \ll m^2$. In these limits the vacuum beta function becomes as in (6),

$$\beta^s_{G,\,\rm UV} = -\frac{1}{4\pi}\Big(\xi - \frac{1}{6}\Big) - \frac{1}{4\pi}\,\frac{m^2}{q^2}\Big[1 - 2\xi\ln\Big(\frac{q^2}{m^2}\Big)\Big] + \ldots \tag{23}$$

$$\beta^s_{G,\,\rm IR} = -\frac{1}{24\pi}\Big(\xi - \frac{1}{5}\Big)\frac{q^2}{m^2} + \ldots \tag{24}$$

The first expression shows that at the leading order of the UV expansion we recover the standard $\overline{\rm MS}$ result for $2D$ (see, e.g., [13]),

$$\beta^s_{G,\,\overline{\rm MS}} = -\frac{1}{4\pi}\Big(\xi - \frac{1}{6}\Big). \tag{25}$$

At the same time, in the IR we observe that for sufficiently small q^2 the vacuum parameter stops running with a quadratic decoupling term, as it should be according to the Appelquist and Carazzone theorem [24]. It is remarkable that this result has been achieved for the beta function of the R-term, that should be regarded as a non-trivial result [2].

5. Dirac spinors

Let us denote the dimension of the Clifford algebra $d_{\rm f} = {\rm tr}\hat{1}$ and remember that in the general even dimension D we have $d_{\rm f} = 2^{D/2}$ (see, e.g., [29,30] for an introduction to fermions in curved space). Furthermore, in $D = 2$ the trace of the square of the commutator of covariant derivatives is

$${\rm tr}\;\Omega^2_{\mu\nu} = -\frac{d_{\rm f}}{8}\,R^2. \tag{26}$$

Using these results and the general expression for the heat kernel solution, we arrive at the fermion contribution in the form

$$\Gamma_{\rm f}[g] = -\frac{1}{2}{\rm Tr}_{\rm f}\ln\Big(-\Delta_g + \frac{1}{4}R + m^2\Big) \tag{27}$$
$$= \frac{d_{\rm f}}{4\pi}\int {\rm d}^2x\sqrt{g}\,\Big\{\frac{1}{2\,\bar\epsilon}\Big(-m^2 - \frac{1}{12}R\Big) - \frac{m^2}{2} + \frac{1}{2}B_{\rm f}(z)R + \frac{1}{24}R\,\frac{C_{\rm f}(z)}{\Delta_g}\,R\Big\},$$

in which

$$B_{\rm f}(z) = -\frac{1}{18} - \frac{2}{3}\frac{Y}{a^2} \quad \text{and} \quad C_{\rm f}(z) = \frac{1}{2}(1-3Y) + \frac{6Y}{a^2}. \tag{28}$$

Likewise the scalar case of the previous section, in the UV limit $m \to 0$ the form factor $C_{\rm f}$ interpolates the central charge of a single fermionic degree of freedom, namely $C_{\rm f}(z) \to 1/2$. However, no running similar to that in Eq. (22) takes place in the case of fermions (similarly with vectors, as we will show in the next subsection), because in these cases there are no non-minimal interactions with external metric such as ξ, and therefore there is no violation of *global* scale invariance besides the mass.

The unique vacuum beta function is given by the expression

$$\beta_G^{\rm f} = \frac{d_{\rm f}}{8\pi}\left\{\frac{1-3Y}{6} + \frac{2Y}{a^2}\right\}. \tag{29}$$

In the UV ($q^2 \gg m^2$) and IR ($q^2 \ll m^2$) regimes we meet the following limits:

$$\beta_{G,\,\rm UV}^{\rm f} = \beta_{G,\,\overline{\rm MS}}^{\rm f} + \mathcal{O}\Big(\frac{m^2}{q^2}\Big), \quad \text{with} \quad \beta_{G,\,\overline{\rm MS}}^{\rm f} = \frac{d_{\rm f}}{48\pi}, \tag{30}$$

$$\beta_{G,\,\rm IR}^{\rm f} = \frac{d_{\rm f}}{480\pi}\frac{q^2}{m^2} + \mathcal{O}\Big(\frac{q^2}{m^2}\Big). \tag{31}$$

As it was expected on the general grounds and in full analogy with the $4D$ result [3], the high-energy limit shows a nice correspondence with the $\overline{\rm MS}$ scheme beta function. At low energies we have once more a gravitational $2D$ version of the decoupling theorem [24]. Once again, the remnant IR running is related to the non-local terms which are of the *second* order in curvature, while the divergence is a local *first* order in curvature expression, in accordance with the Weinberg theorem [31,32] and power counting. This result is as remarkable as the one for the scalar field, because in $4D$ it has not yet been achieved [2].

6. Massive and massless vector fields

For the massive Proca vector field one can use the general expression (10) and essentially repeat the computations described in the previous section. We find

$$\begin{aligned}\Gamma_{\rm p}[g] &= \frac{1}{2}\,{\rm Tr_v}\ln\Big(-\Delta_g + {\rm Ric} + m^2\Big) - \frac{1}{2}\,{\rm Tr_s}\ln\Big(-\Delta_g + m^2\Big)\\ &= \frac{1}{4\pi}\int {\rm d}^2x\sqrt{g}\,\Big\{\frac{1}{2\,\bar{\epsilon}}\Big(m^2 + \frac{5}{6}R\Big) + \frac{m^2}{2} + \frac{1}{2}\,B_{\rm p}(z)R\\ &\quad + \frac{1}{24}\,R\,\frac{C_{\rm p}(z)}{\Delta_g}R\Big\},\end{aligned} \tag{32}$$

in which the form factors for the Proca field are defined

$$\begin{aligned}B_{\rm p}(z) &= \frac{1}{18} + \frac{3}{2}\,Y + \frac{2Y}{3a^2}\\ C_{\rm p}(z) &= -\frac{1}{2} + 3Y - \frac{3}{8}\,a^2(Y-1) - \frac{6Y}{a^2}.\end{aligned} \tag{33}$$

As one might expect from a standard counting of the degrees of freedom of a Proca field, the function $C_{\rm p}(z) \to 1$ in the UV.

The beta function comes from the nonlocal term that is linear in the curvature scalar likewise (19). It is given by

$$\beta_G^{\rm p} = \frac{1}{8\pi}\Big\{-\frac{1}{6} - Y + \frac{3}{8}\,a^2(Y-1) - \frac{2Y}{a^2}\Big\}, \tag{34}$$

which is again a general expression valid at all energies scales. In the UV ($q^2 \gg m^2$) and IR ($q^2 \ll m^2$) it boils down to the simpler results

$$\beta_{G,\,\rm UV}^{\rm p} = \beta_{G,\,\overline{\rm MS}}^{\rm p} + \mathcal{O}\Big(\frac{m^2}{q^2}\Big), \quad \text{with} \quad \beta_{G,\,\overline{\rm MS}}^{\rm p} = -\frac{5}{24\pi}, \tag{35}$$

$$\beta_{G,\,\rm IR}^{\rm p} = -\frac{1}{30\pi}\frac{q^2}{m^2} + \mathcal{O}\Big(\frac{q^2}{m^2}\Big). \tag{36}$$

The last formula completes our determination of the $2D$ gravitational version of the Appelquist and Carazzone theorem [24] including massive Proca fields.

In the UV limit (35) one observes a difference with the well-known result for the gauge massless vector

$$\beta_{G,\,\overline{\rm MS}}^{\rm g} = -\frac{1}{4\pi}. \tag{37}$$

This difference is nothing else than the discontinuity in the massless limit of the quantum contribution of the Proca field, which has been discussed earlier in $4D$ [3,25] and we now observe in the $2D$ case. The origin of this difference is in the fact that while a massive Proca field requires only one "compensating" scalar degree of freedom, instead a massless gauge field requires two, which are better known as Faddeev–Popov ghosts.

7. Conclusions

The nonlocal form factors of the vacuum effective action have been derived for several types of $2D$ massive fields in an approximation which includes all the terms up to the second order in curvature. The results have been obtained by means of the heat kernel approach. An equivalent derivation based on Feynman diagrams for the scalar field case is deferred to Appendix B.

The form factors do not become logarithmic in the UV (or massless) limit, because the divergences appear at the first order in the Ricci scalar. However, one can recover the UV limit successfully, because the terms of the second order in curvature reduce to the Polyakov action in the case of $m \to 0$ for fermions and vectors, while for the scalar field such a limit is achieved only for the minimal version, with $\xi = 0$, which is known to be classically conformally invariant in $2D$.

In the massless limit of the nonminimal scalar we find a modified version of the Polyakov action, which includes a qualitatively new logarithmic form factor. The new logarithm can speculatively be interpreted as a kind of IR renormalization group running that does not have a direct relation to the UV divergences.

In the $\overline{\rm MS}$ renormalization scheme β-functions related to the divergences that appear in the linear part in the curvature are reproduced as the UV limit of more general expressions corresponding to the momentum subtraction scheme. At the opposite end of the energy scale, in the IR, we meet a $2D$ version of the gravitational decoupling theorem very similar to the $4D$ counterpart [2,3]. The main difference between the results in $4D$ and $2D$ is that in the latter case the UV limit reproduces the anomaly-induced action and IR limit shows decoupling in the beta functions *without* the usual correlation between divergences and logarithmic form factors.

We believe that our result are instructive for better understanding how one can explore the running of Newton and cosmological constants, including the cases in which the form factors are formally irrelevant [2]. Let us remind the Reader that the question on whether there is a remnant IR running of these parameters or not has potentially interesting cosmological applications [33,34] and can be relevant in astrophysics too (see, e.g., [35,36]). For these reasons the present work can be seen as a small step forward in understanding how the information concerning the part of the vacuum effective action that is linear in the curvature can be used to interpolate between UV and IR physics through a renormalization group running based on a physical scale.

Appendix A. Non-local heat kernel expansion

In this appendix we provide more details on the form of the non-local expansion of the heat kernel which was originally devel-

oped in [15,16]. We follow however the notation of [4]. Consider an operator of Laplace type

$$\mathcal{O} = -\Delta_g + E\,, \tag{38}$$

acting on a general vector bundle which equipped with a connection and which lives over a Riemaniann manifold with metric $g_{\mu\nu}$. The operator is the sum of the Laplacian $\Delta_g = \nabla^2 = g^{\mu\nu}\nabla_\mu\nabla_\nu$ and an endomorphism over the bundle. In general the connection has curvature $\Omega_{\mu\nu} = \left[\nabla_\mu, \nabla_\nu\right]$ and might even contain a Levi-Civita contribution if part of the vector bundle is obtained as a tensor product of tangent or co-tangent bundles.

The heat kernel $\mathcal{H}(s; x, x')$ is defined as the solution of the following Cauchy problem with respect to the proper time s:

$$\begin{aligned}&(\partial_s + \mathcal{O}_x)\,\mathcal{H}(s;\,x,x') = 0\\ &\mathcal{H}(s;\,x,x') = \delta(x,x')\end{aligned} \tag{39}$$

with $\delta(x, x')$ the Dirac delta over the manifold [37].

The trace of the coincidence limit of the heat kernel $\mathcal{H}(s;\,x,x)$ admits an expansion in terms of Riemaniann and connection's curvatures and of the endomorphism. The expansion can be computed unambiguously over asymptotically flat manifolds [16,17,4]. Up to second order in the curvatures it has the form

$$\begin{aligned}&\mathrm{tr}\,\mathcal{H}(s,x,x)\\ &= \frac{1}{(4\pi s)^{D/2}}\int \mathrm{d}^D x\sqrt{g}\,\mathrm{tr}\Big\{\mathbf{1} + s\big[G_E(-s\Delta_g)E + G_R(-s\Delta_g)R\big]\\ &\quad + s^2\Big[RF_R(-s\Delta_g)R + R^{\mu\nu}F_{Ric}(-s\Delta_g)R_{\mu\nu} + EF_E(-s\Delta_g)E\\ &\quad + EF_{RE}(-s\Delta_g)R + \Omega^{\mu\nu}F_\Omega(-s\Delta_g)\Omega_{\mu\nu}\Big]\Big\} + \mathcal{O}\left(\mathcal{R}^3\right)\,,\end{aligned} \tag{40}$$

in which $\mathcal{O}\left(\mathcal{R}^3\right)$ indicates a nonlocal, but consistent, curvature expansion to the third order as well explained in [16]. The above formula is given without specific boundary conditions at infinity for the first order in the curvatures, while integration by parts is used for the second order (see [4,26] for more details on this derivation).

The nonlocal functions of Δ_g appearing in the expansion are known as form factors of the heat kernel. They are obtained as

$$G_E(x) = -f(x)\,, \tag{41}$$

$$G_R(x) = \frac{f(x)}{4} + \frac{f(x)-1}{2x}\,, \tag{42}$$

for the terms that are linear in the curvatures, and

$$F_{Ric}(x) = \frac{1}{6x} + \frac{f(x)-1}{x^2}\,, \tag{43}$$

$$F_R(x) = -\frac{7}{48x} + \frac{f(x)}{32} + \frac{f(x)}{8x} - \frac{f(x)-1}{8x^2}\,, \tag{44}$$

$$F_{RE}(x) = -\frac{f(x)}{4} - \frac{f(x)-1}{2x}\,, \tag{45}$$

$$F_E(x) = \frac{f(x)}{2}\,, \tag{46}$$

$$F_\Omega(x) = -\frac{f(x)-1}{2x}\,, \tag{47}$$

for the terms which are quadratic in the curvatures. All form factors are expressed via the basic form factor

$$f(x) = \int_0^1 d\alpha\, \mathrm{e}^{-\alpha(1-\alpha)x}\,. \tag{48}$$

These form factors admit well-defined expansions both for large and small values of the parameter s (see the discussion of Appendix C), and are used in the main text to obtain the related nonlocal structures which appear in the vacuum effective action of matter fields.

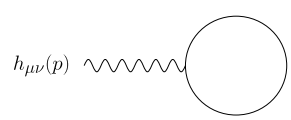
Appendix B. Curvature renormalization from Feynman diagrams

The standard perturbative approach to the renormalization of surface terms such as power series composed by $(-\Delta_g)^n R$ is particularly difficult if Feynman diagrams and an expansion of the metric around flat space $g_{\mu\nu} = \delta_{\mu\nu} + h_{\mu\nu}$ are adopted. Let us comment on the nature of this difficulty and how we solve it in the present work. The interested reader can consult [4] for more details.

Momentum conservation generically constrains external lines in the fluctuation $h_{\mu\nu}$ to have zero incoming momentum in one-point functions. This happens, in particular, for diagrams which would otherwise renormalize the scalar curvature itself. In fact, the first vertex in $h_{\mu\nu}$ is

$$\frac{1}{\sqrt{g}}\frac{\delta}{\delta h_{\mu\nu}}\int d^2x\sqrt{g}R\bigg|_{\delta_{\mu\nu}} = \int \frac{d^2p}{(2\pi)^2}\,e^{ipx}\left(p^2 g^{\mu\nu} - p^\mu p^\nu\right). \tag{49}$$

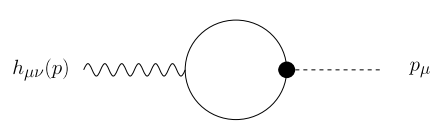
The first tensor structure in this expression receives contributions from the one-point function of the volume term too (cosmological constant), but the second one is only sensitive to operators such as $(-\Delta_g)^n R$ (see Appendix of [4]). It is clear that one cannot see any renormalization of this vertex if the following diagram is considered



(50)

and the momentum p_μ is constrained to be zero by the conservation law.

In order to be able to retain a nonzero incoming momentum p_μ we insert a new "identity" vertex $\mathbf{1}$ in the theory which is trivially defined by momentum conservation. Then, instead of (50) we can consider the following diagram:



(51)

In coordinate space this diagram contributes to

$$\int \mathrm{d}^d x\sqrt{g}\,\mathbf{1}\,g(-\Delta_g)R, \tag{52}$$

in which $g(-\Delta_g)$ is a function to be uniquely determined by loop integration of the diagram itself. E.g., taking the limit $\mathbf{1} \to 1$ in the scalar case one can use the new diagram to renormalize the nonlocal contribution to the Einstein–Hilbert term.

The integral can be easily computed introducing a Feynman parameter $0 \le \alpha \le 1$ and exponentiating the combined propagators with a proper time parameter s. We display only the minimal $\xi = 0$ case for the sake of simplicity. The final result is an integral over the parameters s and α

$$\text{Eq. (51)} \;\propto\; I(p^2)\,p_\mu p_\nu + \dots\,, \tag{53}$$

in which we determine

$$I(p^2) = -\frac{1}{(4\pi)^{d/2}} \int \mathrm{d}s\, s^{1-d/2}\, \mathrm{e}^{-m^2 s} \left\{ \frac{1}{2sp^2} - \frac{2+sp^2}{4sp^2} f(sp^2) \right\}. \tag{54}$$

In this expression the dots stand for further contributions proportional to the background metric (which are needed for the renormalization of the cosmological constant) and $f(x)$ is the basic form factor (48) of the heat kernel. The function

$$I(p^2) = \int \mathrm{d}s\, \frac{1}{(4\pi s)^{d/2}} s\, G_R(sp^2) \tag{55}$$

gives the nonlocal contribution to the Einstein–Hilbert term. One can see that the integrand is exactly the form factor associated with R in the nonlocal expansion (42) (see also [17]). From this point onward, the computation of the vacuum effective action can straightforwardly follow the main text because it coincides with what is obtained from the heat kernel expansion.

Appendix C. On the non-analyticity of expression (22)

In this appendix we elaborate in more detail the reasons why the integrand of (22) is singular in the massless limit. We follow the steps of Refs. [15,16] in which it is shown that the effective two dimensional action is analytical only in the case of a conformal scalar field.

The analyticity of the effective action is related to the convergence of the integral (3) in the upper limit. In this limit the behavior of (15) is determined by the expansion of the form factors to large values of s. From the expressions (41)–(46) one can obtain the following asymptotic behaviors of the form factors:

$$G_E(x) = -\frac{2}{x} + \mathcal{O}\left(\frac{1}{x^2}\right), \tag{56}$$

$$G_R(x) = 0 + \mathcal{O}\left(\frac{1}{x^2}\right), \tag{57}$$

and

$$F_{Ric}(x) = \frac{1}{6x} + \mathcal{O}\left(\frac{1}{x^2}\right), \tag{58}$$

$$F_R(x) = -\frac{1}{12x} + \mathcal{O}\left(\frac{1}{x^2}\right), \tag{59}$$

$$F_{RE}(x) = 0 + \mathcal{O}\left(\frac{1}{x^2}\right), \tag{60}$$

$$F_E(x) = \frac{1}{x} + \mathcal{O}\left(\frac{1}{x^2}\right). \tag{61}$$

Combining everything together, the behavior of $\mathcal{H}(s)$ for $s \to \infty$ is

$$\mathcal{H}(s) = s^{-\frac{D}{2}} (\mathcal{R})^0 + s^{-\frac{D}{2}} (\mathcal{R})^1 + s^{-\frac{D}{2}+1} (\mathcal{R})^2 + \mathcal{O}(\mathcal{R})^3, \tag{62}$$

in which $\mathcal{R}$ symbolically represents all kinds of curvature and indices 0, 1, 2 determine the order of expansion.

Comparing the previous expression to $\mathcal{H}(s)$ with Eq. (2.16) in Ref. [16], we note that the presence of the form factor in the linear part in curvature changes the behavior of $\mathcal{H}(s)$ at large s. In the computation of $\Gamma[g]$ the coefficient of the linear term in the curvature is determined by an integral of the type

$$\int^{\infty} \frac{\mathrm{d}s}{s} \frac{1}{(4\pi s)^{D/2}} \left\{ 1 + \mathcal{O}\left(\frac{1}{s}\right) \right\}, \tag{63}$$

while for the quadratic term we have

$$\int^{\infty} \frac{\mathrm{d}s}{s} \frac{s}{(4\pi s)^{D/2}} \left\{ 1 + \mathcal{O}\left(\frac{1}{s}\right) \right\}, \tag{64}$$

when we consider $m = 0$. For $D = 2$ the effective action is analytic until the first order in curvature. On the other hand, in the second order in curvature the analyticity of $\Gamma[g]$ is broken by the presence of the term $1/s$ in the integrand. It is this kind of singularity that appears in the second term of the integral (22).

From this perspective the Polyakov action is a very special case, since it is analytic at second order in the curvature. The reason is that the form factors combine such that the terms behaving like $1/s$ do cancel and the first term of the expansion for a large s becomes $1/s^2$, making $\Gamma[g]$ finite in the IR. From $\mathcal{H}(s)$ and the expansion of the form factors for large s, one can see that the cancellation of the $1/s$ terms occurs only when $\xi = 0$. Only in this case the $m = 0$ effective action is analytic at $D = 2$. On the contrary, the presence of the term with $1/s$ in the case $\xi \neq 0$ leads to a function $\Gamma(1 - D/2)$, which has a pole at $D = 2$. This pole corresponds to the IR divergence related to the singularity of the effective action in $2D$ [16,37].

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