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New phenomena with the $f(R)$-theory of gravitation in a central gravitational field

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Abstract. The $f(R)$-theory (of gravitation) is an extension of Einstein's general theory of relativity (GR) but if a spherically symmetric vacuum solution of the Einstein equation in the GR is always stationary, a spherically symmetric vacuum solution of an $f(R)$-theory is not necessary stationary. This may have interesting consequences. In comparison with the GR, a process such as a planet's motion (its orbital precession and parameters) and a gravitational deflection of light now get a correction which is a constant for a static central field and varies with time for a non-static central field even from a source of a constant mass, unlike the corresponding GR value not changing in the same situation. In particular, a spherically symmetric source may radiate gravitational waves. This phenomenon cannot happen in the GR. The present work is an extended version based on a presentation in the 44th Vietnam conference on theoretical physics (Dong Hoi, 29 July - 01 August 2019).

1. Introduction

The General theory of Relativity (GR) of A. Einstein is a very successful theory of gravitation [1, 2]. This theory has been verified very precisely, in particular, it was once again confirmed triumphantly by recent detections of gravitational waves [3, 4].

The heart of the GR is Einstein's equation

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -kT_{\mu\nu}, \tag{1}$$

$k = \frac{8\pi G}{c^4}$, derived from the Lagrangian $\mathcal{L}_G = R$. However, some cosmological problems require the GR to be extended or modified. One of the modified theories of the GR is called $f(R)$-theory of gravitation.

The $f(R)$-theory of gravitation is based on the Lagrangian $\mathcal{L}_G = f(R)$ leading to the equation

$$f'(R)R_{\mu\nu} - g_{\mu\nu}\Box f'(R) + \nabla_\mu\nabla_\nu f'(R) - \frac{1}{2}f(R)g_{\mu\nu} = -kT_{\mu\nu}, \tag{2}$$

where $f(R)$ is a scalar function of the scalar curvature R, while ∇_μ is a covariant derivative and $\Box = \nabla_\mu\nabla^\mu$. The function $f(R)$ could be, for example, $f(R) = R + \lambda R^2$ or $f(R) = R - \frac{\lambda}{R^n}$, etc.

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To solve (2) in general is problematic. Here, assuming $f(R)$ deviating from R just slightly, we try to solve (2) perturbatively for a central field of a spherically symmetric gravitational source of radius R_0. More details of the procedure can be found in [5, 6].

2. Perturbative spherically symmetric solutions of an $f(R)$-theory

Starting with

$$f(R) = R + \lambda h(R), \quad \lambda h(R) \ll R,$$

where $h(R)$ is a scalar function of R and λ is a parameter (with an appropriate dimension which will be tacit here), we obtain from Eq. (2)

$$R^{\mu}{}_{\nu} - \frac{1}{2}\delta^{\mu}{}_{\nu}R + \lambda h'(R)R^{\mu}{}_{\nu} - \frac{\lambda}{2}\delta^{\mu}{}_{\nu}h(R) - \lambda\delta^{\mu}{}_{\nu}\triangle h'(R) + \lambda\nabla^{\mu}\nabla_{\nu}h'(R) = -kT^{\mu}{}_{\nu}. \tag{3}$$

Using Einstein's equation in the form $R = kT$ and $R^{\mu}{}_{\nu} = -k(T^{\mu}{}_{\nu} - \frac{1}{2}\delta^{\mu}{}_{\nu}T)$ and solving (3) perturbatively, we find the following perturbative solution

$$g_{00}(r,t) = 1 - \frac{kc^2\left(M - \lambda M_1(r,t) - \lambda M_2(r,t)\right)}{4\pi r},$$
$$g_{11}(r,t) = \frac{-1}{g_{00}(r,t)},\ g_{22} = -r^2,\ g_{33} = -r^2 sin^2\theta, \tag{4}$$

where

$$M_1(r,t) = \tfrac{-2\pi}{kc^2}\int_0^r \left[h(kT^0_{\ 0}) + kT^0_{\ 0}h'(kT^0_{\ 0})\right] r'^2 dr',\quad M_2(t) = \tfrac{-4\pi h''(kT^0_{\ 0})}{kc^2}\left[\tfrac{\partial}{\partial t}\tfrac{M}{[R_0(t)]^3}\right]^2 \alpha(t), \tag{5}$$

with

$$\alpha(t) = \frac{3\arcsin[\xi(t)R_0(t)] - \xi(t)R_0(t)\left\{3 + 2[\xi(t)R_0(t)]^2\right\}\sqrt{1-[\xi(t)R_0(t)]^2}}{256\pi^2[\xi(t)]^5\,(3k^2c^2)^{-1}\left(1-[\xi(t)R_0(t)]^2\right)^{3/2}},\ \ \xi^2(t) = \frac{kMc^2}{4\pi[R_0(t)]^3}. \tag{6}$$

Here $R_0(t)$ is the radius of the gravitational source at time t. Let us apply this result to some special cases.

2.1. *Model* $\mathbf{f(R) = R - 2\lambda}$

In this case $h(R) = -2$ and, therefore,

$$g_{00}(r,t) = 1 - \tfrac{kc^2M}{4\pi r} - \tfrac{\lambda r^2}{3},\ g_{11}(r,t) = \tfrac{-1}{1-\frac{kc^2M}{4\pi r}-\frac{\lambda r^2}{3}},\ g_{22} = -r^2,\ g_{33} = -r^2 sin^2\theta. \tag{7}$$

This solution coincides with the exact solution with λ being the cosmological constant.

2.2. *Model* $\mathbf{f(R) = R + \lambda R^b}$, $(\mathbf{b > 0})$

Here $h(R) = R^b$, we have

$$g_{00}(r,t) = 1 - \tfrac{kc^2M_f(t)}{4\pi r}, g_{11}(r,t) = \tfrac{-1}{1-\frac{kc^2M_f(t)}{4\pi r}},\quad g_{22} = -r^2,\quad g_{33} = -r^2 sin^2\theta, \tag{8}$$

with

$$M_f(t) = M - \lambda M_1(t) - \lambda M_2(t), \tag{9}$$
$$M_1(t) = \frac{(b+1)(Mkc^2)^b}{2[\frac{4}{3}\pi R_0^3(t)]^{b-1}kc^2},\quad M_2(t) = \frac{4\pi b(b-1)(Mkc^2)^{b-2}}{[\frac{4}{3}\pi R_0^3(t)]^{b-2}kc^2}\left[\frac{\partial}{\partial t}\frac{M}{[R_0(t)]^3}\right]^2\alpha(t).$$

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2.3. Model $\mathbf{f(R)} = \mathbf{R^{1+\varepsilon}}$ *with* ε *very small*

In this case $\lambda h(R) = R^{1+\varepsilon} - R$, we obtain

$$g_{00}(r,t) = 1 - \frac{kc^2 M_f(t)}{4\pi r}, \quad g_{11}(r,t) = \frac{-1}{1 - \frac{kc^2 M_f(t)}{4\pi r}}, \quad g_{22} = -r^2, \quad g_{33} = -r^2 sin^2\theta \tag{10}$$

where

$$M_f(t) = M - \lambda M_1(t) - \lambda M_2(t), \tag{11}$$
$$\lambda M_1(t) = -M + \frac{(\varepsilon+2)(Mkc^2)^{\varepsilon+1}}{2[\frac{4}{3}\pi R_0^3(t)]^{\varepsilon} kc^2}, \quad \lambda M_2(t) = \frac{4\pi\varepsilon(\varepsilon+1)(Mkc^2)^{\varepsilon-1}}{[\frac{4}{3}\pi R_0^3(t)]^{\varepsilon-1} kc^2} \left[\frac{\partial}{\partial t}\frac{M}{[R_0(t)]^3}\right]^2 \alpha(t).$$

3. Motion in a central field of the $f(R)$-theory

Applying metrics $g_{\mu\nu}$ obtained to the Hamilton-Jacobi equation

$$g^{\mu\nu}\frac{\partial S}{\partial x^\mu}\frac{\partial S}{\partial x^\nu} = m^2c^2,$$

we get a general equation of motion in a central field for an $f(R)$-theory, and, then a planet's orbital precession [6]

$$\Delta\varphi(n) = \frac{6\pi m^2 G^2 [M - \lambda M_1(t_n) - \lambda M_2(t_n)]^2}{c^2 L^2},$$

where L is the planet's angular momentum. Using [2]

$$\frac{L^2}{m^2} = a(1-e^2) G M_f(t) \tag{12}$$

we get

$$\Delta\varphi_{f(R)} = \frac{6\pi G M_f(t)}{c^2 a(1-e^2)}. \tag{13}$$

(for the deflection angle of light, see [5]).

4. Examination of the $f(R)$-theory

We will examine the $f(R)$-theory for $f(R) = R + \lambda R^2$ and $f(R) = R + \frac{\lambda'}{R}$ in two cases: in a static central (gravitational) field and in a non-static central field. For this goal, let us apply the theory to some real gravitational systems, for example, Sun-Mercury (small system) and Sgr A*-S2 (big system).

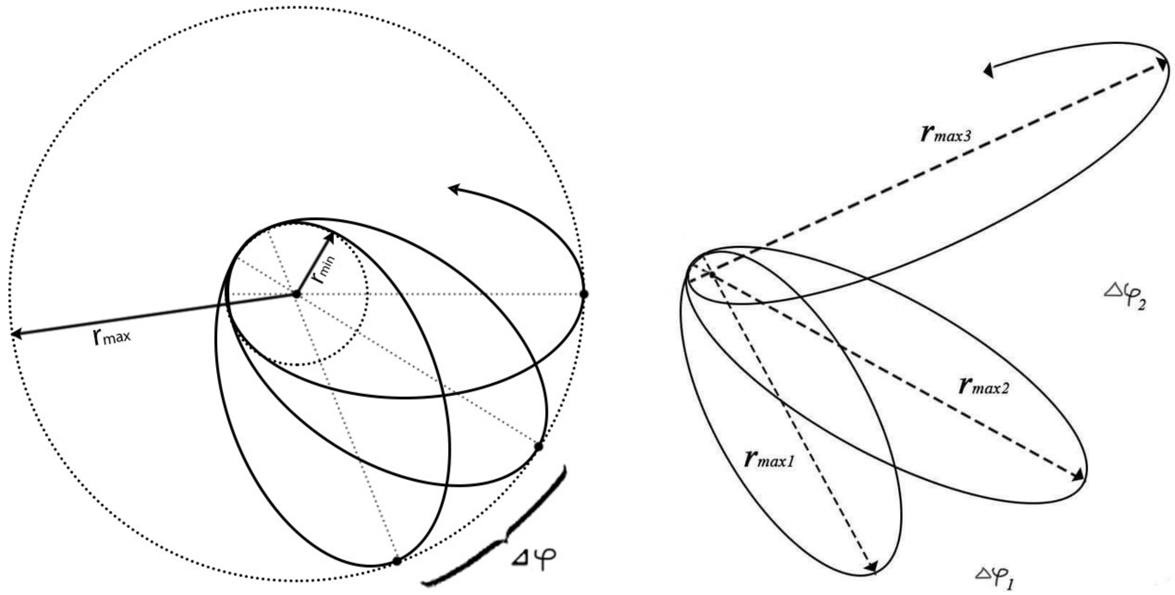

Figure 1. Illustraions of the orbital precession in the GR and in the $f(R)$-theory [5, 6]
.

4.1. Static central field

Let us take Starobinsky's model [8] $f(R) = R + \lambda R^2$ as an example (for other models, see [5, 6]). Using the data [7]

$$\begin{aligned} & c = 299792458\ m/s; \\ & G = 6.67259 \times 10^{-11}\ kg^{-1}m^3s^{-2}; \\ & k = \frac{8\pi G}{c^4} = 2.0761154 \times 10^{-43}\ kg^{-1}m^{-1}s^2; \\ & M \equiv M_\odot = 1.988919 \times 10^{30} kg; \\ & \frac{2GM}{c^2} = 2.95325008 \times 10^3 m; \\ & a = 5.7909175 \times 10^{10} m; \\ & e = 0.20563069; \\ & \Delta\varphi_{obs} = 2\pi(7.98734 \pm 0,00037) \times 10^{-8}\ radian/revolution, \end{aligned} \tag{14}$$

$$\frac{6\pi G}{c^2 a(1-e^2)} = 2.523307 \times 10^{-37}. \tag{15}$$

we obtain the orbital precession of Mercury orbiting around the Sun (the Sun-Mercury system) as follows

- Einstein's value:
  $\Delta\varphi_E = 1.59748694\pi \times 10^{-7}\ radian/rev.$
- Observed value:
  $\Delta\varphi_{obs} = 1.597468\pi \times 10^{-7}\ radian/rev.$

- Deviation:
  $\delta\varphi^{Mer} = \Delta\varphi_{obs} - \Delta\varphi_E = -0.1906\pi \times 10^{-11}\ radian/rev.$

Roughly, if the measurement's error is smaller than the latter, λ should take the value

$$\lambda = 0.296631 \times 10^{18} \tag{16}$$

in order to explain the deviation

$$\Delta\varphi_{f(R)} = \Delta\varphi_{obs} = 1.597468\pi \times 10^{-7}\ radian/rev. \tag{17}$$

and

$$\delta\varphi^{Mer} = \Delta\varphi_{f(R)} - \Delta\varphi_E = \Delta\varphi_{obs} - \Delta\varphi_E = -0.1906\pi \times 10^{-11}\ radian/rev. \tag{18}$$

It is worth noting that the value of λ in (16) satisfies the perturbation condition $\lambda h(R) \ll R$ (see [6]), that is

$$\lambda h(kT^0_0) \ll \frac{6GM}{c^2[R_0]^3}, \tag{19}$$

or, equivalently,

$$\lambda \ll \frac{c^2[R_0]^3}{6GM} = 0.380053 \times 10^{23}. \tag{20}$$

Applying λ given above to a stronger gravitational system, e.g., the system Sgr A*-S2 (with mass $M = 4.31 \times 10^6 M_\odot$) we obtain

- $\Delta\varphi^{S2}_{f(R)} = 1.149305\pi \times 10^{-3}\ radian/rev.$
- $\Delta\varphi^{S2}_{E} = 1.15114\pi \times 10^{-3}\ radian/rev.$
- $\delta\varphi^{S2} = \Delta\varphi^{S2}_{f(R)} - \Delta\varphi^{S2}_{E} = -1.835\pi \times 10^{-6}\ radian/rev.$

We do not consider $f(R) = R + \lambda'/R$ for this case as it, compared with the GR, does not give a new correction (to the orbital precession). To check the theory it is necessary to work in a non-static field.

4.2. Non-static central field

The theory $f(R) = R + \lambda R^2$ is examined above in a static central field and its examination can be repeated straightforwardly for a non-static case. In this case we will examine one more theory, namely, the theory $f(R) = R + \lambda'/R$. As the value of λ, assumed to be universal and, therefore, applicable to the present case, is already estimated in (16), now we estimate the value of λ'. The perturbation condition $\lambda h(R) \ll R$ applied to the theory $f(R) = R + \lambda'/R$ gives

$$\lambda' \ll \frac{9}{[R_0]^6}\left(\frac{2GM}{c^2}\right)^2 = 6.923265 \times 10^{-46}. \tag{21}$$

Next, we estimate the correction to the orbital parameters (eccentricity and axes), compared with their classical values, of a planet moving around a collapsing star with the beginning radius R_0 (before the collapse) and the final radius $R_0(t)$ in a moment t (during the collapsing process or at the end of the collapse). Here, we consider a collapsing star as example but it is possible to consider an expanding (exploding) star. In order to have data for reference we will take a

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Sun-similar star and its Mercury. Below, we will see how an orbit of a planet (Mercury) would change under the star (Sun) contraction keeping its spherical form. In this situation the GR gives no effect unlike the $f(R)$ theory predicting new interesting phenomena (such as corrections to a planet's orbital eccentricity and axes, gravitational waves, etc.) which can be used for testing an $f(R)$ theory.

A star of the Sun's size having a radius of the order

$$R_0 \approx 6.957 \times 10^8 \ m, \tag{22}$$

would collapse to a white dwarf [1] of the Earth's size (or smaller) with a radius of the order

$$R_0(t) \approx 6.371 \times 10^6 \ m. \tag{23}$$

The radius change

$$\Delta R_0(t) = R_0(t) - R_0 = -689329000 \ m \tag{24}$$

would happen for the free falling (assumed) time interval [9]

$$\begin{aligned} \Delta t \quad &= -\left(\frac{8\pi G\rho_0}{3}\right)^{-1/2} \int_1^{\frac{R_0(t)}{R_0}} \left(\frac{\zeta}{1-\zeta}\right)^{1/2} d\zeta \\ &= \left(\frac{3\pi}{32G\rho_0}\right)^{1/2} \left(1 + 3.45 \times 10^{-4}\right) \\ &= \left(\frac{4\pi^2 (R_0)^3}{32GM}\right)^{1/2} \left(1 + 3.45 \times 10^{-4}\right) \\ &= 1769.83 \ s, \end{aligned} \tag{25}$$

with $\frac{R_0(t)}{R_0} = 9.158 \times 10^{-3}$. From here we can estimate the average free falling speed as

$$\frac{|\Delta R_0(t)|}{\Delta t} = 389488.82 \ m/s. \tag{26}$$

Now, let us calculate some parameters of a planet's (Mercury's) orbit in an $f(R)$ theory, where, the star's (Sun's) mass M is replaced by an effective mass M_f [5]. This mass would change with a value ΔM_f during the star contraction until its total collapse. The semi-major axis a and the eccentricity e of the planet's elliptical orbit in an $f(R)$ theory are calculated by the formulas

$$a = \frac{GmM_f(t)}{2|E|}, \tag{27}$$

$$e \approx \sqrt{1 - \frac{2|E|L^2}{G^2 m^3 M_f^2}}, \tag{28}$$

where L is the angular momentum which is a conserved quantity but the energy E is not conserved for a non-static field. To calculate the energy change during the star contraction (until the total collapse) we use the approximation $E \approx \frac{-m(GmM_f)^2}{2L^2}$ (for a circular orbit), thus,

$$\Delta|E| = \frac{2|E|}{M_f} \Delta M_f. \tag{29}$$

[1] More precisely, according to the standard theory of the star evolution, a Sun-type star would first become a red giant before collapsing to a white dwarf.

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Therefore, the planet's orbital semi-major axis and eccentricity would change by the quantities

$$\Delta a = \frac{Gm}{2|E|}\left(\Delta M_f - \frac{M_f \Delta|E|}{|E|}\right) = -\frac{Gm}{2|E|}\Delta M_f = -\frac{L^2 \Delta M_f}{G m^2 M_f^2}, \tag{30}$$

$$\Delta e = -\frac{L^2}{G^2 m^3 M_f^2} \frac{\left(\Delta|E| - \frac{2|E|\Delta M_f}{M_f}\right)}{e} = 0, \tag{31}$$

or with (12) taken into account these changes become

$$\Delta a(t) = a(t) - a_0 = -a_0(1-e^2)\frac{\Delta M_f(t)}{M_f}, \tag{32}$$

$$\Delta e = 0, \tag{33}$$

where a_0 and $a(t)$ are the semi-major axis before the contraction and after the collapse of the star, respectively.

4.2.1. Model $f(R) = R + \lambda R^2$:

Assuming that the star's mass M remains unchanged during the collapse process (in reality, some gravitational radiation and other matter loss may be possible), we calculate the change of the components M_1 and M_2 of M_f during the collapsing (in this $f(R)$-theory M_1 and M_2 have dimension of $[mass/length^2]$ which will be tacit below).

Let us start with calculating the change ΔM_1 of M_1. It is easy to see that

$$\Delta M_1(t) = M_1(t) - M_1(0) = 1.02213108 \times 10^{14} \tag{34}$$

where

$$M_1(0) = \frac{9M^2 kc^2}{8\pi [R_0]^3} = 78498929.12, \tag{35}$$

is the value of M_1 before the collapse starting ($t = 0$), and

$$M_1(t) = \frac{9M^2 kc^2}{8\pi [R_0(t)]^3} = 1.02213186 \times 10^{14}, \tag{36}$$

is the value of M_1 immediately before the end of the collapse. In order to calculate the change ΔM_2 of M_2 we must first calculate $\xi(t)$ and $\alpha(t)$. Using (6), (14) and (23) we get

$$\xi(t) = \sqrt{\frac{kMc^2}{4\pi [R_0(t)]^3}} = 3.37939 \times 10^{-9} \tag{37}$$

and

$$\alpha(t) = 4.5029183 \times 10^{-36}. \tag{38}$$

With the approximation $\frac{\partial}{\partial t} R_0(t) \approx \frac{\Delta R_0(t)}{\Delta t}$,

$$\left[\frac{\partial}{\partial t}\frac{M}{[R_0(t)]^3}\right]^2 \alpha(t) = \left(\frac{-3M}{[R_0(t)]^4}\frac{\Delta R_0(t)}{\Delta t}\right)^2 \alpha(t) = 8.959833 \times 10^{-18}, \tag{39}$$

M_2 takes the value

$$M_2 = \frac{8\pi}{kc^2}\left[\frac{\partial}{\partial t}\frac{M}{[R_0(t)]^3}\right]^2 \alpha(t) = 1.206832 \times 10^{10}. \tag{40}$$

Combining (36) and (40) we have

$$M_f(t) = M(t) - \lambda M_1(t) - \lambda M_2(t) = -2.833426 \times 10^{31}\ kg, \tag{41}$$

$$\Delta M_f = M_f(t) - M_f = -3.03231558 \times 10^{31}\ kg. \tag{42}$$

Inserting this result in (30) we get

$$\Delta a(t) = a(t) - a = 0.4166264 \times 10^{10}\ m. \tag{43}$$

In comparison with the GR, i.e., comparing (43) with (14), we see that the semi-major axis $a(t)$ of the Mercury-like planet in the $f(R)$-theory would increase with 7.19% after the collapse of the Sun-like star. The sign minus in (41) and (42) shows something like an "anti-gravitational" effect which could be a strong argument for verifying the present model.

4.2.2. Model $f(R) = R + \frac{\lambda'}{R}$:

In this model, following similar calculations as above it is not difficult to get for a Sun-Mercury-like system

$$\Delta M_f(t) \simeq 0, \tag{44}$$

that is, there is no sensitive correction to the GR.

5. Conclusion

The $f(R)$-theory is a modified theory of gravitation which makes correction to the general theory of relativity and may replace the latter in explaining new cosmological observations.

We have shown that the $f(R)$-theory allows a non-static spherically symmetric solution and predicts (non-static in general) corrections to cosmological observations (such as orbital precessions and deformations, deflections of light, etc.). Another prediction which is about gravitational radiations of a (non-static) spherically symmetric source, a phenomenon not possible in the general theory of relativity, can be also considered, but it is a subject of a later work being in progress. A testing measurement (observation) may not be easy at the present technical level but we hope it can be done in a not very far future.

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