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Statistical Measures and Complexity of Supersymmetric Polynomials in Quantum Mechanics

Vasil Avramov [1], Hristo Dimov [1,2], Miroslav Radomirov [1], Radoslav C. Rashkov [1,3,*] and Tsvetan Vetsov [1]

[1] Department of Physics, Sofia University, 5 J. Bourchier Blvd., 1164 Sofia, Bulgaria; v.avramov@phys.uni-sofia.bg (V.A.); h_dimov@phys.uni-sofia.bg (H.D.); radomirov@phys.uni-sofia.bg (M.R.); vetsov@phys.uni-sofia.bg (T.V.)
[2] The Bogoliubov Laboratory of Theoretical Physics, Joint Institute for Nuclear Research (JINR), 141980 Dubna, Russia
[3] Institute for Theoretical Physics, Vienna University of Technology, Wiedner Hauptstr. 8–10, 1040 Vienna, Austria
* Correspondence: rash@phys.uni-sofia.bg

Abstract

We study information-theoretic and complexity measures for the Dunkl-supersymmetric harmonic oscillator to identify the effect of supersymmetry on these quantities. Using the Rakhmanov probability density of the Dunkl-SUSY functions, we analyze the Shannon entropy, spreading measures (Heller, Rényi, and Fisher lengths), and several statistical and dynamical complexities. The Shannon entropy is obtained both asymptotically and in closed analytic form, showing that supersymmetry does not affect the leading large-n scaling. In contrast, spreading measures reveal enhanced localization of the SUSY eigenstates relative to the standard harmonic oscillator. Finally, we find that LMC and Fisher–Shannon complexities are higher in the supersymmetric case.

Keywords: Dunkl-supersymmetric oscillator; Shannon entropy; large-n asymptotics; spreading lengths; frequency moments; statistical complexity; quantum harmonic oscillator

MSC: 81Q80; 81Q60; 81Q20

1. Introduction

Information-theoretic and statistical measures provide powerful tools for analyzing quantum systems, as they reveal localization, oscillatory structure, and internal organization beyond spectral information. In quantum mechanics the probability density of a stationary state provides a natural object for applying entropy, information, and complexity measures, allowing one to quantify how quantum states spread, localize, and fluctuate.

Standard descriptors such as variances or higher-order moments only partially characterize delocalization and often miss fine structural features such as nodal patterns and local oscillations. Information-theoretic quantities—particularly the Shannon, Rényi, and Fisher measures—offer a more complete framework [1–10]. Entropy-based measures [3–7] quantify global spreading, while Fisher information [5,11] captures local gradients and oscillatory structure.

Closely related are the associated spreading lengths [1,2], which represent effective spatial scales of a quantum state. The Shannon length [8–10] characterizes the effective support of the Rakhmanov probability density, Rényi lengths [2,8,10] interpolate between peak-dominated and extended regimes, and the Fisher length [1,2,12] probes local sharpness. In

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exactly solvable systems whose eigenfunctions involve classical orthogonal polynomials (Hermite, Laguerre, Jacobi, and Chebyshev) [6,8–10], these quantities describe the evolution from localized ground states to semiclassical states with increasing quantum number.

Complementary measures focus on localization and order, including disequilibrium [13], Onicescu information energy [14], and Fisher information [5,11,15], which quantify clustering and structural organization of probability densities. Their combination with entropy-based quantities leads to statistical complexity measures [1,2], such as the Crámer–Rao [16], Fisher–Shannon [17], Jensen–Shannon [18,19], and López-Ruiz–Mancini–Calvet (LMC) complexities [20,21], which characterize the balance between global spreading and local structure.

In polynomial quantum systems these quantities arise naturally from the Rakhmanov probability density [22–24]:

$$\rho_n(x) = w(x)p_n^2(x), \tag{1}$$

where $p_n(x)$ is an orthonormal polynomial of degree n. This probability density encodes the intrinsic spatial and oscillatory structure of the polynomial and depends only on the orthogonality measure and polynomial degree. Information-theoretic measures derived from $\rho_n(x)$ therefore provide intrinsic descriptions of quantum complexity and connect naturally with the asymptotic theory of orthogonal polynomials.

Recently, supersymmetric (SUSY) quantum mechanics has generated new families of exactly solvable models where the standard quantum harmonic oscillator (QHO) structure is enriched through supersymmetric transformations [25–31]. In this framework, while supersymmetry provides a rigorous mathematical symmetry for solving complex potentials, it also manifests physically through spectral degeneracy and distinct interference patterns in the wavefunctions. These interference effects, occurring between neighboring quantum levels, result in measurable differences in the information-theoretic properties of the system [32]. At the polynomial level, these constructions lead to supersymmetric polynomials defined as coherent linear combinations of adjacent classical orthogonal polynomials, reflecting the physical coupling of states inherent in the SUSY architecture.

From an information-theoretic perspective this raises a natural question of how the presence of supersymmetry affects the behavior of the statistical and complexity properties. In particular, does supersymmetric mixing modify the macroscopic probability structure of the system, or only its finer oscillatory features?

To address this question, we investigate the Rakhmanov probability density and associated information-theoretic measures for the Dunkl-supersymmetric (Dunkl-SUSY) oscillator with a shape-invariant potential [27]. The eigenfunctions of this system consist of orthonormal combinations of even and odd Hermite polynomials, which incorporate the constructive and destructive interference induced by supersymmetric mixing. This framework allows us to quantify the impact of supersymmetry on the statistical and information-theoretic descriptors of quantum states.

Our approach is further motivated by the recent studies on Dunkl systems [33–35], which utilize Shannon and Rényi entropies to characterize wavefunction localization and spectral degeneracy in spherical coordinates. Moreover, the use of generalized Hermite and exceptional Laguerre polynomials in quantifying the complexity of Dunkl oscillators [36,37] highlights the fundamental role of these orthogonal structures in our analysis.

The paper is organized as follows. In Section 2 we present the orthogonal solutions for the Dunkl-supersymmetric oscillator introduced in [27]. In Section 3, we analyze the Shannon entropy of the Dunkl-supersymmetric oscillator and consider its large-n asymptotics. In Section 4, we investigate the spreading lengths and frequency moments, including the Heller, Rényi and Fisher lengths. In Section 5, we study several notions of statistical and dynamical complexity for the Dunkl-supersymmetric system, including the

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LMC and Fisher–Shannon complexities. Where appropriate, our findings are compared to standard results for the quantum harmonic oscillator [8]. Finally, in Section 6 we summarize our results.

2. Dunkl-Supersymmetric Oscillator

We consider a specific realization of a Dunkl-supersymmetric oscillator, as introduced in [27]. The general form of the Dunkl-SUSY Lagrangian with a shape-invariant potential (shape-invariant potentials form a distinguished class of exactly solvable quantum-mechanical potentials in supersymmetric quantum mechanics (SUSY QM)—their defining property is that the supersymmetric partner potentials share the same functional form, differing only by a parameter transformation and an additive constant, and this property allows the full energy spectrum and eigenstates to be obtained algebraically and naturally leads to structures governed by classical orthogonal polynomials—for the Dunkl-SUSY oscillator, see [27]) $v(x)$ is

$$\mathcal{L} = \partial_x \mathcal{R} + v(x)(\mathbb{I} - \mathcal{R}), \tag{2}$$

where $\mathcal{R}$ denotes the reflection operator and $\mathbb{I}$ is the identity operator. The choice of the potential $v(x)$ determines the family of eigenfunctions $Q_n(x)$ of the operator $\mathcal{L}$, referred to as Dunkl-supersymmetric orthogonal functions. These functions can be expressed in terms of the classical Hermite, Laguerre, or Jacobi polynomials.

In the present work, we focus on the shifted oscillator potential $v(x) = s^2 x$, where $s > 0$ is a real coupling constant. In this case, the Lagrangian reduces to

$$\mathcal{L} = \partial_x \mathcal{R} + s^2 x(\mathbb{I} - \mathcal{R}). \tag{3}$$

The corresponding eigenfunctions $Q_n(x)$ of $\mathcal{L}$ are related to the orthonormal Hermite polynomials as follows:

$$Q_{\pm n}(sx) := \frac{1}{\sqrt{2}}\left(\hat{H}_{2n}(sx) \pm \hat{H}_{2n-1}(sx)\right), \quad n \geq 1, \quad E_n = s\sqrt{2n}, \tag{4}$$

where E_n denotes the corresponding energy eigenvalues. The normalized Hermite polynomials $\hat{H}_n(x)$ are related to the standard Hermite polynomials $H_n(x)$ through

$$\hat{H}_n(sx) = \frac{\sqrt{s}\, H_n(sx)}{\sqrt{2^n n! \sqrt{\pi}}}, \tag{5}$$

and satisfy the normalization condition

$$\int_{-\infty}^{\infty} \hat{H}_n(sx)\, \hat{H}_m(sx)\, e^{-s^2 x^2}\, dx = \delta_{nm}. \tag{6}$$

As a consequence, the functions $Q_n(x)$ are orthonormal with respect to the same weight function $\omega(x) = e^{-s^2 x^2}$, namely

$$\int_{-\infty}^{\infty} Q_n(sx)\, Q_m(sx)\, e^{-s^2 x^2}\, dx = \delta_{nm}. \tag{7}$$

For simplicity, and without loss of generality, we set $s = 1$.

The Rakhmanov probability density function $\rho_n(x)$, generated by the Dunkl-SUSY orthonormal functions $Q_{\pm n}(x)$ (4) is given by

$$\rho_{\pm n}(x) = p^2_{\pm n}(x)\omega(x), \quad \omega(x) = e^{-x^2}, \quad p_{\pm n}(x) \equiv Q_{\pm n}(x). \tag{8}$$

We can now proceed with the analysis of intrinsic information-theoretic measures associated with the Rakhmanov probability density $\rho_n(x)$ generated by the Dunkl-SUSY orthonormal functions $Q_{\pm n}(x)$ defined in Equation (4).

3. Shannon Entropy of the Dunkl-Supersymmetric Oscillator

This section derives the Shannon entropy for the SUSY oscillator in both large-n asymptotic and exact analytic forms, demonstrating that the measure remains insensitive to supersymmetry up to leading order when compared to standard Hermite polynomials.

3.1. Asymptotics at Large n

Proposition 1. *The Shannon entropy of the Dunkl-supersymmetric oscillator states,* $Q_{\pm n}(x) = \frac{1}{\sqrt{2}}[\hat{H}_{2n}(x) \pm \hat{H}_{2n-1}(x)]$*, is asymptotically equivalent to the entropy of the standard harmonic oscillator eigenfunctions* $\hat{H}_n(x)$ *in the large-*n *limit.*

Proof. In order to prove this statement, let us recall the definition the Shannon entropy with respect to a probability density $\rho_n(x)$ as follows:

$$S_{\text{Shannon}} = -\int_{-\infty}^{\infty} \rho_n(x) \ln \rho_n(x)\, dx. \tag{9}$$

The form of the Rakhmanov probability density (8) of the Dunkl-supersymmetric oscillator allows us to split the integral into two contributions as follows:

$$S_{\text{SUSY}} = -\int_{-\infty}^{+\infty} Q_{\pm n}^2 e^{-x^2} \ln(Q_{\pm n}^2 e^{-x^2})\, dx = I_1 + I_2, \tag{10}$$

where we used the notation S_{SUSY} for the Shannon entropy of the Dunkl-supersymmetric oscillator. The first contribution I_1 yields the second moment of the position x with respect to $\rho_n(x)$ and can be easily computed as follows:

$$I_1 = \int_{-\infty}^{\infty} x^2 \frac{e^{-x^2}}{2} \left(\hat{H}_{2n}(x) \pm \hat{H}_{2n-1}(x)\right)^2 dx = \int_{-\infty}^{\infty} x^2 \rho_n(x) dx \equiv \langle x^2 \rangle = 2n. \tag{11}$$

The second contribution is given by

$$\begin{aligned} I_2 &= -\int_{-\infty}^{\infty} e^{-x^2} Q_{\pm n}^2(x) \ln Q_{\pm n}^2(x)\, dx \\ &= -\int_{-\infty}^{\infty} \frac{e^{-x^2}}{2} \left(\hat{H}_{2n}(x) \pm \hat{H}_{2n-1}(x)\right)^2 \ln\left[\frac{1}{2}\left(\hat{H}_{2n}(x) \pm \hat{H}_{2n-1}(x)\right)^2\right] dx. \end{aligned} \tag{12}$$

This integral is not straightforward to compute. To extract useful information, however, we can consider its large n asymptotics. More specifically, we consider the weak asymptotics using the following approximation [7,38]:

$$H_n^2(x) e^{-x^2} \approx \frac{1}{\pi\sqrt{2n - x^2}}, \quad |x| < \sqrt{2n}. \tag{13}$$

Inserting this expression into our SUSY Rakhmanov probability density (8), we obtain the following:

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$$\rho_n(x) = Q_n^2(x)e^{-x^2} \approx \frac{1}{2\pi}\left(\frac{1}{\pi\sqrt{4n-x^2}} + \frac{1}{\pi\sqrt{4n-2-x^2}} + \frac{1}{\pi\sqrt[4]{(4n-x^2)}\sqrt[4]{(4n-2-x^2)}}\right). \tag{14}$$

For large n, the denominators can be assumed to be the same; however, this implies that the entire asymptotic probability normalizes to 2, which can be remedied by introducing an additional factor of $1/2$. Thus, the supersymmetric Rakhmanov probability density can be approximated as

$$Q_n^2(x)e^{-x^2} \approx \frac{1}{\pi\sqrt{4n-x^2}}, \quad |x| < 2\sqrt{n}. \tag{15}$$

Applying this asymptotic, we obtain the following approximation for I_2:

$$I_2 \approx -\int_{-2\sqrt{n}}^{2\sqrt{n}} \frac{1}{\pi\sqrt{4n-x^2}} \ln\left(\frac{e^{x^2}}{\pi\sqrt{4n-x^2}}\right) dx. \tag{16}$$

The above expression can be decomposed into the following three integrals [7]:

$$\int_{-2\sqrt{n}}^{2\sqrt{n}} \frac{-x^2}{\pi\sqrt{4n-x^2}}\, dx = -2n, \tag{17}$$

$$\int_{-2\sqrt{n}}^{2\sqrt{n}} \frac{\ln\pi}{\pi\sqrt{4n-x^2}}\, dx = \ln\pi, \tag{18}$$

$$\int_{-2\sqrt{n}}^{2\sqrt{n}} \frac{\ln\sqrt{4n-x^2}}{\pi\sqrt{4n-x^2}}\, dx = \frac{1}{2}\ln n. \tag{19}$$

Combining these results together with (11), we obtain the following asymptotic approximation for the Shannon entropy of the supersymmetric polynomials:

$$S_{\text{SUSY}} \approx \ln\sqrt{n} + \ln\pi + \mathcal{O}(1). \tag{20}$$

This asymptotic scaling is equivalent to the one found for the Hermite polynomial in [3,7]. Thus Proposition 1 holds. In essence, for large n the difference in the entropies is of the order of a constant. Therefore, supersymmetry is not captured by the leading asymptotic order of the Shannon entropy. □

3.2. Analytic Expression in Terms of Hypergeometric Functions

Proposition 2. *There is a closed analytic form of the Shannon entropy of the Dunkl-supersymmetric oscillator in terms of generalized hypergeometric functions.*

Proof. We can find a closed analytic form of the integral I_2 from (12) and thus for S_{SUSY} (10). For this purpose, we define the following relations for the Hermite polynomials:

$$(\hat{H}_{2n} \pm \hat{H}_{2n-1})^2 = \sum_{\lambda} b_\lambda x^\lambda, \tag{21}$$

$$(\hat{H}_{2n} \pm \hat{H}_{2n-1}) = c_{\max}^{(2n)} \prod_i (x - x_i), \quad c_{\max}^{(2n)} = \frac{2^n}{\pi^{1/4}\sqrt{(2n)!}}, \tag{22}$$

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where b_λ are known expansion coefficients of the Hermite polynomials, x_i are the roots of the linear combination of Hermite polynomials $(\hat{H}_{2n} \pm \hat{H}_{2n-1})$, and $c_{\max}^{(2n)}$ is the highest order coefficient of $\hat{H}_{2n}$. Hence I_2 from (12) transforms to

$$I_2 = \frac{1}{2}\ln 2\int_{-\infty}^{\infty} e^{-x^2}(\hat{H}_{2n} \pm \hat{H}_{2n-1})^2 dx - \int_{-\infty}^{\infty} e^{-x^2}(\hat{H}_{2n} \pm \hat{H}_{2n-1})^2\Big(\ln c_{\max}^{(2n)} + \sum_{i=0}^{2n}\ln|x-x_i|\Big)dx. \quad (23)$$

The first integral yields

$$\frac{1}{2}\ln 2\int_{-\infty}^{\infty} e^{-x^2}(\hat{H}_{2n} \pm \hat{H}_{2n-1})^2 dx = \ln 2. \quad (24)$$

The second part of I_2 also has a direct solution as follows:

$$-\ln c_{\max}^{(2n)}\int_{-\infty}^{\infty} e^{-x^2}(\hat{H}_{2n} \pm \hat{H}_{2n-1})^2\, dx = -2\ln c_{\max}^{(2n)}. \quad (25)$$

The third part is a more complicated function given by

$$-\int_{-\infty}^{\infty} e^{-x^2}(\hat{H}_{2n} \pm \hat{H}_{2n-1})^2\sum_{i=0}^{2n}\ln|x-x_i|\, dx = -\sum_{\lambda=0}^{2n}\sum_{i=0}^{2n} b_\lambda \mathcal{I}_{\lambda,i}, \quad (26)$$

where we have introduced the following integrals:

$$\mathcal{I}_{\lambda,i} = \int_{-\infty}^{\infty} e^{-x^2}x^\lambda \ln|x-x_i|\, dx. \quad (27)$$

The solutions are given in terms of the following generalized hypergeometric functions:

$$\begin{aligned}\mathcal{I}_{\lambda,i} = {} & \frac{1+(-1)^\lambda}{4}\Gamma\left(\frac{\lambda+1}{2}\right)\psi\left(\frac{\lambda+1}{2}\right) + \frac{x_i^2\pi}{2}\cos\left(\frac{\pi\lambda}{2}\right){}_2\tilde{F}_2\left(1,1;2,\frac{3}{2}-\frac{\lambda}{2};-x_i^2\right) \\ & + \frac{x_i\pi}{4}\sin\left(\frac{\pi\lambda}{2}\right)\left(\sqrt{\pi}x_i^2\,{}_2\tilde{F}_2\left(1,\frac{3}{2};\frac{5}{2},2-\frac{\lambda}{2};-x_i^2\right) - \frac{4}{\Gamma\left(1-\frac{\lambda}{2}\right)}\right),\end{aligned} \quad (28)$$

where ψ is the digamma function. Substituting Equations (24)–(26) for I_2 and the result (11) for I_1 in Equation (10), we obtain the following analytic form of the Shannon entropy for the Dunkl-SUSY oscillator:

$$S_{\mathrm{SUSY}} = 2n + \ln 2 - 2\ln c_{\max}^{(2n)} - \sum_{\lambda=0}^{2n}\sum_{i=0}^{2n} b_\lambda \mathcal{I}_{\lambda,i}. \quad (29)$$

Note that this expression is particularly useful of identifying the leading and the subleading asymptotic terms of the Shannon entropy. In this context, we can relate this expression to the logarithmic potential integrals related to the Hermite polynomials. □

3.3. Entropy in Terms of Logarithmic Potentials

Proposition 3. *The Shannon entropy of the Dunkl-supersymmetric oscillator, S_{SUSY}, is related to the entropy of the standard quantum harmonic oscillator S via the following identity:*

$$S_{\mathrm{SUSY}} = S - \frac{1}{2} + \ln 2 - \ln\left[c_{max}^{(n)}\left(c_{max}^{(2n)}\right)^2\right] + n + \frac{1}{2}\sum_{i=0}^{2n}[V_{2n}(x_i) + V_{2n-1}(x_i) - 2V_n(\tilde{x}_i) + \partial_{x_i}V_{2n}(x_i)],$$

where $c_{max}^{(n)}$ and $c_{max}^{(2n)}$ denote the leading coefficients of the Hermite polynomials H_n and H_{2n}, respectively. The terms $V_k(x_i)$ represent the logarithmic potentials associated with the zeros x_i of the polynomial $(\hat{H}_{2n} \pm \hat{H}_{2n-1})$, while $\tilde{x}_i$ correspond to the roots of the Hermite polynomial $\hat{H}_n(x)$.

Proof. To prove the above proposition, we first recall the expression for the Shannon entropy of the Hermite polynomials defined via the logarithmic potentials $V_n(\tilde{x}_i)$ [5,39] as follows:

$$S = -\int_{-\infty}^{\infty} \hat{H}_n^2(x) e^{-x^2} \ln\left(\hat{H}_n(x) e^{-x^2}\right) dx = n + \frac{1}{2} + \ln c_{\max}^{(n)} + \sum_{i=0}^{n} V_n(\tilde{x}_i), \tag{30}$$

where the potentials $V_n(\tilde{x}_i)$ are given by

$$V_n(\tilde{x}_i) = -\int_{-\infty}^{\infty} \hat{H}_n^2(x) e^{-x^2} \ln|x - \tilde{x}_i|\, dx. \tag{31}$$

The latter integral has a closed analytic form in terms of hypergeometric functions [5] as follows:

$$\begin{aligned} V_n(\tilde{x}_i) = 2^n n! \sqrt{\pi} \left[\frac{\gamma}{2} + \ln 2 - x_i^2 \, {}_2F_2\left(1, 1; \frac{3}{2}, 2; -\tilde{x}_i^2\right) \right] \\ + \frac{1}{2} \sum_{k=1}^{n} \binom{n}{k} \frac{(-1)^k 2^k}{k} \, {}_1F_1\left(k; \frac{1}{2}; -\tilde{x}_i^2\right). \end{aligned} \tag{32}$$

Now, let us reconsider the integral I_2 from (23). The $\ln 2$ and $-2\ln[c_{\max}^{(2n)}]$ contribution remains the same. However, the non-trivial contribution comes from the logarithmic part as follows:

$$-\sum_{i=0}^{2n} \int_{-\infty}^{\infty} \left(\hat{H}_{2n}(x) \pm \hat{H}_{2n-1}(x)\right)^2 e^{-x^2} \ln|x - x_i|\, dx = \sum_{i=0}^{2n} J_n(x_i). \tag{33}$$

We can write the J_n integrals in (33) such as

$$J_n(x_i) = -\int_{-\infty}^{\infty} e^{-x^2} \left[H_{2n}^2(x) + H_{2n-1}^2(x) + 2H_{2n}(x) H_{2n-1}(x) \right] \ln|x - x_i|\, dx. \tag{34}$$

The first two terms are trivially related to V_{2n} and V_{2n-1} as follows:

$$J_n(x_i) = V_{2n}(x_i) + V_{2n-1}(x_i) - \int_{-\infty}^{\infty} e^{-x^2} [2H_{2n}(x) H_{2n-1}(x)] \ln|x - x_i|\, dx. \tag{35}$$

The final integral can be reduced to a tractable form by integrating by parts and employing the recurrence relations. Explicitly, it reduces to

$$-\int_{-\infty}^{\infty} e^{-x^2} [2H_{2n}(x) H_{2n-1}(x)] \ln|x - x_i|\, dx = \partial_{x_i} V_{2n}(x_i), \tag{36}$$

where ∂_{x_i} denotes the derivative of the logarithmic potential with respect to its respective root x_i. Combining with the other contributions to the entropy from Equation (29), we obtain the following result:

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$$S_{\text{SUSY}} = 2n + \ln 2 - 2\ln c_{\max}^{(2n)} + \frac{1}{2}\sum_{i=0}^{2n}[V_{2n}(x_i) + V_{2n-1}(x_i) + \partial_{x_i}V_{2n}(x_i)]. \tag{37}$$

Using the expression of the Shannon entropy S for the Hermite polynomials from (30) we can write S_{SUSY} in the following desired form:

$$S_{\text{SUSY}} = S - \frac{1}{2} + \ln 2 - \ln\left[c_{max}^{(n)}(c_{max}^{(2n)})^2\right]$$
$$+ n + \frac{1}{2}\sum_{i=0}^{2n}[V_{2n}(x_i) + V_{2n-1}(x_i) - 2V_n(\tilde{x}_i) + \partial_{x_i}V_{2n}(x_i)], \tag{38}$$

This confirms Proposition 3. From this expression, one can extract the large-n expansion of the entropy. In particular, the potential combination in the brackets generates the leading logarithmic term $\ln\sqrt{n}$ in (20), together with subleading corrections. □

4. Spreading Lengths

This section analyzes the spreading and localization properties of Dunkl-SUSY oscillator states via Heller, Rényi, and Fisher measures, comparing them to the standard quantum harmonic oscillator to quantify the impact of supersymmetry on spatial distribution.

4.1. Heller Length

Proposition 4. *The Heller length L_H of the Dunkl-SUSY system can be expressed in terms of the Hermite product coefficients $\widetilde{C}_p$ defined in Equation (A11) as*

$$L_H^{-1}[\rho_n] = \frac{1}{4\sqrt{2}}\sum_{k=0}^{4}\binom{4}{k}\sum_{p=0}^{n(4-k)}\widetilde{C}_p^{(2n\ldots4-k)}\,\widetilde{C}_{p-4n+2nk-\frac{k}{2}}^{(2n-1\ldots k)}.$$

Proof. By definition, the Heller length $L_H[\rho_n]$ (also known as the Rényi entropy-based length of order 2) essentially measures the "effective width" or the spread of the probability density across a given interval Δ as follows:

$$L_H[\rho_n] := \left(\int_\Delta \rho_n^2(x)dx\right)^{-1}. \tag{39}$$

In the Dunkl-SUSY case $\Delta = (-\infty, \infty)$ and the Heller length becomes

$$L_H^{-1}[\rho_n] = \int_{-\infty}^{\infty}\frac{e^{-2x^2}}{2}\left(\hat{H}_{2n}(x) \pm \hat{H}_{2n-1}\right)^4 dx. \tag{40}$$

Expanding the binomial and introducing new variable $x = y/\sqrt{2}$, we obtain

$$L_H^{-1}[\rho_n] = \int_{-\infty}^{\infty}\frac{e^{-y^2}}{4\sqrt{2}}\sum_{k=0}^{4}\binom{4}{k}(-1)^k\hat{H}_{2n}^{4-k}\left(\frac{y}{\sqrt{2}}\right)\hat{H}_{2n-1}^{k}\left(\frac{y}{\sqrt{2}}\right)dy. \tag{41}$$

Applying the expansion (A11), we find

$$\hat{H}_{2n}^{4-k}\left(\frac{y}{\sqrt{2}}\right) = \sum_{p=0}^{n(4-k)}\widetilde{C}_p^{(2n\ldots4-k)}\hat{H}_{2n(4-k)-2p}(y), \tag{42}$$

$$\hat{H}_{2n-1}^{k}\left(\frac{y}{\sqrt{2}}\right) = \sum_{d=0}^{\lfloor\frac{(2n-1)k}{2}\rfloor}\widetilde{C}_d^{(2n-1\ldots k)}\hat{H}_{(2n-1)k-2d}(y). \tag{43}$$

Inserting these expressions into (41) and taking into account the orthonormality of the Hermite polynomial, we obtain the result in Proposition 4 as follows:

$$L_H^{-1}[\rho_n] = \frac{1}{4\sqrt{2}} \sum_{k=0}^{4} \binom{4}{k} \sum_{p=0}^{n(4-k)} \widetilde{C}_p^{(2n\ldots4-k)}\, \widetilde{C}_{p-4n+2nk-\frac{k}{2}}^{(2n-1\ldots k)}, \tag{44}$$

where the summation is over even powers of k. □

Note that the Heller length L_H is the $L_{q=2}^R$ Rényi length. A comparison between the Heller lengths L_2^R for the standard Hermite polynomials and the Dunkl-SUSY states is presented in Figure 1. We observe that the Heller length L_2^R for the Dunkl-SUSY system (represented by the dotted blue curve) is sensitive to supersymmetry. Specifically, it remains consistently smaller than its counterpart for the standard QHO (the dotted red curve) as n increases.

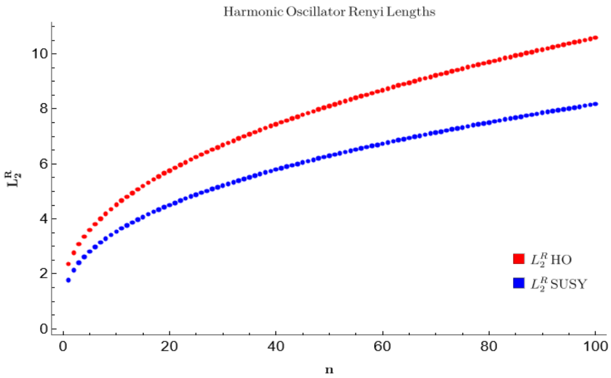


Figure 1. Comparison of Rényi (Heller) lengths L_2^R for QHO and Dynkl-SUSY.

4.2. Generalized Frequency Moments

Proposition 5. *The generalized frequency (entropic) moment, $W_q[\rho_n] = \int_{-\infty}^{\infty} \rho_n^q(x)\,dx$, for the SUSY-Dunkl oscillator can be expressed in terms of the Hermite product coefficients $\widetilde{C}_p$ defined in* (A11) *as follows:*

$$W_q[\rho_n] = \frac{1}{2^q\sqrt{q}} \sum_{k=0}^{2q} \binom{2q}{k} \sum_{p=0}^{n(2q-k)} \widetilde{C}_p^{(2n\ldots4-k)}\, \widetilde{C}_{p-2nq+2nk-\frac{k}{2}}^{(2n-1\ldots k)}. \tag{45}$$

Proof. We can apply similar approach to what we did for the Heller length. The generalized frequency moment for the for the SUSY-Dunkl oscillator is given by the following integral (this integral measures the average "height" of the probability density $\rho_n(x)$ raised to the power q; specifically, for $q > 1$, the integral emphasizes the peaks of the distribution, where the particle is most likely to be, whereas for $q < 1$, it emphasizes the tails, i.e., the spatial spreading of the distribution):

$$W_q[\rho_n] = \int_{-\infty}^{\infty} \frac{e^{-qx^2}}{2^q} \left(\hat{H}_{2n}(x) \pm \hat{H}_{2n-1}(x)\right)^{2q} dx. \tag{46}$$

Expanding the power and changing variables to $x = y/\sqrt{q}$, we obtain

$$W_q[\rho_n] = \int_{-\infty}^{\infty} \frac{e^{-y^2}}{2^q\sqrt{q}} \sum_{k=0}^{2q} \binom{2q}{k} \hat{H}_{2n}^{2q-k}\left(\frac{y}{\sqrt{q}}\right) \hat{H}_{2n-1}^{k}\left(\frac{y}{\sqrt{q}}\right) dy. \tag{47}$$

Applying the expansion (A11), one finds

$$\hat{H}_{2n}^{2q-k}\left(\frac{y}{\sqrt{q}}\right) = \sum_{p=0}^{n(4-k)} \widetilde{C}_p^{(2n\ldots 2q-k)} \hat{H}_{2n(2q-k)-2p}(y), \tag{48}$$

$$\hat{H}_{2n-1}^{k}\left(\frac{y}{\sqrt{q}}\right) = \sum_{d=0}^{\lfloor \frac{(2n-1)k}{2} \rfloor} \widetilde{C}_d^{(2n-1\ldots k)} \hat{H}_{(2n-1)k-2d}(y). \tag{49}$$

Substituting into the original expression and using the orthonormality relation (A1) for Hermite polynomials, we obtain

$$W_q[\rho_n] = \frac{1}{2^q\sqrt{q}} \sum_{k=0}^{2q} \binom{2q}{k} \sum_{p=0}^{n(2q-k)} \widetilde{C}_p^{(2n\ldots 4-k)} \widetilde{C}_{p-2nq+2nk-\frac{k}{2}}^{(2n-1\ldots k)}. \tag{50}$$

This shows that Proposition 5 is correct. □

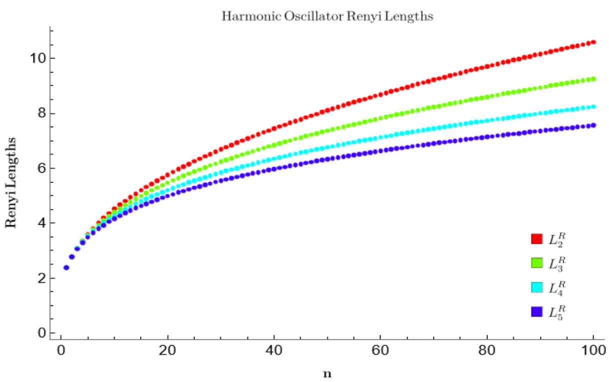
4.3. Rényi Lengths

By definition the Rényi lengths L_q^R are given by the exponent of Réyni entropies (the Rényi entropy $R_q[\rho]$ generalizes the Shannon entropy $S = -\int \rho \ln \rho \, dx$, to which it converges in the limit $q \to 1$; physically, it serves as a measure of the uncertainty or "lack of information" regarding the particle's spatial position by logarithmically scaling the frequency moment $W_q[\rho]$) R_q or by the frequency (entropic) moments W_q as follows:

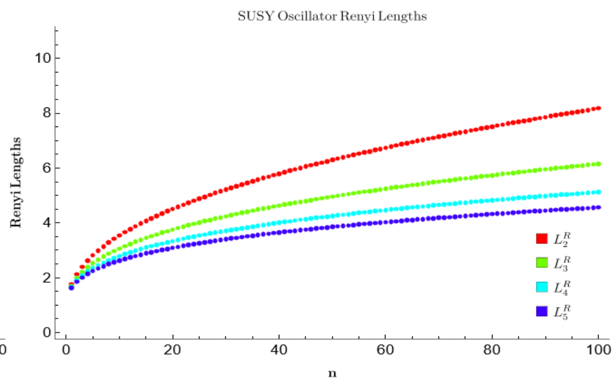
$$L_q^R = e^{R_q[\rho]} = (W_q[\rho])^{\frac{1}{1-q}}, \quad R_q[\rho] = \frac{1}{1-q} \ln W_q[\rho] = \frac{1}{1-q} \int_\Delta \rho_n^q(x) dx. \tag{51}$$

These lengths represent the "effective width" of the probability distribution, i.e., how "spread out" the wavefunctions are. We perform a numerical comparison of the Rényi lengths L_q^R of the Dunkl-SUSY oscillator with those of the standard quantum harmonic oscillator. We show that L_q^R for the SUSY states differs from the standard QHO, thus the spreading properties of the ocillator are physically altered by the presence of supersymmetry.

The results are presented in Figure 2a,b. It is evident that the Rényi lengths of the supersymmetric oscillator are consistently smaller than those of the standard oscillator for particular values of q, indicating that the corresponding states are more localized. One can confidently assume that Rényi lengths are sensitive to the presence of supersymmetry.



(**a**)



(**b**)

Figure 2. Comparison of Rényi lengths L_q^R for $q \in [2, 5]$ of the QHO and Dunkl-SUSY oscillator. The numerical comparison shows that Rényi lengths are sensitive to the presence of supersymmetry. (**a**) Rényi lengths for the QHO. (**b**) Rényi lengths for the Dunkl-SUSY HO.

4.4. Fisher Length

Proposition 6. *The Fisher length $L_F^{(Q)}$ for the Dunkl–SUSY polynomial is given by*

$$L_F^{(Q)} = \frac{1}{2\sqrt{2n}} = \frac{1}{2}\sqrt{2+\frac{1}{n}}\,F_L^{(H)}, \tag{52}$$

where $F_L^{(H)}$ is the Fisher length of the standard QHO.

Proof. The Fisher length L_F is a measure of the "local" spreading of a probability density (a smaller Fisher length indicates a more "sharply peaked" or rapidly changing distribution, implying higher localized precision). Unlike the Rényi lengths (which measure global width), the Fisher length is derived from the Fisher Information (F), which involves the gradient of the probability density. The classical Fisher information is defined as

$$F = \int_{-\infty}^{\infty} \left(\frac{\partial \ln \rho_n(x)}{\partial x}\right)^2 \rho_n(x)dx = \int_{-\infty}^{\infty} \frac{\rho_n'^2(x)}{\rho_n(x)}dx. \tag{53}$$

For the QHO Hermitian polynomials $\rho_n(x) = \hat{H}_n^2(x)e^{-x^2}$ and $\rho_n'(x) = 2\hat{H}_n(x)e^{-x^2}\left[\hat{H}_n'(x) - x\hat{H}_n(x)\right]$. The Fisher information yields

$$F_H = 4\int_{-\infty}^{\infty} \left[\hat{H}_n'(x) - x\hat{H}_n(x)\right]^2 e^{-x^2}dx = 2(2n+1). \tag{54}$$

For the Dunkl-SUSY polynomials, we have the following Fisher information:

$$F_Q = 4\int_{-\infty}^{\infty} \left[Q_n'(x) - xQ_n(x)\right]^2 e^{-x^2}dx = 8n. \tag{55}$$

The corresponding Fisher lengths are

$$L_F^{(H)} = \frac{1}{\sqrt{F_H}} = \frac{1}{\sqrt{2(2n+1)}}, \quad L_F^{(Q)} = \frac{1}{\sqrt{F_Q}} = \frac{1}{2\sqrt{2n}} = \frac{1}{2}\sqrt{2+\frac{1}{n}}\,F_L^{(H)}, \quad L_F^{(H)} > L_F^{(Q)}. \tag{56}$$

This shows that Proposition 6 holds. The result suggests that the "localization" of particles in the Dunkl-SUSY system is fundamentally different from the standard case due to the presence of the Dunkl operator and the superpartner states. Therefore, the identity proves that the Fisher length is highly sensitive to supersymmetry. Even as $n \to \infty$, the ratio between the two lengths does not converge to 1, but rather to a constant factor of $1/\sqrt{2}$. □

5. Complexities

Complexity represents a family of measures quantifying structural organization by balancing order and disorder—ranging from the perfect order of a Dirac delta to the maximum disorder of a uniform distribution—which we here evaluate using several standard metrics for the Rakhmanov probability density of the orthonormal Dunkl-supersymmetric functions $Q_{\pm n}(x)$ from (4).

5.1. LMC Complexity

The López-Ruiz–Mancini–Calbet (LMC) complexity [20] provides a system-independent measure of structural organization by balancing global entropy with a disequilibrium term that quantifies deviations from uniformity. This framework distinguishes genuinely structured quantum states from those that are either perfectly ordered or maximally disordered, making it a robust tool for identifying transitions in complex and chaotic systems. The LMC complexity is defined by the following product:

$$\mathrm{LMC}[\rho_n] = S_{\mathrm{Shannon}}[\rho_n] \times W_2[\rho_n], \tag{57}$$

where S_{Shannon} is the Shannon entropy (9) and W_2 is the disequilibrium or the second entropic moment (46).

The LMC complexity for the QHO has already been studied in [20,21,39]. Comparison of the measure for the first 100 polynomials in the case of the QHO and the Dunkl-SUSY oscillator is depicted in Figure 3. One notes that this measure is sensitive to the supersymmetry. Furthermore, as the supersymmetric LMC complexity is higher than the standard QHO complexity, it suggests the existence of more complex structures retaining some degree of randomness.

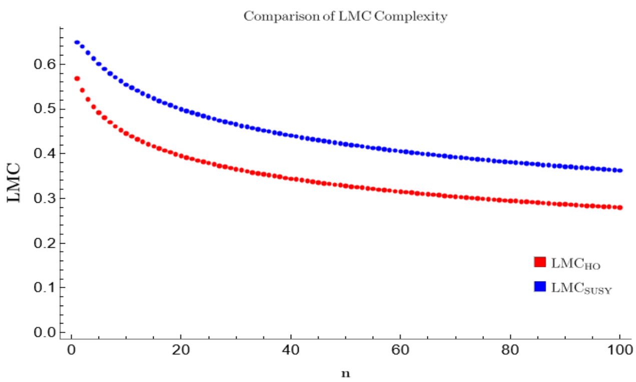


Figure 3. Comparison of LMC complexities for QHO and Dunkl-SUSY HO. It shows that LMC is sensitive to supersymmetry.

5.2. Fisher–Shannon Product Complexity

The Fisher–Shannon complexity is defined by [17,40]

$$J[\rho_n] = \frac{1}{2\pi e} F[\rho_n] \times L_S[\rho_n], \tag{58}$$

where $L_S = e^{S_{\mathrm{Shannon}}[\rho_n]}$ is the Shannon length (entropic power). This quantity is a composite information-theoretical measure that serves as a robust indicator of statistical complexity by linking two complementary descriptions of a probability distribution. Rather than relying on a single metric, this product synthesizes a local measure of intrinsic accuracy or sharpness with a global measure of smoothness or spreading, allowing for a more comprehensive characterization of the localization and delocalization features of a probability density than either component can provide individually.

We numerically compare the Fisher–Shannon information complexity for QHO and Dunkl-SUSY oscillator in Figure 4. Similar to the previous measures the Fisher–Shannon information complexity is also sensitive to supersymmetry.

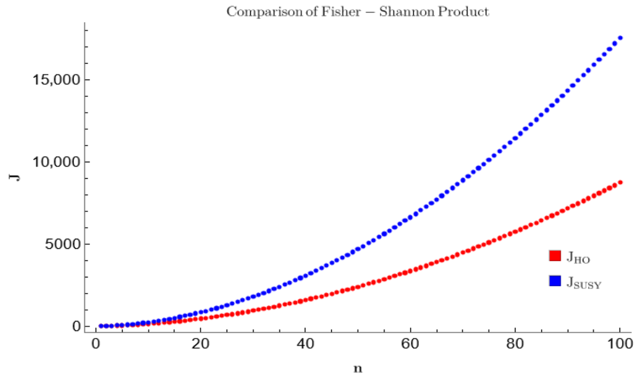


Figure 4. Comparison of Fisher–Shannon information complexity for QHO and Dunkl-SUSY HO.

6. Conclusions

In this work, we have performed a comprehensive information-theoretic and complexity-based analysis of the Dunkl-supersymmetric harmonic oscillator. Our primary objective was to quantify the impact of supersymmetry beyond traditional spectral properties, utilizing the Rakhmanov probability density associated with orthonormal Dunkl-SUSY functions. By investigating a broad class of measures probing localization, spreading, and complexity, we systematically compared the properties of this system with those of the canonical quantum harmonic oscillator (QHO).

Our analysis of the Shannon entropy, supported by both large-n asymptotic derivations and exact expressions via the logarithmic potential formalism, revealed that its leading-order scaling coincides with that of the standard Hermite polynomials. This suggests that dominant entropic growth is a universal feature insensitive to supersymmetry at the leading order. However, finer structural differences associated with the SUSY framework are encoded in the sub-leading terms.

In contrast, spreading measures—specifically the Heller length, generalized frequency moments, Rényi lengths, and the Fisher length—clearly distinguish the Dunkl-SUSY oscillator from the QHO. We observed that Rényi and Fisher lengths are systematically smaller in the supersymmetric case, indicating an enhanced localization of the SUSY eigenstates. The sensitivity of the information-theoretic measures and complexities to supersymmetry (SUSY) has been summarized in Table 1.

Building upon these findings, we evaluated several measures of statistical complexity. Both the LMC complexity and the Fisher–Shannon product revealed that the Dunkl-SUSY oscillator exhibits higher complexity than the QHO. This indicates a richer structural organization characterized by a specific balance between enhanced localization and residual disorder.

Collectively, these results establish a clear hierarchy in the sensitivity of information-theoretic measures to supersymmetry. While global entropic quantities capture universal features, finer measures of spatial localization and statistical complexity provide robust indicators of the structural impact of supersymmetry. Interestingly, our numerical results suggest that the discrepancy between Rényi lengths of the two systems decreases with increasing order q. However, the current lack of analytical large-n and large-q asymptotics precludes a rigorous confirmation of this trend at this stage.

Finally, the framework studied here opens avenues for extending such diagnostics to more complex settings, including holographic systems and non-integrable deformations. In such cases, these information-theoretic measures may reveal qualitatively new behaviors regarding the interplay between symmetry and quantum information.

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Table 1. Sensitivity of information-theoretic measures and complexity to supersymmetry (SUSY).

Measure	**Sensitivity to SUSY**
Shannon entropy	Yes (for small n) No (at leading order for large n) Yes (at sub-leading orders)
Heller Length (L_2^R)	Yes
Rényi Lengths	Yes
Fisher Length	Yes
LMC Complexity	Yes
Fisher–Shannon product	Yes

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Appendix A. Useful Hermite Polynomial Identities

Here we present some useful equalities that we use throughout the work. We use the following notation, standard Hermite polynomials are denoted $H_n(x)$, while normalized Hermite polynomials are denoted $\hat{H}_n(x)$. Explicitly they are related using the following relation:

$$\hat{H}_n(x) = \frac{H_n(x)}{\sqrt{2^n\, n!\, \sqrt{\pi}}}, \quad \int_{-\infty}^{\infty} \hat{H}_n(x)\hat{H}_m(x)\, e^{-x^2} dx = \delta_{nm}. \tag{A1}$$

The standard Hermite polynomials H satisfy the following recurrence relations:

$$H_n(-x) = (-1)^n H_n(x), \tag{A2}$$

$$H'_n(x) = 2nH_{n-1}(x), \tag{A3}$$

$$H_{n+1}(x) = 2xH_n(x) - H'_n(x) = 2xH_n(x) - 2nH_{n-1}(x). \tag{A4}$$

For the normalized Hermite polynomials one has

$$\hat{H}'_n(x) = \sqrt{2n}\hat{H}_{n-1}(x), \tag{A5}$$

$$\sqrt{2(n+1)}\hat{H}_{n+1}(x) = 2x\hat{H}_n(x) - \hat{H}'_n(x) = 2x\hat{H}_n(x) - \sqrt{2n}\hat{H}_{n-1}(x). \tag{A6}$$

Other useful identities are for the product of an arbitrary number of Hermite polynomials [41]. For the normalized polynomials one finds

$$H_{m_1}(a_1x)\dots H_{m_k}(a_kx) = \sum_{2p\le M} C_p^{(m_1,\dots m_k)} \frac{H_{M-2p}(x)}{(M-2p)!}, \quad M = \sum_{i=1}^{k} m_i, \tag{A7}$$

where the coefficients $C_p^{(m_1,\dots,m_k)}$ are of the following form:

$$C_p^{(m_1,\dots,m_k)} := \sum_{R\le p} (-1)^R \frac{(M-2R)!}{(p-R)!} \prod_{i=1}^{k} \frac{(m_i)!\, a_i^{m_i-2r_i}}{(r_i)!(m_i-2r_i)!}, \quad R = \sum_{i=1}^{k} r_i. \tag{A8}$$

By applying the definition of our normalized Hermite polynomials $\hat{H}$, we can find a new version the existing formula, i.e.,

$$\hat{H}_{m_1}(a_1x)\dots \hat{H}_{m_k}(a_kx) = \sum_{2p\le M} \hat{C}_p^{(m_1,\dots,m_k)} \frac{\hat{H}_{M-2p}(x)}{2^p \pi^{(k-1)/4}\sqrt{(M-2p)!}}, \tag{A9}$$

where we can define the new coefficients $\hat{C}_p^{(m_1,\dots,m_k)}$ as

$$\hat{C}_p^{(m_1,\dots,m_k)} := \sum_{R\le p} (-1)^R \frac{(M-2R)!}{(p-R)!} \prod_{i=1}^{k} \frac{\sqrt{(m_i)!}\, a_i^{m_i-2r_i}}{(r_i)!(m_i-2r_i)!}. \tag{A10}$$

We can also define a new quantity $\widetilde{C}_p^{(m_1,\dots,m_k)}$:

$$\widetilde{C}_p^{(m_1,\dots,m_k)} := \frac{1}{2^p \pi^{(k-1)/4}\sqrt{(M-2p)!}} \sum_{R\le p} (-1)^R \frac{(M-2R)!}{(p-R)!} \prod_{i=1}^{k} \frac{\sqrt{(m_i)!}\, a_i^{m_i-2r_i}}{(r_i)!(m_i-2r_i)!}, \tag{A11}$$

thus

$$\hat{H}_{m_1}(a_1x)\dots \hat{H}_{m_k}(a_kx) = \sum_{2p<M} \widetilde{C}_p^{(m_1,\dots,m_k)} \hat{H}_{M-2p}(x). \tag{A12}$$

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