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Special Hermitian structures on suspensions

Anna Fino[1,2] · Gueo Grantcharov[2] · Misha Verbitsky[3]

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Abstract

Motivated by the construction based on topological suspension of a family of compact non-Kähler complex manifolds with trivial canonical bundle given by Qin and Wang (Geom Topol 22:2115–2144, 2018), we study toric suspensions of balanced manifolds by holomorphic automorphisms. In particular, we show that toric suspensions of Calabi–Yau manifolds are balanced. We also prove that suspensions associated with hyperbolic automorphisms of hyperkähler manifolds do not admit any pluriclosed, astheno-Kähler or p-pluriclosed Hermitian metric. Moreover, we consider natural extensions for hypercomplex manifolds, providing some explicit examples of compact holomorphic symplectic and hypercomplex non-Kähler manifolds. We also show that a modified suspension construction provides examples with pluriclosed metrics.

Contents

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✉ Anna Fino
annamaria.fino@unito.it; afino@fiu.edu

Gueo Grantcharov
grantchg@fiu.edu

Misha Verbitsky
verbit@impa.br

1 Dipartimento di Matematica G. Peano, Universitá di Torino, via Carlo Alberto 10, 10123 Turin, Italy

2 Department of Mathematics and Statistics, Florida International University, Miami, FL 33199, USA

3 Instituto Nacional de Matemática Pura e Aplicada (IMPA), Estrada Dona Castorina, 110 Jardim Botânico, Rio de Janeiro, RJ CEP 22460-320, Brazil

1 Introduction

Finding examples of special non-Kähler metrics on compact complex manifolds has become a question of increasing interest in recent years. It is partly due to developments in Physics related to Hull–Strominger system [33, 43] and generalized geometry [29, 30, 32]. In [42] a new class of examples of non-Kähler manifolds with trivial canonical bundle and nice topological properties have been introduced. It is based on the topological suspension construction.

Given in general a smooth manifold M and a diffeomorphism f of M, the mapping torus (or suspension) of f is defined to be the quotient M_f of the product $M \times \mathbb{R}$ by the $\mathbb{Z}$-action defined by

$$(p, t) \to (f^{-n}(p), t + n).$$

As a consequence dt defines a nonsingular closed 1-form on M_f tangent to the fibration

$$M_f \longrightarrow S^1 = \mathbb{R}/\mathbb{Z}.$$

Moreover, the vector field $\frac{\partial}{\partial t}$ on $M \times \mathbb{R}$ defines a vector field on M_f, the suspension of the diffeomorphism f. There is a natural correspondence between the orbits of f and the trajectories of the vector field. Mapping tori have been used in [37] to construct examples of co-symplectic and co-Kähler manifolds.

The suspension construction can be extended to complex manifolds in the following way. Given a complex manifold M, a set of commuting holomorphic automorphisms $f_j, j = 1, \dots, 2k$, of M and a lattice $\Lambda \subset \mathbb{C}^k$ of rank 2k, generated by $\xi_1, \dots, \xi_{2k}$, one can define an action of $\mathbb{Z}^{2k} = \langle \xi_1, \dots, \xi_{2k} \rangle$ on $M \times \mathbb{C}^k$ via $\varphi_j(m, z) = (f_j(m), z + \xi_j)$. The quotient of $M \times \mathbb{C}^k$ by the action of $\mathbb{Z}^{2k}$ is called the toric suspension of $(M, f_1, \dots, f_{2k})$. In particular, if f is an automorphism of a complex manifold M and $T^2 = \mathbb{C}/\mathbb{Z}^2$ an elliptic curve, one can construct the complex suspension of f as the toric suspension $S(f)$ of M associated with the pair (f, Id_M). In the present paper we study the metric properties of the constructed manifolds, like the existence of balanced metrics, that is, Hermitian metrics with co-closed fundamental form. We also extend the construction to produce hypercomplex manifolds with special metric properties.

Using a different construction related to automorphisms of 3-dimensional Sasakian manifolds, we construct suspensions admitting pluriclosed metrics, that is, the Hermitian metrics with $\partial\overline{\partial}$-closed fundamental forms.

In Sects. 2 and 3 we present the necessary information on hyperkähler manifolds and toric suspension construction. In Sect. 4 we prove that the complex toric suspension of a balanced manifold M by two commuting holomorphic diffeomorphisms preserving a volume form is balanced. As a corollary we show that if M is a Calabi–Yau manifold and f is an automorphism of M preserving the holomorphic volume form, then the complex suspension $S(f)$ has trivial canonical bundle and admits a balanced metric.

In Sect. 5 we show that the balanced manifolds constructed using any hyperbolic automorphism of hyperkähler manifolds do not admit any p-pluriclosed and locally conformally Kähler (LCK) metric. In Sect. 6 we recover the construction in [42] as toric suspension of a Kummer surface and we generalize it to suspension of the Hilbert scheme of points on Kummer surfaces. In Sect. 7 we discuss the natural extensions of toric suspensions on hypercomplex manifolds and their metric structures. As an application we construct explicit examples of compact holomorphic symplectic and hypercomplex non-Kähler manifolds. The examples are in fact pseudo-hyperkähler and admit quaternionic balanced metric, but no hyperkähler with torsion (HKT) metric.

Finally in Sect. 8 we show how using automorphisms of Sasakian and Kähler manifolds it is also possible to construct suspensions admitting pluriclosed metrics recovering a recent example constructed in [19], as a compact 3-step solvmanifold.

2 Hyperkähler manifolds and their automorphisms

Here we introduce the necessary background materials on hyperkähler geometry. We follow [3, 4, 6, 9, 34].

2.1 Hyperkähler manifolds and the BBF form

Definition 2.1 A **hyperkähler manifold** is a compact, Kähler, holomorphically symplectic manifold.

Definition 2.2 A hyperkähler manifold M is called to be **of maximal holonomy** (also: simple, or IHS) if $\pi_1(M) = 0$ and $H^{2,0}(M) = \mathbb{C}$.

Theorem 2.3 (Bogomolov's decomposition [7]) *Any hyperkähler manifold admits a finite covering which is a product of a torus and several hyperkähler manifolds of maximal holonomy.*

Remark 2.4 From now on all hyperkähler manifolds are assumed to be of maximal holonomy.

Theorem 2.5 (Fujiki [21]) *Let M be a hyperkähler manifold, $\eta \in H^2(M,\mathbb{Z})$, and $n = \frac{\dim_{\mathbb{C}} M}{2}$. Then $\int_M \eta^{2n} = cq(\eta,\eta)^n$, where q is a primitive integer non-degenerate quadratic form on $H^2(M,\mathbb{Z})$, and $c > 0$ is a rational number depending only on M.*

Definition 2.6 This primitive integral quadratic form q on $H^2(M,\mathbb{Z})$ is called **Bogomolov–Beauville–Fujiki form**, or **BBF form**. It is defined by the Fujiki's relation uniquely, up to a sign. The sign is determined from the following formula (Bogomolov, Beauville, see [4])

$$\lambda q(\eta,\eta) = \int_M \eta\wedge\eta\wedge\Omega^{n-1}\wedge\overline{\Omega}^{n-1} - \frac{n-1}{n}\left(\int_M \eta\wedge\Omega^{n-1}\wedge\overline{\Omega}^{n}\right)\left(\int_M \eta\wedge\Omega^{n}\wedge\overline{\Omega}^{n-1}\right),$$

where Ω is the holomorphic symplectic form on M and $\lambda > 0$.

Remark 2.7 The BBF form q has signature $(3, b_2-3)$ when extended on $H^2(M,\mathbb{R})$. It is negative definite on primitive forms, and positive definite on $\langle\Omega,\overline{\Omega},\omega\rangle$, where ω is a Kähler form. On (1,1)-forms η it can be written as $q(\eta,\eta) = c\int_M \eta\wedge\eta\wedge\Omega^{n-1}\wedge\overline{\Omega}^{n-1}$, where c is a constant.

2.2 Classification of automorphisms of hyperkähler manifolds

Remark 2.8 The indefinite orthogonal group $O(m,n)$, $m,n>0$, is the Lie group of all linear transformations of an l-dimensional real vector space that leave invariant a nondegenerate, symmetric bilinear form q of signature (m,n), where $l = m+n$. $O(m,n)$ has 4 connected components. We denote the connected component of 1 by $SO^+(m,n)$. We call a vector v **positive** if its square is positive, i.e. if $q(v,v) > 0$.

Definition 2.9 Let V be a real vector space of dimension $n+1$ with a quadratic form q of signature $(1,n)$, $\mathrm{Pos}(V) = \{x\in V \mid q(x,x) > 0\}$ its **positive cone**, and $\mathbb{P}^+V$ be the projectivization of $\mathrm{Pos}(V)$. Denote by g any $SO(V)$-invariant Riemannian structure on $\mathbb{P}^+V$ (it is easy to see that g is unique up to a constant multiplier). Then $(\mathbb{P}^+V, g)$ is called **hyperbolic space**, and $SO^+(V)$ **the group of oriented hyperbolic isometries**.

Theorem 2.10 *Let $n > 0$, and $\alpha\in SO^+(1,n)$ be an isometry acting on V. Then one and only one of the following three cases occurs*

(i) *α has an eigenvector x with $q(x) > 0$ (α is* **"elliptic isometry"***);*
(ii) *α has two eigenvectors x and y such that $q(x,x) = q(y,y) = 0$ and real eigenvalues λ_x and $\lambda_y = \lambda_x^{-1}$ satisfying $|\lambda_x| > 1$ and all other eigenvalues have absolute value one (α is* **"hyperbolic isometry"***, or* **loxodromic isometry***);*
(iii) *α has a unique (up to a constant) eigenvector x with $q(x,x) = 0$ with eigenvalue 1, and no fixed points on $\mathbb{P}^+V$ (α is* **"parabolic isometry"***).*

For a proof see for instance [34] or [25, Chapter 5].

Remark 2.11 All eigenvalues of elliptic and parabolic isometries have absolute value 1. Hyperbolic and elliptic isometries are semisimple (that is, diagonalizable over $\mathbb{C}$).

Definition 2.12 Notice that any complex automorphism of a hyperkähler manifold acts by isometry on the space $H^{1,1}(M,\mathbb{R})$ with the BBF metric which has signature $(1, b_2-3)$. A complex automorphism f of a hyperkähler manifold M is called **elliptic (parabolic, hyperbolic)** if the induced action f^* of f is elliptic (parabolic, hyperbolic) on $H^{1,1}(M,\mathbb{R})$.

Further on we shall need the following lemma.

Lemma 2.13 *Let M be a hyperkähler manifold, $f : M \longrightarrow M$ a hyperbolic automorphism, and $\eta \in H^{1,1}(M,\mathbb{R})$ a non-zero f^*-invariant class. Then $q(\eta,\eta) < 0$.*

Proof Let v_+, v_- be eigenvectors of f^* with the real eigenvalues $\lambda > 1$ and λ^{-1}. Then any invariant vector of f^* belongs to $\langle v_+, v_-\rangle^\perp$. However, q is negative definite on the space spanned by the other eigenvectors because signature of q on $H^{1,1}(M,\mathbb{R})$ is $(1, b_2-3)$. □

3 Toric suspensions

3.1 Toric suspensions: definition and basic properties

Definition 3.1 Let M be a complex manifold, and $f_1,\dots,f_{2k} \in \mathrm{Aut}(M)$ a set of commuting holomorphic automorphisms of M. Let $\Lambda \subset \mathbb{C}^k$ be a lattice of rank $2k$, generated by $\xi_1,\dots,\xi_{2k}$. Define an action of $\mathbb{Z}^{2k} = \langle\varphi_1,\dots,\varphi_{2k}\rangle$ on $M \times \mathbb{C}^k$ via $\varphi_j(m,z) = (f_j(m), z+\xi_j)$. In other words, $\mathbb{Z}^{2k}$ acts on $\mathbb{C}^k$ as a shift by the corresponding element of Λ and on M as an automorphism obtained as an appropriate product of f_i. The quotient $(M\times\mathbb{C}^k)/\mathbb{Z}^{2k}$ is called **the toric suspension** of $(M, f_1,\dots,f_{2k})$.

Remark 3.2 The toric suspension is clearly complex analytic, holomorphically fibered over the torus $\mathbb{C}^k/\Lambda$, but not necessarily Kähler.

Theorem 3.3 *Let $S(M, f_1,\dots,f_{2k})$ be a toric suspension, with M a compact Kähler manifold. Then $S(M, f_1,\dots,f_{2k})$ is Kähler if and only if there is a Kähler class $[\omega] \in H^{1,1}(M)$ such that $f_i^*([\omega]) = [\omega]$.*

Proof See the proof of Theorem 3.4.1 in the paper [38]. □

3.2 Hyperbolic suspensions

The following definition is motivated by the classification of the automorphism groups of hyperbolic manifolds, such as a K3 surface.

Definition 3.4 Let $f : M \longrightarrow M$ be an automorphism of a compact complex manifold of Kähler type (i.e. admitting a Kähler metric). We say that f is **a hyperbolic automorphism** if the induced action of f on $H^{1,1}(M,\mathbb{R})$ has a unique (up to a constant) eigenvector η with eigenvalue $\lambda > 1$.

We list some immediate properties of hyperbolic automorphisms.

Proposition 3.5 *Let $f : M \longrightarrow M$ be a hyperbolic automorphism of a compact complex manifold of Kähler type, and $\eta \in H^{1,1}(M)$ an eigenvector with an eigenvalue $f^*\eta = \lambda\eta$ such that $\lambda > 1$. Denote by $\mathrm{Kah}(M) \subset H^{1,1}(M, \mathbb{R})$ the Kähler cone of M. Then*

(i) *η belongs to the closure of the Kähler cone.*
(ii) *$\int_M \eta^n = 0$, where $n = \dim_{\mathbb{C}} M$. In particular, $\eta \notin \mathrm{Kah}(M)$.*
(iii) *The action of f on $\mathrm{Kah}(M)$ has no fixed points.*

Proof Let $S \subset H^{1,1}(M, \mathbb{R})$ be the sum of all eigenspaces of f on $H^{1,1}(M, \mathbb{R})$ with eigenvalues different from λ. Since λ is the biggest eigenvalue, for any $v \in H^{1,1}(M, \mathbb{R})\backslash S$, one has $\lim_i \frac{(f^*)^i(v)}{\lambda^i} = c\eta$ where c is a nonzero constant. Since $\mathrm{Kah}(M)$ is open, this is also true for general Kähler class ω. We obtained η as a limit of Kähler forms. This proves (i).

To see (ii), we notice that $\int_M \eta^n = \int_M f^*(\eta)^n = \lambda^n \int_M \eta^n$.

To obtain (iii), assume that f fixes a Kähler class ω on M. Then f is an elliptic isometry on $H^{1,1}(M, \mathbb{R})$, but by Theorem 2.10 f can not be hyperbolic, giving a contradiction. □

Remark 3.6 Since a hyperbolic automorphism of a hyperkähler manifold preserves its Kähler cone, and the eigenvector x with $|\lambda_x| > 1$ sits on the boundary of the Kähler cone (Proposition 3.5), the number λ_x is positive.

Definition 3.7 Let $f : M \longrightarrow M$ be an automorphism of a compact complex manifold of Kähler type, and $T^2 = \mathbb{C}/\mathbb{Z}^2$ an elliptic curve. Consider a toric suspension $S(f)$ of M associated with the pair (f, Id_M). This manifold is called **a complex suspension of** f. We call $S(f)$ **a hyperbolic suspension** if f is hyperbolic.

Remark 3.8 The toric suspension $S(f)$ of M associated with the pair (f, Id_M) can be viewed as the product manifold $M_f \times S^1$, where M_f is the mapping torus of M by f obtained as the quotient of $M \times \mathbb{R}$ by the $\mathbb{Z}$-action

$$(p, t) \to (f^{-n}(p), t + n).$$

If (t, s) are local coordinates on $\mathbb{R} \times S^1$, then $\frac{\partial}{\partial t}$ on $M \times \mathbb{R}$ defines a vector field X_f on $S(f)$ called the **suspension vector field** of f (see [26]). Note that the vector field $X_f - i\frac{\partial}{\partial s}$ on $S(f)$ is holomorphic. Moreover the vector fields X_f, $\frac{\partial}{\partial s}$ provide a natural splitting $TS(f) = T_{vert}S(f) \oplus \pi^*TE$, which defines a flat Ehresmann connection on $S(f)$, which we call the **standard connection**. We will denote by θ the associated connection 1-form such $\theta + \sqrt{-1}\, ds$ is a $(1, 0)$-form with respect to the complex structure on $S(f)$.

Remark 3.9 By Proposition 3.5 (iii) and Theorem 3.3, a hyperbolic suspension is never Kähler.

4 Balanced metrics on Calabi–Yau suspensions

Balanced metrics were introduced in [39]. For further properties and examples see e.g. [1, 2] and [23].

Definition 4.1 Let (M, I, h) be a complex Hermitian manifold, $\dim_{\mathbb{C}} M = n$, and ω the fundamental $(1, 1)$-form associated to h. We say that h is **balanced** if ω^{n-1} is closed.

The main result of the present Section is the following theorem.

Theorem 4.2 *Let M be a balanced compact manifold of complex dimension n and $f_1, f_2 \in Aut(M)$ two commuting holomorphic automorphisms preserving a volume form V. Denote by $\pi : S \to E$ the corresponding suspension over an elliptic curve E. Assume that M is balanced. Then S is also balanced.*

Proof Let ω_E be a Kähler form on E. Recall that a smooth fibration $\pi : S \to E$ over an elliptic curve is called **essential** [39] if $\pi^*(\omega_E)$ is not Aeppli exact, i.e. $\pi^*(\omega_E)$ cannot be equal to $\overline{\partial}\alpha + \partial\overline{\alpha}$, for any $(1, 0)$-form α. Michelsohn [39] proves that the total space S of an essential fibration with balanced fibers over a complex curve is balanced. To prove Theorem 4.2 it remains only to show that $\pi^*(\omega_E)$ is not Aeppli exact.

Since V is f_j-invariant, $j = 1, 2$, we may extend V to a form V_h on S vanishing on horizontal vector fields of this Ehresmann connection. Then the form V_h is of type (n, n), positive and closed. Since V_h vanishes on any horizontal vector, the form $\pi^*(\omega_E) \wedge V_h$ is of maximal degree and positive, so $\int_S \pi^*\omega_E \wedge V_h > 0$. To prove the theorem by contradiction assume that $\pi^*(\omega_E)$ is Aeppli exact. However by Stokes Theorem we would have $\int_S \pi^*\omega_E \wedge V_h = 0$, which is impossible. □

5 Hyperbolic holomorphically symplectic suspensions

5.1 Hyperbolic holomorphically symplectic suspensions

Definition 5.1 Let M be a hyperkähler manifold and $f : M \longrightarrow M$ a hyperbolic automorphism (as in Definition 3.4) preserving the holomorphic symplectic form. Denote by S the corresponding hyperbolic suspension, fibered over T^2 with fiber M. Then S is called **a hyperbolic holomorphically symplectic suspension**.

Similarly, if M is a Calabi–Yau manifold and f is a hyperbolic automorphism of M preserving the complex holomorphic volume form, we will call $S(f)$ a **Calabi–Yau hyperbolic suspension**.

Proposition 5.2 *Let S be a hyperbolic holomorphically symplectic suspension or a Calabi–Yau hyperbolic suspension. Then S is balanced and non-Kähler Calabi–Yau.*

Proof In both cases there exists an invariant non-vanishing holomorphic section Θ of the canonical bundle of M. Therefore, $V := \Theta \wedge \overline{\Theta}$ is a f-invariant volume on

M. By Theorem 4.2 S is balanced. By Remark 3.9 S is non-Kähler. Moreover, the form $(\theta + \sqrt{-1}\, ds) \wedge \Theta$ (Remark 3.8) is a non-vanishing holomorphic section of the canonical bundle of S. □

5.2 Balanced, pluriclosed and LCK Hermitian metrics

The study of special Hermitian metrics posed also the question of compatibility between different structures of non-Kähler type. We recall the conjecture in [22] according to which a compact complex manifold admitting both a pluriclosed, i.e. whose Hermitian form ω satisfies $dd^c\omega = 0$, and a balanced metric is Kähler. This has been already proven for specific cases in the papers [10, 18, 20, 22–24, 41, 46]. A similar question was posed in [44] (see also [14]) for a compact complex manifold of complex dimension n admitting a balanced metric and an astheno-Kähler metric, i.e. whose Hermitian form satisfies $dd^c\omega^{n-2} = 0$. A negative answer to this question was given in [17, 35]. For conjectures related to the existence of locally conformally Kähler metrics - the ones that satisfies $d\omega = \theta \wedge \omega$, see the book [40].

Based on the previous discussion one can formulate the following general conjecture:

Conjecture 5.3 Let X be a compact complex manifold, $n := \dim_{\mathbb{C}} X > 2$. Assume that two of the following assumptions occur.

(i) X admits a Hermitian form ω which is locally conformally Kähler, that is, satisfies $d\omega = \theta \wedge \omega$.
(ii) X admits a Hermitian form ω which is balanced.
(iii) X admits a Hermitian form ω which is p-pluriclosed, that is, satisfies $dd^c(\omega^p) = 0$, for $p = 1, 2, \ldots, n-3$ if $n > 3$ or for $p = 1$ if $n = 3$.

Then X admits a Kähler structure.

In this section, we prove this conjecture when X is a suspension of a hyperkähler manifold M associated with a hyperbolic automorphism of M. The non-existence of locally, but not globally, conformally Kähler metric on these examples follows from Proposition 37.8 in [40].

5.3 Strongly positive and weakly positive (p, p)-currents

Here we recall that a (p, p)-*current* on a complex manifold X is an element of the Frechet space dual to the space of $(n-p, n-p)$ complex forms $\Lambda^{n-p,n-p}(X)$. In the compact case, the space of (p, p)-currents can be identified with the space of (p, p)-forms with distribution coefficients and the duality is given by integration. So for any (p, p)-current T and a form α of type $(n-p, n-p)$ we have

$$\langle T, \alpha \rangle = \int_X T \wedge \alpha.$$

The operators d and d^c can be extended to (p, p)-currents by using the duality induced by the integration, i.e., dT and d^cT are respectively defined via the relations

$$\langle dT, \beta\rangle = -\int_X T \wedge d\beta, \quad \langle d^cT, \beta\rangle = -\int_X T \wedge d^c\beta.$$

We recall now the definition of a positive (p, p)-form (see e.g. [12, Chapter 3]).

Definition 5.4 A **weakly positive** (**strictly weakly positive**) (p, p)-form on a complex manifold X is a real (p, p)-form η such that for any complex subspace $V \subset TM$, $\dim_C V = p$, the restriction $\eta\mid_V$ is a non-negative volume form (positive volume form). Weakly positive condition is equivalent to

$$i^p \eta(v_1, \overline{v}_1, v_2, \overline{v}_2, \ldots v_p, \overline{v}_p) \geqslant 0,$$

for every tangent vectors $v_1, \ldots, v_p \in T_x^{1,0}X$. A real (p, p)-form η is called **strongly positive** (**strictly strongly positive**) if it can be locally expressed as a sum

$$\eta = i^p \sum_{j_1,\ldots,j_p} \eta_{j_1\ldots j_p} \xi_{j_1} \wedge \overline{\xi}_{j_1} \wedge \ldots \wedge \xi_{j_p} \wedge \xi_{\overline{j}_p},$$

running over the set of p-tuples $\xi_{j_1}, \xi_{j_2}, \ldots, \xi_{j_p}$ of $(1, 0)$-forms, with $\eta_{j_1\ldots j_p} \geqslant 0$ $(\eta_{j_1\ldots j_p} > 0)$.

All strongly positive forms are also weakly positive. The strongly positive and the weakly positive forms form closed, convex cones in the space of real (p, p)-forms, see for instance [12, Chapter 3]. These two cones are dual with respect to the natural pairing

$$\Lambda_x^{p,p}(X, \mathbb{R}) \times \Lambda_x^{n-p,n-p}(X, \mathbb{R}) \to \mathbb{R}.$$

For $(1, 1)$-forms and $(n-1, n-1)$-forms, the strong positivity is equivalent to weak positivity. Finally, a product of a weakly positive form and a strongly positive one is always weakly positive (however, a product of two weakly positive forms may be not weakly positive). A product of strongly positive forms is still strongly positive.

A strongly/weakly positive (p, p)-current is a current taking non-negative values on weakly/strongly positive compactly supported $(n-p, n-p)$-forms.

Definition 5.5 A (p, p)-current T is called **weakly positive** if

$$i^{n-p} \int_X T \wedge \alpha_1 \wedge \overline{\alpha_1} \wedge \ldots \alpha_{n-p} \wedge \overline{\alpha}_{n-p} \geqslant 0,$$

for every (1,0)-forms $\alpha_1, \ldots \alpha_{n-p}$ with inequality being strict for at least one choice of α_i's. The current T is called **strongly positive** if the inequality is strict for every non-zero $\alpha_1 \wedge \overline{\alpha_1} \wedge \ldots \alpha_{n-p} \wedge \overline{\alpha}_{n-p}$.

Definition 5.6 A (p, p)-current T is called **strictly strongly positive** (resp. **strictly weakly positive**) if $T > \varepsilon\omega$ for a strictly strongly positive (resp. strictly weakly positive) (p, p)-form ω and a positive number ε.

Claim 5.7 The cone of strongly positive (p, p)-currents is dual to the cone of strictly weakly positive (p, p)-forms, the cone of weakly positive (p, p)-currents is dual to the cone of strictly strongly positive (p, p)-forms.

The main result of this section is the following

Theorem 5.8 *Let $f \in \mathrm{Aut}(M)$ be a hyperbolic automorphism of a hyperkähler manifold, and denote by $\pi : S \to E$ the suspension $S(f)$ of (M, f). Then S admits a dd^c-exact, strongly positive (p, p)-current β for any $p = 2, 3, \dots, n-1$, where $n := \dim_{\mathbb{C}} M$.*

We prove this theorem in Sect. 5.4. Theorem 5.8 immediately implies the following.

Corollary 5.9 *Let S be a hyperbolic suspension over a hyperkähler manifold M, as in Theorem 5.8. Then S does not admit a dd^c-closed strictly weakly positive $(n-p+1, n-p+1)$-form U for $p = 2, 3, \dots, n-1$. In particular, S is not k-pluriclosed for any $k = 1, 2, \dots, n-1$.*

Proof Let $\beta = dd^c\alpha$ be a current introduced in Theorem 5.8. If U is dd^c-closed strictly weakly positive $(n-p+1, n-p+1)$-form U, we have $0 < \int_M U \wedge \beta = \int_M dd^c U \wedge \alpha = 0$, which is impossible. ◻

5.4 Hyperbolic automorphisms and Cantat–Dingh–Sibony currents

Let f be a hyperbolic automorphism of a hyperkähler manifold M, $\dim_{\mathbb{C}} M = n$, and $p = 1, 2, \dots, n-1$, and denote by λ its unique eigenvalue which satisfies $|\lambda| > 1$.

Recall that **the mass** of a positive (p, p)-current v on a Kähler manifold M is $\int_M v \wedge \omega^{n-p}$. Since f preserves the Kähler cone, it preserves the positive cone of M, hence $\lambda > 1$. The action of f^* on $H^{2p}(M)$ has λ^p as the maximal eigenvalue [8], hence the mass of $\frac{1}{\lambda^{pk}}(f^*)^k\omega^p$ is bounded. Moreover, the set of positive (p, p)-currents of bounded mass is compact [12, Chapter 3].

Therefore the sequence $\{\frac{1}{\lambda^{pk}}(f^*)^k\omega^p\}_{k=1,\dots,\infty}$ has a limit point. The eigenspace corresponding to λ^p in $H^{p,p}(M)$ has multiplicity 1, as shown in [8]. By [13, Theorem 4.3.1], the limit of a subsequence $\lim_k \frac{1}{\lambda^{pk}}(f^*)^k\omega^p$ is a unique positive (p, p)-current σ which satisfies $f^*\sigma = \lambda\sigma$. We call it **the Cantat–Dingh–Sibony current** (Cantat prove this result for (1,1)-currents on a K3 surface [9], and Dingh-Sibony for all dimensions).

Using the decomposition $TS = T_{vert}S \oplus \pi^*TE$ induced by the flat Ehresmann connection on S, we can consider the bundle $\mathbb{D} := D_\pi^{p,p}(S)$ of fiberwise currents as a local system on E; the monodromy of this local system is given by the map $v \mapsto f^*v$. Identifying local systems and flat bundles, we can consider $\mathbb{D}$ as a bundle with flat connection.

Consider a real line bundle $L \subset \mathbb{D}$ spanned by the Cantat–Dingh–Sibony current σ. This line bundle is preserved by the natural flat connection on $\mathbb{D}$, and its monodromy map is multiplication by λ. Choose a trivialization of L such that the corresponding connection 1-form θ satisfies $d\theta = 0$ and $d(I\theta) = 0$. Using the decomposition $TS = T_{vert}S \oplus \pi^* TE$, we can embed the sections of $\mathbb{D}$ into the space of (p, p)-currents on S. Let α be the current on S associated with the section of $L \subset \mathbb{D}$ constructed above. Since σ is a limit of closed currents, $d\sigma = 0$ and we have $d\alpha = \alpha \wedge \theta$, and $\beta := dd^c\alpha = \alpha \wedge \theta \wedge I\theta$. The current β is dd^c-exact. Since β is a limit of the wedge power of strongly positive (1,1)-forms, it is strongly positive. This proves Theorem 5.8.

6 Examples of suspensions of hyperkähler manifolds

We briefly recall the examples of suspensions of Kummer K3 surfaces from [42] first.

Take the complex 2-torus $\mathbb{T}$ given by the quotient of $\mathbb{C}^2$ by the standard lattice generated by the unit vectors $(1, 0)$, $(i, 0)$, $(0, 1)$, $(0, i)$. Consider the involution of $\mathbb{C}^2$ given by multiplication by -1, i.e. $(z_1, z_2) \to (-z_1, -z_2)$. The involution descends to an involution σ of the torus $\mathbb{T}$ with 16 fixed points $p_1, \dots, p_{16}$. The quotient space $\mathbb{T}/\langle 1, \sigma\rangle$ has 16 double points. The singularities can be resolved by blowing the singularities up, yieldings a smooth compact surface containing 16 mutually disjoint smooth rational curves C_j. This is the Kummer surface Km associated to $\mathbb{T}$. There is an alternative description of the Kummer surface. Let X denote the surface obtaining by blowing up $\mathbb{T}$ at each of the points $p_1, \dots, p_{16}$. Let $E_j \cong \mathbb{P}^1$ be the exceptional divisor over p_j. The involution σ of $\mathbb{T}$ lifts to an involution τ of X with the fixed set $E = E_1 \cup \dots \cup E_{16}$. The eingevalues of the differential of τ at every points of E are ± 1. So the quotient $X/\langle 1, \tau\rangle$ is smooth and contains 16 rational (-2)-curves $C_j \cong \mathbb{P}^1$, the images of the rational (-1)-curves E_j in X. The quotient is a Kummer surface Km. Let $\hat{\mathbb{C}}^2$ be the surface obtained by blowing up $\mathbb{C}^2$ at every point of the discrete set $\pi^{-1}(\{p_1, \dots, p_{16}\})$, where $\pi : \mathbb{C}^2 \to \mathbb{T}$ is the quotient map, we have the following diagram

$$\begin{array}{ccccc} \hat{\mathbb{C}}^2 & \longrightarrow & X & \longrightarrow & \mathrm{Km} \\ \downarrow & & \downarrow & & \downarrow \tilde{\pi} \\ \mathbb{C}^2 & \xrightarrow{\pi} & \mathbb{T} & \longrightarrow & \mathbb{T}/\langle 1, \sigma\rangle \end{array}$$

By the Lefschetz Theorem on $(1, 1)$-forms we have that the Picard group of Km is isomorphic to $H^2(\mathrm{Km}, \mathbb{Z}) \cap H^{1,1}(\mathrm{Km})$, so the rank of the Picard group of Km is 20. Moreover, the Picard group of Km is generated by the 16 exceptional divisors E_i and by the pull-back by $\tilde{\pi}$ of divisors on $\mathbb{T}/\langle 1, \sigma\rangle$.

The canonical $(2, 0)$-form $dz_1 \wedge dz_2$ on $\mathbb{C}^2$ induces a nowhere vanishing $(2, 0)$-form on $\mathbb{T}$. Therefore, the pullback of this form on X induces a holomorphic $(2, 0)$-form on the Kummer surface.

Let $A \in SL(2, \mathbb{Z}+\sqrt{-1}\mathbb{Z})$ be a matrix with $|tr(A)| > 2$, so that it is diagonalizable with eigenvalues λ, λ^{-1}. Let dv_1, dv_2 be respectively the associated eigenvectors of the induced map on $H^1(\mathbb{T}, \mathbb{C}) \cong \Lambda^1(\mathbb{R}^4)$. Denote by A also the induced map on $\Lambda^k(\mathbb{R}^4)$. Then A preserves the holomorphic (2,0)-form $dv_1 \wedge dv_2$ on $\mathbb{T}$ and the divisor

$D = \sum_{i=1}^{16} E_i$. So it defines a holomorphic transformation φ_A on Km preserving the induced holomorphic (2, 0)-form. In particular the $\mathbb{Z}$-action on $\mathbb{T} \times \mathbb{R} \times S^1$ generated by

$$f : (p, x, y) \to (A(p), x + 1, y) \tag{6.1}$$

extends to an action on $\mathrm{Km} \times \mathbb{R} \times S^1$, generated by the hyperbolic automorphism f. The quotient is a compact complex manifold $S(\mathrm{Km}, \mathrm{f})$ with trivial canonical bundle and satisfying the hard Lefschetz property, such that its real homotopy type is formal as shown in [42].

In a similar way we can construct hyperbolic automorphisms preserving the holomorphic symplectic form on higher-dimensional hyperkähler manifolds arising as Hilbert scheme of points on Km. More precisely, $f : \mathrm{Km} \to \mathrm{Km}$ extends to $f^{[n]} : \mathrm{Km}^{[n]} \to \mathrm{Km}^{[n]}$ on the Hilbert scheme of order n of Km in a natural way: to a zero-dimensional subscheme $Z \subset \mathrm{Km}$ we assign $f(Z)$. According to Beauville (and PhD thesis by P. Beri [5]), $f^{[n]}$ preserves the holomorphic symplectic form if and only if f does. Now we can construct the suspension $S(\mathrm{Km}^{[n]}, f^{[n]})$ using $f^{[n]}$ and obtain:

Theorem 6.1 *The space $S(\mathrm{Km}^{[n]}, f^{[n]})$ for $n \geqslant 1$ is a non-Kähler compact complex manifold with trivial canonical bundle which admits a balanced metric and it is not k-pluriclosed for any $k = 1, 2, \ldots, 2n - 1$.*

Proof The fact that it is balanced follows from Proposition 5.2 and Corollary 5.9. □

The metric in the examples above is not explicit. But if we consider the suspension over the real 4-torus $\mathbb{T}^4$ we can define such metric explicitly. Denote by v_1 and v_2 the eigenforms of the map A on $H^1(\mathbb{T}^4, \mathbb{C})$ induced by the matrix A as above and by x and y respectively coordinates on $\mathbb{R}$ and S^1. Then $A(dv_1 \wedge \overline{dv_1}) = |\lambda|^2 dv_1 \wedge \overline{dv_1}$ and $A(dv_2 \wedge \overline{dv_2}) = |\lambda|^{-2} dv_2 \wedge \overline{dv_2}$. Consider the differential forms on $\mathbb{T}^4 \times \mathbb{R} \times S^1$ given by $\alpha_1 = |\lambda|^{-2x} dv_1 \wedge d\overline{v}_1$ and $\alpha_2 = |\lambda|^{2x} dv_2 \wedge d\overline{v}_2$. The forms α_1 and α_2 are invariant under the action in (6.1). Moreover $\alpha_1 + \alpha_2$ descends to a weakly positive definite (2, 2)-form on the suspension $S(\mathbb{T}^4, f)$ of the 4-torus defined by this action. By the observation of Michelson [39] $S(\mathbb{T}^4, f)$ admits a balanced metric. We can directly check that

$$\omega = |\lambda|^{-2x} dv_1 \wedge d\overline{v_1} + |\lambda|^{2x} dv_2 \wedge d\overline{v_2} + dx \wedge dy$$

is invariant and satisfies $d\omega^2 = 0$. Hence it defines a balanced metric.

Remark 6.2 We restrict ourselves here to the more explicitly described examples, but many of the known compact hyperkähler manifolds admit hyperbolic automorphisms. We expect that the topological properties of $S(\mathrm{Km}, \mathrm{f})$ from [42] are also valid for $S(\mathrm{Km}^{[n]}, f^{[n]})$. Note that the manifold $A(\mathbb{T}^4)$ can be also described as the almost abelian solvmanifold $M^6(c)$ in [11] (see also Section 3 in [15]). By Theorem 4.1 in [18] the associated almost abelian Lie algebra, which is isomorphic to $\mathfrak{b}_6$ in the notation of [11], admits a balanced metric.

7 Holomorphically symplectic and hypercomplex structures on toric suspensions with 4-dimensional base

In this section we show how the toric suspensions could be used to construct examples of compact holomorphic symplectic and hypercomplex non-Kähler manifolds. The examples are in fact pseudo-hyperkähler. We also discuss their metric structure.

7.1 General construction and example

We consider a toric suspension of M, where M is a real 8-torus, over a real 4-torus base. More precisely we consider $S = S(T^8, f, id, id, id)$, where f is a diffeomorphism of T^8 defined by a matrix $A \in SL(8, \mathbb{R})$. We choose A as in the following Lemma:

Lemma 7.1 *Consider on $\mathbb{R}^8$ the hypercomplex structure (I, J, K) defined, in terms of the standard basis $(e_1, \ldots, e_8)$, by*

$$\begin{aligned} &Ie_1 = e_3, \quad Ie_2 = e_4, \quad Ie_5 = -e_7,\ Ie_6 = -e_8, \\ &Je_1 = -e_5,\ Je_2 = -e_6,\ Je_3 = -e_7\ \ Je_4 = -e_8, \end{aligned}$$

and the pseudo hyperhermitian metric

$$h = (e^1)^2 - (e^2)^2 + (e^3)^2 - (e^4)^2 + (e^5)^2 - (e^6)^2 + (e^7)^2 - (e^8)^2.$$

Then there exists an integer matrix

$$A = \begin{pmatrix} 1 & 0 & 1 & 0 & -1 & -1 & 0 & 1 \\ 0 & -1 & 0 & -1 & -1 & 0 & 1 & 1 \\ -1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & -1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & -1 & 1 & 0 & 1 & 0 \\ 1 & 0 & -1 & -1 & 0 & -1 & 0 & -1 \\ 0 & -1 & -1 & -1 & -1 & 0 & 1 & 0 \\ -1 & -1 & -1 & 0 & 0 & 1 & 0 & -1 \end{pmatrix} \in SL(8, \mathbb{Z})$$

preserving the pseudo hyperhermitian structure (I, J, K, h) and such $A^k \neq Id$, for every $k > 1$.

Proof Since A commutes with the matrices associated to I and J with respect to the standard basis $(e_1, \ldots, e_8)$ and $A^t H A = H$, where H is the matrix associated to h, we have that A preserves the pseudo hyperhermitian structure (I, J, K, h). Moreover, $A^k \neq Id$, for every $k > 1$, since A has eigenvalues $\pm i(1 + \sqrt{2})$ and $\pm i(\sqrt{2} - 1)$ of multiplicity two. □

Recall that an indefinite or pseudo hyperhermitian metric is called **pseudo-hyperkähler** if the corresponding fundamental forms are closed. In particular, the Lemma claims that A preserves the pseudo-hyperkähler structure on $\mathbb{R}^8$ and the lattice generated by $e_1, .., e_8$. Such a matrix defines a hyperbolic diffeomorphism f_A of

T^8 which preserves the pseudo-hyperkahler structure and in particular both the hypercomplex structure and the holomorphic symplectic form. We note that any such matrix which preserves a positive-definite metric, has all eigenvalues on the unit circle so cannot be hyperbolic. To formulate the next Theorem, recall that for every hypercomplex manifold there exists a unique torsion free connection preserving the hypercomplex structure, which is called the **Obata connection**.

Theorem 7.2 *Let f_A be the diffeomorphism of T^8 defined by the matrix A and $S = S(T^8, f_A, id, id, id)$ the hyperbolic toric suspension, where T^8 denotes the 8-dimensional real torus obtained as a quotient of $\mathbb{R}^8$ by the standard lattice. Then S admits a pseudo-hyperkähler structure (I, J, K, h). In particular for the complex structure I, S carries both hypercomplex and holomorphic symplectic structures, but no Kähler metrics. Moreover, the Obata connection of the hypercomplex structure is flat.*

Proof Since f_A preserves both the standard hypercomplex and holomorphic symplectic structures on $\mathbb{R}^8$, the previous structures descend to T^8. Then the action $(p, t) \to (f_A^{-n}(p), t + n)$ of $\mathbb{Z}$ on $T^8 \times \mathbb{R}^4$ preserves the induced natural structures obtained as product of the ones on T^8 and the canonical hyperkähler structure on $\mathbb{R}^4$. Moreover, the hypercomplex structure on $T^8 \times \mathbb{R}^4$ is clearly compatible with flat Obata connection. As a consequence all structures descend to the quotient S. The fact that S is non-Kähler follows from Theorem 3.3, because f_A does not preserve any Kähler class. □

Remark 7.3 Note that pseudo-hyperkähler structures on 12-dimensional compact solvmanifolds are constructed in [47] and are associated to the almost abelian Lie algebras

$$\Psi_I(\Psi_J(\mathfrak{g})) = \mathrm{span}_{\mathbb{R}}\{U_1^1, U_1^2, U_1^3, U_1^4, V_1^1, V_2^1, V_1^2, V_2^2, V_1^3, V_2^3, V_1^4, V_2^4\}$$

with Lie bracket

$$[U_1^1, V_j^h] = c_{1j}^1 V_1^h + c_{1j}^2 V_2^h, \quad j = 1, 2, \;\; h = 1, 2, 3, 4,$$

and hypercomplex structure (I, J, K) defined by

$$\begin{array}{l} IU_1^1 = U_1^2,\; IV_1^1 = V_1^2,\; IV_2^1 = V_2^2,\; IU_1^4 = U_1^3,\; IV_1^4 = V_1^3,\; IV_2^4 = V_2^3, \\ JU_1^1 = U_1^3,\; JV_1^1 = V_1^3,\; JV_2^1 = V_2^3,\; JU_1^2 = U_1^4,\; IV_1^2 = V_1^4,\; IV_2^2 = V_2^4. \end{array}$$

So in particular the associated solvable Lie group is a semidirect product of the form $(\mathbb{R} \ltimes \mathbb{R}^8) \times \mathbb{R}^3$. In the notation of [47] $\mathfrak{g} = \mathfrak{a} \ltimes \mathfrak{b}$, with $\mathfrak{a} = \mathrm{span}_{\mathbb{R}}\{U_1^1\}$ and $\mathfrak{b} = \mathrm{span}_{\mathbb{R}}\{V_1^1, V_2^1\}$ and by Theorem 6.6 in [47] if $\mathfrak{b}$ has a non-degenerate 2-form which is closed on $\mathfrak{g}$, then $(\Psi_I(\Psi_J(\mathfrak{g})), I, J, K)$ admits a compatible pseudo-hyperkähler structure. The previous condition is satisfied if $c_{11}^1 = -c_{12}^2$. The hyperbolic toric suspension $S = S(T^8, f_A, id, id, id)$ corresponds to the compact solvmanifold constructed as a quotient of the solvable Lie group H whose Lie algebra is $\mathfrak{h} := \Psi_I(\Psi_J(\mathfrak{g}))$

with $ad_{U_1^1} = \mathrm{diag}(1, -1, 1, -1, 1, -1, 1, -1)$. If we consider the basis

$$\begin{array}{l} f_1 = V_1^1,\ f_2 = V_2^1\ \ f_3 = V_1^2,\ f_4 = V_2^2,\ \ f_5 = V_1^3,\ \ f_6 = V_2^3, \\ f_7 = V_1^4,\ f_8 = V_2^4,\ f_9 = U_1^1,\ f_{10} = U_1^2,\ f_{11} = U_1^3,\ f_{12} = U_1^4, \end{array}$$

the structure equations of $\mathfrak{h}$ are

$$\begin{array}{l} df^i = f^i \wedge f^9,\ i = 1, 3, 5, 7, \quad df^j = -f^j \wedge f^9,\ j = 2, 4, 6, 7, \\ df^k = 0, k = 9, \dots 12. \end{array} \tag{7.1}$$

The pseudo-hyperkähler structure on $\mathfrak{h}$ is given by $(I, J, K, \omega_I, \omega_J, \omega_K)$, where

$$\begin{array}{l} \omega_I = 2(-f^1 \wedge f^2 - f^3 \wedge f^4 + f^5 \wedge f^6 + f^7 \wedge f^8 + f^9 \wedge f^{10} - f^{11} \wedge f^{12}), \\ \omega_J = 2(f^1 \wedge f^8 + f^4 \wedge f^5 - f^2 \wedge f^7 - f^3 \wedge f^6 + f^9 \wedge f^{11} + f^{10} \wedge f^{12}), \\ \omega_K = 2(f^1 \wedge f^6 - f^4 \wedge f^7 - f^2 \wedge f^5 + f^3 \wedge f^8 - f^9 \wedge f^{12} + f^{10} \wedge f^{11}). \end{array}$$

With respect to the basis of (1, 0)-forms with respect to I

$$\begin{array}{l} \eta_1 = f^1 + i f^3,\ \eta_2 = f^2 + i f^4,\ \eta_3 = f^5 - i f^7, \\ \eta_4 = f^6 - i f^8,\ \eta_5 = f^9 + i f^{10},\ \eta_6 = f^{11} - f^{12} \end{array} \tag{7.2}$$

we have

$$J\eta_1 = \overline{\eta}_3,\ J\eta_2 = \overline{\eta}_4,\ J\eta_3 = -\overline{\eta}_1,\ J\eta_4 = -\overline{\eta}_2,\ J\eta_5 = \overline{\eta}_6,\ J\eta_6 = -\overline{\eta}_5 \tag{7.3}$$

and the associated (2, 0)-form $\omega_J + i\omega_K$ is given by

$$\omega_J + i\omega_K = 2(\eta_5 \wedge \eta_6 + i\eta_1 \wedge \eta_4 - i\eta_2 \wedge \eta_3).$$

Note that

$$J(\omega_J + i\omega_K) = 2(\overline{\eta}_5 \wedge \overline{\eta}_6 - i\overline{\eta}_1 \wedge \overline{\eta}_4 + i\overline{\eta}_2 \wedge \overline{\eta}_3)$$

and that the two (4, 0)-forms $\eta_1 \wedge \eta_3 \wedge \eta_5 \wedge \eta_6$ and $\eta_2 \wedge \eta_4 \wedge \eta_5 \wedge \eta_6$ are both ∂-exact.

7.2 Compatible metric structures

On a hypercomplex manifold (M, I, J, K) we always have a **hyper-Hermitian** (positive-definite) metric, that is a metric compatible with the complex structures I, J, K. When the fundamental forms $\omega_I, \omega_J, \omega_K$ are closed the metric is hyperkähler, but in the example constructed in Theorem 7.2 such metric doesn't exist. A generalization of hyperkähler condition is the condition $\partial\Omega = 0$, where $\Omega = \omega_J + i\omega_K$, in which case the metric is called **hyperkähler with torsion** (shortly HKT) [28]. In fact one can characterize the HKT condition in terms of Ω: if there is a (2, 0) form Ω with respect to I, such that $\partial\Omega = 0$, $\Omega(JX, JY) = -\overline{\Omega(JX, JY)}$, and $\Omega(X, J\overline{X}) > 0$ for every non-zero (1, 0) vector field X, then the metric defined by $g(X, Y) = Re\Omega(X, J\overline{Y})$ is HKT. The HKT metric is a good candidate for a quaternionic analog of Kähler metrics in complex geometry - it arises from a local quaternionic-subharmonic potential and gives rise to a Hodge theory (see [28] and [45]). The existence or non-existence of HKT metrics, in 8-dimensional case depends on purely holomorphic data (see [27]). In hypercomplex geometry the analog of the balanced condition is called **quaternionic balanced** (see [36]) and such metric satisfies $\partial(\Omega^{n-1}) = 0$, where $2n$ is the complex dimension of the manifold. We have the following:

Theorem 7.4 *The hypercomplex manifold* $S = S(T^8, f_A, id, id, id)$ *from Theorem 7.2 admits a quaternionic balanced metric, but admits no HKT metrics.*

Proof We consider the solvmanifold model of S from Remark 7.3. We use the same (1, 0)-forms η_i, $1 \leqslant i \leqslant 6$, and complex structure J as in (7.2) and (7.3). Note that the hypercomplex structure has Obata holonomy in $SL(n, \mathbb{H})$, since it has a closed and real (6, 0)-form. From the structure equations (7.1) we see that the (2, 0)-form form $\Omega = \eta_1 \wedge \eta_3 + \eta_2 \wedge \eta_4 + \eta_5 \wedge \eta_6$ satisfies the condition

$$\partial\Omega^2 = \partial(\eta_1 \wedge \eta_3 \wedge \eta_5 \wedge \eta_6 + \eta_2 \wedge \eta_4 \wedge \eta_5 \wedge \eta_6 + \eta_1 \wedge \eta_2 \wedge \eta_3 \wedge \eta_4) = 0,$$

but $\partial\Omega \neq 0$, so Ω defines a quaternionic-balanced metric. On the other side, by averaging argument (see [16]), if there is any HKT metric, then there exists an invariant one. Working by contradiction, we assume that there is a (2, 0)-form $\tilde{\Omega}$ with $\partial(\tilde{\Omega}) = 0$ which is J-anti-invariant and positive, so defines an HKT metric. Then $\tilde{\Omega}$ has the form

$$\tilde{\Omega} = \sum_{\alpha,\beta} a_{\alpha\overline{\beta}}\, \eta_\alpha \wedge J\overline{\eta_\beta},$$

where $a_{\alpha\overline{\beta}}$ is a Hermitian and positive definite matrix. Now we can adapt the Harvey–Lawson property for $SL(n, \mathbb{H})$ manifolds from [27] and use it explicitly. Since the (4, 0)-form $\alpha = \eta_1 \wedge \eta_3 \wedge \eta_5 \wedge \eta_6 + \eta_2 \wedge \eta_4 \wedge \eta_5 \wedge \eta_6$ is ∂-exact and the (6, 0)-form $\beta = \eta_1 \wedge \eta_2 \wedge \eta_3 \wedge \eta_4 \wedge \eta_5 \wedge \eta_6$ is closed, then we have

$$\int_M \tilde{\Omega} \wedge \alpha \wedge \overline{\beta} = 0$$

by integration by parts. On the other side $vol = \beta \wedge \overline{\beta}$ is a volume form and $\tilde{\Omega} \wedge \alpha \wedge \overline{\beta} = (a_{1\overline{1}} + a_{2\overline{2}})vol > 0$, so

$$\int_M \tilde{\Omega} \wedge \alpha \wedge \overline{\beta} > 0$$

and we get a contradiction. □

8 Pluriclosed metrics from suspensions

We recall the following

Definition 8.1 Let (M, I) be a complex manifold. We say that an I-Hermitian metric h is **pluriclosed** (or **SKT**) if its fundamental form is $\partial\overline{\partial}$-closed.

Using automorphisms of Kähler manifolds it is also possible to construct suspensions admitting pluriclosed metrics in the following way.

Let $(X^3, \xi, \eta, \varphi, \Phi)$ be a 3-dimensional Sasakian manifold and (Y^{2n}, I, g, ω) be a Kähler manifold of complex dimension n.

On the product $X^3 \times Y^{2n} \times \mathbb{R}$ we can define a complex structure $\tilde{I}$ such that $\tilde{I}(\xi) = \frac{\partial}{\partial_t}$, where t is the coordinate on $\mathbb{R}$ and $\tilde{I} = I$ on Y^{2n}. The 2-form

$$\tilde{\omega} = \eta \wedge dt + d\eta + \omega$$

is then a positive $(1, 1)$-form on $(X^3 \times Y^{2n} \times \mathbb{R}, \tilde{I})$. Since $d\eta = \Phi$, by a direct computation we obtain

$$d\tilde{\omega} = \Phi \wedge dt$$

and

$$dT^B = d(\tilde{I} d\tilde{\omega}) = -d(\Phi \wedge \eta) = 0,$$

where $T^B = \tilde{I} d\tilde{\omega}$ is the so-called **Bismut torsion form**. Note that $\Phi \wedge \eta$ is closed since it is a 3-form on X^3. Therefore we have the following

Theorem 8.2 *Let $(X^3, \xi, \eta, \varphi, \Phi)$ be a Sasakian 3-dimensional manifold, (Y^{2n}, I, g, ω) a Kähler manifold and $f = (f_1, f_2)$ a diffeomorphism of $X^3 \times Y^{2n}$ such that f_1 is a diffeomorphism of X^3 preserving the Sasakian structure $(\xi, \eta, \varphi, \Phi)$ and f_2 is a diffeomorphism of Y^{2n} preserving the Kähler structure (I, g, ω). Then the suspension of $X^3 \times Y^{2n}$ by f is non-Kähler and has a pluriclosed metric.*

To prove that the suspension of $(X^3, \xi, \eta, \varphi, \Phi)$ by f is non-Kähler we can use the Harvey–Lawson characterization of non-Kähler manifolds [31] and the observation that $d\eta = \Phi$ is a non-zero (weakly) positive and exact $(1, 1)$-form.

Example 8.3 An application of the previous construction gives the example of compact solvmanifold constructed in [19]. More precisely, let G be the simply connected 3-step solvable Lie group with structure equations

$$\begin{cases} de^1 = e^2 \wedge e^3, \\ de^2 = -e^2 \wedge e^8, \\ de^3 = e^3 \wedge e^8, \\ de^4 = b\, e^5 \wedge e^8, \\ de^5 = -b\, e^4 \wedge e^8, \\ de^6 = b\, e^7 \wedge e^8 \\ de^7 = -b\, e^6 \wedge e^8, \\ de^8 = 0, \end{cases}$$

with $b = \frac{2\pi}{\log(2+\sqrt{3})}$. By [19] G has the left-invariant complex structure

$$Ie_1 = -e_2,\ Ie_3 = e_8,\ Ie_4 = e_5,\ Ie_6 = e_7,$$

and admits a compact quotient by a lattice Γ. The I-Hermitian metric $g = \sum_{i=1}^{8}(e^i)^2$ is pluriclosed since the Bismut torsion 3-form $T^B = Id\omega$ is the closed 3-form $-e^1 \wedge e^2 \wedge e^3$. The compact solvmanifold $\Gamma\backslash G$ can be obtained as a suspension of the product of the 3-Sasakian manifold given by the compact quotient of the real 3-dimensional Heisenberg group by a lattice and the standard torus T^4. Moreover, the compact solvmanifold can be viewed also as the total space of a bundle over a circle with fibre a circle bundle over a 6-torus.

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