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Master Thesis

A Study in Three Dimensional Gravity and Chern-Simons Theory

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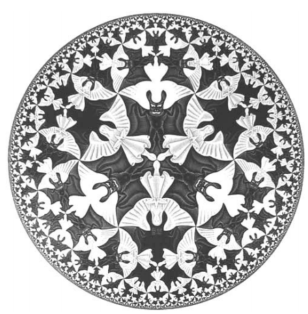

Abstract

First, we introduce some basic facts of manifolds with constant curvature. In particular, we are interested in the geometry of the Anti-de-Sitter spacetime in 2+1 dimensions. Such a spacetime is a solution to Einstein equations in vacuum with negative cosmological constant. Our proposal is that all physically acceptable AdS_3 geometries are completely determined by their spacial slices, once the boundary conditions are specified. Although this proposal seems to contradict the fact that the AdS is not global hyperbolic, we simply assume it is correct for our case of 3D gravity. By studying the classification of Möbius transformation groups acting as isometries on spacial slices of the global AdS_3, we can, in principle, exhaust all possible solutions to Einstein equations in vacuum with negative cosmological constant. In this thesis, however, we only focus on solutions whose spacial slices are quotients of Poincare disks modulo cyclic discrete subgroups of Möbius groups, which enable us to find their moduli spaces. One example of such a spacetime is the BTZ black hole in Lorentzian signature. Some attempts to visualize these geometries are made in this thesis. To determine the coupling constant of three dimensional gravity, we introduce an equivalent Chern-Simons formalism for the Einstein-Hilbert action. The gravitational coupling constant is then a dimensionless parameter, which is quantized for topological reasons. Preliminary materials about fiber bundles and Chern classes introduced in section 2.3 and section 2.4 pave the way for introducing the Chern-Simons formalism for our discussions. Finally, we try to investigate the dual CFT_2 of 3D gravity. We compute its partition function and provide a possible model of its CFT_2.

Acknowledgement

First and foremost, I would like to thank my thesis supervisor, professor Jan de Boer, who led me into this topic. He gave me much freedom of choosing directions in AdS_3/CFT_2. I am thankful to him for our more than forty meetings from which I have learned many important things that I could never learn from anywhere else. I thank him for giving me so many inspirations of physics and enlightening explanations on the Dirac's constraints which I got stuck for several times before our conversations. I thank him for spending so much time on my thesis and his enormous patience on my research project even when I did not have any progress in my second year. I also would like to thank him for writing me reference letters of a winter school on string theory in 2013 and a summer school on quantum gravity in 2012. Without his guidance and help, I would never know how to think of physics and I would never have the opportunity to attend the winter school and summer school. I want to thank him for spending much time reading my thesis carefully and correcting many mistakes.

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Contents

1 Introduction

In three dimensions the Riemann tensor can be expressed in terms of metric and Ricci tensor.

$$R_{\alpha\beta\gamma\delta} = g_{\alpha\gamma}R_{\beta\delta} + g_{\beta\delta}R_{\alpha\gamma} - g_{\beta\gamma}R_{\alpha\delta} - g_{\alpha\delta}R_{\beta\gamma} - \frac{1}{2}(g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma})R \tag{1}$$

which is simply a consequence of the fact that in three dimensions, we have a natural isomorphism $T_p^*(M) \cong \bigwedge^2 T_p^*(M)$ via the Hodge star duality. By solving the vacuum Einstein's equations,

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 0 \tag{2}$$

we see that the on-shell Riemann tensor can be written as multiples of metric solutions.

$$R_{\alpha\beta\gamma\delta} = \Lambda(g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma}) \tag{3}$$

In differential geometry, manifold satisfying this property are called space of constant curvature, which is defined as follows

Definition: Metric $g_{\mu\nu}$ is called constant curvature metric if there exist a constant K such that

$$R_{\alpha\beta\gamma\delta} = 2Kg_{\gamma[\alpha}g_{\beta]\delta} \tag{4}$$

The constant K is usually called the sectional curvature and one can easily check that it is proportional to scalar curvature

$$R = Kn(n-1) \tag{5}$$

A worth mentioning property of manifolds with constant curvature is that if two such manifolds M and N have the same dimensions, K value and the same signature, then they have the same local geometry [2]. Roughly speaking, two spacetimes (M, g) and (N, h) have the same local geometry if there is a local diffeomorphism ϕ whose pull-back satisfies $\phi^*(g) = h$. Thus, for spacetimes of constant curvature, if we only consider the local geometries and ignore the global topologies, there are in total three types in Lorentzian signature and in Euclidean signature, respectively.

Three Lorentzian Spaces

1. **de-Sitter spacetime dS_n, who has positive constant curvature.**
2. **Minkowski spacetime $\mathbb{R}^{1,n-1}$, who has zero curvature.**
3. **Anti-de-Sitter spacetime AdS_n, who has negative constant curvature.**

Three Euclidean Spaces

1. **Sphere $\mathbb{S}^n$, who has positive constant curvature.**
2. **Euclidean space $\mathbb{R}^n$, who has zero curvature.**
3. **Euclidean AdS_n Space (Hyperbolic Space) $\mathbb{H}^n$, who has negative constant curvature.**

Moreover, one can show that a spacetime of constant curvature has maximal number of local symmetries [2]; In n dimensions, the local isometry of such an n-manifold is generated by $\frac{n(n+1)}{2}$ local killing vectors [2]. For $\mathbb{R}^{1,n-1}$, AdS_n, $\mathbb{S}^n$, $\mathbb{R}^n$ and $\mathbb{H}^n$, if their corresponding local killing vectors are also globally defined, we call them global $\mathbb{R}^{1,n-1}$, AdS_n, $\mathbb{S}^n$, $\mathbb{R}^n$ and $\mathbb{H}^n$, respectively. dS_n is a special one because it has no global killing vectors. We call an n-manifold a local AdS_n spacetime if it has the local geometry everywhere as a global AdS manifold. In what follows, AdS will always be referred to as the global AdS. For hyperbolic spaces as well as AdS spacetimes, a well-known fact is that they do not have topological boundaries; In many cases, they are not compact manifolds. This is easy to understand because spaces that are maximally symmetric look the same everywhere from any perspective. In the AdS/CFT correspondence, the boundary of an AdS that we refer to as is a comformal boundary, in the sense that it is a topological boundary of the conformally compactified AdS spacetime. Roughly speaking, at the conformal boundary of a manifold, we do not care about the length or the area but rather the angle between two vectors. Rescaling any quantities defined on the boundary does not alter the conformal geometry and physics at the boundary.

Definition: Let M be a compact manifold whose boundary is ∂M and interior is M^0. We say M^0 is conformally compact if we can find a smooth function χ on M satisfying $\chi \neq 0$ on M^0 but $\chi = 0$, $d\chi \neq 0$ on ∂M. If the interior M^0 has a metric g_{ab}, then χg_{ab} is a metric on M. We call ∂M the conformal boundary of M^0 and compact manifold M the conformal compactification of M^0.

Complete (Semi-)Riemannian manifolds of constant curvature are also homogenous spaces [2]. A manifold M is called homogeneous if there exists a Lie group G acting on M continuously and transitively. Maximally symmetric spaces can always be written as a coset space of Lie groups because of the following theorem.

Theorem: If a group G acts on a topological space M transitively, and a subgroup $H \subset G$ is the stablizer of a point $p \in M$, there is a one-to-one map λ: $G/H \mapsto M$ defined by $\lambda(gH) = gp$ where $g \in G$.

At first glance, this theory looks trivial, since all the classical solutions of the same value of curvature are equivalent up to a local coordinate transformation. Fortunately, we are still allowed to do local identifications to obtain some interesting global topologies. For example, in two dimensions in Euclidean signature, one can easily imagine three types of flat solutions: a **plane**, a **cylinder** and a **torus**, whose fundamental groups are [0], $\mathbb{Z}$ and $\mathbb{Z} \oplus \mathbb{Z}$, respectively.

In 1992, Máximo Bañados, Claudio Teitelboim and Jorge Zanelli showed that in three dimensions with negative cosmological constant, there exists a black hole solution, which is an asymptotic AdS_3 spacetime [43]. This black hole solution can be obtained by doing local identifications of a pure AdS_3. In addition, J. D. Brown and Marc Henneaux showed that for an asymptotic AdS_3 spacetime, its asymptotic isometry is a direct sum of two copies of virasoro algebra, which strongly suggests that this three dimensional gravity has a CFT_2 dual living on its conformal boundary [20]. This was the first evidence of the AdS/CFT conjecture proposed by Juan Maldecena.

In n dimensional spacetime, gravitational fields have $n(n-3)$ degrees of freedom [2]. In four dimensions, there are 4 degrees of freedom, in which two come from the two polarizations of gravitational waves and the other two from their conjugate momenta. It is clear that in three dimensions, there are no gravitational waves. In this sense, this theory is a topological field theory. It is well-known that the classical three dimensional gravity is actually equivalent to the Chern-Simons theory whose connection A is living in some Lie algebra depending on the sign of consmological constant [16]. In this thesis, we will use the property of second Chern-Class to show how it determines the possible values of gravitational coupling constant.

Quantum gravity is difficult because it is not renormalizable. For pure gravity in four dimensions,

$$I = \frac{1}{16\pi G} \int_M d^4x \sqrt{-\det g}(R - 2\Lambda) \tag{6}$$

Setting $\hbar = c = 1$ means that length is inverse of mass; G has dimension of of length-squared (i.e. $[G] = L^2$). The only possible counter terms for one-loop correction to be expected are integrals of R^2, $R_{\mu\nu}R^{\mu\nu}$ and $R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}$. Therefore, one-loop counter term for the Einstein-Hilbert Lagrangian takes the form

$$\Delta\mathcal{L} = \sqrt{g}(\alpha R^2 + \beta R_{ab}R^{ab} + \gamma R_{abcd}R^{abcd}) \tag{7}$$

However, for pure gravity, we know that on the 'mass-shell', we have $R = 0$ and $R_{ab} = 0$ because of Einsteins equations. This implies that the first two terms can be re-written as

$$\Delta g_{ab} \left(EOM\right)^{ab}, \tag{8}$$

where Δg_{ab} is some arbitrary function of g_{ab} and the above expression vanishes on-shell [39]. From this expression we see that we can redefine the field $g_{ab} \to g_{ab} + \Delta g_{ab}$ so that the first two terms can be absorbed into the original Lagrangian. Hence, we can call such terms unphysical counter terms. For the third term, we know that for compact closed 4D manifold without boundary, the Euler characteristic

$$\int_M d^4x\sqrt{g}(R^2 - 4R_{ab}R^{ab} + R_{abcd}R^{abcd}) \tag{9}$$

is topological invariant. Then we can also absorb the last term into the original Lagrangian. Therefore, in four dimensions, pure gravity is one-loop exact [39]. But adding such unphysical counter terms does not eliminate divergences at higher-loop level. We would need an infinite number of counter terms to eliminate all divergences at arbitrary order of loops. This means that in four dimension, pure gravity is non-renormalizable. This theory should be studied as a sub-theory of a much larger theory. For example, in some supergravity theories, we may have fewer divergences [44]. In string theory, we can see all the higher derivative terms, whose coupling constants are determined by the string length [45]. Since in general relativity, we consider physics at very large scale, those higher order terms are irrelevant operators that do not survive in long distance. Einstein-Hilbert gravity is, therefore, a low energy effective field theory.

Because any gauge theory has self interactions, for pure gravity in three dimensions, we should also concern its renormalizability, even though this theory seems trivial. Since $\hbar = c = 1$ implies that $[G] = L$. One may also think that 3D gravity is non-renormalizable. This is, however, incorrect. The first consideration is that in three dimensions, possible counter terms are the Riemann scalar tensor R and the cosmological constant Λ because their integrals are the only possible dimensionless quantities we can have in three dimensions. Since in three dimensions, the Riemann tensor is completely determined by the metric, Ricci tensor and the scalar tensor, adding these counter terms is equivalent to redefining the metric $g_{\mu\nu} \to g_{\mu\nu} + aR_{\mu\nu} + bRg_{\mu\nu} + \cdots$. Thus, 3D quantum gravity is finite. The renormalization of pure gravity in three dimensions is equivalent to renormalization of cosmological constant itself. However, three dimensional gravity, as a topological field theory, has a very special feature that is different from ordinary quantum field theory such as ϕ^4 theory. We will see that by redefining fields, the coupling constant appear in Lagrangian is, in fact, a dimensionless constant l/G. We will see that it can only take discrete values due to topological constraints and thus there is no running coupling constant for this quantum theory. The gravitational coupling constant is determined by topological constraints. From the above analysis, it is hopeful to find a 3D quantum gravity theory.

2 Preliminary

2.1 Hyperbolic Geometry

On an Euclidean plane, the fifth postulate claims that there is exacly one geodesic through a given point parallel with a given geodesic disjoint from that point. From nineteenth century it gradually became clear that one can have a self-consistent theory of geometry where the original fifth postulate is not valid anymore.

One of such geometries is called the hyperbolic geometry, which has negative constant Riemann scalar curvature and Euclidean signature. In the following sections we will see that it is also the analytic continuation of AdS geometry and has three well-known models called Poincare's upper-half space, Poincare's unit ball and Lorentzian model, respectively. Essentially, the three different models describe the same geometric structure. i.e. the Riemannian structure together with its conformal structure at boundary. From a geometric aspect, they are simply the same topological manifold but one is different from another by a different embedding.

In Lorentzian model, a global hyperbolic space is a submanifold M embedded in $n+1$ Minkowski spacetime with metric

$$ds^2 = -dV^2 + (dX^1)^2 + \cdots + (dX^n)^2 \tag{10}$$

such that $\text{codim}(M) = 1$ and the embedding equation is given by

$$-V^2 + (X^1)^2 + \cdots + (X^n)^2 = -1 \tag{11}$$

It's orientation-preserving isometry group is $SO(1, n)$, which is generated by $\dfrac{n(n-1)}{2}$ rotations $X^i\partial_{X^j} - X^j\partial_{X^i}$ in X-plane and n boosts $V\partial_{X^i} + X^i\partial_V$.

In two dimensions, the Poincare's upper-half plane is given by $\mathbb{H}^2 = \{z \in \mathbb{C} : \Im z > 0\}$, with the metric

$$ds^2 = \frac{|dz|^2}{(\Im z)^2} \tag{12}$$

Another model is called Poincare's unit disc. $\mathbb{D}^2 = \{z \in \mathbb{C} : |z| < 1\}$ with the metric

$$ds^2 = \frac{4|dz|^2}{1 - |z|^2} \tag{13}$$

Conformal boundary of the upper-half plane is the real axis plus $i\infty$, which is equivalent to the conformal boundary circle of unit disc. In the upper-half plane model, geodesics are cicles centered at the conformal boundary [8]. While in the disc model, geodesics are arcs of circles or diameters orthogonal to its conformal boundary [8]. Each arc tending to its conformal boundary has infinte length. Suppose a free particle falling in a hyperbolic space, it will never reach the boundary at infinity. It can be proved that the above two models with the given metrics are both of constant negative curvature [8]. The two models are related with

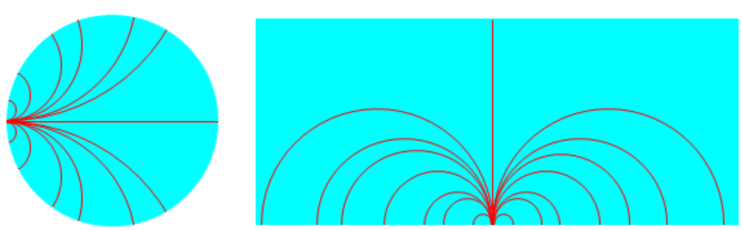


Figure 1: Geodesics in Poincare's Models

each other via a linear fractional transformation

$$\begin{pmatrix} i & 1 \\ 1 & i \end{pmatrix}(z) = \frac{iz+1}{z+i} \tag{14}$$

This transformation has a natural extension mapping the conformal boundary from one to another. For this reason, we do not distinguish the two models and simply denote a global two dimensional hyperbolic space as $\mathbb{H}^2$, whose conformal boundary is denoted by $\mathbb{S}^1 = \partial\mathbb{H}^2$.

The isometry group of $\mathbb{H}^2$ is $PSL(2,\mathbb{R})$, which is the real Möbius transformation. To see this, we first consider how the Möbius transformations acts on the Poincare's upper half plane. Let $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$ then the Möbius transformation

$$z = x + iy \mapsto w = \frac{az+b}{cz+d} = u + iv \tag{15}$$

The inverse map is

$$z = \frac{b - dw}{-a + cw} \tag{16}$$

If we substitue the transformation into the metric

$$d\tilde{s}^2 = \frac{|dw|^2}{(\Im w)^2} = \frac{du^2 + dv^2}{v^2}, \tag{17}$$

we get

$$\begin{aligned} d\tilde{s}^2 &= \frac{du^2 + dv^2}{v^2} = \frac{4|dw|^2}{|w - \bar{w}|^2} = \frac{4(ad-bc)^2|dz|^2}{|(az+b)(c\bar{z}+d) - (a\bar{z}+b)(cz+d)|^2} \\ &= \frac{dx^2 + dy^2}{y^2} = \frac{dz^2}{(\Im z)^2} = ds^2 \end{aligned} \tag{18}$$

In the above calculations, we didn't use the condition $ad - bc = 1$. In fact, the transformation preserves the metric for any $ad - bc > 0$. However, we can always rescale the matrix so that $ad - bc = 1$ holds. In Lorentzian model, we associate each point (x, y, z) of $\mathbb{H}^2$ with a matrix $\begin{pmatrix} z-y & x \\ x & z+y \end{pmatrix}$ and consider an action

$$\begin{pmatrix} z-y & x \\ x & z+y \end{pmatrix} \longmapsto A \begin{pmatrix} z-y & x \\ x & z+y \end{pmatrix} A^T \tag{19}$$

where $A \in SL_2(\mathbb{R})$, we can see that the isometry of this hyperboloid is $SO(2,1) = SL_2(\mathbb{R})/\mathbb{Z}_2 = PSL_2(\mathbb{R})$. Therefore, the isometry group of $\mathbb{H}^2$ is indeed $PSL_2(\mathbb{R})$.

It is useful to introduce the following coordinates for Poincare disc [25]. The first one is given by

$$\begin{cases} X = \sinh\chi\cos\phi \\ Y = \sinh\chi\sin\phi \\ V = \cosh\chi \end{cases} \tag{20}$$

with induced metric $ds^2 = d\chi^2 + \sinh^2\chi d\phi^2$. By introducing $\sinh\chi = r$, we have

$$ds^2 = \frac{dr^2}{1+r^2} + r^2 d\theta^2 \tag{21}$$

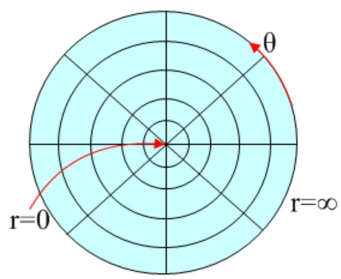


Figure 2: Constant θ are geodesics. Constant r, for $\theta \in (0, 2\pi]$ are not geodesics but rather isometric. i.e. $\dfrac{\partial}{\partial\theta}$ is a killing vector field.

Another coordinate is given by

$$\begin{cases} X = \sinh\rho \\ Y = \cosh\rho\sinh\omega \\ V = \cosh\rho\cosh\omega \end{cases} \tag{22}$$

with

$$ds^2 = d\rho^2 + \cosh^2\rho d\omega^2. \tag{23}$$

By setting $\cosh\rho = r$, we have

$$ds^2 = \frac{dr^2}{r^2-1} + r^2 d\omega^2 \tag{24}$$

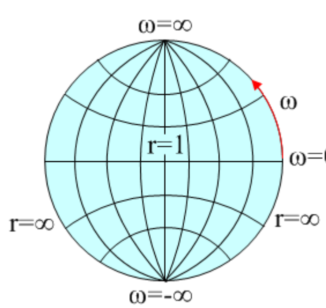


Figure 3: This coordinate describes $r > 1$ and $\omega \in (-\infty, +\infty)$.

Finally, we introduce a special coordinate

$$\begin{cases} X = e^{-\sigma}\mu \\ Y = \sinh\sigma + e^{-\sigma}\dfrac{\mu^2}{2} \\ V = \cosh\sigma + e^{-\sigma}\dfrac{\mu^2}{2} \end{cases} \tag{25}$$

with

$$ds^2 = d\sigma^2 + e^{-\sigma}\frac{\mu^2}{2}. \tag{26}$$

We define $e^{-\sigma} = r$, then the metric becomes

$$ds^2 = \frac{dr^2}{r^2} + r^2 d\mu^2. \tag{27}$$

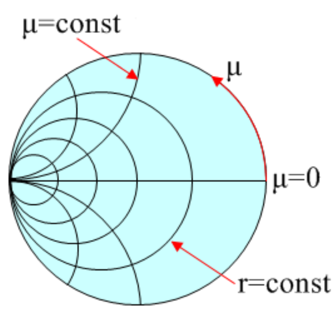


Figure 4: Each μ =const are geodesics arcs tending to conformal infinity. Constant r curves are not geodesic but isometric.

In three dimensions, we also have the Poincare's upper-half-space model $\{(z, u) : z \in \mathbb{C}, u > 0\}$ with the metric

$$ds^2 = \frac{|dz|^2 + du^2}{u^2} \tag{28}$$

as well as the unit ball model $\{x \in \mathbb{R}^3 : |x|^2 < 1\}$ with the metric

$$ds^2 = \frac{4|dx|^2}{1 - |x|^2} \tag{29}$$

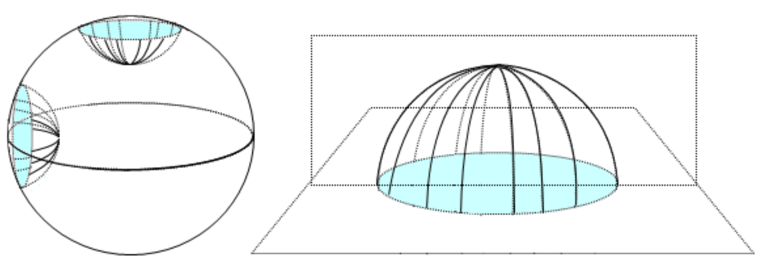


Figure 5: Geodesics in 3D Models

The conformal boundary of a global hyperbolic space is a two-sphere $\mathbb{S}^2$, which can be identified as $\mathbb{CP}^1$. In the upper-half space model, its geodesics are hemi-circles centered at conformal boundary. In the Poincare's ball model, its geodesics are arcs of circles orthogonal to the boundary sphere. The picture depicts totally geodesic surfaces in each model. Each geodesic connecting two end points on the conformal boundary has infinite length. The isometry group is $SO(3,1)$, which is the same as $SL(2,\mathbb{C})/\mathbb{Z}_2$. To see how it acts on $\mathbb{H}^3$, we write the hyperboloid as $\det(g) = 1$ with

$$g = \begin{pmatrix} U - X^1 & iV + X^2 \\ -iV + X^2 & U + X^1 \end{pmatrix} \in SL(2,\mathbb{C})/SU(2) \tag{30}$$

The metric is exactly the Killing-Cartan metric $ds^2 = \mathrm{Tr}\left(g^{-1}dgg^{-1}dg\right)$ of the quotient Lie group [8]. The action is

$$A \begin{pmatrix} U - X^1 & iV + X^2 \\ -iV + X^2 & U + X^1 \end{pmatrix} A^\dagger \tag{31}$$

where $A \in PSL(2,\mathbb{C})$.

2.2 Uniformization of Riemann Surfaces

It is necessary to have a brief introduction to the uniformization of Riemann surfaces because it is closely related with the geometry of BTZ black holes in Lorentzian signature. From uniformization theorem, every simply connected Riemann surface is conformally equivalent to one of three types: a **Riemann sphere** $\mathbb{CP}^1$, a **complex plane** $\mathbb{C}$ and a **Poincare upper-half plane** $\mathbb{H}^2$, corresponding to two-manifolds with positive constant curvature, flat and negative constant curvature, respectively. More specifically, every Riemann surface can be obtained as a quotient space of one of the three types of simply connected Riemann surfaces $\mathbb{C}$, $\mathbb{CP}^1$ or $\mathbb{H}^2$ by a discrete subgroup, which acts freely, of biholomorphic automorphisms of $\mathbb{C}$, $\mathbb{CP}^1$ or $\mathbb{H}^2$, respectively.

Definition: A group G of homeomorphic self-mapping of a manifold M is discontinuous if for any compact subset $U \subset M$, there are at most finitely many elements $g \in G$ such that $g(U) \cap U \neq \emptyset$.

It is easy to see that the biholomorphic automorphisms of $\mathbb{C}$, $\mathbb{CP}^1$ and $\mathbb{H}^2$ are given by

-when $z \in \mathbb{C}$,

$$\sigma(z) = az + b, \quad a \in \mathbb{C}^*, b \in \mathbb{C} \tag{32}$$

-when $z \in \mathbb{CP}^1$,

$$\sigma(z) = \frac{az+b}{cz+d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL_2(\mathbb{C}), \tag{33}$$

-when $z \in \mathbb{H}^2$,

$$\sigma(z) = \frac{az+b}{cz+d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL_2(\mathbb{R}), \tag{34}$$

where in the last case, the group of biholomorphic automorphism is its isometry group. From Gauss-Bonnet theorem

$$\int_X R = 2\pi(2 - 2g) \tag{35}$$

where X is a compact closed two dimensional manifold with genus g, we see that there are restrictions to the topology of the quotient space that we may construct. For example, we can only make a torus from complex plane. This kind of Riemann surfaces are usually called elliptic curves. If constructing a compact closed Riemann surface with genus higher than 1, we can only use $\mathbb{H}^2$, otherwise we would encounter singularities.

For example, we choose a discrete subgroup of isometry of $\mathbb{C}$ generated by two elements (a, b) such that

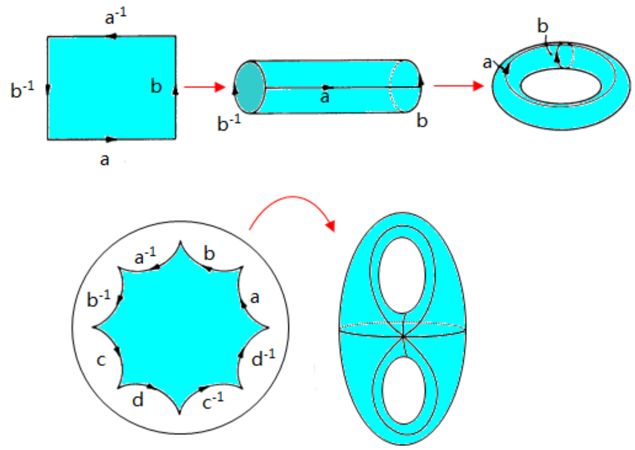


Figure 6: Riemann surfaces of genus 1 and of genus 2

$aba^{-1}b^{-1} = \mathrm{i_d}$, i.e. $< a, b >= \mathbb{Z} \oplus \mathbb{Z}$. Then the quotient space $\mathbb{C}/\mathbb{Z} \oplus \mathbb{Z}$ is a torus. (The identity $aba^{-1}b^{-1} = \mathrm{i_d}$ is a consequence of the fact that the loop corresponding to this product is contractible.) If we choose a discrete subgroup of $PSL(2, \mathbb{C})$ that is generated by four elements (a, b, c, d) such that $aba^{-1}b^{-1}cdc^{-1}d^{-1} = \mathrm{i_d}$, then the quotient space is $\mathbb{H}^2/ < a, b, c, d >$, which is a compact Riemann surface of genus $g = 2$. It is easy to see that these discrete groups are exacly the first fundamental groups of these Riemann surfaces. The fundamental domains are the regions in which no two points are in the same orbit of isometries. In two dimensions, it is natural to choose the fundamental domains to be enclosed by geodesics because geodesics are always mapped to geodesics by isometries. If we did not choose geodesics as the boundary of the fundamental domain, then the quotient space would have singularities. For example, in string theory, we learned that the fundamental domain of $SL(2, \mathbb{Z})$ on the Poincare upper-half plane is an orbifold with two conical singularities and a 'cusp' at infinity. Thus the quotient space $\mathbb{H}/SL(2, \mathbb{Z})$ is not a compact Riemann surface with genus higher than 1. We are also interested in non-compact Riemann

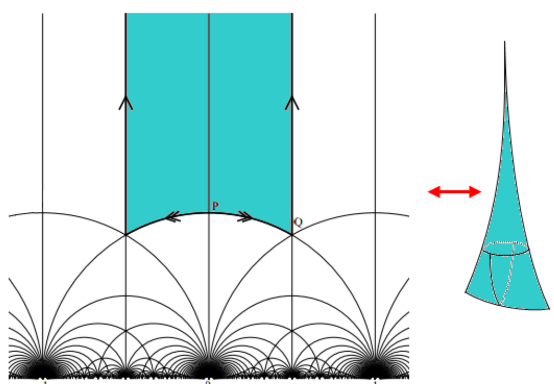


Figure 7: Modular curve $\mathbb{H}^2/SL(2, \mathbb{Z}) = \mathbb{H}^2/ < S, T|S^2 = i_d, (ST)^3 = i_d >$ is generated by two elements S and T. It has a cusp point at $i\infty$ and two conical singularities at points P and Q.

surfaces of constant negative curvature. Riemann surfaces of constant negative curvature are quotient spaces of Poincare discs modulo discrete subgroups of Möbius transformations $SL(2, \mathbb{R})$, which are usually called the Fuchsian groups Γ. Since these quoient spaces can be non-compact, the fundamental domains may not only enlosed by geodesics in the bulk, but also some conformal boundary components if the Riemann surface is non-compact. In three dimensions, we are also interested in non-compact 3-hyperbolic manifolds that have comformal boundaries. Their associated discrete subgroups of isometries are called Kleinian groups. These groups are very useful in the discussion of 3D Euclidean gravity in section 4. To begin with, let us review some basic facts of Möbius transformations.

The elements of a Fuchsian group are categorized into four types:
0. trivial if and only if $\sigma = \pm 1 \in \Gamma$
1. elliptic if and only if $|\mathrm{Tr}(\sigma)| < 2$
2. parabolic if and only if $|\mathrm{Tr}(\sigma)| = 2$
3. hyperbolic if and only if $|\mathrm{Tr}(\sigma)| > 2$
The elements of a Kleinian group are also classified in a similar way:
0. trivial if and only if $\sigma = \pm 1 \in \Gamma$
1. elliptic if and only if $\mathrm{Tr}(\sigma)$ is real and $|\mathrm{Tr}(\sigma)| < 2$
2. parabolic if and only if $\mathrm{Tr}(\sigma)$ is real and $|\mathrm{Tr}(\sigma)| = 2$
3. hyperbolic if and only if $\mathrm{Tr}(\sigma)$ is real and $|\mathrm{Tr}(\sigma)| > 2$
4. loxodromic if and only if $|\mathrm{Tr}(\sigma)| \in \mathbb{C} - \mathbb{R}$

The action of Fuchsian (Klein) group on $\mathbb{CP}^1$ and $\mathbb{H}^2$ are defined by

$$\sigma(z) = \frac{az+b}{cz+d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL_2(\mathbb{C}), \qquad for \;\; z \in \mathbb{CP}^1$$
$$\sigma(z) = \frac{az+b}{cz+d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL_2(\mathbb{R}), \qquad for \;\; z \in \mathbb{H}^2 \tag{36}$$

The real Möbius transformations act on upper-half plane $\mathbb{H}^2$ as isometries. While the complex Möbius transformations act on $\mathbb{CP}^1$ as biholomorphic self-mappings (or biholomorphic automorphisms, which are also called conformal transformations). Previously we showed that complex Möbius transformations act on $\mathbb{H}^3$ as isometries. i.e. we have the following isomorphisms $\mathrm{Aut}(\mathbb{S}^2) = \mathrm{Aut}(\partial(\mathbb{H}^3)) \simeq PSL(2,\mathbb{C}) = \mathrm{Isom}(\mathbb{H}^3)$. Any isometry of the bulk has a correponding conformal map acting on the boundary. This is a trivial example of the Euclidean version of AdS_3/CFT_2 correspondence. If a discrete subgroup of the isometry acts on the bulk, then there is a one-to-one corresponding discrete subgroup of holomorphic map on the boundary. We list some examples of different types of Kleinian groups in the following table, where L is a nonzero real number, $\theta \in (0, 2\pi]$, a is an arbitrary complex number and λ is a complex number such that $|\lambda| \neq 1$.

Transformation	Representative	Effect
Elliptic	$\begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix}$	$z \mapsto e^{i\theta} z$
Parabolic	$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$	$z \mapsto z + a$
Hyperbolic	$\begin{pmatrix} e^{L} & 0 \\ 0 & e^{-L} \end{pmatrix}$	$z \mapsto e^{2L} z$
Loxodromic	$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$	$z \mapsto \lambda^2 z$

A Fuchsian group element acting on τ, the modular parameter of Poincare upper-half plane, is given by $(a\tau + b)/(c\tau + d)$. Infinitesimally, the matrix is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \tag{37}$$

with $\alpha + \delta + 0$. Then, the fixed points of its action is given by the equation

$$\frac{a\tau + b}{c\tau + d} = \tau \tag{38}$$

or $c\tau^2 + (d-a)\tau - b = 0$, from which we see that if it is parabolic, there is a single fixed point on the real axis; if it is hyperbolic, then it has two fixed points on real axis; if it is elliptic, then it has a fixed point inside $\mathbb{H}^2$. We should also extend the transformations at $i\infty$. For example, the parabolic transformation in the above table is a translation, which fixes $i\infty$. To see how this classification is related with the trace, we do an exponential map of the infinitesimal generator of Möbius transformation

$$\mathrm{Tr}\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right] = \mathrm{Tr}\left[\exp\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\right] \tag{39}$$

which is a sum of the exponential of the eigenvalues of the generator. It is easy to compute that the eigenvalues are $k = \pm\sqrt{\alpha\delta - \beta\gamma}$. So the trace formula is

$$\mathrm{Tr}\left[\exp\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\right] = e^{\sqrt{\beta\gamma-\alpha\delta}} + e^{-\sqrt{\beta\gamma-\alpha\delta}} \tag{40}$$

Infinitesimally, the discriminant of the quadratic equation (38) is given by $\Delta = 4\beta\gamma - 4\alpha\delta$. Hence, we have the classification given by the trace formula shown as below

$$\mathrm{Tr}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = e^{\sqrt{\Delta}/2} + e^{-\sqrt{\Delta}/2} \tag{41}$$

with

$$\begin{cases} \Delta > 0 \Leftrightarrow \mathrm{Tr}(\sigma) > 2 \\ \Delta = 0 \Leftrightarrow \mathrm{Tr}(\sigma) = 2 \\ \Delta < 0 \Leftrightarrow \mathrm{Tr}(\sigma) < 2 \end{cases} \tag{42}$$

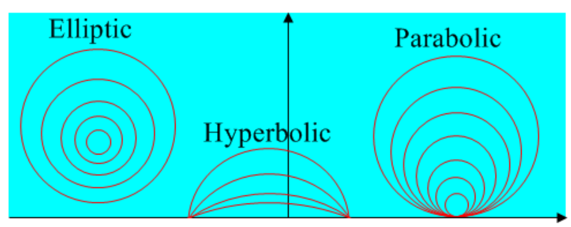


Figure 8: Möbius transformations in upper-half plane

In the Poincare's unit disk model, those curves are illustrated in the following figure. The red lines are

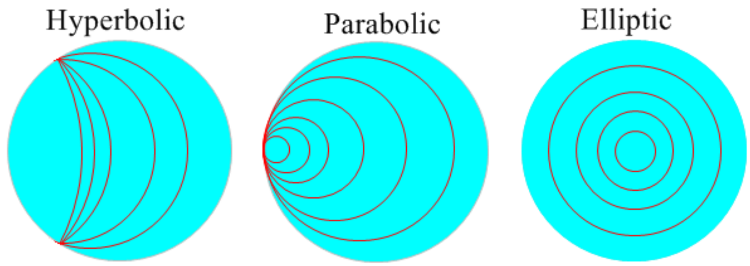


Figure 9: Möbius transformations in Poincare discs

orbits of Möbius transformations acting on Poincare discs.

In section 3, we will see that the spacial slices of a AdS_3 manifold are exactly Poincare discs. If the quotient of the AdS_3 is taken to be time-independent, then the discrete isometry group acting on AdS_3 induces discrete a Möbius transformation acting on each Poincare disk. Our assumption is that once the boundary condition of a physically possible local AdS_3 manifold (which means that it cannot contain closed timelike circle) is fixed, its geometry and global topology is totally determined by the geometry and topology of a single spacial slice of it. However, we are not able to prove that our assumption is correct.

If we assume it is indeed correct, then we only need to study the geometry of those two dimensional surfaces. If the Möbius transformation were generated by a hyperbolic element, then it would have two fixed points on the boundary; If it were generated by an parobolic element, it would have a single fixed point on the boundary; If it were generated by an elliptic element, then it would have a singular point in the bulk. We are mainly interested in these cyclic Fuchsian groups denoted by $<\gamma>$, where γ is the generator, because we will see that these Riemann surfaces are strongly related with BTZ black holes in Lorentzian signature [26] [27] [28] [29] [31]. Quotient Spaces of form $\mathbb{D}^2/<\gamma>$ resemble the following shaded regions followed by identifications along their boundary geodesics inside $\mathbb{D}^2$. The right most disk is $\mathbb{D}/<1>$.

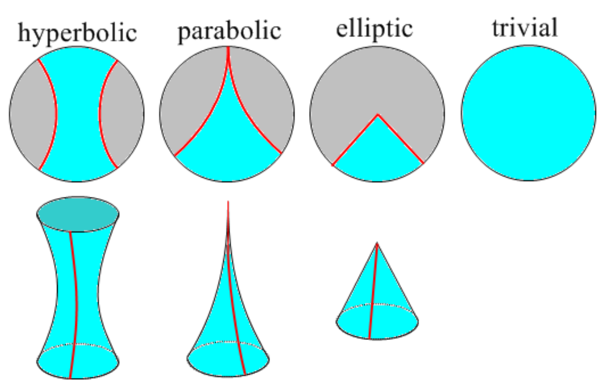


Figure 10: Fundamental domains

Noting that any infinite cyclic group is isomorphic to the group of addition of integers $\mathbb{Z}$; Any finite cyclic group is isomorphic to $\mathbb{Z}_n$, the above quotient spaces are either $\mathbb{D}^2/\mathbb{Z}$ or $\mathbb{D}^2/\mathbb{Z}_n$. A theorem from hyperbolic geometry claims that all hyperbolic and parabolic cyclic subgroups of $SL_2(\mathbb{R})$ are Fuchsian; Any elliptic cyclic subgroup is Fuchsian iff it is finite [9] [11]. Hence, a hyperbolic discrete subgroup that is isomorphic to integers must be of the following form

$$W = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}, \cdots \right\} \tag{43}$$

where we can choose $\alpha > 1$ so that it is a cyclic discrete subgroup. A parabolic discrete subgroup that is isomorphic to $\mathbb{Z}$ is the translation by integers. It is given by

$$\Gamma_\infty = \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \right\}_{n \in \mathbb{Z}} \tag{44}$$

An elliptic motion is of the form

$$Y = \left\{ \begin{pmatrix} \cos(2\pi/n) & \sin(2\pi/n) \\ -\sin(2\pi/n) & \cos(-2\pi/n) \end{pmatrix}, \cdots \right\} \tag{45}$$

One question we need to answer is that how many parameter we have to use to parametrize these quotient surfaces i.e. the dimension of their moduli space. Although we are only studying the cyclic cases for BTZ black holes, it is still useful to elaborate what we mean by the moduli of Riemann surfaces. Instead of using hard mathematics to show the dimension formula, we used very elementary method, which is worth knowning to many people. First, we consider a generic Riemann surface $\mathbb{D}^2/\Gamma$ of genus $g > 1$ with n cusps and m boundary circles, where Γ is the corresponding discrete subgroup of $PSL(2, \mathbb{R})$ which creates g handles, n cusps as well as m boundaries. Such a group must be generated by $2g$ hyperbolic generators which correspond to the $2g$ geodesic hemi-circles centered at $\partial\mathbb{D}^2$, n parabolic generators which correspond to n cusps on the conformal boundary $\partial\mathbb{D}^2$, and m hyperbolic generators corresponding to m intervals on $\partial\mathbb{D}^2$. We denote the $2g$ hyperbolic generators by $\{A_i, B_i\}$ for $i = 1, \cdots, g$, n parabolic generators by C_j for $j = 1, \cdots, n$ and m hyperbolic generators for boundary intervals by D_k, for $k = 1, \cdots, m$. Since the loop is contractible, up to a permutation of products of generators, they should satisfy the following identity [65] [23].

$$\prod_{i=1}^{g} A_i B_i A_i^{-1} B_i^{-1} \prod_{j=1}^{n} C_j \prod_{k=1}^{m} D_k = \mathrm{i_d} \tag{46}$$

Remark: When only considering the dimensionality of parameter space of a type of quotient surfaces, it is no danger to change the order of products among $[A_i, B_i]$, C_j and D_k. These generators of the discrete subgroup of isometry generate the fundamental group of the Riemann surface. i.e. $\pi_1(S_{g,n,m}) =< A, B, C, D >$. Since each generator is in $PSL(2, \mathbb{R})$, which is a three dimensional group manifold, $2g + m$ hyperbolic generators have $6g + 3m$ degrees of freedom. The n parabolic generators have $2n$ degrees of freedoms since we have n constraints from the trace condition for parabolic transformations. The identity above provides us with three independent constraint equations. We also need to consider the fact that $SL(2, \mathbb{R})$ manifold admits a foliation by poincare discs, $\mathbb{D}^2 = SL(2, \mathbb{R})/SO(2)$, which will be explained in later chapters. Using this foliation, we have

$$\mathbb{D}^2/\Gamma = \Gamma \backslash SL(2, \mathbb{R})/SO(2) \tag{47}$$

Consider an arbitrary element $\gamma \in PSL(2, \mathbb{R})$, we have

$$\gamma \Gamma \gamma^{-1} \backslash SL(2, \mathbb{R})/SO(2) = \gamma \Gamma \backslash SL(2, \mathbb{R})/SO(2) \tag{48}$$

since $PSL(2, \mathbb{R})$ is the isometry of Poincare disc. Noting that $\gamma\Gamma$ is simply Γ itself, we have an equivalence class

$$\gamma \Gamma \gamma^{-1} \sim \Gamma \tag{49}$$

from which we can eliminate three degrees of freedom. Hence we need exacly $6g - 6 + 2n + 3m$ real numbers to parametrize the set of isometry class of the quotient surfaces with genus g and n cusp punctures together with m boundaries. We denote the moduli space by $\mathcal{M}_{g,n,m}$. The dimension formula

$\dim_{\mathbb{R}} \mathcal{M}_{g,n,m} = 6g - 6 + 2n + 3m$ is valid only when $g > 1$.

For cyclic cases (i.e. $\Gamma = \mathbb{Z}$ or $\mathbb{Z}_n$), we can still define the 'moduli' as the isometry classes. For the hyperbolic case, the 'moduli space' is given by the hyperbolic class of $PSL(2,\mathbb{R})$. This class can be found by observing Figure 3, where the metric is $ds^2 = \dfrac{dr^2}{r^2-1} + r^2 d\omega^2$. Removing the two grey shaded regions, it is apparent that gluing along two geodesics of constant-ω can be parametrized by the shortest distance between the two constant-ω geodesics, which is a positive number. We call such a parameter the 'mass parameter' denoted by L, because we will see that it is related with the mass of a BTZ black hole. Hence, for hyperbolic

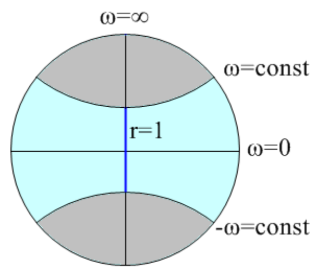


Figure 11: The shortest distance between constant ω and $-\omega$ geodesics is the length of the blue interval.

case, the moduli space can be identified as $\mathbb{R}_{>0}$. For parabolic case, the metric is $ds^2 = \dfrac{dr^2}{r^2} + r^2 d\mu^2$. Suppose we glue two geodesics $\mu = -\pi a$ and $\mu = \pi a$, where $a > 0$. i.e. the fundamental domain is given by the identification $\mu \sim \mu + 2\pi a$. We can define $a\tilde{\mu} = \mu$ so that in terms of $\tilde{\mu}$ coordinate, the periodicity is 2π. This extra factor can again be absorbed by redefining r by $\tilde{r} = ar$, rendering the metric invariant. i.e. $d\tilde{s}^2 = \dfrac{d\tilde{r}^2}{\tilde{r}^2} + \tilde{r}^2 d\tilde{\mu}^2$. Therefore, there is no degree of freedom to make a cusp cone. Hence, the moduli space in parabolic case is a single point. If we apply a similar rescaling procedure to the hyperbolic case, the metric is not invariant. Under the transformation

$$\omega \to a\omega, \quad r \to \frac{r}{a} \tag{50}$$

the metric becomes $ds^2 = \dfrac{dr^2}{r^2 - a} + r^2 d\omega^2$. For the elliptic case, the isometry classes of cones is parametrized by the deficit angle, which is $2\pi/n$, $n \in \mathbb{Z}_{>0}$. We can also consider an m-sheeted branched cover of $\mathbb{D}^2$, from which we may have a deficit angle $2\pi \dfrac{m}{n}$, which runs in $\mathbb{Q}/\mathbb{Z}$. Therefore, the moduli space of $\mathbb{D}^2/\mathbb{Z}_n$ with one marked point (the fixed point, which is also the branching point) is given by $\mathbb{Q}/\mathbb{Z}$, which is dense in circle $\mathbb{S}^1$. However, a cone can also be obtained by a local identification, whose corresponding deficit angle is an irrational number. Such a cone is not obtained by taking quotient, but can be deemed as a limit of a series of rational cones. For this reason, we claim that for elliptic case with a market point, the moduli space is a circle $\mathbb{S}^1$.

The above results agree with the Iwasawa decomposition of $SL(2,\mathbb{R})$, which claims that we have a decomposition $SL(2,\mathbb{R}) = KAN$, where

$$K = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \middle| \theta \in (0, 2\pi] \right\}, \quad A = \left\{ \begin{pmatrix} e^L & 0 \\ 0 & e^{-L} \end{pmatrix} \middle| L > 0 \right\}, \quad N = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \middle| x \in \mathbb{R} \right\}. \tag{51}$$

For every $g \in SL(2,\mathbb{R})$, there is a unique representation as $g = kan$, where $k \in K$, $a \in A$ and $n \in N$. Using this decomposition, it is easy to find representatives for conjugate classes of $PSL(2,\mathbb{R})$:

-elliptic class

$$[g] = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \tag{52}$$

-hyperbolic class

$$[g] = \begin{pmatrix} e^L & 0 \\ 0 & e^{-L} \end{pmatrix}, \tag{53}$$

-parabolic class

$$[g] = \begin{pmatrix} 1 & \pm 1 \\ 0 & 1 \end{pmatrix}, \tag{54}$$

from which we clearly see that for the elliptic case, the modulus is $\theta \in (0, 2\pi]$; for the hyperbolic case, the modulus is $u > 0$. This 'mass' parameter is related with the trace by

$$Tr(g) = 2\cosh(L) \tag{55}$$

For the parabolic case, it seems that the moduli space contains two distinct points, which is a contradiction with our previous result. Nevertheless, the matrix acting on $z \in \mathbb{H}^2$ is simply a shift $z \to z + 1$ or $z \to z - 1$. The fundamental domains are the same in both cases.

2.3 Basic Cohomology Theory

2.3.1 Simplicial Homology

We assume readers are familiar with free Abelian groups, homotopy groups and simplexes. The materials contained in this section is mainly copied from [5]. We first introduce some basic concepts of homology group of simplexes. Let $p_0, \cdots, p_r$ be points in $\mathbb{R}^n$ for $n > r$, an r-simplex $\sigma_r =< p_0 \cdots p_r >$ is expressed as

$$\sigma_r = \left\{ x \in \mathbb{R}^n | x = \sum_{i=0}^{r} c_i p_i, c_i \geq 0, \sum_{i=1}^{r} c_i = 1 \right\} \tag{56}$$

For $0 \leq q \leq r$, then we can choose a q-simplex $< p_{i_0}, \cdots, p_{i_q} >$, which is called a q-face of the original r-simplex and we denote $\sigma_q \leq \sigma_r$.

Definition: Let K be a number of simplexes in $\mathbb{R}^n$. If they satisfy the following conditions, we say that the set K is a simplicial complex.
(i) an arbitrary face of a simplex in K belongs to K.
(ii) if σ' and σ are two simplexes in K, the intersection $\sigma \cap \sigma'$ is either empty set of a common face of them.
the dimension of a simplicial complex is defined to be the largest dimension of simplexes in it.

For a topological space X, if there exists a simplicial complex K and a homeomorphism f: $K \mapsto X$, we say X is triangulable and the pair (K, f) is called its triangulation. For a manifold, it can be proved that it is always possible to associate it with a triangulation, though this is not unique. In the following discussion, we need simplexes to be oriented. In other words, we define

$$(p_i p_j p_k p_l) = \text{sgn}(P)(p_0 p_1 p_2 p_3) \tag{57}$$

where we use $(\cdots)$ to denote oriented simplexes. To extract topological information of a manifold, we first associated it with a triangulation, then we can find topological invariant from the simplicial complex.

Definition: Let I_r be the number of r-simplexes in K. The r-Chain group $C_r(K)$ of a simplicial complex K is a free Abelian group generated by oriented r-simplexes of K. In particular, if $r \geq \dim(K)$, then $C_r(K)$ is defined to be 0. An element c in $C_r(K)$ is called an r-chain, which is expressed as follows

$$c = \sum_{i=1}^{I_r} c_i \sigma_{r,i}, \quad c_i \in \mathbb{Z} \tag{58}$$

From this expression, we see that the group structure is given by a sum

$$c + c' = \sum (c_i + c'_i)\sigma_{r,i} \tag{59}$$

Hence an r-chain group $C_r(K)$ is a free Abelian group of rank I_r

$$C_r(K) = \underbrace{\mathbb{Z} \oplus \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{I_r} \tag{60}$$

The chain group has a subgroup which consists of simplexes that are boundary of some other simplexes. The boundary operator is defined as follows.

Definition: Let $\sigma_r = (p_0 \cdots p_r)$ be an oriented r-simplex. The boundary operator ∂_r acting on σ_r gives an $(r-1)$-chain defined by

$$\partial_r \sigma_r = \sum_{i=0}^{r} (-1)^i (p_0 \cdots \hat{p}_i \cdots p_r) \tag{61}$$

where the point $\hat{p}_i$ is omitted. This operator is linear, in the sense that when it acts on a chain of $C_r(K)$, it acts summand-wise

$$\partial_r c = \sum_i c_i \partial_r \sigma_{r,i} \tag{62}$$

Accordingly, ∂_r is defined as a map

$$\partial_r \colon C_r(K) \mapsto C_{r-1}(K) \tag{63}$$

whose image is called the boundary of the preimage.

Let K be a simplicial complex of dimension n. We can find a sequence of free Abelian groups and homomorphisms,

$$0 \xrightarrow{i} C_n(K) \xrightarrow{\partial_n} C_{n-1}(K) \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} C_2(K) \xrightarrow{\partial_1} C_1(K) \xrightarrow{\partial_0} 0 \tag{64}$$

where $i \colon 0 \hookrightarrow C_n(K)$ is an inclusion. This sequence is called a chain complex associated with K and is denoted by $C(K)$. We can easily check that neither the kernal nor the image of a boundary operator is topological invariant. However, we can construct a quotien subgroup that is topological invaiant. To begin with, we define the following subgroups.

Definition: If $c \in C_r(K)$ satisfies $\partial_r c = 0$ i.e. $c \in \ker(\partial_r)$, then c is called an r-cycle. In other words, cycles are those who does not have boundaries. The set of r-cycles is denoted by $Z_r(K)$, which is a subgroup of $C_r(K)$.

Definition: If $c \in C_r(K)$ is given by $c = \partial_{r+1} f$ for some $f \in C_{r+1}(K)$, i.e. $c \in \operatorname{im}(\partial_{r+1})$, we say c is an r-boundary. The set of r-boundaries in $C_r(K)$ is denoted by $B_r(K)$, which is also a subgroup of $C_r(K)$.

It is easy to see that $\partial_r \circ \partial_{r+1} = 0$. Hence we can define a quotient group

$$H_r(K) = Z_r(K)/B_r(K) \tag{65}$$

called the rth homology group of simplicial complex K. Remark: It is necessary to impose that $H_r(K) = 0$ for $r > \dim(K)$ and $r < 0$. This group only depends on the topology of the simplicial complex. In particular, if K is connected, then $H_0(K) = \mathbb{Z}$.

2.3.2 de Rham Cohomology

In this section, we study the cohomology theory of differential forms on manifolds. First, we define r-chain, r-cycle and r-boundary in an n-dimensional manifold M. Let σ_r be an r-simplex in $\mathbb{R}^n$ and left $f\colon \sigma_r \mapsto M$ be a smooth map. We denote the image of σ_r in M by s_r and call it a singular r-simplex. Let $\{s_{r,i}\}$ be the set of r-simplexes in M, we define r-chain in M by a sum with $\mathbb{R}$-coefficients

$$c = \sum_i a_i s_{r,i}, \quad a_i \in \mathbb{R} \tag{66}$$

r-chains form a chain group $C_r(M)$ of M. We requires that $\partial s_r = f(\partial \sigma_r)$. It is a set of $(r-1)$-simplexes in M and is called the boundary of s_r. We have

$$\partial\colon C_r(M) \mapsto C_{r-1}(M) \tag{67}$$

and $\partial^2 = \partial \circ \partial = 0$. In a similar way, we can define the cycle group $C_r(M)$ and boundary group $B_r(M)$. The singular homology group of M is defined by $H_r(M) = Z_r(M)/B_r(M)$.

Theorem (Stoke): Let $\omega \in \Omega^{r-1}(M)$ and $c \in C_r(M)$, then

$$\int_c d\omega = \int_{\partial c} \omega \tag{68}$$

From this theorem, we can construct a duality between holomogy and cohomology.

Definition: Let M be an n-dimensional manifold. The set of closed r-forms is called the rth cocycle group, denoted by $Z^r(M) = \ker d_{r+1}$. The set of exact r-forms is called the rth coboundary group, denoted by $B^r(M) = \mathrm{im} d_r$. We call the following sequence

$$0 \overset{i}{\rightarrow} \Omega^0(M) \overset{d_1}{\rightarrow} \Omega_1(M) \overset{d_2}{\rightarrow} \cdots \overset{d_{n-1}}{\rightarrow} \Omega_{n-1}(M) \overset{d_n}{\rightarrow} \Omega_n(M) \overset{d_{n+1}}{\rightarrow} 0 \tag{69}$$

a de Rham complex $\Omega^*(M)$.

Since $d^2 = d \circ d = 0$, we have $Z^r(M) \supset B^r(M)$. Consequently, we can define the cohomology group of M.

$$H^r(M;\mathbb{R}) = Z^r(M)/B^r(M) \tag{70}$$

Remark: if $r < 0$ or $r > \dim(M)$, then we require the cohomology group to be trivial. We may also consider de Rham cohomology with integer coefficients $H^r(M;\mathbb{Z})$.

Theorem: If M has m connected components, then its zeroth de Rham colomology is given by

$$H^0(M;\mathbb{R}) = \underbrace{\mathbb{R} \oplus \mathbb{R} \oplus \cdots \oplus \mathbb{R}}_{m} \tag{71}$$

Hence it is specified by m real numbers.

Examples: For n-sphere, the de Rham cohomology is given by

$$H^k(\$^n) = \begin{cases} \mathbb{R} & k = 0, n \\ 0 & k \neq 0, n \end{cases} \tag{72}$$

For punctured Euclidean space we have

$$H^k(\mathbb{R}^n \setminus \{0\}) = \begin{cases} \mathbb{R} & k = 0, n-1 \\ 0 & k \neq 0, n-1 \end{cases} = H^k(\$^{n-1}) \tag{73}$$

The above two examples are for non-contractible manifolds. For a contractible open subset of $\mathbb{R}^n$, according to Poincare lemma, any closed form on this open set is also exact. Hence if open subset $U \subset M$ is contracible, we have

$$H^k(U) = \begin{cases} 0 & 1 \leq k \leq \dim M \\ \mathbb{R} & k = 0 \end{cases} \tag{74}$$

In particular, we have $H^r(\mathbb{R}^n) = 0$ and $H^0(\mathbb{R}^n) = \mathbb{R}$.

Theorem: de Rham cohomology groups are diffeomorphism invariants.

Theorem: Let X and Y be smooth manifolds with Y smoothly contractible. Then $H^k(X \times Y) = H^k(X)$ for every k. Two manifolds of the same smooth homotopic type have the same de Rham cohomology groups.

Theorem: Let X be a compact, connected, oriented, closed n-manifold. Then $H^n(X) = \mathbb{R}$. Furthermore, it can be proved that no compact, connected, closed orientable manifold is contractible.

Theorem: If M is a contractible manifold, then $H^k(M) = 0$ for all $k \neq 0$.

The advantage of cohomology theory is that it in fact has a ring structure. If $[\omega] \in H^q(M)$ and $[\eta] \in H^p(M)$, then we define a product of the two classes

$$[\omega] \wedge [\eta] = [\omega \wedge \eta] \tag{75}$$

It is easy to check that such a product is well-defined and so we can define the de Rham cohomology ring as

$$H^*(M) = \bigoplus_{r=1}^{\infty} H^r(M) \tag{76}$$

in which the wedge product $\wedge$: $H^*(M) \times H^*(M) \mapsto H^*(M)$ is closed.

Let M be an m-dimensional manifold. Take $c \in C_r(M)$ and $\omega \in \Omega^r(M)$, where $1 \leq r \leq m$. We can define the integral of differential forms on cycles as an inner-product $C_r(M) \times \Omega^r(M) \mapsto \mathbb{R}$

$$c, \omega \mapsto (c, \omega) = \int_c \omega. \tag{77}$$

Clearly, this inner-product is bilinear. i.e.

$$(c + c', \omega) = \int_{c+c'} \omega = \int_c \omega + \int_{c'} \omega = (c, \omega) + (c', \omega) \tag{78}$$

$$(c, \omega + \omega') = \int_c (\omega + \omega') = \int_c \omega + \int_c \omega' = (c, \omega) + (c\omega') \tag{79}$$

for c, $c' \in C_r(M)$ and ω, $\omega' \in \Omega^r(M)$. In other words, we can re-interpret Stoke's theorem as

$$(c, d\omega) = (\partial c, \omega) \tag{80}$$

In this sense, the de Rham differential is the adjoint operator of the boundary operator. From this, we can establish a duality between homology and cohomology groups. This is called the de Rham theorem.

Definition: If M is a compact manifold, $H_r(M)$ and $H^r(M)$ are finitely generated. The map $H_r(M) \times H^r(M) \mapsto \mathbb{R}$ is bilinear and non-degenerate.

We call the integral $\int_c \omega$ for a cycle c and a closed form ω a period. From Stoke's theorem, this integral vanishes when cycle c is a boundary or when ω is exact. We call the topological invariant $\dim H_r(M; \mathbb{R}) = \dim H^r(M; \mathbb{R})$ the rth betti number, which is certainly an integer. We denote this integer by k. Then from de Rham theorem, we can easily prove that for c_1, $c_2, \cdots$, $c_k \in Z_r(M)$ such that $[c_i] \neq [c_j]$,
(1) a closed r-form ω is exact if and only if

$$\int_{c_i} \omega = 0 \quad (1 \leq i \leq k) \tag{81}$$

(2) we can always choose a set of dual basis $\{[\omega_j]\}$ of $H^r(M)$ such that

$$\int_{c_i} \omega_j = \delta_{ij} \tag{82}$$

In other words, there always exists a closed r-form ω such that

$$\int_{c_i} \omega = b_i \quad (1 \leq i \leq k) \tag{83}$$

for any set of real numbers b_1, $b_2, \cdots$, b_k.

Let X and Y be two closed, connected oriented m-dimensional manifolds, $[c]$ being a homology class on X, represented by an r-cycle $c \in Z_r(X)$ and $[\omega]$ being the de Rham cohomology class on Y, represented by a closed r-form $\omega \in Z^r(Y)$. By definition, for a smooth map $f\colon X \mapsto Y$, one has

$$(f_*[c], [\omega]) = ([c], f^*[\omega]) \tag{84}$$

where f_* and f^* are induced maps on chains and forms. In particular, $f_*[X]$ must be integral multiple of $[Y]$. This is because under the map $f\colon X \mapsto Y$, the number of times that the push-forward of $[X]$ wraps around $[Y]$ can only be an integer. This is called the degree of mapping f or the winding number, which is denoted by $\deg f$. That is, we have

$$f_*[X] = \deg f[Y]. \tag{85}$$

From this equation, we see that

$$\deg f \int_Y \phi = \int_X f^* \phi \tag{86}$$

for any m-form ϕ on Y.

2.4 Fiber Bundle

2.4.1 Introduction

Most of the materials in this section are based on [1] [2] [3] [7] [13] [14]. One can find more details from them.

Definition: Let B, M and F be smooth manifolds. Let G be a Lie group, which has a left action on F. Let π: $B \mapsto M$ be a smooth projection. We call the structure (B, M, π, F) a smooth fiber bundle over M with structure group G if the following three conditions are satisfied

(a) There exists an open cover $\{U_\alpha | \alpha \in I\}$ such that for each $\alpha \in I$, there is a smooth diffeomorphism ϕ_α: $U_\alpha \times F \mapsto \pi^{-1}[U_\alpha]$ satisfying

$$\pi \circ \phi_\alpha(x, f) = x \tag{87}$$

for $\forall (x, f) \in U_\alpha \times F$.

(b) For each $x \in U_\alpha$ and arbitrary $f \in F$, denote $\phi_{\alpha,x}(f) = \phi_\alpha(x, f)$, then the map $\phi_{\alpha,x}$: $F \mapsto \pi^{-1}[x]$ is a smooth diffeomorphism, and when $x \in U_\alpha \cap U_\beta \neq \emptyset$, the smooth diffeomorphism $\phi_{\alpha,x}^{-1} \circ \phi_{\beta,x}$: $F \mapsto F$ is an element of Lie group G, denoted by $g_{\alpha\beta}(x)$, acting on F.

$$\phi_{\alpha,x}^{-1} \circ \phi_{\beta,x}(f) = g_{\alpha\beta}(x) f \tag{88}$$

for $\forall f \in F$.

(c) When $U_\alpha \cap U_\beta \neq \emptyset$, the map $g_{\alpha\beta}$: $U_\alpha \cap U_\beta \mapsto G$ is smooth.

We call the manifold B as the total space, M as the base space, F as the typical fiber, π as the projection and G as the structure group. We call the inverse map of ϕ_α, T_α: $\pi^{-1}[U_\alpha] \mapsto U_\alpha \times F$ the local trivialization of B, and function $g_{\alpha\beta}$ the transition function.

Theorem: Let M and F be two smooth manifolds. A Lie group G has left action on F. If there exist an open cover $\{U_\alpha | \alpha \in I\}$, such that for arbitrary α, $\beta \in I$, when $U_\alpha \cap U_\beta \neq \emptyset$, we have a smooth function $g_{\alpha\beta}$: $U_\alpha \cap U_\beta \mapsto G$ satisfying

(1) $g_{\alpha\alpha}(x) = e$ for $\forall \alpha \in I$, $\forall x \in U_\alpha$, where e is the identity element of G.

(2) $\forall \alpha,\beta,\gamma \in I$, when $U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset$,

$$g_{\alpha\beta}(x) g_{\beta\gamma}(x) g_{\gamma\alpha} = e \tag{89}$$

for $\forall x \in U_\alpha \cap U_\beta \cap U_\gamma$, then there exist a smooth fiber bundle structure (B, M, π, G), whose transition function is given by $g_{\alpha\beta}$.

Definition: Let (B, M, π, F) be a smooth fiber bundle over M, U be an open subset of M. If there exists a smooth map σ: $U \mapsto B$ satisfying $\pi \circ \sigma = \text{id}$: $U \mapsto U$, then σ is called a local smooth section of B over U. The set of smooth sections over M is denoted by $\Gamma(B)$.

Definition: Let (B, M, π, F) and $(\tilde{B}, M, \tilde{\pi}, \tilde{F})$ be two smooth fiber bundles whose structure group are both G. If they have the same transition function $g_{\alpha\beta}$: $U_\alpha \cap U_\beta \mapsto G$, then we call B and $\tilde{B}$ two associated fiber bundles.

Definition: Let E and M be two smooth manifolds, π: $E \mapsto M$ be smooth surjective map, and let V be an q-dimensional vector space over a field $\mathbb{R}$ or $\mathbb{C}$. If there exist an open cover $\{U_\alpha | \alpha \in I\}$ and a set of maps $\{\phi_\alpha | \alpha \in I\}$ satisfying

(1) ϕ_α: $U_\alpha \times V \mapsto \pi^{-1}[U_\alpha]$ is a smooth diffeomorphism, and for $\forall x \in U_\alpha$, $v \in V$, we have

$$\pi \circ \phi_\alpha(x, v) = x; \tag{90}$$

(2) For any $x \in U_\alpha$, denoting $\phi_{\alpha,x}(v) = \phi_\alpha(x, v)$, the map $\phi_{\alpha,x}(v)$: $V \mapsto \pi^{-1}[x]$ is diffeomorphism, and when $x \in U_\alpha \cap U_\beta \neq \emptyset$, the function

$$g_{\beta\alpha}(x) = \phi^{-1}_{\beta,x} \circ \phi_{\alpha,x} \tag{91}$$

is an linear isomorphism $V \mapsto V$, (i.e. $g_{\beta\alpha}(x) \in GL(q)$) and is smooth as a function $g_{\beta\alpha}$: $U_\alpha \cap U_\beta \mapsto GL(q)$, we call the structure (E, M, π) a vector bundle of rank-q over M. The function $g_{\alpha\beta}$ is called its transition function and its local trivialization is given by the inverse of ϕ_α, $T_\alpha : \pi^{-1}[U_\alpha] \mapsto U_\alpha \times V$. Similarly, its local smooth section is defined by σ: $U \mapsto E$, where $U \subset M$ is open in M, such that

$$\pi \circ \sigma = \mathrm{id} \tag{92}$$

is an identity map $U \mapsto U$. We denote the set of smooth sections of E over M by $\Gamma(E)$, which is a $C^\infty(M)$-module. But when $\Gamma(E)$ is regarded as a vector space over $\mathbb{R}$ or $\mathbb{C}$, it is infinite dimensional.

An example of vector bundle is the tangent bundle $T(M)$ over manifold M, whose fiber at each point $x \in M$ is its tangent space $T_x(M)$. The union of tangent spaces all over the base space M is its tangent bundle. Another example that we will encounter is the complex line bundle $\mathcal{L}(M)$, whose typical fiber is $\mathbb{C}$. When viewing complex plane $\mathbb{C}$ as the representation of a circle group, the complex line bundle that has $U(1)$ structure group becomes the associated vector bundle of a $U(1)$-principal bundle, which is introduced in the following definition.

Definition: Let M be a manifold and G a Lie group. A principal G-bundle over M consists of

(a) a Manifold P together with a free right action of G on P

$$G \times P \mapsto P, \quad (p, g) \mapsto R_g(p) = pg, \quad p \in P,\ g \in G \tag{93}$$

(b) a surjective map π: $P \mapsto M$ which is G-invariant, (i.e. $\pi(pg) = \pi(p)$ for all $p \in P$ and $g \in G$) satisfying local triviality condition: for each $x \in M$, there exists an open neighborhood U of x and a diffeomorphism

$$T_U : \pi^{-1}[U] \mapsto U \times G, \tag{94}$$

which locally is of form

$$T_U(p) = (\pi(p), S_U(p)) \tag{95}$$

for $\forall p \in \pi^{-1}[U]$, where map S_U: $\pi^{-1}[U] \mapsto G$ is G-equivariant, that is,

$$S_U(pg) = S_U(p)g \tag{96}$$

for all $p \in P$ and $g \in G$.

A principal G-bundle is a smooth manifold whose typical fiber is the same as its structure group G. For each fixed $p \in P$, the right action R: $P \times G \mapsto P$ induces a diffeomorphism which sends elements in G to the orbit $\pi^{-1}[\pi(p)]$, i.e. R_p: $G \mapsto \pi^{-1}[\pi(p)] \subset P$. In other words, R_p: $G \mapsto P$ is an embedding, and each fiber $\pi^{-1}[\pi(p)]$ can be regarded as a copy of the Lie group manifold G. The map R_p also brings the group structure to each fiber $\pi^{-1}[\pi(p)]$. But this group structure depends on the choice of point $p \in P$. Therefore, we cannot say that each fiber over a point $x \in M$ is the same as the typical fiber G.

Definition: Let T_U: $\pi^{-1}[U] \mapsto U \times G$ and T_V: $\pi^{-1}[V] \mapsto V \times G$ be two local trivializations of principal bundle $P(M, G)$, such that $U \cap V \neq \emptyset$. A map g_{UV}: $U \cap V \mapsto G$ is called transition function from T_V to T_U if

$$g_{UV}(x) = S_U(p)S_V(p)^{-1} \tag{97}$$

for any $x \in U \cap V$ and p satisfies $\pi(p) = x$.

Remark: the above definition is independent of the choice of point p in fiber $\pi^{-1}[x]$.

Theorem: Transition function g_{UV} has the following properties

(a) $g_{UU}(x) = e$, $\forall x \in U \cap V$;

(b) $g_{VU}(x) = g_{UV}(x)^{-1}$, $\forall x \in U \cap V$;

(c) $g_{UV}(x)g_{VW}(x)g_{WU}(x) = \text{id}$, $\forall x \in U \cap V \cap W$.

Definition: Let $P(M, G)$ be a principal bundle, and U be an open subset of M. A C^∞ map σ: $U \mapsto P$ is a local smooth section if

$$\pi(\sigma(x)) = x \tag{98}$$

for $\forall x \in U$.

Theorem: There is a one-to-one correspondence between local trivialization and local smooth section.

$$\sigma_V(x) = \sigma_U(x)g_{UV}(x) \tag{99}$$

when $x \in U \cap V$.

Definition: Let M be an n-dimensional manifold, T_xM be its tangent space at $x \in M$. Let (U, ϕ) be a local coordinate chart on M, with coordinate written as $\{x^\mu\}$. Let $\{e_\mu(x)\}$ be a frame of T_xM.

$$e_\mu(x) = e^\nu{}_\mu(x)\frac{\partial}{\partial x^\nu} \tag{100}$$

such that $\det(e) \neq 0$. Denoting the set of frames on M by $Fr(M)$, the set

$$F(M, GL(n)) = \{(x, e_\mu)|x \in M, e_\mu(x) \in Fr_x(M)\} \tag{101}$$

is called a frame bundle $F(M)$ over M, whose local chart is given by local diffeomorphism

$$\tilde{\phi} : \{(x, e_\mu) \in F(M)|x \in U, e_\mu(x) \in Fr_x(M)\} \mapsto \mathbb{R}^{n+n^2}, \tag{102}$$

The right action of $GL(n, \mathbb{R})$ acting on $F(M)$ is given by

$$g(x, e_\mu) = (x, e_\nu g^\nu{}_\mu) \tag{103}$$

where $g^\nu{}_\mu$ is an entry of $g \in GL(n, \mathbb{R})$. It has a natural surjective projection $\pi : F(M) \mapsto M$ such that

$$\pi(x, e_\mu) = x \tag{104}$$

and has a local trivialization $T_U : \pi^{-1}[U] \mapsto U \times GL(n, \mathbb{R})$ by assigning $T_U(x, e_\mu) = (x, h)$, where $h = S_U(x, e_\mu) \in GL(n, \mathbb{R})$ such that

$$h^\nu{}_\mu \frac{\partial}{\partial x^\nu} = e_\mu \tag{105}$$

Hence a frame bundle is a principal $GL(n, \mathbb{R})$-bundle, with structure group $GL(n, \mathbb{R})$.

Definition: Let M and $\tilde{M}$ be two manifolds. Let $B(M, G)$ and $\tilde{B}(\tilde{M}), \tilde{G})$ be principal bundles over M and $\tilde{M}$, respectively. If there exists a smooth map Φ: $B \mapsto \tilde{B}$ together with a Lie group morphism ϕ: $G \mapsto \tilde{G}$ such that for $\forall b \in B$ and $g \in G$, the following identity hold

$$\Phi(bg) = \Phi(b)\phi(g), \tag{106}$$

we call Φ: $B \mapsto \tilde{B}$ a bundle morphism. In particular, if Φ is an embedding, ϕ is an injective Lie group morphism, we call B a subbundle of $\tilde{B}$.

Definition: Let $(B(M, G))$ be a principal G-bundle over M and $K \subset G$ be a Lie subgroup of G. If there exist a principal K-bundle $\tilde{B}(M, K)$ over M, and a bundle morphism Φ: $\tilde{B}(M, K) \mapsto B(M, G)$, which induces a map $\Phi^b = \pi \circ \Phi \circ \tilde{\pi}^{-1}$: $M \mapsto M$ as an identity map on M, we say that the bundle $\tilde{B}$ is the reduction of bundle B.

For example, if manifold M admits a Lorentzian structure, we can talk about orthogonal tangent vectors on M and their normalization. In three dimension, if M has a Lorentzian structure $(-1, +1, +1)$, and we denote the orthogonal normalized frame by $\{\hat{e}_\mu\}$, then the frame bundle $F(M) = \{(x, \hat{e}_\mu)|x \in M\}$ becomes a principal $SO(2, 1)$-bundle over M.

Theorem: Let (B, M, π, G) be a principal G-bundle, F be a smooth manifold. G has a left action on F. We define a quotient space

$$\tilde{B} = B \times_G F = (B \times f)/\sim \tag{107}$$

where the equivalence relation is such that for (b, f), $(\tilde{b}, \tilde{f}) \in B \times F$, $(b, f) \sim (\tilde{b}, \tilde{f})$ iff there exist $g \in G$ such that

$$b = \tilde{b}g, \quad f = g^{-1}\tilde{f}. \tag{108}$$

Denoting the equivalent class as $[(b, f)]$, then $(\tilde{B}, M, \tilde{\pi}, F)$ is an associated bundle of (B, M, π, G), whose projection $\tilde{\pi}$: $\tilde{B} \mapsto M$ is given by

$$\tilde{\pi}([(b, f)]) = \pi(b) \tag{109}$$

When the typical fiber F is replaced by a vector space V, and ρ: $G \mapsto GL(V)$ is a representation of G on V, we define the equivalence relation as $(b, v) \sim (\tilde{b}, \tilde{v})$ iff $\exists g \in G$ such that $(\tilde{b}, \tilde{v}) = (bg, \rho(g^{-1})v)$, and define a projection $\tilde{\phi}$: $B \times_\rho V \mapsto M$, by

$$\tilde{\pi}([(b, v)]) = \pi(b) \tag{110}$$

then the quotient space $E = B \times_\rho V = B \times V/\sim$ becomes an associated vector bundle of principal G-bundle over M.

For instance, the associated vector bundle of a frame bundle $F(M)$ is the tangent bundle $T(M)$. When there is a Lorentzian structure on the base space M, i.e. $F(M)$ is an $SO(2, 1)$-principal bundle, then we have an associated vector bundle over M whose transition functions are elements in $SO(2, 1)$. Another example is complex vector bundle $E \mapsto M$, whose typical fiber is n-dimensional complex vector space $\mathbb{C}^n$. It has structure group $GL(n, \mathbb{C})$. If we can consider a Hermtian structure on manifold M, the structure group is then $U(n)$. We have mentioned that a complex line bundle can be viewed as an associated vector bundle of $U(1)$-principal bundle, which is a reduction bundle of $GL(1, \mathbb{C})$-principal bundle. A generic complex line bundle has structure group $GL(1, \mathbb{C}) = \mathbb{C}^*$, which can be reduced to the circle bundle.

By definition, for any principal bundle (P, M, π, G), we can always find a subspace of T_pP defined by

$$\mathrm{Ver}_p = \{X \in T_pP | \pi_*(X) = 0\}\,, \tag{111}$$

which is called a **vertical subspace** of T_pP. Apparently, vertical subspace is a vector space consists of vectors $X \in T_pP$ that are tangent to the fiber $\pi^{-1}[\pi(p)]$. i.e. $\mathrm{Ver}_p = T_p\pi^{-1}[\pi(p)]$. Since each fiber is diffeomorphic to the typical fiber, which for principal bundle is Lie group G, it is reasonable to believe that the vertical subspace is isomorphic to the Lie algebra $\mathfrak{g}$ of G.

Theorem: Let (P, M, π, G) be a principal bundle, Ver_p be a vertical subspace at point $p \in P$. Let $\mathfrak{g}$ be the Lie algebra of structure group G. Then there is an isomorphism $\mathrm{Ver}_p \simeq \mathfrak{g}$. This isomorphism is exactly the push-forward R_{p*}.

Definition: For a fixed $\mathcal{A} \in \mathfrak{g}$, at each point $p \in P$, we attach a vertical vector $\mathcal{A}^*_p$ defined by

$$\mathcal{A}^*_p = R_{p*}\mathcal{A} \tag{112}$$

for $\forall p \in P$. Hence each Lie algebra element $\mathcal{A}$ can generate a vertical vector field living in P, which is called a fundamental vector field induced by $\mathcal{A} \in \mathfrak{g}$.

Theorem: Let T_U be a local trivialization, $x \in U$. The diffeomorphism S_U: $\pi^{-1}[x] \mapsto G$ induces a push-forward S_{U*} which maps fundamental vector fields on $\pi^{-1}[x]$ to left-invariant vector fields on G.

$$S_{U*}\mathcal{A}^* = \bar{\mathcal{A}} \tag{113}$$

where $\bar{\mathcal{A}}$ is a left invariant vector field on G. Since the set of all left-invariant vector fields on a Lie group is identified as its Lie algebra, this theorem shows that there is a one-to-one correspondence between the set of all fundamental vector fields on the fiber over a point with the Lie algebra of the structure group. Although a vertical vector generates a left-invariant vector field on G, itself is not invariant under the right action of G

Theorem: Under the right action, a vertical vector field $\mathcal{A}^*$ transforms in the following way

$$R_{g*}\mathcal{A}^*_p = \left(\mathrm{Ad}_{g^{-1}}\mathcal{A}\right)^*_{pg} \tag{114}$$

where $\forall p \in P$, $g \in G$ and $\mathcal{A} \in \mathfrak{g}$.

To further identify the set of fundamental vector fields on fiber $\pi^{-1}[\pi(p)]$ with Lie algebra $\mathfrak{g}$ under the isomorphism R_{p*}: $\mathfrak{g} \mapsto \mathrm{Ver}_p$, we need to compute the commutators of two fundamental vector fields. The result shows that

$$[\mathcal{A}^*, \mathcal{B}^*] = [\mathcal{A}, \mathcal{B}]^* \tag{115}$$

where $[\mathcal{A}^*, \mathcal{B}^*]$ is the commutator of two vector fields $\mathcal{A}^*$ and $\mathcal{B}^*$ while $[\mathcal{A}, \mathcal{B}]$ is the Lie bracket of $\mathcal{A}$, $\mathcal{B} \in \mathfrak{g}$. Therefore, this is indeed a Lie-algebra isomorphism.

Definition a: Let (P, M, π, G) be a principal bundle with structure group G and canonical prejection $\pi : P \to M$. For each point $p \in P$, we define the subspace Ver_p of the tangent space T_pP satisfying

$$\mathrm{Ver}_p = \{X \in T_pP \bigm| \pi_*(X) = 0\} \tag{116}$$

The connection on P is given by a subspace $\mathrm{Hor}_p \subset T_pP$ such that

(a) $T_pP = \text{Hor}_p \oplus \text{Ver}_p$

(b) $R_{g^*}[\text{Hor}_p] = \text{Hor}_{pg}$, where $g \in G$

(c) Hor_p is an n-dimensional smooth distribution on P

Definition b: A connection on a principal bundle (P, M, π, G) is a $C^\infty(M)$ $\mathfrak{g}$-valued 1-form ω satisfying

(a) $\omega_p(\mathcal{A}^*_p) = \mathcal{A}$, $\forall \mathcal{A} \in \mathfrak{g}$ and $\forall p \in P$.

(b) $\omega_{pg}(R_{g*}X) = \text{Ad}_{g^{-1}}\omega_p(X)$, $\forall p \in P$, $g \in G$ and $X \in T_pP$.

Definition c: Let (P, M, π, G) be a principal bundle with canonical projection $\pi : P \to M$. Let $\{U_i\}_{i\in I}$ be a collection of open subsets of M. For any two local trivializations

$$T_U : \pi^{-1}[U] \to U \times G \quad \text{and} \quad T_V : \pi^{-1}[V] \to V \times G$$

associated with local smooth sections σ_U and σ_V, with transition function g_{UV} and $U \cup V \neq \emptyset$. if there exist $\mathfrak{g}$-valued 1-form ω satisfying

$$\omega|_V = g^{-1}_{UV}\omega|_U g_{UV} + g^{-1}_{UV} dg_{UV} \tag{117}$$

,where $\sigma^*_U\omega = \omega|_U$. If (P, M, π, G) is a frame bundle with structure group $G = SO(n, 1)$ over an $n + 1$ dimensional spacetime, we say $\sigma^*\omega$ defined above is the spin-connection on M.

It can be proved that the above three definitions are equivalent. A connection 1-form ω defined on principal bundle (P, M, π, G) can always be defined as a 1-form on base space M by using pull-back induced by a local section, $\sigma^*\omega$, called local gauge potential, which is usually denoted as $\sigma^*\omega = A$. Connection as a horizontal smooth distribution on P is globally defined on fiber bundle, but once a connection as a Lie algebra-valued 1-form descends onto base space M as $A = \sigma^*\omega$, it is locally defined. In physics, choosing a local smooth section is called a choice of local gauge. The transition functions form a group, $\mathcal{G} = \text{Hom}(M, G)$, called gauge group. Furthermore, it can be shown that the equivalence between the above three definitions of connection implies the following theorem.

Theorem: Let ω be the connection given by **definition c**, then the space Hor_p given by **definition a**is simply

$$\text{Hor}_p = \{X \in T_pP \;\big|\; \omega_p(X) = 0\} \tag{118}$$

for $\forall p \in P$.

Definition: Let (P, M, π, G) be a principal bundle with structure group G. It's connection is given by a $\mathfrak{g}$-valued 1-form ω. Let $p \in P$ and $v, w \in T_pP$. A curvature 2-form Ω is defined by

$$\Omega_p(v, w) = (d\omega)_p \circ \text{Hor}(v, w) = (d\omega)_p(v^H, w^H) \tag{119}$$

In differential geomtry, the differential operator given by $d\omega \circ \text{Hor}$ is called the covariant exterior differential and is denoted by $d_\nabla\omega = d\omega \circ \text{Hor}$.

Theorem (Cartan's 1st Structure Equation):

$$\Omega = d\omega + \omega \wedge \omega \tag{120}$$

A curvature 2-form Ω is globally defined on principal bundle (P, M, π, G). If σ is a local smooth section, then the pull-back $\sigma^*\Omega = F$ is a $\mathfrak{g}$-valued 2-form defined on base space M, which is called the local field strength. Once connection 1-form is descended on base space, we still have

$$F = dA + A \wedge A \tag{121}$$

i.e. the Cartan's 1st structure equation still holds locally on M.

Theorem: Let Hor be a connection on a principal bundle (P, M, π, G), and $x \in M$. Let γ: $(-\epsilon, \epsilon) \mapsto M$ be a smooth curve on base space M, such that $\gamma(0) = x$. Then for $\forall p \in \pi^{-1}[x]$, there exists a unique smooth curve $\tilde{\gamma}$: $(-\epsilon, \epsilon) \mapsto P$, such that $\tilde{\gamma}(0) = p$, $\pi(\tilde{\gamma}(t)) = \gamma(t)$ and $\dfrac{d}{dt}\tilde{\gamma} \in \text{Hor}(\gamma(t))$. We say $\tilde{\gamma}(t)$ the horizontal lift of curve $\gamma(t)$, or parallel transport from point $\gamma(0)$ to $\gamma(t)$.

Connections, curvature form and horizontal lift on vector bundles can be defined in similar ways. The only difference is to replace typical fiber by a vector space V, which is the representation space of structure Lie group. A connection on a vector bundle is defined as follows.

Definition: Let (E, M, π) be a vector bundle over M, $\Gamma(E)$ be the set of smooth sections and $\mathcal{X}$ is the set of vector fields on M. A connection on E is a map ∇: $\Gamma(E) \times \mathcal{X}(E) \mapsto \Gamma(E)$, such that

(a) $\nabla_{X+fY}\xi = \nabla_X\xi + f\nabla_Y\xi$

(b) $\nabla_X(\xi + \lambda\eta) = \nabla_X + \lambda\nabla_X\eta$

(c) $\nabla_X(f\xi) = X(f)\xi + f\nabla_X\xi$

for $\forall X$,$Y \in \Gamma(E)$, ξ,$\eta \in \mathcal{X}(M)$, $\lambda \in \mathbb{R}$ and $f \in C^\infty(M)$.

Definition: Let (P, M, π, G) be a principal bundle with connection Hor, and connection 1-form ω. Let V be a vector space and ρ: $G \mapsto GL(V)$ be a representation of Lie group G. Then Hor induces a connection ∇ on associated vector bundle $E = P \times_\rho V$. Let s be a local smooth section of E, $\gamma(t)$ be a smooth curve on M, who has horizontal lift $\tilde{\gamma}$ on P. Then $s(t)|_{\gamma(t)} = [(\tilde{\gamma}(t), v(t))]$ is the restriction of local section s on $\gamma(t)$. The induced connection on E is given by

$$\nabla_{\dot{\gamma}} s(t) = [(\tilde{\gamma}(t), \frac{d}{dt}v(t))] \tag{122}$$

The geometric significance of the above formula is that we can think of the principal bundle P as a frame bundle. A point in this bundle is a frame over a point in M. We choose the horizontal lift $\tilde{\gamma}(t)$ in principal bundle P as a parallel frame as a 'reference' over curve γ in M. Then time dependent vector $v(t) \in V$ is the 'component' of vector field $s(t)$ in such a frame. Hence the covariant derivative of $s(t)$ along γ is simply the derivative of its component. Moreover, the representation ρ: $G \mapsto GL(V)$ induces a push-forward ρ_*: $\mathfrak{g} \mapsto \mathfrak{gl}(V)$. If ω is the connection 1-form on principal bundle P, and σ is a local smooth section of P, then we can define the connection 1-form on its associated vector bundle $P \times_\rho V$ in the following way.

$$\rho_*(\sigma^*(\omega)) = \sigma^*(\rho_*(\omega)) \tag{123}$$

The above 1-form defined on M is $\mathfrak{gl}(V)$-valued. Let $\{e_a\}$ be a basis of vector spce V. Local components of the above 1-form satisfy

$$(\rho_*(\omega))(e_a) = \omega_a^b e_b, \quad (\rho_*(\sigma^*(\omega)))(e_a) = \tilde{\omega}_a^b e_b \tag{124}$$

Let $\{s(t)_a\}$ be a local frame on associated vector bundle $E \tilde{\mapsto} M$, the induced covariant derivative is given by

$$\nabla s(t)_a = \tilde{\omega}_a^b(t) s_b(t) \tag{125}$$

If base space M has a Lorentzian structure, then we still call $\tilde{\omega}$ a spin connection on M since the connection 1-form $\tilde{\omega}$ on associated vector bundle E induced by ω on principal bundle are totally equivalent.

Definition (Cartan's 2nd Structure Equation): Let (F, M, π, G) be a frame bundle over an n-dimensional manifold M with structure group G and connection 1-form ω. Let $\{e^a(x)\}$ be a basis of T_x^*M, where $x \in M$. We define a $\mathbb{R}^n$-valued 2-form, called torsion form, of $F(M)$ by equation

$$\Theta = de + \omega \wedge e. \tag{126}$$

By using the same formula, we can define the torsion tensor on its associated vector bundle $T(M)$. If there is a smooth local section σ, and we denote $T = \sigma^*(\Theta)$ as the local torsion form on M, then we have

$$T = de + \omega \wedge e \tag{127}$$

It is easy to see that the locally the curvature form and torsion form given by Cartan's formulae on associated vector bundle agree with the original definitions, i.e. $F = \frac{1}{2}[\nabla_a, \nabla_b] dx^a \wedge dx^b$, $T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$ for $X, Y \in \mathcal{X}(M)$.

Theorem (Bianchi Identities): Let $F(M)$ be a frame bundle over M, with connection 1-form ω. Let $\{e^a(x)\}$ be a basis of T_x^*M. We denote its curvature by Ω and denote its torsion by T, then they satisfy the following identities.

$$d_\nabla \Omega = d_\nabla \circ d_\nabla \omega = d\Omega + \omega \wedge \Omega - \Omega \wedge \omega = 0 \tag{128}$$

$$d\Theta + \omega \wedge \Theta = \Omega \wedge e \tag{129}$$

The above equations also hold for vector bundle $T(M)$. If we have a local smooth section σ, denoting $\sigma^*\Omega = F$, $\sigma^*\Theta = T$ and $\sigma^*\omega = A$, then we would have

$$d_\nabla F = d_\nabla \circ d_\nabla A = dF + A \wedge F - F \wedge A = 0 \tag{130}$$

$$dT + A \wedge T = F \wedge e \tag{131}$$

That is to say, the Bianchi identities also hold locally.

Theorem: Let Ω be the curvature form of a principal bundle (P, M, π, G) with structure group G that has right action R_g for $g \in G$. Then we have the following identity

$$R_g^* \Omega = \mathrm{Ad}_{g^{-1}} \circ \Omega \tag{132}$$

If U_i and U_j are two open subsets of M such that $U_i \cap U_j \neq \emptyset$, with transition function $g_{ij}\colon U_i \cap U_j \mapsto G$, and $\sigma_i\colon U_i \mapsto P$ is a smooth local section, denoting the local field strength as $F = \sigma^*\Omega$, then the field strength transforms under gauge transformation in the following way,

$$F_j = g_{ij}^{-1} F_i g_{ij}. \tag{133}$$

If (P, M, π, G) is a frame bundle which possesses torsion form Θ, under right action of G, the torsion form transforms as

$$R_g^*(\Theta) = g^{-1}\Theta, \tag{134}$$

which means that under right action, the torsion form transforms equivariantly.

Noting that definition of dreibeins gives rise of the concept of frame fields on tangent bundle $T(M)$, we can think of frames as some orthogonal basis of our tangent vectors on spacetime, i.e.

$$V(x) = V^\mu(x)\partial_\mu = V^a(x)e_a(x) = V^a(x)e_a^\mu(x)\partial_\mu \tag{135}$$

The spin-connection defined on $F(M)$ induces a corresponding connection on its associated bundle $T(M)$. This is given by:

$$\nabla_{\dot{\gamma}} V(\gamma(t))|_{t=0} = e_a(0)\frac{d}{dt}\Big|_{t=0} V^a(t) \tag{136}$$

From the above equation, it can be proved that the covariant derivative given by the connection satisfies the following equation

$$\nabla_\mu e_a(\gamma(t)) = \omega^b_{\mu a} e_b(\gamma(t)) \tag{137}$$

We can derive the formula for covariant derivative corresponding to spin-connection acting on an arbitrary vector field $E(x)$. It is given by

$$\nabla_\mu E(x) = (\partial_\mu E^a(x))e_a(x) + \omega^a_{\mu b}(x)E^b(x)e_a(x), \tag{138}$$

If we impose a further requirement that the spin-connection must be compatible with flat metric and torsion-free, then in principle, this covariant derivative should be equivalent with the covariant derivative of levi-civita connection, i.e.

$$D_\mu V(x) = (\partial_\mu V^\alpha(x))\partial_\alpha + \Gamma^\rho_{\mu\beta}V^\beta(x)\partial_\rho = (\partial_\mu V^a(x))e_a + \omega^a_{\mu b}V^b(x)e_a \tag{139}$$

From equation (135), we find the relations between spin-connection ω and levi-civita connection Γ:

$$\Gamma^\nu_{\mu\lambda} = e^\nu_a(\partial_\mu e^a_\lambda) + e^\nu_a e^b_\lambda \omega^a_{\mu b} \tag{140}$$

$$\omega^a_{\mu b} = e^a_\nu e^\lambda_b \Gamma^\nu_{\mu\lambda} - e^\lambda_b(\partial_\mu e^a_\lambda) \tag{141}$$

It is worth mentioning that we introduced two types of quantities on a vector bundle. The first one is $\mathfrak{g}$-valued differential forms such as connection and curvature. The other one is torsion, which is just an ordinary differential form. Connection and curvature are called End(E)-valued differential forms while torsion is called E-valued differential form. These notations will give us great advantages in many discussions.

Definition: Let $E(M)$ be a vector bundle over manifold M equipped with connection D, we define E-valued p-form ω to be a section of $E \otimes \bigwedge^p T^*M$. i.e. $\omega \in \Gamma\left(E \otimes \bigwedge^p T^*M\right)$. If ρ is an ordinary differential form on M, and s is a smooth section on E, then we define the covariant exterior differential by

$$d_D\left(s \otimes \rho\right) = Ds \wedge \rho + s \otimes d\rho \tag{142}$$

The E-valued differential form is well-defined iff we assume that $(s \otimes \rho) \wedge \mu = s \otimes (\rho \wedge \mu)$ for any ordinary differential form μ.

Definition: We define the wedge product of an End(E)-valued form $A \otimes \rho$ and E-valued form $s \otimes \lambda$, and the wedge product of $A \otimes \rho$ with another End(E)-valued form $B \otimes \mu$

$$(A \otimes \rho) \wedge (s \otimes \lambda) = A(s) \otimes (\rho \wedge \lambda) \tag{143}$$

$$(A \otimes \rho) \wedge (B \otimes \mu) = AB \otimes (\rho \wedge \mu) \tag{144}$$

It is easy to prove that using the above notations, we have for any E-valued form η,

$$d_D^2 \eta = F \wedge \eta \tag{145}$$

where $F = \frac{1}{2}[D_\mu, D_\nu]dx^\mu \wedge dx^\nu$ is the curvature of D. Then it is easy to see that the Bianchi identity is given by $d_D F = 0$. Physicists are interested in these mathematical formalism because it can be used to generalize the Maxwell equations naturally to Yang-Mills equations

$$d_D F = 0 \quad \star d_D \star F = J$$

The covariant exterior differential operator on vector bundle $E(M)$ still satisfies the Leibniz law

$$d_D (A \wedge B) = d_D A \wedge B + (-1)^p A \wedge d_D B \tag{146}$$

Noting that End(E) can be regarded as a matrix (or algebra element in some representation), we can define the commutators between two End(E)-valued differential forms $A = A^a T_a$ and $B = B^b J_b$, where A^a together with B^a are ordinary differential forms; T_a and J_a are some matrices.

$$[A, B] = A \wedge B - (-1)^{pq} B \wedge A = A^a \wedge B^b [T_a, J_b] \tag{147}$$

This commutator satisfies $[A, B] = -(-1)^{pq}[B, A]$.

Using the above conventions, we can simplify the covariant exterior differentials. For any E-valued form ω and End(E)-valued form η, their covariant exterior differentials are given by

$$d_D \omega = d\omega + A \wedge \omega \quad \text{and} \quad d_D \eta = d\eta + [A, \eta]$$

The latter formula is correct for any End(E)-valued differential form except connection 1-form. The covariant exterior differential of 1-form A is given by $d_D A = dA + A \wedge A = dA + \frac{1}{2}[A, A]$. With the foregoing introduction to covariant exterior differential operators, we can easily prove the following theorems that are extremely important.

Theorem: Let $E \mapsto M$ be a vector bundle over M. A is an End(E)-valued p-form and B is an End(E)-valued q-form, then we have

$$\text{Tr}\,(A \wedge B) = (-1)^{pq} \text{Tr}\,(B \wedge A) \tag{148}$$

which is called the graded cyclic property of trace. It further implies that Tr$[A, B] = 0$.

Theorem: Let D be the connection on vector bundle $E \mapsto M$ and B be given above, then we have

$$\text{Tr}\,(d_D B) = \text{Tr}\,(dB) = d\text{Tr}\,(B) \tag{149}$$

In other words, we can exchange the order of trace and exterior differential.

Theorem: If M is an oriented n dimensional manifold, A and B are the End(E)-valued forms given above, with $p + q = n - 1$, then

$$\int_M \text{Tr}\,(d_D A \wedge B) = (-1)^{p+1} \int_M \text{Tr}\,(A \wedge d_D B) \tag{150}$$

and if M is (semi)-Riemannian, with $p + q = n$, then

$$\int_M \text{Tr}\,(A \wedge \star B) = \int_M \text{Tr}\,(B \wedge \star A) \tag{151}$$

From the definition of commutators, we can easily derive the following formula that is extremely important. Let $F = dA + A \wedge A$, where A is the connection on $E \mapsto M$, if A is parametized by s, we have the variation of curvature F given by

$$\delta F = \frac{d}{ds} (dA_s + A_s \wedge A_s)_{s=0} = d\delta A + \delta A \wedge A + A \wedge \delta A = d\delta A + [A, \delta A] = d_D \delta A \tag{152}$$

2.4.2 Hopf Fibration and Classifying Spaces

There are many examples of fiber bundles. For example, a cylinder is a bundle space whose typical fiber is the $\mathbb{R}$ and its base space is a circle. We can cut throught two parallel straight lines in $\mathbb{R}^2$ and then glue the two sides of the infinitely long strip together. This bundle is obviously trivial since it is globally a product $\mathbb{S}^1 \times \mathbb{R}$. Alternatively, we can construct a non-trivial bundle such as a Möbius strip. The difference is that in this case, we have to twist the strip by $n\pi$ angle and then glue the two sides together. In other words, the integer n meansures how far the bundle deviates from a trivial one. The very first example of a non-trivial circle bundle over a 2-sphere is called Hopf fibration, discovered by Heinz Hopf in 1931, which shows that a 3-sphere has a principal bundle $\mathbb{S}^3 \mapsto \mathbb{S}^2$ structure, whose typical fiber is a circle $\mathbb{S}^1$. This fiber bundle is definitely not trivial. To exhibit the principal fiber bundle structure of a 3-sphere, we use the embedding

$$a^2 + b^2 + c^2 + d^2 = 1 \tag{153}$$

We define the projection map $\mathbb{S}^3 \mapsto \mathbb{S}^2$

$$x = a^2 + b^2 - c^2 - d^2, \quad y = 2(ad + bc), \quad z = 2(bd - ac) \tag{154}$$

From this projection we see that $x^2 + y^2 + z^2 = 1$. If we denote $u = a + ib$ and $v = c - id$, then the above embedding becomes $|u|^2 + |v|^2 = 1$, from which we can observe that any $U(1)$-action preserves the projection. In other words, $U(1) \hookrightarrow \mathbb{S}^3 \to \mathbb{S}^2$ becomes a $U(1)$-principal bundle and is not trivial. Another important fact about $U(1)$ Hopf fibration is that the construction $U(1) \hookrightarrow \mathbb{S}^{2n+1} \mapsto \mathbb{CP}^n$ is in fact the restriction of a tautological line bundle over $\mathbb{CP}^n$ to the unit sphere in $\mathbb{C}^{n+1}$, which is very easy to see.

In general, we can consider the complex hopf fibration $\mathbb{S}^{2n+1} \mapsto \mathbb{CP}^n$ as a principal $U(1)$-bundle. From this we can obtain a tower of hopf fibrations by two series of inclusions

$$\mathbb{S}^3 \subset \mathbb{S}^5 \subset \mathbb{S}^7 \subset \cdots$$
$$\mathbb{CP}^1 \subset \mathbb{CP}^2 \subset \mathbb{CP}^3 \subset \cdots$$

Taking the limit we get space $\mathbb{S}^\infty$ as a principal $U(1)$-bundle over $\mathbb{CP}^\infty$. We will see that this is closely related with $U(1)$-gauge theory, i.e. electromagnetic fields. This space is essential for the discussion in Dirac monopole and its quantization. It is a surprise that although $\mathbb{CP}^n$ is topologically a hypersphere, it becomes different in the limit $n \to \infty$. An infinite dimensional sphere $\mathbb{S}^\infty$ is contractible, while $\mathbb{CP}^\infty$ is not.

In the $SU(2)$ case, we have a similar construction of towers of inclusions. This is obtained from quaternionic Hopf fibration $\mathbb{S}^{4n+3} \mapsto \mathbb{HP}^n$, which is a principal $SU(2)$-bundle. By considering the towers

$$\mathbb{S}^7 \subset \mathbb{S}^{11} \subset \mathbb{S}^{15} \subset \cdots$$
$$\mathbb{HP}^1 \subset \mathbb{HP}^2 \subset \mathbb{HP}^3 \subset \cdots$$

we get a principal $SU(2)$-bundle $\mathbb{S}^\infty \mapsto \mathbb{HP}^\infty$. This case is used in Yang-Mills theory and the quantization of $SU(2)$-instantons.

Definition: Let G be a Lie group. The classification of principal G-bundles over a manifold M is achieved by the classifying spaces. A topological space $B_k(G)$ is said to be k-classifying for G if the following conditions hold:
1. There exists a contractible space $E_k(G)$ on which G acts freely and $B_k(G)$ is the quotient of $E_k(G)$ under this G-action such that

$$E_k(G) \mapsto B_k(G) \tag{155}$$

is a principal G-bundle.
2. Given a manifold M of $\dim(M) \leq k$ and a principal bundle $P(M, G)$, there exists a continuous map

$f\colon M \mapsto B_k(G)$ such that the pull-back $f^*(E_k(G))$ to M is a principal bundle with structure group G that is isomorphic to P.

Theorem: Let G be a Lie group with $\pi_0(G) < \infty$ and k a positive integer. Then there exists a principal G-bundle $E^k(B^k, G)$ with connection θ^k, which is k-universal for all principal G-bundles with connections. i.e For any compact manifold M with $\dim M < k$ and a principal G-bundle $P(M, G)$ with a connection A, there exists a map $f\colon M \mapsto B^k$, defined up to homotopy such that P is the pull-back of E^k to M by f. Then we have the following commutative diagram:

$$\begin{array}{ccc} P & \xrightarrow{\hat{f}} & E^k \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi^k} \\ M & \xrightarrow[f]{} & B^k \end{array}$$

where π and π^k are bundle projections. Any principal bundle P with connection form A can be constructed as an induced bundle $P = f^*(E^k)$ with $A = (\hat{f})^*\theta^k$. In practice, we usually take $k \to \infty$ and denote the large k limit of $E^k(B^k, G)$ and $B^k(G)$ by EG and BG, respectively. We call BG the classifying space of Lie group G and EG the universal principal G-bundle. We can prove that for $U(1)$ and $SU(2)$ case, they are precisely given by the towers of Hopf fibrations.

2.4.3 Dirac Quantization and Chern Class

In physics, the most important example of principal $U(1)$-bundle over a sphere (or a complex line bundle over sphere) is the classical electrodynamics. In this theory, we denote the local gauge potential and local field strength as

$$s^*\omega = A, \quad s^*\Omega = F \tag{156}$$

where s is a local smooth section, ω is the connection on principal $U(1)$-bundle $P(M)$ and Ω is the curvature form on this bundle. Let U_i and U_j be two open subset of M, $s_i\colon U_i \mapsto P$ and $s_j\colon U_j \mapsto P$ be two local smooth sections with $U_i \cap U_j \neq \emptyset$. We denote the transition function by g_{ij}. Since $U(1)$ group is Abelian, we have

$$F_j = g_{ij}^{-1} F_i g_{ij} = F_i \tag{157}$$

Let Λ be the generator of $U(1)$ group, so we have $g_{ij}(x) = e^{\Lambda_{ij}(x)}$. Consequently, $g_{ij}^{-1} dg_{ij} = d\Lambda_{ij}$ and

$$A_j = A_i + d\Lambda_{ij} \tag{158}$$

Hence, for Abelian gauge theories, although gauge potentials do not agree on intersections, the corresponding local field strength is in fact globally defined throughout spacetime. In physics, the field strength is usually denoted as follows

$$\left(F^{\alpha\beta}\right) = \begin{pmatrix} 0 & E^1 & E^2 & E^3 \\ -E^1 & 0 & B^3 & -B^2 \\ -E^2 & -B^3 & 0 & B^1 \\ -E^3 & B^2 & -B^1 & 0 \end{pmatrix} \tag{159}$$

If we apply a hodge star operator, then we exchange electric field and magnetic field

$$\left(*F_{\alpha\beta}\right) = \begin{pmatrix} 0 & B^1 & B^2 & B^3 \\ -B^1 & 0 & E^3 & -E^2 \\ -B^2 & -E^3 & 0 & E^1 \\ -B^3 & E^2 & -E^1 & 0 \end{pmatrix} \tag{160}$$

We consider a simple case, the Coulomb field. i.e. a static electric point charge which carries charge n localed at origin in $\mathbb{R}^{1,3}$. From elementary physics, we learned that the field strength of the electric field is given by

$$F = \frac{n}{\rho^3}\left(x^1dx^1 + x^2dx^2 + x^3dx^3\right)\wedge dx^0, \tag{161}$$

from which we can find an expression of the gauge potential, $A = -\frac{n}{\rho}dx^0$. Clearly, this gauge potential happens to be globally defined on $\mathbb{R}^{1,3} - \left\{\left(x^0, 0, 0, 0\right)\right\}$. In other words, this principle $U(1)$ bundle is trivial and the class $[F]$ is a trivial element in the cohomology group of $\mathbb{R}^{1,3} - \left\{\left(x^0, 0, 0, 0\right)\right\}$. Since the charge is static, it is relatively convenient to focus on spacial slice at a constant time. i.e. We only study the cohomology group of $\mathbb{S}^2$. We have seen that it is given by

$$H^k(\mathbb{S}^2) = \begin{cases}\mathbb{R} & k = 0, 2\\ 0 & k = 1\end{cases} \tag{162}$$

Next we need to find out the non-trivial elements in this cohomology group. This is given by magnetic charge. It is clear that for the above given field strength F, its dual field is given by

$$*F = \frac{n}{\rho^3}\left(x^1dx^2\wedge dx^3 - x^2dx^1\wedge dx^3 + x^3dx^1\wedge dx^2\right) \tag{163}$$

If we restrict this field strength on a unit sphere enclosing the magnetic charge at the origin, the field strength reduces to

$$*F|_{\mathbb{S}^2} = n\left(x^1dx^2\wedge dx^3 - x^2dx^1\wedge dx^3 + x^3dx^1\wedge dx^2\right) \tag{164}$$

This is nothing but the standard volume form on $\mathbb{S}^2$, whose integral gives the area 4π. In conclusion, the integral $\int [F]$ fails to detect the electric charge enclosed by the sphere but can detect the magnetic charge in it. For this reason, only magnetic charge can be chosen as a candidate which may encode information about the topology of the principal $U(1)$-bundle.

From the above analysis, we only need to consider a static magnetic charge g, whose field strengh is given by

$$(*F_{\alpha\beta}) = \frac{g}{\rho^3}\begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & x^3 & -x^2\\ 0 & -x^3 & 0 & x^1\\ 0 & x^2 & -x^1 & 0\end{pmatrix} \tag{165}$$

Thus,

$$*F = \frac{g}{\rho^3}\left(x^1dx^2\wedge dx^3 - x^2dx^1\wedge dx^3 + x^3dx^1\wedge dx^2\right) \tag{166}$$

In accordance with notations in standard textbooks, we denote the dual field strength $*F$ by F for magnetic field strength, which plays a role as $*F$ for a Coulomb field. Using spherical coordinate, the field strength is given by

$$F = g\sin\theta d\theta\wedge d\phi \tag{167}$$

Noticing that this expression is independent of ρ and x^0, we can simply restrict our discussion on a unit sphere $\rho = 1$ and take a constant time slice. i.e. We construct a principal $U(1)$-bundle over $\mathbb{S}^2 = \mathbb{CP}^1$. Since the gauge potential of this field is not globally defined, we can choose two local gauge potential

$$A_N = g(1-\cos\theta)d\phi, \quad A_S = -g(1+\cos\theta)d\phi, \tag{168}$$

each of which is well-defined on northern hemi-sphere or southern hemi-sphere. It is necessary that such two local gauge potentials should be related with each other on the equator by a $U(1)$-gauge transformation, denoted by $g_{NS}(\phi) = e^{\Lambda(\phi)}$. (i.e. $g_{NS}\colon \mathbb{S}^1 \mapsto \mathbb{R}$) Therefore we have

$$A_N = g_{NS}^{-1}A_S g_{NS} + g_{NS}^{-1}dg_{NS} = A_S + d\Lambda(\phi). \tag{169}$$

From the expressions of gauge potentials, we also have $A_N - A_S = 2gd\phi$. Noticing that on the equator, the transition function g_{NS} is well-defined only if $g_{NS}(\phi) = g_{NS}(\phi + 2\pi)$, we have

$$\Lambda(2\pi) - \Lambda(0) = \int_0^{2\pi} d\Lambda = \int_0^{2\pi} 2gd\phi = 4\pi g = 2in\pi \tag{170}$$

where $n \in \mathbb{Z}$. The transition function is thus given by $g_{NS} = e^{in\phi}$. In other words, the magnetic charge can only take discrete values, $-ig = \frac{n}{2}$. When $n = 0$, the charge vanishes and the monopole bundle is trivial. Therefore this integer measures how much this principal bundle $U(1) \hookrightarrow P \mapsto \mathbb{S}^2$ is twisted compared with a trivial bundle. This is our first example of characterist class of a principal bundle. We call it the first Chern class of a $U(1)$ bundle, denoted by c_1. Since it is closed, it is a representative of the cohomology class on sphere. A simple calculation shows that

$$\int_{\mathbb{S}^2} c_1 = \frac{1}{2\pi} \int_{\mathbb{S}^2} iF = 2ig \in \mathbb{Z} \tag{171}$$

We call the integer n the Chern-number, which is topological invariant. When $n = 0$, the monopole bundle is trivial. Furthermore, from the expression of the transition function, we can prove that when $n = \pm 1$, the monopole bundle is Hopf fibration. i.e. The total space is $\mathbb{S}^3$. By a similar computation, we find that for $SU(2)$-gauge theory, the instanton is also quanzied. We can pick up a four dimensional sphere $\mathbb{S}^4$ and define two local gauge potentials A_N on the northern hemi-sphere and A_S on the southern hemi-sphere, respectively. On the equator $\mathbb{S}^3$, the $SU(2)$-transition function gives a map $\mathbb{S}^3 \mapsto SU(2)$, which is the third homotopy group $\pi_3(SU(2)) = \mathbb{Z}$. This winding number again is associated with a topological charge which is called $SU(2)$-instanton. This integer measures the non-triviality of a principal $SU(2)$-bundle over $\mathbb{S}^4$. Its corresponding cohomology class is given by $\mathrm{Tr}(F \wedge F)$. This is an example of the second Chern class.

Before giving the definition of Chern class, we first introduce the invariant polynomials. Let $M(k, \mathbb{C})$ be the set of complex $k \times k$ matrices. Let $S^r(M(k, \mathbb{C}))$ denote the vector space of symmetric r-linear $\mathbb{C}$-valued functions on $M(k, \mathbb{C})$. That is,

$$\tilde{P} \colon \underbrace{M(k, \mathbb{C}) \times \cdots \times M(M, \mathbb{C})}_{r} \mapsto \mathbb{C} \tag{172}$$

is an element in S^r if it satisfies

$$\tilde{P}(a_1, \cdots, a_i, \cdots, a_j, \cdots, a_r) = \tilde{P}(a_1, \cdots, a_j, \cdots, a_j, \cdots, a_r) \tag{173}$$

where $a_i \in M(k, \mathbb{C})$. By defining a product of $\tilde{P} \in S^p$ and $\tilde{Q} \in S^q$,

$$\tilde{P}\tilde{Q}(a_1, \cdots, a_{p+q}) = \frac{1}{(p+q)!} \sum_\sigma \tilde{P}\left(a_{\sigma(1)}, \cdots, a_{\sigma(p)}\right) \tilde{Q}\left(a_{\sigma(p+1)}, \cdots, a_{\sigma(p+q)}\right) \tag{174}$$

the formal sum $S^*(M(k, \mathbb{C})) = \overset{\infty}{\underset{r=0}{\oplus}} S^r(M(k, \mathbb{C}))$ becomes an algebra. When we restrict our discussion on Lie algebras, an element in $S^*(\mathfrak{g})$ is said to be invariant if for any $g \in G$ and $A_i \in \mathfrak{g}$, $\tilde{P}$ satisfies

$$\tilde{P}(\mathrm{Ad}_g A_1, \cdots, \mathrm{Ad}_g A_r) = \tilde{P}(A_1, \cdots, A_r) \tag{175}$$

For instance, we may take $\tilde{P}$ to be symmetrized trace

$$\tilde{P}(A_1, \cdots, A_r) = \pi_S \mathrm{Tr}(A_1, \cdots, A_r) \tag{176}$$

If we denote the set of G-invariant members of $S^r(\mathfrak{g})$ by $I^r(G)$, then we have a subalgebra $I^*(G) = \overset{\infty}{\underset{r=0}{\oplus}} I^r(G)$.

Next, we extend the definition of invariant polynomials on principal bundles. Let $P(M,G)$ be a principal G-bundle over M. We define the invariant polynomial of Lie algebra valued differential forms

$$\tilde{P}(A_1\eta_1, \cdots, A_r\eta_r) = \eta_1 \wedge \cdots \wedge \eta_r \tilde{P}\left(A_1, \cdots, A_r\right), \tag{177}$$

where $A_i \in \mathfrak{g}$ and $\eta_i \in \Omega^{p_i}(M)$.

Theorem (Chern-Weil): Let $\tilde{P}$ be an G-invariant polynomial. Let $P(M,G)$ be a principal bundle. We denote its curvature forms corresponding to connections A and A' by F and F', respectively. Then we have
(a) $d\tilde{P}(F) = 0$
(b) $\tilde{P}(F') - \tilde{P}(F)$ is exact.

That is to say, invariant polynomials of curvature form are elements in de Rham cohomology. In fact, the map $I^*(G) \mapsto H^*(M)$ is a homomorphism, which is called **Weil homomorphism**.

Definition: Let $P(M,G)$ be a principal G-bundle. Let A be the connection 1-form and F is its corresponding curvature form. The total Chern class is defined by

$$c(F) = \det\left(I + \frac{iF}{2\pi}\right) \tag{178}$$

If we are interested in a complex vector bundle $E \mapsto M$, we only need to replace A and F from the above definition by corresponding connection and curvature on the associated bundle.

One can check that the above expression is indeed a sum of invariant polynomials. To see this, we notice that F is a two form, therefore $c(F)$ is a direct sum of forms of even degrees.

$$c(F) = 1 + c_1(F) + c_2(F) + \cdots \tag{179}$$

For example, an easy computation shows that

$$\begin{cases} c_0(F) = 1 \\ c_1(F) = \dfrac{i}{2\pi}\mathrm{Tr}(F) \\ c_2(F) = \dfrac{1}{2}\left(i/2\pi\right)^2\left\{\mathrm{Tr}(F)\wedge\mathrm{Tr}(F) - \mathrm{Tr}(F\wedge F)\right\} \end{cases} \tag{180}$$

We call c_i the ith Chern class. Remark: For a complex vector bundle $E \mapsto M$ with structure group $GL(n,\mathbb{C})$, we have mentioned that this can be reduced to a $U(n)$-bundle. We see that we only need to consider the Chern class of $U(n)$-bundle since any $E \mapsto M$ is isomorphic to some $U(n)$-bundle. If our gauge group is $U(1)$, then it is clear that the only non-trivial class is the first Chern class. Hence if we are talking about a complex line bundle, it has only first Chern class. If we are interested in $SU(n)$ gauge group, then we have $\mathrm{Tr}(F) = 0$, therefore in that case, the expression of higher order Chern classes can take very simply expressions. For example, for $SU(2)$ gauge theory, we have

$$\begin{cases} c_0(F) = 1 \\ c_1(F) = 0 \\ c_2(F) = \dfrac{1}{2}\left(i/2\pi\right)^2\mathrm{Tr}(F\wedge F) \end{cases} \tag{181}$$

We have seen that the Chern class of a principal bundle $P(M,G)$ lives in the de Rham cohomology $H^*(M;\mathbb{C})$. In what follows, we will show that Chern class of a complex line bundle (or a $U(1)$-principal bundle) belongs to integral cohomology $H^*(M;\mathbb{Z})$.

We consider a compact manifold M. If M is contractible, then its cohomology class is trivial except $H^0(M)$. Hence we restrict our discussion on compact closed manifold. To begin with, let us consider a two dimensional compact closed manifold X and its $U(1)$ principal bundle P, whose curvature is denoted by F'. Let us consider the following integral,

$$\frac{1}{2\pi}\int_X F'. \tag{182}$$

From classification theorem of $U(1)$-bundles, we can work in its classying space $\mathbb{CP}^\infty$, whose total space is an infinitely dimensional sphere. The principal bundle P is given by the pull-back of a map $f\colon X \mapsto \mathbb{CP}^N$ up to homotopy, for a large enough positive integer N. Now we turn to work in a 2-cycle in $\mathbb{CP}^\infty$. It is convenient to choose it to be $\mathbb{CP}^1$, which is topologically a unit sphere. We denote the curvature of this Hopf fibration by F. Then F' defined on P is the pull-back of F. i.e. $F' = f^*F$. The integral can thus be written as

$$\frac{1}{2\pi}\int_X f^*F = \frac{\deg f}{2\pi}\int_{\mathbb{CP}^1} F. \tag{183}$$

We have already shown that on the right hand side the integral associated with a Hopf fibration corresponds to a unit magnetic charge. Also notice that the degree of mapping is an integer. Therefore the integral

$$\frac{1}{2\pi}\int_X F'. \tag{184}$$

can only take integer values. In general, this manifold X can be a 2-cycle of an arbitrary compact manifold M.

In higher dimensions, the computation is similar. For example, if we consider a four dimensional manifold M and curvature form denoted by F, the integrand of

$$\frac{1}{4\pi^2}\int_M F\wedge F \tag{185}$$

is square of first Chern class $c_1(\mathcal{L})^2$ (second Chern class and higher order Chern classes of a complex line bundle vanish.). From the above computation, we see that $c_1(\mathcal{L})^2 = -1/4\pi^2[F]^2 \in H(M,\mathbb{Z})$ is integral cohomology class; The above integral over a 4-manifold can only be integer-valued.

There is another approach to show that the Chern class of complex line bundle is integral cohomology. This is given by the axiom of Chern classes.

Axiom 1. For each complex vector bundle E over M and for each integer $i \geq 0$, the i-th Chern class $c_i(E) \in H^{2i}(M,\mathbb{R})$ is given, and $c_0(E) = 1$.

Axiom 2(Naturality). Let E be a complex vector bundle over M and $f\colon N \to M$ a differentiable map. Then

$$c(f^{-1}E) = f^*(c(E)) \in H^*(N,\mathbb{R}) \tag{186}$$

where $f^{-1}(E)$ is the complex vector bundle over N induced by f from E.

Axiom 3. Let E_1, $\cdots$, E_n be complex line bundles over M, i.e., complex vector bundles with fiber $\mathbb{C}$. Let $E_1 \oplus \cdots \oplus E_n$ be their Witney sum, i.e.,

$$E_1 \oplus \cdots \oplus E_n = d^{-1}(E_1 \times \cdots \times E_n) \tag{187}$$

where $d\colon M \mapsto M \times \cdots \times M$. Then we have

$$c(E_1 \oplus \cdots \oplus E_n) = c(E_1) \wedge \cdots \wedge c(E_n). \tag{188}$$

This is often called the Witney sum formula.

Axiom 4(Normalization). $c_1(E_1)$ is the generator of $H^2(\mathbb{CP}^1, \mathbb{Z})$. i.e., the integral of $c_1(E_1)$ over 2-cycle $\mathbb{CP}^1$ equals 1.

One can check that the definition $c(E) = \det(I + \frac{iF}{2\pi})$ satisfies the above axioms. In particular, to see it satisfies the third axiom, we denote $P_1, \cdots, P_n$ the associated $\mathbb{C}^*$-principal bundles of $E_1, \cdots, E_n$. For each index i, let A_i and F_i be connection form and curvature form on P_i. $P_1 \times \cdots \times P_n$ is a principal bundle over $M \times \cdots \times M$ with structure group $\mathbb{C}^* \times \cdots \times \mathbb{C}^*$. The map $d\colon M \mapsto M \times \cdots \times M$ induces a principal bundle $P = d^{-1}(P_1 \times \cdots \times P_n)$ over M with structure group $\mathbb{C}^* \times \cdots \times \mathbb{C}^*$, which is a subgroup of $GL(n, \mathbb{C})$ consisting of diagonal matrices. We denote the corresponding principal bundle of Witney sum $E = E_1 \oplus \cdots \oplus E_n$ by Q. Then principal bundle Q has structure group $GL(n, \mathbb{C})$ because the Witney sum E has typical fiber $\mathbb{C}^n$. It is clear that Q contains P as a subbundle. We denote the connection and curvature on P by A and F, respectively, and let $p_i\colon P \mapsto P_i$ be the projection, then we have

$$A = A_1^* + \cdots + A_n^*, \quad F = F_1^* + \cdots + F_n^* \tag{189}$$

where $A_i^* = p_i^*(A_i)$ and $F_i^* = p_i^*(F_i)$. Let ω and Ω be the connection and curvature on Q. Then the restriction of

$$\det\left(I + \frac{i\Omega}{2\pi}\right) \tag{190}$$

to P is equal to

$$\left(1 + \frac{iF_1^*}{2\pi}\right) \wedge \cdots \wedge \left(1 + \frac{iF_2^*}{2\pi}\right), \tag{191}$$

which establishes the Witney sum formula. To show that the normalization, we use tautological bundle $\mathbb{C}^2 - \{0\}$, which is a $\mathbb{C}^*$-principal bundle over $\mathbb{CP}^1$. It has a natural associated complex line bundle E_1 over sphere. Since Chern class is independent of the choice of connection, we may choose connection form on $\mathbb{C}^2 - \{0\}$ as

$$A = \frac{\bar{z}^0 dz^0 + \bar{z}^1 dz^1}{\bar{z}^0 z^0 + \bar{z}^1 z^1} \tag{192}$$

Then the curature is given by

$$F = dA = \frac{(\bar{z}^0 z^0 + \bar{z}^1 z^1)(d\bar{z}^0 \wedge dz^0 + d\bar{z}^1 \wedge dz^1) - (\bar{z}^0 dz^0 + \bar{z}^1 dz^1) \wedge (\bar{z}^0 dz^0 + \bar{z}^1 dz^1)}{(\bar{z}^0 z^0 + \bar{z}^1 z^1)^2} \tag{193}$$

Working on a local chart $z^0 \neq 0$, if we set $w = z^1/z^0$, we obtain

$$F = \frac{d\bar{w} \wedge dw}{(1 + w\bar{w})^2} \tag{194}$$

so the local expression of first Chern class is

$$c_1(E_1) = \frac{i}{2\pi} \frac{d\bar{w} \wedge dw}{(1 + w\bar{w})^2} \tag{195}$$

From this expression, the normalization of Chern class is manifest.

2.4.4 Chern-Simon Theory

Chern-Simons theory is a three-dimensional topological quantum field theory whose action is given by Chern-Simons 3-form. Given a manifold M in odd dimensions, with a Lie algebra valued 1-form $A = A^a T_a$ over it, we can define a family of Lie algebra valued forms, called Chern-Simons form. We also define the curvature 2-form corresponding to A by $F = dA + A \wedge A$.

In one dimension, a Chern-Simons form is given by

$$\omega_1 = \mathrm{Tr}\,(A) \tag{196}$$

In three dimensions, it is given by

$$\omega_3 = \mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right) = \mathrm{Tr}\left(F \wedge A - \frac{1}{3} A \wedge A \wedge A\right) \tag{197}$$

In five dimensions, it is given by

$$\omega_5 = \mathrm{Tr}\left(F \wedge F \wedge A - \frac{1}{2} F \wedge A \wedge A \wedge A + \frac{1}{10} A \wedge A \wedge A \wedge A \wedge A\right) \tag{198}$$

In general, for a $2k-1$ dimensional manifold with 1-form A, a Chern-Simons form is defined by

$$d\omega_{2k-1} = \mathrm{Tr}\left(\overbrace{F \wedge \cdots \wedge F}^{k}\right) = \mathrm{Tr}\left(F^k\right) \tag{199}$$

where the term $[\mathrm{Tr}\left(F^k\right)]$ is the kth Chern class. For example, in three dimensions, Chern-Simons 3-form is defined as $d\omega_3 = \mathrm{Tr}\,(F \wedge F)$. This can be checked by using the identity (149).

$$\begin{aligned} d\mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right) &= \mathrm{Tr}\left(d\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right)\right) \\ = \mathrm{Tr}\left(d\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right)\right) &= \mathrm{Tr}\,(dA \wedge dA + 2A \wedge A \wedge dA) \end{aligned} \tag{200}$$

But because of the cyclic property of trace, $\mathrm{Tr}\,(A \wedge A \wedge A \wedge A) = 0$,

$$\begin{aligned} &d\mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right) = \mathrm{Tr}\,\{(dA + A \wedge A) \wedge (dA + A \wedge A)\} \\ &= \mathrm{Tr}\,(F \wedge F) \end{aligned} \tag{201}$$

The following calculation shows that the equation of motion of Chern-Simons action implies that its solution is a flat connection.

$$\begin{aligned} \delta I &= \delta \int_M \mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right) \\ &= \int_M \mathrm{Tr}\left(\delta A \wedge dA + A \wedge d\delta A + \frac{2}{3}\left(\delta A \wedge A \wedge A + A \wedge \delta A \wedge A + A \wedge A \wedge \delta A\right)\right) \\ &= \int_M \mathrm{Tr}\,(\delta A \wedge dA + dA \wedge \delta A - d(A \wedge \delta A) + 2\delta A \wedge A \wedge A) \\ &= \int_M \mathrm{Tr}\,(2\delta A \wedge (dA + A \wedge A)) - \int_{\partial M} \mathrm{Tr}\,(A \wedge \delta A) \end{aligned} \tag{202}$$

In the third line, we have used integration by parts and stokes theorem. In the last line, we used cyclic property of trace. If M has no boundary, then $\dfrac{\delta I}{\delta A} = dA + A \wedge A = 0$.

The Chern-Simons action is well-defined in the sense that it is invariant under orientation-preserving diffeomorphisms of M. This can easily been seen since it is defined in a coordinate independent way. But a physically well-defined theory also requires that its action is invariant under gauge transformation. For Chern-Simons theory, it is invariant under small gauge transformation. Suppose A is a connection defined on a bundle space over M, whose local transition function is given by g. i.e. under a gauge transformation g, the potential A transforms into $B = g^{-1}Ag + g^{-1}dg$. Suppose this gauge transformation is parametrized by parameter s. i.e. $g = g_s$, then, infinitesimally,

$$\delta A = \frac{d}{ds}\Big|_{s=0} B = \frac{d}{ds}\left(g_s^{-1}Ag_s + g_s^{-1}dg_s\right)_{s=0} \tag{203}$$

Noting that $\frac{d}{ds}\left(g_s^{-1}g_s\right) = 0$, we have $\delta g^{-1} = -\delta g = -T$, where T is some element of the Lie algebra of the gauge group, we have

$$\delta A = dT + [A, T] \tag{204}$$

and

$$\delta I = \frac{d}{ds}\Big|_{s=0} I[B] = 0 \tag{205}$$

However, it is not invariant under large gauge transformation. The special thing about Chern-Simons theory is that it can be gauge invariant quantum mechanically. Under a large gauge transformation given by $B = g^{-1}Ag + g^{-1}dg$, the Chern-Simons action transforms into

$$\begin{aligned} I[B] &= \int_M \mathrm{Tr}\left(A \wedge dA + \frac{2}{3}A \wedge A \wedge A\right) \\ &= \int_M \mathrm{Tr}\left(A \wedge dA + \frac{2}{3}A \wedge A \wedge A\right) - \int_{\partial M} \mathrm{Tr}\left(dgg^{-1} \wedge A\right) - \frac{1}{3}\int_M \mathrm{Tr}\left(g^{-1}dg \wedge g^{-1}dg \wedge g^{-1}dg\right) \end{aligned} \tag{206}$$

If M has no boundary, then we can get rid of the second term. The last term is called Wess-Zumino-Witten term. We will see that for some Lie algebras, this term is a winding number for topological reasons. Once we naively quantize the Chern-Simons action

$$Z = \int_{A/\mathcal{G}} \mathcal{D}A \exp \int_M \mathrm{Tr}\left(A \wedge dA + \frac{2}{3}A \wedge A \wedge A\right) \tag{207}$$

it is clear that this partition function is indeed invariant under gauge transformation.

Consider the case when M has a boundary, the functional derivative of Chern-Simons action is not well-defined due to the boundary term $\int_{\partial M} \mathrm{Tr}\left(A \wedge \delta A\right)$. For example, we can consider the conformally compactified $\widetilde{AdS}_3$, whose topological boundary is a cylinder. If we impose a boundary condition such that the gauge field A vanishes at innfinity, then it leads to a trivial theory. To address this problem, we should modify the Chern-Simons action so that it has well-defined functional derivative. We can choose a complex structure on ∂M and consider a counter term

$$I_{bdry} = \frac{k}{4\pi}\int_{\partial M} \mathrm{Tr}\left(A_z A_{\bar{z}}\right) dz \wedge d\bar{z} \tag{208}$$

with a weaker boundary condition $\delta A_z = 0$. Then our modified Chern-Simons action will be

$$I[A] = \frac{k}{4\pi}\int_M \mathrm{Tr}\left(A \wedge dA + \frac{2}{3}A \wedge A \wedge A\right) + \frac{k}{4\pi}\int_{\partial M} \mathrm{Tr}\left(A_z A_{\bar{z}}\right) dz \wedge d\bar{z} \tag{209}$$

Under a large gauge transformation $B = g^{-1}dg + g^{-1}Ag$, the boundary terms becomes

$$\begin{aligned} I_{bdry}[B] &= \frac{k}{4\pi}\int_{\partial M} \mathrm{Tr}\left(A_z A_{\bar z}\right) dz\wedge d\bar z + \frac{k}{4\pi}\int_{\partial M}\mathrm{Tr}\left(\partial g g^{-1}\wedge\bar\partial g g^{-1} + \partial g g^{-1}\wedge A_{\bar z}d\bar z - \bar\partial g g^{-1}\wedge A_z dz\right) \\ &= I_{bdry}[A] + \frac{k}{4\pi}\int_{\partial M}\mathrm{Tr}\left(\partial g g^{-1}\wedge\bar\partial g g^{-1} + \partial g g^{-1}\wedge A_{\bar z}d\bar z - \bar\partial g g^{-1}\wedge A_z dz\right) \end{aligned} \tag{210}$$

where the differential operators ∂ and $\bar\partial$ defined on the boundary Riemann surface satisfying $d = \partial + \bar\partial$. Combining with equation (206), we find that under a large gauge transformation, the modified Chern-Simons action transforms as

$$\begin{aligned} I[B] &= I[A] + \frac{k}{4\pi}\int_{\partial M}\mathrm{Tr}\left(g^{-1}\partial g\wedge g^{-1}\bar\partial g + 2g^{-1}\bar\partial g\wedge A_z dz\right) + \frac{k}{12\pi}\int_M \mathrm{Tr}\left(g^{-1}dg\wedge g^{-1}dg\wedge g^{-1}dg\right) \\ &= I[A] + I_{WZW}[g, A_z] \end{aligned} \tag{211}$$

The second term is called Wess-Zumino-Witten term.

2.5 Dirac's Constraint System

This section is mainly based on [2]. One can find more interesting discussions on the internet.

2.5.1 Introduction

It was Dirac who first introduced a theory to deal with the quantization of system with constraints and gauge symmetry. To learn the Hamiltonian approach to gravity, first we have to fully understand Dirac's constrained system. In classical mechanics, the bridge from Lagrangian mechanics to Hamiltonian mechanics is Legendre transformation given by

$$p = \frac{\partial L}{\partial \dot q} \tag{212}$$

from which we can solve $\dot q$ in terms of q and p by theorem of implicit function. This can be done if and only if

$$\det\left(\frac{\partial^2 L}{\partial \dot q^a \dot q^b}\right) \neq 0 \tag{213}$$

where a and b run from 1 to $N = \dim(M)$. But if the lagrangian is singular

$$\det\left(\frac{\partial^2 L}{\partial \dot q^a \partial \dot q^b}\right) = 0 \tag{214}$$

then the Hamiltonian cannot be obtained via a Legendre transformation. This kind of lagrangians are usually associated with gauge theory. For example,

$$L = \frac{1}{2}\left(\left(\dot X - Z\right)^2 + \left(\dot Y - Z\right)^2\right) \tag{215}$$

we have

$$P_X = \dot X - Z \quad P_Y = \dot Y - Z \quad P_Z = 0 \tag{216}$$

The system is unaffected by a gauge transformation

$$X \to X + \epsilon \quad Y \to Y + \epsilon \quad Z \to Z + \dot\epsilon \tag{217}$$

for some arbitrary function $\epsilon(t)$. Dirac's idea was that, from equation (214), in principle, we can still find an invertible block that has largest rank from the Hessian matrix

$$\mathrm{Hess}(L) = \frac{\partial^2 L}{\partial \dot q^a \partial \dot q^b} \tag{218}$$

Let's denote this invertible block by

$$\Lambda = \left(\frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} \right) \tag{219}$$

where i and j run from $1 \leq I$ to $J \leq N$. Since Λ is invertible, we can solve $\dot{q}^i$ as a function of q^a and p^b, where a and b run from 1 to N. Then if we plug the solutions back into the equation

$$p_k = \frac{\partial L}{\partial \dot{q}^k} \tag{220}$$

for $k \neq I \cdots J$, $\dot{q}^k$s are killed. In principle, if we locally solve Z $\dot{q}^i$s out of N $\dot{q}^a$s, the ultimate equations we find are $M = N - Z$ constraints

$$\phi_m(q,p) = 0, \quad m = 1, \ldots, M \tag{221}$$

That is to say, the phase space is constrained and conjugate variables are not independent. To describe the time evolution of a particle on this constrained surface embedded in phase space, we need to modify our Hamiltonian equations. These equations are as follows

$$\begin{cases} \dot{q}^a = \dfrac{\partial H}{\partial p_a} + \lambda^m(t) \dfrac{\partial \phi_m}{\partial p_a} \\ \dot{p}_a = -\dfrac{\partial H}{\partial q^a} - \lambda^m(t) \dfrac{\partial \phi_m}{\partial q^a} \end{cases} \tag{222}$$

where the functions $\lambda_m(t)$ are called Lagrangian multipliers.

We have $2N$ unknown p and q to be solved, with M Lagrangian multipliers to be determined. In principle, with the above $2N$ equations of time evolution and M constraints equations, we can solve all the funtions $p(t)$, $q(t)$ and $\lambda(t)$.

2.5.2 General Theory

Starting with a theory with a singular Lagrangian $L(q, \dot{q})$, there are M irreducible constraints

$$\phi_m(q,p), \quad m = 1, \ldots, M \tag{223}$$

that are due to their lagrangian. We denote $\Pi = \{\phi_m(q,p) | m = 1, \ldots, M\}$. These constraints are called **primary constraints**. They are stemmed from the property of Lagrangian in a natural way without using equations of motion. Once equations of motion hold, these primary constraints should always be satisfied at any time so that our theory is self-consistent, so we have

$$\dot{\phi}_m = \{\phi_m, H\} + \{\phi_m, \phi_n\} \lambda^n = 0 \tag{224}$$

These conditions may or may not imply some further restrictions on the canonical variables or conditions on λ^m. If they indeed give us further restrictions on the undertermined variables, then clearly, these new constraints hold if and only if the equations of motion are satisfied. We call these new constraints **secondary constraints** because they are not directly inherited from the Langrangian function itself, but rather from the time evolution of primary constraints. Once secondary constraints are imposed, we still need to check the consistency of these new constraints. Repeating the steps for a finite number of times, we will end up with a certain number of secondary constraints, which are denoted by $\Sigma = \{\phi_k(q,p) = 0 | k = M+1, \ldots, M+K\}$. Having a complete set of constraints, $\Gamma = \{\phi_c(q,p) = 0 | c = 1, \ldots, M, \ldots, M+K\}$ including primary and secondary constraints, that is, $\Gamma = \Pi \cup \Sigma$, the equation

$$\{\phi_c, H\} + \lambda^m \{\phi_c, \phi_m\} = 0 \tag{225}$$

hold universally for any constraint ϕ_c which can either be primary or secondary, and with any primary constraint ϕ_m. The above equations is a set of inhomogeneous linear equations of λ_m. Now we hope that we can solve these equations. Suppose that these multipliers are solved completely, then plugging these solutions back into modified Hamiltonian equations, we have

$$\begin{cases} \dot{q}^a = \dfrac{\partial H}{\partial p_a} + \lambda^m(t)\dfrac{\partial \phi_m}{\partial p_a} = F^a(q(t), p(t)) \\ \dot{p}_a = -\dfrac{\partial H}{\partial q^a} - \lambda^m(t)\dfrac{\partial \phi_m}{\partial q^a} = G^a(q(t), p(t)) \end{cases} \tag{226}$$

where F and G are some functions on phase space. We have $2N$ independent first order ordinary differential equations with $2N$ variables, which can be solved exacly. Whether the set of inhomogeneous linear equations () have solutions only depends on the invertiblity of the matrix formed by $\Omega_{cm} = \{\phi_c, \phi_m\}$. But obviously, there is no guarantee that the matrix is invertible. Even though, we can still find an invertible block that has the largest rank. To this end, we classify the constraints into two kinds. First, if the Poisson bracket of a constraint denoted by $\gamma_d \in \Gamma$ with all constraints ϕ_c vanish on-shell, then this constraint γ_d is called **first class constraints**. Otherwise, a constraint denoted by $\chi_\alpha \in \Gamma$ is **second class constraint**, that is, if there exist at least one constraint, say $\phi_n \in \Gamma$, such that $\{\chi_\alpha, \phi_n\} \neq 0$. Then all the entries of the matrix $\Omega_{cd} = \{\phi_c, \phi_d\}$ can be expressed in the following way

$$\{\gamma_c, \gamma_d\} = 0, \quad \{\gamma_c, \chi_\alpha\} = 0, \quad \{\chi_\alpha, \chi_\beta\} = \Omega_{\alpha\beta} \tag{227}$$

The above commutations partition the set Γ into two parts, the first class constraints A and second class constraints B, such that $\Gamma = A \cup B$. A crucial property of first class constraints is that they are closed under Poisson bracket. i.e. they themselves form a Lie algebra that leaves the constraint surface invariant.

$$\{\gamma_c, \gamma_d\} = \mathcal{C}^e{}_{cd}\gamma_e = 0|_{on\ \ shell} \tag{228}$$

where $\mathcal{C}^e{}_{cd}$ is the structure constant of the Lie algebra generated by first class constraints. In some cases, linear combination of a set of second class constraints can turn out to be a first class constraint. If some primary constraints $\phi(q,p)$ are also in first class, then from the modified Hamiltonian equation, we see that

$$\delta p = -\delta\lambda\frac{\partial \phi}{\partial q} = \delta\lambda\,\{\phi, p\} \quad \delta q = \delta\lambda\frac{\partial \phi}{\partial p} = \delta\lambda\,\{\phi, q\} \tag{229}$$

Clearly, first class primary constraints are the generators of gauge symmetries. In many cases, all first class constraints are shown to be the generators of gauge symmetries. Hence the Lie algebra $\{\gamma_c, \gamma_d\} = \mathcal{C}^e{}_{cd}\gamma_e$ is isomorphic to the Lie algebra of Gauge group. This is called the 'Dirac' conjecture, which has been proved to be wrong. For example, in general relativity, first class constraints can 'intertwine' the gauge symmetry and dynamics. In quantum mechanics, the second class constraints will be troublesome because their non-commutativity leads to inconsistency of measurement. The fact that anti-symmetric matrices have inverse iff it is in even dimensional implies that the set of constraints Γ always contains an even number of second class constraints. Let's now denote the number of second class constraints in Γ by $2s = |B|$. Then we have $M + K - 2s$ first class constraints together with $2s$ second class constraints. With the intertible block, we still can solve as many q and p as possible. But some undetermined variables, say λ_j, will certainly appear in the solutions. This clearly cannot be the untimate result. Those redundant degrees of freedom are often associated with gauge symmetries, which should be fixed by a choice of gauge condition. Secondly, the algebra of classical observables relies on the symplectic structure on phase space. i.e. the phase space can only be even dimensional. Starting from $2N + M$ variables, we ended up with $M + K$ constraints. Let us denote this reduced constriant surface, which is embedded into the original phase space, as S. From the above information, it is not yet enough to claim that the dimension of this constraint surface must be even. Geometrically, the gauge symmetries define a fibration on the constraint surface $S \subset T^*M$. Each point (q,p) on S is equivalent to any point on an orbit passing through (q,p). Therefore, the set of physical points is actually $S/\sim$. A gauge fixing condition $g(q,p) = 0$ should cut each gauge orbit once and only once. Thus it is reasonable to require that $\{g, \phi\} \neq 0$ for any $\phi(q,p) \in A$, since otherwise $g(q,p)$ will also generate the gauge symmetry. This requirement implies that the gauge fixing conditions together with all the fist class constraints form second class constraints. Consequently, once gauge fixing are imposed, all the first class constraints are eliminated. On this final constraint surface, which is usually called **reduced phase space**, we can introduce the Dirac bracket

$$\{F(q,p), G(q,p)\}_D = \{F, G\}_{PB} - \{F, \chi_\alpha\}_{PB} \Omega^{\alpha\beta} \{\chi_\beta, G\}_{PB} \tag{230}$$

It fulfills all the requirements antisymmetry, Leibniz law and Jacobi identity. It is easy to see that the Dirac bracket of anything with any second class constraint is zero; Dirac bracket of any two first class functions (i.e. gauge invariant functions) coincide with their Poisson bracket. For these reasons, there will be no obstacles for quantization constrainted systems by replacing Dirac bracket with quantum commutators.

In classical field theory, the treatment of constraints are a bit subtle since we have infinite degrees of freedom. From the Lagrangian density of classical fields, the primary constraints usually appear as some functions $C(\vec{x})$ defined on spacial slice Σ_t. But in Dirac's algorithm, constraints should be functions defined on phase space $\mathcal{P} = T^*M$. i.e.

$$\phi_m(q,p): \ \mathcal{P}_{q,p} \mapsto \mathbb{R} \tag{231}$$

Thus, for classical field theory with Hamiltonian $\mathcal{H}\left(\varphi_I(\vec{x}), \pi^I(\vec{x})\right)$, which has infinite degrees of freedom, our definition of constraints should be replaced by a functional

$$C_\xi\left[\varphi_I, \pi^I\right]: \ \mathcal{P}_{\varphi,\pi} \mapsto \mathbb{R} \tag{232}$$

where ξ is an arbitrary scalar field, called **test function** or **smearing function**, defined on Σ_t which satisfying appropriate boundary conditions and the phase space is infinite dimensional, $\dim \mathcal{P}_{\varphi,\pi} = \infty^2$. Specifically, contraint functional for classical field theory is defined as

$$C_\xi = \int_{\Sigma_t} \xi C(\vec{x}) \tag{233}$$

For first class constraints, if they generate gauge symmetry, say G, then, in principle, they should be isomorphic to the Lie algebra $\mathfrak{g}$ of gauge group. Suppose we have a set of test functions $\{\xi, \zeta \cdots\}$ that are $\mathfrak{g}$-valued, with a Lie bracket $[\xi, \zeta]$. Then the functional Poisson brackets of the first class constraints should satisfy

$$\{C_\xi, C_\zeta\} = C([\xi, \zeta]) \tag{234}$$

That is, if the test functions are spanned by Lie algebra elements X_i, then we have first class constraints $C_i = \int_{\Sigma_t} X_i C(\vec{x})$ satisfying

$$\{C_i, C_j\} = \mathcal{C}^k{}_{ij} C_k \tag{235}$$

From the preceding equations, we see that first class constraints who generate gauge symmetry are representations of the Lie algebra $\mathfrak{g}$ of the gauge group G. From a mathematical aspect, these constraints functionals can be regarded as a momentum map $\xi \mapsto C_\xi$. In gauge theory, this algebra is often called smeared algebra.

An interesting property of the smeared algebra is that in two dimensions, it becomes a vanishing central extension of $\mathfrak{g}$. Suppose that spacetime is a two dimensional manifold without boundary, then it's spacial slice is a topological circle. Thus, we need to impose a periodic boundary condition on constraints $C(\phi)$ and test function $\xi(\phi)$. The first class constraints are then written as

$$C[\xi] = \oint_{\mathbb{S}^1} Tr(\xi(\phi) C(\phi)) = \oint_{\mathbb{S}^1} \xi^a(\phi) C_a(\phi) \tag{236}$$

We can write this integral by using Fourier series

$$\xi^a = \sum_{r=-\infty}^{+\infty} \xi_r^a e^{ir\phi} \tag{237}$$

and

$$C_a = \sum_{n=-\infty}^{+\infty} C_a^n e^{in\phi} \tag{238}$$

Then it is easy to see that the integral becomes

$$C[\xi] = \sum_n \xi_a^{-n} C_n^a \tag{239}$$

We see that the smeared algebra becomes

$$\left\{C_n^a, C_m^b\right\} = \mathcal{C}_c^{ab} C_{m+n}^c, \tag{240}$$

which is an affine extension of $\mathfrak{g}$ with vanishing central charge. It is not clear whether this form of smeared algebra has any physical significance. In three dimension, the smeared algebra becomes more interesting. Suppose that the spacetime is a three dimensional manifold with a boundary cylinder. The smeared integral may not be functional differentiable because of boundary terms. We can improve the definition in the following way

$$C[\xi] = \int_\Sigma \xi^a(\phi) C_a(\phi) + Q[\xi] \tag{241}$$

where the term Q is added to cancel boundary terms produced from variation of the bulk term. Since this boundary term is defined up to a constant, we are doing a projective representation of Lie algebra $\mathfrak{g}$. The Poisson bracket should, in general, take the following form

$$\{C[\xi], C[\eta]\} = C[\sigma(\xi, \eta)] + K[\xi, \eta] \tag{242}$$

where the last term is a central term. The expression of $\sigma(\xi, \eta)$ depend on the boundary conditions we impose. After gauge fixing, all first class constraints are eliminated. We usually find that the boundary terms form the algebra

$$\{Q[\xi], Q[\eta]\} = Q[\sigma(\xi, \eta)] + K[\xi, \eta] \tag{243}$$

This is called the surface algebra, which is the key idea of many discussions on AdS_3/CFT_2 correspondence and Chern-Simons theory. For a Chern-Simons theory, the surface charge is realized as Kac-Moody algebra, which is its affine extension [47]. To see this, suppose the topology of spacetime is a product $\mathbb{D} \times \mathbb{R}$. Then we can write the Chern-Simons action in the following way

$$I = \int dt \int_\Sigma d^2x \epsilon^{ij} \mathrm{Tr}\left(\dot{A}_i A_j + A_0 F_{ij}\right) + B(\partial\Sigma \times \mathbb{R}). \tag{244}$$

Clearly, A_0^a is a Lagrangian multiplier and A_i^a are dynamical fields. Furthermore, it is easy to see that A_i is both dynamical field and its conjugate momentum field, satisfying Poisson bracket

$$\left\{A_i^a(x), A_j^b(y)\right\} = \frac{1}{2} g^{ab} \epsilon_{ij} \delta(x, y) \tag{245}$$

Hence, a generic Poisson bracket of two arbitrary functional is

$$\{G, H\} = \sum_{ij} \frac{1}{4} \int \epsilon^{ij} \mathrm{Tr}\left(\frac{\delta G}{\delta A_i}\frac{\delta H}{\delta A_j} - \frac{\delta G}{\delta A_j}\frac{\delta H}{\delta A_i}\right) \tag{246}$$

The primary constraint is given by

$$C^a = \epsilon^{ij} F^a_{ij} = 0, \tag{247}$$

which is an analogue of Gauss law in electrodynamics, satisfying the algebra

$$\{C_a(x), C_b(y)\} = f^c_{ab} C_c(x)\delta(x,y) \tag{248}$$

Defining the smeared algebra by

$$C[\xi] = \int_\Sigma \xi^a C_a + Q[\xi] \tag{249}$$

so we have

$$\delta C = \int_\Sigma \epsilon^{ij} \xi_a((d_A)_i \delta A^a_j) + \delta Q = -\int_\Sigma \epsilon^{ij}((d_A)_i \xi_a)\delta A^a_j + \int_{\partial\Sigma} \xi_a \delta A^a + \delta Q \tag{250}$$

where we have used (152), (149) and integrated by parts, and we have

$$\delta Q = -\int_{\partial\Sigma} \xi_a \delta A^a \tag{251}$$

From the above expressions, we can find that the smeared algebra is

$$\{C[\eta], C[\lambda]\} = 2\int_\Sigma (d_A \eta_a) \wedge (d_A \lambda^a) = \int_\Sigma [\eta,\lambda]^a C_a + 2\int_{\partial\Sigma} \eta_a (d_A \lambda^a) \tag{252}$$

where we used (246) and (145). If we suppose that the surface charge is integrable. i.e.,

$$Q = -\int_{\partial\Sigma} \xi_a A^a \tag{253}$$

Then the boundary integral becomes

$$\int_{\partial\Sigma} \eta_a (d_A \lambda^a) = \int_{\partial\Sigma} \eta_a d\lambda^a + \int_{\partial\Sigma} \eta_a [A,\lambda]^a = \int_{\partial\Sigma} \eta_a d\lambda^a + Q[[\eta,\lambda]]. \tag{254}$$

Bañados claims that after gauge fixing, the algebra of surface charge satisfies the algebra [47]

$$\{Q[\eta], Q[\lambda]\}_D = Q[[\eta,\lambda]] + 2\int \eta_a d\lambda^a \tag{255}$$

, which I do not understand at all. The interpretation of the above equation is that the gauge fixed surface charges are the generators of residual gauge symmetries, which seems similar to Brown and Henneaux's computation of asymptotic symmetry of AdS_3 [20]. From my perspective, this should be a hypothesis. It is worth mentioning that there are alternative ways to show that the algebra of surface charges is indeed given by (255) [66]. In [66], the surface charges are the Noether charges associated with gauge transformations at the boundary when gauge fields are pure gauges. Using the same trick of Fourier series as we did in two dimensional case,

$$A^a(\phi) = \sum_n T^a_n e^{in\phi}, \quad \xi_a(\phi) = \sum_r \xi^r_a e^{ir\phi} \tag{256}$$

surprising thing happens. It turns out that this algebra is an exact affine central extension [47]

$$\{T^a_n, T^b_m\}_D = -f^{ab}_c T^c_{n+m} + in g^{ab}\delta_{0,n+m} \tag{257}$$

This suggests that the boundary dynamics of Chern-Simons theory is a two dimensional conformal field theory.

2.5.3 Canonical Quantization

There are two equivalent approaches to quantize the constrained systems. First one is to find all the constraints and gauge fixing conditions and get the reduced phase space. Then, the quantum observables are promoted as operators from those who are gauge invariant functions defined on the reduced phase space. Another way is to directly replace the Dirac bracket of two classical observables $\{F,G\}_D$ by $i\hbar[\hat{F},\hat{D}]+o(\hbar^2)$. The second class constraints are enforced as operator identities. Since their Dirac brackets vanish globally, they will not bring any difficulties. However, the first class constraints are implemented as 'weak identities', that is, imposed on quantum states. The quantum constraint $\hat{\gamma}_d|\psi\rangle = 0$ defines the physical states. In other words, we define our Hilbert space as the kernel of first class constraints operators $\mathcal{H} = \ker\hat{A}$. Unphysical states should be excluded out of Hilbert space in quantum mechanics.

From mathematical aspect, the quantization of classical mechanics with some Lie group G as its symmetry, is an unitary projective representation $PGL(V)$. This projective can be lifted to a linear representation of the central extension of G. As a result, the quantization is often accompanied with central charge.

3 AdS_3 Spacetime

3.1 AdS Geometry

A global Lorentzian AdS spacetime is defined as an submanifold M of codimension 1 given by

$$-U^2 - V^2 + (X^1)^2 + \cdots + (X^{n-1})^2 = -l^2, \tag{258}$$

embedded in an $n+1$ dimensional flat manifold $\mathbb{R}^{2,n-1}$ with the metric given by

$$ds^2 = -(dU)^2 - (dV)^2 + (dX^1)^2 + \cdots + (dX^{n-1})^2 \tag{259}$$

It has negative Riemann scalar curvature, whose metric is the induced metric from $\mathbb{E}^{2,n-1}$. The parameter l is some positive number called AdS radius. If we plug the induced metric into Einstein's equations, then it shows that the cosmological constant should be $\Lambda = -\frac{(n-1)(n-2)l^{-2}}{2}$. In particular, for AdS_3, the cosmological constant is related to the AdS radius via $\Lambda = -1/l^2$.

In the following discussion, we will take $l = 1$. The embedding equation becomes

$$-U^2 - V^2 + (X^1)^2 + \cdots + (X^{n-1})^2 = -1, \tag{260}$$

from which we see that M has a killing vector $U\partial_V - V\partial_U$ generating the rotation in $U - V$ plane; $2(n-1)$ killing vectors $U\partial_{X^i} + X^i\partial_U$ and $V\partial_{X^i} + X^i\partial_V$ generating the boosts in X directions; $(n-2)(n-1)/2$ killing vectors $X^i\partial_{X^j} - X^j\partial_{X^i}$ generating the rotations in $n-1$ dimensional X plane. Therefore, The isometry group for global AdS_n is just $SO(2, n-1)$. From the embedding equation, fixing each X^i, $U^2 + V^2$ is given by a positive constant. In other words, AdS_n has a topology $\mathbb{S}^1 \times \mathbb{R}^{n-1}$. If we choose a point $p \in AdS_n$ that is invariant under boosts in $U - X$ plane, i.e. It' stabilizer is given by $SO(1, n-1)$, then AdS_n is a coset $SO(2, n-1)/SO(1, n-1)$.

For an AdS_3, a solution of embedding

$$-U^2 - V^2 + X^2 + Y^2 = -1 \tag{261}$$

is given by

$$\begin{cases} U = \cosh\rho\cos t' \\ V = \cosh\rho\sin t' \\ X = \sinh\rho\cos\theta \\ Y = \sinh\rho\sin\theta \end{cases} \tag{262}$$

where $\rho \in (0,\infty)$ and $\theta \in (0,2\pi]$. This coordinate patch covers the whole the AdS_3 spacetime and is usually called the global AdS_3 coordinate. It is clear that the time coordinate t' is in fact periodic, winding around the $\mathbb{S}^1$ circle, which violates the causality. Instead, we usually use its universal covering space $\widetilde{AdS_3}$ in which the t' coordinate is unwrapped. Its topology then becomes $\mathbb{R}^3$. In my thesis, we will explicitly distinguish AdS_3 and its universal cover $\widetilde{AdS_3}$.

An AdS_3 spacetime as a coset space $SO(2,2)/SO(1,2)$ is isomorphic to the group manifold $SL(2,\mathbb{R})$. This can be easily seen if we associate the column in $\mathbb{R}^{2,2}$ with a real matrix in the following way [15]:

$$\begin{pmatrix} U \\ V \\ X \\ Y \end{pmatrix} \longleftrightarrow \begin{pmatrix} U-X & Y-V \\ Y+V & U+X \end{pmatrix} \tag{263}$$

such that $\begin{pmatrix} U-X & Y-V \\ Y+V & U+X \end{pmatrix} \in SL_2(\mathbb{R})$. The metric on this group manifold is given by the Killing-Cartan metric

$$ds^2 = \frac{1}{2}\mathrm{Tr}(g^{-1}dgg^{-1}dg), \tag{264}$$

The Killing-Cartan metric is invariant under two independent global right and left actions $g \to hg$, $g \to gh$, where $h \in SL_2(\mathbb{R})$. In other words, viewing the AdS_3 spacetime as a group manifold of $SL_2(\mathbb{R})$, whose fundamental group is $\pi_1(SL(2,\mathbb{R})) = \mathbb{Z}$, its geometry is invariant under an the action by the group $SL_2(\mathbb{R}) \times SL_2(\mathbb{R})$. One should keep in mind that the group $SL_2(\mathbb{R}) \times SL_2(\mathbb{R})$ has a center $\mathbb{Z}_2$, which acts on the AdS_3 trivially. Thus, the isometry group of AdS_3 is $(SL_2(\mathbb{R}) \times SL_2(\mathbb{R}))/\mathbb{Z}_2 = SO(2,2)$.

The conformal boundary of an AdS_3 is a limit set where the metric blows up. i.e. $\rho \to \infty$. To find this conformal boundary, let us see what happens to the embedding equation at the conformal boundary. From embedding, if we rescale all the components

$$X = \lambda\tilde{X}, \quad Y = \lambda\tilde{Y}, \quad U = \lambda\tilde{U}, \quad V = \lambda\tilde{V} \tag{265}$$

then the embedding equation becomes $-\tilde{U}^2 - \tilde{V}^2 + \tilde{X}^2 + \tilde{Y}^2 = -\frac{1}{\lambda^2}$. We denote the metric diag$(-1,-1,+1,+1)$ by η_{ab} and (U,V,X,Y) by X^a, then as long as we approaches to infinity, i.e. $\lambda \to \infty$, the embedding equation becomes $\eta_{ab}\tilde{X}^a\tilde{X}^b = 0$. Therefore, the conformal boundary is given by the quotient

$$\left\{\eta_{ab}\tilde{X}^a\tilde{X}^b = 0\right\} / \sim \tag{266}$$

where $\sim$ is the equivalence relation $\tilde{X} \sim \lambda\tilde{X}$. For example, the conformal boundary of a global AdS_2 is a circle. Any two points along a ray passing through the center of the cone are to be identified due to the equivalence relation. Hence the conformal boundary can be deemed as the black circle depicted in the above picture. The conformal boundary of AdS_3 is a torus because the equivalence relation gives a product of two independent circles. This can also be seen from metric of global AdS_3 spacetime, which is introduced as follows.

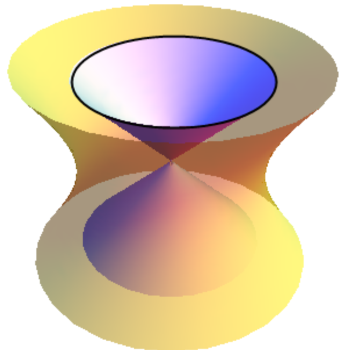


Figure 12: AdS_2 can be viewed as a hyperboloid embedded in a three dimensional space, which is the yellow outer surface in the above figure. The cone inside approaches to the hyperboloid at infinity.

Plugging the above solution to the metric $ds^2 = -dU^2 - dV^2 + dX^2 + dY^2$, we have the induced metric for a global AdS_3.

$$ds^2 = -\cosh^2\rho dt'^2 + \sinh^2\rho d\theta^2 + d\rho^2. \tag{267}$$

Setting dt' to be 0, the metric is a Poincare disc, from which we see that an AdS_3 manifold admits a foliation $\mathbb{H}^2 \times \mathbb{S}$. i.e. There is a foliation for an AdS_3 such that each of its equal time spacial slice is a hyperbolic disc. In addition, this hyperbolic slice is a totally geodesic submanifold. Setting $\cosh\rho = \frac{1}{\cos\xi}$, we find that the metric becomes

$$ds^2 = \frac{1}{\cos^2\xi}\left(-dt^2 + d\xi^2 + \sin^2\xi d\theta^2\right) \tag{268}$$

Then we define $\hat{t} \pm \hat{\xi} = \tan\frac{t+\xi}{2}$, the metric is finally in the form [23]

$$ds^2 = \frac{4\cos^2\frac{t+\xi}{2}\cos^2\frac{t-\xi}{2}}{\cos^2\xi}\left(-d\hat{t}^2 + d\hat{\xi}^2 + \hat{\xi}^2 d\theta^2\right) \tag{269}$$

from which we recognize that the conformally compactified global $\widetilde{AdS_3}$ is indeed a topological solid cylinder, whose conformal boundary is $\xi = \frac{\pi}{2}$, or equivalently, $\rho \to \infty$.

By defining $\sinh\rho = r$, the metric of a global AdS_3 becomes

$$ds^2 = -\left(1 + r^2\right)dt'^2 + \frac{dr^2}{1+r^2} + r^2 d\theta^2 \tag{270}$$

If we consider a radially moving particle, whose effective Lagrangian is given by

$$L = \frac{1}{2}\left[-(1+r^2)\dot{t}'^2 + \frac{\dot{r}^2}{1+r^2}\right], \tag{271}$$

from which we see that its energy is a constant of motion.

$$E = \frac{\partial L}{\partial \dot{t}'} = -(1+r^2)\dot{t}' \tag{272}$$

Consequently, the condition $v^\mu v_\mu = 1$, 0 and -1 for 4-velocity $v = \dfrac{\partial}{\partial \tau}$ implies that

$$\begin{cases} v^\mu v_\mu = -1, \quad r = \pm\sqrt{E^2+1}\sin(\tau - \tau_0) \\ v^\mu v_\mu = 0, \quad r = \pm|E|(\tau-\tau_0) \\ v^\mu v_\mu = 1, \quad r = \pm\sqrt{E^2-1}\sinh(\tau-\tau_0) \end{cases} \tag{273}$$

The first equation tells us that free massive particles in an AdS_3 always oscillate in the bulk and will never reach the conformal boundary. Nevertheless, photons will reach the conformal boundary at a finite time.

Another useful coordinate called Poincare patch, which is related with the ambient space $\{U, V, X, Y\}$ via [23]

$$x = \frac{Y}{U+X}, \quad y = \frac{V}{U+X}, \quad z = \frac{1}{U+X}. \tag{274}$$

Plugging the above expressions into the metric $ds^2 = -dU^2 - dV^2 + dX^2 + dY^2$, we obtain the induced metric in Poincare patch

$$ds^2 = \frac{1}{z^2}\left(dx^2 - dy^2 + dz^2\right), \tag{275}$$

from which we can recognize that the Lorentzian Poincare patch is simply the Poincare's upper-half space model in Lorentzian signature $\mathbb{H}^{2,1}$. Such a coordinate patch covers only a part of a global AdS (which is usually dipicted as Poincare wedge). It's conformal boundary is $z = 0$ slice.

An important feature of the AdS spacetime is that it is not globally hyperbolic. i.e. The spacial slices from the above foliation are not Cauchy hypersurfaces. We can see this from the embedding equation of AdS_3,

$$-U^2 - V^2 + X^2 + Y^2 = -1. \tag{276}$$

Consider an initial surface $t' = 0$, i.e. $V = 0$, the embedding becomes

$$-U^2 + X^2 + Y^2 = -1, \tag{277}$$

which is a hyperboloid embedded in a Lorentzian space with a metric

$$ds^2 = -dU^2 + dX^2 + dY^2 \tag{278}$$

This is a Lorentzian model for a global hyperbolic 2-manifold, which can be identified as a Poincare disc. It has constant negative curvature and Euclidean metric. Let us call it a 'Cauchy surface' of AdS_3 spacetime, and see how it evolves in time. As time passes, this hypersurface is determined by the embedding

$$-U^2 + X^2 + Y^2 = -1 + V^2. \tag{279}$$

As a result, its 'Cauchy development' breaks down at $t' = \pi/2$ in the future, where the embedding hyperboloid becomes a lightcone. In order to give readers some intuition, we forget the θ-direction, suppressing this dimension, then the manifold is reduced to AdS_2, whose embedding equation is given by

$$-U^2 - V^2 + X^2 = -1 \tag{280}$$

Consider an initial surface $t' = 0$, i.e. $V = 0$, the embedding becomes

$$-U^2 + X^2 = -1, \tag{281}$$

This hyperboloid is the 'Cauchy surface' of AdS_2. It's time evolution is illustrated in the following graph.

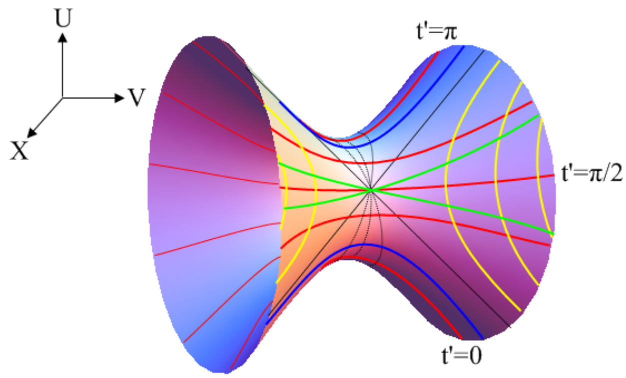


Figure 13: AdS_2

In the figure, the hyperbolic slices at $t' = 0$, $\pi/2$ and π are red lines; Two black straight lines represent the lightlike hypersurface where the 'Cauchy development' breaks down; The yellow lines represent timelike geodesics that meet the lightlike hypersurface at spacial infinity; The green lines are spacelike geodesics that intersect with each other at $X = 0$. At $t' = 0$, the initial slice is given by $V = 0$, denoted by the bottom red line. It's time evolution is represented by the blue lines, which finally coincide with lightlike black lines when $t' = \pi/2$. Since the AdS spacetime is conformally flat, we can draw its Penrose diagram. From the

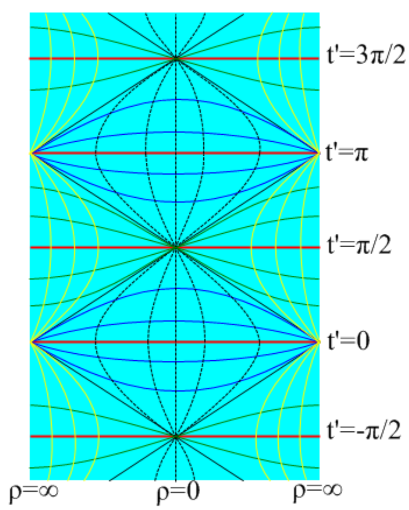


Figure 14: Penrose Diagram of AdS Spacetime

Penrose diagram, we see that information initialized at time $t' = 0$ cannot fully determine the hypersurface at $t' = \pi$. In contrast, for globally hyperbolic spacetimes such as Minkowski spacetime, the evolution of spacial slice can be fully determined by its past. For AdS spacetime, we need to specify boundary conditions to determine the time evolution.

3.2 Lorentzian BTZ Black Hole

It is a surprise that even there are no gravitons in three dimensions, there do exist black hole solutions when cosmological constant is negative. If consmological constant is 0 or positive, there are no black hole solutions [26] [27] [28] [29] [31]. The most obvious reason is that, in the AdS spacetime, timelike geodesics are oscillating in the bulk. In other words, the negative cosmological constant can 'create' an attractive force.

On the other hand, we mentined at the beginning that scalar curvature is a constant in three dimensions, one may wonder how BTZ black holes can be real black holes. It is true that in three dimensions, scalar curvature of any solutions to Einstein equations should be constant. That is to say, there is no curvature singularity in the BTZ solution. We call such solutions black holes in the sense that there are typical causal structures resemble the cases in higher dimensions. i.e. They have event horizons, which are null surfaces that are the boundaries of the past of asymptotic infinities. Once a particle enters the region enclosed by the even horizon, it can never escape to asymptotic infinity. These solutions also have spacetime singularities. As mentioned before, these singularities are not curvature singularities, but are of Misner-type [48]. i.e. They are the end of the spacetime. Particles approaching a Misner singularity inside horizon will finally reach it and stay there forever; It will have nowhere else to go. To avoid extremal black hole solutions, we also need singularities hidden behind event horizons.

Since in three dimensions, solutions can only be constructed by doing local identifications of the AdS_3, BTZ black holes should also be obtained by doing identification in the bulk along integral curves of killing vectors, say ξ. To avoid possibility of time-travel, we should not conider the case when killing vector ξ is timelike. If we glue two points along a timelike direction together, there will be a closed timelike circle. Therefore, if BTZ black holes are quotient spaces of the AdS_3 modulo some discrete subgroups of isometry $SO(2,2)$, then the BTZ group can only be generated by spacelike killing vectors. Since an AdS_3 manifold is the Lie group $SL(2,\mathbb{R})$, the generators must be some elements in Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ as well.

We can consider a hypersurface defined by constant norm of killing vector ξ, i.e. $f(x) = \xi^\mu \xi_\mu = C$, where $C \in \mathbb{R}$ [49].

$$\nabla_\alpha f(x) = \nabla_\alpha \left(\xi_\mu \xi^\mu\right) = 2\xi^\mu \nabla_\alpha \xi_\mu \tag{282}$$

From the above equation, we have

$$\xi^\alpha \nabla_\alpha f(x) = 0 \tag{283}$$

because ξ is a killing vector. We see that the killing vectors that 'create' a black hole should always tangent to the hypersurface $f(x) = C$. In other words, killing vectors that we are interested in always map this hypersurface to itself. In particular, its isometry should also leave singularities invariant. Hence we should define the singularities of BTZ black hole as follows:

Definition: Singularity $\mathcal{S}$ of a BTZ black hole is a subspace of AdS_3 where a spacelike killing vector ξ vanishes.

From the above definition, we can see that there is another constraint to the killing vector. If $\mathcal{S}$ tends to asymptotic infinity, then the killing vector at infinity must either tangent to the boundary cylinder if $\mathcal{S}$ is timelike or vanishes if $\mathcal{S}$ is spacelike so that it preserves the boundary condition.

In this section, we set $l = 8G = \hbar = c = 1$ except when restoring units is necessary. In Schwarzschild-like coordinates, the BTZ metric is given by

$$ds^2 = -\frac{\left(r^2 - r_+^2\right)\left(r^2 - r_-^2\right)}{r^2} dt^2 + \frac{r^2}{\left(r^2 - r_+^2\right)\left(r^2 - r_-^2\right)} dr^2 + r^2 \left(d\phi + \frac{r_+ r_-}{r^2} dt\right)^2 \tag{284}$$

where $-\pi < \phi < \pi$ and $0 < r < r_+$ describes the interior the black hole and $r > r_+$ describes the exterior black hole. It also has an interior horizon at $r = r_-$ and an ergosurface $r_{eg} = \left(r_+^2 + r_-^2\right)^{1/2}$, where the g_{00} component vanishes closely analoguos to Kerr solution in $3+1$ dimensions. As mentioned earlier, singularity

$r = 0$ at the center of a BTZ black hole is not curvature singularity, but rather **inextendible** singularity. If we define

$$M = r_+^2 + r_-^2, \quad J = 2r_+ r_- \tag{285}$$

then the metric becomes

$$ds^2 = -\left(-M + r^2 + \frac{J^2}{4r^2}\right)dt^2 + \frac{dr^2}{-M + r^2 + \dfrac{J^2}{4r^2}} + r^2\left(d\phi + \frac{J}{2r^2}dt\right)^2 \tag{286}$$

Setting $\hbar = c = 1$ means that G has dimension of of length (i.e. $[G] = L$) and angular momentum J is dimensionless. If we write l and G explicitly, the metric takes the form

$$ds^2 = -\left(-8GM + \frac{r^2}{l^2} + \frac{16G^2J^2}{r^2}\right)dt^2 + \frac{dr^2}{-8GM + \dfrac{r^2}{l^2} + \dfrac{16G^2J^2}{r^2}} + r^2\left(d\phi + \frac{4GJ}{r^2}dt\right)^2 \tag{287}$$

The even horizon and inner horizon are given by

$$r_\pm^2 = 4GMl^2\left\{1 \pm \left[1 - \left(\frac{J}{Ml}\right)^2\right]^{1/2}\right\} \tag{288}$$

or

$$M = \frac{r_+^2 + r_-^2}{8Gl^2}, \quad J = \frac{r_+ r_-}{4Gl} \tag{289}$$

We conclude that BTZ black holes are totally determined by two parameters mass M and angular momentum J, which agrees with No-Hair theorem. In order to find all classical contributions to the quantum gravity of the AdS_3, we need to answer if pure AdS_3 and BTZ black holes are the only physical solutions. Since we are working in three dimensions, the answer to the above question relies only on boundary condition and how the discrete isometry acts on pure the AdS_3.

We mentioned that a pure AdS_3 as a group manifold $SL(2,\mathbb{R})$ admits a foliation $\mathbb{H}^2 \times \mathbb{S}^1$. Using the global coordinate, the metric for $\widetilde{AdS}_3$ is

$$ds^2 = -\cosh^2\rho dt'^2 + \sinh^2\rho d\theta^2 + d\rho^2. \tag{290}$$

Each constant t' slice is a hyperbolic surface, which can be reckoned as a Poincare disc. We can consider an isometric action,

$$h_L \begin{pmatrix} U - X & Y - V \\ Y + V & U + X \end{pmatrix} h_R \sim \begin{pmatrix} U - X & Y - V \\ Y + V & U + X \end{pmatrix} \tag{291}$$

where h_L and h_R are two generators of hyperbolic discrete subgroups of $SL(2,\mathbb{R})$. Up to two independent conjugate transformations, we can write

$$h_L = \begin{pmatrix} e^u & 0 \\ 0 & e^{-u} \end{pmatrix}, \quad h_R = \begin{pmatrix} e^v & 0 \\ 0 & e^{-v} \end{pmatrix} \tag{292}$$

for some $u > 0$ and $v > 0$. The isometric identification becomes

$$\begin{pmatrix} e^{u+v}(U - X) & e^{u-v}(Y - V) \\ e^{v-u}(Y + V) & e^{-u-v}(U + X) \end{pmatrix} \sim \begin{pmatrix} U - X & Y - V \\ Y + V & U + X \end{pmatrix} \tag{293}$$

Noting that in global AdS_3 coordinate U and V are time dependent, in general, the above isometric identification may create a closed timelike circle, which inevitably violates causality. This indicates that we have only a limited number of possible quotient spaces to be considered.

The simplest case is when $h_L = h_R = h$. Since h is a diagonal matrix, it satisfies $h = h^T$. At $t' = 0$, the above identification is

$$h \begin{pmatrix} U - X & Y \\ Y & U + X \end{pmatrix} h^T \sim \begin{pmatrix} U - X & Y \\ Y & U + X \end{pmatrix}, \tag{294}$$

which is exactly an isometric identification on the hyperbolic surface in Lorentzian model. The same is true if we set $t' = t'_0$ for some arbitrary constant t'_0. Hence in such a case our problem of seeking for possible quotient spaces of the AdS_3 is significantly reduced to a smaller problem in two dimensions. Once we fix our boundary condition and determine the fundamental domain in a hyperbolic inital spacial slice, the complete three dimensional fundamental domain of this group action in a pure AdS_3 is totally determined by timelike geodesics starting from this initial slice. The fundamental domain can therefore be visualized as a flash of the evolution of the inital slice. We use two sets of local coordinates

$$\begin{cases} U = r\cosh\phi \\ V = \sqrt{r^2 - 1}\sinh t \\ X = r\sinh\phi \\ Y = \sqrt{r^2 - 1}\cosh t \end{cases} \tag{295}$$

for $r > 1$, t and $\phi \in \mathbb{R}$, as well as

$$\begin{cases} U = r\cosh\phi \\ V = \sqrt{-r^2 + 1}\cosh t \\ X = r\sinh\phi \\ Y = \sqrt{-r^2 + 1}\sinh t \end{cases} \tag{296}$$

for $0 < r < 1$ and $\phi \in \mathbb{R}$.

The above two coordinates cover the whole pure AdS_3 and the metric is

$$ds^2 = -\left(r^2 - 1\right) dt^2 + \frac{1}{r^2 - 1} dr^2 + r^2 d\phi^2 \tag{297}$$

This metric looks almost the same as a BTZ black hole of unit mass $M = 1$ except that the range of ϕ is $(-\infty, +\infty)$. The way we write these coordinates for a pure AdS_3 may give a false impression that it is angular coordinate for BTZ black hole. To obtain the true black hole solution, one needs one more step:

$$\phi \sim \phi + 2\pi. \tag{298}$$

This local identification is generated by $\dfrac{\partial}{\partial\phi}$, which equals to $X\partial_U + U\partial_X$. It is clear that this killing vector is hyperbolic. For this unit mass BTZ black hole, since its coordinate is not well-defined along the spacial geodesic $r = 1$, we should compute the horizon length by using new coordinates $\cosh\rho = r$. Then the metric along this geodesic takes the form

$$dl^2 = \cosh^2\rho d\phi^2, \tag{299}$$

from which we see that the horizon length is 2π. We can also consider a general identification,

$$\phi \sim \phi + L \tag{300}$$

It is no danger to set this new parameter satisfying $L = 2\pi a$, for $a \in (0, 1)$ or $a > 1$. By performing a rescaling transformation $\phi \to a\phi$, $r \to r/a$ and $t \to at$, and setting $M = 1/a^2$, i.e.

$$\phi \to \sqrt{\frac{1}{M}}\phi, \quad r \to \sqrt{M}r, \quad t \to \sqrt{\frac{1}{M}}t, \tag{301}$$

the new ϕ-coordinate has the usual periodicity 2π again. The metric in terms of the new coordinates becomes

$$ds^2 = -\left(r^2 - M\right)dt^2 + \frac{dr^2}{r^2 - M} + r^2 d\phi^2, \tag{302}$$

The mass parameter is restored. The horizon after this rescaling is now at $r = \sqrt{M}$, which has length $2\pi\sqrt{M}$. To have a BTZ black hole with angular momentum, we perform a coordinate transformation [26]

$$r^2 = \frac{r'^2 - r_-^2}{r_+^2 - r_-^2}, \quad \begin{pmatrix} t \\ \phi \end{pmatrix} = \begin{pmatrix} r_+ & r_- \\ r_- & r_+ \end{pmatrix} \begin{pmatrix} t' \\ \phi' \end{pmatrix} \tag{303}$$

for $r^2 \geq r_+^2$ and $0 < r_- < r_+ < \infty$ are two positive real numbers. Dropping the 'prime' from new coordinate, the metric becomes

$$ds^2 = -\frac{\left(r^2 - r_+^2\right)\left(r^2 - r_-^2\right)}{r^2} dt^2 + \frac{r^2}{\left(r^2 - r_+^2\right)\left(r^2 - r_-^2\right)} dr^2 + r^2 \left(d\phi + \frac{r_+ r_-}{r^2} dt\right)^2, \tag{304}$$

which equals to

$$ds^2 = -\left(-M + r^2 + \frac{J^2}{4r^2}\right) dt^2 + \frac{dr^2}{-M + r^2 + \frac{J^2}{4r^2}} + r^2 \left(d\phi + \frac{J}{2r^2} dt\right)^2 \tag{305}$$

We see that a BTZ black hole with angular momentum can be constructed by taking the same quotient, after which we introduce an extra parameter r_- as a compensation. In terms of new coordinates, the ambient space $\{U, V, X, Y\}$ has three solutions, each of which represents a region of black hole spacetime [15]. From our assumption, which says that any other time-dependent quotients of the AdS_3 allow closed timelike circles, and the fact that parabolic and elliptic Möbius transformations create singularities, we believe that BTZ black holes are the only physically possible quotients for 3D pure gravity, whose corresponding discrete subgroup of $SO(2,2)$ are cyclic.

$r \geq r_+$

$$\begin{aligned} U &= \sqrt{\frac{r^2 - r_-^2}{r_+^2 - r_-^2}} \cosh(r_+\phi + r_- t), \quad V = \sqrt{\frac{r^2 - r_+^2}{r_+^2 - r_-^2}} \sinh(r_+ t + r_-\phi) \\ X &= \sqrt{\frac{r^2 - r_-^2}{r_+^2 - r_-^2}} \sinh(r_+\phi + r_- t), \quad Y = \sqrt{\frac{r^2 - r_+^2}{r_+^2 - r_-^2}} \cosh(r_+ t + r_-\phi) \end{aligned} \tag{306}$$

$r_- \leq r < r_+$

$$\begin{aligned} U &= \sqrt{\frac{r^2 - r_-^2}{r_+^2 - r_-^2}} \cosh(r_+\phi + r_- t), \quad V = -\sqrt{\frac{-r^2 + r_+^2}{r_+^2 - r_-^2}} \sinh(r_+ t + r_-\phi) \\ X &= \sqrt{\frac{r^2 - r_-^2}{r_+^2 - r_-^2}} \sinh(r_+\phi + r_- t), \quad Y = -\sqrt{\frac{-r^2 + r_+^2}{r_+^2 - r_-^2}} \cosh(r_+ t + r_-\phi) \end{aligned} \tag{307}$$

$0 \leq r < r_-$

$$\begin{aligned} U &= \sqrt{\frac{-r^2 + r_-^2}{r_+^2 - r_-^2}} \cosh(r_+\phi + r_- t), \quad V = -\sqrt{\frac{-r^2 + r_+^2}{r_+^2 - r_-^2}} \sinh(r_+ t + r_-\phi) \\ X &= \sqrt{\frac{-r^2 + r_-^2}{r_+^2 - r_-^2}} \sinh(r_+\phi + r_- t), \quad Y = -\sqrt{\frac{-r^2 + r_+^2}{r_+^2 - r_-^2}} \cosh(r_+ t + r_-\phi) \end{aligned} \tag{308}$$

It is easy to see that identification $\phi \sim \phi + 2\pi$ is equivalent to

$$\rho_L \begin{pmatrix} U - X & Y - V \\ Y + V & U + X \end{pmatrix} \rho_R \sim \begin{pmatrix} U - X & Y - V \\ Y + V & U + X \end{pmatrix}, \tag{309}$$

where the left action and right action are given by

$$\rho_L = \begin{pmatrix} e^{\pi(r_+-r_-)} & 0 \\ 0 & e^{-\pi(r_+-r_-)} \end{pmatrix}, \quad \rho_R = \begin{pmatrix} e^{\pi(r_++r_-)} & 0 \\ 0 & e^{-\pi(r_++r_-)} \end{pmatrix} \tag{310}$$

Therefore, a rotating BTZ black hole can be viewed as a quotien space $\widetilde{AdS}_3/<(\rho_L,\rho_R)>$. The discrete subgroup $<(\rho_L,\rho_R)>$ is hyperbolic, which is generated by a single generator (ρ_L,ρ_R). Hence it is isomorphic to integers $\mathbb{Z}$. The group $\mathbb{Z}\subset SL(2,\mathbb{R})\times SL(2,\mathbb{R})$ (which is called BTZ group) is not a subgroup of only one factor or the other, but a subgroup of the whole product group. It is very subtle when we are saying that a BTZ black hole is a quotient space of a pure AdS_3. Strictly speaking, it is not a quotient space of an AdS_3. The reason is that we want the discrete isometry of an AdS_3 to act on it freely and discontinuously. In our case, the problem is that the killing vector that 'creates' this black hole geometry leaves $r=0$ fixed. But if we exclude the singularity, then the BTZ black hole can be regarded as a quotient space of an $AdS_3/\mathbb{Z}$.

A different representation of the exterior region $r>r_+$ of this quotient geometry is the Lorentzian upper-half space $\mathbb{H}^{2,1}$. Using (274), we obtain

$$\begin{aligned} x &= \sqrt{\frac{r^2-r_+^2}{r^2-r_-^2}}\cosh(r_+t+r_-\phi)\exp\{-r_+\phi-r_-t\} \\ y &= \sqrt{\frac{r^2-r_+^2}{r^2-r_-^2}}\sinh(r_+t+r_-\phi)\exp\{-r_+\phi-r_-t\} \\ z &= \sqrt{\frac{r_+^2-r_-^2}{r^2-r_-^2}}\exp\{-r_+\phi-r_-t\} \end{aligned} \tag{311}$$

that transforms the rotating BTZ metric into the form

$$ds^2 = \frac{1}{z^2}\left(dx^2-dy^2+dz^2\right) \tag{312}$$

The identification in the rotating BTZ coordinate $\phi\sim\phi+2\pi$ requires that

$$(x,y,z)\sim\left(e^{-2\pi r_+}(x\cosh 2\pi r_-+y\sinh 2\pi r_-), e^{-2\pi r_+}(y\cosh 2\pi r_-+x\sinh 2\pi r_-), e^{-2\pi r_+}z\right), \tag{313}$$

from which we see that when $r_-=0$, the change $\phi\to\phi+2\pi$ on $\mathbb{H}^{2,1}$ is simply a dilation. Each constant-y slice is a Poincare's upper-half plane, with the metric $ds^2=\frac{1}{z^2}(dx^2+dz^2)$. We see that this dilation induces a hyperbolic transformation ρ on each constant-y slice. This matrix takes the form

$$\rho = \begin{pmatrix} e^{\pi r_+} & 0 \\ 0 & e^{-\pi r_+} \end{pmatrix}, \tag{314}$$

which fixes two points $(x,z)=(0,0)$ and ∞. For a rotating black hole, whose $r_-\neq 0$, we have a Lorentzian boost on $x-y$ plane.

Since the metric is singular at horizon $r=r_+$, we cannot calculate the length of horizon in BTZ coordinates. We start from metric

$$ds^2 = -\left(r^2-m\right)dt^2+\left(r^2-m\right)^{-1}dr^2+r^2d\phi^2, \tag{315}$$

which is a BTZ black hole of mass m. From previous discussion we know that its horizon is at $r=\sqrt{m}$, whose length is $L=2\pi\sqrt{m}$. Then we introduce new coordinates $\{T,\varphi,R\}$ [26],

$$t = T+\left(\frac{J}{2m}\right)\varphi, \quad \phi=\varphi+\left(\frac{J}{2m}\right)T, \quad R^2=r^2\left(1-\frac{J^2}{4m^2}\right)+\frac{J^2}{4m}, \tag{316}$$

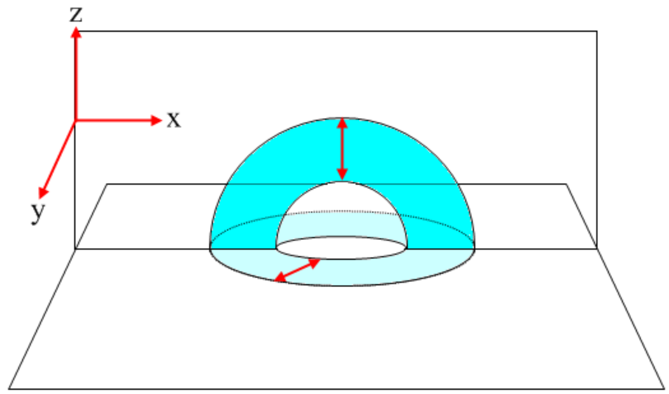


Figure 15: Identification $\phi \sim \phi + 2\pi$ on Lorentzian upper-half space

where $J < 2m$ is a constant. We defining another constant

$$M = m + \frac{J^2}{4m}. \tag{317}$$

In terms of this new coordinate, the non-rotating BTZ black hole metric of mass m can be written as

$$ds^2 = -\left(R^2 - M + \left(\frac{J}{2R}\right)^2\right) dT^2 + \frac{dR^2}{R^2 - M + \left(\frac{J}{2R}\right)^2} + R^2 \left(d\varphi + \frac{JdT}{2R^2}\right)^2 \tag{318}$$

which is exactly the metric of a rotating BTZ black hole with charges M and J. Horizon at $r = \sqrt{m}$ in terms of new coordinates is given by

$$R^2|_{r=\sqrt{m}} = m = \frac{M}{2}\left(1 \pm \sqrt{1 - \frac{J^2}{M^2}}\right) \tag{319}$$

The above two solutions of m are precisely $r_\pm$. Since horizon length should be independent of choice of coordinate, the horizon length of a rotating BTZ black hole with charges $M = r_+^2 + r_-^2$ and $J = 2r_+r_-$ is apparently $L = 2\pi r_+$. It is useful to write G and l explicitly. Remembering that previously we set $8G = 1$ (but now $[G] = [l] = [M]^{-1}$) it is easy to solve $r_\pm$ in terms of M and J

$$\frac{r_+ - r_-}{\sqrt{8Gl}} = \sqrt{lM + J}, \quad \frac{r_+ + r_-}{\sqrt{8Gl}} = \sqrt{lM - J}. \tag{320}$$

The explicit formula of the length of the horizon with G and l restored is

$$L = 2\pi r_+ = \pi\left(\sqrt{8Gl\,(lM + J)} + \sqrt{8Gl\,(lM - J)}\right) \tag{321}$$

From a geometric point of view, a BTZ black hole is a quotient space of a pure AdS_3, but physically, it was shown by Carlip that a BTZ black hole is formed from collapse of matter in three dimensional spacetime, which carries entropy. On the other hand, for an observer far away from the black hole, information of the interior of the black hole is not unveiled by the entropy of that matter because black holes have event horizons. This 'paradox' is resolved by associating entropy with event horizons. From the No-Hair theorem, a stationary black hole in gravitational vacuum is parametrized by its mass M and angular momentum J. For fixed values of M and J, we may still have many different internal microstates of the black hole formation. Therefore, we can imagine that there may have a large amount of information of that black hole blocked by its event horizon, except its mass and angular momentum. For an outside observer, the measure of the missing information hidden behind the horizon can only be accounted for the black hole's entropy. From the dual CFT_2 aspect, we have to associated a BTZ black hole with a mixed state. So far, we are not able to provide further information about such a black hole state. To formulate the black hole entropy, we should first introduce the four laws of black hole mechanics.

Zeroth Law

The horizon has constant surface gravity for a stationary black hole.

First Law

For a black hole near macro-stationary state (thermal equilibrium), the increment of its energy is related to change of its area of horizon A, angular momentum J and other internal charges Q by

$$dM = \frac{\kappa}{8\pi} dA + \Omega dJ + \Phi dQ \tag{322}$$

where κ is its surface gravity, Ω is its angular velocity and Φ is the potential of gauge fields.

Second Law

The area of horizon is non-decreasing in time evolution.

$$\frac{dA}{dt} \geq 0 \tag{323}$$

This law is violated by the discovery of Hawking radiation, which causes decrease of mass and area in the process of evaporation. Hawking radiation is a result of the quantum fluctuation of the vacuum near the event horizon.

Third Law

A black hole with vanishing surface gravity is not possible.

Comparing the above laws of black holes with the laws of thermodynamics, we find that the black hole entropy should be proportional to its event horizon area. The non-decreasing of horizon area in classical gravity is an analogue of non-decreasing law in ordinary thermal dynamics. When quantum effect near horizon is taken into consideration, the black hole radiates at temperature $T = \frac{\kappa}{2\pi}$ [50]. The entropy is given by Hawking-Bekenstein formula

$$S_{BH} = \frac{k_B A}{4\ell_P^2} \tag{324}$$

where k_B is the Boltzmanns constant and ℓ_P is planck scale, which is given by $\sqrt{G\hbar/c^3}$. We set $k_B = 1$ throughout this thesis. In three dimensions, the event horizon of a BTZ black hole is one-dimensional and so the horizon area should be replaced by horizon length, denoted by L.

To cope with the difficulties from the second law of black hole mechanics, we introduce a generalized second law.

Generalized Second Law

The sum of matter entropy outside a black hole and the black hole entropy never decreases in a spontaneous process. In equations

$$\delta S_o + \delta S_{BH} \geq 0 \tag{325}$$

In the radiation process, the generalized second law indicates that the emergent Hawking radiation entropy outstrips the decrement of the black hole entropy.

In this section, we are only interested in the classical geometry of BTZ black holes so the length of horizon must be a constant in time evolution that depends on its mass M and angular momentum J. Since a black hole with mass M and angular momentum J can be transformed to a static black hole of unit mass, we only need to study the metric (297). In terms of global AdS_3 coordinate, the singularity $r = 0$, or $U = X = 0$, is given by $\theta = 0$ and $t' = \pi/2$. The horizon $r = 1$ at $t = 0$, or $V = Y = 0$, is given by $t' = 0$, $\theta = \pi/2$. The initial slice and future (past) singularities are depicted in the graph [26] [27] [28] [29] [31]. At $t' = 0$,

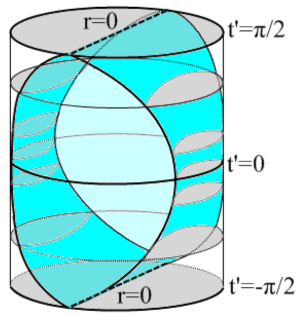


Figure 16: Fundamental domain of Lorentzian BTZ black hole in AdS_3

the fundamental domain on spacial slice is the shaded region, enclosed by two hemi-circles geodesics $\phi = 0$. The red line is the inital horizon $r = 1$. The evolution of this initial slice is totally determined by geodesics starting from the shaded region. Finally these geodesics will collapse at singularity because in AdS_3, timelike geodesics are attractive and meet each other at $t' = \pi/2$. On the initial slice, there are two asymptotic AdS_3

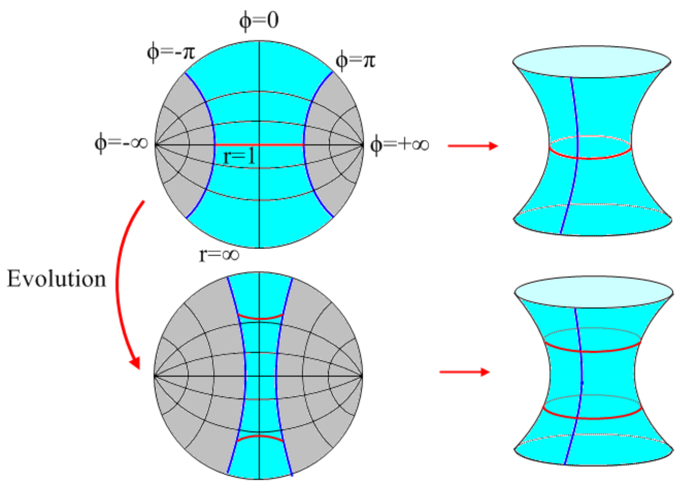


Figure 17: Initial Slice

regions isolated by the event horizon. The fundamental region is the shaded region followed by identifying hemi-circles $\phi = -\pi$ and $\phi = \pi$. Topologically, it is a cylinder. To see how this hyperbolic surface evolve in time, we can find the coordinate transformation between the global AdS_3 coordinates and the black hole coordinates. From (262) and (295), we have

$$U^2 - X^2 = r^2 = \cosh^2\rho \cos^2 t' - \sinh^2\rho \sin^2\theta \tag{326}$$

$$\tanh\phi = \tanh\rho \frac{\sin\theta}{\cos t'} \tag{327}$$

and

$$\tanh t = \coth \rho \frac{\sin t'}{\cos \theta} \tag{328}$$

The fundamental domain of the BTZ group should be bounded by $\phi = \pi$ and $\phi = -\pi$ surfaces embedded in the global AdS_3 in time direction. Two surfaces $\phi = \pi$ and $\phi = -\pi$ finally meet each other at the singularity, where the killing vector $\frac{\partial}{\partial \phi}$ vanishes. The spacial cylinder started from past singularity, followed by expansion from $t' = -\pi/2$ to $t' = 0$ and finally shrinked to singularity at future. For any stationary black hole, its event horizon is a null hypersurface, which is a lightlike totally geodesic submanifold. For an AdS_3 manifold, we have mentioned that each constant time slice $\mathbb{H}^2$ is a totally geodesic submanifold. It can be proved that the intersection of any two totally geodesic submanifolds is itself a totally geodesic submanifold. Therefore, spacial slices of event horizon of a BTZ black should also be a totally geodesic subspace. But since it is a codimensional 2 subspace, it is a spacelike geodesic in the global AdS_3. The evolution of the initial spacial slice of the horizon is determined by lightlike geodesics starting from the spacial geodesic circle $r = 1$, whose 'world sheet' is the event horizon of the BTZ black hole. Using Hawking entropy formula,

$$S = \frac{L}{4G} = 2\pi \left(\sqrt{\frac{l}{8G}(lM + J)} + \sqrt{\frac{l}{8G}(lM - J)} \right) \tag{329}$$

For a non-rotating BTZ black hole, the entropy is

$$S = 4\pi \sqrt{\frac{l^2 M}{8G}} \tag{330}$$

we see that the entropy is related with its mass M and the length of its initial horizon, which is a constant. In our case, since we are studying a black hole of unit mass, we take $M = 1$. To maintain L being a constant, the horizon splits into two circles from $t' = 0$ to $t' = \pi/2$ and will finally meet the singularity at asymptotic infinity.

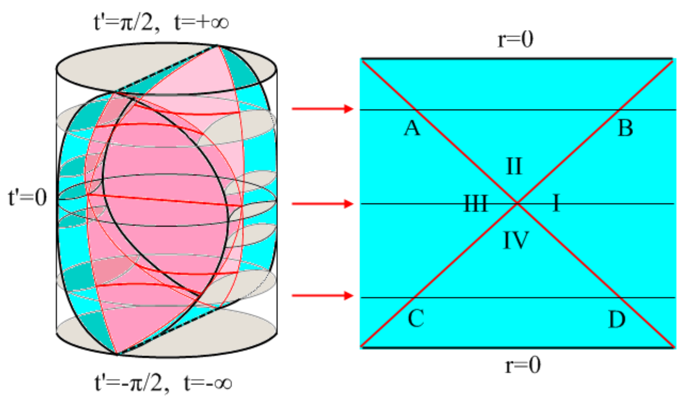


Figure 18: Time evolution of horizon $r = 1$

In the above picture, the solid cylinder is AdS_3 background. Each constant time slice is a Poincare disk. The grey shaded regions should be removed. There are two red surfaces intersecting with each other at $t' = 0$. These two null surfaces are the event horizons of the BTZ black hole. There are several red lines denoting its event horizon at different times. At time $t' = 0$, event horizon is the minimal geodesic (which looks like a straight interval) connecting the two shaded grey regions. It splits into two in time evolution. Finally, in the future horizons collapse into singularity at the asymptotic infinity. If we supress the ϕ-dimension, the fundamental domain gives us the Penrose diagram of the BTZ black hole.

It is easy to see that timelike geodesics outside the horizon finally intersect with the singularity at the asymptotic infinity at AdS-coordinates time $t' = \pi/2$. In contrast, timelike geodesics will ultimately reach at the singularity within a finite AdS-coordinates time. But in the BTZ coordinates, free massive particles outside the horizon will never reach the singularity. Fundamental domain of a BTZ black hole is divided by

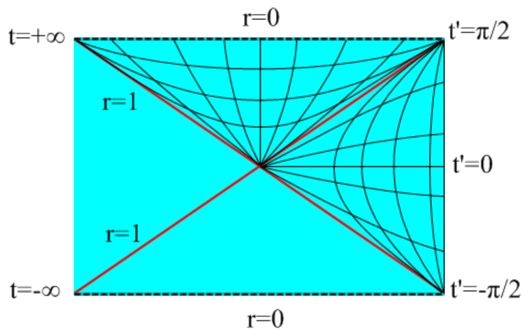


Figure 19: Penrose diagram of BTZ black hole

horizons into four isolated regions I, II, III and IV.

Finally, we consider a very special non-rotating BTZ black hole whose mass $M = 0$.

$$ds^2 = -r^2 dt^2 + \frac{dr^2}{r^2} + r^2 d\phi^2 \tag{331}$$

If we do a coordinate transformation $\rho = 1/r$, then the metric becomes

$$ds^2 = \frac{1}{\rho^2}\left(-dt^2 + d\rho^2 + d\phi^2\right) \tag{332}$$

From the above metric, we clearly see that this new coordinate patch is locally a Poincare patch, which covers a part of the Lorentzian AdS_3. Clearly, a massless BTZ black hole cannot be the a pure AdS since we still have a nontrivial identification $\phi \sim \phi + 2\pi$ on the Poincare patch. The black hole metric (315) goes back to the pure AdS_3 when $M = -1$. Furthermore, on each constant time slice, the induced metric is

$$ds^2 = \frac{1}{\rho^2}(d\rho^2 + d\phi^2), \tag{333}$$

which is locally a Poincare's upper-half plane model. The identification $\phi \sim \phi + 2\pi$ generated by $\frac{\partial}{\partial\phi}$ is clearly parabolic, which has a single fixed point at $\rho = \infty$, or $r = 0$. It's fundamental domain contains cusp point at infinity. Thus massless BTZ black hole should be excluded in our discussion. For a massive extremal BTZ black hole, we can use the same trick, relating it to a quotient of $\mathbb{H}^{2,1}$ to show that it has cusp points. However, coordinates transformation (311) is singular when $r_+ = r_-$. The following transformation can relate extremal BTZ metric to a Lorentzian Poincare form [30]

$$\begin{aligned}
x &= \frac{1}{2}\left(\phi + t - \frac{r_0}{r^2 - r_0^2} + \frac{1}{2r_0} e^{2r_0(\phi - t)}\right) \\
y &= \frac{1}{2}\left(\phi + t - \frac{r_0}{r^2 - r_0^2} - \frac{1}{2r_0} e^{2r_0(\phi - t)}\right) \\
z &= \frac{1}{\sqrt{r^2 - r_0^2}} e^{r_0(\phi - t)}
\end{aligned} \tag{334}$$

It is easy to see that under the translation $\phi \to \phi + 2\pi$, the transformation on constant-y slice is a mixture of a hyperbolic transformation and a parabolic transformation, which give rise to cusp singularities as a massless black hole has.

From previous discussion, the geometry of an AdS_3 manifold is completely determined by its spacial slice, thus the moduli space of non-rotating BTZ black holes is the same as the moduli space of $\mathbb{D}^2/ < \gamma >$. From the introduction to uniformization of Riemann surfaces, we know that if γ is hyperbolic, then the moduli space is $\mathbb{R}_{>0}$; If it is elliptic, then the moduli space is $\mathbb{R}/\mathbb{Z}$. We have mentioned that in elliptic case, the isomorphic classes of cones is $\mathbb{R}/\mathbb{Z}$ only if we fix the invariant point in Poincare disc, while if we unleash that point, we only need to introduce one more parameter. In terms of the AdS_3 geometry, such an orbifold correponds to a particle moving in AdS_3 spacetime. If the invariant point in each spacial slice is fixed, then that particle has no angular momentum. But if we apply a 'boost', then the particle in the AdS_3 can bounce back and forth. The extra parameter can only be the angular momentum of an elliptic BTZ black hole, which runs in $\mathbb{R}$. Clearly, if our assumption is correct, we have exhausted all possible solutions corresponding to cyclic discrete Möbius groups. We find that there are only BTZ black holes with Misner singularities in pure 3D gravity. Our conclusion is that then moduli spaces listed above shows that the AdS_3 spectrum has a mass gap and is bounded from below. Correspondingly, we expect a CFT_2 that has such a mass gap.

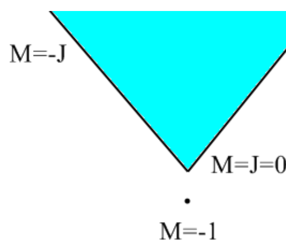


Figure 20: Spectrum of BTZ black holes

From the above analysis, we may conclude that if $-1 < M < 0$, the quotients are generated by elliptic motions; if $M > 0$, they are generated by hyperbolic motions satisfying $|\mathrm{Tr}(\sigma)| > 2$. BTZ black holes of mass $M = 0$ and $M = -1$ corresponds to the Möbius transformations of type $|\mathrm{Tr}(\sigma)| = 2$. The trace is related with the mass of BTZ black hole via the following formula.

Theorem: If h is a hyperbolic element, the translation length L of its action in the upper half-plane is related to the trace of h by

$$|\mathrm{Tr}(h)| = |2\cosh\frac{L}{2}| \tag{335}$$

The translation length is exactly the circumference of the event horizon of a BTZ black hole.

We have already proved the above theorem in the discussion of Iwasawa decomposition of $SL(2,\mathbb{R})$. Similarly, when h is elliptic, its trace is related with the deficit angle θ via

$$|\mathrm{Tr}(h)| = |2\cos\theta| \tag{336}$$

Clearly, for the unipotent class, which corresponds to parabolic transformations, the trace is 2. This critical value of mass is a phase transition point through which global AdS_3 geometry becomes BTZ black hole geometry. Since the isomorphism class of $\mathbb{D}^2/ < \gamma >$ is either $\mathbb{R}_{>0}$ when $M > 0$ or $\mathbb{R}/\mathbb{Z}$ when $-1 < M < 0$, and the BTZ metric becomes a pure AdS_3 when $M = -1$, it is not possible to have BTZ black holes of $M < -1$. From CFT_2 perspective, if $M < -1$, the corresponding CFT_2 is not in unitary representation.

In pure AdS_3, more complicated quotient spaces can be constructed [23] [27] [28] [29]. For instance, we may consider the evolution of a Riemann surface of type (g, n, m) introduced in preliminaries. A $2+1$ dimensional AdS space with such a Riemann surface as its spacial slice is a wormhole. We may also consider a compact universe, whose spacial slice is a Riemann surface of type $(g, 0, 0)$. In general, these different types of AdS_3 universes all start from singularity at past infinity, followed by expansion and finally collapse to singularity at future. In this thesis, we omitted discussion about three dimensional worm holes which are

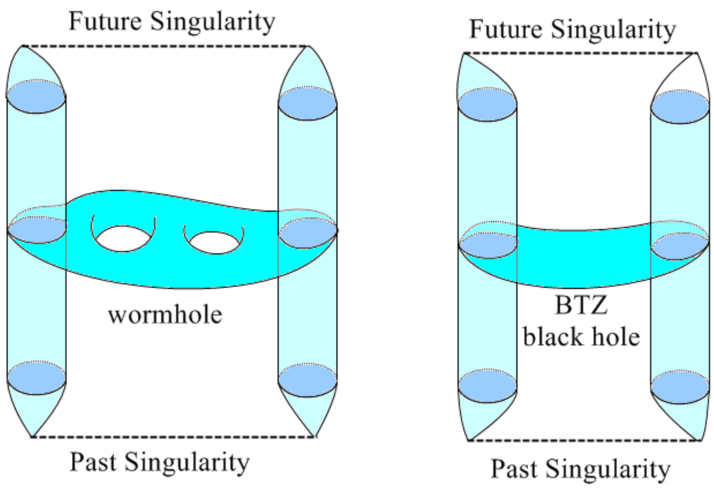


Figure 21: On the left is a three dimensional wormhole of genus 2 with two asymptotic AdS_3 regions, the left one is a BTZ black hole

generated by more than one hyperbolic motions.

3.3 Analytic Continuation

For an $\widetilde{AdS}_3$ spacetime, it is clear that after wick rotation $t \to it$, the embedding equations become

$$\begin{cases} U = \cosh\rho\cosh t' \\ V = i\cosh\rho\sinh t' \\ X^1 = \sinh\rho\cos\phi \\ X^2 = \sinh\rho\sin\phi \end{cases} \tag{337}$$

If we define a new coordinate patch

$$\begin{cases} U' = \cosh\rho\cosh t' \\ V' = \cosh\rho\sinh t' \\ X'^1 = \sinh\rho\cos\phi \\ X'^2 = \sinh\rho\sin\phi \end{cases} \tag{338}$$

then we obtain a hyperboloid $-(U')^2 + (V')^2 + (X'^1)^2 + (X'^2)^2 = -1$ embedded in an Euclidean spacetime, which is a hyperbolic space, whose isometry is $SO(1,3)$. It is important that although the Poincare patch in Lorentzian signature covers a part of the whole an $\widetilde{AdS}_3$ manifold, its Euclidean counter part covers the whole Euclidean spacetime. This is analogous to the fact that Rindler coordinate only covers less than 1/4 of the whole Minkowski spacetime while it covers whole $\mathbb{R}^2$ in Euclidean signature.

Another important analytic continuation is that if we do the following coordinate transformations, $t' \mapsto it$, $r \mapsto ir$ and $\theta \mapsto i\phi$, we see that the pure AdS_3 metric becomes a non-rotating BTZ metric. Although this coordinate transformation is mysterious from a physical aspect, it is very useful in computations of killing vectors of AdS_3 manifolds.

4 Euclidean Saddle Points

4.1 Introduction

In Lorentzian signature, the $\widetilde{AdS_3}$ is a universal covering space, from which all black hole and wormhole solutions can be constructed by making quotient, modding out some discrete subgroups of isometry $SO(2,2)$. In Euclidean signature, the universal covering space can either be Poincare's upper-half space or Poincare's unit ball. Our convention for upper-half space is that

$$ds^2 = \frac{1}{z^2}(dx^2 + dy^2 + dz^2). \tag{339}$$

It conformal boundary is $z = 0$ slice plus ∞, which is Riemann sphere $\mathbb{CP}^1$. On $z = 0$ slice, there is a natural complex structure $w = x + iy$ and $\bar{w} = x - iy$. The 4-dimensional ambient space $\{U, V, X, Y\}$ is still related with this metric via

$$x = \frac{Y}{U+X}, \quad y = \frac{V}{U+X}, \quad z = \frac{1}{U+X} \tag{340}$$

The isometry group is $PSL(2,\mathbb{C})$. The group $SL(2,\mathbb{C})$ also has a Iwasawa decomposition

$$SL(2,\mathbb{C}) = KAN, \tag{341}$$

where $K = SU(2)$, $A = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$, for $t \in \mathbb{R}$, and $N = \begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix}$, for $w \in \mathbb{C}$. By definition, subgroup N is clearly parabolic, whose cyclic discrete subgroup generates a one dimensional lattice Λ on each constant-z slice. Apparently, a quotient space $\mathbb{H}^3/\Lambda$ has topology of a solid cylinder, but with its center circle removed. This center circle corresponds to $z \to \infty$ in the upper-half space, where the metric vanishes. i.e. the quotient space contains a cusp line, which is analogues to the cusp point of modular curve. From a physical aspect, if we regard this solid torus as an Euclidean 3D universe, on the one hand, it has one end at its conformal boundary $z = 0$; on the other hand it has another end at the cusp circle. While in physics, we want our spacetime to have only one end at asymptotic infinity, therefore, we should never consider such a quotient space [18].

Geometries of different kinds of hyperbolic three-spaces have been well-known to physicists, due to Thurston [46]. The following theorem, due to Curt McMullen implies that seeking for quotient spaces of $\mathbb{H}^3$ can be completely determined by conformal structure on their conformal boundaries [51]. Therefore, we only need to investigate different types Kleinian groups acting on $\mathbb{CP}^1$.

Theorem: Let M be a topological 3-manifold. Let $GF(M)$ denote the space of hyperbolic 3-manifold that are homeomorphic to M. As long as M admits at least one hyperbolic realization, there is a one-to-one correspondence between hyperbolic structures on M and conformal structures on ∂M. i.e.

$$GF(M) \simeq \text{Teich}(\partial M) \tag{342}$$

where $\text{Teich}(\partial M)$ is the Teichmüller space of ∂M, which is the universal covering space of the moduli space of ∂M.

4.2 Schottky Uniformization

Let us first investigate the Poincare's upper-half space model. When the Kleinian group is generated by loxodromic motions, the quotient spaces become handlebodies whose conformal boundaries are annuli in the $z = 0$ slice. Using the complex structure, this slice is complex plane $\mathbb{C}$. In the following pictures, the upper hemi-spheres in the Poincare's upper half space are totally geodesic surfaces. Gluing two hemi-spheres together is to make a quotient space of $\mathbb{H}^3$ modulo an action generated by a loxodromic element [23] [31]. The identification in the first figure is given by an action of a cyclic Kleinian group. While in the second figure, there are two loxodromic generators. It would be abstruse if we view the above pictures as handle

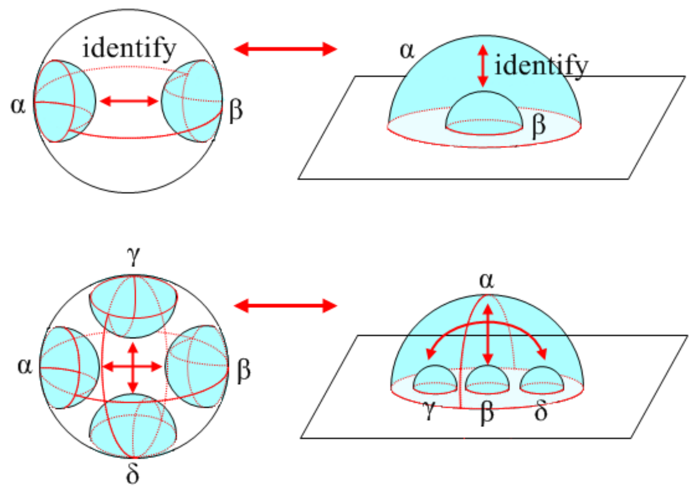


Figure 22: Fundamental domains in three dimensions

bodies. To help readers visualize them, we use the Poincare's unit ball model. The procedure of identification is illustrated in the following picture.

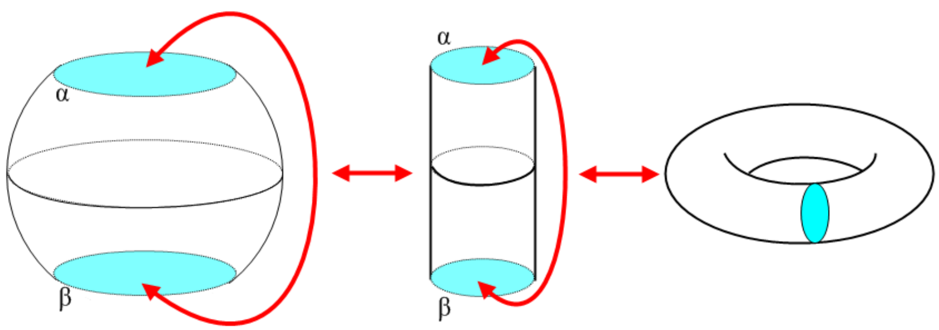


Figure 23

In the above picture, we identify the two shaded discs inside $\mathbb{H}^3$, whose boundaries are closed curves α and β on the conformal boundary $\mathbb{CP}^1$. The generator of the corresponding Möbius transformation sends the one shaded disc to another. The result curve $\alpha \sim \beta$ is a contractible cycle of the solid torus on the right hand side.

Handlebodies with higher genus are created by more than one loxodromic elements. These manifolds are three dimensional Euclidean wormholes, which are out of the range covered in my thesis. A more generic

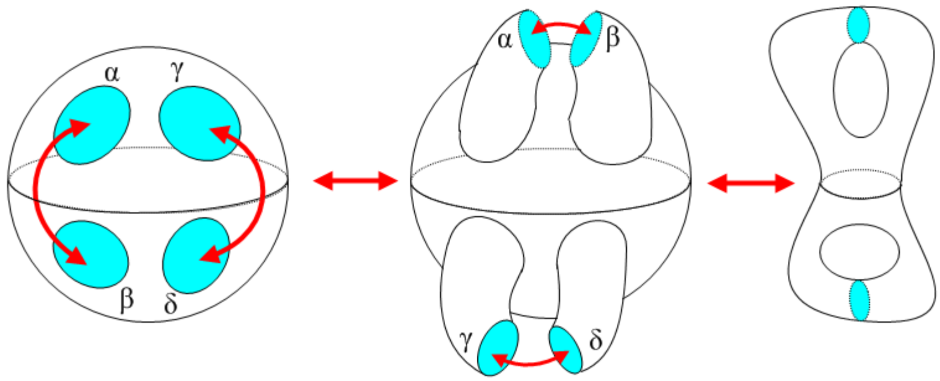


Figure 24: The first loxodromic generator sends the shaded disc enclosed by α to the one enclosed by β and the second loxodromic generator sends the shaded disc enclosed by γ to the one enclosed by δ. The fundamental domain of such a group action is a handlebody of genu $g = 2$.

case is when the fundamental domain has genus g, whose corresponding group action is generated by g loxodromic generators. Such a finitely generated free group is called Schottky group [31] [52]. This group can be defined in the following way.

Definition: For some fixed point $p \in \mathbb{CP}^1$, each Jordan curve not passing through p divides the Riemann sphere into two pieces, and we call the piece containing p the exterior of the curve, and the other piece its interior. Assume we have $2g$ disjoint Jordan curves C_1, $\tilde{C}_1$, ... C_g, $\tilde{C}_g$ in $\mathbb{CP}^1$ with disjoint interiors. A Schottky group is a Kleinian group generated by transformations γ_i taking the exterior of C_i onto the interior of $\tilde{C}_i$.

The quotient space constructed in the above way is $\mathbb{H}^3/ < \gamma_i >$, whose conformal boundary in $\mathbb{CP}^1$ is the region which is exterior to all Jordan curves. The main result of Schottky uniformization of compact Riemann surfaces is that every compact Riemann surface can be built as a quotient surface of $\mathbb{CP}^1$ by actions of a Schottky group, which is proved by Koebe [53]. From the AdS/CFT's perspective, we believe that the bulk geometry of hyperbolic 3-spaces should be completely determined by the conformal structures on their conformal boundaries. Therefore we can roughly regard the degrees of freedom of the conformal structure on the boundary as the degrees of freedom of gravity in the bulk. For each compact Riemann surface, we may associate it with an Euclidean AdS_3 gravity. As we already know that such Riemann surfaces can be constructed as quotient spaces via some action of Schottky group. We call the space of elements that generate the Schottky group (up to Möbius transformations) the Schottky space. We may therefore relate the Schottky space with the space of Euclidean bulk geometries, and with the moduli space of conformal boundaries. It can be proved that the dimension of Schottky space is $6g-6$. One may naively think that this space is the moduli space of the boundary Riemann surfaces. However, there is a subtlety here. Although they have the same dimensions, it can be proved that the Schottky space is actually the universal cover of $\mathcal{M}_g$. i.e. The Schottky space is the Teichmüller space of compact Riemann surfaces of genus g. This is precisely the result from the theorem given by Curt McMullen.

4.3 Euclidean BTZ Black Hole and Thermal AdS_3

Computations in this section is mainly based on [18] [15] [6]. In my thesis, we are only interested in conformal field theory living on a torus, which has a finite temperature. Topologically, a torus can be made by removing two non-intersecting closed curves α and β on $\mathbb{CP}^1$, followed by an identification $\alpha \sim \beta$. From

Brouwer's fixed-point theorem, Mobiös transformation on $\mathbb{CP}^1$ always have fixed points on it. For a discrete subgroup Γ, we define the limit set $\Lambda(\Gamma) \subset \mathbb{CP}^1$, which is the set of fixed points of non-trivial elements of Γ on $\mathbb{CP}^1$. If $\mathbb{H}^3/\Gamma$ is not compact, we define an open set $\Omega(\Gamma) = \mathbb{CP}^1 - \Lambda(\Gamma)$. The conformal boundary of the spacetime that we are interested in is the quotient space $\Omega(\Gamma)/\Gamma$. If Γ does not contain any elliptic elements, then Γ acts on $\mathbb{H}^3$ fixed-point freely and is called torsion-free Klein group.

Let us denote the quotient space as $\Sigma = \Omega(\Gamma)/\Gamma$, whose fundamental group is $\pi_1(\Sigma) = \mathbb{Z} \oplus \mathbb{Z}$. This implies that the fundamental group of $\Omega(\Gamma)$is a subgroup of the fundamental group of Σ. Possible subgroups of $\mathbb{Z} \oplus \mathbb{Z}$ are listed below:

1. $\pi_1(\Omega(\Gamma))$ is a infinite subgroup of finite index, which is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$.
2. $\pi_1(\Omega(\Gamma))$ is isomorphic to either $\mathbb{Z}$ or $\mathbb{Z} \times \mathbb{Z}_n$.
3. $\pi_1(\Omega(\Gamma))$ is an trivial subgroup.

The first case should not be considered here because $\Omega(\Gamma)$ is a finite cover of Σ and so is itself still a Riemann surface with genus 1. But a torus can never be a subset of a Riemann sphere. In case 3, $\Omega(\Gamma)$ is the universal covering space of Σ, thus it is isomorphic to $\mathbb{C}$. In other words, it is the completment of ∞ in the Riemann sphere. Γ is therefore, a discrete subgroup which fixes the ∞. For this reason, in case 3, any element in Γ must be in the form of upper triangular matrices.

$$\begin{pmatrix} \lambda & w \\ 0 & \lambda^{-1} \end{pmatrix} \tag{343}$$

Furthermore, since $\Omega(\Gamma)$ is simply-connected, $\pi_1(\Omega(\Gamma)) = 0$. From the identity $\pi_1(\Omega(\Gamma)/\Gamma) = \pi_0(\Gamma) = \Gamma$, we conclude that in this case, Γ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$, which is generated by the following two independent matrices

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$$

We may reduce the label of this group by the ratio $b/a = \nu$ for $\Im\nu > 0$. Then the equivalent relation given by the group action is

$$w \sim w + m + n\nu \tag{344}$$

where w is the complex coordinated on the conformal boundary and $m, n \in \mathbb{Z}$. The quotient space is a torus. But this is not what we want since the group is generated by parabolic elements. Hence we have the 'cusp' line at the center circle of the solid torus. The last one left is case 2. In the case that the fundamental group of $\Omega(\Gamma)$ is $\mathbb{Z}$, it is a topological cylinder $\mathbb{R} \times \mathbb{S}$. It's conformal structure is determined by the punctured complex plane (which is the $z = 0$ slice of upper-half space) with complex structure $w = x + iy$. This surface can be regarded as a Riemann sphere without its south and north poles. (i.e. Γ fixes two points ($w = 0$ and $w = \infty$) on the conformal boundary). Therefore, this group, which is isomorphic to $\mathbb{Z}$, is generated by

$$W = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \tag{345}$$

where α is some complex number such that $|\alpha| > 1$. We set α to be $e^{i\pi\tau}$. It acts on $\mathbb{H}^3$ freely and discontinuously except the origin point $\{x = 0, y = 0, z = 0\}$. The quotient space Σ is a punctured complex plane modulo the group generated by W. On a punctured complex plane, it is more convenient if we apply a new coordinate $w = e^{2\pi i\mu}$. We can see that the modulo is given by the following two identifications

$$\mu \sim \mu + 1 \text{ and the one given by } W\text{: } \mu \sim \mu + \frac{\log \alpha}{\pi i}$$

This is a torus that we are interested in.

When the fundamental group of $\Omega(\Gamma)$ is $\mathbb{Z}\times\mathbb{Z}_n$, the conformal boundary of the corresponding quotient space is still the torus as the one from last case. This can be seen if we use the isometric action (31), for $A = kW$, where

$$k = \begin{pmatrix} e^{\frac{2\pi i}{n}} & 0 \\ 0 & e^{-\frac{2\pi i}{n}} \end{pmatrix} \tag{346}$$

with some integer n. The matrix k generates the group $\mathbb{Z}_n$, which is clearly a discrete subgroup of $SU(2)$ from the Iwasawa decomposition of $SL(2,\mathbb{C})$. The isometric action is

$$kW \begin{pmatrix} U-X & iV+Y \\ -iV+Y & U+X \end{pmatrix} (kW)^\dagger \tag{347}$$

Since the action of W is isometric, we may denote

$$W \begin{pmatrix} U-X & iV+Y \\ -iV+Y & U+X \end{pmatrix} W^\dagger = \begin{pmatrix} \tilde{U}-\tilde{X} & i\tilde{V}+\tilde{Y} \\ -i\tilde{V}+\tilde{Y} & \tilde{U}+\tilde{X} \end{pmatrix} \tag{348}$$

The identification

$$W \begin{pmatrix} U-X & iV+Y \\ -iV+Y & U+X \end{pmatrix} W^\dagger \sim \begin{pmatrix} U-X & iV+Y \\ -iV+Y & U+X \end{pmatrix} \tag{349}$$

gives us the boundary torus mentioned above. We define complex structure $w = x+iy$ on each constant-z slice, from the subsequent action

$$\begin{aligned} k \begin{pmatrix} U-X & iV+Y \\ -iV+Y & U+X \end{pmatrix} k^\dagger &= \begin{pmatrix} e^{\frac{2\pi i}{n}} & 0 \\ 0 & e^{-\frac{2\pi i}{n}} \end{pmatrix} \begin{pmatrix} U-X & iV+Y \\ -iV+Y & U+X \end{pmatrix} \begin{pmatrix} e^{\frac{2\pi i}{n}} & 0 \\ 0 & e^{-\frac{2\pi i}{n}} \end{pmatrix}^\dagger \\ &= \begin{pmatrix} U-X & e^{\frac{4\pi i}{n}}(iV+Y) \\ e^{-\frac{4\pi i}{n}}(-iV+Y) & U+X \end{pmatrix} \sim \begin{pmatrix} U-X & iV+Y \\ -iV+Y & U+X \end{pmatrix}, \end{aligned} \tag{350}$$

we see that this is equivalent to identifications

$$w = x+iy = \frac{Y+iV}{U+X} \sim e^{\frac{4\pi i}{n}} \frac{Y+iV}{U+X} = e^{\frac{4\pi i}{n}} w \tag{351}$$

and

$$\bar{w} = x-iy = \frac{Y-iV}{U+X} \sim e^{\frac{-4\pi i}{n}} \frac{Y-iV}{U+X} = e^{\frac{-4\pi i}{n}} \bar{w} \tag{352}$$

which leaves z-coordinate invariant. Consequently, the conformal boundary is still the same torus created by action of W. However, the identifications on each constant-z plane produces a deficit angle $\theta = 2\pi(1-\frac{1}{n})$ at $x = y = 0$. Thus, the quotient space $\mathbb{H}^3/\mathbb{Z}\times\mathbb{Z}_n$ is a solid torus whose center circle has conical singularities. It is well-known that at classical level, conical singularity of codimensional 2 for 3D gravity represents orbit of a massive particle. Therefore, such quotient space does not correspond to solutions of pure gravity.

From the above analysis, we see that in Euclidean signature, only loxodromic motion and hyperbolic motions are possible to make a torus to be the conformal boundary of pure three dimensional gravity. Hence, on the bulk side, Euclidean three dimensional gravity is simply $(\mathbb{H}^3)^*/\mathbb{Z}$. As mentiened previously, its conformal boundary is Riemann surface with genus 1. We defined a complex modulus μ of this surface by $w = e^{2\pi i\mu}$. Then, any other good boundary Riemann surfaces can be obtained by an $SL(2,\mathbb{Z})$ action, $\mu \mapsto (a\mu+b)/(c\mu+d)$ with integers $ad-bc = 1$. From Bezout's lemma, for any given pair (c,d), the pair (a,b) is uniquely determined up to an equivalent relation $(a,b) \sim (a,b)+\mathbb{Z}(c,d)$. A further deduction shows that the equation $ad-bc = 1$ for given a pair of integers (c,d) has integer solution if and only if (c,d)

are coprime. Thus all the saddle points of pure three dimensional gravity are labeled by a pair of coprime integers, denoted by $M_{c,d}$. From the equivalent relation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \sim \begin{pmatrix} a+nc & b+nd \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{353}$$

we see that the set of saddle points are actually the quotient group $\Gamma_\infty \backslash SL(2,\mathbb{Z})$, where Γ_∞ is translation group. Hence it is isomorphic to $SL(2,\mathbb{Z})/\mathbb{Z}$. In particular, we will see that thermal AdS_3 is $M_{0,1}$ while the BTZ black hole is $M_{1,0}$.

To understand the geometric difference between these different solid tori, we can pick up a pair of 1-cycles given by (α,β) as the canonical homology basis of a torus. Here, the symbol (α,β) is antisymmetric, which represent whether the two cycles intersect with each other or not. If they do not intersect, then we claim that $(\alpha,\beta)=0$ and if they do intersect, we set $(\alpha,\beta)=\pm 1$. This product of 1-cycles form a matrix

$$\begin{cases} (\alpha,\alpha)=(\beta,\beta)=0 \\ (\alpha,\beta)=-(\beta,\alpha)=1 \end{cases} \leftrightarrow \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{354}$$

Apparently, transformations that leave this matrix invariant is $Sp(2,\mathbb{Z})$, called the symplectic group of integers. This group has two generators which geometrically correspond to dehn twists on the torus. For example, we can consider a dehn twist D_α created by slicing the torus along α-cycle, then we twist the edge on the one side by 2π, and then glue along the two sides back together. Under such a dehn twist, the original β becomes sum of α and β. i.e.

$$D_\alpha(\alpha)=\alpha, \quad D_\alpha(\beta)=\alpha+\beta \tag{355}$$

In terms of matrix realizations, they are

$$D_\alpha = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad D_\beta = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \tag{356}$$

In general, if we pick up an Euclidean saddle point with given two 1-cycles (α,β) satisfying (), and we can

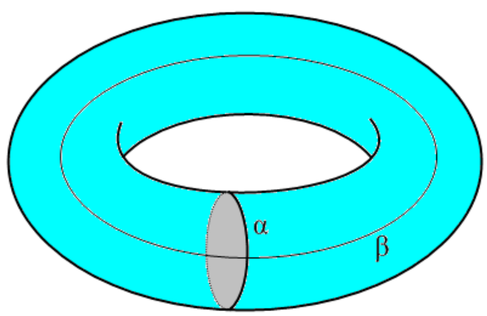


Figure 25: Two 1-cycles of a torus

choose another pair of 1-cycles (A,B) as a new basis

$$(A,B)=1, \quad (A,A)=(B,B)=0 \tag{357}$$

such that A is chosen to be a primitive contractible 1-cycle. For convenience, we set $A=c\alpha+d\beta$ for (c,d) coprime integers. Then $(A,B)=1$ is determined by $B=a\alpha+b\beta$ only when $ad-bc=1$, for a, b integers. In other words, we pick up a fixed boundary torus, these saddle points represent different ways to fill in the bulk.

From the Lorentzian BTZ black hole

$$ds^2 = -\left(r^2 - (r_+^2 + r_-^2) + \frac{(2r_+r_-)^2}{4r^2}\right)dt^2 + \frac{dr^2}{r^2 - (r_+^2 + r_-^2) + \frac{(2r_+r_-)^2}{4r^2}} + r^2\left(d\phi + \frac{2r_+r_-}{2r^2}dt\right)^2, \quad (358)$$

with identification $\phi \sim \phi + 2\pi$, the wick rotation $t \to it_E$ automatically requires an analytic continuation $r_- \to ir_E^-$ in order to let the Euclidean metric be real valued. The Euclidean black hole metric is

$$ds^2 = \left(r^2 - (r_+^2 - (r_E^-)^2) - \frac{(2r_+r_E^-)^2}{4r^2}\right)dt_E^2 + \frac{dr^2}{r^2 - (r_+^2 - (r_E^-)^2) - \frac{(2r_+r_E^-)^2}{4r^2}} + r^2\left(d\phi - \frac{2r_+r_E^-}{2r^2}dt_E\right)^2, \quad (359)$$

with $\phi \sim \phi + 2\pi$. In Euclidean signature, we have $M = (r^+)^2 - (r_E^-)^2$ and $J_E = 2r_E^-r^+$. This metric can also be obtained via an appropriate coordinate transformation from a static Euclidean black hole with unit mass. Using the analytic continuation (338), it is easy to see that an Euclidean static BTZ blakc hole of unit mass is given by

$$\begin{cases} U = r\cosh\phi \\ V = \sqrt{r^2-1}\sin t \\ X = r\sinh\phi \\ Y = \sqrt{r^2-1}\cos t \end{cases} \quad (360)$$

Remark: In Euclidean signature, it does not make any sense to talk about interior of black holes. By performing a corresponding coordinate transformation, the rotating Euclidean black hole is given by equations

$$\begin{cases} U = \sqrt{\frac{r^2 + (r_E^-)^2}{r_+^2 + (r_E^-)^2}}\cosh(r_+\phi - r_E^-t_E) \\ V = \sqrt{\frac{r^2 + r_+^2}{r_+^2 + (r_E^-)^2}}\sin(r_+t_E + r_E^-\phi) \\ X = \sqrt{\frac{r^2 + (r_E^-)^2}{r_+^2 + (r_E^-)^2}}\sinh(r_+\phi - r_E^-t_E) \\ Y = \sqrt{\frac{r^2 + r_+^2}{r_+^2 + (r_E^-)^2}}\cos(r_+t_E + r_E^-\phi) \end{cases} \quad (361)$$

It is related with upper-half space model $ds^2 = \frac{1}{z^2}(dx^2 + dy^2 + dz^2)$ via

$$\begin{aligned} x &= \sqrt{\frac{r^2 - r_+^2}{r^2 + (r_E^-)^2}}\cos(r_+t_E + r_E^-\phi)\exp\left\{-r_+\phi + r_E^-t_E\right\} \\ y &= \sqrt{\frac{r^2 - r_+^2}{r^2 + (r_E^-)^2}}\sin(r_+t_E + r_E^-\phi)\exp\left\{-r_+\phi + r_E^-t_E\right\} \\ z &= \sqrt{\frac{r_+^2 + (r_E^-)^2}{r^2 + (r_E^-)^2}}\exp\left\{-r_+\phi + r_E^-t_E\right\} \end{aligned} \quad (362)$$

It is easy to see that the identification $\phi \sim \phi + 2\pi$ is equivalent to

$$\gamma\begin{pmatrix} U - X & iV + Y \\ -iV + Y & U + X \end{pmatrix}\gamma^\dagger \sim \begin{pmatrix} U - X & iV + Y \\ -iV + Y & U + X \end{pmatrix}, \quad (363)$$

where

$$\gamma = \begin{pmatrix} e^{\pi(r_+ + ir_E^-)} & 0 \\ 0 & e^{-\pi(r_+ + ir_E^-)} \end{pmatrix}. \quad (364)$$

Therefore, the Euclidean BTZ black hole is indeed a quotient space of $\mathbb{H}^3$ modding out a cyclic discrete subgroup generated by a loxodromic Möbius transformation. However, this is not the whole story. We have mentioned that in Lorentzian signature, under the translation $\phi \to \phi + 2\pi$, $(x, y, z) \in \mathbb{H}^{2,1}$ undergoes a dilation, followed by a Lorentzian boost in $x - y$ plane which produces no singularities. While in Euclidean signature, $x - y$ plane has positive signature. Under the action $\phi \sim \phi + 2\pi$, $(x, y, z) \in \mathbb{H}^3$ should have

$$(x, y, z) \sim e^{2\pi r_+} \left(x \cos(2\pi r_E^-) + y \sin(2\pi r_E^-), y \cos(2\pi r_E^-) - x \sin(2\pi r_E^-), z \right) \tag{365}$$

i.e. the identification is a dilation followed by a rotation in $x-y$ plane, which may produce conical singularity on z-axis. To avoid having singularity at $x = y = 0$, we first perform another coordinate transformation

$$(x, y, z) = (R \cos\theta \cos\chi, R \sin\theta \cos\chi, R \sin\chi) \tag{366}$$

The identification $\phi \sim \phi + 2\pi$ is

$$(R, \theta, \chi) \sim (R e^{2\pi r_+}, \theta + 2\pi r_E^-, \chi) \tag{367}$$

From the above identification, we see that the fundamental domain of Euclidean BTZ black hole in $\mathbb{H}^3$ is the region between two hemispheres $R = 1$ and $R = e^{2\pi r_+}$. The identification is performed by a $2\pi r_E^-$ rotation, followed by gluing the two hemispheres. The new coordinate transformation is smooth at z-axis only if $\theta \sim \theta + 2\pi$. This is equivalent to the identification

$$(-\phi, t_E) \sim (-\phi + \Theta, t_E + \beta) \tag{368}$$

where

$$\Theta = \frac{-2\pi r_E^-}{r_+^2 + (r_E^-)^2}, \quad \beta = \frac{2\pi r_+}{r_+^2 + (r_E^-)^2} \tag{369}$$

In other words, the smoothness of Euclidean BTZ black hole requires that it has temperature

$$T = \frac{r_+^2 + (r_E^-)^2}{2\pi r_+} \tag{370}$$

From the complex structure on each constant-z slice of $\mathbb{H}$, we see that for

$$w = \sqrt{\frac{r^2 - r_+^2}{r^2 + (r_+^-)^2}} \exp\left\{ (r_+ - i r_+^-)(-\phi + i t_E) \right\}, \tag{371}$$

the identification

$$-\phi + i t_E \sim (-\phi + i t_E) + (\Theta + i\beta) = -\phi + i t_E + \frac{2\pi i}{r_+ - i r_E^-} \tag{372}$$

reflects a trivial fact that $w = e^{2\pi i} w$. $-\phi + i t_E$ is the complex structure on the conformal boundary of Euclidean BTZ black hole. We define

$$\Theta + i\beta = 2\pi\tau. \tag{373}$$

i.e. $\tau = \dfrac{i}{r_+ - i r_E^-}$. This parameter τ is the modular parameter of the boundary torus of Euclidean black hole.

Thermal AdS_3 is

$$ds_E^2 = \left(1 + \tilde{r}^2\right) d\tilde{t}_E^2 + \tilde{r}^2 d\tilde{\phi}^2 + \frac{d\tilde{r}^2}{1 + \tilde{r}^2} \tag{374}$$

where we have $\tilde{\phi} \in (0, 2\pi]$. Each Euclidean saddle point is related with another by a global diffeomorphism. We, therefore, can transform a thermal AdS_3 metric to an Euclidean rotating BTZ metric with parameters r_+ and r_E^- via the following transformation

$$\begin{cases} \tilde{t}_E = r_E^- t_E - r_+ \phi \\ \tilde{\phi} = -r_+ t_E + r_E^- \phi \\ \tilde{r}^2 = \dfrac{r^2 - r_+^2}{r_+^2 + (r_E^-)^2} \end{cases} \tag{375}$$

Then it is easy to see that the identification (363) of Euclidean BTZ black hole implies that $\tilde{\phi} \sim \tilde{\phi} + 2\pi$ for thermal AdS_3, which agrees with the metric (374). The identification $\phi \sim \phi - 2\pi$ implies that

$$-\tilde{\phi} \sim -\tilde{\phi} + 2\pi r_E^-, \quad i\tilde{t}_E \sim i\tilde{t}_E + 2\pi r_+ \tag{376}$$

or

$$i\tilde{t}_E - \tilde{\phi} \sim i\tilde{t}_E - \tilde{\phi} + 2\pi(ir_+ + r_E^-). \tag{377}$$

We define the modular parameter of thermal AdS_3 as

$$\tilde{\tau} = ir_+ + r_E^- \tag{378}$$

It is clear that modular paramters of Euclidean BTZ black hole and thermal AdS_3 is related with each other via

$$\tau = \frac{-1}{\tilde{\tau}} \tag{379}$$

Therefore the conformal boundaries of thermal AdS_3 and BTZ black hole are related via S-transformation. Each point in the black strip $|\tau| > 1$, $\frac{-1}{2} < \Re\tau < \frac{1}{2}$ represents the conformal boundary of an Euclidean BTZ

Figure 26: Tessellation (Picture Copied from Book 'Outer Circles, An Introduction to Hyperbolic 3-Manifolds, Albert Marden)

black hole, which is sent to a point in the white 'triangle', which is a thermal AdS_3. Points from other pieces represent conformal boundaries of other Euclidean saddle points. These infinitely many regions in upper-half plane related one from another via modular tranformation all together form a tesselation of the infinite strip $\frac{-1}{2} < \Re\tau < \frac{1}{2}$. Other parts of the upper-half plane are related with this strip by T-transformation, which should be modded out to avoid overcounting.

5 Lagrangian Formalism

5.1 Splitting of Spacetime and Extrinsic Curvature

Most of the calculations in this chapter are copied from the book [2] [1]. One can find similar introductions in any advanced textbook on general relativity. For any 3-manifold M, the $2+1$ splitting requires that there exist a smooth function $t : M \mapsto \mathbb{R}$ such that each $t = constant$ defines a spacelike hypersurface Σ_t. On the spacial hypersurface Σ_t, we have a conormal 1-form $(dt)_a$. We call it conormal because for any vector w^b that is tangent to Σ_t, we have $(dt)_a w^a = 0$, that is, at any point $p \in M$, the 1-form $(dt)_a$ is normal to any tangent vector in $T_p\Sigma_t$. This is true without the existence of metric on M. If we can find any two conormal 1-forms n_a and m_a on Σ_t, it is not hard to see that $n_a = \lambda m_a$, where λ is some constant number. But if we have defined metric g_{ab} on M, then we can always find a vector n^a that is normal to the hypersurface Σ_t by means of the metric. It is easy to see that for any conormal 1-form n_a, the corresponding vector $n^a = g^{ab} n_b$ is normal.

In general relativity, the splitting of spacetime will make sense if and only if we can chose a proper reference frame. A reference frame can be defined as a smooth time-like future-directed vector field, whose each integral curve $\gamma(s)$ intersects with Σ_t once and only once. These integral curves are the worldlines of a set of observers in M. Suppose these integral curves are generated by vector field $t^a = \left(\frac{dx^\mu}{ds}\right)^a$ on M. The definition of reference frame requires that $t^a(dt)_a \neq 0$. For simplicity, we can set $t^a(dt)_a = 1$. As a result, $t = s$, so the worldlines of observers are parametrized by t, which is identified as coordinate-time.

In general, t^a is not normal to spacial slice Σ_t. We can decompose this vector field in the following way

$$t^a = Nn^a + N^a \tag{380}$$

where n^a is the normal vector of Σ_t with unit length and N is some scalar field on M. The condition $t^a(dt)_a = 1$ implies that $n_a = -N\partial_a t$. By setting $x^0 = t$, we have $N^0 = 0$. In other words, the decomposition is

$$t^a = Nn^a + \vec{N} \tag{381}$$

The above equations has a strong geometric interpretation. The scalar function N generates the evolution in time direction, called **lapse**, while the spacial vector $\vec{N}$ generates the deformation in spacial directions, called **shift**. Let $N^i = N^a(dx^i)_a$, where the indix i runs in spacial indices 1 and 2, $N_a = g_{ab}N^b$, then the metric

$$g_{ab} = g_{\mu\nu} dx^\mu \otimes dx^\nu = g_{\mu\nu}(dx^\mu)_a(dx^\nu)_b \tag{382}$$

can be written in the following way

$$ds^2 = -N^2 dt^2 + g_{ij}\left(N^i dt + dx^i\right)\left(N^j dt + dx^j\right) \tag{383}$$

The induced metric on hypersurface Σ_t is $h_{ab} = g_{ab} + n_a n_b$. Plugging the above formula for g_{ab}, we have

$$\sqrt{|g|} = N\sqrt{h} \tag{384}$$

Under this decomposition, any vector $v^a \in T_pM$ can be decomposed into a component tangent to Σ_t and a normal component that is proportional to n^a.

$$v^a = \left(-g_{bc}v^c n^b n^a\right) + \left(v^a + g_{cb}v^c n^b n^a\right) \tag{385}$$

or

$$v = -g(v,n)n + (v + g(v,n)n) \tag{386}$$

In particular, for any two arbitrary vector fields v and u defined on Σ_t, we can decompose the $\nabla_u v$ into a normal part and a tangent part.

$$\nabla_u v = -g\left(\nabla_u v, n\right)n + \left(\nabla_u v + g\left(\nabla_u v, n\right)n\right) \tag{387}$$

The first term measures how much a vector fails to tangent to Σ_t after we parallel translate in some direction, that is, how much the hypersurface Σ_t is bended in M. We call the first term

$$-g\left(\nabla_u v, n\right)n = K(u,v)n \tag{388}$$

the extrinsic curvature of Σ_t in M. The second term is often written as

$$\nabla_u v + g\left(\nabla_u v, n\right)n = {}^2\nabla_u v = D_u v \tag{389}$$

because the operator D turns our to be the Levi-Civita connection on spacial slice Σ_t with respect to the induced metric $h_{ab} = g_{ab} + n_a n_b$ on Σ_t. We can easily check that $D_u v$ is $C^\infty(\Sigma_t)$-linear with respect to vector field u and $\mathbb{R}$-linear with respect to v. In addition, it satisfies the Leibniz law

$$D_v(fw) = \nabla_v(fw) + g(n, \nabla_v(fw))\, n = v(f)\, w + f D_v w \tag{390}$$

for any $f \in C^\infty(\Sigma_t)$ and any $v,\, w \in \mathrm{Vect}(\Sigma_t)$. We can check that it is metric preserving

$$u(g(v,w)) = g(\nabla_u v, w) + g(v, \nabla_u w) = g(D_u v, w) + g(v, D_u w) \tag{391}$$

for any u, v and $w \in \mathrm{Vect}(\Sigma_t)$ since $g(n,v) = g(w,n) = 0$. Finally, it is clear that the operator D is torsion free because

$$D_u v - D_v u = [u,v] \tag{392}$$

for any u and $v \in \mathrm{Vect}(\Sigma_t)$.

From the definition of extrinsic curvature, we can easily see that it is symmetric and can be expressed in an alternative way

$$K(u,v) = -g(\nabla_u v, n) = g(\nabla_u n, v) \tag{393}$$

This alternative expression gives us another way of looking at extrinsic curvature; it measures how much a unit normal vector n rotates in the direction of v when being parallel translated in the direction of u. It is not hard to show that extrinsic tensor K_{ab} has the following properties

$$K_{ab} = h_a^c h_b^d \nabla_c n_d = h_a^c \nabla_c n_b \tag{394}$$

and

$$K_{ab} = \frac{1}{2}\mathcal{L}_n h_{ab} \tag{395}$$

where $h_b^a = g^{ac} h_{cb} = g^{ac}(g_{cb} + n_c n_d) = \delta_b^a + n^a n_b$ is the projection tensor. It maps the tensor $\nabla_c n_d$ to its spacial component on Σ_t. This projection is necessary because $\nabla_c n_d$ is defined on M but K_{ab} should be defined only on Σ_t. Consequently, for any normal vector n^a on Σ_t, we have $K_{ab} n^b = 0$. Obviously, the induced metric on Σ_t also has a similar property, $h_{ab} n^b = 0$. In general, we call a tensor T_b^a a **spacial tensor** if it satisfies one of the following conditions

(a) $n_a T_b^a = 0, \quad n^b T_b^a = 0$

(b) $T_b^a = h_c^a h_b^d T_d^c$

For spacial tensors, since they are defined on spacial slices, it makes sense only if we use h_{ab} and h^{ab} to lower or raise indices. Clearly, both the induced metric h_{ab} and extrinsic curvature K_{ab} are defined on spacial slice Σ_t and are spacial tensors. It can be proved that h_{ab} and K_{ab} also satisfy the second condition. By using projection tensor, we can re-write the definition of the operator D as

$$D_c T_b^a = h_d^a h_b^e h_c^f \nabla_f T_e^d \tag{396}$$

for any spacial tensor T_b^a. It is easy to show that this new definition indeed agrees with (). Now this operator D gives the intrinsic Riemann curvature of the spacial slice ${}^2R^d{}_{abc}\omega_d = 2D_{[a}D_{b]}\omega_c$, for any spacial 1-form $\omega_c \in \Omega^1(\Sigma_t)$. It turns out that the intrinsic Riemann tensor of Σ_t is related with its extrinsic curvature K_{ab} and the spacial components of the instrinsic Riemann tensor $R^a{}_{bcd}\Big|_{\Sigma_t}$ via **Gauss-Codazzi equations**

$${}^2R^d{}_{abc} = h_a^e h_b^f h_c^l h_m^d R^m{}_{efl} - 2K_{c[a}K_{b]}^d \tag{397}$$

The proof is extremely tedious and thus is omitted. Readers who are interested in can have a try by self or find answer from google. From the above equations, we see that ${}^2R^d{}_{abc} = h_a^e h_b^f h_c^l h_m^d R^m{}_{efl}$ when the extrinsic curvature vanishes everywhere.

Given a spacial slice Σ_t embedded in spacetime M, if we know the induced metric h_{ab}, which gives us the local geometry of Σ_t, and the extrinsic curvature K_{ab}, which tells us how Σ_t is bended as a hypersurface in M, then have enough information about this surface Σ_t. To study the whole manifold M, we also need to know the evolution of this spacial slice. We hope to find the differential equation of the time derivative of h_{ab} and K_{ab} that agree with Einstein's equations. With given initial data of h_{ab} and K_{ab} (the Cauchy data), we will be able to know the history of our universe.

The time derivative of a spacial vector $w^a \in \mathrm{Vect}(\Sigma_t)$ is defined as a Lie derivative in t^a direction.

$$\dot{w}^a = \mathcal{L}_t w^a \tag{398}$$

This definition is natural because the pull-back $\gamma^*(w^a)$, where the flow γ is generated by t^a, is still a tangent vector field on Σ_t. But for a unit normal n^a on Σ_t, its pull-back $\gamma^*(n^a)$ is not necessarily still a normal vector for obvious reasons. In other word, if we naively define the time derivative of a 1-form w_a, which is dual to the tangent vector w^a, as $\mathcal{L}_t w_a$, then we see that

$$n^a \mathcal{L}_t w_a = \mathcal{L}_t (n^a w_a) - (\mathcal{L}_t n^a) w_a = \mathcal{L}_t g(n, w) - (\mathcal{L}_t n^a) w_a = -(\mathcal{L}_t n^a) w_a \neq 0 \tag{399}$$

where we used the fact that n^a is normal to tangent vector w^a and so $g(n, w) = 0$, that is $\mathcal{L}_t w_a$ is no longer a spacial 1-form. For this reason, we have to project this derivative onto the spacial slice and define the time derivative of a spacial 1-form as

$$\dot{w}_a = h_a^b \mathcal{L}_t w_b \tag{400}$$

From now on, whenever we say the time derivative of a spacial tensor T_b^a, we always need a projection tensor and denote

$$\tilde{\mathcal{L}}_t T_b^a = h_c^a h_b^d \mathcal{L}_t T_d^c \tag{401}$$

Using the decomposition $t^a = N n^a + N^a$, we have

$$\tilde{\mathcal{L}}_t = N \tilde{L}_n + \tilde{L}_{\vec{N}} \tag{402}$$

So the time derivative of metric h_{ab} is given by

$$\begin{aligned} \dot{h}_{ab} &= \tilde{\mathcal{L}}_t h_{ab} = N \tilde{\mathcal{L}}_n h_{ab} + \tilde{\mathcal{L}}_{\vec{N}} h_{ab} = 2N K_{ab} + \tilde{\mathcal{L}}_{\vec{N}} h_{ab} \\ &= 2N K_{ab} + 2 D_{(a} N_{b)} \end{aligned} \tag{403}$$

where we have used killing equation and the fact that D is compactible with h_{ab} in the last line. We use the above formula to calculate the time derivative of K_{ab} and it works out to be

$$\dot{K}_{ab} = N h_a^c h_b^d R_{cd} - {}^2R_{ab} N + 2N K_a^c K_{cb} - N K K_{ab} + D_a D_b N + \tilde{\mathcal{L}}_{\vec{N}} K_{ab} \tag{404}$$

where we denote K as the trace of K_{ab}, i.e. $K = \mathrm{Tr}(K) = h^{ab} K_{ab}$, and ${}^2R_{ab}$ is the intrinsic Ricci tensor on spacial slice Σ_t. But since the procedure of the calculation is extremely tedious, we omit here. The time evolution is fully determined by equation (403) and (404). Note that these two equations hold without $G_{ab} = 0$, the Einstein's equations. For gravity in vacuum, $G_{ab} = 0$ implies $R_{ab} = 0$. So the equations of time evolution are given by

$$\begin{cases} \dot{h}_{ab} = 2N K_{ab} + 2 D_{(a} N_{b)} \\ \dot{K}_{ab} = -{}^2R_{ab} N + 2N K_a^c K_{cb} - N K K_{ab} + D_a D_b N + \tilde{\mathcal{L}}_{\vec{N}} K_{ab} \end{cases} \tag{405}$$

It is not hard to see that the above equations agrees with the spacial parts of Einstein's equations $G_{ab} h_c^a h_d^b = 0$. From the Gauss-Codazzi equations, we can prove the following identity. Since the proof is very long, we will not show it here.

Lemma : The time components of Einstein tensor is related with extrinsic curvature K_{ab} and intrinsic Riemann scalar of Σ_t via the following identity

$$2G_{ab}n^a n^b = {}^2R - K_{ab}K^{ab} + K^2 \tag{406}$$

From this identity, using the equation $G_{ab} = R_{ab} - Rg_{ab}/2$, we see that

$$R = 2\left(G_{ab}n^a n^b - R_{ab}n^a n^b\right) = \left({}^2R - K_{ab}K^{ab} + K^2\right) - 2R_{ab}n^a n^b \tag{407}$$

The last term on the RHS of the above equation is

$$\begin{aligned} R_{ab}n^a n^b &= n^a R^c{}_{acb} n^b = -n^a \left(\nabla_a \nabla_c - \nabla_c \nabla_a\right) n^c \\ &= -\nabla_a \left(n^a \nabla_c n^c\right) + \left(\nabla_a n^a\right) \nabla_c n^c + \nabla_c \left(n^a \nabla_a n^c\right) - \left(\nabla_c n^a\right) \nabla_a n^c \\ &= K^2 - K_{ac}K^{ac} - \nabla_a \left(n^a \nabla_c n^c\right) + \nabla_c \left(n^a \nabla_a n^c\right) \end{aligned} \tag{408}$$

In the last step, we have used the identity $\left(\nabla_c n_a\right) n^a = 0$, due to the fact that vector n^a has unit length, and so

$$\begin{aligned} \left(\nabla^a n_d\right) \nabla^d n_a &= \left(\nabla^a n_d\right) \nabla^d n_a + n^a n_c \left(\nabla^c n_d\right) \nabla^d n_a + n^b n_d \left(\nabla^a n_b\right) \nabla^d n_a + n^a n^b n_c n_d \left(\nabla^c n_b\right) \nabla^d n_a \\ &= \left(\delta^a_c + n^a n_c\right) \left(\delta^b_d + n^b n_d\right) \left(\nabla^c n_b\right) \nabla^d n_a \\ &= h^a_c h^b_d \left(\nabla^c n_b\right) \nabla^d n_a = K^a_b K^b_a = K^{ab} K_{ab} \end{aligned} \tag{409}$$

Using the lemma () and equation (), we conclude that the Lagrangian of Einstein-Hilbert action is given by

$$\mathcal{L} = \sqrt{h} N \left\{{}^2R + K_{ab}K^{ab} - K^2 - 2\Lambda + 2\left[\nabla_a \left(n^a \nabla_c n^c\right) - \nabla_c \left(n^a \nabla_a n^c\right)\right]\right\} \tag{410}$$

We denote the quantity ${}^2R + K_{ab}K^{ab} - K^2 = C$. It can be regarded as a spacial function defined on spacial slice Σ_t. But once we consider the time evolution of h_{ab}, this spacial function C depends on $\dot{h}_{ab}$ explicitly via K_{ab}. From the Lagrangian, we also find that

$$\frac{\partial \mathcal{L}}{\partial \dot{N}} = 0, \quad \frac{\partial \mathcal{L}}{\partial \dot{\vec{N}}} = 0 \tag{411}$$

Therefore, N_a cannot be dynamical variables for gravity. We claim that the dynamical variables of gravity are given by h_{ab} and K_{ab}. Now we only need to re-express the other parts of Einstein's equations in terms of h_{ab}, $\dot{h}_{ab}$ and K_{ab}, $\dot{K}_{ab}$.

5.2 Boundary Terms

For a topological field theory, there is no dynamic in the bulk but the gauge transformations on the conformal boundary become dynamical. In Hamiltonian formalism of gravity theory, boundary terms such as ADM charges carry important physics. In Lagrangian formalism, we also have boundary terms and have to add some conter terms to cancel the those boundary terms so that the action has well-defined functional derivative. The Lagrangian of gravity is given by

$$\mathcal{L} = \sqrt{g} g^{ab} R_{ab} \tag{412}$$

the variation is

$$\delta \mathcal{L} = \left.\frac{d\mathcal{L}}{d\epsilon}\right|_{\epsilon=0} = \left(\sqrt{g} g^{ab} \frac{dR_{ab}}{d\epsilon} + R \frac{d\sqrt{g}}{d\epsilon} + \sqrt{g} R_{ab} \frac{dg^{ab}}{d\epsilon}\right)_{\epsilon=0} \tag{413}$$

If we assume that spacetime has no topological boundary, or gravitational fields vanish at the boundary, we can drop all the boundary terms, the above variation will give us the Einstein equations. Now let's see what happens if spacetime has a boundary and we keep all the boundary terms.

Theorem 1: Let ∇^1 and ∇^2 be two linear connections on manifold M. The difference between them defines a $(2,1)$ tensor field A such that

$$A(X,Y) = \nabla^2_X Y - \nabla^1_X Y \tag{414}$$

for any two vector fields X and Y on M. If ∇^1 and ∇^2 are torsion free, then the tensor A is symmetric with respect to the two lower indices.

Theorem 2: For any two given metric fields g_{ab} and $\tilde{g}_{ab}$ defined on manifold M, if ∇ and $\tilde{\nabla}$ are linear connections on M that are compatable with g_{ab} and $\tilde{g}_{ab}$, respectively, then from **theorem 1**, there exists a tensor C^c_{ab} such that

$$\tilde{\nabla}_a \omega_b = \nabla_a \omega_b - C^c_{ab}\omega_c \tag{415}$$

for any 1-form ω_a. The tensor C^c_{ab} satisfies

$$C^c_{ab} = \frac{1}{2}\tilde{g}^{cd}\left(\nabla_a \tilde{g}_{bd} + \nabla_b \tilde{g}_{ad} - \nabla_d \tilde{g}_{ab}\right) \tag{416}$$

From the above two theorems, we can compute the Riemann tensors with respect to the two different metric

$$\tilde{\nabla}_{[a}\tilde{\nabla}_{b]}\omega_c = \frac{1}{2}\tilde{R}^d{}_{abc}\omega_d \tag{417}$$

$$\nabla_{[a}\nabla_{b]}\omega_c = \frac{1}{2}R^d{}_{abc}\omega_d \tag{418}$$

From (), we have

$$\begin{aligned}
&\tilde{\nabla}_a\left(\tilde{\nabla}_b\omega_c\right) = \nabla_a\left(\tilde{\nabla}_b\omega_c\right) - C^e_{ab}\tilde{\nabla}_e\omega_c - C^e_{ac}\tilde{\nabla}_b\omega_e \\
&= \nabla_a\left(\nabla_b\omega_c - C^d_{bc}\omega_d\right) - C^e_{ab}\tilde{\nabla}_e\omega_c - C^e_{ac}\left(\nabla_b\omega_e - C^d_{be}\omega_d\right) \\
&= \left(\nabla_a\nabla_b\omega_c - \omega_d\nabla_a C^d_{bc} - C^d_{bc}\nabla_a\omega_d\right) - C^e_{ab}\tilde{\nabla}_e\omega_c - C^e_{ac}\nabla_b\omega_e + C^e_{ac}C^d_{be}\omega_d
\end{aligned} \tag{419}$$

Using the defination of Riemann tensor, we have

$$\begin{aligned}
&\tilde{R}^d{}_{abc}\omega_d = R^d{}_{abc}\omega_d - 2\omega_d\nabla_{[d}C^d_{b]c} - 2C^d_{c[b}\nabla_{a]}\omega_d - 2C^e_{c[a}\nabla_{b]}\omega_e + 2C^e_{c[a}C^d_{b]e}\omega_d \\
&= R^d{}_{abc}\omega_d - 2\omega_d\nabla_{[a}C^d_{b]c} + 2C^e_{c[a}C^d_{b]e}\omega_d, \qquad \forall\omega_d \in \Omega^1(M) \\
&\Rightarrow \tilde{R}^d{}_{abc} = R^d{}_{abc} - 2\nabla_{[a}C^d_{b]c} + 2C^e_{c[a}C^d_{b]e}
\end{aligned} \tag{420}$$

Now we set $\tilde{g}_{ab} = g_{ab}(\epsilon)$ with associated connection $\tilde{\nabla}$, we have

$$\left(\nabla_a - \tilde{\nabla}_a\right)\omega_b = C^c_{ab}(\epsilon)\omega_c \tag{421}$$

and

$$C^c_{ab}(0) = 0 \tag{422}$$

Plugging into the formula for Riemann tensor (420), we have

$$\begin{aligned}
&R^d{}_{abc}(\epsilon) = R^d{}_{abc} - 2\nabla_{[a}C^d_{b]c}(\epsilon) + 2C^e_{c[a}(\epsilon)C^d_{b]e}(\epsilon) \\
&\Rightarrow R_{ac}(\epsilon) = R_{ac} - 2\nabla_{[a}C^b_{b]c}(\epsilon) + 2C^e_{c[a}(\epsilon)C^b_{b]e}(\epsilon)
\end{aligned} \tag{423}$$

Since $C^a_{ab}(0) = 0$, we have

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0}\left(C^e_{c[a}(\epsilon)C^d_{b]e}(\epsilon)\right) = 0 \tag{424}$$

and so $\delta R_{ac} = -2\nabla_{[a}\delta C^b_{b]c}$.

By theorem 2,

$$C^b_{ac}(\epsilon) = \frac{1}{2}g^{bd}(\epsilon)[\nabla_a g_{cd}(\epsilon) + \nabla_c g_{ad}(\epsilon) - \nabla_d g_{ac}(\epsilon)] \tag{425}$$

, where $\nabla_a g_{bc} = \nabla_a g_{bc}(0) = 0$, we have the variation

$$\begin{aligned}\delta C^b_{ac} &= \frac{1}{2}\left\{g^{bd}(\epsilon)\left(\nabla_a \frac{dg_{cd}(\epsilon)}{d\epsilon} + \nabla_c \frac{dg_{ad}(\epsilon)}{d\epsilon} - \nabla_d \frac{dg_{ac}(\epsilon)}{d\epsilon}\right)\right\}_{\epsilon=0} \\ &= \frac{1}{2}g^{bd}\left(\nabla_a \delta g_{cd} + \nabla_c \delta g_{ad} - \nabla_d \delta g_{ac}\right)\end{aligned} \tag{426}$$

and

$$\begin{aligned}\delta C^b_{bc} &= \frac{1}{2}g^{bd}\left(\nabla_c \delta g_{bd} + 2\nabla_{[b}\delta g_{d]c}\right) \\ &= \frac{1}{2}g^{bd}\nabla_c \delta g_{bd}\end{aligned} \tag{427}$$

Therefore, we obtain

$$\begin{aligned}\delta R_{ac} &= \frac{1}{2}g^{bd}\left(\nabla_b\nabla_a \delta g_{cd} + \nabla_b\nabla_c \delta g_{ad} - \nabla_b\nabla_d \delta g_{ac} - \nabla_a\nabla_c \delta g_{bd}\right) \\ \Rightarrow g^{ac}\delta R_{ac} &= \frac{1}{2}\left(\nabla^d\nabla^c \delta g_{cd} + \nabla^d\nabla^a \delta g_{ad} - g^{ac}\nabla^d\nabla_d \delta g_{ac} - g^{bd}\nabla^c\nabla_c \delta g_{bd}\right) \\ &= \nabla^a\left(\nabla^b \delta g_{ab} - g^{bc}\nabla_a \delta g_{bc}\right)\end{aligned} \tag{428}$$

This is the boundary term for Einstein-Hilbert action that we often dropped in general relativity. We denote this boundary term as

$$v_a = \nabla^b \delta g_{ab} - g^{bc}\nabla_a \delta g_{bc} \tag{429}$$

Then, the variation of Einstein-Hilbert action is

$$\delta I = \int_M d^3x\left(\sqrt{g}\nabla^a v_a + \sqrt{g}G_{ab}\delta g^{ab}\right) \tag{430}$$

By the Gauss-theorem,

$$\int_M \nabla^a v_a = \oint_{\partial M} \tilde{n}^a v_a \tag{431}$$

where the vector $\tilde{n}^a$ is unit normal to the topological boundary hypersurface ∂M. Here we assumed that M is a compact manifold with a boundary. For AdS spacetime, it is not compact. To obtain meaningful physics, we usually take M as the conformally compactified AdS manifold. But since the metric blows up at this conformal boundary, the surface integral above does not make sense. Before eliciting renormalization, let us pretend that we have a metric (which should be understood as a process of limitation) on this boundary ∂M. It is natural to assume that the variation of metric δg_{ab} vanishes on this boundary, where the induced metric is

$$\tilde{h}_{ab} = g_{ab} + \tilde{n}_a\tilde{n}_b \tag{432}$$

Then the boundary term is

$$\begin{aligned}v_a\tilde{n}^a &= \tilde{n}^a g^{bc}\left(\nabla_c \delta g_{ab} - \nabla_a \delta g_{bc}\right) = \tilde{n}^a\left(\tilde{h}^{bc} + \tilde{n}^b\tilde{n}^c\right)\left(\nabla_c \delta g_{ab} - \nabla_a \delta g_{bc}\right) \\ &= \tilde{n}^a\tilde{h}^{bc}\left(\nabla_c \delta g_{ab} - \nabla_a \delta g_{bc}\right) + \tilde{n}^a\tilde{n}^b\tilde{n}^c\left(\nabla_c \delta g_{ab} - \nabla_a \delta g_{bc}\right) \\ &= \tilde{n}^a\tilde{h}^{bc}\left(\nabla_c \delta g_{ab} - \nabla_a \delta g_{bc}\right) + 2\tilde{n}^b\tilde{n}^{(a}\tilde{n}^{c)}\nabla_{[c}\delta g_{a]b} \\ &= \tilde{n}^a\tilde{h}^{bc}\left(\nabla_c \delta g_{ab} - \nabla_a \delta g_{bc}\right) \\ &= -\tilde{h}^{bc}\tilde{n}^a\nabla_a \delta g_{bc}\end{aligned} \tag{433}$$

The term $\tilde{h}^{bc}\nabla_c\delta g_{ab}$ vanishes because we assumed that $\delta g_{ab}|_{\partial M} = 0$ but the derivative $\tilde{h}^{bc}\nabla_c$ is along the tangent direction of ∂M. From (394), the extrinsic curvature on ∂M is $\tilde{K}_{ab} = \tilde{h}^c_a\nabla_c\tilde{n}_b$, $\tilde{K} = \tilde{h}^{ab}\tilde{K}_{ab} = \tilde{h}^c_b\nabla_c\tilde{n}^b$, so we have

$$\begin{aligned}\tilde{K}(\epsilon) &= \tilde{h}^{ab}(\epsilon)\tilde{K}_{ab}(\epsilon) = \tilde{h}^a{}_b(\epsilon)\tilde{\nabla}_a\tilde{n}^b(\epsilon) \\ &= \tilde{h}^a{}_b(\epsilon)\left(\nabla_a\tilde{n}^b(\epsilon) + C^b_{ac}(\epsilon)\tilde{n}^c(\epsilon)\right)\end{aligned} \tag{434}$$

But on the boundary we assumed that $\delta g_{ab} = 0$, so $\tilde{h}^{ab}(\epsilon) = \tilde{h}^{ab}$. $\tilde{n}^a$ is only defined on ∂M, so it should not depend on parameter ϵ.

$$\begin{aligned}\tilde{K}(\epsilon) &= \tilde{h}^{ab}\tilde{K}_{ab}(\epsilon) = \tilde{h}^a{}_b\tilde{\nabla}_a\tilde{n}^b \\ &= \tilde{h}^a{}_b\left(\nabla_a\tilde{n}^b + C^b_{ac}(\epsilon)\tilde{n}^c\right)\end{aligned} \tag{435}$$

Then the variation of extrinsic curvature is

$$\begin{aligned}\delta\tilde{K}(\epsilon) &= \tilde{h}^a{}_b\left(\delta C^b_{ac}\right)\tilde{n}^c \\ &= \frac{1}{2}\tilde{n}^c\tilde{h}^a{}_b g^{bd}\left(\nabla_a\delta g_{cd} - \nabla_d\delta g_{ac} + \nabla_c\delta g_{ad}\right) \\ &= \frac{1}{2}\tilde{n}^c\tilde{h}^a{}_b g^{bd}\left(2\nabla_{[a}\delta g_{d]c} + \nabla_c\delta g_{ad}\right) \\ &= \frac{1}{2}\tilde{n}^c\tilde{h}^{ad}\nabla_c\delta g_{ad}\end{aligned} \tag{436}$$

We see that the variation of the extrinsic curvature on the boundary is just the boundary term (433). In conclusion, by a reasonable boundary condition, the variation of Einstein-Hilbert action is

$$\delta I = -2\oint_{\partial M}\delta\tilde{K} + \int_M d^3x\sqrt{g}G_{ab}\delta g^{ab} \tag{437}$$

For this reason, the Einstein-Hilbert action will have a functional derivative if we modify the action in the following way

$$I_{EH} = \frac{1}{16\pi G}\int_M (R - 2\Lambda) + \frac{1}{8\pi G}\oint_{\partial M}\tilde{K} \tag{438}$$

The second term is called Hawking-Gibbons term. It appears as a surface charge which plays an important role for the theory of quasi-local energy-momentum tensor from Brown and York.

$$T^{ij} = \frac{2}{\sqrt{h}}\frac{\delta I_{EH}}{\delta h_{ij}} \tag{439}$$

For AdS_3 spacetime, this energy-momentum tensor can be regarded as the energy-momentum tensor for our CFT_2 dual living at infitiny. As we already know that the metric diverges on $\partial\,(AdS)$. A direct consequence is that the Brown-York tensor (439) also diverges. Therefore, we need to find the counter terms for renormalization. This is called the holographic renormalization. In the previous section, the Lagrangian of Einstein-Hilbert action can be expressed in terms of ${}^2R$ and extrinsic curvature K_{ab} on Σ_t, together with a unit vector n^a normal to Σ_t, if we have a foliation of spacetime M

$$\mathcal{L} = \sqrt{h}N\left\{{}^2R + K_{ab}K^{ab} - K^2 - 2\Lambda + 2\left[\nabla_a\left(n^a\nabla_c n^c\right) - \nabla_c\left(n^a\nabla_a n^c\right)\right]\right\} \tag{440}$$

the last term is a total divergence and thus is a boundary term in the action

$$\begin{aligned}I &= \int_M\left({}^2R + K_{ab}K^{ab} - K^2 - 2\Lambda\right) + 2\int_M\nabla_a\left(n^a\nabla_b n^b - n^b\nabla_b n^a\right) \\ &= \int_M\left({}^2R + K_{ab}K^{ab} - K^2 - 2\Lambda\right) + 2\oint_{\partial M}\tilde{n}_a\left(n^a\nabla_b n^b - n^b\nabla_b n^a\right)\end{aligned} \tag{441}$$

But from the equation $\tilde{K} = \tilde{h}_b^c \nabla_c \tilde{n}^b = \left(\delta_b^c + \tilde{n}^c \tilde{n}_b\right) \nabla_c \tilde{n}^b$, we see that the Hawking-Gibbons term is

$$2 \oint_{\partial M} \left(\nabla_b \tilde{n}^b + \tilde{n}_a \tilde{n}^b \nabla_b \tilde{n}^a\right) \tag{442}$$

Thus if M has no boundary, the we won't have any trouble with these boundary terms at all. For example, for asymptotic flat spacetime, the modified Lagrangian density is simply $\mathcal{L} = {}^2R + K_{ab}K^{ab} - K^2$. If the boundary is so special that it is spacelike and $\tilde{n}^a = n^a$ on ∂M, then the boundary term in (441) and the Hawking-Gibbons term cancel with each other. Such a universe has compact closed spacial slices that are sandwiched by two spacial slices which correspond to the initial space and final space. But for comformally compactified $\widetilde{AdS}_3$, with a boundary cylinder at infinity, the Hawking-Gibbons term and the boundary term do not cancel with each other.

To get out of this dilemma, we use a new foliation of $\widetilde{AdS}_3$ manifold, which is not foliated by constant time slices of the $2+1$ splitting, but constant radius slices. It is well-known that for any spacetime, its metric can always be recasted into the form (Gaussian normal coordinates)

$$ds^2 = d\rho^2 + h_{ij}(x, \rho) dx^i dx^j \tag{443}$$

by some coordinate transformation. In this coordinate, our convention is that ρ is unbounded and the conformal boundary is at $\rho = -\infty$. The center line in $\widetilde{AdS}_3$ is at $\rho = +\infty$. The extrinsic curvature of constant ρ-slice is

$$K_{ij} = \frac{1}{2} \partial_\rho h_{ij} \tag{444}$$

and indices of tensors defined on constant ρ-slices are raised and lowered by h_{ij} as we we have seen previously. In what follows, we use the symbol ${}^2R$ to denote the intrinsic curvature of constant ρ-slice, instead of the one of constant time slice. In such a foliation, it is not hard to see that at conformal infinity, $\tilde{n}^a = -n^a$. This is true both in Lorentzian signature and in Euclidean signature. Then we see that the first term in the integrand of the boundary term of Lagrangian cancels with the first term in Hawking-Gibbons term. For the second term, since we have

$$n_a n^b \left(\nabla_b n^a\right) = \frac{1}{2} n^b \nabla_b \left(n_a n^a\right) = 0 \tag{445}$$

Hence in such a foliation, we can get rid of the boundary term by adding Hawking-Gibbons term. The modified Lagrangian has a well-defined functional derivative and is given by a simple form

$$I = \frac{1}{16\pi G} \int_M d^2x d\rho \sqrt{g} \left({}^2R + K_{ab}K^{ab} - K^2 - 2\Lambda\right) \tag{446}$$

We say that this action has a well-defined functional derivative meaning that its variation with respect to the total metric g_{ab} produces no boundary terms. Although we derived such a simple expression by using Gaussian normal coordinate, it is valid in all coordinates because the expression is coordinate independent. To derive the Brown-York tensor, we consider the variation of this action with respect to h_{ij}. This functional derivative produces a boundary term. Once the equations of motion are satisfied, the bulk term vanishes. The variation gives

$$\delta I = -\frac{1}{16\pi G} \int_{\partial M} d^2x \sqrt{h} \left(K^{ij} - K h^{ij}\right) \delta h_{ij} \tag{447}$$

The term in the parenthesis is the Brown-York tensor, which corresponds to the stress-energy tensor of its dual CFT_2. To fit with the standard conventions, we define the Brown-York tensor as

$$T^{ij} = -\frac{1}{8\pi G} \left(K^{ij} - K h^{ij}\right) \tag{448}$$

As we mentioned before, this tensor has to be renormalized by adding counter terms. It's divergence comes from large ρ limit.

6 Hamiltonian Formalism

6.1 ADM Formalism

The Hamiltonian formalism of gravity is also called ADM formalism. Calculations on ADM formalism in this section are mainly copied from [2] [1]. One can also find good introductions on ADM formalism from 'General Relativity' written by Wald. ADM formalism of gravity is also called ADM formalism, whose Hamiltonian is given by

$$H = \frac{1}{16\pi G}\int_{\Sigma_t}\left(NC + N_b C^b\right) + \frac{1}{6\pi G}\int_{\Sigma_t} N\Lambda + (\cdots) \tag{449}$$

where $\cdots$ denoted some counter terms so that it has well-defined functional derivatives. For asymptotic flat spacetime, it is possible that the boundary terms vanish at spacial infinity. For example, it is true for a Minkowski spacetime. But for AdS spacetime, this is not possible in time-space splitting. Let us first see the ADM formalism without boundary terms and without the contribution from cosmological constant. The Hamiltonian is given by

$$H = \frac{1}{16\pi G}\int_{\Sigma_t}\left(NC + N_b C^b\right) \tag{450}$$

where the constraints are given by

$$C = -{}^2R + h^{-1}\left(\pi^{ab}\pi_{ab} - \frac{1}{2}\pi^2\right), \quad C^b = -2D_a\left(h^{-1/2}\pi^{ab}\right) \tag{451}$$

They are all first class constraints and have no secondary constraints. Clearly, the first term of Hamiltonian of gravity vanishes on-shell if we ignore boundary terms. As a result, we would end up with a canonical theory of gravity that has no dynamics. This is true not only in three dimensions, but in any dimensions. The expression of Hamiltonian shows that the lapse N and shift $\vec{N}$ are the test functions for the constraints

$$C_N = \int_{\Sigma_t} NC \quad and \quad C_{\vec{N}} = \int_{\Sigma_t} N^i C_i, \tag{452}$$

respectively, where we use 'i' to denote spacial index. We can compute the Poisson brackets of these constraints for any two sets of test functions N, $\vec{N}$ with N' and $\vec{N}'$. The results are

$$\begin{cases} \{C(\vec{N}), C(\vec{N}')\} = C([\vec{N}, \vec{N}']) \\ \{C(\vec{N}), C(N'\} = C(\vec{N}N') \\ \{C(N), C(N')\} = C(\left(N\partial^i N' - N'\partial^i N\right)\partial_i) \end{cases} \tag{453}$$

They are called **Dirac-Bergmann algebra**. The 'structure' inside the bracket ( ) of C is called **surface deformation algebra**, from which we recognize that the first identity corresponds to gauge symmetry of gravitational fields. The Lie bracket of vector fields $[\vec{N}, \vec{N}']$ is the commutator of the Lie algebra of diffeomorphisms on spacial slice Σ_t. Hence, the vector constraints $C_{\vec{N}}$ are the generators of gauge symmetry Diff (Σ_t). But clearly, in some arbitrary dimensions, gravitational fields have more than just gauge symmetry. This is one of the biggest difficulties that we have to tackle with in quantum gravity. For many reasons, it is more convenient to consider a more general theory. Suppose we have a general foliation of M given by a vector field ξ such that the 'lapse' and 'shift' are given by $\tilde{\xi}$ and $\tilde{\xi}^i$. That is, we have the folloing projections

$$\tilde{\xi} = N\xi^t, \quad \tilde{\xi}^i = \xi^i + N^i\xi^t \tag{454}$$

The corresponding constraints are denoted by $\mathcal{H}_0 = \mathcal{H}$ and $\mathcal{H}_i$. The total constraint is then given by

$$C[\tilde{\xi};\tilde{\xi}^i] = \int_{\Sigma_t}\left(\tilde{\xi}\mathcal{H} + \tilde{\xi}^i\mathcal{H}_i\right) = \int_{\Sigma_t}\tilde{\xi}^\mu\mathcal{H}_\mu \tag{455}$$

For any two constraints $C[\tilde{\xi};\tilde{\xi}^i]$ and $C[\tilde{\zeta};\tilde{\zeta}^i]$, their Poisson bracket of the two is given by

$$\{C[\tilde{\xi};\tilde{\xi}^i], C[\tilde{\zeta};\tilde{\zeta}^i]\} = C[\tilde{\eta};\tilde{\eta}^i] \tag{456}$$

where $\tilde{\eta} = \tilde{\xi}^i\partial_i\tilde{\zeta} - \tilde{\zeta}^i\partial_i\tilde{\xi}$ and $\tilde{\eta}^k = \tilde{\xi}^i\partial_i\tilde{\zeta}^k - \tilde{\zeta}^i\partial_i\tilde{\xi}^k + (\tilde{\xi}\partial^k\tilde{\zeta} - \tilde{\zeta}\partial^k\tilde{\xi})$.

If spacetime has a boundary and we do not add counter terms to the Hamiltonian, the functional derivatives of Hamiltonian also give us boundary terms

$$\begin{aligned}\frac{1}{16\pi G}\delta Q[N;N^i] &= \frac{1}{16\pi G}\oint_{\partial\Sigma_t} s_l G^{ijkl}\left(N\nabla_k\delta h_{ij} - (\nabla_k N)\,\delta h_{ij}\right) + \\ &+ \frac{1}{16\pi G}\oint_{\partial\Sigma_t} s_l\left[2N_k\delta\pi^{kl} + \left(2N^k\pi^{jl} - N^l\pi^{jk}\right)\delta h_{jk}\right]\end{aligned} \tag{457}$$

where $G^{abcd} = h^{c(a}h^{b)d} - h^{ab}h^{cd}$ is often called the 'supermetric', and s_l is the unit normal vector of spacial boundary $\partial\Sigma_t$. The notation used above would cause confusions because we denote the surface charge as a functional of N^μ. One should, therefore, keep in mind that the functional derivatives are with respect to canonical variables h_{ij} and π_{ij}. For three dimensional gravity, it is just a circle. Adding all the contributions from boundary terms and counter terms Q, the total Hamiltonian of gravity will be

$$H = \frac{1}{16\pi G}\int_{\Sigma_t}\left(NC + N^iC_i\right) + \frac{1}{8\pi G}\int_{\Sigma_t} N\Lambda + \frac{1}{16\pi G}Q[N;N^i] \tag{458}$$

The term Q is called the ADM charge of gravity. For black hole geometries, ADM charges are their mass M and angular momentum J, which agree with Komar integrals in ordinary general relativity theory. For three dimensional gravity, ADM charge gives us non-trivial dynamics of AdS_3.

Under the spacetime decomposition $t^a = Nn^a + N^a$, the time evolution of dynamical variable is given by the equation

$$\dot{h}_{ab} = \tilde{\mathcal{L}}_t h_{ab} = \left\{h_{ab}, H[N;\vec{N}]\right\}_{PB} \tag{459}$$

where we define

$$H[N;\vec{N}] = \int_{\Sigma_t}\left(NC + N^iC_i\right) + Q[N;N^i] \tag{460}$$

The variation δQ cancels the boundary terms of $\delta H[N;\vec{N}]$. The physics of the above expression is that the surface charge gives a non-trivial dynamics of gravity. Without such boundary terms, the energy vanishes on-shell. For example, if we take $N = 1$ for BTZ black hole metric with mass M and angular momentum J, we should find that $H[1,0] = Q[1,0] = M$. If we take $N_\phi = 1$, then it should be that $H[0,1] = Q[0,1] = J$.

Consider a more general space-time vector ξ, which generated the diffeomorphism via the Lie derivative $\delta h_{ab} = \mathcal{L}_\xi h_{ab}$. In the Hamilton's canonical approach, the associated canonical generator which generate the same evolution $\delta h_{ab} = \{h_{ab}, H[\xi]\}$ is given by

$$H[\xi] = \int_{\Sigma_t}\tilde{\xi}^\mu\mathcal{H}_\mu + Q[\xi] \tag{461}$$

where the term $Q[\xi]$ is a surface term whose variation precisely cancels the boundary terms produced by the integral of bulk constraints. By a similar calculation, we find that

$$\delta Q[\xi] = \oint_{\partial\Sigma_t} s_l G^{ijkl}\left(\tilde{\xi}\nabla_k\delta h_{ij} - \left(\nabla_k\tilde{\xi}\right)\delta h_{ij}\right) + \oint_{\Sigma_t} s_l\left[2\tilde{\xi}_k\delta\pi^{kl} + \left(2\tilde{\xi}^k\pi^{ij} - \tilde{\xi}^l\pi^{jk}\right)\delta h_{jk}\right] \tag{462}$$

6.2 Asymptotic Symmetry Group

An important property of BTZ blackhole is that it is an asymptotic pure AdS_3. Brown and Henneaux found that for any asymptotic AdS_3 gravity, its classical asymptotic symmetry is the conformal group in two dimensions. Furthermore, the Dirac algebra associated with the asymptotic conformal killing vectors is an central extension of the conformal algebra, with central charge $3l/2G$. The asymptotic boundary condition given by Brown and Henneaux is

$$\begin{cases} g_{tt} = -\dfrac{r^2}{l^2} + o(1) \\ g_{tr} = o\left(\dfrac{1}{r^3}\right) \\ g_{t\phi} = o(1) \\ g_{rr} = \dfrac{l^2}{r^2} + o\left(\dfrac{1}{r^4}\right) \\ g_{r\phi} = o\left(\dfrac{1}{r^3}\right) \\ g_{\phi\phi} = r^2 + o(1) \end{cases} \tag{463}$$

and so

$$N = \frac{r}{l} + o\left(\frac{1}{r}\right), \quad N^r = o\left(\frac{1}{r}\right), \quad N^\phi = o\left(\frac{1}{r^2}\right) \tag{464}$$

The boundary conditions for asymptotic AdS_3 can be seen from the asymptotic behaviour of AdS_3 or BTZ black hole metrics. It is definitely not unique, but their corresponding asymptotic symmetries must be the same up to adding some subleading terms. The asymptotic symmetry is given by a vector fields ξ satisfying

$$\begin{cases} \mathcal{L}_\xi g_{tt} = o(1), \quad \mathcal{L}_\xi g_{tr} = o\left(\dfrac{1}{r^3}\right), \quad \mathcal{L}_\xi g_{t\phi} = o(1) \\ \mathcal{L}_\xi g_{rr} = o\left(\dfrac{1}{r^4}\right), \quad \mathcal{L}_\xi g_{r\phi} = o\left(\dfrac{1}{r^3}\right), \quad \mathcal{L}_\xi g_{\phi\phi} = o(1) \end{cases} \tag{465}$$

In general, subleading terms may also depend on t and ϕ. We can always factor out an arbitrary function $f_{\mu\nu}(t,\phi)$ from the above subleading terms. i.e.

$$\begin{cases} \delta_\xi g_{tt} = f_{tt}(t,\phi), \quad \delta_\xi g_{tr} = \dfrac{f_{tr}(t,\phi)}{r^3}, \quad \delta_\xi g_{t\phi} = f_{t\phi}(t,\phi) \\ \delta_\xi g_{rr} = \dfrac{f_{rr}(t,\phi)}{r^4}, \quad \delta_\xi g_{r\phi} = \dfrac{f_{r\phi}(t,\phi)}{r^3}, \quad \delta_\xi g_{\phi\phi} = f_{\phi\phi}(t,\phi) \end{cases} \tag{466}$$

These equations tell us how the asymptotic AdS_3 metrics change under asymptotic symmetry vectors. The above equations should be understood as asymptotic conformal killing equations. They are conformal because of the ambiguity of rescaling a factor on the right hand side. The solutions to these equations are given by

$$\begin{cases} \xi^t = l\left(T + \bar{T}\right) + \dfrac{l^3}{2r^2}\left(\partial_z^2 T + \partial_{\bar{z}}^2 \bar{T}\right) + o\left(\dfrac{1}{r^4}\right) \\ \xi^r = -r\left(\partial_z T + \partial_{\bar{z}} \bar{T}\right) + o\left(\dfrac{1}{r}\right) \\ \xi^\phi = T - \bar{T} - \dfrac{l^2}{2r^2}\left(\partial_z^2 T - \partial_{\bar{z}}^2 \bar{T}\right) + o\left(\dfrac{1}{r^4}\right) \end{cases} \tag{467}$$

where we use the notation

$$\begin{aligned} \partial_z &= \frac{1}{2}\left(l\partial_t + \partial_\phi\right), \quad \partial_{\bar{z}} = \frac{1}{2}\left(l\partial_t - \partial_\phi\right) \\ T &= T\left(\frac{t}{l} + \phi\right), \quad \bar{T} = \bar{T}\left(\frac{t}{l} - \phi\right) \end{aligned} \tag{468}$$

These equations show that there is clearly a periodicity in function T and $\bar{T}$ so we can do the Fourier expansions $T = \frac{1}{2}\sum_n T_n e^{inz}$ and $\bar{T} = \frac{1}{2}\sum_n \bar{T}_n e^{in\bar{z}}$. The vector field $\xi = \xi^\mu \partial_\mu$ can be decomposed as a sum of two terms

$$\begin{aligned} \xi &= \xi^\mu \partial_\mu = \xi^t \partial_t + \xi^\phi \partial_\phi + \xi^r \partial_r \\ &= \left(2T + \frac{l^2}{r^2}\partial_z^2 T\right)\partial_z - r\left(\partial_z T\right)\partial_r + \left(2\bar{T} + \frac{l^2}{r^2}\partial_{\bar{z}}^2 \bar{T}\right)\partial_{\bar{z}} - r\left(\partial_{\bar{z}}\bar{T}\right)\partial_r + (subleading\ \ terms) \end{aligned} \tag{469}$$

where the first term depends only on T while the second term depends only on $\bar{T}$. We define the Fourier expansion of vector field ξ as

$$\begin{aligned} \xi = \xi^\mu \partial_\mu &= \sum_n e^{inz}\xi_n + \sum_m e^{im\bar{z}}\bar{\xi}_m = \sum_n T_n l_n + \sum_m \bar{T}_m \bar{l}_m + \cdots \\ &= \sum_n T_n e^{inz}\left(\partial_z - \frac{l^2 n^2}{2r^2}\partial_z - \frac{inr}{2}\partial_r\right) + \sum_m \bar{T}_m e^{im\bar{z}}\left(\partial_{\bar{z}} - \frac{l^2 m^2}{2r^2}\partial_{\bar{z}} - \frac{imr}{2}\partial_r\right) + \cdots \end{aligned} \tag{470}$$

where the $\cdots$ is the subleading terms and we denote

$$l_n = e^{inz}\left(\partial_z - \frac{l^2 n^2}{2r^2}\partial_z - \frac{inr}{2}\partial_r\right), \quad \bar{l}_m = e^{im\bar{z}}\left(\partial_{\bar{z}} - \frac{l^2 m^2}{2r^2}\partial_{\bar{z}} - \frac{imr}{2}\partial_r\right) \tag{471}$$

It is easy to see that the above two differential operators form two copies of de Witt algebra

$$[l_m, l_n] = i(m-n)l_{m+n}, \quad [\bar{l}_m, \bar{l}_n] = i(m-n)\bar{l}_{m+n}, \quad [l_m, \bar{l}_n] = 0 \tag{472}$$

In other words, the asymptotic symmetry of asymptotic AdS_3 is given by conformal group in two diemensions, which is generated by holomorphic and anti-holomorphic de Witt algebras, $\mathfrak{diff}(\mathbb{S}^1)$.

For the Dirac algebra regarding the asymptotic conformal killing vectors that vanish very rapidly along with the growth of radius r (i.e. the subleading terms in (467)), say ξ and η, the surface charge vanishes asymptotically because integrating equation (462) leads to $Q[\xi] = 0$ asymptotically. i.e. They correspond to trivial gauges. The strategy is that first we want to shown that for pure gauges, the Poisson bracket is [20]

$$\left\{\int_{\Sigma_t}\tilde{\xi}^\mu \mathcal{H}_\mu, \int_{\Sigma_t}\tilde{\eta}^\nu \mathcal{H}_\nu\right\} = \int d^2x\left[[\xi,\eta]^\mu_{SD} + \delta_\eta \xi^\mu - \delta_\xi \eta^\mu - \int d^2y\left\{\xi^\mu(x), \eta^\nu(y)\right\}\mathcal{H}_\nu(y)\right]\mathcal{H}_\mu(x) \tag{473}$$

which equals to $\int_{\Sigma_t}[\tilde{\xi},\tilde{\eta}]^\mu_{SD}\mathcal{H}_\mu$ asymptotically for pure gauges, where the commutator of surface deformation algebra $[\tilde{\xi},\tilde{\eta}]_{SD}$ is given by (453) and the projection (454). More specifically, we have

$$\begin{aligned} [\tilde{\xi},\tilde{\eta}]^t_{SD} &= \left(\tilde{\xi}^i + N^i \tilde{\xi}^t\right)\partial_i \tilde{\xi}^t + \frac{\partial_i N}{N}\tilde{\xi}^i \tilde{\eta}^t - (\tilde{\xi} \leftrightarrow \tilde{\eta}) \\ [\tilde{\xi},\tilde{\eta}]^i_{SD} &= N^2 \tilde{\xi}^t \partial^i \eta^t + \left(\partial_j N^j - \frac{N^i \partial_j N}{N}\right)\tilde{\xi}^j \tilde{\eta}^t + \left(\tilde{\xi}^j + N^j \tilde{\xi}^t\right)\partial_j \tilde{\eta}^j - (\tilde{\xi} \leftrightarrow \tilde{\eta}) \end{aligned} \tag{474}$$

The result shows that $\int_{\Sigma_t}\tilde{\xi}^\mu \mathcal{H}_\mu$ is a representation of the surface deformation algebra for asymptotic symmetry generated by trivial gauges. We can use the asymptotic condition (464) for lapse N, shift N^i and the full asymptotic conformal killing vectors in (467) to further calculate the above commutations. i.e. the commutations for generic ξ and η satisfying (467). It turns out that we again end up with the test vector field

$$\zeta^\mu(x) = [\xi,\eta]^\mu_{SD}(x) + \delta_\eta \xi^\mu(x) - \delta_\xi \eta^\mu(x) - \int d^2y\left\{\xi^\mu(x), \eta^\nu(y)\right\}\mathcal{H}_\nu(y)$$

as already shown in (473) for pure gauges. The last three terms in (473) for general ξ and η still only contribute to higher orders of $1/r$ that vanishes rapidly. i.e. they are not the leading terms. The reason is simply that their Poisson commutators as well as their derivatives decrease much faster than the first term of (473) for generic vectors satisfying (467). The final step is to show that the surface deformation algebra for general gauges ξ and η (the full conformal algebra of (467)) equals to the Lie bracket asymptotically to leading order of $1/r$. The calculation is indeed surprisingly simplified by the asymptotic conditions (467). The surface deformation algebra $[\xi,\eta]_{SD}$ for generic gauges turns out to coincide with an exact conformal algebra $[\xi,\eta]$, which is the Lie bracket of two vectors, to the leading order of $1/r$. That is to say, for generic asymptotic killing vectors ξ and η, we have

$$\left\{\int_{\Sigma_t} \tilde{\xi}^\mu \mathcal{H}_\mu, \int_{\Sigma_t} \tilde{\eta}^\nu \mathcal{H}_\nu\right\} = \int_{\Sigma_t} [\tilde{\xi},\tilde{\eta}]^\mu \mathcal{H}_\mu \tag{475}$$

or, in terms of Fourier modes,

$$\left\{C[l^\pm_m], C[l^\pm_n]]\right\}_{PB} = i(m-n)C[l^\pm_{m+n}] \tag{476}$$

where we use $\pm$ to denote the holomorphic and anti-holomorphic sectors.

Finally, we assume that if we consider non-vanishing surface term (461), the Dirac algebra for the total asymptotic symmetry is generalized into the form

$$\{H[\xi], H[\eta]\} = H[\zeta] + K[\xi,\eta] \tag{477}$$

with $H[\xi] = C[\xi] + Q[\xi]$. Obviously, we do not need to consider the commutator of surface charge with the bulk constraint. The equation (475) implies that the commutator remains to be

$$\left\{H[l^\pm_m], H[l^\pm_n]]\right\}_{PB} = i(m-n)H[l^\pm_{m+n}] + K[l^\pm_m, l^\pm_n] \tag{478}$$

This suggests that $H[\xi]$ is regarded as a projective representation for the asymptotic symmetry. Quite obviously, the equation (475) also implies that

$$\{Q[\xi], Q[\eta]\}_{PB} = Q[[\xi,\eta]] + K[\xi,\eta], \quad \text{or} \quad \delta_\eta Q[\xi] = Q[[\xi,\eta]] + K[\xi,\eta]. \tag{479}$$

This means that the surface charges are not only a projective representation of the asymptotic symmetry of the AdS_3, but the generators of the residual gauge symmetry, which gives non-trivial dynamics at the asymptotic boundary, after the gauge fixing conditions are imposed. Furthermore, for any surface charge, say $Q[\xi]$, it is clear that $\delta_\eta Q[\xi] = [Q[\xi], Q[\eta]] = 0$ for any trivial gauge η. i.e. the Lie algebra (denoted by $\mathfrak{h}$) of pure gauge symmetry $\mathcal{H}$ looks like an ideal of the Lie algebra (denoted by $\mathfrak{g}$) of the residual of allowed symmetry $\mathcal{G}$; The asymptotic symmetry group $\mathcal{ASG}$ of AdS_3, which is deemed as the symmetry group of the dual CFT_2 is, therefore, a quotient group $\mathcal{G}/\mathcal{H}$. Using this result we conclude that when imposing constraints $C[\xi] = 0$, we have the following identity hold on-shell

$$\left\{Q[l^\pm_m], Q[l^\pm_n]]\right\}_{PB} = i(m-n)Q[l^\pm_{m+n}] + K[l^\pm_m, l^\pm_n] \tag{480}$$

Since $Q[\xi]$ is defined up to a constant, we can always shift its value to $Q = 0$ at $t = 0$ in the AdS_3 coordinate. Then on the $t = 0$ slice, we have

$$K[l^\pm_m, l^\pm_n] = \left\{Q[l^\pm_m], Q[l^\pm_n]\right\}_{PB} = \delta_{l^\pm_n} Q[l^\pm_m] = \left\{Q[l^\pm_m], Q[l^\pm_n]\right\}_D \tag{481}$$

where we used the property of Dirac bracket that it concides with Poisson bracket for first class constraints. Using the formula (462), we can compute this central charge by doing the integral. The result is

$$K[l^\pm_m, l^\pm_n] = 2\pi i l n(n^2-1)\delta_{m+n,0} \tag{482}$$

To find out the true value of the central charge for the asymptotic symmetry of AdS_3 gravity generated by the total Hamiltonian we still need to restore the factor $1/16\pi G$ that we had dropped. We simply re-scale the generator $H[\xi]$ and so

$$\frac{1}{16\pi G} K[l^\pm_m, l^\pm_n] = \frac{l}{8G} i n(n^2-1)\delta_{m+n,0} \tag{483}$$

The quantization is by passing from Dirac bracket $\{\ ,\ \}_D$ to the quantum commutator $-i[\ ,\ ]$, we would get the quantum Virasoro algebra

$$[L_m^{\pm}, L_n^{\pm}] = (m-n)L_{m+n}^{\pm} + \frac{c}{12}m(m^2-1)\delta_{m+n,0}, \quad [L_m^{\pm}, L_n^{\mp}] = 0 \tag{484}$$

where we denote $L_m^{\pm} = Q[l_m^{\pm}]$. From the above commutators, we find the famous central charge

$$c = \frac{3l}{2G} \tag{485}$$

In three dimensions, there are no local degrees of freedom, thereby having no propagating gravitons. However, in quantum AdS_3, we have to sum over 'small fluctuations' around classical AdS_3, under the given boundary condition, say Dirichlet boundary condition. From Brown and Henneaux's computation, these small fluctuations should be understood via a two dimensional conformal field theory. The exactly global symmetry of AdS_3 is $SO(2,2)$, which locally splits into $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$. This symmetry is exactly the symmetry generated by $\{L_{-1}, L_0, L_1\}$ and $\left\{\bar{L}_{-1}, \bar{L}_0, \bar{L}_1\right\}$ that leave the CFT_2 vacuum invariant. To see this, we first compute the six independent killing vector fields for pure AdS_3. We start from the six global killing vectors given in $\{U, V, Z, Y\}$ coordinate, plugging the solutions of embedding equations for pure AdS_3, we find that for pure AdS_3, there are the following global killing vectors

$$\begin{cases} \xi_1 = \partial_t \\ \xi_2 = \partial_\phi \\ \xi_3 = \dfrac{-r\sin t\cos\phi}{\sqrt{1+r^2}}\partial_t + \sqrt{1+r^2}\cos t\cos\phi\partial_r + \dfrac{\sqrt{1+r^2}}{r}\sin t\sin\phi\partial_\phi \\ \xi_4 = \dfrac{-r\sin t\sin\phi}{\sqrt{1+r^2}}\partial_t + \sqrt{1+r^2}\cos t\sin\phi\partial_r + \dfrac{\sqrt{1+r^2}}{r}\sin t\cos\phi\partial_\phi \\ \xi_5 = \cdots \\ \xi_6 = \cdots \end{cases} \tag{486}$$

whose linear combinations give $\{l_0, l_{\pm 1}\}$ and $\left\{\bar{l}_0, \bar{l}_{\pm 1}\right\}$ up to some subleading terms. i.e. vectors ξ at large r limit are equivalent to the $\{l_0, l_{\pm 1}\}$ and $\left\{\bar{l}_0, \bar{l}_{\pm 1}\right\}$. This asymptotic equivalence is, in fact, a Lie algebra isomorphism. For BTZ black hole, we can also do a similar computation by using the embedding equations. However, the difference is that there are two embeddings, one of which is the interior geometry while the other is the exterior geometry. For this reason, we may still obtain six killing vectors in BTZ coordinates but only two of them are global killing vectors. These two vectors are ∂_t and ∂_ϕ. The reason is that a BTZ black hole as a quotient of the pure AdS_3 has a different topology, which inherits only parts of the global symmetry from the pure AdS_3. For example, a torus as a quotient space of a complex plane has fundamental group $\pi_1\left(\mathbb{T}^2\right) = \mathbb{Z} \oplus \mathbb{Z}$. The two global translational symmetries ∂_x and ∂_y on complex plane becomes two independent rotational symmetries on that torus, while the global rotational symmetry $x\partial_y - y\partial_x$ is locally preserved but globally broken on the torus since the two local identifications on complex plane breaks the periodicity of the original rotation killing vector field on complex plane. In this sense, doing local identifications will eliminate a certain number of global symmetries.

Since classically a pure AdS_3 has the largest number of global symmetries and has the lowest energy (because it has energy $M = -1$), the corresponding quantum state of the pure AdS_3 is reckoned to be the ground state of the quantum AdS_3 gravity. To find such a CFT_2 dual to the quantum gravity, we need to specify the representation of the Virasoro algebra given by Brown and Henneaux. In highest weight representations, the Virasoro operators L_n in holomorphic sector and $\bar{L}_n$ in anti-holomorphic sector for $n > 0$ annihilate heighest states $|h, \bar{h}\rangle$. In particular, the ground state $|\Omega\rangle$, which has the lowest energy and should be killed by the largest number of Virasoso operators, is annihilated by L_n for $n \geq -1$. We have mentioned that the three generators $\{L_{-1}, L_0, L_{+1}\}$ form a Lie subalgebra $\mathfrak{sl}(2,\mathbb{R})$. In other words, the vacuum of CFT_2 has a stabilizer $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$, which agrees with the classical limit. We should notice that the asymptotic symmetry computed by Brown and Henneaux is purely classical. Although we see that the asymptotic symmetry of the AdS_3 has a central term, this virasoro algebra has not yet been quantized. But we already see that vectors ξ act on the space of all asymptotic AdS_3 metrics transitively. Consequently, the classical asymptotic symmetry group $\widehat{\text{Diff}}(\mathbb{S}^1) \times \widehat{\text{Diff}}(\mathbb{S}^1)$ generated by classical surface charges $L_n^{\pm} = Q[l_n^{\pm}]$ acts on the classical phase space of asymptotic AdS_3 gravity transitively and has a Lie subalgebra $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$ that leaves the metric invariant, we therefore claim that there is a fixed point in the classical phase space of the asymptotic AdS_3 whose stablizer is $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$. Hence, we identify the classical phase space as a homogeneous space $\widehat{\text{Diff}}(\mathbb{S}^1) \times \widehat{\text{Diff}}(\mathbb{S}^1)/SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$. i.e.

$$\mathcal{P} \simeq \left(\widehat{\text{Diff}}(\mathbb{S}^1)/SL(2,\mathbb{R})\right) \times \left(\widehat{\text{Diff}}(\mathbb{S}^1)/SL(2,\mathbb{R})\right) \tag{487}$$

Such a phase space is exactly what we expected since this phase space splits into a product and thereby has a structure of a trivial cotangent bundle, whose typical fiber is the same as its base space. In this phase space, momentum space and position space are canonically the same. This can be seen from the Chern-Simons formulation of three dimensional gravity. In ordinary quantum mechanics, momentum is roughly the time derivative of position, while in Chern-Simons theory, the canonical momentum of gauge potential A is itself.

To specify the unitary representation of the CFT_2, the first question we must address is whether this representation is irreducible or reducible. Since a BTZ black hole as a quotient of the pure AdS_3 has higher energy, one may guess that a BTZ black hole is an thermal ensemble of descendant states of vacuum $|\Omega\rangle$. If that is true, the theory must be a single irreducible representation. In fact, we will show that this is impossible because the Verma module of vacuum cannot explain the entropy of a BTZ black hole. Another possibility is that the CFT_2 is in reducible representation, which contains more than one highest weight states. In such as theory, the BTZ black hole can be explained by all highest weight states $|h,\bar{h}\rangle$ including the vacuum and their descendant states.

7 Gravitational Action

In the general relativity, metric $g_{\mu\nu}$ on base manifold M plays a fundamental role and the levi-civita connection $\Gamma^{\gamma}_{\mu\nu}$ is determined by the metric. In sloppy language, the connection on tangent bundle $T(M)$ descending on M defines an affine connection on M. While in gauge field theory, connection A, which is defined on pincipal bundle $P(M)$ over M, plays a fundamental role. Palatini action is simply a formalism of Einstein-Hilbert action in terms of dreibein(or frame) e and connection A defined on the frame bundle $F(M)$, which is a principal bundle, where both dreibein and connection play fundamental roles. For this reason, gravity can be regarded as a gauge theory on frame bundle. In this theory, each fiber over a point $\{x^{\mu}\}$ is a collection of dreibeins $e^{\mu}_a(x)$ such that

$$g^{\mu\nu}(x) = e^{\mu}_a(x) e^{\nu}_b(x) \eta^{ab} \tag{488}$$

and we use the 'internal metric' η_{ab} to raise and lower Latin indices. And we define co-frames via the following equations

$$e^a_{\mu} e^{\mu}_b = \delta^a_b \quad e^a_{\mu} e^{\nu}_a = \delta^{\nu}_{\mu} \tag{489}$$

The definitation of a dreibein has ambiguity since it is defined up to a local Lorentz transformation. Therefore, the Latin index a naturally carries a representation of Lorentz group $SO(2,1)$. That is to say, the structure group of $F(M)$ is $SO(2,1)$. In what follows, we will use the equivalence between Einstein-Hilbert action and Palatini action. The proof of this equivalence and the details of calculations are given in appendix.

7.1 Chern-Simons Actions for 3D Gravity

All the calculations in this section are based on [16] [17]. Consider the case when cosmologival constant is 0, whose action is

$$I = \epsilon_{abc} \int_M e^a \wedge F^{bc}, \tag{490}$$

Its equations of motion are given by variation of the above action.

$$\delta I = \epsilon_{abc} \int_M \delta e^a \wedge \left(d\omega^{bc} + \omega^b{}_d \wedge \omega^{dc}\right) + \int_M \left\{\epsilon_{abc} de^a \delta^b_e + 2\epsilon^{ab}{}_c e_a \wedge \omega_{bd}\delta^d_e\right\} \wedge \delta\omega^{ec} \tag{491}$$

Details of this variation can be found in appendix. Since variation $\delta\omega$ and δe are arbitrary, $\delta I = 0$ implies that $F^{bc} = 0$ and $\epsilon_{cae} de^a + 2\epsilon_{cab} e^a \wedge \omega^b{}_e = 0$. Then, using the identity

$$\epsilon^{a_1 \cdots a_p a_{p+1} \cdots a_n} \epsilon_{a_1 \cdots a_p b_{p+1} \cdots b_n} = (-1)^s (n-p)! p! \delta^{[a_{p+1}}_{b_{p+1}} \cdots \delta^{a_n]}_{b_n}, \tag{492}$$

where s is the signature of metric, which in our case is 1, we have the equations of motion:

$$d\omega + \omega \wedge \omega = 0 \quad \text{or} \quad d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b = 0 \tag{493}$$

$$de + \omega \wedge e = 0 \quad \text{or} \quad de^a + \omega^a{}_c \wedge e^c = 0 \tag{494}$$

The first one shows that the ω is a flat $\mathfrak{so}_{\mathbb{R}}(2,1)$-connection. The second one is saying that our theory is torsion-free. If we put dreibein e and spin-connection ω together into a matrix

$$\begin{pmatrix} \omega & e \\ 0 & 0 \end{pmatrix}, \tag{495}$$

where the ω fills out the first 3×3 block and e occupies the last column, then the two solutions together imply that pair (e, ω) is a flat $\mathfrak{iso}_{\mathbb{R}}(2,1)$-connection, where $\mathfrak{iso}_{\mathbb{R}}(2,1)$ is the Poincare algebra in three dimension and e generates the space-time translations.

$$d\begin{pmatrix} \omega & e \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \omega & e \\ 0 & 0 \end{pmatrix} \wedge \begin{pmatrix} \omega & e \\ 0 & 0 \end{pmatrix} = 0 \tag{496}$$

This is perhaps the very first observation that led to Witten's shocking discovery in 1989, which will be introduced. From the above discussion, we can even promote the Palatini action into a more elegant form. First, we introduce a Lie-algebra valued quantity. We define $\mathbf{e} = e^a P_a$, where P_a is the generator of spacetime translation, whose fundamental representation is given by the following matrices

$$P_0 = \begin{pmatrix} 0&0&0&1 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix} \quad P_1 = \begin{pmatrix} 0&0&0&0 \\ 0&0&0&1 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix} \quad P_2 = \begin{pmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&1 \\ 0&0&0&0 \end{pmatrix}$$

Similarty, the the spin connection $\boldsymbol{\omega} = \frac{1}{2}\omega^a{}_b J_a{}^b = \frac{1}{2}\omega^{ab} J_{ab} = \frac{1}{2}\epsilon_a{}^{bc}\omega^a{}_b J_c$. The fundamental representation of Lorentz generators J_a are given by

$$J_0 = \begin{pmatrix} 0&0&0&0 \\ 0&0&-1&0 \\ 0&1&0&0 \\ 0&0&0&0 \end{pmatrix} \quad J_1 = \begin{pmatrix} 0&0&-1&0 \\ 0&0&0&0 \\ -1&0&0&0 \\ 0&0&0&0 \end{pmatrix} \quad J_2 = \begin{pmatrix} 0&1&0&0 \\ 1&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix}$$

A small computation shows the commutation relations

$$[J_a, J_b] = \epsilon^c{}_{ab} J_c \quad [J_a, P_b] = \epsilon^c{}_{ab} P_c \quad [P_a, P_b] = 0 \tag{497}$$

One might think think that the Casimir element is $J_a J^a + P_a P^a$. But this cannot be correct. Casimir is only defined for semi-simple Lie algebra. In our case, we hope to find a generalized Casimir element from the universal enveloping algebra $\mathcal{U}(\mathfrak{iso}_{\mathbb{R}}(2,1))$ that commutes with every generator of $\mathfrak{iso}_{\mathbb{R}}(2,1)$. The element $J_a J^a + P_a P^a$ fails this basic requirement of being a Casimir element. However, from the commutation relations of $\mathfrak{iso}_{\mathbb{R}}(2,1)$, we have

$$
\begin{aligned}
&[P^a J_a, P_b] = 0 \\
&[P^a J_a, J_b] = 0
\end{aligned} \tag{498}
$$

This strongly suggest that the element $P^a J_a$ can be a candidate of our Casimir element of $\mathfrak{iso}_{\mathbb{R}}(2,1)$. A special feature of the algebra $\mathfrak{iso}_{\mathbb{R}}(2,1)$ is that in three dimensions, the subalgebra of translations has three dimensions, the same as the one of Lorentz rotations. Together with the Casimir element, it implies that there is a 'inner-product' $\langle J_a, P_b\rangle = \delta_{ab}$. The pair (e,ω) can thus be written as $\omega^a J_a + e^a P_a = \boldsymbol{\omega} + \mathbf{e}$. If we denote the hodge star dual of F by $\epsilon^a{}_{bc}F^{bc} = F^a$, and $\mathbf{F} = F^a J_a$, then the Palatini action is given by

$$
I = \epsilon_{abc}\int_M e^a \wedge F^{bc} = \int_M \langle \mathbf{e} \wedge \mathbf{F}\rangle \tag{499}
$$

As mentioned earlier, we will see that this action is actually a Chern-Simons action

$$
\int_M \langle A \wedge dA + \frac{2}{3} A \wedge A \wedge A\rangle \tag{500}
$$

where the connection is exacly $A = \omega^a J_a + e^a P_a$ in Lorentzian signature, or $\mathfrak{iso}(3,\mathbb{R})$-valued in Euclidean signature. The symbol $\langle , \rangle$ here can be viewed as an inner-product, or 'killing form' of the Poincare algebra, which is assumed to be bilinear and symmetric.

$$
\langle J_a, P_a\rangle = \delta_{ab}, \quad \langle J_a, J_b\rangle = \langle P_a, P_b\rangle = 0 \tag{501}
$$

However, this 'killing form' seems quite awkward since the Poincare algebra has a nontrivial radical and thus has no non-degenerate killing form. Therefore, we should ask whether this inner product is indeed non-degenerate and invariant so that it can give us the correct kinetic energy of gauge fields. Furthermore, one can easily check that this set of inner-product cannot be the killing form in the common sense because $\mathrm{Tr}(J_a J_b) = 2\eta_{ab}$ and $\mathrm{Tr}(J_a P_b) = \mathrm{Tr}(P_a P_b) = 0$. Luckily, using the commutation relations, we can see that the inner-product is indeed invariant.

$$
\begin{aligned}
&\langle [J_a, J_b], P_c\rangle = \langle J_a, [J_b, P_c]\rangle, \quad \langle [J_a, P_b], P_c\rangle = \langle J_a, [P_b, P_c]\rangle \\
&\langle [J_a, P_b], J_c\rangle = \langle J_a, [P_b, J_c]\rangle, \quad \langle [P_a, J_b], P_c\rangle = \langle P_a, [J_b, P_c]\rangle
\end{aligned}
$$

As a result, the new form of Palatini action is indeed valid. In fact, there is an exact expression for the above 'inner-product', which is given by $\langle A, B\rangle = \mathrm{Tr}(A \star B) = (A)^j{}_i (\star B)^i{}_j$, where $\star$ is the hodge star operator acting on B entry-wise. Specifically, we have $(\star B)^i{}_j = \frac{1}{2}\epsilon^i{}_{jk}{}^l B^k{}_l$. This 'killing form' is indeed symmetric and satisfies all the equations given above. By using the generalized 'inner-product', we can generalize the Chern-Simons action.

$$
\begin{aligned}
I &= \frac{k}{4\pi}\int_M \langle A \wedge dA + \frac{2}{3} A \wedge A \wedge A\rangle \\
&= \frac{k}{4\pi}\int_M \left\{ A^a \wedge dA^b \langle T_a T_b\rangle + \frac{1}{3}\epsilon_{ab}{}^d A^a \wedge A^b \wedge A^c \langle T_d T_c\rangle \right\} \\
&= \frac{k}{4\pi}\int_M \left\{ A^a \wedge dA_a + \frac{1}{3}\epsilon_{abc} A^a \wedge A^b \wedge A^c \right\}
\end{aligned} \tag{502}
$$

Taking $k = \dfrac{1}{4G}$, where G is the gravitational constant, and $A = \omega^a J_a + e^a P_a$, with $\langle J_a, P_b\rangle = \delta_{ab}$, $\langle J_a, J_b\rangle = \langle P_a, P_b\rangle = 0$, then

$$
I = \frac{k}{4\pi}\int_M \langle A \wedge dA + \frac{2}{3} A \wedge A \wedge A\rangle = \frac{1}{16\pi G}\int_M e^a \wedge F_a \tag{503}
$$

For the theory with non-vanishing cosmological constant, we use the identity $\det(g) = -\det^2(e^{-1})$, we have

$$\int_M \sqrt{g} d^3x = \frac{1}{3!}\epsilon_{abc} \int e^a \wedge e^b \wedge e^c \tag{504}$$

Plugging into the Einstein-Hilbert action with cosmological constant, we have the following Palatini-action

$$\int_M d^3x \sqrt{g}\,(R - 2\Lambda) = \epsilon_{abc} \left(\int_M e^a \wedge F^{bc} - \frac{\Lambda}{3} \int_M e^a \wedge e^b \wedge e^c \right) \tag{505}$$

The equations of motion for this action are

$$d\omega + \omega \wedge \omega = \Lambda e \wedge e \tag{506}$$

$$de + \omega \wedge e = 0 \tag{507}$$

This action can also be expressed in a more compact form by employing some Lie algebras. To begin with, we generalize the commutation relations (497) to

$$[J_a, J_b] = \epsilon^c{}_{ab} J_c \quad [J_a, P_b] = \epsilon^c{}_{ab} P_c \quad [P_a, P_b] = \epsilon^c{}_{ab} J_c \tag{508}$$

This is in fact the $\mathfrak{so}_{\mathbb{R}}(2,2)$ algebra with the following generators

$$P_0 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad P_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad P_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

and

$$J_0 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad J_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad J_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

along with $\mathrm{Tr}(J_a J_b) = -2\eta_{ab}$, $\mathrm{Tr}(J_a P_b) = 0$ and $\mathrm{Tr}(P_a P_b) = -2\eta_{ab}$. The commutation relations also show that

$$\begin{aligned} [P^a J_a, P_b] &= 0 \\ [P^a J_a, J_b] &= 0 \end{aligned} \tag{509}$$

This suggests that for $\mathfrak{so}_{\mathbb{R}}(2,2)$, we have a set of 'inner-products'

$$\langle J_a, P_b \rangle = 2\eta_{ab}, \quad \langle J_a, J_b \rangle = \langle P_a, P_b \rangle = 0 \tag{510}$$

Just like the case for Poincare algebra, the explicit expression for this 'inner-product' is given by $\langle A, B \rangle = (A)^i{}_k (\star B)^k{}_i$. In Euclidean signature, $\mathfrak{so}_{\mathbb{R}}(2,2)$ is replaced by $\mathfrak{so}(3,1)_{\mathbb{C}} \simeq \mathfrak{su}(2)_{\mathbb{C}} \oplus \mathfrak{su}(2)_{\mathbb{C}}$. The generators are given by

$$P_0 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad P_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad P_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

and

$$J_0 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad J_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad J_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The 'inner-products' in Enclidean signature is still the same. But we should be careful that unlike in Lorentzian signature the indices of entries in the matrices are lowered or raised by the metric diag$(-1,-1,+1,+1)$, in Euclidean signature, they are lowered or raised by metric diag$(-1,-1,-1,+1)$. Using the above Lie algebra-valued 1-forms, the Palatini action (505) is

$$I = \frac{1}{2}\int_M \langle \boldsymbol{e}\wedge \boldsymbol{F}\rangle - \frac{\Lambda}{3}\int_M \langle \boldsymbol{e}\wedge\boldsymbol{e}\wedge\boldsymbol{e}\rangle \tag{511}$$

By setting

$$A = \begin{pmatrix} 0 & \omega^0{}_1 & \omega^0{}_2 & e^0 \\ -\omega^0{}_1 & 0 & \omega^1{}_2 & e^1 \\ \omega^0{}_2 & \omega^1{}_2 & 0 & e^2 \\ e^0 & e^1 & -e^2 & 0 \end{pmatrix}$$

or setting

$$A = \begin{pmatrix} 0 & \omega^0{}_1 & \omega^0{}_2 & e^0 \\ \omega^0{}_1 & 0 & \omega^1{}_2 & e^1 \\ \omega^0{}_2 & -\omega^1{}_2 & 0 & e^2 \\ e^0 & -e^1 & -e^2 & 0 \end{pmatrix}$$

in Euclidean signature, it is easy to show that the equations of motion (506) and (507) are equivalent to the following equation

$$dA + A\wedge A = 0 \tag{512}$$

It implies that AdS_3 gravity can still be reformulated as an action whose equation of motion is saying that connection A is flat. It is reasonable to hope that the Palatini action for AdS_3 gravity turns out to be equivalent to a Chern-Simons action. Indeed, choosing connection $A = \omega + \frac{1}{l}\boldsymbol{e}$ and the 'inner-product' (510), with $k = \frac{l}{4G}$. Then the Chern-Simons action becomes

$$I = \frac{1}{8\pi G}\int_M \left\{\langle \boldsymbol{e}\wedge\boldsymbol{F}\rangle - \frac{1}{2}d\langle\boldsymbol{\omega}\wedge\boldsymbol{e}\rangle + \frac{1}{3l^2}\langle\boldsymbol{e}\wedge\boldsymbol{e}\wedge\boldsymbol{e}\rangle\right\} = \frac{1}{16\pi G}\int_M d^3x\sqrt{g}\left(R+\frac{2}{l^2}\right) + \frac{1}{8\pi G}\oint_{\partial M}\tilde{K} \tag{513}$$

If we chose the 'inner-product' of $\mathfrak{so}(2,2)$ as its trace, an explicit calculation shows

$$I = \int_M \mathrm{Tr}\left(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\right) = \int_M \mathrm{Tr}\left(\boldsymbol{\omega}\wedge d\boldsymbol{\omega} + \frac{2}{3}\boldsymbol{\omega}\wedge\boldsymbol{\omega}\wedge\boldsymbol{\omega} + \frac{1}{l^2}\boldsymbol{e}\wedge\boldsymbol{T}\right) \tag{514}$$

Therefore, adding a Chern-Simons action $\int_M \mathrm{Tr}\left(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\right)$ to Palatini action will leads to massive topological gravity.

We should be careful that in many cases this action does not make any sense since the integration does not converge. The Einstein-Hilbert action for AdS spacetime is an integral of constant curvature over a non-compact manifold whose volume is not finite. In Euclidean signature, the manifold is a non-compact hyperbolic manifold. In section (8), we will introduce the holographic renormalization for AdS action that is developed by Graham, Fefferman and Skenderis. We will find the counter term for AdS action by using the Fefferman-Graham expansion. In what follows, whenever we use the Einstein-Hilbert action for AdS spacetime, we refer to as the finite part of the Einstein-Hilbert action.

$$I = \mathrm{FP}_{\epsilon\to 0}\int_{AdS} R = \mathrm{FP}_{\epsilon\to 0}\int_{AdS}\langle A\wedge dA + \frac{2}{3}A\wedge A\wedge A\rangle \tag{515}$$

7.2 Further Identification

Under a small gauge transformation, $\delta A = dT + [A, T]$, where $T = \rho^a P_a + \tau^a J_a$ for some real parameters ρ and τ is some element in the Lie algebra $\mathfrak{so}_{\mathbb{R}}(2,2)$, the Chern-Simons action has been shown to be invariant. Under such a infinitesimal gauge transformation, we can compute how the Einstein-Hilbert action transforms. From the equation of small gauge transformation, we can identify the following formulae as the infinitesimal transformation of fields e and ω.

$$\delta\omega^a = -\left(d\tau^a + \epsilon^a{}_{bc}\omega^b\tau^c + \frac{1}{l^2}\epsilon^a{}_{bc}e^b\rho^c\right) \tag{516}$$

$$\delta e^a = -\left(d\rho^a + \epsilon^a{}_{bc}\omega^b\rho^c + \epsilon^a{}_{bc}e^b\tau^c\right) \tag{517}$$

Plugging them into the variation of Einstein-Hilbert action, we find that the Einstein-Hilbert action is indeed invariant under such small gauge transformations. In other words, from a physical point of view, it is reasonable to add the above topological interation term to the action for pure gravity to have a more general theory. This invariance is still not enough to fully identify the classical AdS_3 gravity as a Chern-Simons action. At the moment, the physical significance of this small gauge transformation in Einstein-Hilbert action is still not clear. It is very important that Einstein-Hilbert action is invariant under diffeomorphism. Under a local Lorentz transformation acting on dreibein e and ω, the Palatini action should be invariant, but the Chern-Simons action is defined in a coordinate independent way. Thus, we may hope that the small gauge transformation is ultimately related with diffeomorphism or local Lorentzian transformations. Using the Cartan's identity of Lie derivative $\mathcal{L}_v = i_v \circ d + d \circ i_v$, where v is a vector field and $\mathcal{L}_v$ is a Lie derivative along v direction (or variation in v direction in physicists' language), we can find that the small gauge transformations given above are equivalent to diffeomorphisms and local Lorentzian transformations iff the field equations $F = \Lambda e \wedge e$ and $T = 0$ hold. For example, under a diffeomorphism generated by a vector field $v^\alpha\partial_\alpha$, the field e and ω transform to

$$\tilde{\delta}\omega^a_\mu = \mathcal{L}_v\omega^a_\mu = -v^\alpha\left(\partial_\alpha\omega^a_\mu - \partial_\mu\omega^a_\alpha\right) - \partial_\mu\left(v^\alpha\omega^a_\alpha\right) \tag{518}$$

$$\tilde{\delta}e^a_\mu = \mathcal{L}_v e^a_\mu = -v^\alpha\left(\partial_\alpha e^a_\mu - \partial_\mu e^a_\alpha\right) - \partial_\mu\left(v^\alpha e^a_\alpha\right) \tag{519}$$

Then taking subtraction of the two variations, we have

$$\tilde{\delta}\omega - \delta\omega = \# T + \cdots \tag{520}$$

and

$$\tilde{\delta}e - \delta e = \#\left(F - \Lambda e \wedge e\right) + \cdots \tag{521}$$

where we use the symbol # denotes some factor and $\cdots$ represents terms of diffeomorphisms and local Lorentzian transformations. Our conclusion is that the classical three dimensional gravity is equivalent with a Chern-Simons gauge theory at this stage.

7.3 Coupling Constant

Zamolodchikov c-theorem concludes that in a continuous family of CFT_2s, the central charge is a constant [17] [54]. From Brown and Henneaux's computation, the ratio l/G fully determines the value central charge c. The paramter l/G can only take some specific values, provided that there exists a CFT_2 dual of pure AdS_3 gravity. To this end, we introduce an alternative expression for gravitational Chern-Simons action found by Witten.

The fact that in Lorentzian signature, the gauge group $SO(2,2)$ locally splits into $SO(2,1) \times SO(2,1)$, or in Euclidean signature, however, the correct gauge group should not be $SO(3,1)$, but $SL(2,\mathbb{C})$. The Lie algebra $so(3,1)_{\mathbb{C}} \simeq \mathfrak{sl}(2,\mathbb{C})_{\mathbb{C}} \simeq \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})$ is also isomorphic to $\mathfrak{su}(2)_{\mathbb{C}} \oplus \mathfrak{su}(2)_{\mathbb{C}}$. These isomorphsims imply that for negative cosmological constant $\Lambda = -1/l^2 < 0$, we can define the following connections,

$$A_L^a = \omega^a - \frac{1}{l} e^a \quad and \quad A_R^a = \omega^a + \frac{1}{l} e^a \tag{522}$$

with two new Lie algebras

$$(J_L)_a = \frac{1}{2} (J_a - P_a) \quad and \quad (J_R)_a = \frac{1}{2} (J_a + P_a) \tag{523}$$

or by setting

$$A_-^a = \omega^a - i\sqrt{\Lambda} e^a \quad and \quad A_+^a = \omega^a + i\sqrt{\Lambda} e^a \tag{524}$$

with Lie algebra

$$(J_-)_a = \frac{1}{2} (J_a - iP_a) \quad and \quad (J_+)_a = \frac{1}{2} (J_a + iP_a) \tag{525}$$

in Euclidean signature, then the following commutation relations show that J_L and J_R form a direct sum of two real Lie algebras $\mathfrak{so}_{\mathbb{R}}(2,1)$, or $\mathfrak{su}(2)_{\mathbb{C}} \oplus \mathfrak{su}(2)_{\mathbb{C}}$ with complex gauge potential in Euclidean signature.

$$[(J_L)_a, (J_L)_b] = \epsilon^c{}_{ab} (J_L)_c \quad [(J_R)_a, (J_L)_b] = \epsilon^c{}_{ab} (J_L)_c \quad [(J_L)_a, (J_R)_b] = 0 \tag{526}$$

The 'inner-product' is just the trace (i.e. the standard killing form), which will be easy to handle. In the latter case the Euclidean gauge potential is $\mathfrak{sl}(2,\mathbb{C})$-valued, it is also the one for dS_3 action in Lorentzian signature whose isometry is $SO(3,1)$. A general theory with $\mathfrak{so}_{\mathbb{R}}(2,1) \oplus \mathfrak{so}_{\mathbb{R}}(2,1)$ gauge symmetry is in the following form

$$\begin{aligned} I &= k_L I_L + k_R I_R \\ &= \frac{k_L}{4\pi} \int_M \mathrm{Tr} \left(A_L \wedge dA_L + \frac{2}{3} A_L \wedge A_L \wedge A_L \right) + \frac{k_R}{4\pi} \int_M \mathrm{Tr} \left(A_R \wedge dA_R + \frac{2}{3} A_R \wedge A_R \wedge A_R \right) \end{aligned} \tag{527}$$

in which both A_L and A_R are $\mathfrak{so}_{\mathbb{R}}(2,1)$-valued. The magic of three dimensional gravity is that if we decompose the above action into the following form,

$$\frac{k_L + k_R}{2} (I_L - I_R) + \frac{k_L - k_R}{2} (I_L + I_R) \tag{528}$$

then the first term is equal to Einstein-Hilbert action precisely if

$$k_L + k_R = \frac{l}{8G} \tag{529}$$

and the last term is proportional to the topological interaction term that we discussed in the last section. The coupling constants for Einstein-Hilbert action and topological interation term are proportional to $k_L + k_R$ and $k_L - k_R$, respectively. They are in general, independent. But pure three dimensional gravity requires that the last term must vanish. i.e. $k_L = k_R$. For this reason, we set $k = l/16G$. In other words, the value of central charge is $c = 24k$.

In Euclidean signature, it is quite easy to determine the value of k since the real Lie group $SU(2)$ is the compact real form of $SL(2,\mathbb{C})$. When we do the Dirac quantization, we expect to use the fact that $SL(2,\mathbb{C})$ is contractible onto $SU(2)$. Because $\pi_3 (SU(2) \times SU(2)) = \mathbb{Z} \times \mathbb{Z}$, we may hope that the Dirac quantization works for for $SU(2)$ Chern-Simons theory. Using the fact that complement of a solid torus in 3-sphere is another solid torus, we take a conformally compactified spacetime $\mathbb{T}^2$, denoted by X, and another $\mathbb{T}^2$, denoted by Y. Gluing them together with their boundaries identified in opposite orientation, then we have a 3-sphere $\mathbb{S}^3 = X \sqcup_{\mathbb{T}^2} Y$, which is a compact and closed manifold. This 3-sphere has a natural extension to be the boundary of a 4-ball, which can be deemed as a four dimensional hemi-sphere. i.e. $\partial\mathbb{D}^4 = \mathbb{S}^3$. Then we glue the two hemi-four-sphere together to make a four dimensional compact closed manifold. The boundary terms introduced in previous section will not be problematic for the following reasons. The modified Chern-Simons actions on the two solid toruses are given by

$$I_X = \frac{k}{4\pi}\int_X \mathrm{Tr}\left(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\right) + \frac{k}{4\pi}\oint_{\partial X}\mathrm{Tr}\left(A_z A_{\bar{z}}\right) dz\wedge d\bar{z} \tag{530}$$

$$I_Y = \frac{k}{4\pi}\int_Y \mathrm{Tr}\left(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\right) + \frac{k}{4\pi}\oint_{\partial Y}\mathrm{Tr}\left(A_z A_{\bar{z}}\right) d\bar{z}\wedge dz \tag{531}$$

the two boundary terms cancel with each other because we glue X and Y together with their boundary identified in opposite orientation. So we end up with a well-defined Chern-Simons action

$$I_{CS}[A] = \frac{k}{4\pi}\int_{\mathbb{S}^3}\mathrm{Tr}\left(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\right) \tag{532}$$

on a three dimensional sphere $\mathbb{S}^3$, without boundary term. Since $\pi_3\left(SU(2)\right) = \mathbb{Z}$, the second Chern class $[\mathrm{Tr}\left(F\wedge F\right)]$ of an $SU(2)$-principal bundle over $\mathbb{S}^4$ is integral cohomology. Hence k is quantized in Euclidean signature.

We expect that k is still quantized for similar reasons in Lorentzian signature. The gauge group $SO(2,2)$ locally splits into $SO(2,1)\times SO(2,1)$. Although the third homotopy of $SO(2,1)$ is trivial. i.e. $\pi_3\left(SO(2,1)\right) = \pi_3\left(U(1)\right) = 0$, the non-compact group $SO(2,1)$ has its maximal compact subgroup $SO(2)$, which is ismorphic to a circle $U(1)$. Noticing that we can regard a $U(1)$-bundle as the reduction of an $SO(2,1)$-bundle. We may define the Chern class of $SO(2,1)$-bundle as the pull-back of $U(1)$-bundle. Since the Lie algebra $\mathfrak{so}_{\mathbb{R}}(2,1)$ is traceless (i.e. $\mathrm{Tr}F = 0$) and Chern classes of $U(1)$-bundle of degree higher than 1 vanish, we define the second Chern class of this $SO(2,1)$-bundle as the square of the first Chern class of $U(1)$-bundle. Therefore the quantization of coupling can be deduced from the one for $U(1)$ gauge theory. We already know that the square of the first Chern class $c_1\left(\mathcal{L}\right)^2$ of a line bundle over $\mathbb{S}^4$ is integral cohomology. In Lorentzian signature, all solutions are quotient spaces of universal covering space $\widetilde{AdS}_3$. This manifold is an solid Lorentzian cylinder, which can be regarded as a solid torus whose non-contractible 1-cycle has 'infinitely large radius'. We can do a similar surgery, gluing two $\widetilde{AdS}_3$ manifolds together with their boundaries identified in opposite orientations. This gives us a manifold with topology $\mathbb{R}^4$. After a one point compactification, we obtain a base space $\mathbb{S}^4$.

If we only consider local expressions of gauge potentials, then there is no difference whether we are doing an $SL(2,\mathbb{R})$ theory or a $SO(2,1)$ theory since their Lie algebras are isomorphic, $\mathfrak{sl}(2,\mathbb{R}) \simeq \mathfrak{so}_{\mathbb{R}}(2,1)$. One may consider the Pontrjagin class by constructing a real vector bundle over a 4-manifold whose typical fiber is given by the fundamental representation of $SL(2,\mathbb{R})$. However, since $SL(2,\mathbb{R})$ is the double cover of $SO(2,1)$, such a gauge group provides the theory with a spin structure. Hence it does not correspond to pure gravity. Concerning the fact that we already reduced the quantization condition for coupling constant to the one for $U(1)$ charge, replacing the gauge group $SO(2,1)$ by its n-fold cover means that the magnetic charge takes values in $n\mathbb{Z}$. Hence on the compact closed 4-manifold, the integral becomes

$$\frac{1}{4\pi^2}\int \mathrm{Tr}\left(F\wedge F\right) \in n^2\mathbb{Z}, \tag{533}$$

where we have to multiply it with an extra 2 factor due to the embedding $U(1)\hookrightarrow SO(2,1)$, whose Killing form gives a factor of 2. This implies that the quantization condition of coupling constant k would be

$$k \in \frac{\mathbb{Z}}{n^2} \tag{534}$$

For three dimensional gravity, we are working in $SO(2,1)\times SO(2,1)$ gauge theory. We consider $U(1)\times U(1)$ gauge theory with two independent potentials A and B and a Chern-Simons action

$$I[M]=\frac{k_L}{2\pi}\int_M A\wedge dA-\frac{k_R}{2\pi}\int_M B\wedge dB, \tag{535}$$

or extending the field and define

$$I[\$^4]=\int_{\$^4}\left(\frac{k_L}{2\pi}F_A\wedge F_A-\frac{k_R}{2\pi}F_B\wedge F_B\right) \tag{536}$$

In this theory, the integral cohomology classes $[x]=\left[\frac{F_A}{2\pi}\right]$ and $[y]=\left[\frac{F_B}{2\pi}\right]$ generated a two dimensional charge lattice. We denote the generator corresponding to $[x]$ by $(1,0)$ and the other one by $(0,1)$. If we consider a diagonal cover, meaning that the magnetic charges corresponding to A field and B field both take vlues in $n\mathbb{Z}$, then we need to add a vector $(\frac{1}{n},\frac{1}{n})$ into the charge lattice. As a result, in diagonal cover, there exists an integral cohomology class $[z]$ such that $[x]-[y]=n[z]$ for some integer n. We have [17]

$$I[\$^4]=2\pi\left(k_L-k_R\right)\int_{\$^4}y^2+2\pi k_L\int_{\$^4}\left(n^2z^2+2nyz\right) \tag{537}$$

The quantization condition becomes

$$k_L\in\begin{cases}\mathbb{Z}/n, & n\ \ is\ \ odd\\ \mathbb{Z}/(2n), & n\ \ is\ \ even\end{cases} \tag{538}$$

$$k_L-k_R\in\mathbb{Z} \tag{539}$$

In our discussion, the correct gauge group for pure gravity should be $SO(2,1)\times SO(2,1)$ and thereby $k_L=k_R\in\mathbb{Z}$.

8 Holography

8.1 Holographic Renormalization

Calculations in this section is mainly based on [21] [22] [24]. In [21], Kostas Skenderis used a theorem due to Charles Fefferman and Robin Graham [55] obtained an expansion of asymptotic AdS spacetime metric. The main result of Fefferman and Graham is that in a finite neighborhood of the conformal boundary, the metric of asymptotic local AdS_{n+1} spacetime has the form

$$ds^2=z^{-2}\left(dz^2+g_{ij}dx^idx^j\right) \tag{540}$$

where the conformal boundary is localed at $z=0$ and g_{ij} is regular on the boundary. For odd n, one has

$$\begin{aligned}g(x,z)&=g_0+z^2g_2+\cdots+z^{n-1}g_{n-1}+\cdots\\&+z^ng_n+z^{n+1}g_{n+1}+\cdots\end{aligned} \tag{541}$$

The expansion is in even powers of z up to order $n-1$.

For even n, one has

$$\begin{aligned}g(x,z)&=g_0+z^2g_2+\cdots+z^{n-2}g_{n-2}+\cdots\\&+z^ng_n+z^nh\log z+z^{n+1}g_{n+1}+\cdots\end{aligned} \tag{542}$$

The expansion is even powers of z up to order $n-2$ and the log term is related with conformal anomaly. But for AdS_3, i.e. $n=2$, we do not have logarithm term because the conformal anomaly is the Euler characteristic of conformal boundary. In general, the Einstein-Hilbert action for AdS_{1+n} manifold is divergent because it is integral of a negative constant function over a non-compact manifold whose volume is not finite.

$$\frac{1}{16\pi G}\int_M (R-2\Lambda) - \frac{1}{8\pi G}\int_{\partial M}\tilde{K} \tag{543}$$

Using the above expansions and take regularization $z \to \epsilon$, we have

$$\frac{1}{16\pi G}\int_M \sqrt{\det g_0}\left(P(\epsilon^{-1}) - a\log\epsilon\right) \tag{544}$$

where $P(\epsilon^{-1})$ is a polynomial of ϵ^{-1} and a can be computed from g_0. We remove this infinities by adding counter terms.

For AdS_3, the Fefferman-Graham expansion is

$$ds^2 = z^{-2}\left(dz^2 + g_{(0)ij}dx^i dx^j + z^2 g_{(2)ij}dx^i dx^j + z^4 g_{(4)ij}dx^i dx^j\right) \tag{545}$$

where the fourth order term can be solved in terms of g_0 and g_2 by using Einstein's equations. In order to match conventions we used for AdS_3 geometries, we perform a coordinate transformation $z = \dfrac{1}{r}$ so that the conformal boundary is at $r \to +\infty$. Then the Fefferman-Graham expansion takes the form

$$ds^2 = \frac{dr^2}{r^2} + h_{ij}dx^i dx^j. \tag{546}$$

Comparing this expression with the metrics of pure AdS_3 and BTZ black hole, we see that both pure AdS_3 metric and BTZ metric take this form in large r limit. In large r limit, the metric of pure AdS_3 as well as the one of BTZ black hole grow as r^2. To fit with the coordinate we used in the discussion of Lagrangian density of $\widetilde{AdS}_3$, we recover the AdS-radius l and perform a further coordinate transformation $r = e^{\sigma/l}$, and write the Fefferman-Graham expansion of h_{ij} in the following way

$$h_{ij} = e^{2\sigma/l}h_{ij}^{(0)} + h_{ij}^{(2)} + o\left(e^{-2\sigma/l}\right) \tag{547}$$

In the large r limit near conformal infinity, we can omit the subleading terms in the above expansion. The leading contribution is from the first term $e^{2\sigma/l}h_{ij}^{(0)}$. This piece of metric is determined up to a Weyl transformation since we always have freedom to redefine σ. Hence this metric $h_{ij}^{(0)}$ should be identified with the metric of the dual CFT_2. Upon a Weyl transformation, this metric is held fixed in order to make sense of the boundary CFT, while we allow the subleading terms in the expansion to vary. Consequently, we fix the temperature of our CFT. We let subleading terms vary meaning that we allow energy fluctuation. In other words, an Euclidean BTZ black hole can be regarded as a canonical ensemble. It's mass should be given by

$$M(\beta) = \frac{\mathrm{Tr}\left(\hat{H}e^{-\beta\hat{H}}\right)}{\mathrm{Tr}\left(e^{-\beta\hat{H}}\right)} \tag{548}$$

However, this is already awkward if we recall the definition of Brown-York tensor, which we claimed as the stress-energy tensor for the dual CFT_2. From (547), we see that the variation of modified Lagrangian always contain the variation of subleading terms. But we hope to have a variation only with respect to $h^{(0)}$ term. This problem can be solved once we renormalize the gravitational action.

8.2 Renormalization of AdS_3 Actions

In three dimensions, computations shows that the second order term satisfies a simple relation

$$\mathrm{Tr}\left(h^{(2)}\right) = \frac{l^2}{2} R^{(0)} \tag{549}$$

where $R^{(0)}$ is the curvature corresponds to $h_{ij}^{(0)}$ on the conformal boundary and indices are raised and lowered with $h^{(0)}$. If we do the computation in Euclidean signature and work it out in Poincare's upper-half space model $ds^2 = \left(dx^2 + dy^2 + dz^2\right)/z^2$, we can see that the divergence comes from the limit $z \to 0$; The divergence is roughly the 'area' of the boundary. The counterterm is therefore

$$I_{ct} = -\frac{1}{8\pi G l} \int_{\partial M} d^2 x \sqrt{h} \tag{550}$$

After adding this counterterm, the renormalized 3D action has a well-behaved functional derivative with respect to $h^{(0)}$. The variation is [24]

$$\delta\left(I + I_{ct}\right) = \frac{1}{16\pi G l} \int_{\partial M} \sqrt{h^{(0)}} \left(h_{ij}^{(2)} - \mathrm{Tr}\left(h^{(2)}\right) h_{ij}^{(0)}\right) \delta h_{(0)}^{ij}. \tag{551}$$

We define the renormalized Brown-York tensor as

$$T_{ij} = \frac{1}{8\pi G l} \left(h_{ij}^{(2)} - \mathrm{Tr}\left(h^{(2)}\right) h_{ij}^{(0)}\right), \tag{552}$$

from which we can find that it is not traceless.

$$\mathrm{Tr}(T) = -\frac{1}{8\pi G l} \mathrm{Tr}\left(h^{(2)}\right) = -\frac{l}{16\pi G} R^{(0)} \tag{553}$$

This is exactly the Weyl anomaly for CFT_2 (i.e. $\mathrm{Tr}(T) = -\frac{c}{24\pi} R$ for $c = \frac{3l}{2G}$ given by Brown and Henneaux's computation.), given by a topological number of the conformal boundary. The presence of this anomaly implies that we need to fix and specify an $h^{(0)}$ as the 'representative' of the boundary metric. To simplify formulations, we choose this boundary metric to be a flat metric and work in Euclidean signature after performing a wick rotation $u = \phi + t_E/l$.

$$h_{ij}^{(0)} dx^i dx^j = du d\bar{u} \tag{554}$$

with $u \sim u + 2\pi \sim u + 2\pi\tau$, where τ is defined as the parameter of loxodromic subgroups generating a thermal AdS_3 or an Euclidean BTZ black hole. The stress-energy tensor is therefore given by

$$T_{uu} = \frac{1}{8\pi G l} h_{uu}^{(2)}, \quad T_{\bar{u}\bar{u}} = \frac{1}{8\pi G l} h_{\bar{u}\bar{u}}^{(2)} \tag{555}$$

Remark: When we are working on an infinite cylinder or a torus, we should also shift L_0 and $\bar{L}_0$ of Brown and Henneaux by $\frac{c}{24}$. The Virasoro generators of our CFT_2 are

$$L_n - \frac{c}{24}\delta_{n,0} = \oint du e^{-inu} T_{uu}, \quad \bar{L}_n - \frac{c}{24}\delta_{n,0} = \oint d\bar{u} e^{in\bar{u}} T_{\bar{u}\bar{u}} \tag{556}$$

For an AdS_3 black hole, its angular momentum and mass are related with the eigenvalues of L_0 and $\bar{L}_0$ via

$$L_0 - \frac{c}{24} = \frac{1}{2}\left(Ml - J\right), \quad \bar{L}_0 - \frac{c}{24} = \frac{1}{2}\left(Ml + J\right) \tag{557}$$

To compute the vacuum energy ($M = -1$ for unit mass BTZ black hole), we take $L_0 = \bar{L}_0 = 0$. We see that vacuum energy is $M = \frac{-1}{8G}$, which is negative, as a result of negative cosmological constant.

To evaluate the renormalized gravitational action, we need to somehow 'integrate' the variation. To this end we introduce a new coordinate,

$$v = \frac{i - \bar{\tau}}{\tau - \bar{\tau}} u - \frac{i - \tau}{\tau - \bar{\tau}} \bar{u} \tag{558}$$

This new coordinate has period $v \sim v + 2\pi \sim v + 2\pi i$ [24]. The metric $ds^2 = dud\bar{u}$ becomes

$$ds^2 = \left| \frac{1 - i\tau}{2} dv + \frac{1 + i\tau}{2} d\bar{v} \right|^2 \tag{559}$$

In this coordinate, under the variation of the parameter τ, the variation of action is [24]

$$\delta \left(I + I_{ct} \right) = 4i\pi^2 \left(T_{\bar{u}\bar{u}} \delta\bar{\tau} - T_{uu} \delta\tau \right) \tag{560}$$

For thermal AdS_3 (i.e. $L_0 = \bar{L}_0 = 0$), we have $T_{uu} = T_{\bar{u}\bar{u}} = \frac{-c}{48\pi}$. Integrating the above differential equation, we get

$$I_{thermal} = \frac{i\pi}{12} \left(c\tau - c\bar{\tau} \right) \tag{561}$$

After an S-transformation, the renormalized action for BTZ black hole is

$$I_{BTZ} = -\frac{i\pi}{12} \left(\frac{c}{\tau} - \frac{c}{\bar{\tau}} \right) \tag{562}$$

For a CFT_2 at finite temperature, the period of time circle is roughly the inserve of temperature, which is $\Im(\tau)$. Taking $\Im(\tau) \to 0$, i.e. very high temperature, from the above expression we see that the BTZ black hole dominate the partition function.

9 Conformal Field Theory

9.1 Partition Function

From holographic principle, we believe that there is a kind of duality between an n dimensional CFT and a quantum gravity in $n+1$ dimensions. On the gravitional side, the Euclidean path integral of the quantum gravity should be in the following form

$$\int_{\partial M = T^2} Dg e^{-S[g]} = \sum_M \exp\left\{ -kS_0 + S_1 + k^{-1} S_2 + \cdots \right\} \tag{563}$$

The boundary torus indicates that we are working at a finite temperature. The first time on the right hand side is sum of exponential of Euclidean actions. This is roughly speaking, the semi-classical approximation of the Euclidean path integral. In Chern-Simons theory, the level $1/k$ plays a role of $\hbar$ in quantum mechanics. The subsequent terms on the right hand side are loop corrections to the Hilbert-Einstein action. One may ask if this expansion series is exact. The answer is negative in general. Hawking showed that on the left hand side, the Euclidean action can be made arbitrarily negative [56]. Therefore the left hand side can not be convergent. The correct way to understand gravitational path integral is to consider the contribution from classical actions as well as small perturbations around classical actions.

We have seen the renormalized actions for a thermal AdS_3 as well as for a BTZ black hole. In general, we should consider contributions coming from all Euclidean saddle points related with each other via modular transformations.

$$I(\tau,\bar{\tau}) = \frac{i\pi}{12}\left[c_{L,R}\frac{a\tau+b}{c\tau+d} - c_{L,R}\frac{a\bar{\tau}+b}{c\bar{\tau}+d}\right] \tag{564}$$

where a, b, c and d are integers satisfying $ad-bc=1$ and we denote the central charge by $c_{L,R}$ in order to distinguish it from the parameter of modular group. Hence one may expect that the partition function of quantum gravity of AdS_3 should at least contain the sum

$$\sum_{\Gamma_\infty\backslash SL(2,\mathbb{Z})} \exp\left[-\frac{i\pi c_{L,R}}{12}\frac{a\tau+b}{c\tau+d} + \frac{i\pi c_{L,R}}{12}\frac{a\bar{\tau}+b}{c\bar{\tau}+d}\right] \tag{565}$$

However, even if we ignore the fact that this summation does not converge, this expression is already physically unacceptable because there is no obvious reason why the above summation can be taken into an expansion of the form

$$\sum N_{nm} q^n \bar{q}^m \tag{566}$$

On the CFT side, a generic partition function should take the form

$$\sum_{h\bar{h}} q^h \bar{q}^{\bar{h}} \prod_{n>0} \frac{1}{(1-q^n)(1-\bar{q}^n)} \tag{567}$$

where we sum over all highest weight states $|h,\bar{h}\rangle$. In particular, the ground state $|\Omega\rangle$ must be annihilated by all $L_{n>-2}$ and $\bar{L}_{n>-2}$. In fact, there is another problem that we should concern. The partition function of CFT, denoted by Z_C is therefore related with the Euclidean path integral of the quantum gravity Z_G. i.e.

$$Z_C \sim Z_G \tag{568}$$

However, we don't know whether the two functions on the two sides are precisely equal or they are related via some transformations. It is essential to understand the exact relationship between the two partition functions. This question was first considered by Robbert Dijkgraaf, Juan Maldacena, Gregory Moore and Erik Verlinde in [19] in $D1/D5$ system, where they focused on the duality between IIB string theory on $AdS_3 \times \mathbb{S}^3 \times K3$ and the dual conformal field theory. Their idea was inherited by Jan Manschot in [32]. The exact relationship between the partition functions on the two sides is called the Fareytail transformation. We will come back to this point in the last section.

9.2 Verma Module of $\hat{1}$

Our first possibility is that the dual CFT_2 is in a irreducible representation. Our computations in this case are based on [18]. In this theory, we only need to compute the Verma module of vacuum, which is given by

$$Z_{0,1}(\tau,\bar{\tau}) = |\bar{q}q|^{-k} \frac{1}{\prod_{n=2}^{\infty} |1-q^n|^2} \tag{569}$$

This is not a modular invariant expression and so can not be the partition function of the whole theory. A complete partition function should also contain contributions from other saddle points that are related with $Z_{0,1}$ by modular transformations. In Euclidean signature, free energy generating the connected diagram is related with partition function via

$$W(\tau,\bar{\tau}) = -\frac{1}{k}\ln(Z(\tau,\bar{\tau})) \tag{570}$$

This free energy is regarded as the effective action that includes contributions from classical action as well as quantum corrections from loop diagrams. We should keep in mind that this is only a heuristic quantum field theory because in three dimensions, there are no propagating degree of freedom and thus we don't have Feynman propagators. In general, an effective action should be in the form

$$W(\tau,\bar{\tau}) = I + \sum_{l=1}^{\infty} \frac{W_l(\tau,\bar{\tau})}{k^l} \tag{571}$$

For the contribution from $Z_{0,1}$, we have

$$W(\tau,\bar{\tau}) = -\ln|q\bar{q}| - \frac{1}{k}\sum_{n=2}^{\infty} \ln(|1-q^n|^2), \tag{572}$$

from which we find that this theory is in fact 1-loop exact since there is no higher order of $1/k$ appearing in the above expression.

In section (4), we saw that saddle points are labelled by a pair of coprime integers (c, d). All manifold $M_{c,d}$ are diffeomorphic to each other. We can calculate the associated partition function for every Euclidean saddle points

$$Z_{c,d}(\tau,\bar{\tau}) = Z_{0,1}\left(\frac{a\tau+b}{c\tau+d}, \frac{a\bar{\tau}+b}{c\bar{\tau}+d}\right) \tag{573}$$

where integers a and b satisfy $ad - bc = 1$. The partition function from all contributions should be

$$Z(\tau,\bar{\tau}) = \sum_{c,d} Z_{c,d}(\tau,\bar{\tau}) = \sum_{c,d} Z_{0,1}\left(\frac{a\tau+b}{c\tau+d}, \frac{a\bar{\tau}+b}{c\bar{\tau}+d}\right) \tag{574}$$

where (c, d) are coprime integers and we choose $c \geq 0$ because the modular transformations are projective.

To avoid duplication, in what follows, we will omit the anti-holomorphic sector in the expression of partition function and simply write it as

$$Z(\tau) = \sum_{c,d} Z_{0,1}(\gamma\tau) \tag{575}$$

where $\gamma \in \Gamma_\infty \backslash PSL(2,\mathbb{Z})$. For convenience, we introduce the Dedekind η-function defined as

$$\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n) \tag{576}$$

The partition function can be written as

$$Z_{0,1}(\tau) = \frac{|q\bar{q}|^{-(k-1/24)}|1-q|^2}{|\eta(\tau)|^2} \tag{577}$$

This expression is easier to handle because of the well-known fact that $(\Im(\tau))^{1/2}|\eta(\tau)|^2$ is modular invariant. Modular invariant factor in the summand can be factored out of parentheses. To this end we write

$$Z(\tau) = \frac{1}{\sqrt{\Im\tau}|\eta(\tau)|^2}\sum_{c,d}\left(\sqrt{\Im\tau}|q\bar{q}|^{\frac{1}{24}-k}|1-q|^2\right)\Big|_\gamma \tag{578}$$

where $(\cdots)|_\gamma$ denotes the modular transformation of the expression $(\cdots)$ in the parentheses. Expanding the above expression, we have

$$Z(\tau) = \frac{1}{\sqrt{\Im\tau}|\eta(\tau)|^2}\sum_{c,d}\left\{\sqrt{\Im\tau}\left(|q\bar{q}|^{-k+\frac{1}{24}} - \bar{q}^{-k+\frac{1}{24}}q^{-k+\frac{25}{24}} - q^{-k+\frac{1}{24}}\bar{q}^{-k+\frac{25}{24}} + \bar{q}^{-k+\frac{25}{24}}q^{-k+\frac{25}{24}}\right)\right\}\Big|_\gamma \tag{579}$$

Each term in the above summation is of the form

$$E(\tau; n, m) = \sum_{c,d} \left(\sqrt{\Im\tau} \frac{1}{q^n} \frac{1}{\bar{q}^m} \right) \Big|_\gamma \tag{580}$$

with $n - m$ equals to either 0 or ± 1 and $q = e^{2\pi i\tau}$. We set $\kappa = n + m$ and $\mu = m - n$, and use the identity $\Im(\gamma\tau) = \dfrac{\Im\tau}{|c\tau + d|^2}$, then each term in the summand can be written as

$$E(\tau; \kappa, \mu) = \sqrt{\Im\tau} \sum_{c,d} |c\tau + d|^{-1} \exp\left\{2\pi\kappa \Im\gamma\tau + 2\pi i\mu \Re\gamma\tau\right\} \tag{581}$$

The term $E(\tau; \kappa, \mu)$ is the well-known Poincare series. The total partition function is

$$Z(\tau) = \frac{1}{\sqrt{\Im\tau}|\eta(\tau)|^2} \left(E(\tau; 2k - \frac{1}{12}, 0) + E(\tau; 2k + 2 - \frac{1}{12}, 0) - E(\tau; 2K + 1 - \frac{1}{12}, 1) - E(\tau; 2k + 1 - \frac{1}{12}, -1) \right) \tag{582}$$

Each term in this summation does not converge because for large c, d summand goes to a non-zero constant. The trick is to use Riemann-ζ-regularizaion. We do an analytic continuation that let the Poincare series depends on an extra parameter s

$$E(\tau; n, m; s) = \sum_{c,d} \left(\left(\sqrt{\Im\tau} \right)^s \frac{1}{q^n} \frac{1}{\bar{q}^m} \right) \Big|_\gamma \tag{583}$$

This new series converges for $\Re s > 1$, hence our original series concerning the case $s = 1/2$ converges.

We expect that the partition function should be

$$Z = \mathrm{Tr} e^{-\beta \hat{H} - i\xi \hat{J}} \tag{584}$$

Since angular momentum is a unitary representation of $\mathfrak{u}(1)$-algebra, its eigenvalues must be integers. Therefore the expected partition function should be

$$Z = \sum_{n\in\mathbb{Z}} e^{in\xi} \mathrm{Tr}_{\mathcal{H}_n} \left(e^{-\beta\hat{H}} \right) \tag{585}$$

where we take trace over the eigenpace $\mathcal{H}_n$ of $\hat{J}$, on which $\hat{J}$ acts with eigenvalue n. More precisely, the partition function is the trace of exponential of the operator $-\beta\hat{H} \otimes \hat{1}_J - i\xi \hat{1}_H \otimes \hat{J}$. To have a convergent partition function, in each subspace $\mathrm{Tr} e^{-\beta\hat{H}}$ should be convergent, provided that for each eigenvalue E, states of energy no greater than E must be finitely many. Let E_* be an arbitrary energy level. Energy states lower than E_* are finitely many and are denoted by E_1, ... , E_m. Then the partition function would look like

$$Z = \sum_{n\in\mathbb{Z}} e^{in\xi} \left\{ \sum_{j=1}^{m} e^{-\beta E_j} + o\left(e^{-\beta E_*}\right) \right\} \tag{586}$$

Unfortunately, the computation of Poincare series shows that the partition function is

$$\begin{aligned} Z = Z_{0,1} + \frac{1}{|\eta|^2} \Bigg(& -6 + \frac{(\pi^3 - 6\pi)(11 + 24k)}{9\zeta(3)} y^{-1} \\ & + \frac{5(53\pi^6 - 882\pi^2) + 528(\pi^6 - 90\pi^2)k + 576(\pi^6 - 90\pi^2)k^2}{2430\zeta(5)} y^{-2} + o(y^{-3}) \Bigg) \end{aligned} \tag{587}$$

where we set $\tau = x + iy$ [18]. This shows that the dual CFT cannot be in a single irreducible representation and there are other contributions that we excluded in the above sum.

9.3 Kleins j-invariant and $ECFT$

From calculations in last section, we conclude that the dual CFT_2 must be in reducible representation. Witten first suggested that the dual CFT_2 for $c = 24$ should be holomorphic factorized and is given by the Frenkel Lepowsky Meurman construction [17] [37] [38] [41]. It is now known as the Monstrous Moonshine conjecture that there is a certain conformal field theory on torus which has the Monster group as its symmetry; In particular, in holomorphic sector, its partition function is given by the Klein J-invariant. Any two complex tori $\mathbb{C}/\Lambda_1$ and $\mathbb{C}/\Lambda_2$ are isomorphic to each other if and only if there exists a nonzero complex number c such that the two lattices satisfy $\Lambda_2 = c\Lambda_1$. Roughly speaking, the set of all elliptic curves modulo the isomophism is the moduli space of elliptic curves. The moduli space of elliptic curves is therefore characterized by lattices in $\mathbb{C}$. Since distinct lattices are different from each other by an $SL(2,\mathbb{Z})$ transformation, it turns out that the set of isomorphism classes of complex tori is a quotient $\mathbb{H}^2/SL(2,\mathbb{Z})$. Topologically, this quotient space is a punctured Riemann sphere (it also has two conical singularities). The Klein J-invariant is a modular function defined as

$$j = \frac{1728E_4^3}{E_4^3 - E_6^2} \tag{588}$$

where the series $E_{2k}(\tau) = \sum_{(m,n)} \frac{1}{(m\tau + n)^{2k}}$ for coprime integers (m,n) absolutely converges to a holomorphic function of τ in the upper-half plane. This sum E_{2k} is called the Eisenstein series. Under an $SL(2,\mathbb{Z})$ action, it transforms as

$$E_{2k}(\gamma\tau) = (c\tau + d)^{2k} E_{2k}(\tau) \tag{589}$$

where $\gamma\tau = (a\tau + b)/(c\tau + d)$ for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$, and we call the even number $2k$ the weight of Eisenstein series. It can be proved that the function $j(\tau)$ is of weight 0 and is modular invariant, i.e.

$$j(\gamma\tau) = j(\tau) \tag{590}$$

for $\tau \in \mathbb{H}^2$ and the j-function is holomorphic on $\mathbb{H}^2$ but has a pole of first order at the cusp infinity ∞. In addition, $j(\tau)$ is bijective from moduli space of elliptic curves over $\mathbb{C}$ to complex numbers. In other words, for every complex number $z \in \mathbb{C}$, there is a unique τ in the fundamental region, which corresponds to an isomorphism class of elliptic curves, such that $z = j(\tau)$. The most remarkable feature of this function is its Laurent series in terms of $q = \exp(2\pi i\tau)$, which is

$$j(\tau) = \frac{1}{q} + 744 + 196884q + 21493760q^2 + 864299970q^3 + 20245856256q^4 + \cdots \tag{591}$$

It diverges at $q = 0$. It is more convenient to use the J-function

$$J(q) = j - 744 = \frac{1}{q} + 196884q + 21493760q^2 + 864299970q^3 + 20245856256q^4 + \cdots \tag{592}$$

instead of j-function.

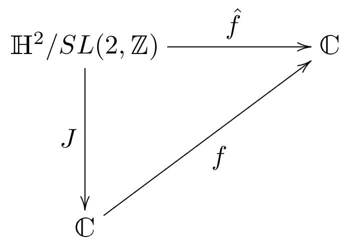
Theorem: If a meromorphic function f only has a pole at ∞, then it must be a polynomial.

Proof: Locally the function f can be written as an expansion

$$f = \cdots + a_1 z + \cdots + a_n z^n \tag{593}$$

Consider the function $g = f - a_n z^n - \cdots - a_1 z$, it is holomorphic on $\mathbb{CP}^1$. From Liouville's theorem, the function g must be a constant. Hence, f is a polynomial.

The above theorem implies that the partition function under the Frenkel Lepowsky Meurman construction is a polynomial of Klein's J-function. The reason goes as follows. Any modular invariant function $f(z) = f(\gamma z)$ induces a funtion $\hat{f}$ on the fundamental domain $\mathbb{H}^2/SL(2,\mathbb{Z})$. By comparing the poles of J-function and of partition function $Z(q)$, we see that the partition function can be constructed by using the following commutative diagram



where $f = \hat{f} \circ J^{-1}$. The partition function $Z(q)$ as a polynomial of Klein's J-function should therefore have a pole $J = \infty$. Witten claimed that the partition function should have a pole at $q = 0$ of order k that corresponding to $J = \infty$, which implies that $Z(q)$ must be a polynomial in J of degree k.

$$Z(q) = \sum_{r=0}^{k} f_r J^r \tag{594}$$

For example, for $k = 2$, we have

$$Z_2(q) = J(q)^2 - 393767 = q^{-2} + 1 + 42987520q + 40491909396q^2 + \cdots \tag{595}$$

where the coefficients 1 and -393767 are easily determined by the requirement that $Z(q)$ should have an expansion of the form $q^{-k}\prod_{n=2}^{\infty}\frac{1}{1-q^n} + o(q)$. We can recognize that the first term is simply the holomorphic sector of descendants of vacuum, which corresponds to pure AdS_3. While the second term is the contribution from a BTZ black hole. Similarly, for $k = 4$, we have

$$Z_4(q) = q^{-4} + q^{-2} + q^{-1} + 2 + 81026609428q + 1604671292452452276q^2 \tag{596}$$

For $c = 24$, i.e. $k = 1$, the partition function is exactly given by $Z(q) = J(q)$. The most fascinating fact about this function is that all the magics are hidden behind its horrible looking coefficients. The first nontrivial coefficient 196884 equals to $1 + 196883$. The number 196883 is exact the smallest dimension of a nontrivial representation of the Monster group $\mathbb{M}$. To be more specific, the theory is constructed such that there is a commutative algebra (called Griess algebra) structure on an 196884 dimensional vector space over $\mathbb{R}$, whose automorphsim group is given by $\mathbb{M}$. In this theory, the Monster group fixes an one dimensional subspace and acts irreducibly on the 196883 dimensional complement [37] [38] [41]. The finite simple groups are completely classified into 18 countable infinite families together with 26 exceptional groups, called sporadic groups [38] [57]. The Monster group is the largest one of these sporadic groups [38] [57]. It has order

$$|\mathbb{M}| = 2^{46} \cdot 3^{20} \cdot 5^{9} \cdot 7^{6} \cdot 11^{2} \cdot 13^{3} \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71 \tag{597}$$

In sloppy language, finite simple groups are to finite groups what semi-simple Lie algebra are to Lie algebras. The Monster group plays a similar role in finite simple group theory that Lie groups of $\mathbb{E}_8(\mathbb{C})$ type do in Lie group theory. In 1978, MacKay observed an intriguing phenomenon, which is that

$$\begin{aligned} 196884 &= 196883 + 1 \\ 21493760 &= 21296876 + 196883 + 1 \\ 864299970 &= 842609326 + 21296876 + 2 \cdot 196883 + 2 \cdot 1 \end{aligned} \tag{598}$$

The numbers 1, 196883, 21296876 and 842609326 are the dimensions of irreducible representations of $\mathbb{M}$. i.e. The coefficients of the q-expansion of J function are related with irreducible representations of Monster group [38]. This led to the Monstrous moonshine conjecture made by John Conway and Simon P. Norton in 1979 [38]. Let ρ_0, ρ_1,$\cdots$ be irreducible representations of $\mathbb{M}$, ordered by dimension. Then the q-expansion of J function is really hinting that there is an infinite dimensional graded $\mathbb{M}$-module

$$V^\sharp = V_{-1} \oplus V_1 \oplus V_2 \oplus \cdots = \rho_0 \oplus (\rho_1 \oplus \rho_0) \oplus (\rho_2 \oplus \rho_1 \oplus \rho_0) \oplus \cdots \tag{599}$$

and J function is given by

$$J(q) = \dim(V_{-1})q^{-1} + \sum_{i=1}^{\infty} \dim(V_i)q^i = \mathrm{ch}_{V_{-1}}(g)q^{-1} + \sum_{i=1}^{\infty} \mathrm{ch}_{V_i}(g)q^i \tag{600}$$

for each element $g \in \mathbb{M}$. The right hand side is called MacKay-Thompson series [38]. Igor Frenkel, James Lepowsky and Arne Meurman explicitly constructed such a graded module and showed that the vector space they constructed, called Moonshine module $V^\sharp$, has additional algebraic structure, which is Vertex Operator Algebra (VOA) in conformal field theory [38]. This modle $V^\sharp = \oplus V_n$ gives bilinear maps from $V_i \times V_j$ to V_k. The special case is maps from $V_2 \times V_2$ to V_2, which is called the Griess product [38] [37] [41] [58]. We say that Monter group is the automorphism group in the sense that its action preserves the Griess product. In bosonic string theory, this algebra can be constructed as a conformal field theory describing 24 compactified bosons [38].

In FLM interpretation, the CFT has an identity operator 1, which corresponds to its unique vacuum $|\Omega\rangle$. It has 196884 operators of dimension 2, one of which is an stress tensor, which is a 2nd order secondary operator of identity 1, while they others are primary fields transforming in the representation of $\mathbb{M}$. We denote their associated highest weight states by $|II\rangle_i$, for $i = 1, \cdots, 196883$. It has infinitely many consecutive primary operators, whose highest weight states are denoted by $|III\rangle_j$, $|IV\rangle_k$, $|V\rangle_l$,$\cdots$, where the ranges of indices $i, j, k, l, \cdots$ are given by dimensional of irreducible representations of the Monster group. In a broader sense, the Moonshine module is a bridge connecting algebraic structures with theory of modular invariant. Witten suggested that in AdS_3/CFT_2 correspondence, the 196883 primaries are deemed as the generators of a BTZ black hole, the one operator left corresponds the boundary gravitons. The FLM construction yields a duality between AdS_3 quantum gravity and extremal holomorphic CFT, which was introduced by G. Höhn. We call it extremal CFT for lacking of primaries in low energy. In this conformal field theory, the lowest dimensional highest weight state above vacuum has conformal dimension $k+1$ with central charge $c = 24k$, $k \in \mathbb{Z}_{>0}$. In the theory of holomorphic VOA, it is known that for $c \leq 16$, we have three theories [41]:

(1) $c = 0$ and $V = \mathbb{C}1$.

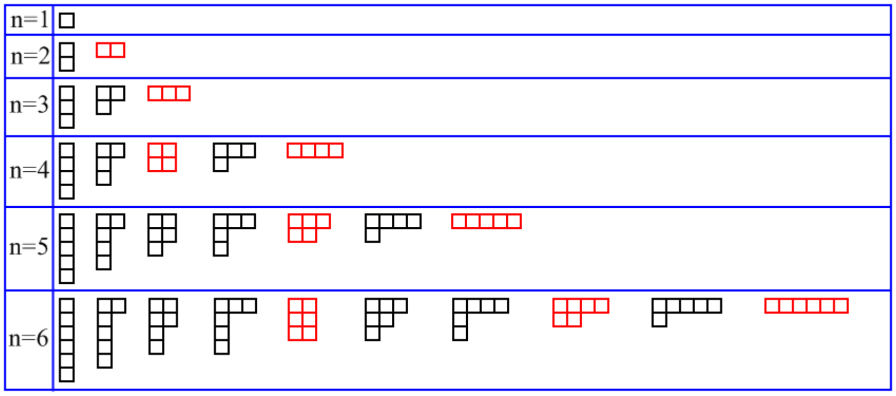
(2) $c = 8$ and $V = V_{\mathbb{E}_8}$ is the $\mathbb{E}_8$-lattice theory.

(3) $c = 16$ and V is $V_{\mathbb{E}_8 \perp \mathbb{E}_8}$-lattice theory.

For $c = 24k$, it is not known whether such theories exist except when $k = 1$. It was conjectured that there are in total 71 holomorphic VOAs with $c = 24$, from which 39 are known to exist [17] [34]. In Schellekens' classification, among the 71 CFTs, 70 have current algebras. For pure AdS_3/CFT_2, we want a CFT that contains no current algebra. The unique theory satisfying this feature is the FLM construction. In Monstrous moonshine, J-function is usually written in terms of $\mathbb{M}$ characters as shown above. While in FLM's construction, J-function as a partition function of CFT_2 should be expressed in terms of Virasoro characters. i.e. the coefficients of its q-expansion should be decomposed by dimensions of (irreducible) Verma modules of $|\Omega\rangle$, $|II\rangle$, $|III\rangle$, $|IV\rangle$, $|V\rangle$,$\cdots$ The moonshine module shouled be of the form [59]

$$V^\sharp = \bigoplus_{n=0}^{\infty} (V_n \otimes W(24, h)) \tag{601}$$

where each V_n is a represenation of the Monster group, while $W(24,h)$ is the irreducible Virasoro representation with conformal weight h and central charge 24. The best way to calculate the numbers of independent states in Verma modules $W(24,h)$ is to use Young Tableaux. For highest weight states above vacuum, the



number of independent states at energy level n is given by partition $p(n)$. However, for the vacuum state, since it is annihilated by L_{-1}, the number of independent states is given by partition which does not contain 1, which are those red ones in the above table. From the Virasoro charaters

$$J(q) = \sum_n d_n \chi_n(q) \tag{602}$$

we can solve the first few terms, which are given by $d_0 = 1 = \dim\rho_0$, $d_2 = 196883 = \dim\rho_1$, $d_3 = \dim\rho_2$, $d_4 = \dim\rho_3$, $d_{n\geq 5} \geq \dim\rho_{n-1}$, where d_i is the dimension of Verma module of Virasoro algebra.

This dual CFT can also explain the entropy of BTZ black holes. We can use the entropy formula (329), which can be written as

$$S = \pi(l/2G)^{1/2}(\sqrt{Ml-J} + \sqrt{Ml+J}) = 4\pi\sqrt{k}(\sqrt{L_0} + \sqrt{\bar{L}_0}), \tag{603}$$

where we used $l/G = 16k$. In quantum mechanics, the entropy measures the microscopic degeneracy of a macroscopic state. In our case, the degeneracy of 196883 highest weight states (or 196883 primary fields) give an entropy $\ln 196883 \simeq 12.19$. On the gravitational side, we take $k = 1$ and $L_0 = 1$, the Bekenstein-Hawking entropy is approximately 12.57. For $k = 4$, taking $L_0 = 1$, the entropy computed from extremal CFT is $\ln 81026609426 \simeq 25.12$. The Bekenstein-Hawking entropy is about 25.13. In semi-classical limit $k \to \infty$, the two computations coincide with each other because Bekenstein-Hawking entropy is derived in semi-classical approximation. We interpret 196883 primaries for $k = 1$ as the generators of one BTZ black hole; the following numbers $21493759, \cdots$ corresponding to $L_0 > 1$ are the degeneracy of many BTZ black holes. If this is really the CFT_2 dual of AdS_3 pure gravity, then three dimensional gravity would surprisingly have a discrete symmetry $\mathbb{M}$.

To match this partition function with the Euclidean path integral of AdS_3, we utilize the Rademacher expansion of J function [32],

$$J(q) = -12 + \lim_{K\to\infty} \frac{1}{2} \sum_{|c|\leq K} \sum_{\substack{|d|\leq K \\ (c,d)=1}} \exp 2\pi i \left(-\frac{a\tau+b}{c\tau+d}\right) - \exp\left(-2\pi i \frac{a}{c}\right)(1-\delta_{c,0}), \quad ad - bc = 1 \tag{604}$$

The first term is the summation can be interpreted as the contribution from classical Euclidean actions in holomorphic sector. The last term $\exp\left(-2\pi i \frac{a}{c}\right)(1-\delta_{c,0})$ is necessary for convergence, but hard to be interpreted from gravitational aspect. The way to relate this expansion to the Euclidean path integral of quantum gravity is relied on Fareytail transform, which was pointed out by Jan Manschot. First of all, it is necessary to introduce what modular form is.

Definition: A modular form of weight k for the modular group $SL(2,\mathbb{Z})$ is a complex-valued function f on the upper-half plane $\mathbb{H}^2$ satisfying three conditions:

(1) f is holomorphic on $\mathbb{H}^2$.

(2) For any $\tau \in \mathbb{H}^2$, the equation

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k f(\tau) \tag{605}$$

holds, where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$.

(3) f is required to be holomorphic at the cusp $\tau \to i\infty$.

We see that an Eisenstein series $E_{2k}(\tau)$ is a modular forms of weight $2k$. If we forget the third condition and allow f to be a meromorphic function which is not holomorphic at cusp, we call such a function f non-entire modular form. For example, for $c = 24$, the CFT partition function is given by J-invariant, which is a non-entire modular form of weight 0 and has a pole at $i\infty$. For a given (non-entire) modular form f of weight w, the Fareytail transform is defined as [19] [12]

$$D_F(f) = \left(q\frac{\partial}{\partial q}\right)^{1-w} f \tag{606}$$

For (non-entire) modular forms of weight zero, the Fareytail transform is given by its derivative.

$$D = \frac{1}{2\pi i}\frac{d}{d\tau} \tag{607}$$

Hence the inverse Fareytail transform gives back $J(q)$ up to a constant term. It can be shown that after the Fareytail transformation, we have [32]

$$DJ(\tau) = \frac{-1}{2}\sum_{\Gamma_\infty \backslash SL(2,\mathbb{Z})} \frac{\exp 2\pi i\left(-\frac{a\tau+b}{c\tau+d}\right)}{(c\tau+d)^2} \tag{608}$$

By defining $\mathfrak{M}(c\tau+d) = \dfrac{-1}{2}(c\tau+d)^2$, we can express the above partition function as

$$Z_G(\tau) = \sum_{M_{c,d}} e^{-S} = DJ(\tau) = \sum_{\Gamma_\infty \backslash SL(2,\mathbb{Z})} \mathfrak{M}(c\tau+d)\exp 2\pi i\left(-\frac{a\tau+b}{c\tau+d}\right) \tag{609}$$

where $\mathfrak{M}(c\tau+d)$ is some measure factor. This summation is of form of a Poincare series and can be interpreted as a sum over geometries in the holomorphic sector. It is a semi-classical approximation that we have already seen. Since we assumed that the extremal CFT is holomorphically factorized, the full partition function of quantum AdS_3 gravity would be

$$Z_G(\tau,\bar{\tau}) = \left(\sum_{\Gamma_\infty \backslash SL(2,\mathbb{Z})} \mathfrak{M}(c\tau+d)\exp 2\pi i\left(-\frac{a\tau+b}{c\tau+d}\right)\right)\left(\sum_{\Gamma_\infty \backslash SL(2,\mathbb{Z})} \mathfrak{M}(\tilde{c}\bar{\tau}+\tilde{d})\exp 2\pi i\left(-\frac{\tilde{a}\bar{\tau}+\tilde{b}}{\tilde{c}\bar{\tau}+\tilde{d}}\right)\right), \tag{610}$$

from which we see that if this were the correct partition function for quantum gravity, we would have complex saddle points because only when $(c,d) = (\tilde{c},\tilde{d})$ we would have real-valued Euclidean saddle points. Moreover, from a more physical aspect, this cannot be the dual CFT_2. First, it was shown that the Monster symmetry is killed at larger value of k [61]. Secondly, it was shown that for $k \geq 42$, such theories cannot exist [62]. We have seen that when $k = 1$, the gravitational action in Lorentzian signature correspond to a magnetic monopole of unit strength. For a magnetic charge with strength $n > 1$, the corresponding circle bundle over $\mathbb{S}^2$ is not a Hopf fibration anymore. Its total space has structure of lense space, $\mathbb{S}^3/\mathbb{Z}_n$. The gap of the spectrum requires that there are no primary fields of weight in the interval $0 < h < k+1$, which seems to be very hard to be satisfied in the semi-classical limit $k \to \infty$.

Although we have successfully explained the entropy of BTZ black holes in extremal CFT, this theory of quantum AdS_3 itself is still speculative. We are still not able to find a reliable theory of three dimensional quantum gravity. One possibility is that perhaps for some unknown reasons, complex geometries should be included in our theory. The most obvious way to make such a generalization is to complexify the equation of motion $R_{ab} = -\Lambda g_{ab}$. However, Witten and Maloney have not found any solutions that depend on the pair of modular parameters in the above product of sums [18]. A pessimistic aspect is that there is no CFT_2 dual for pure AdS_3 at all.

10 Outlook

A natural generalization of three dimensional pure gravity is massive topological gravity, whose interaction term is given by a Chern-Simons action. It was shown by Witten that 3D topological field theories are intimately related with CFTs in two dimensions. Since we have not found a candidate CFT dual to pure AdS_3 quantum gravity, one may hope that there exist dualities between CFTs and quantum versions of various massive topological gravity. After decades of investigations, research in this topic has flourished and become fruitful. The most well-known theory is called chiral gravity introduced by Wei Li, Wei Song and Andrew Strominger [63] [64]. It is chiral, in the sense that it has only one copy of Virasoro algebra [63] [64]. We consider a massive topological gravity whose action is

$$I = \frac{1}{16\pi G}\left[\int d^3x\sqrt{g}\left(R + \frac{2}{l^2} + \frac{1}{\mu}I_{CS}\right)\right], \tag{611}$$

where I_{CS} is a gravitational Chern-Simons action

$$I_{CS} = \frac{1}{2}\int d^3x\sqrt{g}\epsilon^{abc}\Gamma^d_{ae}\left(\partial_b\Gamma^e_{dc} + \frac{2}{3}\Gamma^e_{bf}\Gamma^f_{cd}\right). \tag{612}$$

By doing variation, we find that [63]

$$R_{ab} - \frac{1}{2}Rg_{ab} - \frac{1}{l^2}g_{ab} + \frac{1}{\mu}C_{ab} = 0, \tag{613}$$

C_{ab} is the Cotton tensor. Since the Chern-Simons interaction term is not invariant under parity transformation, one should expect that in such a theory, the central charges of left moving and right moving sectors are not equal, while still satisfying $c_R + c_L = 3l/G$. It can be proved that [63]

$$c_L = \frac{3l}{2G}(1 - \frac{1}{\mu l}) \quad c_R = \frac{3l}{2G}(1 + \frac{1}{\mu l}) \tag{614}$$

Chiral gravity is defined by taking the limit $\mu l \to 1$ while keeping the Brown Henneaux's boundary conditions. This implies that

$$c_L = 0 \quad c_R = \frac{3l}{G} \tag{615}$$

Hence we only have right-moving excitations.

To compute the partition function, we need to wick rotate the classical action and determine the Euclidean saddle points. We fix the conformal boundary to be a torus. The Euclidean bulk action is

$$I = \frac{1}{16\pi G}\int d^3x\sqrt{g}\left(R+\frac{2}{l^2}\right)+\frac{il}{16\pi G}\int d^3x\sqrt{g}\epsilon^{abc}\Gamma^d_{ae}\left(\partial_b\Gamma^e_{dc}+\frac{2}{3}\Gamma^e_{bf}\Gamma^f_{cd}\right). \tag{616}$$

Remark: In Euclidean path integral, this extra factor i is exactly what we should expect because otherwise the Euclidean action is not invariant under large gauge transformations. This is analogous to WZW theory. We can construct a compact closed 4-manifold. The Chern-Simons term is extended to

$$\int R_{abcd}R^{abcd} \tag{617}$$

which is a topological term. The equation of motion is [64]

$$R_{ab}-\frac{1}{2}Rg_{ab}-\frac{1}{l^2}g_{ab}+ilC_{ab}=0 \tag{618}$$

Since we require that Euclidean saddle points are real metric, Euclidean saddle points satisfy the above equation should obey

$$R_{ab}-\frac{1}{2}Rg_{ab}-\frac{1}{l^2}g_{ab}=0 \tag{619}$$

Hence Euclidean saddle points of chiral gravity are the same as those of pure 3D gravity. But since in this theory, we turned off the anti-holomorphic sector, the partition function of the dual CFT should be given by J-invariant.

$$Z(\tau)=\mathrm{Tr}_{\mathcal{H}}q^{L_0}=q^{-1}+196884q+\cdots \tag{620}$$

The Hilbert space is spanned by eigenstates of L_0, containing vacuum $|\Omega\rangle$ and other highest weights $|h\rangle$, together with their descendants $L_{-n_1}\cdots L_{-n_k}|\Omega\rangle$ and $L_{-n_1}\cdots L_{-n_k}|h\rangle$. Since in this theory, we have only holomorphic sector, the level of quantum state is the eigenvalue of the sum of mass and angular momentum. Denoting the degeneracy of a given level E_n by $N(E_n)$, then a partition funtion of a conformal field theory should take the form

$$Z(\tau)=\sum_{n\in\mathbb{Z}}N(E_n)q^{E_n}. \tag{621}$$

We have already seen that this indeed agrees with the expression of partition function of extremal CFT. The non-trivial coefficients of the q-expansion of Klein's J-invariant are the microscopic degeneracy of BTZ black holes plus boundary gravitons in holomorphic sector. In last section, we have shown that for $k=1$, logarithm of $N(E)-1$ is approximately the Bekenstein-Hawking entropy $S=4\pi\sqrt{M}=4\pi\sqrt{L_0}$. i.e.

$$N(E)-1\sim e^{4\pi\sqrt{kE}} \tag{622}$$

In fact, there is an exact formula from technique of Rademacher expansion, including all the quantum corrections to Bekenstein-Hawking entropy [32]. This is

$$J(\tau)=q^{-1}+\sum_{n=1}^{\infty}c(n)q^n, \tag{623}$$

where the Fourier coecient $c(n)$ is given by

$$c(n)=\frac{2\pi}{\sqrt{n}}\sum_{n=1}^{\infty}\frac{K_m(n)}{m}I_1\left(\frac{4\pi\sqrt{n}}{m}\right). \tag{624}$$

$K_m(n)$ is the Kloomsterman sum

$$K_m(n) = \sum_{d\in(\mathbb{Z}/n\mathbb{Z})^*} \exp\left(\frac{2\pi i(nd+\bar{d})}{m}\right), \quad d\bar{d} = 1 \;\; mod \;\; m. \tag{625}$$

and $I_\nu(z)$ is

$$I_\nu(z) = \frac{(z/2)^\nu}{2\pi i}\int_{c-i\infty}^{c+i\infty} t^{-\nu-1}e^{t+\frac{z^2}{4t}}dt, \quad c>0, \;\; \Re(v)>0 \tag{626}$$

which is the Bessel function.

There are many other research areas that I should have investigated in my thesis if I had had much more time, including higher spin theory and three dimensional supergravity. Higher spin theory is a generalization of gravitational Chern-Simons action whose gauge group is replaced by $SL(N,\mathbb{R})\times SL(N,\mathbb{R})$. Such a theory contains three dimensional gravity and higher spin fields. Another interesting aspect of three dimensional gravity is new boundary condition of asymptotic AdS_3 [40], which has raised attention from many researchers.

11 Appendix

Proof: We have identity

$$\frac{1}{3!}\epsilon_{\alpha\beta\gamma}\epsilon^{abc}e_a^\alpha e_b^\beta e_c^\gamma = \det(e) \tag{627}$$

Multiplying a factor $\epsilon^{\mu\nu\rho}$ on both sides of the above equation, we have

$$\begin{aligned}&\epsilon^{\mu\nu\rho}\det(e) = \frac{1}{3!}\epsilon^{\mu\nu\rho}\epsilon_{\alpha\beta\gamma}\epsilon^{abc}e_a^\alpha e_b^\beta e_c^\gamma = \delta_\alpha^{[\mu}\delta_\beta^\nu\delta_\gamma^{\rho]}\epsilon^{abc}e_a^\alpha e_b^\beta e_c^\gamma\\ &= \epsilon^{abc}e_a^\mu e_b^\nu e_c^\rho\end{aligned} \tag{628}$$

Concerning that ϵ tensor is fully antisymmetric, the above equation implies that

$$\begin{aligned}\epsilon^{abc}e_a^\mu e_b^\nu e_c^\rho e_\mu^i e_\nu^j &= \epsilon^{[abc]}e_a^\mu e_b^\nu e_c^\rho e_\mu^i e_\nu^j = \epsilon^{abc}e_{[a}^\mu e_b^\nu e_{c]}^\rho e_\mu^i e_\nu^j = \epsilon^{abc}e_a^{[\mu}e_b^\nu e_c^{\rho]}e_\mu^i e_\nu^j\\ &= \epsilon^{abc}e_{[a}^\mu e_b^\nu e_{c]}^\rho e_\mu^i e_\nu^j = \epsilon^{[abc]}e_a^\mu e_b^\nu e_c^\rho e_\mu^i e_\nu^j = \epsilon^{abc}e_a^\mu e_b^\nu e_c^\rho e_\mu^i e_\nu^j = \epsilon^{abc}\delta_a^i\delta_b^j e_c^\rho\end{aligned} \tag{629}$$

and

$$\begin{aligned}&\epsilon^{abc}e_a^{[\mu}e_b^\nu e_c^{\rho]}e_\mu^i e_\nu^j = \epsilon^{\mu\nu\rho}\epsilon^{abc}e_a^{[0}e_b^1 e_c^{2]}e_\mu^i e_\nu^j = \epsilon^{\mu\nu\rho}\epsilon^{abc}e_{[a}^0 e_b^1 e_{c]}^2 e_\mu^i e_\nu^j\\ &= \epsilon^{\mu\nu\rho}\epsilon^{[abc]}e_a^0 e_b^1 e_c^2 e_\mu^i e_\nu^j = \epsilon^{\mu\nu\rho}\epsilon^{abc}e_a^0 e_b^1 e_c^2 e_\mu^i e_\nu^j = \epsilon^{\mu\nu\rho}\det(e)e_\mu^i e_\nu^j\end{aligned} \tag{630}$$

so

$$\epsilon^{ijc}e_c^\rho = \epsilon^{\mu\nu\rho}\det(e)e_\mu^i e_\nu^j \tag{631}$$

or

$$\epsilon_{ijc}e_\rho^c = \epsilon_{\mu\nu\rho}\det(e^{-1})e_i^\mu e_j^\nu \tag{632}$$

For any given anti-symmetric tensor $F^{ab} = \frac{1}{2}F_{\mu\nu}^{ab}dx^\mu\wedge dx^\nu$, applying the above results, we have the following

$$\begin{aligned}&\frac{1}{2}\epsilon^{\alpha\beta\gamma}\epsilon_{abc}e_\alpha^a F_{\beta\gamma}^{bc} = \frac{1}{2}\epsilon^{\alpha\beta\gamma}F_{\beta\gamma}^{bc}\epsilon_{\mu\nu\alpha}\det(e^{-1})e_b^\mu e_c^\nu = \frac{1}{2}\delta_\mu^{[\beta}\delta_\nu^{\gamma]}\det(e^{-1})e_b^{[\beta}e_c^{\gamma]}F_{\beta\gamma}^{bc}\\ &= \det(e^{-1})e_b^{[\beta}e_c^{\gamma]}F_{\beta\gamma}^{bc} = \det(e^{-1})e_b^\beta e_b^\gamma F_{\beta\gamma}^{bc}\end{aligned} \tag{633}$$

From the definition of spin-connection, it can easily be proved that

$$\det(g) = -\det{}^2(e^{-1}), \quad F^{ab}_{\mu\nu} = e^{a\rho}e^{b\sigma}R_{\mu\nu\rho\sigma}$$

Plugging these two identities into Einstein-Hilbert action. we have

$$\begin{aligned}
&\int_M d^3x\sqrt{g}R = \int_M d^3x\sqrt{g}g^{\mu\nu}R_{\mu\nu} = \int_M d^3x \det(e^{-1})e^\mu_a e^{\nu a}R_{\mu\rho\nu\sigma}e^\rho_b e^{\sigma b} \\
&= \int_M d^3x \det(e^{-1})e^\mu_a F^{ab}_{\mu\rho}e^\rho_b = \frac{1}{2}\int_M d^3x\epsilon^{\alpha\beta\gamma}\epsilon_{abc}e^a_\alpha F^{bc}_{\beta\gamma} \\
&= \frac{1}{2}\int_M \epsilon^{\alpha\beta\gamma}\epsilon_{abc}e^a_\alpha F^{bc}_{\beta\gamma}dx^0\wedge dx^1\wedge dx^2 \\
&= \frac{1}{2}\int_M \epsilon_{abc}e^a_\alpha F^{bc}_{\beta\gamma}dx^\alpha\wedge dx^\beta\wedge dx^\gamma = \epsilon_{abc}\int_M e^a\wedge F^{bc}
\end{aligned} \tag{634}$$

Variation:

$$\begin{aligned}
\delta I &= \delta\left\{\epsilon_{abc}\int_M e^a\wedge F^{bc}\right\} = \epsilon_{abc}\delta\int_M e^a\wedge\left(d\omega^{bc}+\omega^b{}_d\wedge\omega^{dc}\right) \\
&= \epsilon_{abc}\int_M \delta e^a\wedge\left(d\omega^{bc}+\omega^b{}_d\wedge\omega^{dc}\right) + \epsilon_{abc}\int_M e^a\wedge\left(d\delta\omega^{bc}+\delta\omega^b{}_d\wedge\omega^{dc}+\omega^b{}_d\wedge\delta\omega^{dc}\right)
\end{aligned} \tag{635}$$

Using integration by parts, together with stokes theorem on the second term above, and assuming that the field e and ω vanishes at infinity, we have

$$\begin{aligned}
\delta I &= \epsilon_{abc}\int_M \delta e^a\wedge\left(d\omega^{bc}+\omega^b{}_d\wedge\omega^{dc}\right) + \epsilon_{abc}\int_M\left\{de^a\wedge\delta\omega^{bc} - e^a\wedge\omega_d{}^c\wedge\delta\omega^{bd} + e^a\wedge\omega^b{}_d\wedge\delta\omega^{dc}\right\} \\
&= \epsilon_{abc}\int_M \delta e^a\wedge\left(d\omega^{bc}+\omega^b{}_d\wedge\omega^{dc}\right) + \epsilon_{abc}\int_M\left\{de^a\wedge\delta\omega^{bc} + e^a\wedge\left(\omega^b{}_d\wedge\delta\omega^{dc} - \omega_d{}^b\wedge\delta\omega^{dc}\right)\right\} \\
&= \epsilon_{abc}\int_M \delta e^a\wedge\left(d\omega^{bc}+\omega^b{}_d\wedge\omega^{dc}\right) + \epsilon_{abc}\int_M de^a\wedge\delta\omega^{bc} + \epsilon_a{}^b{}_c\int_M e^a\wedge\left(\omega_{bd}\wedge\delta\omega^{dc} - \omega_{db}\wedge\delta\omega^{dc}\right)
\end{aligned} \tag{636}$$

The fact that spin connection ω is $\mathfrak{so}(2,1)$-valued implies that its component $\omega^a{}_b$ is pseudo-antisymmetric. i.e. $\eta_{ac}\omega^a{}_b = \omega_{cb} = -\eta_{bd}\omega^d{}_c = -\omega_{bc}$. So we have

$$\begin{aligned}
\delta I &= \epsilon_{abc}\int_M \delta e^a\wedge\left(d\omega^{bc}+\omega^b{}_d\wedge\omega^{dc}\right) + \epsilon_{abc}\int_M de^a\wedge\delta\omega^{bc} + 2\epsilon^{ab}{}_c\int_M e_a\wedge\omega_{bd}\wedge\delta\omega^{dc} \\
&= \epsilon_{abc}\int_M \delta e^a\wedge\left(d\omega^{bc}+\omega^b{}_d\wedge\omega^{dc}\right) + \epsilon_{abc}\int_M de^a\wedge\delta\omega^{ec}\delta^b_e + 2\epsilon^{ab}{}_c\int_M e_a\wedge\omega_{bd}\wedge\delta\omega^{ec}\delta^d_e \\
&= \epsilon_{abc}\int_M \delta e^a\wedge\left(d\omega^{bc}+\omega^b{}_d\wedge\omega^{dc}\right) + \int_M\left\{\epsilon_{abc}de^a\delta^b_e + 2\epsilon^{ab}{}_c e_a\wedge\omega_{bd}\delta^d_e\right\}\wedge\delta\omega^{ec}
\end{aligned} \tag{637}$$

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