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HULLS OF $\mathbb{Z}_p\mathbb{Z}_p[v]$-CYCLIC CODES AND CONSTRUCTION OF EAQECCS

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Abstract. In this paper, we study the hulls of $\mathbb{Z}_p\mathbb{Z}_p[v]$- cyclic codes for any prime p, where $v^2 = v$. Firstly, we derive the generator polynomials of the hulls of separable cyclic codes (SCCs) over $\mathbb{Z}_p\mathbb{Z}_p[v]$ and determine their dimensions using the concept of a generating function in combinatorics. We also enumerate the SCCs over $\mathbb{Z}_p\mathbb{Z}_p[v]$ with hulls of a fixed dimension. Next, we present the generator polynomials of the hulls of non-separable cyclic codes (NSCCs) over $\mathbb{Z}_2\mathbb{Z}_2[v]$ with coprime odd block lengths and determine their dimensions. Additionally, we enumerate the NSCCs over $\mathbb{Z}_2\mathbb{Z}_2[v]$ with hulls of a fixed dimension. As an application, we use the hulls of NSCCs over $\mathbb{Z}_2\mathbb{Z}_2[v]$ to construct some good entanglement-assisted quantum error-correcting codes (EAQECCs).

1. **Introduction.** To categorize finite projective planes, the idea of the hull of a linear code, which is defined as the intersection of the code and its dual, was first presented in [3]. Since then, the properties of the hulls of linear codes have been thoroughly examined. Also, the dimension of the hull plays a crucial role in determining the complexity of algorithms used to calculate permutation equivalence between two linear codes and compute the automorphism of a given linear code (discussed in [18, 19, 20, 25, 27]). Recently, the hulls of linear codes have also been used in the construction of good EAQECCs, as detailed in [1, 10, 22, 30]. Consequently, the study of hulls and dimensions of the hulls of linear codes over finite fields has gained significant interest. In 1997, Sendrier [26] identified the number of distinct linear codes with hulls of a given dimension. Skersys [29] later derived an expression for the average hull dimension of cyclic codes. Subsequently, in 2018, Jitman and Sangwisut [15] explored the average dimension of the Hermitian hulls of constacyclic codes. Recently, Sangwisut et al. [24] determined dimensions of the Euclidean hull of cyclic and negacyclic codes over $\mathbb{F}_q$ and calculated the number of cyclic codes having Euclidean hulls of a fixed dimension. The average hull dimension of negacyclic codes over $\mathbb{F}_q$ has also been investigated.

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In 2018, Borges et al. [4] conducted the study of double cyclic codes over $\mathbb{Z}_2$. Their study involved treating $\mathbb{Z}_2$-double cyclic codes as $\mathbb{Z}_2[x]$–submodules of $\frac{\mathbb{Z}_2[x]}{\langle x^r-1\rangle} \times \frac{\mathbb{Z}_2[x]}{\langle x^s-1\rangle}$. They successfully determined generating sets and identified numerous optimal codes within this framework. Similarly, Gao et al. [11] thoroughly examined the structural characteristics of double cyclic codes over $\mathbb{Z}_4$. Their research focused on generator polynomials, minimum spanning sets, and the derivation of corresponding dual codes. Furthermore, they found several non-linear optimal codes over $\mathbb{Z}_2$. In 2020, Diao et al. [9] studied the additive-cyclic codes over mixed alphabet $\mathbb{Z}_p\mathbb{Z}_p[v]$. Their study determined the generator polynomials of cyclic codes and their Euclidean dual codes.

Recently, Gao et al. [12] conducted a pioneering study on the hulls of double cyclic codes over $\mathbb{Z}_2$. In their study, they determined the generator polynomials of hulls in both separable and non-separable cases, provided explicit solutions for the hulls of double cyclic codes, and enumerated cyclic codes with hulls of fixed dimensions. Additionally, they constructed efficient EAQECCs using the Euclidean hulls of double cyclic codes over $\mathbb{Z}_2$. Notably, before their research, no studies had investigated the hulls of cyclic codes over mixed alphabets. Inspired by the foundational insights from [9] and [12], we see an opportunity to explore the hulls of $\mathbb{Z}_p\mathbb{Z}_p[v]$-cyclic codes. In this work, we address these gaps by systematically studying the Euclidean hulls of $\mathbb{Z}_p\mathbb{Z}_p[v]$-cyclic codes. Our main contributions are as follows:

- We derive the generator polynomials of the hulls of separable cyclic codes (SCCs) over $\mathbb{Z}_p\mathbb{Z}_p[v]$ and determine their dimensions using generating functions from combinatorics. Additionally, we enumerate SCCs with hulls of a fixed dimension.
- We determine the generator polynomials of the hulls of non-separable cyclic codes (NSCCs) over $\mathbb{Z}_2\mathbb{Z}_2[v]$ with coprime odd block lengths, compute their dimensions, and count the number of NSCCs with hulls of a fixed dimension.
- We construct entanglement-assisted quantum error-correcting codes (EAQECCs) using the hulls of NSCCs over $\mathbb{Z}_2\mathbb{Z}_2[v]$. The constructed EAQECCs demonstrate improved parameters compared to existing codes in the literature.

This paper is structured as follows. Section 2 provides essential background on cyclic codes over $\mathbb{Z}_p$ and $\mathbb{Z}_p\mathbb{Z}_p[v]$, setting the foundation for our study. Section 3 delves into the Euclidean hulls of separable cyclic codes (SCCs) over $\mathbb{Z}_p\mathbb{Z}_p[v]$, where we determine their generator polynomials, dimensions, and enumerate SCCs with hulls of a fixed dimension. In Section 4, we turn our attention to the Euclidean hulls of non-separable cyclic codes (NSCCs) over $\mathbb{Z}_2\mathbb{Z}_2[v]$ with coprime odd block lengths. This section explores their generator polynomials, dimensions, and the enumeration of hulls with a fixed dimension. Section 5 presents a key application of our findings: using the hulls of NSCCs over $\mathbb{Z}_2\mathbb{Z}_2[v]$ to construct efficient entanglement-assisted quantum error-correcting codes (EAQECCs). Our constructions yield codes with better parameters compared to existing EAQECCs in the literature. Finally, Section 6 concludes the paper.

2. **Preliminary.** This section reviews some basic results and prior findings on cyclic codes over $\mathbb{Z}_p$ and $\mathbb{Z}_p\mathbb{Z}_p[v]$. For detailed information about $\mathbb{Z}_p\mathbb{Z}_p[v]$-cyclic codes, we refer to [9].

2.1. $\mathbb{Z}_p$-Cyclic codes.

Definition 2.1. A linear code $\mathfrak{D}$ of length n over $\mathbb{Z}_p$ is called a cyclic code if it satisfies the property

$$(d_0, d_1, \ldots, d_{n-1}) \in \mathfrak{D} \text{ implying } (d_{n-1}, d_0, \ldots, d_{n-2}) \in \mathfrak{D}.$$

The dual code $\mathfrak{D}^\perp$ with respect to the Euclidean inner product is defined to be

$$\mathfrak{D}^\perp = \{ \ \boldsymbol{d} \in \mathfrak{D}^\perp \mid \boldsymbol{c} \cdot \boldsymbol{d} = 0 \ \forall \ \boldsymbol{c} \in \mathfrak{D}\}.$$

The code $\mathfrak{D}^\perp$ is a cyclic code if $\mathfrak{D}$ is a cyclic code. The hull of $\mathfrak{D}$ with respect to the Euclidean inner product is defined as

$$\mathrm{Hull}(\mathfrak{D}) = \mathfrak{D} \cap \mathfrak{D}^\perp.$$

Let $a(x) = a_0 + a_1x + \cdots + a_{\lambda-1}x^{\lambda-1} + x^\lambda$ be a polynomial of degree λ in $\mathbb{Z}_p$, provided $(a_0 \neq 0)$. The polynomial $a^*(x) = a_0^{-1}x^{\deg a(x)}a(\frac{1}{x})$ is called the reciprocal polynomial of $a(x)$. By definition it is clear that $\deg(a(x)) = \deg(a^*(x))$. Moreover, $a(x)$ is called a self-reciprocal polynomial if $a(x) = a^*(x)$.

Theorem 2.2. [14] *Let $\mathfrak{D}_1 = \langle d_1(x)\rangle$ and $\mathfrak{D}_2 = \langle d_2(x)\rangle$ be two cyclic codes of length n over $\mathbb{Z}_p$. Then $\mathfrak{D}_1 \cap \mathfrak{D}_2 = \langle \mathrm{lcm}(d_1(x), d_2(x))\rangle$.*

Let $R = \mathbb{Z}_p + v\mathbb{Z}_p = \{c + vd \mid c, d \in \mathbb{Z}_p\}$ with $v^2 = v$ has p^2 elements. Let

$$\mathbb{Z}_pR = \{(b, r) \mid b \in \mathbb{Z}_p, \ r \in R\}.$$

Any arbitrary element $g \in R$ can be written as $g = \xi_1a_1 + \xi_2a_2$, where $a_1, a_2 \in \mathbb{Z}_p$, $\xi_1 = v$ and $\xi_2 = 1 - v$ and $R = \xi_1\mathbb{Z}_p \oplus \xi_2\mathbb{Z}_p$ (see [2]). Let $\eta : R \to \mathbb{Z}_p$ be defined by

$$\eta(\xi_1a_1 + \xi_2a_2) = a_2.$$

Using map η, the scalar multiplication of R on $\mathbb{Z}_p\mathbb{Z}_p[v]$ is defined as

$$g \star (b, r) = (\eta(g)b, gr).$$

This scalar multiplication $\star$ can be extended over $\mathbb{Z}_p^\lambda \times R^\gamma$ as

$$g \star \mathbf{x} = (\eta(g)b_0, \eta(g)b_1, \ldots, \eta(g)b_{\lambda-1}, gr_0, gr_1, \ldots . gr_{\gamma-1}),$$

where $\mathbf{x} = (b_0, b_1, \ldots, b_{\lambda-1}, r_0, r_1, \ldots, r_{\gamma-1}) \in \mathbb{Z}_p^\lambda \times R^\gamma$. Indeed, the multiplication $\star$ endows $\mathbb{Z}_p^\lambda \times R^\gamma$ with the structure of an R-module.

Definition 2.3. A non-empty R-submodule $\mathfrak{C}$ of $\mathbb{Z}_p^\lambda \times R^\gamma$ is called a $\mathbb{Z}_p\mathbb{Z}_p[v]$-linear code of length (λ, γ).

Let $\phi : R \to \mathbb{Z}_p^2$ be a Gray map defined by $\phi(r) = (a_1, a_2)M$, where $r = \xi_1a_1 + \xi_2a_2 \in R$ and $\mathrm{M} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. Note that this Gray map works only when p is an odd prime. In case of $p = 2$, we define $\phi(r) = (a_1, a_2)$, where $r = \xi_1a_1 + \xi_2a_2 \in \mathbb{Z}_2[v]$. Now, we define a naturally extended Gray map $\Phi : \mathbb{Z}_p^\lambda \times R^\gamma \to \mathbb{Z}_p^{\lambda+2\gamma}$ as

$$\Phi(\boldsymbol{b}, \boldsymbol{r}) = (\boldsymbol{b}, \phi(\boldsymbol{r})),$$

where $\boldsymbol{b} = (b_0, b_1, \ldots, b_{\lambda-1}) \in \mathbb{Z}_p^\lambda$, $\boldsymbol{r} = (r_0, r_1, \ldots, r_{\gamma-1}) \in R^\gamma$ and $\phi(\boldsymbol{r}) = (\phi(r_0), \phi(r_1), \ldots, \phi(r_{n-1}))$. It is clear that the map Φ is $\mathbb{Z}_p$-linear. From this, we have the following result.

Proposition 2.4. *Let $\mathfrak{C}$ be a $\mathbb{Z}_p\mathbb{Z}_p[v]$-linear code of length (λ, γ) with $|\mathfrak{C}| = p^k$. Then $\Phi(\mathfrak{C})$ is a $\mathbb{Z}_p$-linear code with parameters $[\lambda + 2\gamma, k, d_H]$.*

Definition 2.5. A linear code $\mathfrak{C}$ of length (λ, γ) over $\mathbb{Z}_p\mathbb{Z}_p[v]$ is a cyclic code if it satisfies the following condition:

$(b_0, b_1, \ldots, b_{\lambda-1}, r_0, r_1, \ldots, r_{\gamma-1}) \in \mathfrak{C} \Longrightarrow (b_{\lambda-1}, b_0, \ldots, b_{\lambda-2}, r_{\gamma-1}, r_0, \ldots, r_{\gamma-2}) \in \mathfrak{C}$.

Let $\mathfrak{R}_{\lambda,\gamma} = \frac{\mathbb{Z}_p[x]}{\langle x^\lambda - 1\rangle} \times \frac{R[x]}{\langle x^\gamma - 1\rangle}$. Every element in $\mathbb{Z}_p^\lambda \times R^\gamma$ can be identified by using a pair of polynomials in $\mathfrak{R}_{\lambda,\gamma}$ as follows:

$$\begin{aligned}(b_0, b_1, \ldots, b_{\lambda-1}, r_0, r_1, \ldots, r_{\gamma-1}) \rightarrow &(b_0 + b_1x + \cdots + b_{\lambda-1}x^{\lambda-1},\\ &r_0 + r_1x + \cdots + r_{\gamma-1}x^{\gamma-1}) = (b(x), r(x)),\end{aligned}$$

where $b(x) \in \frac{\mathbb{Z}_p[x]}{\langle x^\lambda - 1\rangle}$ and $r(x) \in \frac{R[x]}{\langle x^\gamma - 1\rangle}$, and this is called polynomial representation. This shows that there is an R-module isomorphism between $\mathbb{Z}_p^\lambda \times R^\gamma$ and $\mathfrak{R}_{\lambda,\gamma}$. Moreover, $\mathfrak{R}_{\lambda,\gamma}$ forms an $R[x]$-module. Therefore, any linear code $\mathfrak{C}$ over $\mathbb{Z}_p\mathbb{Z}_p[v]$ is a cyclic code if and only if the polynomial representation of $\mathfrak{C}$ is an $R[x]$-submodule of $\mathfrak{R}_{\lambda,\gamma}$.

We assume $\mathfrak{m} = \mathrm{lcm}(\lambda, \gamma)$ for the rest of this paper. Let $c(x) = (b(x), r(x))$, $c'(x) = (b'(x), r'(x)) \in \mathfrak{R}_{\lambda,\gamma}$. Define a map

$$o : \mathfrak{R}_{\lambda,\gamma} \times \mathfrak{R}_{\lambda,\gamma} \rightarrow \frac{R[x]}{\langle x^{\mathfrak{m}} - 1\rangle},$$

by:

$$\begin{aligned}c(x)\ o\ c'(x) = \xi_2 b(x) x^{\mathfrak{m}-1-\deg(b'(x))} \frac{x^{\mathfrak{m}} - 1}{x^\lambda - 1}(b'^*(x))\\ + r(x)x^{\mathfrak{m}-1-\deg(r'(x))}\frac{x^{\mathfrak{m}} - 1}{x^\gamma - 1}(r'^*(x)) \mod (x^{\mathfrak{m}} - 1).\end{aligned}$$

Theorem 2.6. [9, Lemma 3.3-Lemma 3.6] *Let $\mathfrak{C}$ be a $\mathbb{Z}_p\mathbb{Z}_p[v]$-cyclic code of length (λ, γ). Then*

$$\mathfrak{C} = \langle (m(x) \mid 0), (k(x) \mid \xi_1 l_1(x) + \xi_2 l_2(x))\rangle,$$

where $m(x), k(x) \in \frac{\mathbb{Z}_p[x]}{\langle x^\lambda - 1\rangle}$, and $l_1(x), l_2(x) \in \frac{R[x]}{\langle x^\gamma - 1\rangle}$ with $m(x), k(x) \mid x^\lambda - 1$ and $l_1(x), l_2(x) \mid x^\gamma - 1$. Moreover, $\deg(m(x)) > \deg(k(x))$, $m(x) \mid l_2(x)k(x)$ and $m(x) \mid l_2(x) \gcd(m(x), k(x))$.

Let $\mathfrak{C}_\lambda$ and $\mathfrak{C}_\gamma$ be the canonical projections of the first λ and last γ coordinates, respectively. Then $\mathfrak{C}_\lambda = \langle \gcd(m(x), k(x))\rangle$ and $\mathfrak{C}_\gamma = \langle \xi_1 l_1(x) + \xi_2 l_2(x)\rangle$ are cyclic codes over $\mathbb{Z}_p$ and R, respectively. Moreover, if $\mathfrak{C} = \mathfrak{C}_\lambda \times \mathfrak{C}_\gamma$, then $\mathfrak{C}$ is called a SCC over $\mathbb{Z}_p\mathbb{Z}_p[v]$.

Corollary 2.7. [9] *Let $\mathfrak{C}$ be a SCC over $\mathbb{Z}_p\mathbb{Z}_p[v]$. Then $k(x) = 0$.*

In [9], it has been proved that the dual code $\mathfrak{C}^\perp$ of a cyclic code $\mathfrak{C}$ is also cyclic. Therefore, $\mathfrak{C}^\perp = \langle (\bar{m}(x) \mid 0), (\bar{k}(x) \mid \xi_1\bar{l}_1(x) + \xi_2\bar{l}_2(x))\rangle$, where $\bar{m}(x), \bar{k}(x) \in \frac{\mathbb{Z}_p[x]}{\langle x^\lambda - 1\rangle}$, and
$\bar{l}_1(x), \bar{l}_2(x) \in \frac{R[x]}{\langle x^\gamma - 1\rangle}$ with $\bar{m}(x), \bar{k}(x) \mid x^\lambda - 1$ and $\bar{l}_1(x), \bar{l}_2(x) \mid x^\gamma - 1$.

Theorem 2.8. [9, Theorem 4.5] *Suppose* $\mathfrak{C} = \langle (m(x) \mid 0), (k(x) \mid \xi_1 l_1(x) + \xi_2 l_2(x)) \rangle$ *is a* $\mathbb{Z}_p\mathbb{Z}_p[v]$*-cyclic code, and* $\mathfrak{C}^\perp = \langle (\bar{m}(x) \mid 0), (\bar{k}(x) \mid \xi_1 \bar{l}_1(x) + \xi_2 \bar{l}_2(x)) \rangle$ *is its dual code. Then*

$$\bar{m}(x) = \frac{x^\lambda - 1}{\gcd(m(x), k(x))^*}$$
$$\bar{l}_1(x) = \frac{x^\gamma - 1}{l_1^*(x)}$$
$$\bar{l}_2(x) = \frac{(x^\gamma - 1)\gcd(m(x), k(x))^*}{m^*(x) l_2^*(x)}$$
$$\bar{k}(x) = \frac{(x^\lambda - 1)\mu(x)}{m^*(x)},$$

where $\mu(x) = x^{\mathtt{m} - \deg(l_2(x)) + \deg(k(x))} \left(\frac{k^*(x)}{\gcd(m(x), k(x))^*} \right)^{-1} \mod \left(\frac{m^*(x)}{\gcd(m(x), k(x))^*} \right)$.

Corollary 2.9. *Suppose* $\mathfrak{C} = \langle (m(x) \mid 0), (0 \mid \xi_1 l_1(x) + \xi_2 l_2(x)) \rangle$ *is a SCC over* $\mathbb{Z}_p\mathbb{Z}_p[v]$*. Then* $\mathfrak{C}^\perp = \langle (\bar{m}(x) \mid 0), (0 \mid \xi_1 \bar{l}_1(x) + \xi_2 \bar{l}_2(x)) \rangle$*, where*

$$\bar{m}(x) = \frac{x^\lambda - 1}{m^*(x)},\ \bar{l}_1(x) = \frac{x^\gamma - 1}{l_1^*(x)},\ \bar{l}_2(x) = \frac{x^\gamma - 1}{l_2^*(x)}.$$

Proof. The result follows immediately if we put $k(x) = 0$ in Theorem 2.8. □

2.3. **Factorizations of $x^\lambda - 1$ and $x^\gamma - 1$ over $\mathbb{Z}_p$.** Now, we give the complete factorizations of $x^\lambda - 1$ and $x^\gamma - 1$ over $\mathbb{Z}_p$, where λ and γ are positive integers. For more details, we refer to [24].

Let $\mathrm{ord}_m(n)$ denote the multiplicative order of n modulo m for any coprime positive integers m and n. We say a pair (m, n) is a good pair if there exists $k \in \mathbb{Z}^+$ such that $m \mid n^k + 1$. Otherwise, they are called a bad pair. Consider a function χ, which maps $\mathbb{N} \times \mathbb{N}$ to $\{0, 1\}$ as follows:

$$\chi(m, n) = \begin{cases} 1 \text{ if } & (m, n) \text{ is a bad pair.} \\ 0 \text{ if } & (m, n) \text{ is a good pair ,} \end{cases} \tag{1}$$

Suppose $\lambda = p^{s_1}\bar{\lambda}$ and $\gamma = p^{s_2}\bar{\gamma}$, where $\gcd(p, \bar{\lambda}) = 1 = \gcd(p, \bar{\gamma})$. Therefore, $x^\lambda - 1 = (x^{\bar{\lambda}} - 1)^{p^{s_1}}$ and $x^\gamma - 1 = (x^{\bar{\gamma}} - 1)^{p^{s_2}}$. Now, the factorizations of $x^{\bar{\lambda}} - 1$ and $x^{\bar{\gamma}} - 1$ are given by the following equations:

$$\begin{aligned} x^{\bar{\lambda}} - 1 &= \prod_{\substack{\omega \mid \bar{\lambda} \\ \chi(\omega, p) = 0}} \left(\prod_{e=1}^{\Psi_1(\omega, p)} h_{e\omega}(x) \right) \prod_{\substack{\omega \mid \bar{\lambda} \\ \chi(\omega, p) = 1}} \left(\prod_{e=1}^{\Phi_1(\omega, p)} t_{e\omega}(x) t_{e\omega}^*(x) \right) \\ &= h_1(x) h_2(x) \cdots h_{m_1}(x) t_1(x) t_1^*(x) t_2(x) t_2^*(x) \cdots t_{m_2}(x) t_{m_2}^*(x) \end{aligned} \tag{2}$$

$$\begin{aligned} x^{\bar{\gamma}} - 1 &= \prod_{\substack{\nu \mid \bar{\gamma} \\ \chi(\nu, p) = 0}} \left(\prod_{i=1}^{\Psi_2(\nu, p)} f_{i\nu}(x) \right) \prod_{\substack{\nu \mid \bar{\gamma} \\ \chi(\nu, p) = 1}} \left(\prod_{i=1}^{\Phi_2(\nu, p)} d_{i\nu}(x) d_{i\nu}^*(x) \right) \\ &= f_1(x) f_2(x) \cdots f_{k_1}(x) d_1(x) d_1^*(x) d_2(x) d_2^*(x) \cdots d_{k_2}(x) d_{k_2}^*(x), \end{aligned} \tag{3}$$

where

$$\Psi_1(\omega,p)=\frac{\phi(\omega)}{\operatorname{ord}_\omega(p)},\ \Phi_1(\omega,p)=\frac{\phi(\omega)}{2\operatorname{ord}_\omega(p)},\tag{4}$$

$$\Psi_2(\nu,p)=\frac{\phi(\nu)}{\operatorname{ord}_\nu(p)},\ \Phi_2(\nu,p)=\frac{\phi(\nu)}{2\operatorname{ord}_\nu(p)}.\tag{5}$$

Here $h_{e\omega}(x)$ and $f_{i\nu}(x)$ are monic irreducible self-reciprocal polynomials of degree $\operatorname{ord}_\omega(p)$ and $\operatorname{ord}_\nu(p)$, respectively, $t_{e\omega}(x)t^*_{e\omega}(x)$ and $d_{i\nu}(x)d^*_{i\nu}(x)$ are monic irreducible reciprocal polynomials pairs of degree $\operatorname{ord}_\omega(p)$ and $\operatorname{ord}_\nu(p)$, respectively. ϕ is the Euler function. Moreover, m_1 and k_1 denote the number of self-reciprocal irreducible polynomials in factorizations of $x^{\bar{\lambda}}-1$ and $x^{\bar{\gamma}}-1$ over $\mathbb{Z}_p$, respectively, m_2 and k_2 denote the number of irreducible reciprocal polynomials pairs in factorizations of $x^{\bar{\lambda}}-1$ and $x^{\bar{\gamma}}-1$ over $\mathbb{Z}_p$, respectively.

The complete factorizations of $x^{\lambda}-1$ and $x^{\gamma}-1$ are given by the following equations:

$$\begin{aligned}x^{\lambda}-1&=(x^{\bar{\lambda}}-1)^{p^{s_1}}\\&=\left(\prod_{\substack{\omega\mid\bar{\lambda}\\ \chi(\omega,p)=0}}\left(\prod_{e=1}^{\Psi_1(\omega,p)}h_{e\omega}(x)\right)\prod_{\substack{\omega\mid\bar{\lambda}\\ \chi(\omega,p)=1}}\left(\prod_{e=1}^{\Phi_1(\omega,p)}t_{e\omega}(x)t^*_{e\omega}(x)\right)\right)^{p^{s_1}}\end{aligned}\tag{6}$$

$$\begin{aligned}x^{\gamma}-1&=(x^{\bar{\gamma}}-1)^{p^{s_2}}\\&=\left(\prod_{\substack{\nu\mid\bar{\gamma}\\ \chi(\nu,p)=0}}\left(\prod_{i=1}^{\Psi_2(\nu,p)}f_{i\nu}(x)\right)\prod_{\substack{\nu\mid\bar{\gamma}\\ \chi(\nu,p)=1}}\left(\prod_{i=1}^{\Phi_2(\nu,p)}d_{i\nu}(x)d^*_{i\nu}(x)\right)\right)^{p^{s_2}}.\end{aligned}\tag{7}$$

In the next theorem, we give the values of m_1, m_2, k_1, and k_2 present in Eqs. (2) and (3).

Theorem 2.10. [12, Theorem 2.6] *Let Eqs.* (2) *and* (3) *be the factorizations of* $x^{\bar{\lambda}}-1$ *and* $x^{\bar{\gamma}}-1$. *Then*

$$m_1=\sum_{\omega\mid\bar{\lambda}}(1-\chi(\omega,p))\frac{\phi(\omega)}{\operatorname{ord}_\omega(p)},\ m_2=\frac{1}{2}\sum_{\omega\mid\bar{\lambda}}\chi(\omega,p)\frac{\phi(\omega)}{\operatorname{ord}_\omega(p)},$$

$$k_1=\sum_{\nu\mid\bar{\gamma}}(1-\chi(\nu,p))\frac{\phi(\nu)}{\operatorname{ord}_\nu(p)},\ k_2=\frac{1}{2}\sum_{\nu\mid\bar{\gamma}}\chi(\nu,p)\frac{\phi(\nu)}{\operatorname{ord}_\nu(p)}.$$

3. Euclidean hulls of SCCs over $\mathbb{Z}_p\mathbb{Z}_p[v]$.

3.1. Generator polynomials of hulls of SCCs over $\mathbb{Z}_p\mathbb{Z}_p[v]$. In this subsection, we give the generator polynomials of hulls of SCCs over $\mathbb{Z}_p\mathbb{Z}_p[v]$ and determine their dimensions. First, we give a result discussed by Tian et al. in [31].

Proposition 3.1. [31, Theorem 2] *Let* $\mathfrak{C}_\gamma=\langle\xi_1 l_1(x)+\xi_2 l_2(x)\rangle$ *be a cyclic code over* R *and* $\mathfrak{C}_\gamma^{\perp}=\langle\xi_1\bar{l}_1(x)+\xi_2\bar{l}_2(x)\rangle$ *be its dual. Then*

$$\operatorname{Hull}(\mathfrak{C}_\gamma)=\Big\langle\xi_1\operatorname{lcm}(l_1(x),\bar{l}_1(x))+\xi_2\operatorname{lcm}(l_2(x),\bar{l}_2(x))\Big\rangle.$$

Theorem 3.2. *Let* $\mathfrak{C} = \langle(m(x) \mid 0), (0 \mid \xi_1 l_1(x) + \xi_2 l_2(x))\rangle$ *and its dual* $\mathfrak{C}^\perp = \langle(\bar{m}(x) \mid 0), (0 \mid \xi_1 \bar{l}_1(x) + \xi_2 \bar{l}_2(x))\rangle$ *be SCCs in* $\mathfrak{R}_{\lambda,\gamma}$. *Then*

$$\mathrm{Hull}(\mathfrak{C}) = \Big\langle (\mathrm{lcm}(m(x), \bar{m}(x)) \mid 0)), (0 \mid \xi_1 \,\mathrm{lcm}(l_1(x), \bar{l}_1(x)) + \xi_2 \,\mathrm{lcm}(l_2(x), \bar{l}_2(x)))\Big\rangle.$$

Proof. Since $\mathfrak{C}$ is a SCC in $\mathfrak{R}_{\lambda,\gamma}$, therefore by Corollary 2.7, we have $k(x) = 0$. Thus $\mathfrak{C} = \langle(m(x) \mid 0), (0 \mid \xi_1 l_1(x)+\xi_2 l_2(x))\rangle$, and $\mathfrak{C}^\perp = \langle(\bar{m}(x) \mid 0), (0 \mid \xi_1 \bar{l}_1(x)+\xi_2 \bar{l}_2(x))\rangle$. By Theorem 2.2 and Proposition 3.1, we have

$$\begin{aligned}\mathrm{Hull}(\mathfrak{C}) = \Big\langle & (\mathrm{lcm}(m(x), \bar{m}(x)) \mid 0)), \\ & \big(0 \mid \xi_1 \,\mathrm{lcm}(l_1(x), \bar{l}_1(x)) + \xi_2 \,\mathrm{lcm}(l_2(x), \bar{l}_2(x))\big) \Big\rangle.\end{aligned}$$

□

The following corollary follows directly from the combination of Theorem 3.2 and Corollary 2.9.

Corollary 3.3. *Let* $\mathfrak{C} = \langle(m(x) \mid 0), (0 \mid \xi_1 l_1(x) + \xi_2 l_2(x))\rangle$ *and its dual* $\mathfrak{C}^\perp = \langle(\frac{x^\lambda-1}{m^*(x)} \mid 0), (0 \mid \xi_1 \frac{x^\gamma-1}{l_1^*(x)} + \xi_2 \frac{x^\gamma-1}{l_2^*(x)})\rangle$ *be SCCs in* $\mathfrak{R}_{\lambda,\gamma}$. *Then*

$$\begin{aligned}\mathrm{Hull}(\mathfrak{C}) = \Big\langle & \Big(\mathrm{lcm}(m(x), \frac{x^\lambda - 1}{m^*(x)}) \mid 0\Big), \Big(0 \mid \xi_1 \,\mathrm{lcm}(l_1(x), \frac{x^\gamma-1}{l_1^*(x)}) \\ & + \xi_2 \,\mathrm{lcm}(l_2(x), \frac{x^\gamma - 1}{l_2^*(x)}))\Big)\Big\rangle.\end{aligned}$$

Lemma 3.4. [16, Lemma 3.2] *Suppose s is a non-negative integer and $0 \le n_1, n_2, n_3 \le p^s$ are integers. Then, the following holds:*

1. $0 \le p^s - \max\{n_1, p^s - n_1\} \le p^{s-1}$
2. $0 \le p^{s+1} - \max\{n_2, p^s - n_3\} - \max\{n_3, p^s - n_2\} \le p^s$.

Now, we determine the dimensions of hulls of SCCs in $\mathfrak{R}_{\lambda,\gamma}$.

Theorem 3.5. *Suppose χ, Ψ_1, Φ_1, Ψ_2, and Φ_2 are the same as in Eqs.* (1), (4), *and* (5), *respectively. Then the dimensions of hulls of SCCs in $\mathfrak{R}_{\lambda,\gamma}$ are of the form*

$$\begin{aligned}&\sum_{\omega \mid \bar{\lambda}} (1 - \chi(\omega, p)) \,\mathrm{ord}_\omega(p) u_\omega + \sum_{\omega \mid \bar{\lambda}} \chi(\omega, p) \,\mathrm{ord}_\omega(p) v_\omega \\ &\quad + \sum_{\nu \mid \bar{\gamma}} (1 - \chi(\nu, p)) \,\mathrm{ord}_\nu(p)[x_\nu + z_\nu] + \sum_{\nu \mid \bar{\gamma}} \chi(\nu, p) \,\mathrm{ord}_\nu(p)[y_\nu + w_\nu],\end{aligned} \tag{8}$$

where $0 \le u_\omega \le \Psi_1(\omega, p)\lfloor \frac{p^{s_1}}{2} \rfloor$, $0 \le v_\omega \le \Phi_1(\omega, p) p^{s_1}$, $0 \le x_\nu, z_\nu \le \Psi_2(\nu, p)\lfloor \frac{p^{s_2}}{2} \rfloor$, *and* $0 \le v_\nu, w_\nu \le \Phi_2(\nu, p) p^{s_2}$.

Proof. Since $\mathfrak{C}$ is SCC in $\mathfrak{R}_{\lambda,\gamma}$, then $\mathfrak{C} = \langle(m(x) \mid 0), (0 \mid \xi_1 l_1(x) + \xi_2 l_2(x))\rangle$. From Eqs. (6) and (7), we have

$$m(x) = \prod_{\substack{\omega \mid \bar{\lambda} \\ \chi(\omega,p)=0}} \left(\prod_{e=1}^{\Psi_1(\omega,p)} (h_{e\omega}(x))^{\alpha_{e\omega}} \right) \prod_{\substack{\omega \mid \bar{\lambda} \\ \chi(\omega,p)=1}} \left(\prod_{e=1}^{\Phi_1(\omega,p)} (t_{e\omega}(x))^{\beta_{e\omega}} (t^*_{e\omega}(x))^{\eta_{e\omega}} \right), \tag{9}$$

$$l_1(x) = \prod_{\substack{\nu \mid \bar{\gamma} \\ \chi(\nu,p)=0}} \left(\prod_{i=1}^{\Psi_2(\nu,p)} (f_{i\nu}(x))^{\delta_{i\nu}} \right) \prod_{\substack{\nu \mid \bar{\gamma} \\ \chi(\nu,p)=1}} \left(\prod_{i=1}^{\Phi_2(\nu,p)} (d_{i\nu}(x))^{\tau_{i\nu}} (d^*_{i\nu}(x))^{\rho_{i\nu}} \right), \tag{10}$$

$$l_2(x) = \prod_{\substack{\nu|\bar{\gamma}\\ \chi(\nu,p)=0}} \left(\prod_{i=1}^{\Psi_2(\nu,p)} (f_{i\nu}(x))^{\sigma_{i\nu}} \right) \prod_{\substack{\nu|\bar{\gamma}\\ \chi(\nu,p)=1}} \left(\prod_{i=1}^{\Phi_2(\nu,p)} (d_{i\nu}(x))^{\xi_{i\nu}} (d^*_{i\nu}(x))^{\Delta_{i\nu}} \right), \tag{11}$$

where $0 \le \alpha_{e\omega}, \beta_{e\omega}, \eta_{e,\omega} \le p^{s_1}$ and $0 \le \delta_{i\nu}, \tau_{i\nu}, \rho_{i\nu}, \sigma_{i\nu}, \xi_{i\nu}, \Delta_{i\nu} \le p^{s_2}$. Then

$$m^*(x) = \prod_{\substack{\omega|\bar{\lambda}\\ \chi(\omega,p)=0}} \left(\prod_{e=1}^{\Psi_1(\omega,p)} (h_{e\omega}(x))^{\alpha_{e\omega}} \right) \prod_{\substack{\omega|\bar{\lambda}\\ \chi(\omega,p)=1}} \left(\prod_{e=1}^{\Phi_1(\omega,p)} (t_{e\omega}(x))^{\eta_{e\omega}} (t^*_{e\omega}(x))^{\beta_{e\omega}} \right),$$

$$l_1^*(x) = \prod_{\substack{\nu|\bar{\gamma}\\ \chi(\nu,p)=0}} \left(\prod_{i=1}^{\Psi_2(\nu,p)} (f_{i\nu}(x))^{\delta_{i\nu}} \right) \prod_{\substack{\nu|\bar{\gamma}\\ \chi(\nu,p)=1}} \left(\prod_{i=1}^{\Phi_2(\nu,p)} (d_{i\nu}(x))^{\rho_{i\nu}} (d^*_{i\nu}(x))^{\tau_{i\nu}} \right),$$

$$l_2^*(x) = \prod_{\substack{\nu|\bar{\gamma}\\ \chi(\nu,p)=0}} \left(\prod_{i=1}^{\Psi_2(\nu,p)} (f_{i\nu}(x))^{\sigma_{i\nu}} \right) \prod_{\substack{\nu|\bar{\gamma}\\ \chi(\nu,p)=1}} \left(\prod_{i=1}^{\Phi_2(\nu,p)} (d_{i\nu}(x))^{\Delta_{i\nu}} (d^*_{i\nu}(x))^{\xi_{i\nu}} \right).$$

Hence,

$$\frac{x^\lambda - 1}{m^*(x)} = \prod_{\substack{\omega|\bar{\lambda}\\ \chi(\omega,p)=0}} \left(\prod_{e=1}^{\Psi_1(\omega,p)} (h_{e\omega}(x))^{p^{s_1}-\alpha_{e\omega}} \right) \prod_{\substack{\omega|\bar{\lambda}\\ \chi(\omega,p)=1}} \left(\prod_{e=1}^{\Phi_1(\omega,p)} (t_{e\omega}(x))^{p^{s_1}-\eta_{e\omega}} (t^*_{e\omega}(x))^{p^{s_1}-\beta_{e\omega}} \right)$$

$$\frac{x^\gamma - 1}{l_1^*(x)} = \prod_{\substack{\nu|\bar{\gamma}\\ \chi(\nu,p)=0}} \left(\prod_{i=1}^{\Psi_2(\nu,p)} (f_{i\nu}(x))^{p^{s_2}-\delta_{i\nu}} \right) \prod_{\substack{\nu|\bar{\gamma}\\ \chi(\nu,p)=1}} \left(\prod_{i=1}^{\Phi_2(\nu,p)} (d_{i\nu}(x))^{p^{s_2}-\rho_{i\nu}} (d^*_{i\nu}(x))^{p^{s_2}-\tau_{i\nu}} \right),$$

$$\frac{x^\gamma - 1}{l_2^*(x)} = \prod_{\substack{\nu|\bar{\gamma}\\ \chi(\nu,p)=0}} \left(\prod_{i=1}^{\Psi_2(\nu,p)} (f_{i\nu}(x))^{p^{s_2}-\sigma_{i\nu}} \right) \prod_{\substack{\nu|\bar{\gamma}\\ \chi(\nu,p)=1}} \left(\prod_{i=1}^{\Phi_2(\nu,p)} (d_{i\nu}(x))^{p^{s_2}-\Delta_{i\nu}} (d^*_{i\nu}(x))^{p^{s_2}-\xi_{i\nu}} \right).$$

Now,

$$\begin{aligned} \operatorname{lcm}\left(m(x), \frac{x^\lambda - 1}{m^*(x)}\right) &= \prod_{\substack{\omega|\bar{\lambda}\\ \chi(\omega,p)=0}} \left(\prod_{e=1}^{\Psi_1(\omega,p)} (h_{e\omega}(x))^{\max\{\alpha_{e\omega}, p^{s_1}-\alpha_{e\omega}\}} \right) \\ &\times \prod_{\substack{\omega|\bar{\lambda}\\ \chi(\omega,p)=1}} \left(\prod_{e=1}^{\Phi_1(\omega,p)} (t_{e\omega}(x))^{\max\{\beta_{e\omega}, p^{s_1}-\eta_{e\omega}\}} (t^*_{e\omega}(x))^{\max\{\eta_{e\omega}, p^{s_1}-\beta_{e\omega}\}} \right), \end{aligned} \tag{12}$$

$$\begin{aligned}\operatorname{lcm}\left(l_1(x), \frac{x^\gamma-1}{l_1^*(x)}\right) &= \prod_{\substack{\nu\mid\bar{\gamma}\\ \chi(\nu,p)=0}} \left(\prod_{i=1}^{\Psi_2(\nu,p)} (f_{i\nu}(x))^{\max\{\delta_{i\nu},p^{s_2}-\delta_{i\nu}\}}\right)\\ &\times \prod_{\substack{\nu\mid\bar{\gamma}\\ \chi(\nu,p)=1}} \left(\prod_{i=1}^{\Phi_2(\nu,p)} (d_{i\nu}(x))^{\max\{\tau_{i\nu},p^{s_2}-\rho_{i\nu}\}}(d^*_{i\nu}(x))^{\max\{\rho_{i\nu},p^{s_2}-\tau_{i\nu}\}}\right),\end{aligned}\tag{13}$$

$$\begin{aligned}\operatorname{lcm}\left(l_2(x), \frac{x^\gamma-1}{l_2^*(x)}\right) &= \prod_{\substack{\nu\mid\bar{\gamma}\\ \chi(\nu,p)=0}} \left(\prod_{i=1}^{\Psi_2(\nu,p)} (f_{i\nu}(x))^{\max\{\sigma_{i\nu},p^{s_2}-\sigma_{i\nu}\}}\right)\\ &\times \prod_{\substack{\nu\mid\bar{\gamma}\\ \chi(\nu,p)=1}} \left(\prod_{i=1}^{\Phi_2(\nu,p)} (d_{i\nu}(x))^{\max\{\xi_{i\nu},p^{s_2}-\Delta_{i\nu}\}}(d^*_{i\nu}(x))^{\max\{\Delta_{i\nu},p^{s_2}-\xi_{i\nu}\}}\right),\end{aligned}\tag{14}$$

where

$$\begin{aligned}\left\lfloor\frac{p^{s_1}}{2}\right\rfloor \le \max\{\alpha_{e\omega},p^{s_1}-\alpha_{e\omega}\} \le p^{s_1},\ p^{s_1} \le& \max\{\beta_{e\omega},p^{s_1}-\eta_{e\omega}\}\\ &+\max\{\eta_{e\omega},p^{s_1}-\beta_{e\omega}\}\le 2p^{s_1},\end{aligned}\tag{15}$$

$$\begin{aligned}\left\lfloor\frac{p^{s_2}}{2}\right\rfloor \le \max\{\delta_{i\nu},p^{s_2}-\delta_{i\nu}\} \le p^{s_2},\ p^{s_2} \le& \max\{\tau_{i\nu},p^{s_2}-\rho_{i\nu}\}\\ &+\max\{\rho_{i\nu},p^{s_2}-\tau_{i\nu}\}\le 2p^{s_2},\end{aligned}\tag{16}$$

$$\begin{aligned}\left\lfloor\frac{p^{s_2}}{2}\right\rfloor \le \max\{\sigma_{i\nu},p^{s_2}-\sigma_{i\nu}\} \le p^{s_2},\ p^{s_2} \le& \max\{\xi_{i\nu},p^{s_2}-\Delta_{i\nu}\}\\ &+\max\{\Delta_{i\nu},p^{s_2}-\xi_{i\nu}\}\le 2p^{s_2}.\end{aligned}\tag{17}$$

Since $\mathrm{Hull}(\mathfrak{C}) = \left\langle \left(\operatorname{lcm}(m(x), \frac{x^\lambda-1}{m^*(x)}) \mid 0\right), \left(0 \mid \xi_1 \operatorname{lcm}(l_1(x), \frac{x^\gamma-1}{l_1^*(x)}) + \xi_2 \operatorname{lcm}(l_2(x), \frac{x^\gamma-1}{l_2^*(x)})\right)\right\rangle$. Therefore,

$$\begin{aligned}&\dim(\mathrm{Hull}(\mathfrak{C}))\\ &= \lambda - \deg\left(\operatorname{lcm}\left(m(x), \frac{x^\lambda-1}{m^*(x)}\right)\right) + 2\gamma - \left\{\deg\left(\operatorname{lcm}\left(l_1(x), \frac{x^\gamma-1}{l_1^*(x)}\right)\right)\right.\\ &\quad\left. + \deg\left(\operatorname{lcm}\left(l_2(x), \frac{x^\gamma-1}{l_2^*(x)}\right)\right)\right\}\\ &= \left(\sum_{\substack{\omega\mid\bar{\lambda}\\ \chi(\omega,p)=0}} \operatorname{ord}_\omega(p) \sum_{e=1}^{\Psi_1(\omega,p)} p^{s_1} + \sum_{\substack{\omega\mid\bar{\lambda}\\ \chi(\omega,p)=1}} \operatorname{ord}_\omega(p) \sum_{e=1}^{\Phi_1(\omega,p)} 2p^{s_1} - \sum_{\substack{\omega\mid\bar{\lambda}\\ \chi(\omega,p)=0}} \operatorname{ord}_\omega(p)\right.\\ &\quad \sum_{e=1}^{\Psi_1(\omega,p)} \max\{\alpha_{e\omega},p^{s_1}-\alpha_{e\omega}\} - \sum_{\substack{\omega\mid\bar{\lambda}\\ \chi(\omega,p)=1}} \operatorname{ord}_\omega(p) \sum_{e=1}^{\Phi_1(\omega,p)} (\max\{\beta_{e\omega},p^{s_1}-\eta_{e\omega}\}+\end{aligned}$$

$$\max\{\eta_{e\omega}, p^{s_1}-\beta_{e\omega}\})\Bigg)+\Bigg(\sum_{\substack{\nu|\bar{\gamma}\\ \chi(\nu,p)=0}}\operatorname{ord}_\nu(p)\sum_{i=1}^{\Psi_2(\nu,p)}p^{s_2}+\sum_{\substack{\nu|\bar{\gamma}\\ \chi(\nu,p)=1}}\operatorname{ord}_\nu(p)\sum_{i=1}^{\Phi_2(\nu,p)}2p^{s_2}$$
$$-\sum_{\substack{\nu|\bar{\gamma}\\ \chi(\nu,p)=0}}\operatorname{ord}_\nu(p)\sum_{i=1}^{\Psi_2(\nu,p)}\max\{\delta_{i\nu}, p^{s_2}-\delta_{i\nu}\}-\sum_{\substack{\nu|\bar{\gamma}\\ \chi(\nu,p)=1}}\operatorname{ord}_\nu(p)\sum_{i=1}^{\Phi_2(\nu,p)}(\max\{\tau_{i\nu}, p^{s_2}$$
$$-\rho_{i\nu}\}+\max\{\rho_{i\nu}, p^{s_2}-\tau_{i\nu}\})\Bigg)+\Bigg(\sum_{\substack{\nu|\bar{\gamma}\\ \chi(\nu,p)=0}}\operatorname{ord}_\nu(p)\sum_{i=1}^{\Psi_2(\nu,p)}p^{s_2}+\sum_{\substack{\nu|\bar{\gamma}\\ \chi(\nu,p)=1}}\operatorname{ord}_\nu(p)$$
$$\sum_{i=1}^{\Phi_2(\nu,p)}2p^{s_2}-\sum_{\substack{\nu|\bar{\gamma}\\ \chi(\nu,p)=0}}\operatorname{ord}_\nu(p)\sum_{i=1}^{\Psi_2(\nu,p)}\max\{\sigma_{i\nu}, p^{s_2}-\sigma_{i\nu}\}-\sum_{\substack{\nu|\bar{\gamma}\\ \chi(\nu,p)=1}}\operatorname{ord}_\nu(p)$$
$$\sum_{i=1}^{\Phi_2(\nu,p)}(\max\{\xi_{i\nu}, p^{s_2}-\Delta_{i\nu}\}+\max\{\Delta_{i\nu}, p^{s_2}-\xi_{i\nu}\})\Bigg).$$

Moreover,

$$\begin{aligned}\dim(\operatorname{Hull}(\mathfrak{C}))=&\sum_{\omega|\bar{\lambda}}(1-\chi(\omega,p))\operatorname{ord}_\omega(p)\sum_{e=1}^{\Psi_1(\omega,p)}u_{e\omega}+\sum_{\omega|\bar{\lambda}}\chi(\omega,p)\operatorname{ord}_\omega(p)\sum_{e=1}^{\Phi_1(\omega,p)}v_{e\omega}\\ &+\sum_{\nu|\bar{\gamma}}(1-\chi(\nu,p))\operatorname{ord}_\nu(p)\sum_{i=1}^{\Psi_2(\nu,p)}x_{i\nu}+\sum_{\nu|\bar{\gamma}}\chi(\nu,p)\operatorname{ord}_\nu(p)\sum_{i=1}^{\Phi_2(\nu,p)}y_{i\nu}\\ &+\sum_{\nu|\bar{\gamma}}(1-\chi(\nu,p))\operatorname{ord}_\nu(p)\sum_{i=1}^{\Psi_2(\nu,p)}z_{i\nu}+\sum_{\nu|\bar{\gamma}}\chi(\nu,p)\operatorname{ord}_\nu(p)\sum_{i=1}^{\Phi_2(\nu,p)}w_{i\nu},\end{aligned}\tag{18}$$

where

$$\begin{aligned}u_{e\omega}=p^{s_1}-\max\{\alpha_{e\omega}, p^{s_1}-\alpha_{e\omega}\},\ v_{e\omega}=2p^{s_1}-\max\{\beta_{e\omega}, p^{s_1}-\eta_{e\omega}\}\\ -\max\{\eta_{e\omega}, p^{s_1}-\beta_{e\omega}\},\end{aligned}\tag{19}$$

$$x_{i\nu}=p^{s_2}-\max\{\delta_{i\nu}, p^{s_2}-\delta_{i\nu}\},\ y_{i\nu}=2p^{s_2}-\max\{\tau_{i\nu}, p^{s_2}-\rho_{i\nu}\}-\max\{\rho_{i\nu}, p^{s_2}-\tau_{i\nu}\},\tag{20}$$

$$z_{i\nu}=p^{s_2}-\max\{\sigma_{i\nu}, p^{s_2}-\sigma_{i\nu}\},\ w_{i\nu}=2p^{s_2}-\max\{\xi_{i\nu}, p^{s_2}-\Delta_{i\nu}\}-\max\{\Delta_{i\nu}, p^{s_2}-\xi_{i\nu}\}.\tag{21}$$

By Eqs. (15)-(17) and (19)-(21), we have

$$0\le u_{e\omega}\le\left\lfloor\frac{p^{s_1}}{2}\right\rfloor, 0\le v_{e\omega}\le p^{s_1}, 0\le x_{i\nu}, z_{i\nu}\le\left\lfloor\frac{p^{s_2}}{2}\right\rfloor \text{ and } 0\le y_{i\nu}, w_{i\nu}\le p^{s_2}.\tag{22}$$

Let $u_\omega=\sum_{e=1}^{\Psi_1(\omega,p)}u_{e\omega}$, $v_\omega=\sum_{e=1}^{\Phi_1(\omega,p)}v_{e\omega}$, $x_\nu=\sum_{i=1}^{\Psi_2(\nu,p)}x_{i\nu}$, $y_\nu=\sum_{i=1}^{\Phi_2(\nu,p)}y_{i\nu}$, $z_\nu=\sum_{i=1}^{\Psi_2(\nu,p)}z_{i\nu}$ and $w_\nu=\sum_{i=1}^{\Phi_2(\nu,p)}w_{i\nu}$. Putting these values in Eq.(18), we have

$$\dim(\operatorname{Hull}(\mathfrak{C}))=\sum_{\omega|\bar{\lambda}}(1-\chi(\omega,p))\operatorname{ord}_\omega(p)u_\omega+\sum_{\omega|\bar{\lambda}}\chi(\omega,p)\operatorname{ord}_\omega(p)v_\omega$$

$$+\sum_{\nu|\bar{\gamma}}(1-\chi(\nu,p))\operatorname{ord}_\nu(p)x_\nu+\sum_{\nu|\bar{\gamma}}\chi(\nu,p)\operatorname{ord}_\nu(p)y_\nu$$
$$+\sum_{\nu|\bar{\gamma}}(1-\chi(\nu,p))\operatorname{ord}_\nu(p)z_\nu+\sum_{\nu|\bar{\gamma}}\chi(\nu,p)\operatorname{ord}_\nu(p)w_\nu.$$

From Eq.(22), we get $0\le u_\omega\le \Psi_1(\omega,p)\lfloor\frac{p^{s_1}}{2}\rfloor$, $0\le v_\omega\le\Phi_1(\omega,p)p^{s_1}$, $0\le x_\nu,z_\nu\le \Psi_2(\nu,p)\lfloor\frac{p^{s_2}}{2}\rfloor$, and $0\le y_\nu,w_\nu\le\Phi_2(\nu,p)p^{s_2}$. This proves that the dimensions of hulls of SCCs in $\mathfrak{R}_{\lambda,\gamma}$ satisfy the Eq.(8). □

We can re-write the Eq.(8) appeared in Theorem 3.5 in another form as follows:

$$\begin{aligned}&\sum_{\substack{\omega|\bar{\lambda}\\ \chi(\omega,p)=0}}\operatorname{ord}_\omega(p)u_\omega+\sum_{\substack{\omega|\bar{\lambda}\\ \chi(\omega,p)=1}}\operatorname{ord}_\omega(p)v_\omega+\sum_{\substack{\nu|\bar{\gamma}\\ \chi(\nu,p)=0}}\operatorname{ord}_\nu(p)[x_\nu+z_\nu]\\&+\sum_{\substack{\nu|\bar{\gamma}\\ \chi(\nu,p)=1}}\operatorname{ord}_\nu(p)[y_\nu+w_\nu].\end{aligned}\tag{23}$$

Now, Eq.(23) can be written in more simplified way using a multi-set S defined as $S=\{\Psi_1(\omega,p)\lfloor\frac{p^{s_1}}{2}\rfloor*\operatorname{ord}_\omega(p)\mid\omega|\bar{\lambda}\text{ and }\chi(\omega,p)=0\}\bigcup\{\Phi_1(\omega,p)p^{s_1}*\operatorname{ord}_\omega(p)\mid\omega|\bar{\lambda}$ and $\chi(\omega,p)=1\}\bigcup\{\Psi_2(\nu,p)2\lfloor\frac{p^{s_2}}{2}\rfloor*\operatorname{ord}_\nu(p)\mid\nu|\bar{\gamma}\text{ and }\chi(\nu,p)=0\}$ $\bigcup\{\Phi_2(\nu,p)2p^{s_2}*\operatorname{ord}_\nu(p)\mid\nu|\bar{\gamma}\text{ and }\chi(\nu,p)=1\}$, where $c*d$ in the multi-set is defined as $\underbrace{d,d,\cdots,d}_{\text{c-copies}}$.

Each value in Eq. (23) represents the sum of elements from a subset of S. These values can be determined using the generating function, as discussed in [8]. Alternatively, the dimensions of hulls of SCCs in $\mathfrak{R}_{\lambda,\gamma}$ correspond to the exponents of X in the following expression:

$$\begin{aligned}&\prod_{\substack{\omega|\bar{\lambda}\\ \chi(\omega,p)=0}}\left(\sum_{n=0}^{\Psi_1(\omega,p)\lfloor\frac{p^{s_1}}{2}\rfloor}(X^{\operatorname{ord}_\omega(p)})^n\right)\prod_{\substack{\omega|\bar{\lambda}\\ \chi(\omega,p)=1}}\left(\sum_{n=0}^{\Phi_1(\omega,p)p^{s_1}}(X^{\operatorname{ord}_\omega(p)})^n\right)\\&\prod_{\substack{\nu|\bar{\gamma}\\ \chi(\nu,p)=0}}\left(\sum_{n=0}^{2\Psi_2(\nu,p)\lfloor\frac{p^{s_2}}{2}\rfloor}(X^{\operatorname{ord}_\nu(p)})^n\right)\prod_{\substack{\nu|\bar{\gamma}\\ \chi(\nu,p)=1}}\left(\sum_{n=0}^{2\Phi_2(\nu,p)p^{s_2}}(X^{\operatorname{ord}_\nu(p)})^n\right),\end{aligned}\tag{24}$$

where χ is defined in Eq.(1), Ψ_1,Φ_1 in Eq.(2) and Ψ_2,Φ_2 in Eq.(4).

Example 3.6. Let $p=2$, $\lambda=21$ and $\gamma=15$. Then $\gcd(21,2)=\gcd(15,2)=1$, divisors of 21 are $1,3,7,21$ and divisors of 15 are $1,3,5,15$. Note that $(1,2),(3,2),(5,2)$ are good pairs, and $(7,2),(15,2),(21,2)$ are bad pairs. Since $\operatorname{ord}_7(2)=3,\operatorname{ord}_{15}(2)=4,\operatorname{ord}_{21}(2)=6$. Therefore, by Theorem 3.5, the dimensions of hulls of SCCs of length $(21,15)$ in $\mathfrak{R}_{\lambda,\gamma}$ are of the form

$$3v_7+6v_{21}+4y_{15}+4w_{15},$$

where $0\le v_7,v_{21}\le 1$, and $0\le y_{15},w_{15}\le 1$. Furthermore by Eq.(29), the dimensions of hulls of SCCs of length $(21,15)$ in $\mathfrak{R}_{\lambda,\gamma}$ correspond to the exponent of X in the expression of the generating function $(1+X^3)(1+X^6)(1+X^4+X^8)$. These exponents are $17,14,13,11,10,9,8,7,6,4,3,1,0$.

3.2. **Enumeration of SCCs over $\mathbb{Z}_p\mathbb{Z}_p[v]$ with hulls of a fixed dimension.** Now, we enumerate SCCs over $\mathbb{Z}_p\mathbb{Z}_p[v]$ that have hulls of a fixed dimension, say d. Since Eq. (8) determines the dimensions of hulls of SCCs, therefore, enumeration of SCCs with a given hull dimension d can be obtained by solving the following equation for the variables $u_{e\omega}, v_{e\omega}, x_{i\nu}, y_{i\nu}, z_{i\nu}$, and $w_{i\nu}$

$$\begin{aligned} d = &\sum_{\omega|\bar{\lambda}}(1-\chi(\omega,p))\operatorname{ord}_\omega(p)\sum_{e=1}^{\Psi_1(\omega,p)} u_{e\omega} + \sum_{\omega|\bar{\lambda}}\chi(\omega,p)\operatorname{ord}_\omega(p)\sum_{e=1}^{\Phi_1(\omega,p)} v_{e\omega} \\ &+\sum_{\nu|\bar{\gamma}}(1-\chi(\nu,p))\operatorname{ord}_\nu(p)\sum_{i=1}^{\Psi_2(\nu,p)} x_{i\nu} + \sum_{\nu|\bar{\gamma}}\chi(\nu,p)\operatorname{ord}_\nu(p)\sum_{i=1}^{\Phi_2(\nu,p)} y_{i\nu} \\ &+\sum_{\nu|\bar{\gamma}}(1-\chi(\nu,p))\operatorname{ord}_\nu(p)\sum_{i=1}^{\Psi_2(\nu,p)} z_{i\nu} + \sum_{\nu|\bar{\gamma}}\chi(\nu,p)\operatorname{ord}_\nu(p)\sum_{i=1}^{\Phi_2(\nu,p)} w_{i\nu}, \end{aligned} \tag{25}$$

where $0 \le u_{e\omega} \le \lfloor \frac{p^{s_1}}{2} \rfloor, 0 \le v_{e\omega} \le p^{s_1}, 0 \le x_{i\nu}, z_{i\nu} \le \lfloor \frac{p^{s_2}}{2} \rfloor, 0 \le y_{i\nu}$ and $w_{i\nu} \le p^{s_2}$.

For convenience, let $((u_{e\omega}))$ denote a vector whose entries satisfy $0 \le u_{e\omega} \le \left\lfloor \frac{p^{s_1}}{2} \right\rfloor$, with indices constrained by $\omega \mid \bar{\lambda}$, $\chi(\omega,p)=0$, and $1 \le e \le \Psi_1(\omega,p)$, i.e.,

$$((u_{e\omega})) := (u_{e\omega})_{\omega|\bar{\lambda},\, \chi(\omega,p)=0,\, 1\le e\le \Psi_1(\omega,p)}.$$

Similarly, $((v_{e\omega})) := (v_{e\omega})_{\omega|\bar{\lambda},\chi(\omega,p)=1,1\le e\le \Phi_1(\nu,p)}$, $((x_{i\nu})) := (x_{i\nu})_{\nu|\bar{\gamma},\chi(\omega,p)=0,1\le i\le \Psi_2(\nu,p)}$, $((y_{i\nu})) := (y_{i\nu})_{\nu|\bar{\gamma},\chi(\nu,p)=1,1\le i\le \Phi_2(\nu,p)}$, $((z_{i\nu})) := (z_{i\nu})_{\nu|\bar{\gamma},\chi(\nu,p)=0,1\le i\le \Psi_2(\nu,p)}$, and $((w_{i\nu})) := (w_{i\nu})_{\nu|\bar{\gamma},\chi(\nu,p)=0,1\le i\le \Phi_2(\nu,p)}$.
Let $n = \Big(((u_{e\omega})), ((v_{e\omega})), ((x_{i\nu})), ((y_{i\nu})), ((z_{i\nu})), ((w_{i\nu}))\Big)$ be the concatenation of vectors $((u_{e\omega})), ((v_{e\omega})), ((x_{i\nu})), ((y_{i\nu})), ((z_{i\nu}))$ and $((w_{i\nu}))$.

Theorem 3.7. *Let λ, γ and d be positive integers such that d satisfies the Eq.(8). Then the number of SCCs in $\mathfrak{R}_{\lambda,\gamma}$ whose hulls have dimension d is given by*

$$\begin{aligned} \sum_{n\in h(d)}\Bigg(&\prod_{\substack{\omega|\bar{\lambda}\\ \chi(\omega,p)=0}}\prod_{e=1}^{\Psi_1(\omega,p)} |\{u_{e\omega}, p^{s_1}-u_{e\omega}\}| \prod_{\substack{\omega|\bar{\lambda}\\ \chi(\omega,p)=1}}\prod_{e=1}^{\Phi_1(\omega,p)} p^{1-[\frac{v_{e\omega}}{p^{s_1}}]}(v_{e\omega}+1) \\ &\prod_{\substack{\nu|\bar{\gamma}\\ \chi(\nu,p)=0}}\prod_{i=1}^{\Psi_2(\nu,p)} |\{x_{i\nu}, p^{s_2}-x_{i\nu}\}| \prod_{\substack{\nu|\bar{\gamma}\\ \chi(\nu,p)=1}}\prod_{i=1}^{\Phi_2(\nu,p)} p^{1-[\frac{y_{i\nu}}{p^{s_2}}]}(y_{i\nu}+1) \\ &\prod_{\substack{\nu|\bar{\gamma}\\ \chi(\nu,p)=0}}\prod_{i=1}^{\Psi_2(\nu,p)} |\{z_{i\nu}, p^{s_2}-z_{i\nu}\}| \prod_{\substack{\nu|\bar{\gamma}\\ \chi(\nu,p)=1}}\prod_{i=1}^{\Phi_2(\nu,p)} p^{1-[\frac{w_{i\nu}}{p^{s_2}}]}(w_{i\nu}+1)\Bigg), \end{aligned} \tag{26}$$

where

$$h(d) = \Bigg\{ n \mid \sum_{\omega|\bar{\lambda}}(1-\chi(\omega,p))\operatorname{ord}_\omega(p)\sum_{e=1}^{\Psi_1(\omega,p)} u_{e\omega} + \sum_{\omega|\bar{\lambda}}\chi(\omega,p)\operatorname{ord}_\omega(p)\sum_{e=1}^{\Phi_1(\omega,p)} v_{e\omega}$$
$$+ \sum_{\nu|\bar{\gamma}}(1-\chi(\nu,p))\operatorname{ord}_\nu(p)\sum_{i=1}^{\Psi_2(\nu,p)} x_{i\nu} + \sum_{\nu|\bar{\gamma}}\chi(\nu,p)\operatorname{ord}_\nu(p)\sum_{i=1}^{\Phi_2(\nu,p)} y_{i\nu}$$
$$+ \sum_{\nu|\bar{\gamma}}(1-\chi(\nu,p))\operatorname{ord}_\nu(p)\sum_{i=1}^{\Psi_2(\nu,p)} z_{i\nu} + \sum_{\nu|\bar{\gamma}}\chi(\nu,p)\operatorname{ord}_\nu(p)\sum_{i=1}^{\Phi_2(\nu,p)} w_{i\nu} = d \Bigg\}, \tag{27}$$

and $\chi, \Psi_1, \Phi_1, \Psi_2$ and Φ_2 are as defined previously.

Proof. For $\omega \mid \bar{\lambda}, \chi(\omega,p)=0$ and $1 \le e \le \Psi_1(\omega,p)$, let $u_{e\omega} \in \{0,1,\dots,\lfloor \frac{p^{s_1}}{2} \rfloor\}$ and $\alpha_{e\omega} \in \{0,1,\dots,p^{s_1}\}$. For $\omega \mid \bar{\lambda}, \chi(\omega,p)=1$ and $1 \le e \le \Psi_1(\omega,p)$, let $v_{e\omega}$, $\beta_{e\omega}$, $\eta_{e\omega} \in \{0,1,\dots,p^{s_1}\}$. For $\nu \mid \bar{\gamma}, \chi(\nu,p)=0$ and $1 \le i \le \Psi_2(\nu,p)$, let $x_{i\nu}, z_{i\nu} \in \{0,1,\dots,\lfloor \frac{p^{s_2}}{2} \rfloor\} and \delta_{i\nu},\ \sigma_{i\nu} \in \{0,1,\dots,p^{s_2}\}$. For $\nu \mid \bar{\gamma}, \chi(\omega,p)=1$ and $1 \le e \le \Phi_2(\omega,p)$, let $y_{i\nu}, w_{i\nu},\ \tau_{i\nu},\ \rho_{i\nu},\ \xi_{i\nu}, \Delta_{i\nu} \in \{0,1,\dots,p^{s_2}\}$. For a given n, we aim to find the polynomials $m(x), l_1(x)$ and $l_2(x)$ as in Eqs.(9), (10) and (11) such that the dimension of the hull of a separable $\mathbb{Z}_p\mathbb{Z}_p[v]$-cyclic code generated by $\langle (m(x) \mid 0), (0 \mid \xi_1 l_1(x) + \xi_2 l_2(x))\rangle$ is

$$d = \sum_{\omega|\bar{\lambda}}(1-\chi(\omega,p))\operatorname{ord}_\omega(p)\sum_{e=1}^{\Psi_1(\omega,p)} u_{e\omega} + \sum_{\omega|\bar{\lambda}}\chi(\omega,p)\operatorname{ord}_\omega(p)\sum_{e=1}^{\Phi_1(\omega,p)} v_{e\omega}$$
$$+ \sum_{\nu|\bar{\gamma}}(1-\chi(\nu,p))\operatorname{ord}_\nu(p)\sum_{i=1}^{\Psi_2(\nu,p)} x_{i\nu} + \sum_{\nu|\bar{\gamma}}\chi(\nu,p)\operatorname{ord}_\nu(p)\sum_{i=1}^{\Phi_2(\nu,p)} y_{i\nu} \tag{28}$$
$$+ \sum_{\nu|\bar{\gamma}}(1-\chi(\nu,p))\operatorname{ord}_\nu(p)\sum_{i=1}^{\Psi_2(\nu,p)} z_{i\nu} + \sum_{\nu|\bar{\gamma}}\chi(\nu,p)\operatorname{ord}_\nu(p)\sum_{i=1}^{\Phi_2(\nu,p)} w_{i\nu}.$$

Equivalently, we determine all the values of $\alpha_{e\omega}, \beta_{e\omega}, \eta_{e\omega}, \delta_{i\nu}, \tau_{i\nu}, \rho_{i\nu}, \sigma_{i\nu}, \xi_{i\nu}, \Delta_{i\nu}$ in Eqs.(9), (10) and (11) that satisfy Eqs.(19),(20),(21) and (25).

From Eqs.(19)-(21), we get that $\alpha_{e\omega}$ is either $u_{e\omega}$ or $p^{s_1} - u_{e\omega}$, $\delta_{i\nu}$ is either $x_{i\nu}$ or $p^{s_2} - x_{i\nu}$ and $\sigma_{i\nu}$ is either $z_{i\nu}$ or $p^{s_2} - z_{i\nu}$. Now we calculate the values of $(\beta_{e\omega}, \eta_{e\omega})$, $(\tau_{i\nu}, \rho_{i\nu})$ and $(\xi_{i\nu}, \Delta_{i\nu})$ into two cases for $v_{e\omega}, y_{i\nu}$ and $w_{i\nu}$, respectively.

Case 1: Let $v_{e\omega} = p^{s_1}$, then $\max\{\beta_{e\omega}, p^{s_1} - \eta_{e\omega}\} + \max\{\eta_{e\omega}, p^{s_1} - \beta_{e\omega}\} = p^{s_1}$. Hence, $\beta_{e\omega} + \eta_{e\omega} = p^{s_1}$. Therefore, $(\beta_{e\omega}, \eta_{e\omega}) \in \{(p^{s_1} - k_1, k_1) \mid k_1 \in \{0,1,\dots,p^{s_1}\}\}$.

Case 2: Let $0 \le v_{e\omega} < p^{s_1}$, then $2p^{s_1} - v_{e\omega} = \max\{\beta_{e\omega}, p^{s_1} - \eta_{e\omega}\} + \max\{\eta_{e\omega}, p^{s_1} - \beta_{e\omega}\}$. Therefore $2p^{s_1} - v_{e\omega}$ is either $\beta_{e\omega} + \eta_{e\omega}$ or, $p^{s_1} - \beta_{e\omega} + p^{s_1} - \eta_{e\omega}$. Therefore, $(\beta_{e\omega}, \eta_{e\omega}) \in \{(p^{s_1} - (v_{e\omega} - k_1), (p^{s_1} - k_1) \mid k_1 \in \{0,1,\dots,v_{e\omega}\}\}$ or, $(\beta_{e\omega}, \eta_{e\omega}) \in \{(v_{e\omega} - k_1, k_1) \mid k_1 \in \{0,1,\dots,v_{e\omega}\}\}$. Similarly, we can calculate

- If $y_{i\nu} = p^{s_2}$, then $(\tau_{i\nu}, \rho_{i\nu}) \in \{(p^{s_2} - k_2, k_2) \mid k_2 \in \{0,1,\dots,p^{s_2}\}\}$.
- If $0 \le y_{i\nu} < p^{s_2}$, then $(\tau_{i\nu}, \rho_{i\nu}) \in \{(p^{s_2} - (z_{i\nu} - k_2), (p^{s_2} - k_2) \mid k_2 \in \{0,1,\dots,z_{i\nu}\}\}$ or, $(\tau_{i\nu}, \rho_{i\nu}) \in \{(z_{i\nu} - k_2, k_2) \mid k_2 \in \{0,1,\dots,z_{i\nu}\}\}$.
- If $w_{i\nu} = p^{s_2}$, then $(\xi_{i\nu}, \Delta_{i\nu}) \in \{(p^{s_2} - k_3, k_3) \mid k_3 \in \{0,1,\dots,p^{s_3}\}\}$.
- If $0 \le w_{i\nu} < p^{s_2}$, then $(\xi_{i\nu}, \Delta_{i\nu}) \in \{(p^{s_2} - (w_{i\nu} - k_3), (p^{s_3} - k_3) \mid k_3 \in \{0,1,\dots,w_{i\nu}\}\}$ or, $(\xi_{i\nu}, \Delta_{i\nu}) \in \{(w_{i\nu} - k_3, k_3) \mid k_3 \in \{0,1,\dots,w_{i\nu}\}\}$.

Therefore, for a given n, the number of SCCs such that the dimensions of their hulls satisfy Eq. (27) is

$$\prod_{\substack{\omega|\bar{\lambda}\\ \chi(\omega,p)=0}} \prod_{e=1}^{\Psi_1(\omega,p)} |\{u_{e\omega}, p^{s_1}-u_{e\omega}\}| \prod_{\substack{\omega|\bar{\lambda}\\ \chi(\omega,p)=1}} \prod_{e=1}^{\Phi_1(\omega,p)} p^{1-[\frac{v_{e\omega}}{p^{s_1}}]}(v_{e\omega}+1)$$
$$\prod_{\substack{\omega|\bar{\gamma}\\ \chi(\nu,p)=0}} \prod_{i=1}^{\Psi_2(\nu,p)} |\{x_{i\nu}, p^{s_2}-x_{i\nu}\}| \prod_{\substack{\nu|\bar{\gamma}\\ \chi(\nu,p)=1}} \prod_{i=1}^{\Phi_2(\nu,p)} p^{1-[\frac{y_{i\nu}}{p^{s_2}}]}(y_{i\nu}+1)$$
$$\prod_{\substack{\omega|\bar{\gamma}\\ \chi(\nu,p)=0}} \prod_{i=1}^{\Psi_2(\nu,p)} |\{z_{i\nu}, p^{s_2}-z_{i\nu}\}| \prod_{\substack{\nu|\bar{\gamma}\\ \chi(\nu,p)=1}} \prod_{i=1}^{\Phi_2(\nu,p)} p^{1-[\frac{w_{i\nu}}{p^{s_2}}]}(w_{i\nu}+1).$$

Adding this expression for all values of $n \in h(d)$, we get the desired result. $\square$

The cardinality of $h(d)$ as appeared in Eq. (27), or equivalently, the number of solutions of Eq.(28) is the coefficients of X^d in the generating function

$$\begin{aligned}
&\prod_{\substack{\omega|\bar{\lambda}\\ \chi(\omega,p)=0}} \prod_{e=1}^{\Psi_1(\omega,p)} \left(\sum_{u_{e\omega}=0}^{\lfloor \frac{p^{s_1}}{2} \rfloor} (X^{\mathrm{ord}_\omega(p)})^{u_{e\omega}} \right) \prod_{\substack{\omega|\bar{\lambda}\\ \chi(\omega,p)=1}} \prod_{e=1}^{\Phi_1(\omega,p)} \left(\sum_{v_{e\omega}=0}^{p^{s_1}} (X^{\mathrm{ord}_\omega(p)})^{v_{e\omega}} \right)\\
&\prod_{\substack{\nu|\bar{\gamma}\\ \chi(\nu,p)=0}} \prod_{i=1}^{\Psi_2(\nu,p)} \left(\sum_{x_{i\nu}=0}^{\lfloor \frac{p^{s_2}}{2} \rfloor} (X^{\mathrm{ord}_\nu(p)})^{x_{i\nu}} \right) \prod_{\substack{\nu|\bar{\gamma}\\ \chi(\nu,p)=1}} \prod_{i=1}^{\Phi_2(\nu,p)} \left(\sum_{y_{i\nu}=0}^{p^{s_2}} (X^{\mathrm{ord}_\nu(p)})^{y_{i\nu}} \right) \qquad (29)\\
&\prod_{\substack{\nu|\bar{\gamma}\\ \chi(\nu,p)=0}} \prod_{i=1}^{\Psi_2(\nu,p)} \left(\sum_{z_{i\nu}=0}^{\lfloor \frac{p^{s_2}}{2} \rfloor} (X^{\mathrm{ord}_\nu(p)})^{z_{i\nu}} \right) \prod_{\substack{\nu|\bar{\gamma}\\ \chi(\nu,p)=1}} \prod_{i=1}^{\Phi_2(\nu,p)} \left(\sum_{w_{i\nu}=0}^{p^{s_2}} (X^{\mathrm{ord}_\nu(p)})^{w_{i\nu}} \right).
\end{aligned}$$

Corollary 3.8. *Let λ, γ and d be positive integers such that $\gcd(\lambda, p) = 1 = \gcd(\gamma, p)$ and d satisfies the Eq.(8). Let $\chi, \Psi_1, \Phi_1, \Psi_2$ and Φ_2 be as defined previously. Then number of SCCs in $\mathfrak{R}_{\lambda,\gamma}$ whose hulls have dimension "d" is $2^{m_1+m_2+2(k_1+k_2)}|h(d)|$, where m_1, m_2, k_1, k_2 are defined in Theorem 2.10, and*

$$\begin{aligned}
h(d) = \Bigg\{ &((v_{e\omega}), (y_{i\nu}), (w_{i\nu}))| \sum_{\omega|\bar{\lambda}} \chi(\omega,p)\,\mathrm{ord}_\omega(p) \sum_{e=1}^{\Phi_1(\omega,p)} v_{e\omega} + \sum_{\nu|\bar{\gamma}} \chi(\nu,p)\,\mathrm{ord}_\nu(p)\\
&\sum_{i=1}^{\Phi_2(\nu,p)} y_{i\nu} + \sum_{\nu|\bar{\gamma}} \chi(\nu,p)\,\mathrm{ord}_\nu(p) \sum_{i=1}^{\Phi_2(\nu,p)} w_{i\nu} = d \Bigg\}.
\end{aligned} \qquad (30)$$

Proof. Since $\gcd(\lambda, p) = \gcd(\gamma, p) = 1$, it follows that $p^{s_1} = p^{s_2} = 1$. From the proof of Theorem 3.5, we obtain that $u_{e\omega}, x_{i\nu}, z_{i\nu} = 0$, for all $\omega \mid \lambda$ and $\nu \mid \gamma$, while $v_{e\omega}, y_{i\nu}, w_{i\nu} \in \{0, 1\}$, for all $\omega \mid \lambda$ and $\nu \mid \gamma$, respectively. Consequently, Eq.(26) simplifies to $2^{m_1+m_2+2(k_1+k_2)}|h(d)|$ and Eq. (27) reduced to Eq. (30). $\square$

Example 3.9. In Ex. (3.6), we determine all the possible dimensions of hulls of SCCs of length $(21, 15)$ in $\mathfrak{R}_{\lambda,\gamma}$, which are $17, 14, 13, 11, 10, 9, 8, 7, 6, 4, 3, 1, 0$. Here, we focus on counting the number of SCCs of length $(21, 15)$ in $\mathfrak{R}_{\lambda,\gamma}$ with hulls of dimension 10. Note that

$$h(10) = \{((v_{1,7}), (v_{1,21}), (y_{1,15}), (w_{1,15})) \mid 3v_{1,7} + 6v_{1,21} + 4y_{1,15} + 4w_{1,15} = 10\},$$

where $0 \le v_{1,7}, v_{1,21} \le 1$ and $0 \le y_{1,15}, w_{1,15} \le 1$. Moreover, using Eq.(29) we can determine the cardinality of $h(10)$, which corresponds to the coefficient of X^{10} in the following generating function

$$\begin{aligned}&(1 + X^3)(1 + X^6)(1 + X^4)(1 + X^4)\\ &= X^{17} + X^{14} + 2X^{13} + X^{11}\\ &\quad + 2X^{10} + X^9 + X^8 + 2X^7 + X^6 + 2X^4 + X^3 + 1.\end{aligned}$$

Since the coefficient of X^{10} is 2, it follows that $|h(10)| = 2$, and thus $h(10) = \{((0), (1), (1), (0)),\ ((0), (1), (0), (1))\}$. Putting the elements of $h(10)$ into Eq.(26), we conclude that the total number of SCCs of length $(21, 15)$ in $\mathfrak{R}_{\lambda,\gamma}$ having hulls of dimension 10 is 8192.

4. **Euclidean hulls of NSCCs over $\mathbb{Z}_2\mathbb{Z}_2[v]$.** In this section, we study the hulls of NSCCs over $\mathbb{Z}_2\mathbb{Z}_2[v]$. Throughout, we assume that λ and γ are odd, pairwise coprime integers. We define $\mathfrak{R}_{\lambda,\gamma} = \frac{\mathbb{Z}_2[x]}{\langle x^\lambda - 1\rangle} \times \frac{R[x]}{\langle x^\gamma - 1\rangle}$, where $R = \mathbb{Z}_2 + v\mathbb{Z}_2$ and $v^2 = v$.

Theorem 4.1. [9] *Suppose $\mathfrak{C} = \langle (m(x) \mid 0), (k(x) \mid \xi_1 l_1(x) + \xi_2 l_2(x))\rangle$ is a NSCC in $\mathfrak{R}_{\lambda,\gamma}$, as defined in Theorem 2.6. Then* $\gcd(m(x), \frac{x^\gamma - 1}{l_2(x)}) \neq 1$.

Theorem 4.2. *Let $\mathfrak{C} = \langle (m(x) \mid 0), (k(x) \mid \xi_1 l_1(x) + \xi_2 l_2(x))\rangle$ be a NSCC in $\mathfrak{R}_{\lambda,\gamma}$. Then $\mathfrak{C} = \langle ((x-1)A_1(x) \mid 0), (A_1(x) \mid \xi_1 l_1(x) + \xi_2 l_2(x))\rangle$ such that $A_1(x) \mid x^\lambda - 1, l_1(x) \mid x^\gamma - 1$, $l_2(x) \mid x^\gamma - 1$, $x - 1 \nmid A_1(x)$ and $x - 1 \nmid l_2(x)$.*

Proof. Since λ, γ are pairwise coprime and $\gcd(\lambda, 2) = 1 = \gcd(\gamma, 2)$, it follows that $x^\lambda - 1$ and $x^\gamma - 1$ have only common factor $x - 1$. Now, suppose $x - 1 \mid l_2(x)$. Then, $x - 1 \nmid \frac{x^\gamma - 1}{l_2(x)}$. Consequently, we have $\gcd(m(x), \frac{x^\gamma-1}{l_2(x)}) = 1$. By Theorem 2.6, this implies that $m(x) \mid k(x)$, which leads to a contradiction. Therefore, we conclude that $x - 1 \nmid l_2(x)$ and $x - 1 \mid m(x)$. Let $m(x) = (x-1)A_1(x)$ and $\frac{x^\gamma-1}{l_2(x)} = (x-1)A_2(x)$, where $\gcd(A_1(x), A_2(x)) = 1$. Since $m(x) \mid \frac{x^\gamma-1}{l_2(x)} k(x) = (x-1)A_1(x) \mid (x-1)A_2(x)k_2(x)$, implies $A_1(x) \mid A_2(x)k(x)$, implies $A_1(x) = k(x)$. Therefore, $\mathfrak{C} = \langle ((x-1)A_1(x) \mid 0), (A_1(x) \mid \xi_1 l_1(x) + \xi_2 l_2(x))\rangle$ such that $A_1(x) \mid x^\lambda - 1, l_1(x) \mid x^\gamma - 1$, $l_2(x) \mid x^\gamma - 1$, $x - 1 \nmid A_1(x)$ and $x - 1 \nmid l_2(x)$. ∎

In the following corollary, we determine the generator polynomials of $\mathfrak{C}^\perp$.

Corollary 4.3. *Let $\mathfrak{C} = \langle (m(x) \mid 0), (k(x) \mid \xi_1 l_1(x) + \xi_2 l_2(x))\rangle$ be a NSCC in $\mathfrak{R}_{\lambda,\gamma}$, as defined in Theorem 4.2. Then $\mathfrak{C}^\perp = \langle (\frac{x^\lambda-1}{A_1(x)^*} \mid 0), (\frac{x^\lambda-1}{(x-1)A_1(x)^*} \mid \xi_1 \frac{x^\gamma-1}{l_1^*(x)} + \xi_2 \frac{x^\gamma-1}{(x-1)l_2^*(x)})\rangle$.*

Proof. By Theorem 2.8, $\mathfrak{C}^\perp = \langle (\bar{m}(x) \mid 0), (\bar{k}(x) \mid \xi_1 \bar{l}_1(x) + \xi_2 \bar{l}_2(x))\rangle$, where $\bar{m}(x) = \frac{x^\lambda-1}{\gcd(m(x),k(x))^*}$, $\bar{l}_1(x) = \frac{x^\gamma-1}{l_1^*(x)}$, $\bar{l}_2(x) = \frac{(x^\gamma-1)\gcd(m(x),k(x))^*}{m^*(x)l_2^*(x)}$ and $\bar{k}(x) = \frac{(x^\lambda-1)\mu(x)}{m^*(x)}$, where $\mu(x) = x^{\mathfrak{m}-\deg(l_2(x))+\deg(k(x))}\left(\frac{k^*(x)}{\gcd(m(x),k(x))^*}\right)^{-1}$ mod $\left(\frac{m^*(x)}{\gcd(m(x),k(x))^*}\right)$. Now, $\bar{m}(x) = \frac{x^\lambda-1}{\gcd((x-1)A_1(x),A_1(x))^*} = \frac{x^\lambda-1}{A_1^*(x)}$,$\bar{l}_2(x) = \frac{(x^\gamma-1)\gcd(m(x),k(x))^*}{m^*(x)l_2^*(x)} = \frac{(x^\lambda-1)A_1^*(x)}{(x-1)A_1^*(x)l_2^*(x)} = \frac{x^\gamma-1}{(x-1)l_2^*(x)}$. Moreover, $\mu(x) =$

$x^{\mathfrak{m}-\deg(l_2(x))+\deg(k(x))}\left(\frac{k^*(x)}{\gcd(m(x),k(x))^*}\right)^{-1} \mod \left(\frac{m^*(x)}{\gcd(m(x),k(x))^*}\right)$, i.e., $\mu(x) \equiv 1 \mod (x-1)$, then $\bar{k}(x) = \frac{(x^\lambda-1)}{(x-1)A_1^*(x)}$. This completes the proof. □

Let Π_λ and Π_γ be the canonical projections defined as $\Pi_\lambda : \mathfrak{R}_{\lambda,\gamma} \to \frac{\mathbb{Z}_2[x]}{\langle x^\lambda-1\rangle}$ and $\Pi_\gamma : R_{\lambda,\gamma} \to \frac{\mathfrak{R}[x]}{\langle x^\gamma-1\rangle}$, respectively. As $\mathfrak{C}$ and $\mathfrak{C}^\perp$, are submodules of $\mathfrak{R}_{\lambda,\gamma}$, therefore by Theorem 4.2 and Corollary 4.3, we have $\Pi_\lambda(\mathfrak{C}) = \langle \gcd((x-1)A_1(x), A_1(x))\rangle = \langle A_1(x)\rangle$, $\Pi_\lambda(\mathfrak{C}^\perp) = \langle \frac{x^\lambda-1}{(x-1)A_1^*(x)}\rangle$, $\Pi_\gamma(\mathfrak{C}) = \langle \xi_1 l_1(x) + \xi_2 l_2(x)\rangle$, and $\Pi_\gamma(\mathfrak{C}^\perp) = \langle \xi_1\frac{x^\gamma-1}{l_1^*(x)} + \xi_2\frac{x^\gamma-1}{(x-1)l_2^*(x)}\rangle$.

Proposition 4.4. *Let $\Pi_\gamma : \mathfrak{R}_{\lambda,\gamma} \to \frac{R[x]}{\langle x^\gamma-1\rangle}$ and $\mathfrak{C}$ and $\mathfrak{C}^\perp$ be NSCCs in $\mathfrak{R}_{\lambda,\gamma}$. Then $\Pi_\gamma(\mathfrak{C}\cap\mathfrak{C}^\perp) \subseteq \Pi_\gamma(\mathfrak{C})\cap\Pi_\gamma(\mathfrak{C}^\perp)$.*

Proof. Suppose $b(x) \in \Pi_\gamma(\mathfrak{C}\cap\mathfrak{C}^\perp)$, i.e., there exists $(a(x), b(x)) \in \Pi_\gamma(\mathfrak{C}\cap\mathfrak{C}^\perp)$ such that $\Pi_\gamma(a(x), b(x)) = b(x)$. Thus $b(x) \in \Pi_\gamma(\mathfrak{C})$ and $b(x) \in \Pi_\gamma(\mathfrak{C}^\perp)$. This implies that $b(x) \in \Pi_\gamma(\mathfrak{C})\cap\Pi_\gamma(\mathfrak{C}^\perp)$, i.e, $\Pi_\gamma(\mathfrak{C}\cap\mathfrak{C}^\perp) \subseteq \Pi_\gamma(\mathfrak{C})\cap\Pi_\gamma(\mathfrak{C}^\perp)$. □

Proposition 4.5. *Let $\Pi_\lambda : \mathfrak{R}_{\lambda,\gamma} \to \frac{\mathbb{Z}_2[x]}{\langle x^\lambda-1\rangle}$. Then $\Pi_\lambda(\mathfrak{C}\cap\mathfrak{C}^\perp) \subseteq \Pi_\lambda(\mathfrak{C})\cap\Pi_\lambda(\mathfrak{C}^\perp)$.*

Proof. The proof can be done using similar steps as the proof of Proposition 4.4. □

Lemma 4.6. *If $\Big(h_1(x)\,\mathrm{lcm}(A_1(x), \frac{x^\lambda-1}{(x-1)A_1^*(x)}) \mid \xi_1 h_2(x)\,\mathrm{lcm}(l_1(x), \frac{x^\gamma-1}{l_1(x)}) + \xi_2 h_2(x)\,\mathrm{lcm}(l_2(x), \frac{x^\gamma-1}{(x-1)l_2^*(x)})\Big)$ o $\Big(A_1(x), \xi_1 l_1(x) + \xi_2 l_2(x)\Big) = 0$. Then $h_1(x) + h_2(x) \equiv 0 \mod (x-1)$.*

Proof. Since, $\Big(h_1(x)\,\mathrm{lcm}(A_1(x), \frac{x^\lambda-1}{(x-1)A_1^*(x)}) \mid \xi_1 h_2(x)\,\mathrm{lcm}(l_1(x), \frac{x^\gamma-1}{l_1(x)}) + \xi_2 h_2(x)$ $\mathrm{lcm}(l_2(x), \frac{x^\gamma-1}{(x-1)l_2^*(x)})\Big)$ o $\Big(A_1(x), \xi_1 l_1(x) + \xi_2 l_2(x)\Big) = 0$. Then

$$\begin{aligned}
&\Big(h_1(x)\,\mathrm{lcm}(A_1(x), \frac{x^\lambda-1}{(x-1)A_1^*(x)}) \mid \xi_1 h_2\,\mathrm{lcm}(l_1(x), \frac{x^\gamma-1}{l_1(x)}) \\
&\quad + \xi_2 h_2\,\mathrm{lcm}(l_2(x), \frac{x^\gamma-1}{(x-1)l_2^*(x)})\Big) o \Big(A_1(x), \xi_1 l_1(x) + \xi_2 l_2(x)\Big) \\
&= \xi_2 h_1(x)\,\mathrm{lcm}\left(A_1(x), \frac{x^\lambda-1}{(x-1)A_1^*(x)}\right) A_1^*(x)\frac{x^{\mathfrak{m}}-1}{x^\lambda-1}x^{\mathfrak{m}-1-\deg(A_1(x))} \\
&\quad + \xi_1 h_2(x)\,\mathrm{lcm}\left(l_1(x), \frac{x^\gamma-1}{l_1^*(x)}\right) l_1^*(x)\frac{x^{\mathfrak{m}}-1}{x^\gamma-1}x^{\mathfrak{m}-1-\deg(l_1(x))} \\
&\quad + \xi_2 h_2(x)\,\mathrm{lcm}\left(l_2(x), \frac{x^\gamma-1}{(x-1)l_2^*(x)}\right) l_2^*(x)\frac{x^{\mathfrak{m}}-1}{x^\gamma-1}x^{\mathfrak{m}-1-\deg(l_2(x))} \mod (x^{\mathfrak{m}}-1) \\
&= \xi_2 h_1(x)\,\mathrm{lcm}\left(A_1(x)A_1^*, \frac{x^\lambda-1}{(x-1)}\right)\frac{x^{\mathfrak{m}}-1}{x^\lambda-1}x^{\mathfrak{m}-1-\deg(A_1(x))} + \xi_2 h_2(x) \\
&\quad \mathrm{lcm}\left(l_2(x)l_2^*(x), \frac{x^\gamma-1}{(x-1)}\right)\frac{x^{\mathfrak{m}}-1}{x^\gamma-1}x^{\mathfrak{m}-1-\deg(l_2(x))} \mod (x^{\mathfrak{m}}-1) \\
&= \xi_2 h_1(x)g_1(x)\frac{x^{\mathfrak{m}}-1}{x-1}x^{\mathfrak{m}-1-\deg(A_1(x))} + \xi_2 h_2(x)g_3(x)\frac{x^{\mathfrak{m}}-1}{x-1}x^{\mathfrak{m}-1-\deg(l_2(x))}
\end{aligned}$$

$\mod (x^{\mathfrak{m}}-1)$,

where $g_1(x)=A_1(x)$, if $A_1(x)$ is a self-reciprocal polynomial, otherwise $g_1(x)=1$ and $g_3(x)=l_2(x)$, if $l_2(x)$ is a self-reciprocal polynomial, otherwise $g_3(x)=1$. Thus, either

$$h_1(x)g_1(x)x^{\mathfrak{m}-1-\deg(A_1(x))}+h_2(x)g_3(x)x^{\mathfrak{m}-1-\deg(l_2(x))}\equiv 0 \mod (x^{\mathfrak{m}}-1), \quad (31)$$

or,

$$h_1(x)g_1(x)x^{\mathfrak{m}-1-\deg(A_1(x))}+h_2(x)g_3(x)x^{\mathfrak{m}-1-\deg(l_2(x))}\equiv 0 \mod (x-1). \quad (32)$$

Due to $x-1\mid x^{\mathfrak{m}}-1$, Eq. (31) implies Eq. (32). Because $g_1(x)\equiv 1 \mod (x-1)$, $g_3(x)\equiv 1 \mod (x-1)$ and $x^k\equiv 1 \mod (x-1)$, over $\mathbb{Z}_2$, where $k\in\mathbb{Z}^+$, it follows that $h_1(x)+h_2(x)\equiv 0 \mod (x-1)$. □

Theorem 4.7. *Let $\mathfrak{C}$ be a NSCC and $\mathfrak{C}^\perp$ be its dual in $\mathfrak{R}_{\lambda,\gamma}$, as defined in Theorem 4.2 and Corollary 4.3, respectively. Then*

$$\mathrm{Hull}(\mathfrak{C})=\Bigg\langle \left(\mathrm{lcm}\left((x-1)A_1(x),\frac{x^\lambda-1}{A_1^*(x)}\right)\Big|0\right),\left(\mathrm{lcm}\left(A_1(x),\frac{x^\lambda-1}{(x-1)A_1^*(x)}\right)\right.$$
$$\left.\Big|\xi_1\,\mathrm{lcm}\left(l_1(x),\frac{x^\gamma-1}{l_1^*(x)}\right)+\xi_2\,\mathrm{lcm}\left(l_2(x),\frac{x^\gamma-1}{(x-1)l_2^*(x)}\right)\right)\Bigg\rangle. \quad (33)$$

where $A_1(x)\mid x^\lambda-1, l_1(x)\mid x^\gamma-1, l_2(x)\mid x^\gamma-1$, $x-1\nmid A_1(x)$ and $x-1\nmid l_2(x)$.

Proof. Since $\mathfrak{C}$ and $\mathfrak{C}^\perp$ both are NSCCs in $\mathfrak{R}_{\lambda,\gamma}$, it follows that Hull($\mathfrak{C}$) is also a NSCC in $\mathfrak{R}_{\lambda,\gamma}$. By Theorem 4.2, we can express $\mathrm{Hull}(\mathfrak{C})=\langle((x-1)E(x)\mid 0),(E(x)\mid \xi_1F_1(x)+\xi_2F_2(x))\rangle$, where $E(x)\mid x^\lambda-1, F_1(x)\mid x^\gamma-1, F_2(x)\mid x^\gamma-1$, $x-1\nmid E(x)$ and $x-1\nmid F_2(x)$. We also have, $\Pi_\lambda(\mathrm{Hull}(\mathfrak{C}))=\langle E(x)\rangle$, and $\Pi_\gamma(\mathrm{Hull}(\mathfrak{C}))=\langle \xi_1F_1(x)+\xi_2F_2(x))\rangle$. Since $\Pi_\lambda(\mathfrak{C})=\langle A_1(x)\rangle$, $\Pi_\lambda(\mathfrak{C}^\perp)=\langle\frac{x^\lambda-1}{(x-1)A_1^*(x)}\rangle$, $\Pi_\gamma(\mathfrak{C})=\langle \xi_1l_1(x)+\xi_2l_2(x))\rangle$, and $\Pi_\gamma(\mathfrak{C}^\perp)=\langle\xi_1\frac{x^\gamma-1}{l_1^*(x)}+\xi_2\frac{x^\gamma-1}{(x-1)l_2^*(x)})\rangle$, then by Theorem 2.2, we have $\Pi_\lambda(\mathfrak{C})\cap\Pi_\lambda(\mathfrak{C}^\perp)=\langle \mathrm{lcm}(A_1(x),\frac{x^\lambda-1}{(x-1)A_1^*(x)}))\rangle$, and $\Pi_\gamma(\mathfrak{C})\cap\Pi_\gamma(\mathfrak{C}^\perp)=\langle\xi_1\,\mathrm{lcm}(l_1(x),\frac{x^\gamma-1}{l_1^*(x)})+\xi_2\,\mathrm{lcm}(l_2(x),\frac{x^\gamma-1}{(x-1)l_2^*(x)}))\rangle$. Since $\Pi_\lambda(\mathrm{Hull}(\mathfrak{C}))=\Pi_\lambda(\mathfrak{C}\cap\mathfrak{C}^\perp)\subseteq \Pi_\lambda(\mathfrak{C})\cap\Pi_\lambda(\mathfrak{C}^\perp)$ and $\Pi_\gamma(\mathrm{Hull}(\mathfrak{C}))=\Pi_\gamma(\mathfrak{C}\cap\mathfrak{C}^\perp)\subseteq\Pi_\gamma(\mathfrak{C})\cap\Pi_\gamma(\mathfrak{C}^\perp)$, then $\mathrm{lcm}\left(A_1(x),\frac{x^\lambda-1}{(x-1)A_1^*(x)}\right)\Big|E(x)$ and $\left(\xi_1\,\mathrm{lcm}\left(l_1(x)+\frac{x^\gamma-1}{l_1^*(x)}\right)\right.$ $\left.+\xi_2\,\mathrm{lcm}\left(l_2(x),\frac{x^\gamma-1}{(x-1)l_2^*(x)}\right)\right)\Big|F(x)$. Therefore, $E(x)=h_1(x)\,\mathrm{lcm}\left(A_1(x),\frac{x^\lambda-1}{(x-1)A_1^*(x)}\right)$, $F(x)=h_2(x)\Big(\xi_1\,\mathrm{lcm}\left(l_1(x)+\frac{x^\gamma-1}{l_1^*(x)}\right)+\xi_2\,\mathrm{lcm}\left(l_2(x),\frac{x^\gamma-1}{(x-1)l_2^*(x)}\right)\Big)$. Putting these values in the previous expression of Hull($\mathfrak{C}$), we have

$$\mathrm{Hull}(\mathfrak{C})$$
$$=\Bigg\langle\left(h_1(x)\,\mathrm{lcm}\left((x-1)A_1(x),\frac{x^\lambda-1}{A_1^*(x)}\right)\mid 0\right),\left(h_1(x)\,\mathrm{lcm}\left(A_1(x),\frac{x^\lambda-1}{(x-1)A_1^*(x)}\right)\right|$$
$$\left.\xi_1h_2(x)\,\mathrm{lcm}\left(l_1(x),\frac{x^\gamma-1}{l_1^*(x)}\right)+\xi_2h_2(x)\,\mathrm{lcm}\left(l_2(x),\frac{x^\gamma-1}{(x-1)l_2^*(x)}\right)\right)\Bigg\rangle.$$

Let

$$\mathfrak{C}'=\Bigg\langle\left(\mathrm{lcm}\left((x-1)A_1(x),\frac{x^\lambda-1}{A_1^*(x)}\right)\Big|0\right),\left(\mathrm{lcm}\left(A_1(x),\frac{x^\lambda-1}{(x-1)A_1^*(x)}\right)\right|$$

$$\xi_1 \operatorname{lcm}\left(l_1(x), \frac{x^\gamma-1}{l_1^*(x)}\right) + \xi_2 \operatorname{lcm}\left(l_2(x), \frac{x^\gamma-1}{(x-1)l_2^*(x)}\right)\right)\Big\rangle.$$

We now establish $\mathrm{Hull}(\mathfrak{C}) = \mathfrak{C}'$. Firstly, we prove that $\mathrm{Hull}(\mathfrak{C}) \subseteq \mathfrak{C}'$. Given $\Big(h_1(x)\operatorname{lcm}(A_1(x), \frac{x^\lambda-1}{(x-1)A_1^*(x)}) \mid \xi_1 h_2(x)\operatorname{lcm}(l_1(x), \frac{x^\gamma-1}{l_1(x)}) + \xi_2 h_2(x)\operatorname{lcm}(l_2(x), \frac{x^\gamma-1}{(x-1)l_2^*(x)})\Big) o\Big(A_1(x), \xi_1 l_1(x) + \xi_2 l_2(x)\Big) = 0$, according to Theorem 4.6, $h_1(x) + h_2(x) \equiv 0 \mod (x-1)$. Therefore, $h_1(x) + h_2(x) = p(x)(x-1)$. Now,

$$\begin{aligned}
&\left(h_1(x)\operatorname{lcm}\left(A_1(x), \frac{x^\lambda-1}{(x-1)A_1^*(x)}\right)\Big|\xi_1 h_2(x)\operatorname{lcm}\left(l_1(x), \frac{x^\gamma-1}{l_1(x)}\right)\right.\\
&\quad\left.+\,\xi_2 h_2(x)\operatorname{lcm}\left(l_2(x), \frac{x^\gamma-1}{(x-1)l_2^*(x)}\right)\right)\\
&= p(x)\star\left(\operatorname{lcm}\left((x-1)A_1(x), \frac{x^\lambda-1}{A_1^*(x)}\right)\Big|0\right) + h_2(x)\star\left(\operatorname{lcm}\left(A_1(x),\right.\right.\\
&\quad\left.\left.\frac{x^\lambda-1}{(x-1)A_1^*(x)}\right)\Big|\xi_1\operatorname{lcm}\left(l_1(x), \frac{x^\gamma-1}{l_1(x)}\right) + \xi_2\operatorname{lcm}\left(l_2(x), \frac{x^\gamma-1}{(x-1)l_2^*(x)}\right)\right) \in \mathfrak{C}'.
\end{aligned}$$

This proves that $\mathrm{Hull}(\mathfrak{C}) \subseteq \mathfrak{C}'$.

Now, we prove that $\mathfrak{C}' \subseteq \mathrm{Hull}(\mathfrak{C})$. For $\Big(\operatorname{lcm}\Big(A_1(x), \frac{x^\lambda-1}{(x-1)A_1^*(x)}\Big) \Big| \xi_1\operatorname{lcm}\Big(l_1(x), \frac{x^\gamma-1}{l_1^*(x)}\Big) + \xi_2\operatorname{lcm}\Big(l_2(x), \frac{x^\gamma-1}{(x-1)l_2^*(x)}\Big)\Big) \in \mathfrak{C}'$, and $(A_1(x) \mid \xi_1 l_1(x) + \xi_2 l_2(x)) \in \mathfrak{C}$, we have

$$\begin{aligned}
&\left(\operatorname{lcm}\left(A_1(x), \frac{x^\lambda-1}{(x-1)A_1^*(x)}\right)\Big|\xi_1\operatorname{lcm}\left(l_1(x), \frac{x^\gamma-1}{l_1^*(x)}\right)\right.\\
&\quad\left.+\,\xi_2\operatorname{lcm}\left(l_2(x), \frac{x^\gamma-1}{(x-1)l_2^*(x)}\right)\right) o\Big(A_1(x) \mid \xi_1 l_1(x) + \xi_2 l_2(x)\Big)\\
&= \xi_2\operatorname{lcm}\left(A_1(x), \frac{x^\lambda-1}{(x-1)A_1^*(x)}\right) A_1^*(x)\frac{x^{\mathfrak{m}}-1}{x^\lambda-1}x^{\mathfrak{m}-1-\deg(A_1(x))}\\
&\quad+\,\xi_1\operatorname{lcm}\left(l_1(x), \frac{x^\gamma-1}{l_1^*(x)}\right) l_1^*(x)\frac{x^{\mathfrak{m}}-1}{x^\gamma-1}x^{\mathfrak{m}-1-\deg(l_1(x))}\\
&\quad+\,\xi_2\operatorname{lcm}\left(l_2(x), \frac{x^\gamma-1}{(x-1)l_2^*(x)}\right) l_2^*(x)\frac{x^{\mathfrak{m}}-1}{x^\gamma-1}x^{\mathfrak{m}-1-\deg(l_2(x))} \mod (x^{\mathfrak{m}}-1)\\
&= \xi_2\operatorname{lcm}\left(A_1^*(x)A_1(x), \frac{x^\lambda-1}{(x-1)}\right)\frac{x^{\mathfrak{m}}-1}{x^\lambda-1}x^{\mathfrak{m}-1-\deg(A_1(x))}
\end{aligned}$$

$$+ \xi_1 \operatorname{lcm}\left(l_1^*(x)l_1(x), x^\gamma - 1\right) \frac{x^{\mathfrak{m}}-1}{x^\gamma-1} x^{\mathfrak{m}-1-\deg(l_1(x))}$$

$$+ \xi_2 \operatorname{lcm}\left(l_2^*(x)l_2(x), \frac{x^\gamma-1}{(x-1)}\right) \frac{x^{\mathfrak{m}}-1}{x^\gamma-1} x^{\mathfrak{m}-1-\deg(l_2(x))} \mod (x^{\mathfrak{m}}-1)$$

$$= \xi_2 P(x)\frac{x^{\mathfrak{m}}-1}{x-1} x^{\mathfrak{m}-1-\deg(A_1(x))} + \xi_1 Q_1(x)(x^{\mathfrak{m}}-1)x^{\mathfrak{m}-1-\deg(l_1(x))} + \xi_2 Q_2(x),$$

$$\frac{x^{\mathfrak{m}}-1}{x-1} x^{\mathfrak{m}-1-\deg(l_2(x))} \mod (x^{\mathfrak{m}}-1)$$

$$= \xi_2 \frac{x^{\mathfrak{m}}-1}{x-1}\left(P(x)x^{\mathfrak{m}-1-\deg(A_1(x))} + Q_2(x)x^{\mathfrak{m}-1-\deg(l_2(x))}\right) \mod (x^{\mathfrak{m}}-1),$$

where $P(x) = A_1(x)$ if $A_1(x)$ is a self-reciprocal polynomial, otherwise $P(x) = 1$. Similarly, $Q_1(x) = l_1(x)$ or $Q_1(x) = 1$ and $Q_1(x) = l_2(x)$ or $Q_2(x) = l_2(x)$.

Note that $P(x) \equiv 1 \mod (x-1)$, $Q_2(x) \equiv 1 \mod (x-1)$ over $\mathbb{Z}_2$, $(x^{\mathfrak{m}}-1) \nmid \left(P(x)x^{\mathfrak{m}-1-\deg(A_1(x))} + Q_2(x)x^{\mathfrak{m}-1-\deg(l_2(x))}\right)$ and $(x-1) \mid \left(x^{\mathfrak{m}-1-\deg(A_1(x))} + x^{\mathfrak{m}-1-\deg(l_2(x))}\right)$ over $\mathbb{Z}_2$. Therefore, $\frac{x^{\mathfrak{m}}-1}{x-1}\left(P(x)x^{\mathfrak{m}-1-\deg(A_1(x))} + Q_2(x)x^{\mathfrak{m}-1-\deg(l_2(x))}\right) \equiv 0 \mod (x^{\mathfrak{m}}-1)$. Thus,

$$\left(\operatorname{lcm}\left(A_1(x), \frac{x^\lambda-1}{(x-1)A_1^*(x)}\right) \middle| \xi_1 \operatorname{lcm}\left(l_1(x), \frac{x^\gamma-1}{l_1^*(x)}\right) + \xi_2 \operatorname{lcm}\left(l_2(x), \frac{x^\gamma-1}{(x-1)l_2^*(x)}\right)\right) o \left(A_1(x) \mid \xi_1 l_1(x) + \xi_2 l_2(x)\right) \equiv 0 \mod (x^{\mathfrak{m}}-1).$$

As $\left(A_1(x) \mid \xi_1 l_1(x) + \xi_2 l_2(x)\right) \in \mathfrak{C}$, then $\left(\operatorname{lcm}\left(A_1(x), \frac{x^\lambda-1}{(x-1)A_1^*(x)}\right) \middle| \xi_1 \operatorname{lcm}\left(l_1(x), \frac{x^\gamma-1}{l_1^*(x)}\right) + \xi_2 \operatorname{lcm}\left(l_2(x), \frac{x^\gamma-1}{(x-1)l_2^*(x)}\right)\right) \in \mathfrak{C}^\perp$. Thus, we have $\mathfrak{C}' \subseteq \mathfrak{C}^\perp$.

For $\left(\frac{x^\lambda-1}{(x-1)A_1^*(x)} \middle| \xi_1 \frac{x^\gamma-1}{l_1^*(x)} + \xi_2 \frac{x^\gamma-1}{(x-1)l_2^*(x)}\right) \in \mathfrak{C}^\perp$ and $\left(\operatorname{lcm}\left(A_1(x), \frac{x^\lambda-1}{(x-1)A_1^*(x)}\right) \middle| \xi_1 \operatorname{lcm}\left(l_1(x), \frac{x^\gamma-1}{l_1^*(x)}\right) + \xi_2 \operatorname{lcm}\left(l_2(x), \frac{x^\gamma-1}{(x-1)l_2^*(x)}\right)\right) \in \mathfrak{C}'$, we have

$$\left(\operatorname{lcm}\left(A_1(x), \frac{x^\lambda-1}{(x-1)A_1^*(x)}\right) \middle| \xi_1 \operatorname{lcm}\left(l_1(x), \frac{x^\gamma-1}{l_1^*(x)}\right) + \xi_2 \operatorname{lcm}\left(l_2(x), \frac{x^\gamma-1}{(x-1)l_2^*(x)}\right)\right) o \left(\frac{x^\lambda-1}{(x-1)A_1^*(x)} \middle| \xi_1 \frac{x^\gamma-1}{l_1^*(x)} + \xi_2 \frac{x^\gamma-1}{(x-1)l_2^*(x)}\right)$$

$$= \xi_2 \operatorname{lcm}\left(A_1(x), \frac{x^\lambda-1}{(x-1)A_1^*(x)}\right) \frac{x^{\mathfrak{m}}-1}{x^\lambda-1} x^{\mathfrak{m}-\lambda+\deg(A_1(x))} \left(\frac{x^\lambda-1}{(x-1)A_1(x)}\right)$$

$$+ \xi_1 \operatorname{lcm}\left(l_1(x), \frac{x^\gamma - 1}{l_1^*(x)}\right) \frac{x^{\mathfrak{m}} - 1}{x^\gamma - 1} x^{\mathfrak{m}-1-\gamma+\deg(l_1(x))} \left(\frac{x^\gamma - 1}{(l_1(x)}\right)$$

$$+ \xi_2 \operatorname{lcm}\left(l_2(x), \frac{x^\gamma - 1}{(x-1)l_2^*(x)}\right) \frac{x^{\mathfrak{m}} - 1}{x^\gamma - 1} x^{\mathfrak{m}-\gamma+\deg(l_2(x))} \left(\frac{x^\gamma - 1}{(x-1)l_2(x)}\right)$$

$$\mod (x^{\mathfrak{m}} - 1)$$

$$= \xi_2 \frac{x^{\mathfrak{m}} - 1}{x - 1} (x^{\mathfrak{m}-\lambda+\deg(A_1(x))} + x^{\mathfrak{m}-\gamma+\deg(l_2(x))}) \mod (x^{\mathfrak{m}} - 1).$$

Clearly, $(x^{\mathfrak{m}} - 1) \nmid x^{\mathfrak{m}-\lambda+\deg(A_1(x))} + x^{\mathfrak{m}-\gamma+\deg(l_2(x))}$ and $(x-1) \mid x^{\mathfrak{m}-\lambda+\deg(A_1(x))} + x^{\mathfrak{m}-\gamma+\deg(l_2(x))}$. Therefore, $\frac{x^{\mathfrak{m}}-1}{x-1}(x^{\mathfrak{m}-\lambda+\deg(A_1(x))} + x^{\mathfrak{m}-\gamma+\deg(l_2(x))}) \equiv 0 \mod (x^{\mathfrak{m}} - 1)$. Thus,

$$\left(\operatorname{lcm}\left(A_1(x), \frac{x^\lambda - 1}{(x-1)A_1^*(x)}\right) \middle| \xi_1 \operatorname{lcm}\left(l_1(x), \frac{x^\gamma - 1}{l_1^*(x)}\right)\right.$$
$$\left. + \xi_2 \operatorname{lcm}\left(l_2(x), \frac{x^\gamma - 1}{(x-1)l_2^*(x)}\right)\right) o \left(\frac{x^\lambda - 1}{(x-1)A_1^*(x)} \middle| \xi_1 \frac{x^\gamma - 1}{l_1^*(x)} + \xi_2 \frac{x^\gamma - 1}{(x-1)l_2^*(x)}\right)$$
$$\equiv 0 \mod (x^{\mathfrak{m}} - 1).$$

Since, $\left(\frac{x^\lambda-1}{(x-1)A_1^*(x)} \middle| \xi_1 \frac{x^\gamma-1}{l_1^*(x)} + \xi_2 \frac{x^\gamma-1}{(x-1)l_2^*(x)}\right) \in \mathfrak{C}^\perp$, then $\left(\operatorname{lcm}\left(A_1(x), \frac{x^\lambda-1}{(x-1)A_1^*(x)}\right) \middle| \xi_1 \operatorname{lcm}\left(l_1(x), \frac{x^\gamma-1}{l_1^*(x)}\right) + \xi_2 \operatorname{lcm}\left(l_2(x), \frac{x^\gamma-1}{(x-1)l_2^*(x)}\right)\right) \in \mathfrak{C}$. So $\mathfrak{C}' \subseteq \mathfrak{C}$, and hence $\mathfrak{C}' \subseteq \operatorname{Hull}(\mathfrak{C})$. Thus,

$$\operatorname{Hull}(\mathfrak{C}) = \left\langle \left(\operatorname{lcm}\left((x-1)A_1(x), \frac{x^\lambda - 1}{A_1^*(x)}\right) \middle| 0\right), \left(\operatorname{lcm}\left(A_1(x), \frac{x^\lambda - 1}{(x-1)A_1^*(x)}\right)\right.\right.$$
$$\left.\left. \middle| \xi_1 \operatorname{lcm}\left(l_1(x), \frac{x^\gamma - 1}{l_1^*(x)}\right) + \xi_2 \operatorname{lcm}\left(l_2(x), \frac{x^\gamma - 1}{(x-1)l_2^*(x)}\right)\right)\right\rangle. \qquad \square$$

Theorem 4.8. *Let λ and γ be pairwise coprime odd positive integers. Suppose χ, Ψ_1, Ψ_2, Φ_2, and Φ_2 are the same as in Eqs. (1), (4), and (5), respectively. Then the dimensions of hulls of NSCCs in $\mathfrak{R}_{\lambda,\gamma}$ are of the form*

$$1 + \sum_{\omega|\lambda} \chi(\omega, 2). \operatorname{ord}_\omega(2).v'_\omega + \sum_{\nu|\gamma} \chi(\nu, 2). \operatorname{ord}_\nu(2).y'_\nu + \sum_{\nu|\gamma} \chi(\nu, 2). \operatorname{ord}_\nu(2).w'_\nu, \quad (34)$$

where $0 \leq v'_\omega \leq \Phi_1(\omega, 2)$, $0 \leq y'_\nu, w'_\nu \leq \Phi_2(\nu, 2)$.

Proof. As λ and γ are odd integers, therefore, $\lambda = \bar{\lambda}$ and $\gamma = \bar{\gamma}$. By Theorem 4.2, $\mathfrak{C} = \langle (x-1)A_1(x) \mid 0), (A_1(x) \mid \xi_1 l_1(x) + \xi_2 l_2(x))\rangle$. Then by Eqs. (6) and (7), we have

$$A_1(x) = \prod_{\substack{\omega|\lambda \\ \chi(\omega,2)=0, \omega\neq 1}} \left(\prod_{e=1}^{\Psi_1(\omega,2)} (h_{e\omega}(x))^{\alpha'_{e\omega}}\right) \prod_{\substack{\omega|\lambda \\ \chi(\omega,2)=1}} \left(\prod_{e=1}^{\Phi_1(\omega,2)} (t_{e\omega}(x))^{\beta'_{e\omega}} (t^*_{e\omega}(x))^{\eta'_{e\omega}}\right) \quad (35)$$

$$l_1(x) = \prod_{\substack{\nu|\gamma \\ \chi(\nu,2)=0}} \left(\prod_{i=1}^{\Psi_2(\nu,2)} (f_{i\nu}(x))^{\delta'_{i\nu}} \right) \prod_{\substack{\nu|\gamma \\ \chi(\nu,2)=1}} \left(\prod_{i=1}^{\Phi_2(\nu,2)} (d_{i\nu}(x))^{\tau'_{i\nu}} (d^*_{i\nu}(x))^{\rho'_{i\nu}} \right) \quad (36)$$

$$l_2(x) = \prod_{\substack{\nu|\gamma \\ \chi(\nu,2)=0,\nu\neq 1}} \left(\prod_{i=1}^{\Psi_2(\nu,2)} (f_{i\nu}(x))^{\sigma'_{i\nu}} \right) \prod_{\substack{\nu|\gamma \\ \chi(\nu,2)=1}} \left(\prod_{i=1}^{\Phi_2(\nu,2)} (d_{i\nu}(x))^{\xi'_{i\nu}} (d^*_{i\nu}(x))^{\Delta'_{i\nu}} \right), \quad (37)$$

where $\alpha'_{e,\omega}, \beta'_{e,\omega}, \eta'_{e,\omega} \in \{0,1\}$ and $\delta'_{i\nu}, \tau'_{i\nu}, \rho'_{i\nu}, \sigma'_{i\nu}, \xi'_{i\nu}, \Delta'_{i\nu} \in \{0,1\}$. Now,

$$A^*_1(x) = \prod_{\substack{\omega|\lambda \\ \chi(\omega,2)=0,\omega\neq 1}} \left(\prod_{e=1}^{\Psi_1(\omega,2)} (h_{e\omega}(x))^{\alpha'_{e\omega}} \right) \prod_{\substack{\omega|\lambda \\ \chi(\omega,2)=1}} \left(\prod_{e=1}^{\Phi_1(\omega,2)} (t_{e\omega}(x))^{\eta'_{e\omega}} (t^*_{e\omega}(x))^{\beta'_{e\omega}} \right),$$

$$(x-1)A^*_1(x) = \prod_{\substack{\omega|\lambda \\ \chi(\omega,2)=0}} \left(\prod_{e=1}^{\Psi_1(\omega,2)} (h_{e\omega}(x))^{\alpha'_{e\omega}} \right) \prod_{\substack{\omega|\lambda \\ \chi(\omega,2)=1}} \left(\prod_{e=1}^{\Phi_1(\omega,2)} (t_{e\omega}(x))^{\eta'_{e\omega}} (t^*_{e\omega}(x))^{\beta'_{e\omega}} \right),$$

$$l^*_1(x) = \prod_{\substack{\nu|\gamma \\ \chi(\nu,2)=0}} \left(\prod_{i=1}^{\Psi_2(\nu,2)} (f_{i\nu}(x))^{\delta'_{i\nu}} \right) \prod_{\substack{\nu|\gamma \\ \chi(\nu,2)=1}} \left(\prod_{i=1}^{\Phi_2(\nu,2)} (d_{i\nu}(x))^{\rho'_{i\nu}} (d^*_{i\nu}(x))^{\tau'_{i\nu}} \right),$$

$$l^*_2(x) = \prod_{\substack{\nu|\gamma \\ \chi(\nu,2)=0,\nu\neq 1}} \left(\prod_{i=1}^{\Psi_2(\nu,2)} (f_{i\nu}(x))^{\sigma'_{i\nu}} \right) \prod_{\substack{\nu|\gamma \\ \chi(\nu,2)=1}} \left(\prod_{i=1}^{\Phi_2(\nu,2)} (d_{i\nu}(x))^{\Delta'_{i\nu}} (d^*_{i\nu}(x))^{\xi'_{i\nu}} \right)$$

$$(x-1)l^*_2(x) = \prod_{\substack{\nu|\gamma \\ \chi(\nu,2)=0}} \left(\prod_{i=1}^{\Psi_2(\nu,2)} (f_{i\nu}(x))^{\sigma'_{i\nu}} \right) \prod_{\substack{\nu|\gamma \\ \chi(\nu,2)=1}} \left(\prod_{i=1}^{\Phi_2(\nu,2)} (d_{i\nu}(x))^{\Delta'_{i\nu}} (d^*_{i\nu}(x))^{\xi'_{i\nu}} \right).$$

$$\frac{x^\lambda - 1}{A^*_1(x)} = \prod_{\substack{\omega|\lambda \\ \chi(\omega,2)=0}} \left(\prod_{e=1}^{\Psi_1(\omega,2)} (h_{e\omega}(x))^{1-\alpha'_{e\omega}} \right) \prod_{\substack{\omega|\lambda \\ \chi(\omega,2)=1}} \left(\prod_{e=1}^{\Phi_1(\omega,2)} (t_{e\omega}(x))^{1-\eta'_{e\omega}} (t^*_{e\omega}(x))^{1-\beta'_{e\omega}} \right),$$

$$\frac{x^\lambda - 1}{(x-1)A^*_1(x)} = \prod_{\substack{\omega|\lambda \\ \chi(\omega,2)=0,\omega\neq 1}} \left(\prod_{e=1}^{\Psi_1(\omega,2)} (h_{e\omega}(x))^{1-\alpha'_{e\omega}} \right) \prod_{\substack{\omega|\lambda \\ \chi(\omega,2)=1}} \left(\prod_{e=1}^{\Phi_1(\omega,2)} (t_{e\omega}(x))^{1-\eta'_{e\omega}} (t^*_{e\omega}(x))^{1-\beta'_{e\omega}} \right),$$

$$\frac{x^\gamma - 1}{l^*_1(x)} = \prod_{\substack{\nu|\gamma \\ \chi(\nu,2)=0}} \left(\prod_{i=1}^{\Psi_2(\nu,2)} (f_{i\nu}(x))^{1-\delta'_{i\nu}} \right) \prod_{\substack{\nu|\gamma \\ \chi(\nu,2)=1}} \left(\prod_{i=1}^{\Phi_2(\nu,2)} (d_{i\nu}(x))^{1-\rho'_{i\nu}} (d^*_{i\nu}(x))^{1-\tau'_{i\nu}} \right),$$

$$\frac{x^{\gamma}-1}{l_2^*(x)}=\prod_{\substack{\nu|\gamma\\ \chi(\nu,2)=0,}}\left(\prod_{i=1}^{\Psi_2(\nu,2)}(f_{i\nu}(x))^{1-\sigma'_{i\nu}}\right)\prod_{\substack{\nu|\gamma\\ \chi(\nu,2)=1}}\left(\prod_{i=1}^{\Phi_2(\nu,2)}(d_{i\nu}(x))^{1-\Delta'_{i\nu}}(d^*_{i\nu}(x))^{1-\xi'_{i\nu}}\right),$$

$$\frac{x^{\gamma}-1}{(x-1)l_2^*(x)}=\prod_{\substack{\nu|\gamma\\ \chi(\nu,2)=0,\nu\neq 1}}\left(\prod_{i=1}^{\Psi_2(\nu,2)}(f_{i\nu}(x))^{1-\sigma_{i\nu}}\right)\prod_{\substack{\nu|\gamma\\ \chi(\nu,2)=1}}\left(\prod_{i=1}^{\Phi_2(\nu,2)}(d_{i\nu}(x))^{1-\Delta'_{i\nu}}(d^*_{i\nu}(x))^{1-\xi'_{i\nu}}\right).$$

According to Theorem 4.7,

$$\begin{aligned}G_1 &= \operatorname{lcm}\left((x-1)A_1(x),\frac{x^{\lambda}-1}{A_1^*(x)}\right)\\ &= \prod_{\substack{\omega|\lambda\\ \chi(\omega,2)=0}}\left(\prod_{e=1}^{\Psi_1(\omega,2)}(h_{e\omega}(x))^{\max\{\alpha'_{e\omega},1-\alpha'_{e\omega}\}}\right)\\ &\quad\times\prod_{\substack{\omega|\lambda\\ \chi(\omega,2)=1}}\left(\prod_{e=1}^{\Phi_1(\omega,2)}(t_{e\omega}(x))^{\max\{\beta'_{e\omega},1-\eta'_{e\omega}\}}(t^*_{e\omega}(x))^{\max\{\eta'_{e\omega},1-\beta'_{e\omega}\}}\right),\end{aligned}\tag{38}$$

$$\begin{aligned}G_2 &= \operatorname{lcm}\left(A_1(x),\frac{x^{\lambda}-1}{(x-1)A_1^*(x)}\right)\\ &= \prod_{\substack{\omega|\lambda\\ \chi(\omega,2)=0,\omega\neq 1}}\left(\prod_{e=1}^{\Psi_1(\omega,2)}(h_{e\omega}(x))^{\max\{\alpha'_{e\omega},1-\alpha'_{e\omega}\}}\right)\\ &\quad\times\prod_{\substack{\omega|\lambda\\ \chi(\omega,2)=1}}\left(\prod_{e=1}^{\Phi_1(\omega,2)}(t_{e\omega}(x))^{\max\{\beta'_{e\omega},1-\eta'_{e\omega}\}}(t^*_{e\omega}(x))^{\max\{\eta'_{e\omega},1-\beta'_{e\omega}\}}\right),\end{aligned}\tag{39}$$

$$\begin{aligned}G_3 &= \left(l_1(x),\frac{x^{\gamma}-1}{l_1^*(x)}\right)\\ &= \prod_{\substack{\nu|\gamma\\ \chi(\nu,2)=0}}\left(\prod_{i=1}^{\Psi_2(\nu,2)}(f_{i\nu}(x))^{\max\{\delta'_{i\nu},1-\delta'_{i\nu}\}}\right)\\ &\quad\times\prod_{\substack{\nu|\gamma\\ \chi(\nu,2)=1}}\left(\prod_{i=1}^{\Phi_2(\nu,2)}(d_{i\nu}(x))^{\max\{\tau'_{i\nu},1-\rho'_{i\nu}\}}(d^*_{i\nu}(x))^{\max\{\rho'_{i\nu},1-\tau'_{i\nu}\}}\right),\end{aligned}\tag{40}$$

$$\begin{aligned} G_4 =& \operatorname{lcm}(l_2(x), \frac{x^\gamma - 1}{(x-1)l_2^*(x)}) \\ =& \prod_{\substack{\nu|\gamma \\ \chi(\nu,2)=0, \nu\neq 1}} \left(\prod_{i=1}^{\Psi_2(\nu,2)} (f_{i\nu}(x))^{\max\{\sigma'_{i\nu}, 1-\sigma'_{i\nu}\}} \right) \times \\ & \prod_{\substack{\nu|\gamma \\ \chi(\nu,2)=1}} \left(\prod_{i=1}^{\Phi_2(\nu,2)} (d_{i\nu}(x))^{\max\{\xi'_{i\nu}, 1-\Delta'_{i\nu}\}} (d^*_{i\nu}(x))^{\max\{\Delta'_{i\nu}, 1-\xi'_{i\nu}\}} \right), \end{aligned} \tag{41}$$

where

$$\max\{\alpha'_{e\omega}, 1-\alpha'_{e\omega}\} = 1 \text{ and } 1 \leq \max\{\beta'_{e\omega}, 1-\eta'_{e\omega}\} + \max\{\eta'_{e\omega}, 1-\beta'_{e\omega}\} \leq 2, \tag{42}$$

$$\max\{\delta'_{i\nu}, 1-\delta'_{i\nu}\} = 1 \text{ and } 1 \leq \max\{\tau'_{i\nu}, 1-\rho'_{i\nu}\} + \max\{\rho'_{i\nu}, 1-\tau'_{i\nu}\} \leq 2, \tag{43}$$

$$\max\{\sigma'_{i\nu}, 1-\sigma'_{i\nu}\} = 1 \text{ and } 1 \leq \max\{\xi'_{i\nu}, 1-\Delta'_{i\nu}\} + \max\{\Delta'_{i\nu}, 1-\xi'_{i\nu}\} \leq 2. \tag{44}$$

and

$$\operatorname{Hull}(\mathfrak{C}) = \langle (G_1 \mid 0), (G_2 \mid \xi_1 G_3 + \xi_2 G_4) \rangle. \tag{45}$$

From Eqs. (38)-(41) and Eqs.(42)-(44), we have

$$\begin{aligned} \dim(\operatorname{Hull}(\mathfrak{C})) =& \lambda - \deg(G_1) + 2\gamma - (\deg(G_3) + \deg(G_4)) \\ =& \Bigg(\sum_{\substack{\omega|\lambda \\ \chi(\omega,2)=0}} \operatorname{ord}_\omega(2) \sum_{e=1}^{\Psi_1(\omega,2)} 1 + \sum_{\substack{\omega|\lambda \\ \chi(\omega,2)=1}} \operatorname{ord}_\omega(2) \sum_{e=1}^{\Phi_1(\omega,2)} 2 \\ & - \sum_{\substack{\omega|\lambda \\ \chi(\omega,2)=0}} \operatorname{ord}_\omega(2) \sum_{e=1}^{\Psi_1(\omega,2)} \max\{\alpha'_{e\omega}, 1-\alpha'_{e\omega}\} \\ & - \sum_{\substack{\omega|\lambda \\ \chi(\omega,2)=1}} \operatorname{ord}_\omega(2) \sum_{e=1}^{\Phi_1(\omega,2)} (\max\{\beta'_{e\omega}, 1-\eta'_{e\omega}\} + \max\{\eta'_{e\omega}, 1-\beta'_{e\omega}\}) \Bigg) \\ & + \Bigg(\sum_{\substack{\nu|\gamma \\ \chi(\nu,2)=0}} \operatorname{ord}_\nu(2) \sum_{i=1}^{\Psi_2(\nu,2)} 1 + \sum_{\substack{\nu|\gamma \\ \chi(\nu,2)=1}} \operatorname{ord}_\nu(2) \sum_{i=1}^{\Phi_2(\nu,2)} 2 \\ & - \sum_{\substack{\nu|\gamma \\ \chi(\nu,2)=0}} \operatorname{ord}_\nu(2) \sum_{i=1}^{\Psi_2(\nu,2)} \max\{\delta'_{i\nu}, 1-\delta_{i\nu}\} \\ & - \sum_{\substack{\nu|\gamma \\ \chi(\nu,2)=1}} \operatorname{ord}_\nu(2) \sum_{i=1}^{\Phi_2(\nu,2)} (\max\{\tau'_{i\nu}, 1-\rho'_{i\nu}\} + \max\{\rho'_{i\nu}, 1-\tau'_{i\nu}\}) \Bigg) \\ & + \Bigg(\sum_{\substack{\nu|\gamma \\ \chi(\nu,2)=0}} \operatorname{ord}_\nu(2) \sum_{i=1}^{\Psi_2(\nu,2)} 1 + \sum_{\substack{\nu|\gamma \\ \chi(\nu,2)=1}} \operatorname{ord}_\nu(2) \sum_{i=1}^{\Phi_2(\nu,2)} 2 \end{aligned}$$

$$
\begin{aligned}
&- \sum_{\substack{\nu|\gamma \\ \chi(\nu,2)=0,\nu\neq 1}} \mathrm{ord}_\nu(2) \sum_{i=1}^{\Psi_2(\nu,2)} \max\{\sigma'_{i\nu}, 1-\sigma'_{i\nu}\} \\
&- \sum_{\substack{\nu|\gamma \\ \chi(\nu,2)=1}} \mathrm{ord}_\nu(2) \sum_{i=1}^{\Phi_2(\nu,2)} (\max\{\xi'_{i\nu}, 1-\Delta'_{i\nu}\} + \max\{\Delta'_{i\nu}, 1-\xi'_{i\nu}\})\Bigg) \\
=&\Bigg(\sum_{\substack{\omega|\lambda \\ \chi(\omega,2)=1}} \mathrm{ord}_\omega(2) \sum_{e=1}^{\Phi_1(\omega,2)} 2 - \sum_{\substack{\omega|\lambda \\ \chi(\omega,2)=1}} \mathrm{ord}_\omega(2) \sum_{e=1}^{\Phi_1(\omega,2)} (\max\{\beta'_{e\omega}, 1-\eta'_{e\omega}\} \\
&+ \max\{\eta'_{e\omega}, 1-\beta'_{e\omega}\})\Bigg) + \Bigg(\sum_{\substack{\nu|\gamma \\ \chi(\nu,2)=1}} \mathrm{ord}_\nu(2) \sum_{i=1}^{\Phi_2(\nu,2)} 2 - \sum_{\substack{\nu|\gamma \\ \chi(\nu,2)=1}} \mathrm{ord}_\nu(2) \\
&\sum_{i=1}^{\Phi_2(\nu,2)} (\max\{\tau'_{i\nu}, 1-\rho'_{i\nu}\} + \max\{\rho'_{i\nu}, 1-\tau'_{i\nu}\})\Bigg) + \Bigg(\mathrm{ord}_1(2) \sum_{i=1}^{\Psi_2(1,2)} 1 \\
&+ \sum_{\substack{\nu|\gamma \\ \chi(\nu,2)=0,\nu\neq 1}} \mathrm{ord}_\nu(2) \sum_{i=1}^{\Psi_2(\nu,2)} 1 + \sum_{\substack{\nu|\gamma \\ \chi(\nu,2)=1}} \mathrm{ord}_\nu(2) \sum_{i=1}^{\Phi_2(\nu,2)} 2 - \sum_{\substack{\nu|\gamma \\ \chi(\nu,2)=0,\nu\neq 1}} \mathrm{ord}_\nu(2) \\
&\sum_{i=1}^{\Psi_2(\nu,2)} \max\{\sigma'_{i\nu}, 1-\sigma'_{i\nu}\} - \sum_{\substack{\nu|\gamma \\ \chi(\nu,2)=1}} \mathrm{ord}_\nu(2) \sum_{i=1}^{\Phi_2(\nu,2)} (\max\{\xi'_{i\nu}, 1-\Delta'_{i\nu}\} \\
&\quad + \max\{\Delta'_{i\nu}, 1-\xi'_{i\nu}\})\Bigg) \\
=&\Bigg(\sum_{\substack{\omega|\lambda \\ \chi(\omega,2)=1}} \mathrm{ord}_\omega(2) \sum_{e=1}^{\Phi_1(\omega,2)} 2 - \sum_{\substack{\omega|\lambda \\ \chi(\omega,2)=1}} \mathrm{ord}_\omega(2) \sum_{e=1}^{\Phi_1(\omega,2)} (\max\{\beta'_{e\omega}, 1-\eta'_{e\omega}\} \\
&+ \max\{\eta'_{e\omega}, 1-\beta'_{e\omega}\} + \sum_{\substack{\nu|\gamma \\ \chi(\nu,2)=1}} \mathrm{ord}_\nu(2) \sum_{i=1}^{\Phi_2(\nu,2)} 2 - \sum_{\substack{\nu|\gamma \\ \chi(\nu,2)=1}} \mathrm{ord}_\nu(2) \\
&\sum_{i=1}^{\Phi_2(\nu,2)} (\max\{\tau'_{i\nu}, 1-\rho'_{i\nu}\} + \max\{\rho'_{i\nu}, 1-\tau'_{i\nu}\}) + 1 + \sum_{\substack{\nu|\gamma \\ \chi(\nu,2)=1}} \mathrm{ord}_\nu(2) \sum_{i=1}^{\Phi_2(\nu,2)} 2 \\
&- \sum_{\substack{\nu|\gamma \\ \chi(\nu,2)=1}} \mathrm{ord}_\nu(2) \sum_{i=1}^{\Phi_2(\nu,2)} (\max\{\xi'_{i\nu}, 1-\Delta'_{i\nu}\} + \max\{\Delta'_{i\nu}, 1-\xi'_{i\nu}\})\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
&= 1 + \sum_{\omega|\lambda} \chi(\omega,2).\operatorname{ord}_\omega(2) \sum_{e=1}^{\Phi_1(\omega,2)} v'_{e\omega} + \sum_{\nu|\gamma} \chi(\nu,2).\operatorname{ord}_\nu(2) \sum_{i=1}^{\Phi_2(\nu,2)} y'_{i\nu} \\
&\quad + \sum_{\nu|\gamma} \chi(\nu,2).\operatorname{ord}_\nu(2) \sum_{i=1}^{\Phi_2(\nu,2)} w'_{i\nu},
\end{aligned} \tag{46}
$$

where

$$v'_{e\omega} = 2 - \max\{\beta'_{e\omega}, 1-\eta'_{e\omega}\} - \max\{\eta'_{e\omega}, 1-\beta'_{e\omega}\} \tag{47}$$

$$y'_{i\nu} = 2 - \max\{\tau'_{i\nu}, 1-\rho'_{i\nu}\} - \max\{\rho'_{i\nu}, 1-\tau'_{i\nu}\} \tag{48}$$

$$w'_{i\nu} = 2 - \max\{\xi'_{i\nu}, 1-\Delta'_{i\nu}\} - \max\{\Delta'_{i\nu}, 1-\xi'_{i\nu}\}. \tag{49}$$

By Eqs.(42)-(44) and Eqs.(47)-(49), we have $v'_{e\omega}, y'_{i\nu}, w'_{i\nu} \in \{0,1\}$. Then Eq.(46) becomes

$$
\begin{aligned}
\dim(\mathrm{Hull}(\mathfrak{C})) = &1 + \sum_{\omega|\lambda} \chi(\omega,2).\operatorname{ord}_\omega(2).v'_\omega + \sum_{\nu|\gamma} \chi(\nu,2).\operatorname{ord}_\nu(2).y'_\nu \\
&+ \sum_{\nu|\gamma} \chi(\nu,2).\operatorname{ord}_\nu(2).w'_\nu,
\end{aligned} \tag{50}
$$

where $v'_\omega = \sum_{e=1}^{\Phi_1(\omega,2)} v'_{e\omega}$, $y'_\nu = \sum_{i=1}^{\Phi_2(\nu,2)} y'_{i\nu}$ and $w'_\nu = \sum_{i=1}^{\Phi_2(\nu,2)} w'_{i\nu}$. From Eqs.(47)-(49), we get that $0 \le v'_\omega \le \Phi_1(\omega,2)$, $0 \le y'_\nu \le \Phi_2(\nu,2)$, and $0 \le w'_\nu \le \Phi_2(\nu,2)$. This completes the proof. □

The multi-set S in this case is of the form, $S = \{\Phi_1(\omega,2) * \operatorname{ord}_\omega(2) \mid \omega|\lambda$ and $\chi(\omega,2) = 1\} \bigcup \{1 * \operatorname{ord}_\nu(2) \mid \nu|\gamma, \chi(\nu,2) = 0$ and $\nu = 1\} \bigcup \{2\Phi_2(\nu,2) * \operatorname{ord}_\nu(2) \mid \nu|\gamma$ and $\chi(\nu,2) = 1\}$.

Each value in Eq. (50) represents the sum of elements from a subset of S. These values can be determined using the generating functions, as discussed in [8]. Alternatively, the dimensions of hulls of NSCCs in $\mathfrak{R}_{\lambda,\gamma}$ correspond to the exponents of X in the following expression:

$$\prod_{\substack{\omega|\lambda \\ \chi(\omega,2)=1}} \left(\sum_{n=0}^{\Phi_1(\omega,2)} (X^{\operatorname{ord}_\omega(2)})^n \right) X \prod_{\substack{\nu|\gamma \\ \chi(\nu,2)=1}} \left(\sum_{n=0}^{2\Phi_2(\nu,2)} (X^{\operatorname{ord}_\nu(2)})^n \right), \tag{51}$$

where χ, Φ_1, Φ_2 be as defined previously.

Example 4.9. Let $\lambda = 7$ and $\gamma = 15$. Then divisors of 7 are $1, 7$ and divisors of 15 are $1, 3, 5, 15$. Note that $\chi(1,2), \chi(5,2) = 0$ and $\chi(7,2), \chi(15,2) = 1$. Since $\operatorname{ord}_7(2) = 3, \operatorname{ord}_{15}(2) = 4$, according to Theorem 4.8, the dimensions of hulls of NSCCs of length $(7,15)$ in $\mathfrak{R}_{\lambda,\gamma}$ are of the form

$$1 + 3v'_7 + 4y'_{15} + 4w'_{15},$$

where $0 \le v'_7 \le 1$, and $0 \le y'_{15}, w'_{21} \le 1$. Moreover, by Eq.(51), the dimensions of hulls of NSCCs of length $(7,15)$ in $\mathfrak{R}_{\lambda,\gamma}$ are the exponents of X in the expression $(1+X^3)X(1+X^4+X^8)$, which are $12, 9, 8, 5, 4, 1$.

In the next theorem, we determine the number of NSCCs over $\mathbb{Z}_2\mathbb{Z}_2[u]$ having hulls of a fixed dimension (say d').

Theorem 4.10. *Let λ, γ be odd, pairwise coprime integers and d' be given in the form of Eq.(34). Let χ, Φ_1 and Φ_2 be the same as in Eqs.(1), (4), and (5), respectively. Then the number of NSCCs over $\mathbb{Z}_2\mathbb{Z}_2[v]$ whose hulls have dimension "d'" is $2^{m_2+2k_2}|h(d')|$, where m_2 and k_2 are defined in the Theorem 2.10, and*

$$h(d') = \Bigg\{ ((v'_{e\omega}), (y'_{i\nu}), (w'_{i\nu})) | 1 + \sum_{\omega|\lambda} \chi(\omega, 2) \operatorname{ord}_\omega(2) \sum_{e=1}^{\Phi_1(\omega,2)} v'_{e\omega} + \sum_{\nu|\gamma} \chi(\nu, 2) \operatorname{ord}_\nu(2) \sum_{i=1}^{\Phi_2(\nu,2)} y'_{i\nu} + \sum_{\nu|\gamma} \chi(\nu, 2) \operatorname{ord}_\nu(2) \sum_{i=1}^{\Phi_2(\nu,2)} w'_{i\nu} = d' \Bigg\}. \tag{52}$$

Proof. The proof can be done using the same concept as Corollary 3.8. □

Example 4.11. In Example 4.9, we determine all the possible dimensions of hulls of NSCCs of length $(7, 15)$ in $\mathfrak{R}_{\lambda,\gamma}$, which are $12, 9, 8, 5, 4, 1$. Here, we count the number of NSCCs of length $(7, 15)$ in $\mathfrak{R}_{\lambda,\gamma}$ having hulls of the fixed dimension 4. Note that

$$h(4) = \{((v'_{1,7}), (y'_{1,15}), (w'_{1,15})) \mid 1 + 3v'_{1,7} + 4y'_{1,15} + 4w'_{1,15} = 4\},$$

where $0 \leq v'_{1,7} \leq 1$ and $0 \leq y'_{1,15}, w'_{1,15} \leq 1$. Moreover, we can determine the cardinality of $h(4)$, which is coefficient of X^4 in $(1 + X^3)X(1 + X^4)(1 + X^4) = X^{12} + X^9 + X^8 + X^5 + X^4 + X^3 + X$. Since the coefficient of X^4 is 1, then $|h(4)| = 1$. Therefore, by Theorem 4.10, the number of NSCCs of length $(7, 15)$ in $\mathfrak{R}_{\lambda,\gamma}$ having hulls of fixed dimension 4 is $2^{1+2\cdot1}.|h(4)| = 8$

We list all eight NSCCs of length $(7, 15)$ in $\mathfrak{R}_{\lambda,\gamma}$ whose hulls have the fixed dimension 4 in Table 1.

TABLE 1. NSCCs of length $(7, 15)$ in $\mathfrak{R}_{\lambda,\gamma}$, whose hulls have the fixed dimension 4

Generator of $\mathfrak{C}$	**Generator of** $\mathrm{Hull}(\mathfrak{C})$
$\langle(x^4+x^3+x^2+1 \mid 0), (x^3+x+1 \mid \xi_1(x-1)+\xi_2(x^2+x+1))\rangle$	$\langle(x^4+x^3+x^2+1 \mid 0), (x^3+x+1 \mid \xi_1(x^{15}-1)+\xi_2(x^{14}+x^{13}+\cdots+x+1))\rangle$
$\langle(x^4+x^3+x^2+1 \mid 0), (x^3+x+1 \mid \xi_1(x^4+x^3+x^2+x+1)+\xi_2(x^2+x+1))\rangle$	$\langle(x^4+x^3+x^2+1 \mid 0), (x^3+x+1 \mid \xi_1(x^{15}-1)+\xi_2(x^{14}+x^{13}+\cdots+x+1))\rangle$
$\langle(x^4+x^3+x^2+1 \mid 0), (x^3+x+1 \mid \xi_1(x-1)+\xi_2(x^4+x^3+x^2+x+1))\rangle$	$\langle(x^4+x^3+x^2+1 \mid 0), (x^3+x+1 \mid \xi_1(x^{15}-1)+\xi_2(x^{14}+x^{13}+\cdots+x+1))\rangle$
$\langle(x^4+x^3+x^2+1 \mid 0), (x^3+x+1 \mid \xi_1(x^4+x^3+x^2+x+1)+\xi_2(x^4+x^3+x^2+x+1))\rangle$	$\langle(x^4+x^3+x^2+1 \mid 0), (x^3+x+1 \mid \xi_1(x^{15}-1)+\xi_2(x^{14}+x^{13}+\cdots+x+1))\rangle$
$\langle(x^4+x^2+x+1 \mid 0), (x^3+x^2+1 \mid \xi_1(x-1)+\xi_2(x^2+x+1))\rangle$	$\langle(x^4+x^2+x+1 \mid 0), (x^3+x^2+1 \mid \xi_1(x^{15}-1)+\xi_2(x^{14}+x^{13}+\cdots+x+1))\rangle$
$\langle(x^4+x^2+x+1 \mid 0), (x^3+x^2+1 \mid \xi_1(x^4+x^3+x^2+x+1)+\xi_2(x^2+x+1))\rangle$	$\langle(x^4+x^2+x+1 \mid 0), (x^3+x^2+1 \mid \xi_1(x^{15}-1)+\xi_2(x^{14}+x^{13}+\cdots+x+1))\rangle$
$\langle(x^4+x^2+x+1 \mid 0), (x^3+x^2+1 \mid \xi_1(x-1)+\xi_2(x^4+x^3+x^2+x+1))\rangle$	$\langle(x^4+x^2+x+1 \mid 0), (x^3+x^2+1 \mid \xi_1(x^{15}-1)+\xi_2(x^{14}+x^{13}+\cdots+x+1))\rangle$
$\langle(x^4+x^2+x+1 \mid 0), (x^3+x^2+1 \mid \xi_1(x^4+x^3+x^2+x+1)+\xi_2(x^4+x^3+x^2+x+1))\rangle$	$\langle(x^4+x^2+x+1 \mid 0), (x^3+x^2+1 \mid \xi_1(x^{15}-1)+\xi_2(x^{14}+x^{13}+\cdots+x+1))\rangle$

5. **Construction of EAQECCs.** In quantum information processing and quantum computation, quantum error-correcting codes are crucial [6, 17, 28]. Dual-containing (or self-orthogonal) classical codes can be constructed as ordinary quantum codes using the stabilizer formalism [6]. The dual-containing condition, however, presents a challenge to the advancement of quantum coding theory. If shared entanglement is available between the sender and receiver, Brun et al. [5] showed that entanglement-assisted (EA) stabilizer formalism can be used to generate EAQECCs from non-dual-containing classical codes.

An $[[n,k,d;c]]_q$ EAQECC over a $\mathbb{F}_q$, encodes k information qubits into n channel qubits by means of c pairs of maximally-entangled Bell states (i.e., c ebits) and correct up to $[\frac{d-1}{2}]$ errors, where d is the minimum distance of EAQECC. Notably, when $c=0$, the EAQECC simplifies to a q-array standard $[[n,k,d]]$ QECC. The performance of an $[[n,k,d;c]]_q$ EAQECCs is measured by its rate $\frac{k}{n}$ and net rate $\frac{k-c}{n}$.

Definition 5.1. [23] An $[[n_1,k_1,d_1;c_1]]_q$ EAQECC is said to be better than any other $[[n_2,k_2,d_2;c_2]]_q$ EAQECC if at least one of the following conditions holds:

1. $\frac{k_1}{n_1}>\frac{k_2}{n_2}$, *and* $\frac{k_1-c_1}{n_1}>\frac{k_2-c_2}{n_2}$ *if* $d_1=d_2$, *i.e., larger code rate and net rate with same distance.*
2. $d_1>d_2$, *if* $\frac{k_1-c_1}{n_1}=\frac{k_2-c_2}{n_2}$ *i.e, larger distance with same net rate.*

Theorem 5.2. [5, EA Singleton Bound] *An* $[[n,k,d;c]]_q$ *EAQECC with* $0\le c\le n-1$, *satisfies* $2(d-1)\le n-k+c$.

An EAQECC achieving this singleton bound is called a maximum-distance separable (MDS) EAQECC.

Definition 5.3. [21] An EAQECC with parameters $[[n,k,d;c]]_q$ is called a weakly maximum-distance separable (WMDS) EAQECC if $2d=n-k+c$.

Theorem 5.4. [13] *Let* $\mathfrak{C}$ *be an* $[n,k_1,d]$ *linear code and* $\mathfrak{C}^\perp$ *be its Euclidean dual code with parameters* $[n,k_2,d^\perp]$. *Then* $[[n,k_1-\dim(\mathrm{Hull}(\mathfrak{C})),d,k_2-\dim(\mathrm{Hull}(\mathfrak{C}))]]_q$ *and* $[[n,k_2-\dim(\mathrm{Hull}(\mathfrak{C})),d^\perp,k_1-\dim(\mathrm{Hull}(\mathfrak{C}))]]_q$ *EAQECCs exist.*

In the following example, we construct EAQECCs using hulls of SCCs over $\mathbb{Z}_p\mathbb{Z}_p[v]$.

Example 5.5. Let $q=5$, $\lambda=3,\gamma=5$ and $\mathfrak{C}=\langle(x^2+x+1\mid 0),(0\mid \xi_1(x+4)^4+\xi_2(x+4)^5)\rangle$ be a SCC of length $(3,5)$ in $\mathfrak{R}_{\lambda,\gamma}$, and $\Phi(\mathfrak{C})$ has parameters $[13,2,3]$. Then by Theorem 2.8, $\mathfrak{C}^\perp=\langle(x+4\mid 0),(0\mid \xi_1(x+4)+\xi_2(1))\rangle$ and $\Phi(\mathfrak{C}^\perp)$ has parameters $[13,11,2]$. Moreover, $\mathrm{Hull}(\mathfrak{C})=\langle((x^3-1\mid 0),(0\mid \xi_1(x^4+x^3+x^2+x+1)+\xi_2(x^5-1)))\rangle$ and $\dim(\mathrm{Hull}(\mathfrak{C}))=1$ By Theorem 5.4, we get an EAQECC with parameters $[[13,10,2;1]]_5$. The parameters of this EAQECC satisfy $2d=n-k+c$, i.e., it is a WMDS EAQECC.

Example 5.6. Let $q=7$, $\lambda=11,\gamma=13$ and $\mathfrak{C}=\langle(x+6\mid 0),(0\mid \xi_1(x+1)+\xi_2(1))\rangle$ be a SCC of length $(11,13)$ in $\mathfrak{R}_{\lambda,\gamma}$, and $\Phi(\mathfrak{C})$ has parameters $[37,35,2]$. Then by Theorem 2.8, $\mathfrak{C}^\perp=\langle(x^{10}+x^9+\cdots+x+1\mid 0),(0\mid \xi_1(x^{12}+x^{11}+\cdots+x+1)+\xi_2(x^{13}-1))\rangle$ and $\Phi(\mathfrak{C}^\perp)$ has parameters $[37,2,11]$. Moreover, $\mathrm{Hull}(\mathfrak{C})=\langle((x^{11}-1\mid 0),(0\mid \xi_1(x^{13}-1)+\xi_2(x^{13}-1)))\rangle$ and $\dim(\mathrm{Hull}(\mathfrak{C}))=0$ By Theorem 5.4, we get an EAQECC with parameters $[[37,35,2;2]]_7$. The parameters of this EAQECC satisfy $2d=n-k+c$, i.e., it is a WMDS EAQECC.

Çalişkan et al. in [7] and Liu et al. in [21] have used Euclidean and Hermitian hulls of skew cyclic codes over $\mathbb{F}_2 \times (\mathbb{F}_2 + u\mathbb{F}_2)$, and the Euclidean hulls of cyclic codes over $\mathbb{F}_q + u\mathbb{F}_q$, respectively, to construct some good EAQECCs over $\mathbb{F}_2$. In the following examples, we construct some good EAQECCs using the Euclidean hulls of NSCCs over $\mathbb{Z}_2\mathbb{Z}_2[v]$. Moreover, our EAQECCs have better parameters than those of [7, 21].

Example 5.7. Let $\lambda = 47, \gamma = 21$ and $\mathfrak{C} = \langle((x^{24} + x^{23} + x^{20} + x^{18} + x^{15} + x^{12} + x^{11} + x^9 + x^8 + x^5 + x^4 + 1) \mid 0), ((x^{23} + x^{19} + x^{18} + x^{14} + x^{13} + x^{12} + x^{10} + x^9 + x^7 + x^6 + x^5 + x^3 + x^2 + x + 1) \mid \xi_1(x^{19} + x^{18} + x^{16} + x^{15} + x^{13} + x^{12} + x^{10} + x^9 + x^7 + x^6 + x^4 + x^3 + x + 1) + \xi_2(x^2 + x+)))\rangle$ be a NSCC of length $(47, 21)$ in $\mathfrak{R}_{\lambda,\gamma}$ and $\Phi(\mathfrak{C})$ has parameters $[89, 44, 9]$. Then by Corollary 4.3, $\mathfrak{C}^\perp = \langle((x^{24} + x^{23} + x^{20} + x^{18} + x^{15} + x^{12} + x^{11} + x^9 + x^8 + x^5 + x^4 + 1) \mid 0), ((x^{23}+x^{19}+x^{18}+x^{14}+x^{13}+x^{12}+x^{10}+x^9+x^7+x^6+x^5+x^3+x^2+x+1) \mid \xi_1(x^2+x+1)+\xi_2(x^{18}+x^{15}+x^{12}+x^9+x^6+x^3+1)))\rangle$ and $\Phi(\mathfrak{C}^\perp)$ has parameters $[89, 45, 2]$. Moreover, $\mathrm{Hull}(\mathfrak{C}) = \langle((x^{24}+x^{23}+x^{20}+x^{18}+x^{15}+x^{12}+x^{11}+x^9+x^8+x^5+x^4+1) \mid 0), ((x^{23} + x^{19} + x^{18} + x^{14} + x^{13} + x^{12} + x^{10} + x^9 + x^7 + x^6 + x^5 + x^3 + x^2 + x + 1) \mid \xi_1(x^{21}+1)+\xi_2(x^{20}+x^{19}+\cdots+x+1)))\rangle$ and $\Phi(\mathrm{Hull}(\mathfrak{C}))$ has parameters $[89, 24, 12]$. By Theorem 5.4, we have an EAQECC with parameters $[[89, 20, 9; 21]]_2$. This EAQECC has a better net rate than the net rate of the existing EAQECC $[[90, 10, 9; 80]]_2$ in [21].

Example 5.8. Let $\lambda = 25, \gamma = 33$ and $\mathfrak{C} = \langle((x^5 + 1) \mid 0), ((x^4 + x^3 + x^2 + x + 1) \mid \xi_1(x^{13}+x^{12}+x^{10}+x^9+x^8+x^5+x^4+x^3+x+1)+\xi_2(x^{12}+x^9+x^7+x^6+x^5+x^3+1)))\rangle$ be a NSCC of length $(17, 9)$ in $\mathfrak{R}_{\lambda,\gamma}$ and $\Phi(\mathfrak{C})$ has parameters $[91, 61, 2]$. Then by Corollary 4.3, $\mathfrak{C}^\perp = \langle((x-1)(x^{20}+x^{15}+x^{10}+x^5+1) \mid 0), ((x^{20}+x^{15}+x^{10}+x^5+1) \mid \xi_1(x^{20}+x^{19}+x^{18}+x^{15}+x^{14}+x^{10}+x^6+x^5+x^2+x+1)+\xi_2(x^{20}+x^{19}+x^{18}+x^{15}+x^{14}+x^{10}+x^6+x^5+x^2+x+1))$ and $\Phi(\mathfrak{C}^\perp)$ has parameters $[91, 30, 10]$. Moreover, $\mathrm{Hull}(\mathfrak{C}) = \langle((x^{25} + 1) \mid 0), ((x^{24} + x^{23} + \cdots + x + 1) \mid \xi_1(x^{33} + 1) + \xi_2(x^{32} + x^{33} + \cdots + x + 1)))\rangle$ and $\Phi(\mathrm{Hull}(\mathfrak{C}))$ has parameters $[91, 1, 34]$. By Theorem 5.4, we have an EAQECC with parameters $[[91, 29, 10; 60]]_2$. This EAQECC has a better net rate than the net rate of the existing EAQECC $[[90, 28, 10; 62]]_2$ in [21].

Example 5.9. Let $\lambda = 31, \gamma = 9$ and $\mathfrak{C} = \langle((x - 1)(x^5 + x^3 + 1)(x^5 + x^2 + 1) \mid 0), ((x^5+x^3+1)(x^5+x^2+1) \mid \xi_1(x-1)+\xi_2(x^6+x^3+1)(x^2+x+1)))\rangle$ be an NSCC of length $(31, 9)$ in $\mathfrak{R}_{\lambda,\gamma}$ and $\Phi(\mathfrak{C})$ has parameters $[49, 29, 2]$. Then by Corollary 4.3, $\mathfrak{C}^\perp = \langle((x-1)(x^5+x^3+x^2+x+1)(x^5+x^4+x^2+x+1)(x^5+x^4+x^3+x+1)(x^5+x^4+x^3+x^2+1) \mid 0), ((x^5 + x^3 + x^2 + x + 1)(x^5 + x^4 + x^2 + x + 1)(x^5 + x^4 + x^3 + x + 1)(x^5 + x^4 + x^3 + x^2 + 1) \mid \xi_1(x^8 + x^7 + \cdots + x + 1) + \xi_2 1))\rangle$ and $\Phi(\mathfrak{C}^\perp)$ has parameters $[49, 20, 9]$. Moreover, $\mathrm{Hull}(\mathfrak{C}) = \langle((x^{31}-1) \mid 0), ((x^{30}+x^{29}+\cdots+x+1) \mid \xi_1(x^9-1)+\xi_2(x^8+x^7+\cdots+x+1)))\rangle$ and $\Phi(\mathrm{Hull}(\mathfrak{C}))$ has parameters $[49, 1, 40]$. By Theorem 5.4, we have an EAQECC with parameters $[[49, 19, 9; 28]]_2$. This EAQECC has a better net rate than the net rate of EAQECC $[[48, 9, 9; 39]]_2$ in [7].

6. **Conclusion.** We have characterized the hulls of $\mathbb{Z}_p\mathbb{Z}_p[v]$-cyclic codes for separable and non-separable cases. In Section 3, we determined the generator polynomials in Theorem 3.2 and Corollary 3.3, and derived the solutions of hulls of SCCs over $\mathbb{Z}_p\mathbb{Z}_p[v]$ using the concept of the generating function in combinatorics in Theorem 3.5. We also enumerated SCCs over $\mathbb{Z}_p\mathbb{Z}_p[v]$ having a fixed dimension in Theorem 3.7 and Corollary 3.8. Section 4 focused on NSCCs over $\mathbb{Z}_2\mathbb{Z}_2[v]$ with coprime odd block lengths, where we provided generator polynomials, found the solutions of hulls

of NSCCs over $\mathbb{Z}_2\mathbb{Z}_2[v]$ in Theorem 4.7 and Theorem 4.8, respectively. Moreover, in Theorem 4.10, we enumerated NSCCs over $\mathbb{Z}_2\mathbb{Z}_2[v]$ having hulls of a fixed dimension. In Section 5, we applied our findings to construct good EAQECCs over $\mathbb{Z}_2$ using the hulls of NSCCs over $\mathbb{Z}_2\mathbb{Z}_2[v]$ in Example 5.7 - Example 5.9, having a better rate than some of the existing EAQECCs.

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