

Review Article

The Expanding Zoo of Calabi-Yau Threefolds

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This is a short review of recent constructions of new Calabi-Yau threefolds with small Hodge numbers and/or nontrivial fundamental group, which are of particular interest for model building in the context of heterotic string theory. The two main tools are topological transitions and taking quotients by actions of discrete groups. Both of these techniques can produce new manifolds from existing ones, and they have been used to bring many new specimens to the previously sparse corner of the Calabi-Yau zoo, where both Hodge numbers are small. Two new manifolds are also obtained here from hyperconifold transitions, including the first example with fundamental group S_3 , the smallest non-Abelian group.

1. Introduction

This paper is a short review of recent work on constructing smooth Calabi-Yau threefolds with interesting topological properties, such as small cohomology groups and nontrivial fundamental group. In practice, these two properties often go hand-in-hand, as emphasised in [1, 2].

The majority of known three-dimensional Calabi-Yau manifolds are constructed as complete intersections in higher-dimensional toric varieties [3–8]. Most of the new examples found in recent years are in fact obtained from these via one of two techniques. The first is to take the quotient by a holomorphic action of some finite group. As explained in Section 2, when the group action is fixed-point-free, this is guaranteed to yield another Calabi-Yau manifold, and many Calabi-Yau threefolds with nontrivial fundamental group have been constructed in this way. Several early examples can be found in [9–14], but recent efforts have brought to light many more [2, 15–26], some of which will be discussed later. In the case that the group action has fixed points, it is often possible to resolve the resulting orbifold singularities in such a way as to again obtain a Calabi-Yau manifold. Examples can be found in [27–29]. The second technique is to vary either the complex structure or Kähler moduli of a known space until it becomes singular and then desingularise it by varying the other

type of moduli. Topologically, such a process is a surgery and yields a Calabi-Yau manifold topologically distinct from the original. Two classes of such topological transitions, the conifold and hyperconifold transitions, are discussed in Section 3, and explicit examples of each are given. Conifold transitions have been known for some time to connect many Calabi-Yau threefolds [30–33] and have been used to construct new manifolds in [2, 24, 34, 35]. Hyperconifold transitions were described in [36], and the first examples of new manifolds discovered this way were given in [37]. The examples of Section 3.2 yield two more new manifolds, one of which is the first known with fundamental group S_3 , the smallest non-Abelian group.

The fruit of these labours is that there are now many more known Calabi-Yau threefolds with small Hodge numbers (defined arbitrarily in this paper by $h^{1,1} + h^{2,1} \leq 24$) than were known to the authors of [1]. The number with nontrivial fundamental group has also increased dramatically, thanks largely to Braun’s classification of free group actions on complete intersections in products of projective spaces [23]. (The Hodge numbers of many of these quotients are yet to be calculated, but some are likely to be “small” as defined above.)

The physical motivation for studying such manifolds comes predominantly from heterotic string theory. In this context, a nontrivial fundamental group is necessary to be able to turn on discrete Wilson lines and thus obtain a realistic four-dimensional gauge group. The requirement of small Hodge numbers is not so clear-cut, but it seems advantageous if one wants to appeal to the methods of [38, 39] to stabilise the moduli, and this is currently the only known way to stabilise all (geometric) moduli in heterotic Calabi-Yau backgrounds. Although heterotic model building is not the theme of this review, other recent developments will be mentioned sporadically.

Throughout the paper, an arbitrary Calabi-Yau and its universal cover will be denoted by X and \tilde{X} , respectively, while a particular Calabi-Yau threefold with Hodge numbers $(h^{1,1}, h^{2,1})$ will be denoted by $X^{h^{1,1}, h^{2,1}}$. $X^\#$ will denote a singular member of the family X , and $\hat{X}^\#$ a resolution of such a singular variety.

2. Quotients by Group Actions

2.1. The Calabi-Yau Condition

It is an elementary fact of topology that every manifold X has a simply connected universal covering space \tilde{X} , from which it can be obtained as a quotient by the free action of a group $G \cong \pi_1(X)$. We will write this relationship as $X = \tilde{X}/G$. Although our interest is in (complex) threefolds, we will allow the dimension n of X to be arbitrary through much of this section.

If X is a Calabi-Yau manifold, it is easy to see that its universal cover \tilde{X} is too, by pulling back the complex structure, Kähler form ω , and holomorphic $(n, 0)$ -form Ω under the covering map (for this reason, we will often abuse notation by using the same symbols for these objects on X and \tilde{X}). Only a little more difficult is the converse: under what conditions is $X = \tilde{X}/G$ a Calabi-Yau manifold, given that \tilde{X} is? There are several points to consider (we assume always that G is a finite group).

- (i) X will be a manifold as long as the action of G is fixed-point free. Otherwise, it will have orbifold singularities.
- (ii) It will furthermore be a complex manifold if and only if G acts by biholomorphic maps. In this case, X simply inherits the complex structure of \tilde{X} .

- (iii) To see that X is Kähler, pick any Kähler form ω on \tilde{X} . Now, note that for any element $g \in G$, $g^*\omega$ is also a Kähler form, since $d(g^*\omega) = g^*(d\omega) = 0$, and for any k -dimensional complex submanifold M_k ,

$$\int_{M_k} g^*\omega^k = \int_{g(M_k)} \omega^k > 0. \quad (2.1)$$

We can use this to construct a Kähler form which is invariant under G and, therefore, descends to a Kähler form on X .

$$\omega^G := \frac{1}{|G|} \sum_{g \in G} g^*\omega. \quad (2.2)$$

- (iv) Finally, we must check whether X supports a nowhere-vanishing holomorphic $(n, 0)$ -form. This can only descend from the one on \tilde{X} , so we need to check whether Ω is G -invariant. Note that Ω is the unique (up to scale) element of $H^{n,0}(\tilde{X})$, so since G acts freely, the Atiyah-Bott fixed point formula [40, 41] for any $g \in G \setminus e$ reduces to

$$0 = \sum_{q=0}^n (-1)^q \operatorname{Tr}(H^{n,q}(g)) = \operatorname{Tr}(H^{n,0}(g)) + (-1)^n \operatorname{Tr}(H^{n,n}(g)), \quad (2.3)$$

where $H^*(g)$ denotes the induced action of g on the cohomology H^* . The group $H^{n,n}(\tilde{X})$ is generated by $(\omega^G)^n$, which is invariant, so we conclude that $\operatorname{Tr}(H^{n,0}(g)) = (-1)^{n+1}$, and therefore,

$$g^*\Omega = (-1)^{n+1}\Omega \quad \forall g \in G \setminus e. \quad (2.4)$$

In odd dimensions, therefore, Ω is automatically invariant under free group actions. In even dimensions, on the other hand, this simple calculation shows that there are no multiply-connected Calabi-Yau manifolds.

In summary, if \tilde{X} is a smooth Calabi-Yau threefold, then $X = \tilde{X}/G$ is a Calabi-Yau manifold if and only if G acts freely and holomorphically.

2.1.1. Smoothness

Above, we have simply assumed that the covering space \tilde{X} is smooth. In practice, \tilde{X} usually belongs to a family of spaces of which only a subfamily admits a free G action. It may be the case that although a generic member of \tilde{X} is smooth, members of the symmetric subfamily are all singular, so we never get a smooth quotient. Although this seems to be rare, it does occur for $\mathbb{Z}_8 \times \mathbb{Z}_8$ -symmetric complete intersections of four quadrics in \mathbb{P}^7 [11, 18], $\mathbb{Z}_5 \times \mathbb{Z}_5$ -symmetric complete intersections of five bilinears in $\mathbb{P}^4 \times \mathbb{P}^4$ [2], and a number of other examples, including some found in [23]. The extra condition, that a generic *symmetric* member is smooth, must, therefore, be checked on a case-by-case basis.

Let us start with the special case of complete intersection Calabi-Yau manifolds in smooth ambient spaces. This includes what have traditionally been called the ‘‘CICY’’ manifolds, where the ambient space is a product of projective spaces [3–5] and certain of the

toric hypersurfaces [7, 8]. The complete intersection condition means that if the ambient space has dimension $n + k$, then the Calabi-Yau \tilde{X} is given by the intersection of k hypersurfaces, each given by a single polynomial equation $f_a = 0$. In other words, the number of equations needed to specify \tilde{X} is equal to its codimension. When the ambient space is smooth, it can be covered in affine patches each isomorphic to \mathbb{C}^{n+k} , and the condition for \tilde{X} to be smooth is that $df_1 \wedge \cdots \wedge df_k$ is nonzero at every point on \tilde{X} . This is a very intuitive condition—if it holds, then at any point of \tilde{X} we can choose local coordinates x_1, \dots, x_{n+k} such that $f_a = x_a + \mathcal{O}(x^2)$. Locally, then, \tilde{X} projects biholomorphically onto the linear subspace $x_1 = x_2 = \cdots = x_k = 0$, and is, therefore, smooth. On any affine coordinate patch, the components of the differential form $df_1 \wedge \cdots \wedge df_k$ are just the $k \times k$ minors of the Jacobian matrix $J = (\partial f_a / \partial x_i)$, so the condition is that this matrix has rank k everywhere on \tilde{X} . It is, therefore, necessary to check that there is no simultaneous solution to the equations $f_a = 0$ along with the vanishing of all $k \times k$ minors of J , which is equivalent to the algebraic statement that the ideal generated by the polynomials and the minors is the entire ring $\mathbb{C}[x_1, \dots, x_{n+k}]$. This is checked by calculating a Gröbner basis for the ideal, algorithms for which are implemented in a variety of computer algebra packages [42–44]; a Gröbner basis for the entire ring is just a constant (usually given as 1 or -1 by software).

The more general case of singular ambient spaces or noncomplete intersections is not much harder than the above. Suppose \tilde{X} is not a complete intersection so that it is given by l equations in an $n + k$ -dimensional ambient space, where now we allow $l > k$. (A typical example is the Veronese embedding of \mathbb{P}^2 in \mathbb{P}^5 . If we take homogeneous coordinates z_i for \mathbb{P}^2 , and w_{ij} for \mathbb{P}^5 , where $j \geq i$, then the embedding is given by $w_{ij} = z_i z_j$. The equations needed to specify the image of this map are $w_{ij} w_{kl} - w_{il} w_{kj} = 0$, which amount to six independent equations, whereas the codimension of the embedded surface is only three.) Then, the condition for \tilde{X} to be smooth is still that the rank of the Jacobian be equal to k (the codimension) everywhere on \tilde{X} [45]. The reasoning is the same as before—if this is true, k of the polynomials will provide good local coordinates on the ambient space, allowing us to define a smooth coordinate patch on \tilde{X} .

If some affine patch on the ambient space is singular, it can still be embedded in \mathbb{C}^N for some N , by polynomial equations $F_1 = \cdots = F_K = 0$. The Calabi-Yau is then given by $F_1 = \cdots = F_K = f_1 = \cdots = f_l = 0$, and the condition for smoothness is once again that the Jacobian has rank equal to the codimension, $N - n$, at all points.

For examples of interest, Gröbner basis calculations are often very computationally intensive, since at intermediate stages the number of polynomials, as well as their coefficients, can become extremely large. It is, therefore, convenient to choose integer coefficients for all polynomials and perform the calculation over a finite field \mathbb{F}_p . As explained in [2], if a collection of polynomial equations are inconsistent over \mathbb{F}_p , then they are also inconsistent over \mathbb{C} , so the corresponding variety is smooth.

Note that there does exist a slight variation on the above procedure which still leads to smooth quotient manifolds. It may be the case that although the symmetric manifolds admit a free group action, they are all singular. If, however, these singularities can be resolved in a group-invariant way, the resolved space still admits a free group action, with a smooth quotient. Examples can be found in [15, 18, 21].

The final possibility is that the symmetric manifolds are smooth, but the group action always has fixed points, in which case the quotient space has orbifold singularities. It is frequently possible to resolve these in such a way as to again obtain a Calabi-Yau manifold, but this will not be discussed in detail here.

2.2. Notable Examples

2.2.1. New Three-Generation Manifolds

Calabi-Yau threefolds with Euler number $\chi = \pm 6$ were of particular interest in the early days of string phenomenology, since these give physical models with three generations of fermions via the “standard embedding” compactification of the heterotic string [9]. This typically gives an E_6 grand unified theory, and although the gauge symmetry can be partially broken by Wilson lines, it is impossible to obtain exactly the standard model gauge group in this way [46]. Nevertheless, it was argued by Witten that deformations of the standard embedding, combined with Wilson lines, can give realistic models [47], and this was put on firmer mathematical foundations by Li and Yau [48].

The archetypal example of a three-generation manifold is Yau’s manifold, with fundamental group \mathbb{Z}_3 [10], but recently two new promising three-generation manifolds were constructed in [22]. These are quotients of a manifold $X^{8,44}$ by groups of order twelve, which are the cyclic group \mathbb{Z}_{12} and the non-Abelian group $\text{Dic}_3 \cong \mathbb{Z}_3 \rtimes \mathbb{Z}_4$ (this is generated by two elements, one of order three and one of order four, satisfying $g_4 g_3 g_4^{-1} = g_3^2$), and each has Hodge numbers $(h^{1,1}, h^{2,1}) = (1, 4)$. Unfortunately, it was shown in [49] that the physical model on the non-Abelian quotient does not admit a deformation which yields exactly the field content of the minimal supersymmetric standard model (MSSM) in four dimensions. However, the \mathbb{Z}_{12} quotient allows many more distinct deformations, and the analysis of the corresponding physical models has not been completed.

The covering space $X^{8,44}$ is an anticanonical hypersurface in $dP_6 \times dP_6$, where dP_6 is the del Pezzo surface of degree six, which is \mathbb{P}^2 blown up in three generic points. This surface is rigid and toric, and its fan is shown in Figure 1.

As well as the action of the torus $(\mathbb{C}^*)^2$, dP_6 also admits an action by the dihedral group D_6 , as suggested by its fan. This can be realised as a group of lattice morphisms preserving the fan, generated by an order-six rotation ρ and a reflection σ , with matrix representations

$$\rho = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}, \quad \sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (2.5)$$

The product $dP_6 \times dP_6$ therefore has symmetry group $(D_6 \times D_6) \rtimes \mathbb{Z}_2$, where the extra \mathbb{Z}_2 factor swaps the two copies of the surface. The quotient groups Dic_3 and \mathbb{Z}_{12} are both order-twelve subgroups of this which act transitively on the vertices of the fan. Many more details can be found in [22].

2.2.2. Quotients of the (19, 19) Manifold

The Euler number of a three-dimensional Calabi-Yau manifold is given by the simple formula $\chi = 2(h^{1,1} - h^{2,1})$. If a group G acts freely, then $\chi(\tilde{X}/G) = \chi(\tilde{X})/|G|$, so this gives a simple necessary condition for the existence of such an action: the order of the group must divide $\chi/2$. This usually gives a fairly strong restriction on the groups which can act freely on any given manifold. The only time it gives no restriction is when $\chi = 0$. In this section, we will look at a particular manifold, $X^{19,19}$, which admits free actions by a number of disparate groups, including groups of order five, eight, and nine. For a Calabi-Yau threefold with $\chi \neq 0$, this would imply $|G| \geq 720$.

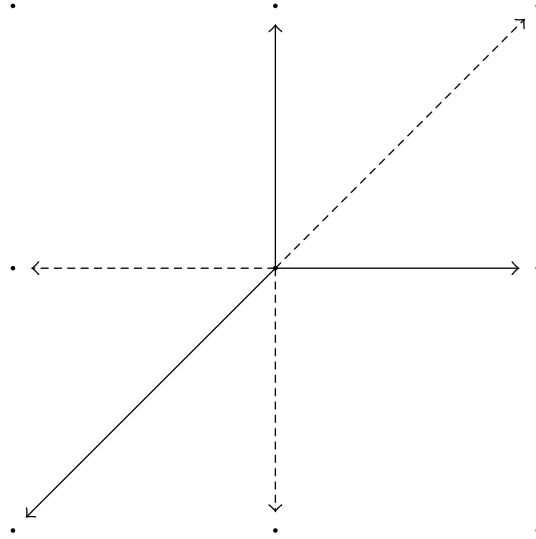


Figure 1: The fan for the toric surface dP_6 . Removing the dashed rays corresponds to the projection to \mathbb{P}^2 . All graphics were produced using [43].

The manifold $X^{19,19}$ can be represented in a number of different ways. Abstractly, it is the fibre product of two rational surfaces, each elliptically fibred over \mathbb{P}^1 [16, 19]. Such a surface is given by blowing up \mathbb{P}^2 at the nine points given by the intersection of two cubic curves; if we take homogeneous coordinates t_0, t_1 on \mathbb{P}^1 and z_0, z_1, z_2 on \mathbb{P}^2 , the corresponding equation is

$$f(z)t_0 + g(z)t_1 = 0, \quad (2.6)$$

where f and g are homogeneous cubic polynomials. We can easily see that this corresponds to \mathbb{P}^2 blown up at the nine points given by $f = g = 0$. Indeed, for any point of \mathbb{P}^2 , where $f \neq 0$ or $g \neq 0$, we get a unique solution for $[t_0 : t_1]$, whereas for $f = g = 0$, the equation is satisfied identically, giving a whole copy of \mathbb{P}^1 . To see that it is also an elliptic fibration over \mathbb{P}^1 , note that for any fixed value of $[t_0 : t_1] \in \mathbb{P}^1$, we get a cubic equation in \mathbb{P}^2 , which defines an elliptic curve.

To get the fibre product of two such surfaces, we introduce another \mathbb{P}^2 , with homogeneous coordinates w_0, w_1, w_2 , and another equation of the form (2.6) over the same \mathbb{P}^1 . The resulting threefold is Calabi-Yau, and has a projection to \mathbb{P}^1 , with typical fibre which is a product of two elliptic curves. In [19], Bouchard and Donagi studied group actions which preserve the elliptic fibration, and found free actions by the groups $\mathbb{Z}_3 \times \mathbb{Z}_3, \mathbb{Z}_4 \times \mathbb{Z}_2, \mathbb{Z}_6 \cong \mathbb{Z}_3 \times \mathbb{Z}_2$ and \mathbb{Z}_5 (as well as all subgroups of these, of course).

Certain of these quotient manifolds have, in fact, played crucial roles in the heterotic string literature. A model with the spectrum of the $U(1)_{B-L}$ -extended supersymmetric standard model was constructed on a quotient by $\mathbb{Z}_3 \times \mathbb{Z}_3$ and studied in [50, 51] (a similar model on the $\mathbb{Z}_3 \times \mathbb{Z}_3$ quotient of the ‘‘bicubic’’, which is related to this manifold by a conifold transition, was found in [52]), while a model with the exact MSSM spectrum exists on a \mathbb{Z}_2 quotient and was described in [53, 54]. In [55, 56], the quotient by a different $\mathbb{Z}_3 \times \mathbb{Z}_3$ action was used as a test case for calculating instanton corrections on manifolds with torsion curves.

Table 1: The Hodge numbers for known quotients of $X^{19,19}$.

$h^{1,1} = h^{2,1}$	Fundamental Group	Reference
11	\mathbb{Z}_2	[2, 19]
7	$\mathbb{Z}_3, \mathbb{Z}_2 \times \mathbb{Z}_2$	[2, 19]
5	\mathbb{Z}_4	[19]
3	$\mathbb{Z}_5, \mathbb{Z}_6, \mathbb{Z}_4 \times \mathbb{Z}_2, Q_8, \mathbb{Z}_3 \times \mathbb{Z}_3$	[2, 19]
2	$\mathbb{Z}_{12}, \text{Dic}_3$	[22, 49]

There are, in fact, further (relatively large) groups which act freely on $X^{19,19}$, which can be easily described using its representation(s) as a CICY. First, we note that the fibre product construction above is equivalent to the rather more prosaic statement that the manifold is a complete intersection of two hypersurfaces in $\mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^2$, of multi-degrees $(1, 3, 0)$ and $(1, 0, 3)$. In the notation of [3, 4], $X^{19,19}$ can, therefore, be specified by the “configuration matrix”

$$\begin{array}{c} \mathbb{P}^1 \\ \mathbb{P}^2 \\ \mathbb{P}^2 \end{array} \begin{pmatrix} 1 & 1 \\ 3 & 0 \\ 0 & 3 \end{pmatrix} \quad (2.7)$$

By utilising various *splittings* and *contractions* (see, e.g., [2, 4, 30, 31]), and checking that the Euler number remains constant, it is easy to show that $X^{19,19}$ can also be specified by the configuration matrices

$$\begin{array}{c} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \end{array} \begin{pmatrix} 1 & 1 \\ 2 & 0 \\ 2 & 0 \\ 0 & 2 \\ 0 & 2 \end{pmatrix}, \quad \begin{array}{c} \mathbb{P}^1 \\ \mathbb{P}^2 \\ \mathbb{P}^2 \\ \mathbb{P}^2 \\ \mathbb{P}^2 \end{array} \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \end{pmatrix}. \quad (2.8)$$

It was shown in [2] that in the first form, $X^{19,19}$ admits a free action of the order-eight quaternion group (denoted in [2] by \mathbb{H} , but more conventionally by Q_8), with elements $\{\pm 1, \pm i, \pm j, \pm k\}$, induced by a linear action of this group on the ambient space.

In the second form, $X^{19,19}$ admits free actions by two groups of order twelve. One is the cyclic group \mathbb{Z}_{12} , and the other is the dicyclic group $\text{Dic}_3 \cong \mathbb{Z}_3 \times \mathbb{Z}_4$ (introduced in Section 2.2.1) [49]. These were in fact discovered via conifold transitions from the corresponding quotients of $X^{8,44}$, an idea reviewed in Section 3.

In summary, $X^{19,19}$ is rather exceptional in that it admits free actions by the groups \mathbb{Z}_{12} , Dic_3 , $\mathbb{Z}_3 \times \mathbb{Z}_3$, $\mathbb{Z}_4 \times \mathbb{Z}_2$, Q_8 , \mathbb{Z}_6 , and \mathbb{Z}_5 . The Hodge numbers for the quotients by all these groups and their subgroups are collected in Table 1.

2.2.3. Manifolds with Hodge Numbers (1,1)

For a long time, the smallest known Hodge numbers of a Calabi-Yau threefold satisfied $h^{1,1} + h^{2,1} = 4$. This record has now been overtaken by Braun’s examples of manifolds with

$(h^{1,1}, h^{2,1}) = (1, 1)$ [25] (as well as Freitag and Salvati Manni's manifold with $(h^{1,1}, h^{2,1}) = (2, 0)$ [29]).

The covering space of Braun's $(1, 1)$ manifolds is a self-mirror manifold $X^{20,20}$. This is realised as an anticanonical hypersurface in the toric fourfold determined by the face fan over the 24-cell, which is a self-dual regular four-dimensional polytope.

There are three different groups of order 24 which act freely on particular smooth one-parameter subfamilies of $X^{20,20}$; these are $\mathbb{Z}_3 \times \mathbb{Z}_8, \mathbb{Z}_3 \times Q_8$, and $SL(2, 3)$. The first two are self-explanatory, while the third is the group of two-by-two matrices of determinant one over the field with three elements. All the groups act via linear transformations on the lattice in which the polytope lives, and act transitively on its vertices. Full details can be found in [25].

2.2.4. Complete Intersections of Four Quadrics in \mathbb{P}^7

A particularly fertile starting point for finding new Calabi-Yau manifolds has been the complete intersection of four quadrics in \mathbb{P}^7 . A smooth member of this family is a Calabi-Yau manifold with Hodge numbers $(h^{1,1}, h^{2,1}) = (1, 65)$. Hua classified free group actions on smooth subfamilies in [18], finding groups of order 2, 4, 8, 16, and 32. The quotients all have $h^{1,1} = 1$, and $h^{2,1} = 33, 17, 9, 5$, and 3, respectively.

Certain nodal families allow free actions of groups of order 64 and, furthermore, have equivariant small resolutions [15, 18]. The resolutions have Hodge numbers $(h^{1,1}, h^{2,1}) = (2, 2)$ and inherit the free group actions. Remarkably, in this case, all the quotients have the same Hodge numbers as the covering space. The quotient by $\mathbb{Z}_8 \times \mathbb{Z}_8$ was investigated as a background for heterotic string theory in [57], but unfortunately, no realistic models were found.

Freitag and Salvati Manni have also constructed a large number of new manifolds by starting with a particular complete intersection $X^\#$ of four quadrics which has 96 nodes and a very large symmetry group [28, 29]. They show that the quotients by many subgroups admit crepant projective resolutions, thereby giving rise to a large number of new Calabi-Yau manifolds. Some of the subgroups of order 2, 4, 8, and 16 act freely on a small resolution of $X^\#$, and the corresponding quotient manifolds are connected to some of Hua's examples by conifold transitions. The manifolds from [29] with small Hodge numbers are listed in the appendix 4, including one with $(h^{1,1}, h^{2,1}) = (2, 0)$, which is, therefore, equal with Braun's manifolds for smallest known Hodge numbers. Note that the theoretical minimum is $(h^{1,1}, h^{2,1}) = (1, 0)$.

3. New Manifolds From Topological Transitions

One fascinating feature of Calabi-Yau threefolds is the interconnectedness of moduli spaces of topologically distinct manifolds. Generally speaking, there are two ways to pass from one smooth Calabi-Yau to another. We may deform the complex structure until a singularity develops and then "resolve" this singularity using the techniques of algebraic geometry, which involves replacing the singular set with new embedded holomorphic curves or surfaces. Alternatively, we may allow certain embedded curves or surfaces to collapse to zero size and then "smooth" the resulting singular space by varying its complex structure. Obviously these two processes are inverses of each other.

Our main interest here is in constructing new smooth Calabi-Yau threefolds via such topological transitions, but first we will indulge in a few comments about the connectedness of the space of all Calabi-Yau threefolds.

The suggestion that all Calabi-Yau threefolds might be connected by topological transitions goes back to [58]. Early work showed that this was true for nearly all examples known at the time [30, 31]. These papers considered conifold transitions, in which the intermediate variety has only nodal singularities; the smoothing process replaces these singular points by three-spheres, while the “small” resolution replaces them by two-spheres (holomorphically embedded). Such singularities were shown to be at finite distance in moduli space [32, 33], and conifold transitions were later shown to be smooth processes in type II string theory [59, 60].

If we wish to connect all Calabi-Yau threefolds, conifold transitions are not sufficient, because they cannot change the fundamental group. To see this, we note that topologically, a conifold transition consists of removing neighbourhoods of some number of copies of S^3 , each with boundary $S^3 \times S^2$, and replacing them with similar neighbourhoods of S^2 . Since all these spaces are simply connected, a simple application of van Kampen’s theorem (see, e.g., [61]) shows that the fundamental group does not change.

There do exist relatively mild topological transitions which can change the fundamental group; these are known as *hyperconifold transitions* and were described by the author in [36, 37]. Here, the singularities of the intermediate variety are finite quotients of a node, and their resolutions are no longer “small”. It is an interesting question whether all Calabi-Yau threefolds can be connected by conifold and hyperconifold transitions.

In the following sections, we will consider these two types of transition separately, mostly through examples. The examples in Section 3.2 actually yield previously unknown manifolds, with Hodge numbers $(h^{1,1}, h^{2,1}) = (2, 5)$ and $(2, 3)$ and fundamental groups \mathbb{Z}_5 and S_3 , respectively.

3.1. Conifold Transitions

In [2, 24], free group actions were followed through conifold transitions, leading to webs of conifold transitions between smooth quotients with the same fundamental group (conifold transitions were also used in [34, 35] to construct new simply connected manifolds). Here, we will just consider a simple example (taken from [2]) which exemplifies the idea.

Consider the well-known family of quintic hypersurfaces in \mathbb{P}^4 , with Hodge numbers $(h^{1,1}, h^{2,1}) = (1, 101)$ and hence Euler number $\chi = -200$. If we take homogeneous coordinates z_0, \dots, z_4 on \mathbb{P}^4 , then an action of \mathbb{Z}_5 can be defined by the generator

$$g_5 : z_i \longrightarrow z_{i+1}. \quad (3.1)$$

Then, there is a smooth family of invariant quintics, given by

$$f = \sum_{ijklm} A_{j-i, k-i, l-i, m-i} z_i z_j z_k z_l z_m = 0. \quad (3.2)$$

For generic coefficients, \mathbb{Z}_5 acts freely, so we get a family of smooth quotients with Hodge numbers $(h^{1,1}, h^{2,1}) = (1, 21)$.

Now, let us consider a nongeneric choice for the coefficients in (3.2) such that f is the determinant of some 5×5 matrix M which is linear in the homogeneous coordinates. If we take the entries of M to be

$$M_{ik} = \sum_j a_{j-i, k-i} z_j, \quad (3.3)$$

then the induced \mathbb{Z}_5 action is $M_{ik} \rightarrow M_{i+1,k+1}$, so the determinant does indeed correspond to an invariant quintic. The action of \mathbb{Z}_5 is still generically fixed-point-free on the family given by $\det M = 0$, but the hypersurfaces are no longer smooth. Indeed, using a computer algebra package, it can be checked that the rank of M drops to three at exactly fifty points on a typical such hypersurface, and that these points are nodes. Furthermore, they fall into ten orbits of five nodes under the \mathbb{Z}_5 action.

We now ask whether these nodes can be resolved in a group-invariant way; if so, the group will still act freely on the resolved manifolds, and we will have constructed a conifold transition between the quotient manifolds. In fact this is easy to do. Introduce a second \mathbb{P}^4 , with homogeneous coordinates w_0, \dots, w_4 , and consider the equations

$$f_i := \sum_k M_{ik} w_k = \sum_{j,k} a_{j-i,k-i} z_j w_k = 0. \quad (3.4)$$

These are five bilinears in $\mathbb{P}^4 \times \mathbb{P}^4$, and it can be checked that they generically define a smooth Calabi-Yau threefold $X^{2,52}$. Since we cannot have $w_i = 0$ for all i , there are only simultaneous solutions to these equations when $\det M = 0$, so this gives a projection from $X^{2,52}$ to nodal members of $X^{1,101}$. At most points, this is one-to-one, but at the fifty points where the rank of M drops to three, we get a whole copy of $\mathbb{P}^1 \subset \mathbb{P}^4$ projecting to a (nodal) point of $X^{1,101}$. In this way we see that we have constructed a conifold transition $X^{1,101} \rightsquigarrow X^{2,52}$.

To see that the free \mathbb{Z}_5 action is preserved by the conifold transition above, it suffices to note that if we extend the action by defining $g_5 : w_i \rightarrow w_{i+1}$, then this induces $f_i \rightarrow f_{i+1}$, implying that the manifolds defined by (3.4) are \mathbb{Z}_5 -invariant. The absence of fixed points follows from the absence of fixed points on the nodal members of $X^{1,101}$ although this can also be checked directly.

Since the conifold transition from $X^{1,101}$ to $X^{2,52}$ can be made \mathbb{Z}_5 -equivariant, it descends to a conifold transition between their quotients, $X^{1,21} \rightsquigarrow X^{2,12}$, where the intermediate variety has ten nodes.

3.2. Hyperconifold Transitions

The conifold transition in the last section illustrates two completely general features of such transitions: the fundamental group does not change, for reasons explained previously, and the intermediate singular variety has multiple nodes [62]. We now turn our attention to a class of transitions for which neither of these statements hold—the so-called hyperconifold transitions introduced in [36]. Here, the intermediate space typically has only one singular point, which is a quotient of a node by some finite cyclic group \mathbb{Z}_N . (Quotients by non-Abelian groups can also occur, but these do not admit a toric description, and their resolutions have not been studied.) These arise naturally when a generically free group action is allowed to develop a fixed point. A \mathbb{Z}_N -hyperconifold transition changes the Hodge numbers according to

$$\delta(h^{1,1}, h^{2,1}) = (N - 1, -1). \quad (3.5)$$

The resolution of a hyperconifold singularity replaces the singular point with a simply connected space, and in this way, we see that the transitions can change the fundamental group. It is worth pausing here to consider this in more detail than has been done in previous papers.

Suppose that we have a smooth quotient $X = \tilde{X}/G$ and deform the complex structure until some order- N element g_N , which generates a subgroup $\langle g_N \rangle \cong \mathbb{Z}_N < G$, develops a fixed point $p \in \tilde{X}$. Then, as described in [36], this point will be singular, and generically a node. In some cases, the group structure implies that other elements will simultaneously develop fixed points, which we can see by taking a group element $g' \in G \setminus \langle g_N \rangle$ and performing an elementary calculation

$$g' g_N g'^{-1} \cdot (g' \cdot p) = g' \cdot (g_N \cdot p) = g' \cdot p. \quad (3.6)$$

So, the point $g' \cdot p \in \tilde{X}$ is fixed by $g' g_N g'^{-1}$. We see that every subgroup conjugate to $\langle g_N \rangle$ also develops a fixed point. All such points are identified by G , so the singular quotient $X^\#$ has only one \mathbb{Z}_N -hyperconifold singularity.

What is the fundamental group of the resolution $\hat{X}^\#$? To calculate this, excise a small ball around each fixed point of $\tilde{X}^\#$ to obtain a smooth space \tilde{X}' on which the whole group G acts freely. We can then quotient by G to obtain X' , with fundamental group G . Finally, we glue in a neighbourhood Σ of the exceptional set of the resolution of the hyperconifold. Σ is simply connected. We now have $\hat{X}^\# = X' \cup \Sigma$, and can use van Kampen's theorem to calculate $\pi_1(\hat{X}^\#)$. Note that the intersection of the two subspaces X' and Σ is homotopy equivalent to $S^3 \times S^2/\mathbb{Z}_N$, since the stabiliser of each point on the covering space was isomorphic to \mathbb{Z}_N . So, we have the data

$$\hat{X}^\# = X' \cup \Sigma, \quad X' \cap \Sigma \stackrel{\text{hom.}}{\cong} S^3 \times S^2/\mathbb{Z}_N, \quad \pi_1(\Sigma) \cong \mathbf{1}, \quad \pi_1(X') \cong G, \quad (3.7)$$

which by van Kampen's theorem immediately implies that $\pi_1(\hat{X}^\#) \cong G/\langle g_N \rangle^G$, where $\langle g_N \rangle^G$ is the smallest normal subgroup of G which contains $\langle g_N \rangle$, usually called the normal closure.

Trivial examples arise when $G = \mathbb{Z}_N \times H$ or $\mathbb{Z}_N \rtimes H$, and the generator of \mathbb{Z}_N develops a fixed point. In this case, the corresponding hyperconifold transition changes the fundamental group from G to H .

3.2.1. Example 1: $X^{1,6} \rightsquigarrow X^{2,5}$

We will first consider an example related to that in Section 3.1. If we demand that the matrix appearing in (3.4) is symmetric, $a_{jk} = a_{kj}$, then the resulting family of threefolds are invariant under a further order-two symmetry, generated by $g_2 : z_i \leftrightarrow w_i$. As shown in [2], this family is still generically smooth, and the entire group $\mathbb{Z}_5 \times \mathbb{Z}_2 \cong \mathbb{Z}_{10}$ acts freely, so we get a smooth quotient family $X^{1,6}$.

Suppose now that we ask for g_2 to develop a fixed point. In the ambient space, it fixes an entire copy of \mathbb{P}^4 , given by $w_i = z_i$ for all i . Choose a single point on this locus (as long as it is not also a fixed point of g_5), say $w_i = z_i = \delta_{i0}$. The evaluation of the defining polynomials at this point is $f_i = c_{-i,-i}$, so it lies on the hypersurface if $c_{i,i} = 0$ for all i . One can check that for arbitrary choices of the other coefficients; this point is a node on the covering space, and there are no other singularities. This, therefore, corresponds to a sub-family of $X^{1,6}$ with an isolated \mathbb{Z}_2 -hyperconifold singularity. Such a singularity has a crepant projective resolution, as described in [36], obtained by a simple blowup of the singular point. This introduces an irreducible exceptional divisor, thus increasing $h^{1,1}$ by one, and since we imposed a single constraint on the complex structure of $X^{1,6}$, (naïvely, it seems that we have imposed five constraints, $c_{i,i} = 0$). However, we had the freedom to choose a generic point on the fixed \mathbb{P}^4 ,

corresponding to a four-parameter choice of possible conditions, so the number of complex structure parameters is actually only reduced by one.) the resolved space has Hodge numbers $(h^{1,1}, h^{2,1}) = (2, 5)$, and its fundamental group is \mathbb{Z}_5 . (Note that this is, in fact, a new “three-generation” manifold, with $\chi = -6$. Unfortunately, \mathbb{Z}_5 -valued Wilson lines cannot perform the symmetry breaking required for a realistic model.)

So we have constructed a \mathbb{Z}_2 -hyperconifold transition $X^{1,6} \xrightarrow{\mathbb{Z}_2} X^{2,5}$, where the fundamental group of the first space is \mathbb{Z}_{10} and that of the second space is \mathbb{Z}_5 .

3.2.2. Example 2: $X^{1,4} \rightsquigarrow X^{2,3}$

For a second example, which will also yield an interesting new manifold, consider the Dic_3 quotient of $X^{8,44}$, described in Section 2.2.1. As shown in [22], there is a codimension-one locus in moduli space, where the unique order-two element of the group develops a fixed point. It is easy to check that on the covering space, this is the only singular point, and is a node. As such, the quotient space $X^{1,4}$ develops a \mathbb{Z}_2 -hyperconifold singularity. Blowing up this point yields a new Calabi-Yau manifold, with Hodge numbers $(h^{1,1}, h^{2,1}) = (2, 3)$, as per the general formula (3.5).

The \mathbb{Z}_2 subgroup of Dic_3 is actually the centre, so it is trivially normal, and the fundamental group of the new manifold $X^{2,3}$ is $\text{Dic}_3/\mathbb{Z}_2$, which is isomorphic to S_3 , the symmetric group on three letters. To see this, recall that Dic_3 is generated by two elements, g_3 and g_4 , of orders three and four, respectively, subject to the relation $g_4 g_3 g_4^{-1} = g_3^2$. So, the \mathbb{Z}_2 subgroup is generated by g_4^2 , meaning that in $\text{Dic}_3/\mathbb{Z}_2$, $g_4^2 \sim e$. To reflect this, we rename g_4 to g_2 and obtain

$$\text{Dic}_3/\mathbb{Z}_2 \cong \langle g_2, g_3 \mid g_2^2 = g_3^3 = e, g_2 g_3 g_2 = g_3^2 \rangle, \quad (3.8)$$

which is the standard presentation of S_3 .

So, in summary, we have constructed a \mathbb{Z}_2 -hyperconifold transition $X^{1,4} \xrightarrow{\mathbb{Z}_2} X^{2,3}$, where the fundamental group of the first space is Dic_3 and that of the second space is S_3 . This is the first known Calabi-Yau threefold with fundamental group S_3 [23].

Appendix

The New-Look Zoo

The techniques reviewed in Sections 2 and 3, along with a few exceptional constructions, have led in recent years to the construction of a relatively large number of new Calabi-Yau threefolds with small Hodge numbers and/or nontrivial fundamental group. A table appeared in [2] of all manifolds known at the time with $h^{1,1} + h^{2,1} \leq 24$. Instead of repeating that list here, only new manifolds discovered since the appearance of [2] are listed in Table 2 and Table 3. Since they are of most relevance for string theory, those with nontrivial fundamental group are listed separately in Table 2, while Table 3 contains new simply connected manifolds and those with fundamental group yet to be calculated. Figure 2 displays the tip of the distribution of manifolds catalogued by their Hodge numbers, showing which values of $(h^{1,1}, h^{2,1})$ satisfying $h^{1,1} + h^{2,1} \leq 24$ are realised by known examples (and their mirrors, which are assumed to exist).

Table 2: Manifolds with small Hodge numbers and $\pi_1 \neq 1$.

(X, Y)	$(h^{1,1}, h^{2,1})$	Manifold	π_1	Reference
(0,24)	(12,12)	$X^{20,20}/\mathbb{Z}_2$	\mathbb{Z}_2	[25]
(-16,18)	(5,13)	(Hypersurface in $\mathbb{P}^1 \times \mathbb{P}^1 \times dP_4)/\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	[26]
(-20,16)	(3,13)	$\begin{matrix} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^2 \\ \mathbb{P}^2 \\ \mathbb{P}^2 \end{matrix} \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix} / \mathbb{Z}_3$	\mathbb{Z}_3	[23, 24]
(-12,16)	(5,11)	$\begin{matrix} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^3 \\ \mathbb{P}^2 \end{matrix} \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix} / \mathbb{Z}_3$	\mathbb{Z}_3	[23, 24]
(0,16)	(8,8)	$X^{20,20}/\mathbb{Z}_3$	\mathbb{Z}_3	[25]
(0,16)	(8,8)	(Toric hypersurface $Y^{20,20})/\mathbb{Z}_3$	\mathbb{Z}_3	[37]
(32,16)	(16,0)	$(\widehat{\mathbb{P}^7 [2 \ 2 \ 2 \ 2 \ 2]})/\mathbb{Z}_2$	\mathbb{Z}_2	[29]
(-14,15)	(4,11)	$\begin{matrix} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^2 \\ \mathbb{P}^2 \\ \mathbb{P}^2 \\ \mathbb{P}^2 \end{matrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix} / \mathbb{Z}_3$	\mathbb{Z}_3	[23, 24]
(-10,15)	(5,10)	$\begin{matrix} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^2 \\ \mathbb{P}^2 \\ \mathbb{P}^2 \end{matrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix} / \mathbb{Z}_3$	\mathbb{Z}_3	[23, 24]
(-12,14)	(4,10)	(Toric hypersurface $X^{8,26})/\mathbb{Z}_3$	\mathbb{Z}_3	[37]
(-8,14)	(5,9)	$\begin{matrix} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^2 \\ \mathbb{P}^2 \\ \mathbb{P}^2 \\ \mathbb{P}^2 \end{matrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix} / \mathbb{Z}_3$	\mathbb{Z}_3	[23, 24]
(-4,14)	(6,8)	$\begin{matrix} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^2 \\ \mathbb{P}^2 \\ \mathbb{P}^2 \end{matrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix} / \mathbb{Z}_3$	\mathbb{Z}_3	[23, 24]
(0,12)	(6,6)	$X^{20,20}/\mathbb{Z}_4$	\mathbb{Z}_4	[25]
(-10,9)	(2,7)	(Hypersurface in $dP_5 \times dP_5)/\mathbb{Z}_5$	\mathbb{Z}_5	[26]
(2,9)	(5,4)	(Toric hypersurface $X^{21,16})/\mathbb{Z}_5$	\mathbb{Z}_5	[37]
(-4,8)	(3,5)	(Hypersurface in $dP_4 \times dP_4)/\mathbb{Z}_4 \times \mathbb{Z}_2$	$\mathbb{Z}_4 \times \mathbb{Z}_2$	[26]
(0,8)	(4,4)	$X^{20,20}/\mathbb{Z}_6$	\mathbb{Z}_6	[25]

Table 2: Continued.

(χ, y)	$(h^{1,1}, h^{2,1})$	Manifold	π_1	Reference
(16,8)	(8,0)	$(\mathbb{P}^7[2\ 2\ 2\ 2]^\# / \{\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_4\})$	$\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_4$	[29]
(-6,7)	(2,5)	$(X^{2,52} / \mathbb{Z}_{10})^\#$	\mathbb{Z}_5	Section 3.2.1
(-8,6)	(1,5)	(Hypersurface in $(\mathbb{P}^1)^4$) / $\mathbb{Z}_8 \times \mathbb{Z}_2$	$\mathbb{Z}_8 \times \mathbb{Z}_2$	[26]
(0,6)	(3,3)	$X^{20,20} / \{\mathbb{Z}_8, Q_8\}$	\mathbb{Z}_8, Q_8	[25]
(-6,5)	(1,4)	$X^{8,44} / \{\text{Dic}_3, \mathbb{Z}_{12}\}$	$\text{Dic}_3, \mathbb{Z}_{12}$	[22]
(-2,5)	(2,3)	$(X^{8,44} / \text{Dic}_3)^\#$	S_3	Section 3.2.2
(0,4)	(2,2)	$X^{19,19} / \{\text{Dic}_3, \mathbb{Z}_{12}\}$	$\text{Dic}_3, \mathbb{Z}_{12}$	[22, 49]
(0,4)	(2,2)	$X^{20,20} / \mathbb{Z}_{12}$	\mathbb{Z}_{12}	[25]
(8,4)	(4,0)	$(\mathbb{P}^7[2\ 2\ 2\ 2]^\# / G), G = 8$	G	[29]
(0,2)	(1,1)	$X^{20,20} / \{SL(2,3), \mathbb{Z}_3 \rtimes \mathbb{Z}_8, \mathbb{Z}_3 \times Q_8\}$	$SL(2,3), \mathbb{Z}_3 \rtimes \mathbb{Z}_8, \mathbb{Z}_3 \times Q_8$	[25]
(4,2)	(2,0)	$(\mathbb{P}^7[2\ 2\ 2\ 2]^\# / G), G = 16$	G	[29]

This table complements the one in [2], and briefly describes the manifolds which have $y = h^{1,1} + h^{2,1} \leq 24$ and nontrivial fundamental group discovered since that paper appeared in 2008. There should still be a number of other manifolds in this region, including quotients from [23] whose Hodge numbers have not yet been calculated, and manifolds obtained from known quotients by hyperconifold transitions [37], of which only a few have so far been written down explicitly. In the ‘‘Manifold’’ column, $X^{20,20}$ denotes the Calabi-Yau toric hypersurface associated to the 24-cell, discussed in [25] and Section 2.2.3, while $X^{19,19}$ refers to the manifold discussed in Section 2.2.2, and $X^{8,44}$ to that in Section 2.2.1. dP_n is the del Pezzo surface of degree n . Multiple quotient groups indicate different quotients with the same Hodge numbers. $X^\#$ denotes a singular member of a generically smooth family, while \hat{X} denotes a resolution of a singular variety X . The column labelled by π_1 gives the fundamental group. For each manifold listed here there should also be a mirror, which is not listed.

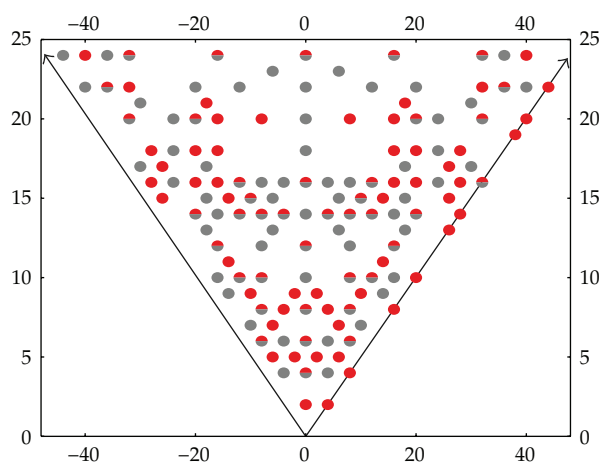


Figure 2: The tip of the distribution of Calabi-Yau threefolds. Grey dots denote manifolds included in [2], while red dots denote newer examples. Split dots indicate multiple occupation of a site. Note that some red and grey dots are also multiply occupied.

Table 3: Other manifolds with small Hodge numbers.

(χ, y)	$(h^{1,1}, h^{2,1})$	Manifold	Reference
(16,24)	(16,8)	—	[29]
(32,24)	(20,4)	—	[29]
(40,24)	(22,2)	—	[29]
(-32, 22)	(3,19)	$(\widehat{\mathbb{P}^4[5]}/D_5)$	[27]
(36,22)	(20,2)	Smoothing of variety obtained by blowing down 18 rational curves on the rigid “Z” manifold.	[35]
(44,22)	(22,0)	—	[29]
(18,21)	(15,6)	Smoothing of variety obtained by blowing down 27 rational curves on the rigid “Z” manifold.	[35]
(-20, 20)	(5,15)	$(\widehat{\mathbb{P}^4[5]}/A_5)$	[27]
(8,20)	(12,8)	—	[29]
(16,20)	(14,6)	—	[29]
(32,20)	(18,2)	—	[29]
(40,20)	(20,0)	—	[29]
(38,19)	(19,0)	—	[29]
(20,18)	(14,4)	—	[29]
(28,18)	(16,2)	—	[29]
(26,17)	(15,2)	—	[29]
(16,16)	(12,4)	—	[29]
(28,16)	(15,1)	—	[29]
(32,16)	(16,0)	—	[29]
(26,15)	(14,1)	—	[29]
(20,14)	(12,2)	—	[29]
(28,14)	(14,0)	—	[29]
(26,13)	(13,0)	—	[29]
(16,12)	(10,2)	—	[29]
(14,11)	(9,2)	—	[29]
(8,10)	(7,3)	—	[29]
(12,10)	(8,2)	—	[29]
(20,10)	(10,0)	—	[29]
(8,8)	(6,2)	—	[29]
(16,8)	(8,0)	—	[29]
(8,4)	(4,0)	—	[29]

This table is the same as that above, except all the manifolds listed either have trivial fundamental group, or a fundamental group which has not been calculated (which is the case for several examples from [29]). The notation is the same as above, and the manifolds with no description are all desingularisations of quotients by various groups of a singular complete intersection of four quadrics in \mathbb{P}^7 [29].

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