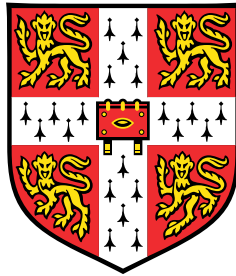


Mathematical Studies on the Asymptotic Behaviour of Gravitational Radiation in General Relativity



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Declaration

I hereby declare that except where specific reference is made to the work of others, the contents of this thesis are original and have not been submitted in whole or in part for consideration for any other degree or qualification in this, or any other university. This thesis is the result of my own work and includes nothing which is the outcome of work done in collaboration except as declared in the preface below and specified in the text.

- Chapter 1 is based on my work “The Case Against Smooth Null Infinity I: Heuristics and Counter-Examples” [Keh21a], which was published in *Annales Henri Poincaré* in 2021 (and first appeared on arXiv.org in May 2021).
- Chapter 2 is based on my work “The Case Against Smooth Null Infinity II: A Logarithmically Modified Price’s Law” [Keh21b], which first appeared on arXiv.org in May 2021 and is accepted for publication in *Advances in Theoretical and Mathematical Physics*.
- Chapter 3 is based on my work “The Case Against Smooth Null Infinity III: Early-Time Asymptotics for Higher ℓ -Modes of Linear Waves on a Schwarzschild Background” [Keh22], which was published in *Annals of PDE* in 2022 and first appeared on arXiv.org in June 2021.
- Chapter 4 is based on the work “On the Relation Between Asymptotic Charges, the Failure of Peeling and Late-time Tails” [GK22], which was published in *Classical and Quantum Gravity* in 2022 and first appeared on arXiv.org in February 2022. **This work was done in collaboration with Dejan Gajic.**
- Chapter 5 is based on an overview paper (“The Case Against Smooth Null Infinity IV: Linearised Gravity—An Overview” [Keh23]) that I will submit for a theme issue of the *Philosophical Transactions of the Royal Society A* associated to the workshop “At the interface of asymptotics, conformal methods and analysis in general relativity (May 2023)”.
- Chapter 6 reflects the progress of an **ongoing collaboration with Hamed Masaoood** [KM23].

Abstract

Mathematical Studies on the Asymptotic Behaviour of Gravitational Radiation

This thesis contains various constructions that attempt to answer and clarify the long-standing question of how to model the asymptotic behaviour of gravitational radiation in astrophysical processes within the theory of general relativity.

It is argued that the correct mathematical setup to address this question is the *scattering problem* (as opposed to the Cauchy problem), where data are posed in the infinite past. The choice for these data is informed by making certain heuristic connections to the Newtonian theory (Post-Newtonian theory). It is then proved that there exists a unique solution that attains these data in the limit, and the asymptotic properties of this solution are analysed.

In this way, it is found that the constructed solutions, which have clear physical interpretations afforded by the connection to the Newtonian theory, neither admit a smooth past null infinity nor a smooth future null infinity (they violate peeling near both), and moreover decay slower towards spatial infinity than typically assumed in the literature. In other words, our constructions show that many of the commonly-used concepts to model isolated physical systems are, in fact, not suitable for this purpose. Furthermore, it is shown that the non-smoothness of future null infinity is, in a certain sense, exactly conserved and even determines the asymptotic behaviour of gravitational radiation at late times.

Our constructions either concern the nonlinear Einstein–Scalar field equations under the assumption of spherical symmetry, or the system of linearised gravity around Schwarzschild with no symmetry assumptions, but the methods employed are capable of generalisation to the nonlinear Einstein vacuum equations.

Leonhard Matti Augustin Kehrberger

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Contents

Chapter 0: Introduction	1
0.1 Historical background and context	1
0.2 Overview of the contents of this thesis	12
0.3 Guide to reading the thesis	16
0.4 Errata to the published versions	18
Chapter 1: Early-time asymptotics for a self-gravitating scalar field in spherical symmetry	19
Part I: Introduction, motivation and summary of the main results	21
1.1 Introduction	21
1.2 Overview of the main results (Thms. 1.2.1–1.2.5) and of upcoming work	27
Part II: Construction of counter-examples to the smoothness of $\mathcal{I}^+$	40
1.3 The Einstein-Maxwell-Scalar field equations in spherical symmetry	41
1.4 Case 1: Initial data posed on an ingoing null hypersurface	46
1.5 Case 2: Boundary data posed on a timelike hypersurface	60
1.6 An application: The scattering problem	102
Appendix 1.A Useful curvature computations	112
Appendix 1.B Constructing the time integral from characteristic initial data	113
Chapter 2: A logarithmically modified Price’s law	115
2.1 Introduction	116
2.2 The geometric setting	122
2.3 Preliminaries	125
2.4 Asymptotics I: The case $I_0^{\log}[\phi] \neq 0$	127
2.5 Time inversion for $I_0^{\log}[\phi] = 0$ and $I_0^{\frac{\log r}{r^3}}[\phi] \neq 0$	138
2.6 Asymptotics II: The case $I_0^{\log}[\phi] = 0$ and $I_0^{\frac{\log r}{r^3}}[\phi] \neq 0$	141
2.7 Proof of Theorem 2.1.1	143
2.8 Comments on higher-order asymptotics	145
2.9 Comments on the non-linear problem	147

Appendix 2.A	Definitions of the main energy norms	149
Chapter 3:	Early-time asymptotics for higher ℓ-modes of linear waves on Schwarzschild	151
3.1	Introduction	153
3.2	Geometric preliminaries	172
3.3	Generalities on the wave equation	174
Part I:	The case $\ell = 1$	179
3.4	Data on an ingoing null hypersurface $\mathcal{C}_{v=1}$	179
3.5	Boundary data on a timelike hypersurface Γ_R	189
3.6	Boundary data on a timelike hypersurface Γ_f	211
Part II:	Generalising to all $\ell \geq 0$	218
3.7	The higher-order Newman–Penrose constants	218
3.8	Boundary data on a timelike hypersurface Γ_R	221
3.9	Data on an ingoing null hypersurface $\mathcal{C}_{v=1}$ I	236
3.10	Data on an ingoing null hypersurface $\mathcal{C}_{v=1}$ II	244
Appendix 3.A	Proofs omitted from the chapter’s main body	260
Chapter 4:	A dictionary for asymptotics near $\mathcal{I}^-$, $\mathcal{I}^+$ and i^+ for all ℓ-modes	265
4.1	Introduction	266
4.2	Coordinates, foliations and conventions	270
4.3	The $f(r)$ -modified Newman–Penrose charges	272
4.4	Deriving asymptotics towards i^+ from asymptotics towards $\mathcal{I}^+$	274
4.5	Deriving asymptotics towards $\mathcal{I}^+$ from physical data near i^-	281
4.6	Completing the dictionary and extending to gravitational perturbations	287
4.7	Further directions	289
4.8	Conclusion	291
Chapter 5:	Early-time asymptotics for linearised gravity—An overview	293
5.1	Introduction	294
5.2	The physical setup for the generation problem	296
5.3	The mathematical setup for the propagation problem	299
5.4	Solving the scattering problem I: Constructing the unique solution	304
5.5	Solving the scattering problem II: Asymptotic analysis of solutions $\alpha^{[s]}$ to the Teukolsky equations	306
5.6	Solving the scattering problem III: Asymptotic analysis of the remainder of (LGS)	311
5.7	The question of late-time asymptotics	312
5.8	Conclusion	313

Chapter 6: Early-time asymptotics for linearised gravity—The details	315
6.1 Introduction	318
6.2 The Schwarzschild family of spacetimes	321
6.3 The linearised Einstein vacuum equations in a double null gauge	329
6.4 Pure gauge and linearised Kerr solutions	337
Part I: Setting up and solving the scattering problem	340
6.5 The general scattering data setup for (6.3.9)–(6.3.32)	340
6.6 Scattering for the Regge–Wheeler equation and for the Teukolsky equation . .	350
6.7 Construction of the (unique) scattering solution to (6.3.9)–(6.3.32)	361
6.8 The physical scattering data	383
Part II: Asymptotic Analysis of the Teukolsky equations and the Regge–Wheeler equations	388
6.9 The approximate conservation laws	389
6.10 Asymptotic analysis of $\alpha^{[s]}$ arising from general scattering data	398
6.11 Asymptotics for $\alpha^{[2]} = r^5 \Omega^{-2} \overset{(i)}{\alpha}$ arising from physical data	414
6.12 Asymptotics for $\Psi^{[\pm 2]}$ arising from physical data	418
6.13 Asymptotics for $\alpha^{[-2]} = r \Omega^2 \overset{(i)}{\underline{\alpha}}$ arising from physical data	419
6.14 Post-Minkowskian expansions of the Teukolsky equation	424
6.15 Remarks on summing up the angular modes	432
Part III: Asymptotic analysis of the remainder of the system	434
6.16 Asymptotics for $\overset{(i)}{\hat{\chi}}$ and $\overset{(i)}{\hat{\underline{\chi}}}$	434
6.17 Deriving asymptotics for $\overset{(i)}{\rho}$, $\overset{(i)}{\sigma}$, $\overset{(i)}{\beta}$ and $\overset{(i)}{\underline{\beta}}$	436
6.18 Deriving asymptotics for the remaining connection coefficients	437
6.19 Deriving asymptotics for the remaining metric coefficients and for $\overset{(i)}{K}$	437
Appendix 6.A Spin-weighted functions, the Newman–Penrose formalism, and the Christodoulou–Klainerman formalism	439
Appendix 6.B Useful integral identities	446
Bibliography	451

Chapter 0

Introduction

We begin by providing historical background and context for the contents of this thesis, a brief overview of its individual chapters, as well as a guide to reading the thesis.

0.1	Historical background and context	1
0.1.1	Isolated systems	1
0.1.2	The works of Bondi, Sachs et al., and the peeling property	3
0.1.3	Penrose’s smooth conformal compactification	4
0.1.4	The case against smooth conformal compactification	6
0.1.5	But why does this matter?	8
0.2	Overview of the contents of this thesis	12
0.2.1	Chapter 1: Early-time asymptotics for a self-gravitating field in spherical symmetry	12
0.2.2	Chapter 2: A logarithmically modified Price’s law	12
0.2.3	Chapter 3: Early-time asymptotics for higher ℓ -modes of linear waves on Schwarzschild	13
0.2.4	Chapter 4: A dictionary between early- and late-time asymptotics for all ℓ -modes	13
0.2.5	Chapter 5: Asymptotics for linearised gravity around Schwarzschild—An overview	14
0.2.6	Chapter 6: Early-time asymptotics for linearised gravity around Schwarzschild—The details	15
0.3	Guide to reading the thesis	16
0.4	Errata to the published versions	18

0.1 Historical background and context

0.1.1 Isolated systems

The desire to study systems in isolation is ubiquitous in physics: Given a physical process, we first identify which parts of its surroundings and interactions we can neglect. We then

seek a mathematical model that treats the process as independent of those surroundings and interactions. Finally, we attempt to study the resulting model—the *isolated system*—using suitable mathematical methods, hoping that its predictions will accurately approximate the actual physical process.

Consider, for instance, the motion of Earth and Sun around each other within Newtonian theory. A simple way to study this in isolation is to disregard all other matter in the universe, i.e. to model the surroundings of Earth and Sun by *vacuum*, and to only consider (Newtonian) gravitational interactions between Earth and Sun. By taking into account further interactions with other planets etc., the resulting predictions can be made more and more accurate; but the conceptual approach to *isolate* the system remains the same.

Two reasons why this conceptual approach is simple within Newtonian theory are that Newtonian theory assumes that matter moves on a fixed geometric background, and that the theory has a simple and unambiguous realisation of vacuum.

In General Relativity, however, the situation is different. The Einstein equations,

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi T_{\mu\nu} + \Lambda g_{\mu\nu}, \quad (0.1.1)$$

are equations for the geometry of an a priori unknown spacetime, and, even when we set the matter part as well as the cosmological constant to 0 (so as to ignore cosmological effects), the remaining Einstein vacuum equations,

$$R_{\mu\nu} = 0, \quad (0.1.2)$$

still form a highly complicated system of geometric PDE with a large space of solutions, the dynamical degrees of freedom of these solutions being carried by gravitational radiation which, if focussed enough, can even create black holes [Chr09]. Intuitively, we want to exclude such events from occurring and expect that the gravitational radiation of isolated systems should disperse at large distances; we want the spacetime to asymptote to the Minkowski spacetime in some very weak sense. Making this precise requires a thorough understanding of the asymptotic structure of gravitational radiation, and, indeed, notions of isolated systems in GR are often masked behind expressions such as “asymptotic flatness”.

In the next few subsections, we will give an overview of (and some commentary on) some of the historically most influential works and approaches to this problem, and explain some short-comings of these approaches. These approaches model isolated systems based on what can be called *the asymptotic problem*. We will then go over certain arguments against these approaches, which also take into account what can be called the *generation problem* and the *propagation problem* of gravitational radiation.

Finally, we will argue that the matter of how to model isolated systems is not only relevant from an epistemological point of view, but that it has very concrete implications for important

mathematical problems and even astrophysical effects such as the asymptotic behaviour of gravitational radiation at late times.

0.1.2 The works of Bondi, Sachs et al., and the peeling property

In a series of papers by Bondi, van der Burg, Metzner and Sachs [Bon60, Sac61, BVdBM62, Sac62b], the authors introduced various ideas whose influence on future works concerning gravitational radiation is difficult to overestimate. While the initial project had the aim of understanding conditions that guarantee that a given asymptotic region of spacetime only features outgoing radiation, its scope quickly widened: The authors essentially opened up a new field of study, centred around the question: To what degree can we understand aspects of spacetime by only analysing its possible asymptotic behaviour, i.e. by its behaviour near infinity? For instance, can we deduce from soft arguments certain symmetries and regularity properties that gravitational radiation or spacetimes should satisfy, without studying the actual “interior” of spacetime? And can this potentially give us insights *a posteriori* into the interior of spacetimes? (Note that several modern approaches to quantum gravity ask exactly this question, with countless conjectured correspondences between the bulk and the boundary of spacetime.)

To go back to the concrete question at hand, the authors argued in [Sac61, BVdBM62] that the gravitational field of (vacuum) spacetimes with no incoming radiation should, just as for linearised gravity around Minkowski (see already §5.2), admit power series expansions in $1/r$ along *outgoing* null geodesics, r denoting the luminosity parameter along these geodesics. Operating under this assumption, a system of coordinates, nowadays referred to as “Bondi coordinates”, was constructed, and it was cautiously suggested in [Sac62a] that the notion of an isolated system/asymptotic flatness might be captured by the existence of such Bondi coordinates. It was further shown that the Weyl curvature tensor “peels” towards $\mathcal{I}^+$ [Sac61, NP62, Sac62b, GK64], a statement which in Newman–Penrose notation can be written as

$$\Psi_i = \Psi_i^0 r^{-5+i} + \mathcal{O}(r^{-6+i}), \quad i = 0, \dots, 4, \quad (0.1.3)$$

where Ψ_i denote the Newman–Penrose scalars [NP62] (they are the extremal components of the Weyl curvature tensor, see eq. (5.2.1) of the thesis for a definition), and Ψ_i^0 are some functions independent of r .¹

At the technical level, this peeling property greatly simplifies the Einstein equations, transforming them into a set of a few relatively simple evolutionary equations and a set of algebraic identities. Its imposition therefore led to a stark increase in activity and progress in studying the previously almost impenetrable Einstein equations.

¹For an overview over the field before the advent of the ideas discussed in the present section, we refer to the lecture notes [Tra02] (from 1958) and references therein.

At the physical level, however, there was debate from early on whether the peeling property captures the no incoming radiation condition, or whether it is even logically related to it (it only is for gravity linearised around the Minkowski spacetime), and several works [CT72, NG82, Win85, CMS95] considered more general expansions of the Weyl curvature tensor than (0.1.3). The approach of *imposing certain asymptotic behaviour* and then studying the consequences, however, remained popular ever since. Furthermore, many of the early discoveries that came out of the general framework set up by Bondi et al., such as the Bondi mass loss formula [Bon60, Sac62b, CK93], or the BMS symmetry group at infinity [BVdBM62, Sac62b, Sac62a, NP66], were soon understood to be somewhat robust with respect to a broad class of violations of peeling.

0.1.3 Penrose’s smooth conformal compactification

Only very briefly after the appearance of [Sac61], a concept of pristine geometric elegance and clarity that, in particular, captures the peeling property discussed above was introduced by Penrose [Pen63, Pen65]. This concept gave a new, very concrete proposal for modelling isolated systems; we paraphrase it here:

An isolated astrophysical system should asymptotically become empty and approach the Minkowski spacetime. The Minkowski spacetime has the property that it can be conformally compactified and, upon compactification, admits smooth conformal null boundaries called future and past null infinity ($\mathcal{I}^+$ and $\mathcal{I}^-$), respectively. The hope is now that spacetimes describing isolated astrophysical processes should share this property. Precisely this idea is captured by the definition of *asymptotic simplicity*: Without going into details, a physical spacetime is called asymptotically simple if there exists another (unphysical) manifold and a conformal factor such that the conformally rescaled spacetime isometrically embeds into the unphysical manifold and can be smoothly extended to its null boundaries. This property is also referred to as the spacetime possessing *a smooth null infinity*.

Penrose then showed that, owing to the conformal properties of the Weyl tensor, spacetimes with a smooth null infinity satisfy the peeling property (0.1.3) near $\mathcal{I}^+$. Similarly, along *ingoing* null geodesics near $\mathcal{I}^-$, they satisfy

$$\Psi_{4-i} = \Psi_i^{4-i, \text{past}} r^{-5+i} + \mathcal{O}(r^{-6+i}), \quad i = 0, \dots, 4. \quad (0.1.4)$$

Note that the original argument of Sachs et al. deducing (0.1.3) from the absence of incoming radiation would similarly, by time-reversal, deduce (0.1.4) from the absence of outgoing radiation.

Since asymptotic simplicity defines a class of *global* spacetimes, it has played a crucial role in understanding various global aspects of spacetimes, for instance related to black holes (which in most textbooks are defined using asymptotic simplicity).²

Again, the conceptual approach of asymptotic simplicity is to understand aspects of spacetime by certain considerations concerning their asymptotic behaviour, which in turn are motivated by soft/formal considerations, that is to say, considerations that do not directly take into account the physical structure of the system under consideration.

Nonetheless, even though there was not yet a direct connection established between the generation of gravitational waves by some physical system and their asymptotic behaviour as posited by peeling or asymptotic simplicity, and even though the conformal irregularity of spatial infinity in the presence of non-trivial ADM mass was later understood to cause trouble to the simultaneous regularity of *both* future *and* past null infinity [SS79, PS81], there was still a class of physical spacetimes for which it was relatively clear that they would possess a smooth null infinity, namely *past-stationary spacetimes* (cf. Fig. 1). Such spacetimes, sometimes referred to as *Bondi bomb spacetimes*, are modelled to be exactly stationary up until some sudden, violent explosion of gravitational waves (which is triggered by something so weak that it is neglected within the model). It is relatively straight-forward to see that such a spacetime satisfies peeling at least until the retarded time at which the bomb explodes. Furthermore, by propagation of regularity arguments, the spacetimes will then continue to satisfy peeling for any finite retarded time. See [Bla87, Fri86].

From the mathematical perspective, this Bondi bomb scenario corresponds to the study of *forward* Cauchy problems where the initial data are assumed to be compactly supported. (Notice that Bondi bomb spacetimes as depicted in Fig. 1, evolving from stationary to nonstationary spacetimes, cannot arise from backwards evolution from Cauchy data for standard matter models; one would have to manually insert the explosion of the bomb into the matter model!)

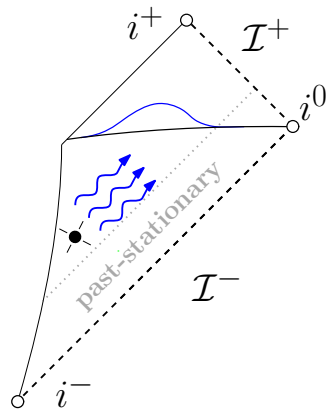


Figure 1 Depiction of a “Bondi bomb” spacetime, i.e. a spacetime that is stationary in the past. The often made assumption of compactly supported initial data corresponds to precisely such a scenario.

²We note in passing that, owing to another condition in the definition of asymptotic simplicity, not even the Schwarzschild spacetime is asymptotically simple due the presence of trapped null geodesics, but this is of no further relevance here.

0.1.4 The case against smooth conformal compactification

The previous two subsections discussed the *asymptotic problem of gravitational radiation* on relatively abstract grounds. We will now, in addition, take into account the *generation* (what is the structure of gravitational radiation *generated* by a physical process?) and the *propagation problem* (how does this radiation *propagate* through spacetime?) of gravitational radiation. Of course, as long as the generation problem is studied at *finite* time, one inevitably has to make a *choice* for the asymptotic behaviour of radiation along that time slice. On the other hand, if we study the generation and propagation problem in the *infinite past*, we will obtain a dynamical prediction on the asymptotic behaviour of gravitational radiation, see Fig. 2!

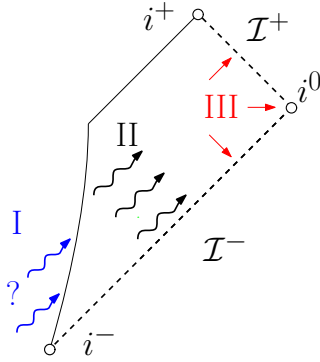


Figure 2 Schematic depiction of the generation problem (I), the propagation problem (II) and the asymptotic problem (III). If problems (I) and (II) are studied in the *infinite past*, they determine the asymptotic behaviour (III) near $\mathcal{I}^+$.

In order to study them at *finite* time, one instead needs to make an *ad hoc* assumption about this asymptotic behaviour near $\mathcal{I}^+$.

One of the most basic physical systems that any notion of isolated system should be able to describe is the two-body problem. Let us, for instance, think about two masses approaching each other from the infinite past. If these masses move at non-relativistic relative speeds, then Newtonian theory provides a reasonable approximation for the movement of these masses. For similar reasons that the hydrogen atom is classically unstable, we are then forced to consider orbits which are unbound in the infinite past, i.e. either approximately parabolic or hyperbolic Keplerian orbits (see [WW79a]).

Next, given this setup, we can attempt to understand the generation of gravitational radiation by these masses coming in from the infinite past using Post-Newtonian approximations for gravity linearised around Minkowski. It then turns out, as was already understood very early on [BP73, WW79b], that the radiation generated in this way fails to satisfy peeling near $\mathcal{I}^-$ (eq. (0.1.4)): The study of the *generation problem* shows that the asymptotic structure of gravitational radiation as posited by peeling or asymptotic simplicity fails, and it *constructively provides a different prediction for this asymptotic structure*. See already §5.2 of chapter 5 for details.

However, the same early arguments, by extending the assumed validity of the Post-Minkowskian predictions all the way to infinite advanced times, found (as they were still made within the framework of linearised gravity around Minkowski) that the future null infinity of such spacetimes should be smooth. The general idea that peeling fails near $\mathcal{I}^-$ but holds

true near $\mathcal{I}^+$ was somewhat substantiated by the evidence of [SS79, PS81] concerning the incompatibility between peeling at $\mathcal{I}^-$ and at $\mathcal{I}^+$.

New light on the matter was shed by Damour in 1986 [Dam86] (see also [IWW84]): In addition to taking into account the generation problem (to find that peeling fails near past null infinity), Damour also analysed the *propagation problem* of gravitational radiation by perturbatively studying a subset of the equations of linearised gravity around Schwarzschild and propagating the asymptotics near past infinity towards $\mathcal{I}^+$. The result of this preliminary analysis was that peeling also fails near future null infinity—the rigorous and complete analysis of this propagation problem is one of the topics of this thesis.

Further influential heuristics against smooth conformal compactification were put forth by Christodoulou in 2002 [Chr02]. To this date, his argument remains the only argument at the level of the full non-linear Einstein equations without symmetry: Christodoulou suggests that any spacetime arising from the class of initial data studied in the monumental work [CK93] must necessarily violate peeling near $\mathcal{I}^-$ provided that there is no incoming radiation from $\mathcal{I}^-$ and provided that the limit of the ingoing shear along $\mathcal{I}^+$ (a.k.a. the News function) decays as predicted by Einstein’s quadrupole formula for a system of N infalling masses following approximately hyperbolic Keplerian orbits in the infinite past. An account of this argument, which, in fact, initiated the work done in this thesis, is given in §1.1.2 of chapter 1.

It is important to understand that Christodoulou’s counterargument against previous notions of isolated systems can be interpreted to be itself based upon a different prescription of modelling isolated systems, the assumption being that isolated systems can be modelled by spacetimes arising from the initial data assumed in [CK93] (cf. the definition of C–K compatible initial data in Remark 1.1.1 in chapter 1). In particular, these spacetimes are assumed to have certain decay towards spatial infinity i^0 .

The mathematically rigorous study of the case against smooth null infinity was then initiated in the series of works that comprise this thesis. While the previous approaches towards modelling isolated systems either imposed certain asymptotic behaviour on gravitational radiation at some finite time, be it towards $\mathcal{I}^+$ (peeling, Bondi coordinates...) or towards i^0 (as in [CK93]), or by setting it to 0 as in the Bondi bomb spacetimes, the approach towards modelling isolated systems taken in the present work is to let them arise as solutions to *scattering problems in the infinite past*—as discussed in the beginning of §0.1.4, this is the only way to avoid having to make assumptions near spacelike of future null infinity. To be concrete, one of the current (not yet resolved) goals of the research program initiated in this thesis is the following:

Open Problem 1 (A mathematical understanding of the propagation problem). *Set up a scattering problem in the infinite past for the non-linear Einstein vacuum equations (0.1.2) with scattering data posed at some finite advanced time v and along the part of past null infinity beyond v .*

Formulate precise assumptions on the scattering data that capture the Post–Newtonian predictions for a system of N infalling masses from the infinite past (the generation problem).

Solve this scattering problem and find the asymptotic properties of the solution, i.e. the asymptotic properties of gravitational radiation, near spacelike infinity and future null infinity (see Fig. 2).

As a precursor to this problem, this thesis contains, amongst other things, work that resolves the corresponding problem for the equations of linearised gravity around Schwarzschild, up to one remaining challenge. We already refer the impatient reader to chapter 5, which gives a detailed overview of this problem, together with a detailed discussion of the Post–Newtonian literature. In fact, in that chapter, we also show that the interpretation of Christodoulou’s argument [Chr02] given in §1.1.2 is not correct. While the existence of a mathematical interpretation of [Chr02] that can be proved remains open (the author spent more than a year of their PhD trying to find one), we present in the same chapter a scattering argument at the level of linearised gravity that fully recovers the predictions and the spirit of [Chr02] and is, in contrast to Christodoulou’s argument, entirely dynamical.

Now, even if we eventually obtain a mathematically complete resolution of the *propagation problem* of gravitational radiation by solving Open Problem 1, the corresponding results will still rest on assumptions how to model the scattering data. While it is of course not entirely unsatisfactory to make assumptions on the scattering data based on Post–Newtonian arguments for the *generation problem*, it would be a most interesting problem to achieve a mathematical understanding of this *generation problem*, i.e. a mathematical understanding of the N -body problem in the infinite past in general relativity:

Open Problem 2 (A mathematical understanding of the generation problem). *Find a suitable matter model describing two or more particle-like mass distributions approaching each other, and solve the corresponding scattering problem for the Einstein equations (0.1.1) (with $\Lambda = 0$) in the infinite past, at least up to some finite advanced time.*

We note that even though chapter 1 of this thesis makes a very small step in the direction of this problem, this second open problem remains largely unaddressed in this thesis.

Remark 0.1.1. The entire discussion up until now took place under the pretence that the cosmological constant Λ can reasonably be set to vanish, and that the astrophysical processes under consideration can thus be viewed to start in the infinite past of a “cosmologically flat” ($\Lambda = 0$) spacetime. The question of how to formalise the limiting process where Λ is sent to 0 is yet another completely different story and remains entirely unaddressed by this thesis.

0.1.5 But why does this matter?

Having given some level of historical commentary over the evolution of models for isolated systems/asymptotic flatness in general relativity, we now outline a few reasons why the general

problem of modelling isolated systems (and thus the contents of this thesis) is relevant from various perspectives:

Firstly, there is the epistemological perspective: If we have a widely-used and accepted concept that aims to model isolated systems, such as that of asymptotic simplicity, we should critically examine its justifications and shortcomings, and understand what kinds of physics can, or can not, be captured by the concept. To be quite concrete, we will learn in this thesis that a spacetime can, somewhat ironically, essentially only be expected to have a smooth future null infinity if there is *no outgoing radiation* (or a fine-tuned mix of outgoing and incoming radiation) in the infinite past; any radiation that reaches spatial infinity will generically lead to an irregular future null infinity. Thus, operating under the assumption of a smooth null infinity excludes any physical scenario where there is radiation in the infinite past and effectively only leaves us with various types of Bondi bomb scenarios.

Secondly, there is the physical perspective: Any physicist used to making approximations wherever possible might now feel inclined to point out that, from a measurement perspective, the violent outburst of gravitational radiation during, say, the merger of two black holes, should completely dominate the radiation that the black holes emitted in the “infinite” past (when they were “infinitely far away from each other”), effectively also turning it into a Bondi bomb scenario. However, as was first pointed out by Kroon [Kro01] and discussed further in [Keh21b, GK22] (chapters 2 and 4 of this thesis), such intuitive arguments must necessarily become incorrect eventually: No matter the strength of any outbursts of gravitational radiation at *finite* time, if one measures for sufficiently long times, the late-time tails of gravitational radiation will be dominated by the effects of the irregularity of $\mathcal{I}^+$ (which in turn arises from the radiation in the infinite past). This is because the late-time decay rates associated to radiation coming from the infinite past and radiation coming from some finite-time explosion are different, and the decay rate of the latter is faster, see chapters 2 and 4 of this thesis. Conversely, quantitatively studying the effects of gravitational radiation in the infinite past is exactly what allows one to control the error made by setting this very radiation to vanish. The methods and results of this thesis can thus also be seen as tools that can be used to try and justify the assumption of past-stationarity.

Thirdly, there is the mathematical perspective: The largest part of mathematical problems in general relativity, e.g. concerning the stability or instability of explicit solutions such as the Minkowski spacetime, singular/regular behaviour in the interior of black holes, or the study of precise asymptotics of gravitational waves, is studied from the perspective of the forward Cauchy problem: Initial data are posed on an asymptotically flat (terminating at i^0) or an asymptotically null (terminating at $\mathcal{I}^+$) hypersurface. In such settings, one is thus forced to make an assumption on the asymptotic decay behaviour of the initial data set. Since there is no *a priori* way to understand the asymptotic decay of an initial data set, any choice for this decay might thus be thought of as an *ad hoc* choice of modelling a class of isolated systems.

Conversely, the scattering approach towards modelling isolated systems taken in this thesis dynamically produces certain decay behaviour towards i^0 or towards $\mathcal{I}^+$, and will therefore affect all of the mathematical problems listed above.

For instance, the well-known and much studied Price’s law (see e.g. [Pri72, Lea86, GPP94, AAG18c, AAG18b, AAG21, MZ22b, Hin22]) governing the late-time behaviour of gravitational radiation has mostly been studied under the assumption of compactly supported initial data (see Fig. 1). It has also been studied under the slightly weaker assumption of initial data satisfying peeling and recently, in [Keh21b] and [GK22] (chapters 2 and 4 of this thesis), under assumptions that violate peeling, and it was shown that this crucially changes the late-time behaviour, cf. our comment two paragraphs above. But since the precise late-time behaviour of gravitational radiation in the exterior of black holes affects the regularity in the interior of black holes, this, too, is influenced by the regularity of $\mathcal{I}^+$ (see e.g. [Daf05a, LO19, Sbi22, MZ22a]). See the introduction of chapter 4 for a more detailed discussion.

The situation is similar for stability studies of explicit solutions to (0.1.2): Let’s recall the monumental work on the stability of Minkowski spacetime [CK93]. In this work, Christodoulou and Klainerman consider perturbations of the Minkowski initial data that satisfy *inter alia* $g_{ij} = (1 + \frac{2M}{r})\delta_{ij} + o(r^{-3/2})$ (g denoting the metric on the initial data surface $\{t = 0\}$), and show that the corresponding solutions disperse and asymptotically approach the Minkowski spacetime with quantitative rates. Since these quantitative rates were slower than predicted by (0.1.3), the results of [CK93] were, in particular, incorrectly interpreted by some to disprove peeling, even though [CK93] only showed that there are certainly larger classes of spacetimes than the peeling spacetimes—the question whether peeling is “true” or not can only be answered by taking into account physical arguments. And indeed, as we will see in this thesis, the initial decay $g_{ij} = (1 + \frac{2M}{r})\delta_{ij} + o(r^{-3/2})$ is itself too strong to model certain physically relevant perturbations—in chapters 5 and 6, we construct spacetimes which, restricted to $t = 0$, asymptotically decay like

$$g_{ij} = (1 + \frac{2M}{r})\delta_{ij} + \mathcal{O}_{\ell \leq 1}(r^{-2}) + \mathcal{O}_{\ell \geq 2}(r^{-1}), \quad (0.1.5)$$

the subscripts $\ell \leq 1$, $\ell \geq 2$ denoting the projections to spherical harmonics with $\ell \leq 1$, $\ell \geq 2$, respectively. Out of all the various proofs of stability of the Minkowski spacetime floating about in the literature (e.g. [CK93, KN03b, LR10, Kei18, HV20, She22, Hin23] and references therein), only Bieri’s class of perturbations [BZ09] is large enough to allow for such slow decay—all other proofs assume in particular that $g = (1 + \frac{2M}{r})\delta + o(r^{-1})$!

Somewhat surprisingly, even though we must thus conclude that most stability works starting from a Cauchy hypersurface terminating at i^0 make assumptions too strong to model a physically relevant class of perturbations, we will find that these very perturbations exhibit sufficiently fast decay towards $\mathcal{I}^+$ so as to still be compatible with most stability works that start

from an asymptotically null initial data hypersurface [DHR19b, DHRT21, KS21]! For further details concerning this, we refer the reader to chapter 5, and the conclusion therein (§5.8).

At the end of this introductory section, we now confess that this introduction is taken from introduction to the work [Keh23], on which also chapter 5 is based: For the reader who is less interested in the mathematical details and constructions of the other parts of this thesis, and more interested in gaining just a better overview of the general problems described over the past few pages, it would therefore not be unreasonable to now skip directly to the relatively short chapter 5, though we advise to still read §0.2 and §0.3 of the present chapter before.

0.2 Overview of the contents of this thesis

In this section, we provide a coarse overview of the individual chapters of this thesis:

0.2.1 Chapter 1: Early-time asymptotics for a self-gravitating field in spherical symmetry

In chapter 1, we first provide an account of Christodoulou’s argument against smooth null infinity (cf. §0.1.4).

We then consider the Einstein-Scalar field system³ under spherical symmetry with a chargeless and massless scalar field ϕ ,

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 2T_{\mu\nu}^{sf} = 2(\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}\partial_\xi\phi\partial^\xi\phi) \quad (0.2.1)$$

and pose a scattering problem with boundary data for it: We pose polynomially decaying boundary data along some fairly general timelike hypersurface Γ terminating at i^- and vanishing data along past null infinity $\mathcal{I}^-$. The timelike boundary can be thought of as the surface of a star, and the data posed along it as radiation emitted by this star. The vanishing of the data at $\mathcal{I}^-$, on the other hand, corresponds to excluding incoming radiation from $\mathcal{I}^-$. We then solve this scattering problem and prove that the solution does not admit a smooth future null infinity by computing the asymptotic expansion of ϕ towards $\mathcal{I}^+$ and showing that it exhibits logarithms, violating the scalar-field analogue of the peeling property (0.1.3).

Several corollaries can be deduced from the results: For instance, we deduce an analogous statement where the scattering data are instead posed on some ingoing null hypersurface terminating at $\mathcal{I}^-$. We also apply the results to the classical linear scattering problem on Schwarzschild: Putting smooth, compactly supported spherically symmetric data for the *linear scalar wave equation* on Schwarzschild

$$\square_g\phi = 0 \quad (0.2.2)$$

on the past event horizon $\mathcal{H}^-$ of Schwarzschild and on $\mathcal{I}^-$, we show that *every* solution that doesn’t vanish identically will eventually feature logarithmic terms in its expansion towards $\mathcal{I}^+$. Finally, we also deduce some surprising statements concerning extremal Reissner–Nordström black holes.

This chapter corresponds to the author’s [Keh21a], with a few minor corrections.

0.2.2 Chapter 2: A logarithmically modified Price’s law

In chapter 2, we restrict our attention to the linear wave equation (0.2.2) on Schwarzschild under spherical symmetry, and prove a relatively simple observation: The violation of peeling

³We also include a Maxwell field, but this does not affect the results.

near $\mathcal{I}^+$ as proved for the constructions of the previous chapter impacts the asymptotics of ϕ at late times. In particular, we prove that the late-time decay rates in this setting are a logarithm slower compared to the celebrated Price's law [Pri72].

This chapter corresponds to the author's [Keh21b] and uses methods developed by Angelopoulos–Aretakis–Gajic in [AAG18c, AAG18b].

0.2.3 Chapter 3: Early-time asymptotics for higher ℓ -modes of linear waves on Schwarzschild

In chapter 3, we continue to investigate the asymptotic behaviour at early times of solutions to the linear wave equation (0.2.2), but we now direct our attention to the behaviour of higher ℓ -modes of ϕ , i.e. we study projections of ϕ onto higher spherical harmonics. We again consider the timelike boundary value problem of chapter 1, i.e. we pose polynomially decaying boundary data along a timelike boundary Γ and we pose vanishing data at $\mathcal{I}^-$. We solve the corresponding scattering problem and analyse the resulting asymptotics towards $\mathcal{I}^+$. As in chapter 1, we also solve the simpler problem where the boundary data along Γ are replaced with characteristic data along an ingoing null hypersurface terminating at $\mathcal{I}^-$.

We prove that if the timelike boundary has constant area-radius and all ℓ -modes ϕ_ℓ along Γ are assumed to have the same decay towards i^- , then higher ℓ -modes decay faster towards $\mathcal{I}^+$ in the sense that the first logarithmic term in the expansion of ϕ_ℓ towards $\mathcal{I}^+$ appears at later orders for higher values of ℓ .

However, if the area radius along Γ grows linearly in time, or if the data are posed along an ingoing null hypersurface instead, and if all ℓ -modes of the data feature the same decay, then this is no longer true, and the first logarithmic term in the expansion of ϕ_ℓ will appear at the same order for (almost) all ℓ . In this setting, all ℓ -modes are thus on the same footing, and one cannot prove asymptotic estimates for the summed solution by simply proving an asymptotic estimate for the first few ℓ -modes and then showing the remaining ℓ -modes to be subleading. This means that the issue of summing these fixed-frequency estimates to obtain an asymptotic estimate for $\phi = \sum_{\ell=0}^{\infty} \phi_\ell$ becomes highly non-trivial. While some commentary on this issue is provided throughout the thesis, its resolution is ongoing work that is not contained in this thesis [KK23].

We will in chapter 5 of the thesis see that the scenario where all ℓ -modes have the same asymptotics towards $\mathcal{I}^+$ has physical relevance.

This chapter corresponds to the author's [Keh22].

0.2.4 Chapter 4: A dictionary between early- and late-time asymptotics for all ℓ -modes

In chapter 2, we showed that the failure of peeling for spherically symmetric solutions ϕ to (0.2.2) near $\mathcal{I}^+$ affects the asymptotics of ϕ at late times. In chapter 4, we systematically generalise

this result to higher ℓ -modes. We introduce the $f(r)$ -modified Newman–Penrose charges $I_\ell^f[\phi]$, which measure the degree to which peeling is violated near $\mathcal{I}^+$ and which are conserved along $\mathcal{I}^+$, and we explain how these conserved charges entirely determine the asymptotics at late times, i.e. towards future timelike infinity i^+ . Combined with the results of chapter 3, which we briefly recap, this provides us with a complete dictionary between asymptotics of fixed-frequency solutions ϕ_ℓ near $\mathcal{I}^-$, $\mathcal{I}^+$ and i^+ .

We also explain a mechanism that extends these results to solutions of the Teukolsky equations with general spin s ,

$$\mathcal{T}_g^{[s]}\Psi^{[|s|-s]} = 0, \quad (0.2.3)$$

which, in particular, describe gravitational perturbations around Schwarzschild if $s = \pm 2$. We then sketch an argument why the late-time asymptotics associated to a system of N infalling masses following approximately hyperbolic Keplerian orbits near i^- are three powers slower than predicted by Price’s law.

This chapter is the outcome of collaboration with Dejan Gajic [GK22].

0.2.5 Chapter 5: Asymptotics for linearised gravity around Schwarzschild—An overview

Chapter 5 is at the heart of the thesis. Having discussed the Einstein–Scalar field system under spherical symmetry in chapter 1 and the linear wave equation on Schwarzschild in chapters 2–4, chapters 5–6 discuss the system of linearised gravity around Schwarzschild, which we will here denote by **(LGS)**. Chapter 5 gives a detailed overview of our problem in this setting.

First, we give an account of the Post-Newtonian framework that is used to understand the generation problem, i.e. we explain how we use Post-Newtonian theory to understand the gravitational radiation emitted by a system of N infalling masses following approximately hyperbolic orbits in the infinite past up to some finite advanced time.

We then utilise the prediction of this Post-Newtonian framework to formulate a scattering setup for **(LGS)**, and we give a rough sketch of how we solve this scattering problem and how we obtain the asymptotic behaviour of the corresponding solution. This will enable us to finally make the direct connection to the discussion of §0.1. In particular, it will enable us to understand the exact way in which the peeling property (0.1.3) is violated. We will also use this to show that, while the predictions of Christodoulou’s argument [Chr02] are confirmed, the mathematical interpretation of this argument as presented in chapter 1 is incorrect.

We conclude by commenting on various consequences that our results have—the conclusion to this chapter (§5.8) can be read as a final conclusion to the introductory §0.1.

This chapter is based on work that has been written by the author and will be submitted for a theme issue of the *Philosophical Transactions of the Royal Society A* associated to the

workshop “At the interface of asymptotics, conformal methods and analysis in general relativity (May 2023)” [Keh23].

0.2.6 Chapter 6: Early-time asymptotics for linearised gravity around Schwarzschild—The details

In chapter 6, we provide the full details corresponding to the overview given in chapter 5. We carefully introduce the system (LGS) , we set up a semi-global scattering problem for it with scattering data posed along $\mathcal{I}^-$ and along some ingoing null hypersurface terminating at $\mathcal{I}^-$, and we solve this scattering problem.

We then define a class of scattering data that describes the exterior of a system of N infalling masses following approximately hyperbolic Keplerian orbits in the infinite past, and we analyse the asymptotic properties of the corresponding scattering solution at early times, i.e. in a neighbourhood of $\mathcal{I}^-$, i^0 and $\mathcal{I}^+$.

For this latter asymptotic analysis, we rely on the fact that the system (LGS) is governed, at a fundamental level, by the Teukolsky (0.2.3) and the Regge–Wheeler equations, which are both similar to the wave equation (0.2.2). This allows us to extend the methods used in chapter 3 in order to find the early-time asymptotics for fixed angular frequency solutions to these equations. The summing of these estimates is left to future work [KK23].

This chapter reflects the progress of an ongoing collaboration with Hamed Masaoood [KM23].

0.3 Guide to reading the thesis

Given the length of the thesis, we provide a small guide to reading it. In general, each chapter of the thesis can be read in an entirely self-contained way and contains a thorough introduction together with an overview of the chapter's structure and contents, so this guide is not necessary for reading the thesis, but we hope it may perhaps give some further insight into its structure.

The long route

If the reader wants to read the entirety of the thesis with all its details, then the most natural route to take is to read the chapters in the order in which they are presented. If the reader follows this route, then they will notice that there is a certain level of overlap between the individual chapters, for instance because some results from earlier chapters will be generalised in later chapters. We here provide some commentary on this:

Chapter 1, being the only one analysing the Einstein–Scalar field system (0.2.1) stands out from the other chapters the most, and there is virtually no overlap between it and the other chapters.

Chapter 2, where we study late-time asymptotics for spherically symmetric solutions to (0.2.2), on the other hand, has significant overlap with chapter 4. It is certainly worth reading the introduction to chapter 2, as well as its ending sections §2.8 and §2.9, but the reading experience of the thesis may benefit from temporarily skipping the main contents of chapter 2.

Chapter 3 treats two different problems, the problem with data on a timelike boundary problem and the problem with data on an ingoing null cone (in both problems, complementary data are posed at $\mathcal{I}^-$.) It is the only part of the thesis where the timelike boundary problem is treated for higher ℓ -modes, generalising some ideas already present in chapter 1 (such as commuting with $(p - 1 - |u|\partial_t)$). However, the treatment of the characteristic problem with data on a null cone is treated in more generality in chapter 6 (for the Teukolsky equation (0.2.3), which includes the wave equation (0.2.2) as a special case), and a quick sketch of the resolution of this problem is given both in chapter 4 for the case of the linear wave equation, and in chapter 5 for the case of the Teukolsky equation. It is worth keeping this in mind when reading chapter 3. Chapter 3 also contains its own guide to reading it in §3.1.4.

The relatively short chapter 4 should be read in full; it significantly generalises the contents of chapter 2. This chapter's goal is to *expose* certain mathematical mechanisms relating early-time asymptotics and late-time asymptotics, and it therefore leaves many technical details to the reader. After having read this chapter, it may prove helpful to go back to the contents of chapter 2 where all these details are given in the case of spherical symmetry.

We strongly recommend to read chapter 5 in full as well. This chapter has essentially no overlap with the previous chapters and gives a detailed overview of chapter 6.

Having read chapter 5, the reader should then have a good idea of the structure of the final chapter, chapter 6, which is essentially just a much more detailed version of chapter 5.

The shorter route

If the reader is interested in the results, the physical relevance of the results, as well as the main mathematical ideas of the present thesis without necessarily wanting to understand all the details, then we strongly recommend skipping chapters 2, 3 and 6, and instead only reading chapters 1, 4 and 5, which already give a very thorough discussion of the main results and ideas presented in the thesis. Indeed, the only problem that is not discussed in chapters 1, 4 and 5 is the problem with timelike boundary data for higher ℓ -modes of solutions to the linear wave equation (0.2.2). The interested reader can read the introduction to chapter 3 for an overview of this problem.

The very short route

For the reader who is very short on time, we recommend reading the very short chapter 5 immediately after having read the present introductory chapter, which will also give a more precise idea of the issues dealt with in the other chapters, so that the reader can then choose where to go from there.

Notation: Finally, we believe that it is helpful to tell the reader that, while each chapter has its own section on coordinates and notation for analytic and geometric objects in order to be self-contained, we consistently use the same notation throughout the thesis. There are three exceptions to this:

- In chapters 2 and 4 (i.e. the chapters where we discuss late-time asymptotics), we use double null coordinates which are related to those of the other sections by a factor of two.
- In chapter 4, we use ψ rather than ϕ to denote the solution to the linear wave equation (0.2.2).
- The quantities $\alpha^{[s]}$ defined in chapters 5 and 6 are related to each other by multiples of the area radius.

0.4 Errata to the published versions

As chapters 1–4 are all based on research papers that are already published or awaiting publication, we here record a few important changes and mistakes that have been fixed compared to the published versions of [Keh21a, Keh21b, Keh22, GK22].

- As explained in this introduction and shown in chapter 5, the mathematical interpretation of Christodoulou’s argument [Chr02] against smooth null infinity as presented in [Keh21a] is incorrect. Nevertheless, the reader will still find the original account of it in chapter 1 because it serves as a useful reference.
- Equation (4.45) of Thm. 4.3 of [Keh21a] contains a sign mistake, it should be $\dots - 6M \dots$ rather than $\dots + 6M \dots$. This sign mistake then propagates through to eq. (6.5) of Theorem 6.1, eq. (6.18) of Theorem 6.2 and eq. (B.2) of Theorem B.1. All these sign mistakes are fixed in chapter 1.
- Equation (3.9) of [GK22] contains a typo: The equation should have its factor $r^{-2\ell-2s}$ replaced by $r^{-2\ell}$. This typo is fixed in chapter 4.
- The paper [GK22] makes an incorrect prediction on the late-time decay of the Newman–Penrose scalar $r\Psi^{[4]}$; it predicts this to decay like u^{-4} along future null infinity. This decay should instead be u^{-3} . This is fixed in chapter 4.

Chapter 1

Early-time asymptotics for a self-gravitating scalar field in spherical symmetry

Abstract

This chapter begins with a brief review of an argument due to Christodoulou [Chr02] stating that Penrose's proposal of smooth conformal compactification of spacetime (or smooth null infinity) fails to accurately capture the structure of gravitational radiation emitted by N infalling masses coming from past timelike infinity i^- .

Modelling gravitational radiation by scalar radiation, we then take a first step towards a rigorous, fully general relativistic understanding of the non-smoothness of null infinity by constructing solutions to the spherically symmetric Einstein-Scalar field equations. Our constructions arise dynamically from polynomially decaying boundary data, $r\phi \sim |t|^{-p}$ as $t \rightarrow -\infty$, on a timelike hypersurface (to be thought of as the surface of a star) and the no incoming radiation condition, $r\partial_v\phi = 0$, on past null infinity. We show that if the initial Hawking mass at past timelike infinity i^- is non-zero, then there exists a constant $C \neq 0$ such that, in the case $p = 1$, we obtain the following asymptotic expansion near $\mathcal{I}^+$, precisely in accordance with the non-smoothness of $\mathcal{I}^+$: $\partial_v(r\phi) = Cr^{-3} \log r + \mathcal{O}(r^{-3})$. Similarly, if $p > 1$, we find constant coefficient logarithmic terms appearing at higher orders in the expansion of $\partial_v(r\phi)$.

Even though these results are obtained in the non-linear setting, we show that the same logarithmic terms appear already in the linear theory, i.e. when considering the spherically symmetric linear wave equation on a fixed Schwarzschild background.

As a corollary, we can apply our results to the classical scattering problem on Schwarzschild: Putting smooth compactly supported scattering data for the linear (or coupled) wave equation on $\mathcal{I}^-$ and on $\mathcal{H}^-$, we find that the asymptotic expansion of $\partial_v(r\phi)$ near $\mathcal{I}^+$ generically contains logarithmic terms at second order, i.e. at order $r^{-4} \log r$.

Part I: Introduction, motivation and summary of the main results	21
1.1 Introduction	21
1.1.1 Historical background	21
1.1.2 Christodoulou's argument against smooth null infinity	22
1.2 Overview of the main results (Thms. 1.2.1–1.2.5) and of upcoming work . . .	27
1.2.1 Construction of counter-examples to smooth null infinity within the Einstein-Scalar field system in spherical symmetry	27
1.2.2 An application: The scattering problem	33
1.2.3 Translating asymptotics near i^0 into asymptotics near i^+	36
1.2.4 Further directions	37
1.2.5 Structure of the chapter	38
Part II: Construction of counter-examples to the smoothness of $\mathcal{I}^+$	40
1.3 The Einstein-Maxwell-Scalar field equations in spherical symmetry	41
1.3.1 The coupled case	41
1.3.2 The Reissner–Nordström/Schwarzschild family of solutions	43
1.3.3 Specialising to the linear case	44
1.3.4 Conventions	45
1.4 Case 1: Initial data posed on an ingoing null hypersurface	46
1.4.1 Assumptions and initial data	46
1.4.2 Coordinates and energy boundedness	48
1.4.3 Sharp upper and lower bounds for $\partial_u(r\phi)$ and $r\phi$	52
1.4.4 Asymptotics of $\partial_v(r\phi)$ near $\mathcal{I}^+$, i^0 and $\mathcal{I}^-$ (Proof of Thm. 1.2.4)	55
1.5 Case 2: Boundary data posed on a timelike hypersurface	60
1.5.1 Overview	60
1.5.2 Preliminary description of the final solution	62
1.5.3 The ambient manifold	63
1.5.4 A local existence result	64
1.5.5 The finite problem: Data on an outgoing null hypersurface $\mathcal{C}_{u_0}$	67
1.5.6 The limiting problem: Sending $\mathcal{C}_{u_0}$ to $\mathcal{I}^-$	74
1.5.7 Refinements	85
1.5.8 Proof of Thm. 1.2.1	97
1.6 An application: The scattering problem	102
1.6.1 Non-linear scattering with a Schwarzschild or Minkowskian i^- (Proof of Thm. 1.2.5)	102
1.6.2 Linear scattering on Schwarzschild	106
1.6.3 Linear scattering on extremal Reissner–Nordström	110

Appendix 1.A Useful curvature computations	112
Appendix 1.B Constructing the time integral from characteristic initial data	113

Part I:

Introduction, motivation and summary of the main results

1.1 Introduction

This work is concerned with the rigorous mathematical analysis of gravitational waves near infinity. In particular, it contains various dynamical constructions of physically motivated example spacetimes that violate the well-known *peeling property* of gravitational radiation and, thus, do not possess a smooth null infinity.

The chapter aims to be accessible to an audience of both mathematicians and physicists. In hopes of achieving this aim, we divided it into two parts, with only the second one containing the actual mathematical proofs.

In the first part (Part I), we give some historical background on the concept of smooth null infinity and review an important argument against smooth null infinity due to Christodoulou, which forms the main motivation for the present work. This is done in section 1.1. Motivated by this argument, we then summarise, explain, and discuss the main results of this work (in the form of mathematical theorems) in section 1.2.

The proofs of these results are then entirely contained in Part II of this chapter, which, in principle, can be read independently of Part I.

1.1.1 Historical background

The first direct detection of gravitational waves a few years ago [Abb16] may not only well be seen as one of the most important experimental achievements in recent times, but also as one of theoretical physics' greatest triumphs. The theoretical analysis of gravitational waves “near infinity”, i.e. far away from an *isolated system* emitting them, has seen its basic ideas set up in the 1960s, in works by Bondi, van der Burg and Metzner [BVdBM62], Sachs [Sac61, Sac62b], Penrose and Newman [NP62], and others. The ideas developed in these works were combined by Penrose's notion of *asymptotic simplicity* [Pen65], a concept that can now be found in most advanced textbooks on general relativity. The idea behind this notion is to characterise the asymptotic behaviour of gravitational radiation by the requirement that the conformal structure of spacetime be smoothly¹ extendable to “null infinity” (denoted by $\mathcal{I}$ and to be thought of as a “boundary of the spacetime”) – the place where gravitational radiation is observed. This

¹In fact, smooth here can be replaced by C^k for, say, $k \geq 4$.

requirement is also referred to as the spacetime possessing a “smooth null infinity”. Implied by this smoothness assumption is, amongst other things, the so-called *Sachs peeling property*. This states that the different components of the Weyl curvature tensor fall off with certain negative integer powers of a certain parameter r (whose role will in our context be played by the area radius function) as null infinity is approached along null geodesics [Pen65].²

Although Penrose’s proposal of smooth null infinity has certainly left a notable impact on the asymptotic analysis of gravitational radiation, its assumptions have been subject to debate ever since. In particular, the implied Sachs peeling property has been a cause of early controversy; in fact, it remained unclear for decades whether there even exist non-trivial dynamical solutions to Einstein’s equations that exhibit the Sachs peeling behaviour or a smooth null infinity. This question has been answered in the affirmative in the case of hyperboloidal initial data in [Fri83, Fri86, ACF92] and, more recently, also in the more interesting case of asymptotically flat initial data in [CD02, Cor07], where a large class of asymptotically simple solutions was constructed by gluing the interior part of initial data to e.g. Schwarzschild initial data in the exterior (using the gluing results of [Cor00]) and then exploiting the domain of dependence property combined with the fact that Schwarzschild initial data lead to a smooth null infinity. See also the recent [GK17] or the survey article [Fri18] and references therein for related works. A similar result with a different approach (based on [CK93]) was obtained in [KN03a], where it was shown that if the initial data decay fast enough towards spatial infinity³, then the evolution of those data satisfies peeling.

While the analyses above show that the class of solutions with smooth $\mathcal{I}$ is non-trivial, they tell us very little about the *physical* relevance of that class. Moreover, several heuristic works [BP73, SS79, WW79b, IWW84, Win85] have hinted at Penrose’s regularity assumptions being too rigid to admit physically relevant systems, and a relation between the non-vanishing of the quadrupole moment of the radiating mass distribution and the failure of $\mathcal{I}$ to be smooth was suggested by Damour using perturbative methods [Dam86]. In fact, there is a much stronger argument against the smoothness of $\mathcal{I}$ due to Christodoulou [Chr02], which we will review now. The core contents and results of the present chapter (which are logically independent from Christodoulou’s argument, but heavily motivated by it) will then be introduced in section 1.2, where we will present various classes of physically motivated counter-examples to smooth null infinity. The reader impatient for the results may wish to skip to section 1.2 directly.

1.1.2 Christodoulou’s argument against smooth null infinity

Perhaps the most striking argument against smooth null infinity comes from the monumental work of Christodoulou and Klainerman on the proof of the global non-linear stability of the Minkowski spacetime [CK93]. The results of this work do not confirm the Sachs peeling

²A more precise statement is given in section 1.1.2 below.

³So fast as to force the angular momentum of the initial data set to vanish.

property; moreover, an argument by Christodoulou [Chr02], which adds to the proof [CK93] a physical assumption on the radiative amplitude on $\mathcal{I}$, shows that this failing of peeling is not a shortcoming of the proof but is, instead, likely to be a true physical effect. It is this argument [Chr02] which gives the present section its name, and which forms the main motivation for the present chapter. Since it does not appear to be widely known, we will give a brief review of it now.

First, let us outline the setup. In the work [CK93], given asymptotically flat vacuum initial data sufficiently close to the Minkowski initial data, two foliations of the dynamical vacuum solution (M, g) – which is shown to remain globally close and quantitatively settle down to the Minkowski spacetime – are constructed: A foliation of maximal hypersurfaces, which are level sets $\mathcal{H}_t$ of a canonical time function t , as well as a foliation of outgoing null hypersurfaces, level sets $\mathcal{C}_u^+$ of a canonical optical function u (to be thought of as retarded time and tending to $-\infty$ as i^0 is approached).

Let now e_4 be a suitable choice of the corresponding generating (outgoing) null geodesic vector field of $\mathcal{C}_u^+$ and e_3 a suitable choice of conjugate incoming null normal such that $g(e_4, e_3) = -2$, let X, Y be vector fields on the spacelike 2-surfaces $S_{t,u} = \mathcal{H}_t \cap \mathcal{C}_u^+$, and let $\not\epsilon$ be the volume form induced on $S_{t,u}$. Then, under the following null decomposition⁴ of the Riemann tensor R ,

$$\begin{aligned} \alpha(X, Y) &:= R(X, e_4, Y, e_4), & \beta(X) &:= R(X, e_4, e_3, e_4), \\ \underline{\alpha}(X, Y) &:= R(X, e_3, Y, e_3), & \underline{\beta}(X) &:= R(X, e_3, e_3, e_4), \\ 4\rho &:= R(e_4, e_3, e_4, e_3), & 2\sigma\not\epsilon(X, Y) &:= R(X, Y, e_3, e_4), \end{aligned} \quad (1.1.1)$$

Penrose's regularity requirements would require the Sachs peeling property to hold, i.e., they would require along each $\mathcal{C}_u^+$ the following decay rates, r denoting the area radius of $S_{t,u}$:

$$\begin{aligned} \alpha &= \mathcal{O}(r^{-5}), & \beta &= \mathcal{O}(r^{-4}), & \rho &= \mathcal{O}(r^{-3}), \\ \sigma &= \mathcal{O}(r^{-3}), & \underline{\beta} &= \mathcal{O}(r^{-2}), & \underline{\alpha} &= \mathcal{O}(r^{-1}). \end{aligned} \quad (1.1.2)$$

In addition, depending on the precise regularity under which the conformal structure of spacetime is assumed to be extendable, Penrose's proposal would imply that all of the above quantities will admit higher-order power series expansions in $1/r$. However, the results of [CK93] only confirm the last four rates of (1.1.2), whereas, for α and β , the following weaker decay results are obtained:

$$\alpha, \beta = \mathcal{O}(r^{-\frac{7}{2}}), \quad (1.1.3)$$

so the peeling hierarchy is *chopped off* at $r^{-7/2}$.

⁴This decomposition is closely related to the decomposition into the Newman–Penrose scalars $\Psi_0, \dots, \Psi_4$.

Now, on the one hand, the rates (1.1.3) are only shown in [CK93] to be upper bounds (i.e. not asymptotics). Moreover, one might think that these upper bounds can be improved if one imposes further conditions on the initial data – for, the data considered in [CK93] are only required to have $\alpha, \beta = \mathcal{O}(r^{-7/2})$ on the initial hypersurface. Indeed, one can slightly adapt the methods of Christodoulou–Klainerman to show that if the initial data decay much faster than assumed in [CK93], the peeling rates (1.1.2) can indeed be recovered [KN03a]. We will return to this at the end of this section.

On the other hand, as remarked before, the fundamental question is not whether there exist initial data which lead to solutions satisfying peeling, but whether *physically relevant spacetimes satisfy peeling*. Evidently, any answer to this latter question must appeal to some additional physical principle. This is exactly what Christodoulou does in [Chr02]. There, he shows that, indeed, the rates (1.1.2) cannot be recovered in several physically relevant systems, making the idea of smooth $\mathcal{I}$ physically implausible. At the core of Christodoulou’s argument lies the assumption that the *Bondi mass* along $\mathcal{I}^+$ decays with the rate predicted by the quadrupole approximation for a system of N infalling masses coming from past infinity, combined with the assumption that there be no incoming radiation from past null infinity.

Remark 1.1.1. Before we move on to explain Christodoulou’s argument, we shall make an important remark. Even though we stressed that one should not derive arguments for or against peeling from sufficiently strong Cauchy data assumptions, but rather appeal to some physical ingredients, we still want to make some initial data assumptions in order to have access to the results of [CK93]. These results, a priori, only hold for evolutions of asymptotically flat vacuum initial data sufficiently close to Minkowski initial data, i.e. data for which, in particular, a certain Sobolev norm $\|\cdot\|_{\text{CK}}$ is small. We shall call such data **C–K small data**.

Of course, C–K small data are not directly suited to describe the evolutions of spacetimes with N infalling masses. However, consider now initial data which are only required to have finite (as opposed to small) $\|\cdot\|_{\text{CK}}$ -norm and to be vacuum only in a neighbourhood of spatial infinity (as opposed to everywhere). We shall call such data **C–K compatible**. Let us explain this terminology: One can now restrict these data to a region, let’s call it the exterior region, sufficiently close to spacelike infinity in a way so that the data in this exterior region are vacuum and have arbitrarily small $\|\cdot\|_{\text{CK}}$ -norm. By a gluing argument, one can then extend these exterior data to interior data whose $\|\cdot\|_{\text{CK}}$ -norm can also be chosen sufficiently small so that the resulting glued data are C–K small. Therefore, the results of [CK93] apply to the (C–K small) glued data, and, thus, by the domain of dependence property, they apply to the domain of dependence of the exterior part of the (C–K compatible) original data, i.e. in a neighbourhood of spacelike infinity containing a piece of null infinity.⁵ It is evolutions of C–K compatible data that we shall make statements on. One can reasonably expect that such evolutions contain a

⁵We note that one should be able to avoid this gluing argument by appealing to the results of [KN03b], see the first remark below Definition 3.6.4 therein. Alternatively, one could also use the results of the more general [Kei18], since that work does not require the constraint equations to be satisfied on data.

large class of physically interesting systems such as that of N infalling masses from the infinite past.

We can now paraphrase⁶ Christodoulou's result [Chr02]:

*Consider all evolutions of C - K compatible initial data which **a)** satisfy on $\mathcal{I}^-$ the no incoming radiation condition and **b)** behave on $\mathcal{I}^+$ as predicted by the quadrupole approximation for N infalling masses. These evolutions do not admit a smooth conformal compactification.*

More precisely, the failure of these evolutions to admit a smooth conformal compactification manifests itself in the asymptotic expansion of β near future null infinity containing logarithmic terms at leading order (namely, at order $r^{-4} \log r$).

Let us briefly expose the main ideas of the proof of the above statement: We recall from [CK93] that the traceless part $\underline{\hat{\chi}}$ of the connection coefficient

$$\underline{\chi}(X, Y) = g(\nabla_X e_3, Y) \quad (1.1.4)$$

tends along any given $\mathcal{C}_u^+$ to

$$\lim_{\mathcal{C}_u^+, r \rightarrow \infty} r \underline{\hat{\chi}} = \Xi(u) \quad (1.1.5)$$

as the area radius function r associated to $S_{t,u}$ tends to infinity. Here, $\Xi(u)$ is a 2-form on the unit sphere $\mathbb{S}^2$ that should be thought of as living on future null infinity and which defines the radiative amplitude per solid angle. The quantity $\underline{\hat{\chi}}$ is often called the ingoing shear of the 2-surfaces $S_{t,u}$, and the limit Ξ is sometimes referred to as *Bondi news*. Indeed, one of the many important corollaries of [CK93] is the *Bondi mass loss formula*: If $M(u)$ denotes the Bondi mass along $\mathcal{I}^+$, then we have

$$\frac{\partial}{\partial u} M(u) = -\frac{1}{16\pi} \int_{\mathbb{S}^2} |\Xi(u, \cdot)|^2. \quad (1.1.6)$$

Now, the quadrupole approximation for N infalling masses predicts that $\partial_u M \sim -|u|^{-4}$ as $u \rightarrow -\infty$ (it is assumed that the relative velocities tend to constant values near the infinite past and that the mass distribution has non-vanishing quadrupole moment) and, thus, in view of (1.1.6), that

$$\lim_{u \rightarrow -\infty} u^2 \Xi =: \Xi^- \neq 0. \quad (1.1.7)$$

Christodoulou's two core observations then are the following: Even though β itself only decays like $r^{-7/2}$ (see (1.1.3)), its derivative in the e_3 -direction decays like r^{-4} as a consequence of the differential Bianchi identities. Schematically, an analysis of Einstein's equations on $\mathcal{I}^+$

⁶As explained in the introductory chapter 0 of this thesis, the statement below is, in fact, not correct. We will come back to it in chapter 5.

moreover reveals that, assuming (1.1.7),

$$\lim_{\mathcal{C}_u^+, r \rightarrow \infty} \partial_u(r^4\beta) = \frac{\mathcal{D}^{(3)}\Xi^-}{|u|} + \dots, \quad (1.1.8)$$

where $\mathcal{D}^{(3)}$ is a third-order differential operator on $\mathbb{S}^2$. The most difficult part of the argument then consists of obtaining a similar estimate for $\partial_u(r^4\beta)$ away from null infinity. Once this is achieved, one can integrate $\partial_u(r^4\beta)$ from initial data ($t = 0$) to obtain schematically (see Figure 1.1 below):

$$(r^4\beta)(u_2, t) - (r^4\beta)(u_1(t), 0) \sim \int_{u_1(t)}^{u_2} \frac{\mathcal{D}^{(3)}\Xi^-}{|u|} du \sim (\log r_{t,u_2} - \log |u_2|) \cdot \mathcal{D}^{(3)}\Xi^-. \quad (1.1.9)$$

Here, r_{t,u_2} denotes the area radius of S_{t,u_2} , and we used that $u_1(t) \sim r_{t,u_2}$.

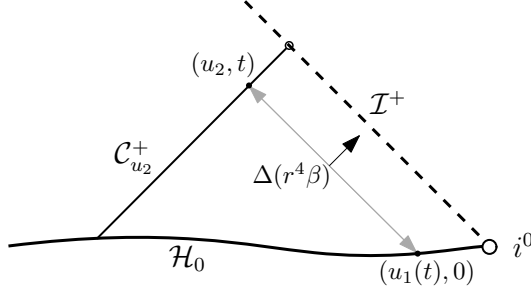


Figure 1.1 Schematic depiction of Christodoulou's argument. Integrating $\partial_u(r^4\beta) \sim |u|^{-1}$ from initial data gives rise to logarithmic terms.

Finally, Christodoulou argues that $r^4\beta$ remains finite on $t = 0$ as a consequence of the *no incoming radiation condition*, which is the statement that the Bondi mass remains constant along past null infinity.

He thus concludes that the peeling property is violated by β , and that one instead has that

$$\beta = B^* r^{-4} (\log r - \log |u|) + \mathcal{O}(r^{-4}) \quad (1.1.10)$$

for a 1-form B^* which encodes physical information about the quadrupole distribution of the infalling matter and which is independent of u .

Similarly, he shows that $\alpha = \mathcal{O}(r^{-4})$, in contrast to the r^{-5} -rate predicted by peeling.

Now, rather than *imposing* (1.1.7), it would of course be desirable to *dynamically derive* the rate (1.1.7) (and thus the failure of peeling) from a suitable scattering setup resembling that of N infalling masses.

In fact, this is exactly what we present in section 1.2.1, albeit for a simpler model. In this context, we will also be able to motivate the following simpler conjectures (cf. Thms. 1.2.4 and 1.2.5):

Conjecture 1.1.1. *Consider the scattering problem for the Einstein vacuum equations with conformally regular data on an ingoing null hypersurface and no incoming radiation from past null infinity. Then, generically, the future development fails to be conformally smooth near $\mathcal{I}^+$.*

Conjecture 1.1.2. *Consider the scattering problem for the Einstein vacuum equations with compactly supported data on $\mathcal{I}^-$ and a Minkowskian i^- . Then, generically, the future development fails to be conformally smooth near $\mathcal{I}^+$.*

To clarify, we do not explicitly conjecture that the leading-order peeling behaviour (1.1.2) is violated, but that there will be logarithmic terms in the expansion of e.g. β or α at some finite, potentially higher order.

Before we move on to the next section, we feel that it may be helpful to comment on the work [KN03a]. There, it is shown that if one works with faster decaying $r^{\frac{1}{2}+\epsilon}$ -weighted C–K data (which have finite $\|r^{\frac{1}{2}+\epsilon} \cdot \|_{\text{CK-norm}}$), then peeling holds for β if $\epsilon > 0$, and also for α if $\epsilon > 1$. So how is this consistent with the above result? Well, one of the results of [KN03a] implies that $r^{\frac{1}{2}+\epsilon}$ -weighted C–K data lead to solutions which have $|\Xi| \leq |u|^{-2-\epsilon}$, hence the data considered in [KN03a] are incompatible with eq. (1.1.7) or, in other words, with the quadrupole approximation of N infalling masses. The same applies to [CD02, Cor07].

1.2 Overview of the main results (Thms. 1.2.1–1.2.5) and of upcoming work

1.2.1 Construction of counter-examples to smooth null infinity within the Einstein-Scalar field system in spherical symmetry

While the argument [Chr02] presented above already forms a serious obstruction to peeling, one would ultimately – in order to develop a fully general relativistic understanding of the non-smoothness of null infinity – like to actually construct solutions to Einstein’s equations that resemble the setup of N infalling masses from past infinity (and which lead to (1.1.7) *dynamically*). That is to say, one would like to understand the semi-global evolution of a configuration of N masses at past timelike infinity with no incoming radiation from $\mathcal{I}^-$. More concretely, one would like to understand the asymptotics of such solutions in a neighbourhood of i^0 containing a piece of $\mathcal{I}^+$.

Of course, the resolution of this problem seems to be quite difficult.

We will therefore, in this chapter, take only a first step towards the resolution of said problem by explicitly constructing a fully general relativistic example system that is based on a simple realisation of infalling masses from past timelike infinity and the no incoming radiation condition; namely, we consider the Einstein-Scalar field equations for a chargeless and massless

scalar field under the assumption of spherical symmetry:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 2T_{\mu\nu} = 2T_{\mu\nu}^{sf}, \quad (1.2.1)$$

with the matter content⁷ given by

$$T_{\mu\nu}^{sf} = \phi_{;\mu}\phi_{;\nu} - \frac{1}{2}g_{\mu\nu}\phi^{;\xi}\phi_{;\xi}. \quad (1.2.2)$$

Here, ϕ denotes the scalar field, $R_{\mu\nu}$ the Ricci tensor, R the scalar curvature of the metric $g_{\mu\nu}$, and “ $;$ ” denotes covariant differentiation.

The assumption of spherical symmetry essentially allows us to write the unknown metric in double null coordinates (u, v) as

$$g = -\Omega^2 du dv + r^2 \gamma, \quad (1.2.3)$$

where γ is the standard metric on the unit sphere $\mathbb{S}^2$, and where Ω and r (the area radius function) are functions depending only on u and v . The spherically symmetric Einstein-Scalar field system thus reduces to a system of hyperbolic partial differential equations for the unknowns Ω , r and ϕ in two dimensions. In practice, it is often convenient to replace Ω in this system with the *Hawking mass* m , which is defined in terms of Ω and r .

We construct for this system data resembling the assumptions of Christodoulou’s argument that lead to a non-smooth future null infinity in the following way:

On past null infinity, to resemble the no incoming radiation condition (for more details on the interpretation of this, see Remark 1.4.1), we set

$$\partial_v(r\phi)|_{\mathcal{I}^-} = 0, \quad (1.2.4)$$

where v is advanced time, see Figure 1.2 below. Note that, in spherical symmetry, it is not possible to have N infalling masses for $N > 1$. We thus have to restrict to a single infalling mass. In particular, there can be no non-vanishing quadrupole moment. To still have some version of “infalling masses” that emit (scalar) radiation, we therefore impose decaying boundary data on a smooth timelike hypersurface⁸ Γ (to be thought of as the surface of a single star) s.t.

$$r\phi|_{\Gamma} = \frac{C}{|t|^{p-1}} + \mathcal{O}\left(\frac{1}{|t|^{p-1+\epsilon}}\right), \quad T(r\phi|_{\Gamma}) = \frac{(p-1)C}{|t|^p} + \mathcal{O}\left(\frac{1}{|t|^{p+\epsilon}}\right), \quad (1.2.5)$$

⁷We can also include a Maxwell field that is coupled to the geometry (and not to the scalar field) in the equations.

⁸The precise conditions on Γ are fairly general and, in particular, admit cases where $r|_{\Gamma}$ tends to a finite or infinite limit. For the derivation of upper bounds, we only require $r|_{\Gamma} > 2M$, where M is defined in (1.2.6). For the derivation of lower bounds, we require the slightly stronger assumption $r|_{\Gamma} > 2.95M$. We expect that this bound can be improved.

where $C \neq 0$ and $p > 1$ are constants, $\mathbf{T}$ is the normalised future-directed vector field generating Γ , and t is its corresponding parameter ($\mathbf{T}(t) = 1$), tending to $-\infty$ as i^- is approached. It will turn out that, in the case $p = 2$, this condition implies the precise analogue of eq. (1.1.7), i.e. the prediction of the quadrupole approximation (see also the Remark 1.4.4). This motivates the case $p = 2$ to be the most interesting one.

Finally, we need the “infalling mass” to be non-vanishing; we thus set the Hawking mass m to be positive initially, i.e. at i^- :

$$m(i^-) = M > 0. \quad (1.2.6)$$

Remark 1.2.1. Note already that conditions (1.2.4) and (1.2.6) are to be understood in a certain limiting sense; indeed, we will construct solutions where $\mathcal{I}^-$ is replaced by an outgoing null hypersurface $\mathcal{C}_{u_n}^+$ at finite retarded time u_n and then show that the solutions to these mixed characteristic-boundary value problems converge to a unique limiting solution as $u_n \rightarrow -\infty$, that is, as $\mathcal{C}_{u_n}^+$ “approaches” $\mathcal{I}^-$. We will then show that the solution constructed in this way is the unique solution to our problem, cf. Remark 1.2.2.

To more clearly state the following rough versions of our results, we remark that, throughout most parts of this work, we work in a globally regular double null coordinate system (u, v) (see Figure 1.2 below) in which $\mathcal{I}^+$ can be identified with $v = \infty$, $\mathcal{I}^-$ can be identified with $u = -\infty$, and which satisfies $u = v$ on Γ and $\partial_v r = 1$ along $\mathcal{I}^-$ (in a limiting sense).

We then have the following theorem (see Thms. 1.5.6 and 1.5.7 for the precise statement):

Theorem 1.2.1. *For sufficiently regular initial/boundary data on $\mathcal{I}^-$ and Γ as above, i.e. obeying eqns. (1.2.4), (1.2.5), (1.2.6), a unique semi-global solution to the spherically symmetric Einstein-Scalar field system exists for sufficiently large negative values of u . Moreover, if $p = 2$, we get the following asymptotic behaviour for the outgoing derivative of the radiation field:⁹*

$$|\partial_v(r\phi)| \sim \begin{cases} \frac{\log r}{r^3}, & u = \text{constant}, v \rightarrow \infty, \\ \frac{1}{r^3}, & v = \text{constant}, u \rightarrow -\infty, \\ \frac{1}{r^3}, & v + u = \text{constant}, v \rightarrow \infty. \end{cases} \quad (1.2.7)$$

More precisely, for fixed values of u , we obtain the following asymptotic expansion as $\mathcal{I}^+$ is approached:

$$\partial_v(r\phi)(u, v) = B^* \frac{\log r - \log |u|}{r^3} + \mathcal{O}(r^{-3}). \quad (1.2.8)$$

Here, $B^* \neq 0$ is a constant independent of u given by

$$B^* = -2M \lim_{u \rightarrow -\infty} |u| r \phi(u, v), \quad (1.2.9)$$

⁹Here, and in the remainder of the chapter, we write $f \sim g$ if there exist positive constants A, B s.t. $Af \leq g \leq Bf$. Similarly, we write $f = \mathcal{O}(g)$ if there exists a constant A s.t. $|f| \leq Ag$.

and the limit above exists and is independent of v .

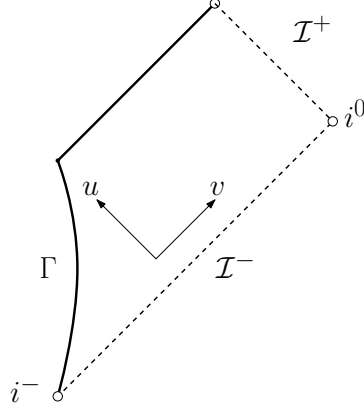


Figure 1.2 The Penrose diagram of the solution of Theorem 1.2.1. We impose polynomially decaying data on a timelike boundary Γ and no incoming radiation from past null infinity $\mathcal{I}^-$. Note that, with our choice of coordinates ($u = v$ on Γ), Γ becomes a straight line.

Remark 1.2.2. The uniqueness in the above statement is with respect to the class of solutions with uniformly bounded Hawking mass. See also Remark 1.5.7.

Theorem 1.2.1 shows that the asymptotic expansion of $\partial_v(r\phi)$ near $\mathcal{I}^+$, which should be thought of as the analogue to β for the wave equation, contains logarithmic terms and, thus, fails to be regular in the conformal picture (i.e. in the variable $1/r$), whereas the expansion near $\mathcal{I}^-$ remains regular.¹⁰

One can moreover show that, for general integer $p > 2$, one instead gets the following expansion for fixed values of u :

$$\partial_v(r\phi)(u, v) = B(u)\frac{1}{r^3} + \dots + B'\frac{\log r}{r^{p+1}} + \mathcal{O}(r^{-p-1}), \quad (1.2.10)$$

where the $\dots$ -terms denote negative integer powers of r , and where $B' \neq 0$ is a constant determined by M and $\lim_{u \rightarrow -\infty} |u|^{p-1} r\phi$, the latter limit again being independent of v .

We can also state the precise analogue of the argument [Chr02] presented in section 1.1.2 for the Einstein-Scalar field system (see Remark 1.4.3):

Theorem 1.2.2. *Suppose a semi-global solution to the spherically symmetric Einstein-Scalar field system with Hawking mass $m \geq c > 0$ for some constant c and $m(\mathcal{I}^-) \equiv M > 0$ and obeying the no incoming radiation condition exists such that, on $\mathcal{I}^+$, $r\phi = \Phi^-|u|^{-1} + \mathcal{O}(|u|^{-1-\epsilon})$. Then, for fixed values of u , we obtain the following asymptotic expansion of $\partial_v(r\phi)$ as $\mathcal{I}^+$ is approached:*

$$\partial_v(r\phi)(u, v) = B^* \frac{\log r - \log |u|}{r^3} + \mathcal{O}(r^{-3}), \quad (1.2.11)$$

where B^* is a constant independent of u given again by $-2M\Phi^-$.

¹⁰Since various ideas regarding the relation between the conformal regularity of $\mathcal{I}^-$ and that of $\mathcal{I}^+$ have been entertained in the literature, we want to point out that, in our setting, the smoothness or non-smoothness of $\mathcal{I}^-$ is completely inconsequential to the smoothness of $\mathcal{I}^+$. See also footnote 12.

Indeed, the main work of this chapter consists of showing that both the lower and upper bounds on the u -decay of $r\phi$ imposed on Γ are propagated all the way up to $\mathcal{I}^+$.¹¹ The limit Φ^- then plays a similar role to Ξ^- from (1.1.7), see already Remark 1.4.4.

We remark that, even though the above theorems are proved for the coupled problem, the methods of the proofs can also be specialised to the linearised problem (see section 1.3.3 of the present chapter or section 11 of [DR05]), i.e. the problem of the wave equation on a fixed Schwarzschild (or Reissner–Nordström) background:

Theorem 1.2.3. *Consider the spherically symmetric wave equation*

$$\nabla^\mu \nabla_\mu \phi = 0 \quad (1.2.12)$$

on a fixed Schwarzschild background with mass $M \neq 0$, where ∇ is the connection induced by the Schwarzschild metric

$$g^{Schw} = - \left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2, \quad (1.2.13)$$

and consider sufficiently regular initial/boundary data as above, i.e. obeying eqns. (1.2.4) and (1.2.5). Then the results of Theorems 1.2.1, 1.2.2 apply.

Notice that the same result does not hold on Minkowski, as we need the spacetime to possess some mass near spatial infinity.

Let us now explain, both despite and due to its simplicity, the main cause for the logarithmic term (focusing now on $p = 2$): The wave equation (derived from $\nabla^\mu T_{\mu\nu}^{sf} = \nabla^\mu R_{\mu\nu} = 0$) then reads

$$\partial_u \partial_v (r\phi) = -2m \frac{(-\partial_u r) \partial_v r}{1 - \frac{2m}{r}} \frac{r\phi}{r^3}. \quad (1.2.14)$$

Assuming that we can propagate upper and lower bounds for $r\phi$ from Γ to null infinity, we have that $r\phi \sim |u|^{-1}$ everywhere. For sufficiently large r , and for sufficiently large negative values of u , we then have that $r(u, v) \sim (v - u)$ and that all other terms appearing in front of the $\frac{r\phi}{r^3}$ -term remain bounded from above, and away from zero, such that integrating (1.2.14) from $\mathcal{I}^-$ gives (we decompose into fractions)

$$\begin{aligned} \partial_v (r\phi)(u, v) &\sim - \int_{-\infty}^u \frac{1}{r(u', v)^3 |u'|} du' \sim \int_{-\infty}^u \frac{1}{(v - u')^3 u'} du' \\ &= \int_{-\infty}^u \frac{1}{v^3} \left(\frac{1}{u'} + \frac{1}{v - u'} + \frac{v}{(v - u')^2} + \frac{v^2}{(v - u')^3} \right) du' = \frac{\log |u| - \log(v - u)}{v^3} + \frac{3v - 2u}{2v^2(v - u)^2}. \end{aligned} \quad (1.2.15)$$

¹¹The propagation of u -decay is somewhat special to spherical symmetry, see also section 1.2.4.1 and chapter 3.

Taking the limit of $v \rightarrow \infty$ while fixing u then, already, suggests the logarithmic term in the asymptotic expansions of Thms. 1.2.1 and 1.2.2. Of course, the calculation above is only a sketch, and many details have been left out.¹²

Let us remark that posing polynomially decaying boundary data on a timelike hypersurface comes with various technical difficulties. For instance, one cannot a priori prescribe the Hawking mass on Γ – in fact, even showing local existence will come with some difficulties – and r -weights cannot be used to infer decay when integrating in the outgoing direction from Γ since r is, in general, allowed to remain bounded on Γ . Both of these difficulties disappear in the characteristic initial value problem, i.e., when one prescribes initial data on an ingoing null hypersurface $\mathcal{C}_{\text{in}}$ terminating at past null infinity (see Figure 1.3) according to

$$r\phi|_{\mathcal{C}_{\text{in}}} = \frac{\Phi^-}{r^{p-1}} + \mathcal{O}\left(\frac{1}{r^{p-1+\epsilon}}\right), \quad \left.\frac{\partial_u(r\phi)}{\partial_u r}\right|_{\mathcal{C}_{\text{in}}} = \frac{(p-1)\Phi^-}{r^p} + \mathcal{O}\left(\frac{1}{r^{p+\epsilon}}\right), \quad (1.2.16)$$

where Φ^- and $p > 1$ are constants, one again sets $\partial_v(r\phi)$ to vanish on past null infinity, and makes the obvious modification to condition (1.2.6):

$$m(\mathcal{C}_{\text{in}} \cap \mathcal{I}^-) = M > 0. \quad (1.2.17)$$

We then obtain the following theorem (see Thm. 1.4.2 for the precise statement):

Theorem 1.2.4. *For sufficiently regular characteristic initial data on $\mathcal{I}^-$ and $\mathcal{C}_{\text{in}}$ as above, i.e. obeying eqns. (1.2.4), (1.2.16), (1.2.17), a unique semi-global solution to the Einstein-Scalar field system in spherical symmetry exists for sufficiently large negative values of u . Moreover, in the case $p = 2$, we obtain the following asymptotic expansion of $\partial_v(r\phi)$ as $\mathcal{I}^+$ is approached along hypersurfaces of constant u :*

$$\partial_v(r\phi)(u, v) = B^* \frac{\log r - \log |u|}{r^3} + \mathcal{O}(r^{-3}), \quad (1.2.18)$$

where B^* is a constant independent of u given by $B^* = -2M\Phi^-$. On the other hand, the expansion near $\mathcal{I}^-$ remains regular, i.e. $\partial_v(r\phi) = \mathcal{O}(r^{-3})$ near $\mathcal{I}^-$.

As before, the same holds true for the linear case, cf. Thm. 1.2.3.

It is this result which motivates Conjecture 1.1.1 from section 1.1.2.

Since the characteristic setup above is much simpler to deal with compared to the case of boundary data on Γ , we shall prove Thm. 1.2.4 first such that the technically more involved timelike case can be understood more easily afterwards. Moreover, it turns out that this setting allows for another interesting motivation or interpretation of our choice of polynomially

¹²To relate to footnote 10, note that one can do a similar calculation if $r\phi \sim |u|^{-\epsilon}$ for some $\epsilon > 0$. In this case, there will be an $r^{-2-\epsilon}$ -term in the asymptotic expansion of $\partial_v(r\phi)$, unless $\epsilon \in \mathbb{Z}$. In other words, one cannot recover conformal regularity near $\mathcal{I}^+$ from a lack of conformal regularity near $\mathcal{I}^-$.

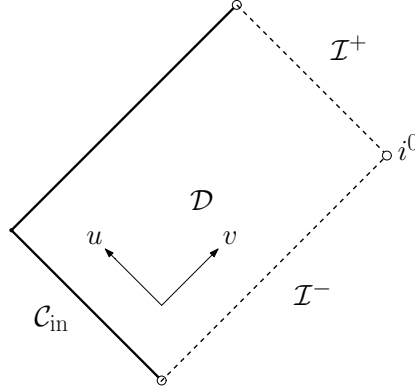


Figure 1.3 The Penrose diagram of the solution of Theorem 1.2.4. We impose polynomially decaying data on an ingoing null hypersurface $\mathcal{C}_{\text{in}}$ and no incoming radiation from past null infinity $\mathcal{I}^-$.

decaying initial data, namely in the context of the *scattering problem* of scalar perturbations of Minkowski or Schwarzschild. We will discuss this in the next section (section 1.2.2).

On the other hand, the problem of timelike boundary data is interesting precisely because of its difficulties and the methods used to deal with them. Indeed, we develop a quite complete understanding of the evolutions of such data in Thm. 1.5.6. Let us point out again that we are not able to work directly with such data, but rather need to consider a sequence of smooth compactly supported data that lead to solutions which can be extended to the past by the vacuum solution. We will show uniform bounds and sharp decay rates for this sequence of solutions. We will then show that these bounds carry over to the limiting solution, which then restricts correctly to the (non-compactly supported) initial boundary data. A major obstacle in obtaining the necessary bounds will be proving decay for $\partial_u(r\phi)$, for which we will need to commute with the timelike generators of Γ . The limiting argument itself proceeds via a careful Grönwall-type argument on the differences of two solutions, thus establishing that the sequence is Cauchy. This method is then also used to infer the uniqueness of the limiting solution. Notice that the logarithmic term of (1.2.8) only appears in the limiting solution, whereas the actual sequence of solutions satisfies peeling. This can be understood already from the heuristic computation (1.2.15).

We refer the reader to the introduction of section 1.5 as well as Theorems 1.5.6 and 1.5.7 (which together contain Thm. 1.2.1) for details.

1.2.2 An application: The scattering problem

1.2.2.1 The scattering problem on Minkowski, Schwarzschild and Reissner–Nordström

In the setting of data on an ingoing null hypersurface, the case $p = 3$ is of independent interest in view of its natural appearance in the scattering problem “on” Minkowski or Schwarzschild (or Reissner–Nordström). If one puts compactly supported data for the scalar field $r\phi = G(v)$ on $\mathcal{I}^-$ and¹³ on the past event horizon $\mathcal{H}^-$, it is not difficult to see that there exists an ingoing null

¹³Note that, since all our results only apply in a region sufficiently close to $\mathcal{I}^-$, it does not make a difference whether we consider compactly supported or vanishing data on $\mathcal{H}^-$.

hypersurface $\mathcal{C}_{\text{in}}$, “intersecting” $\mathcal{I}^-$ to the future of the support of $r\phi|_{\mathcal{I}^-}$, on which eq. (1.2.16) generically holds with $p = 3$ and such that eq. (1.2.17) holds on $\mathcal{C}_{\text{in}} \cap \mathcal{I}^-$. See Figure 1.4. This puts us in the situation of Thm. 1.2.4.

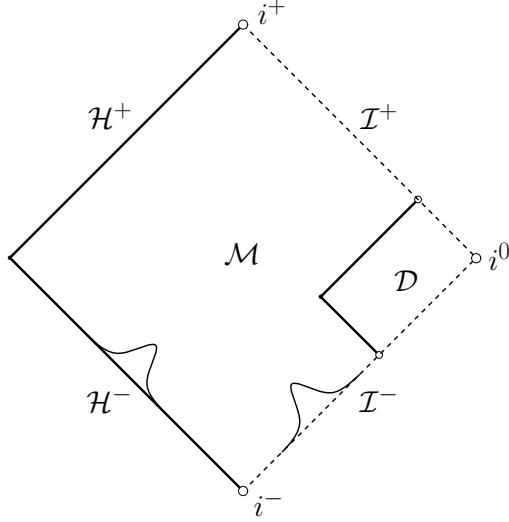


Figure 1.4 The Penrose diagram of Schwarzschild. By Theorem 1.2.5 c), smooth compactly supported scattering data on $\mathcal{H}^-$ and $\mathcal{I}^-$ generically lead to the setup of Theorem 1.2.4 with $p = 3$. The region $\mathcal{D}$ as depicted corresponds to Figure 1.3. As a consequence, the solution fails to be conformally regular on $\mathcal{I}^+$.

However, recall that we required M from eq. (1.2.17) to be strictly positive in order for the log-terms in $\partial_v(r\phi)$ to be non-vanishing: Therefore, while M is positive in both the coupled and the linear problem on Schwarzschild, *one needs to consider the coupled problem* when considering the corresponding problem with a Minkowskian i^- since one needs the scalar field to generate mass along $\mathcal{I}^-$.¹⁴

Remark 1.2.3. Let us quickly explain our terminology: Since we only consider *compactly supported* scattering data, the arising solutions will be identically vacuum in a neighbourhood of i^- . Depending on the setting, we then say that the arising spacetimes either have a *Minkowskian* or a *Schwarzschildian* (with mass $M > 0$) i^- .

We therefore obtain the following result (see Thm. 1.6.1 for the precise version):

Theorem 1.2.5. *Consider either*

a) the non-linear scattering problem for the spherically symmetric Einstein-Scalar field system with a Schwarzschildian i^- (with mass $M > 0$), with vanishing data on $\mathcal{H}^-$ and with smooth compactly supported data $r\phi = G(v)$ on $\mathcal{I}^-$,

or

b) the non-linear scattering problem for the spherically symmetric Einstein-Scalar field system with a Minkowskian i^- , with smooth compactly supported data $r\phi = G(v)$ on $\mathcal{I}^-$,

or

¹⁴In general, the spherically symmetric linear wave equation on a Minkowski background is not very exciting in view of the exact conservation law $\partial_u \partial_v(r\phi) = 0$. Combined with the no incoming radiation condition, this conservation law would force $\partial_v(r\phi)$ to vanish everywhere.

c) the linear scattering problem for the wave equation on a fixed Schwarzschild background with mass $M > 0$, with vanishing data on $\mathcal{H}^-$ and with smooth compactly supported data $r\phi = G(v)$ on $\mathcal{I}^-$.

Then, a unique smooth semi-global solution exists (in fact, in case *c*), this smooth solution exists globally in the exterior of Schwarzschild, and we get, along hypersurfaces of constant u , for sufficiently large negative values of u , the following asymptotic expansion near $\mathcal{I}^+$:

$$\partial_v(r\phi)(u, v) = B(u)\frac{1}{r^3} + B'\frac{\log r - \log |u|}{r^4} + \mathcal{O}(r^{-4}), \quad (1.2.19)$$

where B' is a constant which, in each case, can be explicitly computed from $G(v)$ and is generically non-zero.

Remark 1.2.4. We note that, in the case *b*), if an additional smallness assumption on the data on $\mathcal{I}^-$ is made, then, in fact, the solution is causally geodesically complete, globally regular, and has a complete $\mathcal{I}^+$. This follows from [Chr93]. See also Theorem 1.7 of [LOY18] and the dichotomy of [LO15].

Theorem 1.2.5 suggests that, in the context of the scattering problem, one should generically expect logarithmic terms to appear at the latest at second order in the asymptotic expansions of $\partial_v(r\phi)$ near $\mathcal{I}^+$. This is precisely what motivates our statement of Conjecture 1.1.2 in section 1.1.2.

In the case of the linear wave equation on Schwarzschild, we can make an even stronger statement: There, the condition that G needs to satisfy so that no logarithmic terms appear in the expansion of $\partial_v(r\phi)$ up to order $r^{-(4+n)}$ is that

$$\int G(v)v^m dv = 0 \quad (1.2.20)$$

for all $m \leq n$. In particular, *all* non-trivial smooth compactly supported scattering data on Schwarzschild lead to expansions of $\partial_v(r\phi)$ which eventually fail to be conformally smooth. See already Theorem 1.6.2. We also refer the reader to [DRS18] for a general treatment of the scattering problem on Kerr.

1.2.2.2 The conformal isometry on extremal Reissner–Nordström

Our results can also be applied to the linear wave equation on extremal Reissner–Nordström.¹⁵ In this setting, let us finally draw the reader’s attention to the well-known conformal “mirror” isometry [CT84] on extremal Reissner–Nordström, which implies that all results on the radiation

¹⁵More generally, recall that our results also apply to the coupled Einstein–Maxwell–Scalar field system with a chargeless and massless scalar field and can then – *a fortiori* – be specialised to the linear setting (i.e. to the linear wave equation on Reissner–Nordström) as in Theorem 1.2.3.

field are essentially invariant under interchange of

$$u \longleftrightarrow v, \quad \frac{1}{r} \longleftrightarrow (r - r_+),$$

where r_+ is the value of r at the event horizon. To make this more precise, we recall from [LMRT13] (see also [BF13]) that if ϕ is a solution to the linear wave equation in *outgoing* Eddington–Finkelstein coordinates (u, r) , then, in *ingoing* Eddington–Finkelstein coordinates (v, r') ,

$$\tilde{\phi}(v, r') := \frac{r_+}{r' - r_+} \phi \left(u = v, r = \frac{r_+ r'}{r' - r_+} \right) = \frac{r - r_+}{r_+} \phi \left(u = v, r = \frac{r_+ r'}{r' - r_+} \right) \quad (1.2.21)$$

also is a solution to the wave equation, where, in the above definition, the LHS is evaluated in ingoing and the RHS in outgoing null coordinates. One can directly read off from this that regularity in r' of $\tilde{\phi}$ near the future event horizon $\mathcal{H}^+$ is equivalent to regularity of $r\phi$ in the conformal variable $1/r$ near $\mathcal{I}^+$. In other words, applying this conformal isometry to Theorems 1.2.1–1.2.5, which made statements on the conformal regularity of $r\phi$ near $\mathcal{I}^+$, now produces statements on the *physical* regularity of $\tilde{\phi}$ near the event horizon.

For instance, the mirrored version of Theorem 1.2.5 shows that smooth compactly supported scattering data on $\mathcal{H}^-$ and on $\mathcal{I}^-$ for the linear wave equation on extremal Reissner–Nordström generically lead to solutions ϕ which not only fail to be conformally smooth near $\mathcal{I}^+$, but also fail to be in C^4 near $\mathcal{H}^+$. (See also [AAG20a] for a general scattering theory on extremal Reissner–Nordström.) This is in stark contrast to the scattering problem on Schwarzschild, where, under the same setup, the solution remains smooth up to and including the future event horizon. One can relate this to the absence of a bifurcation sphere in extremal Reissner–Nordström (see also [LK14]). Indeed, if one, instead of posing data on all of $\mathcal{H}^-$, poses compactly supported data on a null hypersurface which coincides with $\mathcal{H}^-$ up to some finite time and which, for sufficiently large u , becomes a timelike boundary intersecting $\mathcal{H}^+$ at some finite v , then the corresponding solution remains smooth.

We will not explore potential implications of this on Strong Cosmic Censorship in this chapter (see however also [Gaj17], where the importance of logarithmic asymptotics for extendibility properties near the inner Cauchy horizon of extremal Reissner–Nordström is discussed).

1.2.3 Translating asymptotics near i^0 into asymptotics near i^+

All the results presented so far hold true in a neighbourhood of i^0 . In chapter 2, we answer the question how the asymptotics for $\partial_v(r\phi)$ obtained near *spacelike infinity* translate into asymptotics for ϕ near *future timelike infinity*. In that work, we restrict to the analysis of the linear wave equation on a fixed Schwarzschild background and focus on the case $p = 2$ of Theorem 1.2.1 (so $r\phi \sim |t|^{-1}$ on data). Smoothly extending the boundary data to the event

horizon, we prove in chapter 2 that the logarithmic asymptotics (1.2.8) imply that the leading-order asymptotics of ϕ on $\mathcal{H}^+$ and of $r\phi$ on $\mathcal{I}^+$ are also logarithmic and entirely determined by the constant $-2M\Phi^-$. For instance, we obtain that $r\phi|_{\mathcal{I}^+} = -2M\Phi^-u^{-2}\log u + \mathcal{O}(u^{-2})$ along $\mathcal{I}^+$ as $u \rightarrow \infty$. In particular, the leading-order asymptotics are independent of the extension of the data to (and towards) the horizon. This gives rise to a *logarithmically modified Price’s law* and, in principle, provides a tool to *directly measure the non-smoothness of $\mathcal{I}^+$* .

Chapter 2 crucially uses methods and results from [AAG18c], [AAG18b].

It would be an interesting problem to show a similar statement for the *coupled* Einstein–Scalar field system considered in the present chapter. See also the works [DR05] and [LO15] in this context.

1.2.4 Further directions

We here outline some further directions which we will pursue in the next few sections or in the future, and which build on the present work.

1.2.4.1 Going beyond spherical symmetry: Higher ℓ -modes

It is natural to ask what happens outside of spherical symmetry in the case of the linear wave equation on a fixed Schwarzschild background (as the coupled problem would be incomparably more difficult): If one decomposes the solution to the wave equation by projecting onto spherical harmonics and works in double null Eddington–Finkelstein coordinates (u, v) , one gets the following generalisation of the spherically symmetric wave equation (1.2.14):

$$\partial_u \partial_v (r\phi_\ell) = -\ell(\ell+1) \left(1 - \frac{2M}{r}\right) \frac{r\phi_\ell}{r^2} + 2M \frac{\partial_u r \partial_v r}{1 - \frac{2M}{r}} \frac{r\phi_\ell}{r^3}, \quad (1.2.22)$$

where ϕ_ℓ is the projection onto the ℓ -th spherical harmonic. The difference from the spherically symmetric case treated so far is obvious: The RHS decays slower for $\ell \neq 0$. Since the good r^{-3} -weight for $\ell = 0$ plays a crucial rule in the proofs of all theorems in the present chapter, one might think that this renders the methods of this chapter useless for higher ℓ -modes. However, one can recover the good r^{-3} -weight by commuting ℓ times with vector fields which, in Eddington–Finkelstein coordinates, to leading order all look like $r^2 \partial_v$.¹⁶ Using these commuted wave equations, one can then adapt the methods of this chapter to obtain similar results for higher ℓ -modes, with logarithms appearing in the expansions of $\partial_v(r\phi_\ell)$ at orders which depend in a more subtle way on the precise setup. We note that these commuted wave equations, which we will dub *approximate conservation laws*, are closely related to the higher-order Newman–Penrose quantities for the scalar wave equation (see also the recent [AAG21]).

¹⁶Recall that on the Minkowski background, i.e. for $M = 0$, the strong Huygens’ principle manifests itself in form of the conservation law $\partial_u(r^{-2\ell-2}(r^2 \partial_v)^{\ell+1}(r\phi_\ell)) = 0$ for the ℓ -th spherically harmonic mode.

We will dedicate chapter 3 to the discussion of higher ℓ -modes. Similarly to chapter 2, we will furthermore discuss the issue of late-time asymptotics for higher ℓ -modes in chapters 3 and 4. It will turn out that, in certain physically reasonable scenarios (such as the scattering problem of Theorem 1.2.5), the usual expectation that higher ℓ -modes decay faster towards i^+ is violated.

1.2.4.2 The wave equation on a fixed Kerr background

Similarly, it would be interesting to understand how the results obtained in this chapter would differ if one were to consider the linear wave equation on a fixed Kerr background. See also the recent [AAG23] and [Hin22], where a generalisation of the well-known Price's law is obtained for Kerr backgrounds.

1.2.4.3 Going from scalar to tensorial waves: The Teukolsky equations

Once the behaviour of higher ℓ -modes is understood in chapter 3, the natural next step towards a resolution of Conjectures 1.1.1 and 1.1.2 is the analysis of the Teukolsky equations of linearised gravity. This is the content of chapters 5 and 6.

1.2.4.4 A resolution of Conjectures 1.1.1, 1.1.2

In turn, once a detailed understanding of the Teukolsky equations is obtained, we will attempt to resolve Conjectures 1.1.1, 1.1.2 for the Einstein vacuum equations themselves. This will, in particular, require a detailed understanding of the scattering problem for the Einstein vacuum equations with a Minkowskian i^- , which we hope to obtain in the not too distant future. In the context of resolving the above conjectures, we will also give a detailed explanation and enhancement of Christodoulou's argument [Chr02], in which we hope to obtain (1.1.7) *dynamically*.

Once this program is completed, one could finally attempt to tackle the actual N -body problem in a fully general relativistic setting, i.e. one could attempt to obtain a result similar to Thm. 1.2.1. Let us however not yet speculate how this would look like. For now, we hope that it suffices to say that one of the most interesting aspects of such a problem would be a rigorous justification for the quadrupole approximation, arguably one of the most important tools in general relativity.

1.2.5 Structure of the chapter

The remainder of this chapter is structured as follows: We first reduce the spherically symmetric Einstein-Scalar field system to a system of first-order equations and set up the notation that we shall henceforth work with in section 1.3. We sketch the specialisation to the linear case, i.e. to the case of the wave equation on a fixed Schwarzschild background, in section 1.3.3. We

construct characteristic initial data as outlined above in section 1.4 and prove Theorem 1.2.4 in section 1.4.4. We deal with the problem of timelike boundary data in section 1.5 and prove Theorem 1.2.1 in section 1.5.8. Section 1.5 can, in principle, be read independently of section 1.4, though we recommend reading it after section 1.4.

The scattering results and, in particular, Theorem 1.2.5 are proved in section 1.6. This section can be read immediately after section 1.4.

Part II:

Construction of spherically symmetric counter-examples to the smoothness of $\mathcal{I}^+$

In this part of the chapter, we will construct two classes of initial data that have a non-smooth future null infinity in the sense that the outgoing derivative of the radiation field $\partial_v(r\phi)$ has an asymptotic expansion near $\mathcal{I}^+$ that contains logarithmic terms at leading order. These examples will be for the spherically symmetric Einstein-Maxwell¹⁷-Scalar field system, with no incoming radiation from $\mathcal{I}^-$ and polynomially decaying initial/boundary data on an ingoing null hypersurface or a timelike hypersurface, respectively. They are motivated by Christodoulou's argument against smooth null infinity, see the introductory remarks in section 1.2.1.

This part of the chapter is structured as follows:

We first reduce the spherically symmetric Einstein-Maxwell-Scalar field system to a system of first-order equations in section 1.3.

We then construct counter-examples to the smoothness of null infinity that have polynomially decaying data on an ingoing null hypersurface in section 1.4.

In section 1.5, we construct counter-examples with polynomially decaying data on a general timelike hypersurface (e.g. on a hypersurface of constant area radius). This latter case will be strictly more difficult than the former, so we advise the reader to first understand the former. Nevertheless, each of the sections can be understood independently of the respective other one.

Our constructions will be fully general relativistic, however, we remark that the non-smoothness of null infinity can already be observed in the linear setting, which we present in sections 1.3.2 and 1.3.3.

We finally discuss implications of our results on the scattering problem on Schwarzschild; in particular, we find that it is essentially impossible for solutions to remain conformally smooth near $\mathcal{I}^+$ if they come from compactly supported scattering data. This is discussed in section 1.6. The reader can skip to this section immediately after having read section 1.4.

More detailed overviews will be given at the beginning of each section.

¹⁷We emphasise that the presence of a Maxwell field is not important to most results, we mainly include it in view of the remarks on the scattering problem on extremal Reissner–Nordström made in the introduction.

1.3 The Einstein-Maxwell-Scalar field equations in spherical symmetry

In this section, we introduce the systems of equations that are considered in this chapter. We write down the spherically symmetric Einstein-Maxwell-Scalar equations in double null coordinates and transform them into a particularly convenient system of first-order equations in section 1.3.1. We then briefly introduce the Reissner–Nordström family of solutions and discuss the linear setting in sections 1.3.2 and 1.3.3.

1.3.1 The coupled case

Throughout this section, we will use the convention that upper case Latin letters denote coordinates on the sphere, whereas lower case Latin letters denote “downstairs”-coordinates. For general spacetime coordinates, we will use Greek letters.

In any double null coordinate system (u, v) , the Einstein equations

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 2T_{\mu\nu} \quad (1.3.1)$$

in spherical symmetry (see [Daf03] and section 3 of [Chr95] for details on the notion of spherical symmetry in this context) can be re-expressed into the following system of equations for the metric

$$g = -\Omega^2 du dv + r^2 \gamma, \quad (1.3.2)$$

where γ is the metric on the unit sphere $\mathbb{S}^2$, r is the area radius function, Ω is a positive function, and where we assume that r, Ω are C^2 :

$$\partial_u \partial_v r = -\frac{\Omega^2}{4r} \left(1 + 4 \frac{\partial_v r \partial_u r}{\Omega^2} \right) + r T_{uv}, \quad (1.3.3)$$

$$\partial_u \partial_v \log \Omega = \frac{\Omega^2}{4r^2} \left(1 + 4 \frac{\partial_v r \partial_u r}{\Omega^2} \right) - T_{uv} - \frac{\Omega^2}{4} g^{AB} T_{AB}, \quad (1.3.4)$$

$$\partial_u (\Omega^{-2} \partial_u r) = -r \Omega^{-2} T_{uu}, \quad (1.3.5)$$

$$\partial_v (\Omega^{-2} \partial_v r) = -r \Omega^{-2} T_{vv}. \quad (1.3.6)$$

The matter system considered in this chapter is represented by the sum of the following two energy momentum tensors:

$$T_{\mu\nu}^{sf} = \phi_{;\mu} \phi_{;\nu} - \frac{1}{2} g_{\mu\nu} \phi^{;\xi} \phi_{;\xi}, \quad (1.3.7)$$

$$T_{\mu\nu}^{em} = F_{\mu\xi} F^{\xi}_{\nu} - \frac{1}{4} g_{\mu\nu} F_{\xi o} F^{\xi o}. \quad (1.3.8)$$

These are in turn governed by the wave equation and the Maxwell equations, respectively, which can compactly be written as $\nabla^\mu T_{\mu\nu}^{sf} = 0 = \nabla^\mu F_{\mu\nu} = \nabla^{\mu*} F_{\mu\nu}$.

One can show that, in spherical symmetry (assuming no magnetic monopoles¹⁸), the electromagnetic contribution decouples and can be computed in terms of a constant e^2 (the electric charge) and r :

$$T_{ab}^{em} = -\frac{e^2}{2r^4} g_{ab}, \quad T_{AB}^{em} = \frac{e^2}{2r^4} g_{AB}. \quad (1.3.9)$$

For more details, see [Daf03]. On the other hand, for the scalar field, one computes directly

$$T_{uu}^{sf} = (\partial_u \phi)^2, \quad T_{vv}^{sf} = (\partial_v \phi)^2, \quad T_{uv}^{sf} = 0, \quad g^{AB} T_{AB}^{sf} = 4\Omega^{-2} \partial_u \phi \partial_v \phi. \quad (1.3.10)$$

In particular, equations (1.3.3), (1.3.4) now read:

$$\partial_u \partial_v r = -\frac{\Omega^2}{4r} \left(1 + 4 \frac{\partial_v r \partial_u r}{\Omega^2} \right) - \frac{\Omega^2}{4r} \frac{e^2}{r^2}, \quad (1.3.11)$$

$$\partial_u \partial_v \log \Omega = \frac{\Omega^2}{4r^2} \left(1 + 4 \frac{\partial_v r \partial_u r}{\Omega^2} \right) - \frac{e^2 \Omega^2}{2r^4} - \partial_u \phi \partial_v \phi. \quad (1.3.12)$$

Moreover, one derives the following wave equation for the scalar field from $\nabla^\mu T_{\mu\nu}^{sf} = 0$:¹⁹

$$r \partial_u \partial_v \phi + \partial_u r \partial_v \phi + \partial_v r \partial_u \phi = 0. \quad (1.3.13)$$

We can transform this second-order system into a system of first-order equations by introducing the *renormalised Hawking mass*:

$$\varpi := m + \frac{e^2}{2r} := \frac{r}{2} (1 - g'(\nabla r, \nabla r)) + \frac{e^2}{2r}, \quad (1.3.14)$$

where g' is the projected metric $g' = -\Omega^2 du dv$ and m denotes the *Hawking mass*. In the remainder of the chapter, we shall write g instead of g' . As we shall see, the renormalised Hawking mass obeys important monotonicity properties and will essentially allow us to do energy (i.e. L^2 -) estimates, which will usually form the starting point for our estimates, which will otherwise be L^1 - or L^∞ -based.

Let us now recall the notation introduced by Christodoulou:

$$\partial_u r = \nu, \quad \partial_v r = \lambda \quad (1.3.15)$$

¹⁸This is an evolutionary consistent assumption. This is to say that if we exclude magnetic monopoles on data, then they cannot arise dynamically. More mathematically, this is the statement that if $F_{AB} = 0$ on data, then $F_{AB} = 0$ everywhere.

¹⁹Note that if $T_{\mu\nu}^{em} = 0$, this would be a consequence of (1.3.1) and the Bianchi identities.

and

$$r\partial_u\phi = \zeta, \quad r\partial_v\phi = \theta. \quad (1.3.16)$$

Moreover, we write

$$\mu := \frac{2m}{r}, \quad \kappa := \frac{\lambda}{1-\mu} = -\frac{1}{4}\Omega^2\nu^{-1}, \quad (1.3.17)$$

where the last equality comes from the definition of m . It is then straightforward to derive equivalence between the system of second-order equations (1.3.5), (1.3.6), (1.3.11)–(1.3.13) and the following system of first-order equations:²⁰

$$\partial_u\varpi = \frac{1}{2}(1-\mu)\frac{\zeta^2}{\nu}, \quad (1.3.18)$$

$$\partial_v\varpi = \frac{1}{2}\frac{\theta^2}{\kappa}, \quad (1.3.19)$$

$$\partial_u\kappa = \frac{1}{r}\frac{\zeta^2}{\nu}\kappa, \quad (1.3.20)$$

$$\partial_u\theta = -\frac{\zeta\lambda}{r}, \quad (1.3.21)$$

$$\partial_v\zeta = -\frac{\theta\nu}{r}. \quad (1.3.22)$$

From these equations, one derives the following two useful wave equations for r and the radiation field $r\phi$:

$$\partial_v\nu = \partial_u\lambda = \partial_u\partial_v r = \frac{2\nu\kappa}{r^2} \left(\varpi - \frac{e^2}{r} \right), \quad (1.3.23)$$

$$\partial_u\partial_v(r\phi) = 2\nu\kappa \left(\varpi - \frac{e^2}{r} \right) \frac{r\phi}{r^3}. \quad (1.3.24)$$

In the sequel, we shall mostly work with equations (1.3.18)–(1.3.24).

1.3.2 The Reissner–Nordström/Schwarzschild family of solutions

If one sets ϕ to vanish identically in the system of equations (1.3.18)–(1.3.22), then, by (a generalisation of) Birkhoff's theorem – which essentially follows from equations (1.3.18), (1.3.19) – all *asymptotically flat* solutions²¹ belong to the well-known Reissner–Nordström family of solutions, which contains as a subfamily the Schwarzschild family (corresponding to the case where also $e^2 = 0$).

²⁰Notice already that, e.g., by controlling ϖ in the u -direction, eq. (1.3.18) gives us an L^2 -estimate for $\partial_u\phi$, assuming sufficient control over ν and $1-\mu$.

²¹In contrast to the $e^2 = 0$ -case, there exist spherically symmetric solutions to the Einstein-Maxwell equations which are not asymptotically flat.

Let us, for the moment, go back to the four-dimensional picture and restrict to the physical parameter range $M \geq 0$, $|e| \leq M$ ($|e| = M$ corresponding to the *extremal* case). Then, the exteriors of this family of spacetimes are given by the family of Lorentzian manifolds $(\mathcal{M}_{M,e}, g_{M,e})$, with

$$\mathcal{M}_{M,e} = \mathbb{R} \times (M + \sqrt{M^2 - e^2}, \infty) \times \mathbb{S}^2$$

covered by the coordinate chart $(t, r, \vartheta, \varphi)$, where $t \in \mathbb{R}$, $r \in (M + \sqrt{M^2 - e^2}, \infty)$, and where ϑ, φ are the standard coordinates on the sphere, and with $g_{M,e}$ given in these coordinates by

$$g_{M,e} = -D(r) dt^2 + \frac{1}{D(r)} dr^2 + r^2 (d\vartheta^2 + \sin^2 \vartheta d\varphi^2). \quad (1.3.25)$$

Here, $D(r)$ is given by $D(r) = 1 - \frac{2M}{r} + \frac{e^2}{r^2}$. By introducing the tortoise coordinate

$$r^*(r) := R + \int_R^r D^{-1}(r') dr' \quad (1.3.26)$$

for some $R > M + \sqrt{M^2 - e^2}$ and further introducing the (Eddington–Finkelstein) coordinates $2u = t - r^*(r)$, $2v = t + r^*(r)$, one can bring the metric into the double null form (1.3.2) with $\Omega^2 = 4D(r)$. One then has $\varpi \equiv M$ and $\lambda = -\nu = D(r)$.

1.3.3 Specialising to the linear case

We claimed in the introduction that the results that we will obtain for the coupled Einstein–Maxwell–Scalar field system can also be applied to the linear case, i.e. to the case of the wave equation (1.3.24) on a fixed Reissner–Nordström background.

In that case, the right-hand sides of eqns. (1.3.18)–(1.3.20) are replaced by zero, whereas the remaining equations remain unchanged, with $\varpi \equiv M$ a constant. This severely simplifies most proofs in the present chapter. However, there is one ingredient that seems to be lost at first sight: the energy estimates (see (1.4.16), (1.4.17))! These are, for instance, used for obtaining preliminary decay, $|\phi| \lesssim r^{-1/2}$, for the scalar field in (1.4.25). However, in the linear case, one can obtain these very estimates (1.4.16), (1.4.17) by an application of the divergence theorem to $\nabla^\mu (T_{\mu\nu}^{sf} \mathbf{K}^\nu)$ in a null rectangle, where $\mathbf{K}$ is the static Killing vector field of the Reissner–Nordström metric (given by ∂_t in (t, r) -coordinates). In fact, the divergence theorem implies that the 1-form (for details, see section 11 of [DR05])

$$\eta := \frac{1}{2}(1 - \mu) \frac{\zeta^2}{\nu} du + \frac{1}{2} \frac{\theta^2}{\kappa} dv \quad (1.3.27)$$

is closed, $d\eta = 0$, and one can thus define a 0-form ϖ' via $d\varpi' = \eta$ and by demanding that $\varpi' = M$ on past null infinity. The quantity ϖ' then obeys the exact same equations as ϖ does in the coupled case. This means that one can repeat all the estimates of the present chapter,

mutatis mutandis, in the uncoupled case. In particular, once we show Theorem 1.2.1 from Part I, Theorem 1.2.3 will follow *a fortiori*.

1.3.4 Conventions

In the remainder of the chapter, we shall typically consider functions defined on some set $\mathcal{D}$. We then write $f \sim g$ if there exist uniform positive constants A, B such that $Af \leq g \leq Bf$ on $\mathcal{D}$. Similarly, we write $f = \mathcal{O}(g)$ if there exists a uniform constant $A > 0$ such that $|f| \leq Ag$. Occasionally, we shall write that $f \sim g$ on some subset of $\mathcal{D}$. In this case, the constants A, B may also depend on the subset. Similarly for $f = \mathcal{O}(g)$.

1.4 Case 1: Initial data posed on an ingoing null hypersurface

In this section, we consider the semi-global characteristic initial value problem with polynomially decaying data on an ingoing null hypersurface and no incoming radiation from past null infinity to the future of that null hypersurface.

As the case of initial data on an ingoing null hypersurface is significantly simpler than that with boundary data on a timelike hypersurface presented in section 1.5, and since, in particular, the relevant local existence theory is well-known, we will only present *a priori estimates* in this section, i.e., we will assume that a sufficiently regular solution that restricts correctly to the initial data, and that “possesses” past and future null infinity as well as no anti-trapped or trapped surfaces, exists, and then show the relevant estimates on this assumed solution. We hope that this will allow the reader to more easily develop a tentative understanding of the main argument. The left out details of the proof of existence will then be dealt with in section 1.5.

We shall first explicitly state our assumptions in section 1.4.1. The middle part of the section will be devoted to showing that the geometric quantities $\nu, \lambda, \kappa, \varpi$ etc. remain bounded for large enough negative values of u in section 1.4.2. We then use the wave equation (1.3.24) to derive sharp decay rates for the scalar field and its derivatives in section 1.4.3. Equipped with these sharp rates, we can then upgrade all the previous estimates on ν, λ etc. to asymptotic estimates. This will finally allow us to obtain an asymptotic expansion of $\partial_v(r\phi)$ near future null infinity in section 1.4.4. This last section is thus also the section where Thm. 1.2.4 is proved (see Thm. 1.4.2).

1.4.1 Assumptions and initial data

1.4.1.1 Global *a priori* assumptions

Let $\mathbb{R}^2$ denote the standard plane, and call its double null coordinates (u, v) . Fix a constant $M > 0$, and assume that we have a rectangle (see Figure 1.5 below)

$$\mathcal{D}_U := (-\infty, U] \times [1, \infty) \subset \mathbb{R}^2 \quad (1.4.1)$$

with $U < -2M$, and denote, for $u \in (-\infty, U]$, the sets $\mathcal{C}_u := \{u\} \times [1, \infty)$ as *outgoing null rays* and, for $v \in [1, \infty)$, the sets $\mathcal{C}_v := (-\infty, U] \times \{v\}$ as *ingoing null rays*. We furthermore write $\mathcal{C}_{v=1} := \mathcal{C}_{\text{in}}$, and we *colloquially refer to* $\{-\infty\} \times [1, \infty)$ as $\mathcal{I}^-$ or *past null infinity*, to $(-\infty, U] \times \{\infty\}$ as $\mathcal{I}^+$ or *future null infinity*, and to $\{-\infty\} \times \{\infty\}$ as i^0 or *spacelike infinity*.

On this rectangle $\mathcal{D}_U$, we assume that a strictly positive C^3 -function $r(u, v)$, a non-negative C^2 -function $m(u, v)$, a C^2 -function $\phi(u, v)$ and a constant $e^2 > 0$ are defined and obey the following properties:

The function r is such that, along each of the ingoing and outgoing null rays, it tends to infinity, i.e., $\sup_{C_u} r(u, v) = \infty$ for all $u \in (-\infty, U]$, and $\sup_{C_v} r(u, v) = \infty$ for all $v \in [1, \infty)$. We moreover assume that, throughout $\mathcal{D}_U$,

$$\partial_u r = \nu < 0, \quad (1.4.2)$$

$$\partial_v r = \lambda > 0, \quad (1.4.3)$$

that $\nu = -1$ along $\mathcal{C}_{\text{in}}$, and that $r(U, 1) = r_1 = -U > 0$. We also assume that $\lim_{u \rightarrow -\infty} \lambda(u, v) = 1$ for all $v \in [1, \infty)$.

Concerning m , we assume that

$$\frac{\lambda}{1 - \mu} = \kappa > 0 \quad (1.4.4)$$

is a strictly positive quantity and that

$$\lim_{u \rightarrow -\infty} m(u, v) = M > 0 \quad (1.4.5)$$

for all $v \in [1, \infty)$.

On the function ϕ , we make the assumptions that, along $\mathcal{C}_{\text{in}}$, it obeys

$$r^p \frac{\partial_u(r\phi)}{\nu} + \frac{\Phi^-}{p-1} = \mathcal{O}(r^{-\epsilon}). \quad (1.4.6)$$

for some constants $\Phi^- \neq 0$, $p > 1$ and $\epsilon \in (0, 1)$, and that

$$\lim_{u \rightarrow -\infty} r\phi(u, v) = 0 = \lim_{u \rightarrow -\infty} \partial_v(r\phi)(u, v) \quad (1.4.7)$$

for all $v \in [1, \infty)$.

Finally, we assume that, throughout $\mathcal{D}_U$, equations (1.3.18)–(1.3.22) hold *pointwise*.

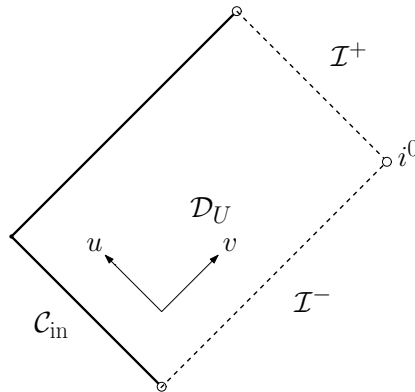


Figure 1.5 The Penrose diagram of $\mathcal{D}_U$. It contains no black or white holes and, correspondingly, no trapped or anti-trapped surfaces (cf. (1.4.2), (1.4.3)). See also [Daf05b] for an explanation of these notions.

The reader familiar with Penrose diagrams may refer to the Penrose diagram above (Figure 1.5), where the geometric content of these assumptions is summarised. The reader

unfamiliar with Penrose diagrams may either ignore this remark or refer to the appendix of [DR05] for a gentle introduction to Penrose diagrams.

1.4.1.2 Retrieving the assumptions

By essentially considering solutions to the spherically symmetric Einstein-Maxwell-Scalar field system with characteristic initial data which satisfy $\nu = -1$ as well as (1.4.6) on an ingoing null hypersurface $\mathcal{C}_{\text{in}}$, and which satisfy $\lambda = 1$, (1.4.5) and (1.4.7) on $\mathcal{I}^-$ (and by a limiting argument), we will, in section 1.5 (cf. Thm. 1.5.3), prove the following:

Proposition 1.4.1. *Given a set $\mathcal{D}_U$ as in (1.4.1), there exists a unique triplet of functions (r, ϕ, m) such that the above assumptions are satisfied, with the uniqueness being understood in the sense of Remark 1.5.7.*

The metric associated to this solution is then given by (1.3.2), with $-\Omega^2 = 4\nu\kappa$.

Remark 1.4.1. In order to see why condition (1.4.7) is the correct interpretation of the no incoming radiation condition, we recall from section 1.1.2 that the statement of no incoming radiation should be interpreted as the Bondi mass along past null infinity being a conserved quantity. In spherical symmetry, the definition of the Bondi mass as limit of the Hawking mass m (or ϖ) is straight-forward. The analogue to the Bondi mass loss formula (1.1.6) then becomes (1.3.18), or, formulated with respect to the past, (1.3.19). We thus see that the analogue to Ξ is given by $\lim_{\mathcal{I}^+} \zeta$ (or by $\lim_{\mathcal{I}^-} \theta$ in the past).

1.4.2 Coordinates and energy boundedness

Note that the following consistency calculation

$$r(u, 1) - r_1 = - \int_u^U \nu \, du' = U - u$$

confirms that, as $\mathcal{I}^-$ is approached along $v = 1$,²² r tends to infinity. Moreover, one sees from this equation that, with this choice of u -coordinate, $u = -r$ along $\mathcal{C}_{\text{in}}$.

In the remainder of the section, we will want to restrict to sufficiently large negative values of u in order to be able to make asymptotic statements. Therefore, we introduce the set

$$\mathcal{D}_{U_0} := \mathcal{D}_U \cap \{u \leq U_0\} \tag{1.4.8}$$

for some sufficiently large negative constant U_0 whose choice will only depend on $M, e^2, \Phi^-, p, \epsilon$ and the implicit constant in the RHS of (1.4.6). Our first restriction on U_0 will be the

²²In the sequel, we will show that $\nu \sim -1$ and $\lambda \sim 1$ throughout $\mathcal{D}_{U_0}$, so one can do a similar consistency calculation for any $\mathcal{C}_v$ or $\mathcal{C}_u$.

following: Since $|u| = r$ along $\mathcal{C}_{\text{in}}$, we shall from now on assume that

$$\frac{|\Phi^-|}{2(p-1)}|u|^{-p} \leq |\partial_u(r\phi)| \leq 2\frac{|\Phi^-|}{(p-1)}|u|^{-p} \quad (1.4.9)$$

along $\mathcal{C}_{\text{in}} \cap \{u \leq U_0\}$. In view of assumption (1.4.6), this indeed holds for sufficiently large values of U_0 .

In a first step, we will now prove energy boundedness along $\mathcal{C}_{\text{in}}$, i.e. bounds on m and ϖ .

Proposition 1.4.2 (Energy boundedness on $\mathcal{C}_{\text{in}} \cap \{u \leq U_0\}$). *For sufficiently large negative values of U_0 , we have along $\mathcal{C}_{\text{in}} \cap \{u \leq U_0\}$, where $\mathcal{C}_{\text{in}}$ is as described in the assumptions of section 1.4.1:*

$$\frac{M}{2} < m, \varpi < 2M. \quad (1.4.10)$$

Proof. We recall the transport equation for ϖ (1.3.18):

$$\partial_u \varpi = \frac{1}{2} \left(1 + \frac{e^2}{r^2} \right) \frac{\zeta^2}{\nu} - \frac{\varpi \zeta^2}{r \nu}.$$

Upon integrating, one sees that, along $\mathcal{C}_{\text{in}}$, the energy ϖ is given by

$$\varpi(u, 1) = M e^{-\int_{-\infty}^u \frac{\zeta^2}{\nu r} d\tilde{u}} + e^{-\int_{-\infty}^u \frac{\zeta^2}{\nu r} d\tilde{u}} \int_{-\infty}^u \frac{1}{2} \left(1 + \frac{e^2}{r^2} \right) \frac{\zeta^2}{\nu} e^{\int_{-\infty}^{u'} \frac{\zeta^2}{\nu r} d\tilde{u}} du'. \quad (1.4.11)$$

Now, observe that, on $\mathcal{C}_{\text{in}}$, we have

$$\zeta = r \partial_u \phi = \partial_u(r\phi) + \phi.$$

Moreover, by (1.4.7) and (1.4.9), we have that

$$|r\phi(u, 1)| = \left| \int_{-\infty}^u \partial_u(r\phi) du' \right| \leq \frac{2|\Phi^-|}{1-p} |u|^{-p+1}.$$

Combining the estimate above with (1.4.9) and applying the triangle inequality, we thus find

$$|\zeta| \leq C_\zeta |u|^{-p},$$

where $C_\zeta = 2|\Phi^-| + \frac{2|\Phi^-|}{p-1}$. Inserting this estimate back into (1.4.11), and using that $r(u, v) \geq |u|$ for all $v \geq 1$ as a consequence of $\lambda > 0$, we thus find

$$\begin{aligned} \varpi(u, 1) &\leq M e^{\int_{-\infty}^{U_0} C_\zeta^2 |\tilde{u}|^{-2p-1} d\tilde{u}} \\ &\quad + e^{\int_{-\infty}^{U_0} C_\zeta^2 |\tilde{u}|^{-2p-1} d\tilde{u}} \int_{-\infty}^{U_0} \frac{1}{2} \left(1 + \frac{e^2}{r^2} \right) C_\zeta^2 d\nu^{-1} |u'|^{-2p} e^{\int_{-\infty}^{u'} C_\zeta^2 |\tilde{u}|^{-2p-1} d\tilde{u}} du'. \end{aligned}$$

For sufficiently large values of U_0 , the RHS can be chosen smaller than $2M$. Similarly, one can make the second term in the RHS of (1.4.11) small enough such that the lower bound for ϖ also follows. The bounds for m then follow from (1.3.14) by again choosing U_0 sufficiently large. $\square$

Equipped with these energy bounds on $\mathcal{C}_{\text{in}}$ (to be thought of as initial data), we can now exploit the monotonicity properties of the (renormalised) Hawking mass to extend these bounds into all of $\mathcal{D}_{U_0}$:

Proposition 1.4.3 (Energy boundedness in $\mathcal{D}_{U_0}$). *For sufficiently large negative values of U_0 , we have the following bounds in all of $\mathcal{D}_{U_0}$, where $\mathcal{D}_{U_0}$, r , m and ϕ are as described in (1.4.8) and in the assumptions of section 1.4.1:*

$$\partial_u \varpi \leq 0, \quad (1.4.12)$$

$$\partial_v \varpi \geq 0. \quad (1.4.13)$$

In particular, we have

$$\frac{M}{2} < m, \varpi, \varpi - \frac{e^2}{r} \leq M. \quad (1.4.14)$$

Moreover, we have

$$0 < d_\mu := \frac{1}{2} < 1 - \mu \leq 1. \quad (1.4.15)$$

Proof. Observe that

$$\kappa = \frac{\lambda}{1 - \mu}$$

is positive by (1.4.4). We thus obtain that $1 - \mu > 0$, so (1.3.18), (1.3.19) imply $\partial_u \varpi \leq 0$, $\partial_v \varpi \geq 0$, respectively. From these monotonicity properties, we obtain the following global energy bounds for all $(u, v) \in \mathcal{D}_{U_0}$ (we recall assumption (1.4.5)):

$$\varpi(U_0, 1) \leq \varpi(u, v) \leq M,$$

so the estimate (1.4.14) for ϖ follows from (1.4.10). Boundedness of m and $\varpi - \frac{e^2}{r}$ again follows by choosing U_0 sufficiently large. To find the positive lower bound for $1 - \mu$, we simply insert the upper bound $m \leq M$ into the definition $\mu = 1 - \frac{2m}{r}$. This gives $1 - \frac{2M}{|U_0|} \leq 1 - \mu$. The bound then follows by choosing $U_0 \leq -4M$. $\square$

The energy estimates

Energy boundedness (Prop. 1.4.3) in particular implies the following two crucial energy estimates by the fundamental theorem of calculus (simply integrate eqns. (1.3.18), (1.3.19)), which hold

throughout $\mathcal{D}_{U_0}$:

$$0 \leq \int_{v_1}^{v_2} \frac{1}{2} \frac{\theta^2}{\kappa}(u, v) dv \leq \frac{M}{2}, \quad (1.4.16)$$

$$0 \leq - \int_{u_1}^{u_2} \frac{1}{2} \frac{(1-\mu)}{\nu} \zeta^2(u, v) du \leq \frac{M}{2}. \quad (1.4.17)$$

Equipped with these energy estimates, we can now control the geometric quantities ν, λ, κ in L^∞ .

Proposition 1.4.4. *For sufficiently large negative values of U_0 , there exist positive constants $d_\kappa, C_\lambda, C_\nu, d_\lambda, d_\nu$, depending only on initial data²³, such that the following inequalities hold throughout all of $\mathcal{D}_{U_0}$, where $\mathcal{D}_{U_0}$, r , m and ϕ are as described in (1.4.8) and in the assumptions of section 1.4.1:*

$$\partial_u \kappa \leq 0, \quad (1.4.18)$$

$$d_\kappa \leq \kappa \leq 1, \quad (1.4.19)$$

$$d_\lambda \leq \lambda \leq C_\lambda, \quad (1.4.20)$$

$$-d_\nu \geq \nu \geq -C_\nu. \quad (1.4.21)$$

Proof. It is clear that $\partial_u \kappa \leq 0$ (see eq. (1.3.20)). Since $\lim_{u \rightarrow -\infty} \kappa = \lim_{u \rightarrow -\infty} \lambda = 1$ by assumption, $\kappa \leq 1$ follows by monotonicity. Moreover, integrating the equation (1.3.20) for $\partial_u \kappa$ in u , we find, for $(u, v) \in \mathcal{D}_{U_0}$,

$$\begin{aligned} \kappa(u, v) &= \kappa(-\infty, v) e^{-\int_{-\infty}^u \frac{1}{r} \frac{\zeta^2}{\nu} du'} \\ &\geq e^{\frac{2}{U_0} \int_{-\infty}^{U_0} \frac{1}{2} \frac{\zeta^2}{\nu} \frac{1-\mu}{d_\mu} du'} \geq e^{-\frac{1}{U_0 d_\mu} M} \geq e^{-\frac{1}{2d_\mu}} =: d_\kappa, \end{aligned}$$

where we used $r \geq -U_0$ and $1 - \mu > d_\mu$ in the second step, and the energy estimate (1.4.17) in the last step. We now immediately get bounds on $\lambda = \kappa(1 - \mu)$: $d_\mu d_\kappa \leq \lambda \leq 1$.

To finally show boundedness for ν , we integrate eq. (1.3.23) from $\mathcal{C}_{\text{in}}$. We find, for $(u, v) \in \mathcal{D}_{U_0}$:

$$|\nu(u, v)| = |\nu(u, 1)| e^{\int_1^v \frac{2\kappa}{r^2} \left(\varpi - \frac{e^2}{r} \right) dv'}. \quad (1.4.22)$$

The bound (1.4.21) then follows in view of

$$\left| \int_1^v \frac{\kappa}{r^2} \left(\varpi - \frac{e^2}{r} \right) dv' \right| \leq \frac{1}{d_\mu} \int_1^v \frac{\lambda}{r^2} M dv' \leq \frac{1}{d_\mu} M \left(\frac{1}{r(u, 1)} - \frac{1}{r(u, v)} \right) \leq \frac{1}{d_\mu} M \cdot \frac{1}{|U_0|}.$$

²³By this, we henceforward mean that the constants depend only on Φ^- , p , M and e^2 . We also recall that the choice of U_0 only depends on the constants Φ^- , p , ϵ , M , e^2 and the implicit constant in the RHS of (1.4.9).

□

1.4.3 Sharp upper and lower bounds for $\partial_u(r\phi)$ and $r\phi$

In this section, we will use the previous results, in particular the energy estimates, to derive sharp upper and lower bounds for $r\phi$ and $\partial_u(r\phi)$.

Theorem 1.4.1. *For sufficiently large negative values of U_0 , there exist positive constants b_1, b_2, B_1, B_2 , depending only on initial data, such that the following estimates hold throughout $\mathcal{D}_{U_0}$, where $\mathcal{D}_{U_0}$, r , m and ϕ are as described in (1.4.8) and in the assumptions of section 1.4.1:*

$$b_1|u|^{-p} \leq |\partial_u(r\phi)| \leq B_1|u|^{-p} \quad (1.4.23)$$

and

$$b_2|u|^{-p+1} \leq |r\phi| \leq B_2|u|^{-p+1}. \quad (1.4.24)$$

In particular, both quantities have a sign.

Proof. We will prove this by integrating the wave equation (1.3.24) along characteristics and using the energy estimates. In a first step, we will integrate $\partial_u\phi = \zeta/r$ along an ingoing null ray starting from $\mathcal{I}^-$ and use Cauchy–Schwarz and the energy estimate to infer weak decay for the scalar field: $|\phi| \lesssim r^{-1/2}$. In a second step, we will integrate the wave equation (1.3.24) along an outgoing null ray starting from $\mathcal{C}_{\text{in}}$, using the decay obtained in step 1, to then infer bounds on $|\partial_u(r\phi)| \lesssim |u|^{-3/2}$. In a third step, we integrate $\partial_u(r\phi)$ from $\mathcal{I}^-$ to improve the decay of the radiation field: $|r\phi| \lesssim |u|^{-1/2}$. We then reiterate steps 2 and 3 until the decay matches that of the initial data on $\mathcal{C}_{\text{in}}$. (Note that one could replace this inductive procedure by a continuity argument; this will be the approach of section 1.5.)

Let now $(u, v) \in \mathcal{D}_{U_0}$. Recalling the no incoming radiation condition (1.4.7), we obtain

$$\begin{aligned} |\phi(u, v)| &= \left| \int_{-\infty}^u \frac{\zeta}{r} du' \right| \\ &\leq \left(\int_{-\infty}^u \frac{\zeta^2}{\nu} (1 - \mu) du' \right)^{\frac{1}{2}} \left(\int_{-\infty}^u \frac{-\nu}{1 - \mu} \frac{1}{r^2} du' \right)^{\frac{1}{2}} \\ &\leq \sqrt{\frac{M}{2}} \sqrt{\frac{1}{d_\mu} \frac{1}{r(u, v)}} \leq C_1 r(u, v)^{-\frac{1}{2}}, \end{aligned} \quad (1.4.25)$$

where we used the energy estimate (1.4.17) in the last estimate.

Next, by integrating the wave equation (1.3.24) from $v = 1$, we get

$$\begin{aligned} |\partial_u(r\phi)(u, v)| &\leq \frac{2|\Phi^-|}{p-1}|u|^{-p} + \left| \int_1^v 2 \frac{\nu}{1-\mu} \left(\varpi - \frac{e^2}{r} \right) \frac{\lambda\phi}{r^2} dv' \right| \\ &\leq \frac{2|\Phi^-|}{p-1}|u|^{-p} + \int_1^v 2 \frac{C_\nu}{d_\mu} M C_1 \partial_v \left(-\frac{2}{3r^{\frac{3}{2}}} \right) dv' \\ &\leq \frac{2|\Phi^-|}{p-1}|u|^{-p} + \frac{4}{3} \frac{C_\nu C_1}{d_\mu} M r(u, 1)^{-\frac{3}{2}}, \end{aligned} \quad (1.4.26)$$

where we used (1.4.9) to estimate the boundary term. But now recall that on $\mathcal{C}_{\text{in}} \cap \{u \leq U_0\}$, i.e. on $v = 1$, we have that r and $|u|$ are comparable²⁴, so we indeed get, for some constant C_2 :

$$|\partial_u(r\phi)|(u, v) \leq C_2 |u|^{-\min(\frac{3}{2}, p)}.$$

In a third step, we integrate this estimate in the u -direction:

$$|r\phi|(u, v) \leq \int_{-\infty}^u |\partial_u(r\phi)| du' \leq \max(2, (p-1)^{-1}) C_2 |u|^{-\min(\frac{1}{2}, p-1)}.$$

This is an improvement over the decay obtained from the energy estimates. We can plug it back into the second step, i.e. into (1.4.26), to get improved decay for $\partial_u(r\phi)$, from which we can then improve the decay for $r\phi$ again. The upper bounds (1.4.23), (1.4.24) then follow inductively.

Moreover, we can use the upper bound $|r\phi| \leq B_2 |u|^{-p+1}$ to infer a lower bound on $\partial_u(r\phi)$: Integrating again the wave equation (1.3.24) as in (1.4.26), and estimating the arising integral according to

$$\begin{aligned} \left| \int_1^v 2 \frac{\nu}{1-\mu} \left(\varpi - \frac{e^2}{r} \right) \frac{\lambda r\phi}{r^3} dv' \right| &\leq \int_1^v \frac{C_\nu}{d_\mu} M B_2 |u|^{-p+1} \partial_v \left(-\frac{1}{r^2} \right) dv' \\ &\leq \frac{C_\nu B_2}{d_\mu} \frac{M}{r(u, 1)^2} |u|^{-p+1} \leq \frac{|\Phi^-|}{4(p-1)} |u|^{-p}, \end{aligned}$$

where the last inequality holds true for large enough U_0 , we obtain the lower bound

$$|\partial_u(r\phi)(u, v)| \geq \frac{|\Phi^-|}{2(p-1)} |u|^{-p} - \frac{|\Phi^-|}{4(p-1)} |u|^{-p} = \frac{|\Phi^-|}{4(p-1)} |u|^{-p}. \quad (1.4.27)$$

In fact, we get the asymptotic statement that

$$\partial_u(r\phi)(u, v) - \partial_u(r\phi)(u, 1) = \mathcal{O}(|u|^{-p-1}). \quad (1.4.28)$$

The lower bound for $r\phi$ then follows by integrating the lower bound for $\partial_u(r\phi)$. $\square$

²⁴This is a property that we will not be able to exploit in the timelike case!

We have now obtained bounds over all relevant quantities. Plugging these back into the previous proofs allows for these bounds to be refined. This is done by following mostly the same steps but replacing all energy estimates with the improved pointwise bounds we now have at our disposal.

Corollary 1.4.1. *For sufficiently large values of U_0 , we have the following asymptotic estimates throughout $\mathcal{D}_{U_0}$, where $\mathcal{D}_{U_0}$, r , m and ϕ are as described in (1.4.8) and in the assumptions of section 1.4.1:*

$$|\varpi(u, v) - M| = \mathcal{O}(|u|^{-2p+1}), \quad (1.4.29)$$

$$|\nu(u, v) + 1| = \mathcal{O}(|u|^{-1}), \quad (1.4.30)$$

$$|\kappa(u, v) - 1| = \mathcal{O}(r^{-1}|u|^{-2p+1}), \quad (1.4.31)$$

$$|\lambda(u, v) - 1| = \mathcal{O}(r^{-1}), \quad (1.4.32)$$

$$|\partial_u(r\phi)(u, v) - \partial_u(r\phi)(1, v)| = \mathcal{O}(|u|^{-p-1}). \quad (1.4.33)$$

In particular, since $|u|^{p-1}r\phi$ takes a limit on initial data as $u \rightarrow -\infty$, it takes the same limit everywhere, that is:

$$\lim_{u \rightarrow -\infty} |u|^{p-1}r\phi(u, 1) = \lim_{u \rightarrow -\infty} |u|^{p-1}r\phi(u, v) = \Phi^- \quad (1.4.34)$$

for all $v \geq 1$. In particular, we then have

$$r\phi(u, v) = \frac{\Phi^-}{|u|^{p-1}} + \mathcal{O}(u^{-p+1-\epsilon}). \quad (1.4.35)$$

Remark 1.4.2. Notice, in particular, that we obtain that $\zeta \sim |u|^{-2}$ if $p = 2$. Using the results below, one can also show that $\zeta = r\partial_u\phi$ attains a limit on $\mathcal{I}^+$ and that $\lim_{\mathcal{I}^+} \zeta \sim |u|^{-2}$ (see Remark 1.4.4). Comparing (1.1.6) with (1.3.18) (see Remark 1.4.1), this can be recognised as the direct analogue of the condition that $|\Xi| \sim |u|^{-2}$ from assumption (1.1.7) from section 1.1.2. In turn, (1.1.7) was motivated by the quadrupole approximation. Thus, the case $p = 2$ reproduces the prediction of the quadrupole approximation. It is therefore the most interesting one from the physical point of view.

Remark 1.4.3. Note that one can still prove the above corollary if one demands assumption (1.4.6) to hold on $\mathcal{I}^+$ rather than on $\mathcal{C}_{\text{in}}$, and if one assumes a positive lower bound on the Hawking mass m . In fact, the only calculation that changes in that case is (1.4.26): One now integrates $\partial_u(r\phi)$ from $v = \infty$ rather than from $v = 1$. Combined with Theorem 1.4.2 below, this explains the statement of Theorem 1.2.2.

We conclude this subsection with the following observation:

Lemma 1.4.1. *For sufficiently large values of U_0 , we have throughout $\mathcal{D}_{U_0}$, where $\mathcal{D}_{U_0}$ and r are as described in (1.4.8) and in the assumptions of sec. 1.4.1:*

$$|r(u, v) - (v - u)| = \mathcal{O}(\log(r)). \quad (1.4.36)$$

Proof. This follows from $r(u, 1) - r(U_0, 1) = U_0 - u$ and the following estimate:

$$r(u, v) - r(u, 1) = \int_1^v \lambda \, dv' = (v - 1) + \mathcal{O}(\log r(u, v)),$$

where we used the asymptotic estimate (1.4.32) for λ . □

1.4.4 Asymptotics of $\partial_v(r\phi)$ near $\mathcal{I}^+$, i^0 and $\mathcal{I}^-$ (Proof of Thm. 1.2.4)

We are now ready to state the main result of this section, namely the asymptotic behaviour of $\partial_v(r\phi)$. Let us first focus on the most interesting case $p = 2$. We have the following theorem:

Theorem 1.4.2. *Let $p = 2$ in eq. (1.4.6), i.e., let $\lim_{u \rightarrow -\infty} |u|r\phi(u, 1) = \Phi^- \neq 0$. Then, for sufficiently large negative values of U_0 , we obtain the following asymptotic behaviour for $\partial_v(r\phi)$ throughout $\mathcal{D}_{U_0}$, where $\mathcal{D}_{U_0}$, r , m and ϕ are as described in (1.4.8) and in the assumptions of section 1.4.1 (in particular, $M \neq 0$):*

$$|\partial_v(r\phi)| \sim \begin{cases} \frac{\log r - \log |u|}{r^3}, & u = \text{constant}, v \rightarrow \infty, \\ \frac{1}{r^3}, & v = \text{constant}, u \rightarrow -\infty, \\ \frac{1}{r^3}, & v + u = \text{constant}, v \rightarrow \infty. \end{cases} \quad (1.4.37)$$

More precisely, for fixed u , we have the following asymptotic expansion as $\mathcal{I}^+$ is approached:

$$\left| \partial_v(r\phi)(u, v) + 2M\Phi^- r^{-3} \left(\log r - \log(|u|) - \frac{3}{2} \right) \right| = \mathcal{O}(r^{-3}|u|^{-\epsilon}). \quad (1.4.38)$$

Combined with Proposition 1.4.1 (and the specialisation to the linear case from section 1.3.3), this theorem proves Thm. 1.2.4 from the introduction.

Proof. We plug the asymptotics from Corollary 1.4.1 as well as the estimate (1.4.36) into the wave equation (1.3.24) to obtain

$$\partial_u \partial_v(r\phi) = \frac{-2M\Phi^-}{r^3|u|} + \mathcal{O}(r^{-3}u^{-1-\epsilon}) = \frac{2M\Phi^-}{(v-u)^3u} + \mathcal{O}\left(r^{-3}|u|^{-1-\epsilon} + r^{-4} \log r |u|^{-1}\right).$$

Integrating the above estimate from past null infinity then gives

$$\partial_v(r\phi)(u, v) - \int_{-\infty}^u \frac{2M\Phi^-}{(v-u')^3u'} \, du' = \mathcal{O}(r^{-3}|u|^{-\epsilon}). \quad (1.4.39)$$

We can calculate the integral on the LHS by decomposing the integrand into fractions:

$$\partial_v(r\phi)(u, v) - 2M\Phi^- \left(\frac{\log|u| - \log(v-u)}{v^3} + \frac{3v-2u}{2v^2(v-u)^2} \right) = \mathcal{O}(r^{-3}|u|^{-\epsilon}). \quad (1.4.40)$$

It is then clear that, for fixed u and $v \rightarrow \infty$, we have, to leading order,

$$\partial_v(r\phi) \sim -\frac{\log r - \log|u|}{r^3}. \quad (1.4.41)$$

On the other hand, for fixed v and $u \rightarrow -\infty$, we have

$$\partial_v(r\phi) \sim -\frac{1}{r^3}, \quad (1.4.42)$$

which can be seen by expanding the logarithm $\log(1 - v/u)$ to third order in powers of v/u .

Lastly, if we take the limit along a spacelike hypersurface, e.g. along $u + v = 0$, we get

$$\partial_v(r\phi) \sim -\frac{1}{r^3}. \quad (1.4.43)$$

□

Remark 1.4.4 (Similarities to Christodoulou's argument). Notice that Φ^- here plays the same role as $\mathcal{P}^{(3)}\Xi^-$ does in Christodoulou's argument. Indeed, recall from Remark 1.4.1 that, in our case, the analogue of the radiative amplitude Ξ is $\lim_{\mathcal{I}^+} \zeta = \lim_{\mathcal{I}^+} r\partial_u\phi$ (this limit exists in view of estimate (1.4.38) and the wave equation (1.3.24)), and that we moreover have

$$\lim_{u \rightarrow -\infty} \lim_{v \rightarrow \infty} u^2 \zeta(u, v) = \lim_{u \rightarrow -\infty} \lim_{v \rightarrow \infty} u^2 (\partial_u(r\phi) - \nu\phi)(u, v) = \Phi^-; \quad (1.4.44)$$

and compare equations (1.4.39), (1.4.40) to equation (1.1.9).

One can generalise the above proof to integer $p > 1$ to find that the asymptotic expansion of $\partial_v(r\phi)$ will contain a logarithmic term with constant coefficient at $(p+1)$ st order. Here, we will demonstrate this explicitly only for the case $p = 3$ since this case is of relevance for the black hole scattering problem, as will be explained in section 1.6. However, we provide a full treatment of general integer p for the uncoupled problem in the appendix 1.B, see Theorem 1.B.1. We also note that, by considering integrals of the type $\int \frac{1}{(v-u)|u|^p} du$ for non-integer p , one can obtain similar results for non-integer p , cf. footnote 12. For instance, if $p \in (1, 2)$, we would obtain that $\partial_v(r\phi) = Cr^{-1-p} + \dots$.

Theorem 1.4.3. *Let $p = 3$ in eq. (1.4.6), i.e., let $\lim_{u \rightarrow -\infty} |u|^2 r\phi(u, 1) = \Phi^- \neq 0$.*

Then, throughout $\mathcal{D}_{U_0}$ and for sufficiently large negative values of U_0 , where $\mathcal{D}_{U_0}$, r , m and ϕ are as described in (1.4.8) and in the assumptions of section 1.4.1 (in particular, $M \neq 0$), we obtain for fixed u the following asymptotic expansion for $\partial_v(r\phi)$ along each $\mathcal{C}_u$ as $\mathcal{I}^+$ is

approached:

$$\left| \partial_v(r\phi)(u, v) - \frac{F(u)}{r^3} - 6M\Phi^- \frac{\log(r) - \log|u|}{r^4} \right| = \mathcal{O}(r^{-4}), \quad (1.4.45)$$

where $F(u)$ is given by

$$F(u) = \int_{-\infty}^u \lim_{v \rightarrow \infty} (2m\nu r\phi)(u', v) du' = \frac{2M\Phi^-}{u} + \mathcal{O}(|u|^{-1-\epsilon}). \quad (1.4.46)$$

Proof. Following the same steps as in the previous proof, we find that

$$\partial_v(r\phi)(u, v) = \mathcal{O}(r^{-3}|u|^{-1}). \quad (1.4.47)$$

In order to write down higher-order terms in the expansion of $\partial_v(r\phi)$, we commute the wave equation with r^3 and integrate:

$$\begin{aligned} r^3 \partial_v(r\phi)(u, v) &= r^3 \partial_v(r\phi)(-\infty, v) + \int_{-\infty}^u \partial_u(r^3 \partial_v(r\phi))(u', v) du' \\ &= \int_{-\infty}^u 3\nu r^2 \partial_v(r\phi)(u', v) du' + \int_{-\infty}^u 2 \left(\varpi - \frac{e^2}{r} \right) \nu \kappa r \phi(u', v) du'. \end{aligned} \quad (1.4.48)$$

Here, we used that, by the above (1.4.47), $r^3 \partial_v(r\phi)$ vanishes as $u \rightarrow -\infty$.

Let's first deal with the second integral from the second line of (1.4.48). Observe that each of the quantities ϖ , ν and $r\phi$ attain a limit on $\mathcal{I}^+$ by monotonicity, and that $\kappa \rightarrow 1$ by Cor. 1.4.1. We write these limits as $\varpi(u, \infty)$ etc. Note, moreover, that $\partial_v \varpi \lesssim r^{-2}u^{-4}$ and $\partial_v \nu \lesssim r^{-2}$ by (1.3.19) and (1.3.23), respectively. We can further show that $\partial_v \kappa \lesssim r^{-2}u^{-6}$ by integrating $\partial_u \partial_v \log \kappa$ in u from $\mathcal{I}^-$, where $\partial_v \log \kappa$ vanishes:²⁵

$$\partial_u \partial_v \log \kappa = \partial_v \partial_u \log \kappa = \partial_v \left(\frac{\zeta^2}{\nu r} \right) = -\frac{\lambda \zeta^2}{\nu r^2} - 2 \frac{\zeta \theta}{r^2} - \frac{2 \left(\varpi - \frac{e^2}{r} \right) \kappa \zeta^2}{\nu r^3}.$$

We can thus apply the fundamental theorem of calculus to write

$$\begin{aligned} &\int_{-\infty}^u 2 \left(\varpi - \frac{e^2}{r} \right) \nu \kappa r \phi(u', v) du' \\ &= \int_{-\infty}^u 2 \varpi \nu r \phi(u', \infty) du' - \int_{-\infty}^u \int_v^\infty \partial_v \left(2 \left(\varpi - \frac{e^2}{r} \right) \nu \kappa r \phi \right) (u', v') dv' du'. \end{aligned} \quad (1.4.49)$$

The first integral on the RHS equals $F(u)$ from (1.4.46) and asymptotically evaluates to $F(u) = \frac{2M\Phi^-}{u} + \mathcal{O}(|u|^{-1-\epsilon})$ as a consequence of Corollary 1.4.1. On the other hand, by the above estimates for the v -derivatives of ϖ , ν , κ and $r\phi$, we can estimate the double integral

²⁵The computation below is the only place where we use that r is C^3 . If one wishes to compute higher-order asymptotics, then more regularity needs to be assumed.

above according to

$$\begin{aligned} \int_{-\infty}^u \int_v^{\infty} \partial_v \left(2 \left(\varpi - \frac{e^2}{r} \right) \nu \kappa r \phi \right) (u', v') dv' du' \\ \lesssim \int_{-\infty}^u \int_v^{\infty} \frac{1}{r^2 |u'|^6} + \frac{1}{r^2 |u'|^2} + \frac{1}{r^2 |u'|^8} + \frac{1}{r^3 |u'|} dv' du' \lesssim \frac{1}{r |u|}. \end{aligned} \quad (1.4.50)$$

Let us now turn our attention to the first integral in the second line of eq. (1.4.48). Plugging in our preliminary estimate (1.4.47) for $\partial_v(r\phi)$, we obtain:

$$\int_{-\infty}^u 3\nu r^2 \partial_v(r\phi)(u', v) du' \lesssim \int_{-\infty}^u \frac{1}{(v-u')|u'|} du' = \frac{\log(v-u) - \log|u|}{v}. \quad (1.4.51)$$

Therefore, combining the three estimates above, we obtain from (1.4.48) the asymptotic estimate:

$$r^3 \partial_v(r\phi)(u, v) - \int_{-\infty}^u 2m\nu r \phi(u', \infty) du' = \mathcal{O} \left(\frac{\log(v-u) - \log|u|}{v} \right). \quad (1.4.52)$$

This is an improvement over the estimate (1.4.47). By inserting this into (1.4.51), we can further improve the estimate (1.4.51) to

$$\begin{aligned} & \int_{-\infty}^u 3\nu r^2 \partial_v(r\phi)(u', v) du' \\ &= \int_{-\infty}^u \frac{-6M\Phi^-}{(v-u')u'} + \mathcal{O}(r^{-1}|u'|^{-1-\epsilon}) + \mathcal{O} \left(\frac{\log(v-u') - \log|u'|}{v(v-u')} \right) du' \\ &= 6M\Phi^- \left(\frac{\log(v-u) - \log|u|}{v} \right) + \mathcal{O}(r^{-1}). \end{aligned} \quad (1.4.53)$$

Here, we used that

$$\int_{-\infty}^u \frac{\log(v-u') - \log|u'|}{v(v-u')} du' = \frac{1}{v} \text{Li}_2 \left(\frac{v}{v-u} \right), \quad (1.4.54)$$

where Li_2 denotes the *dilogarithm*²⁶, which has the two equivalent definitions for $|x| \leq 1$:

$$-\int_0^x \frac{\log(1-y)}{y} dy =: \text{Li}_2(x) := \sum_{k=1}^{\infty} \frac{x^k}{k^2}. \quad (1.4.55)$$

In particular, we thus have, since $0 < v/(v-u) < 1$,

$$\int_{-\infty}^u \frac{\log(v-u') - \log|u'|}{v(v-u')} du' \leq \frac{1}{v} \frac{v}{v-u} \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6} \frac{1}{v-u}. \quad (1.4.56)$$

²⁶See e.g. [AS74], page 1004.

Plugging the above asymptotics (1.4.53) back into eq. (1.4.48) and dividing by r^3 completes the proof. $\square$

Remark 1.4.5 (Higher derivatives). We remark that one can commute the two wave equations for r and $r\phi$ with ∂_v to obtain similar results for higher derivatives. For instance, one gets that $\partial_v \partial_v r \sim r^{-2}$ and, thus, asymptotically,

$$\partial_v^2(r\phi) = -\frac{3}{r}\partial_v(r\phi) + \dots \quad (1.4.57)$$

This fact is of importance for proving higher-order asymptotics for general p using *time integrals*, see also the proof of Theorem 1.6.2, in particular eq. (1.6.20).

Remark 1.4.6. Comparing these results to those of [Chr02] presented in section 1.1.2, one can of course also compute the Weyl curvature tensor W and relate it to ϕ using the Einstein equations. Since we work in spherical symmetry, ρ is the only non-vanishing component of the Weyl tensor $W_{\mu\nu\xi\sigma}$ under the null decomposition (1.1.1). We derive the following formula in Appendix 1.A:

$$W_{vuvu} = -\frac{\Omega^4}{2} \frac{m}{r^3} + \frac{8}{3} \Omega^2 \partial_u \phi \partial_v \phi. \quad (1.4.58)$$

Using the results above, it is thus easy to see that, in the case $p = 2$, the asymptotic expansion of ρ contains a logarithmic term at order $r^{-5} \log r$ (coming from the $\partial_u \phi \partial_v \phi$ -term). We stress, however, that the point of working with the Einstein-Scalar field system is to model the more complicated Bianchi equations (which encode the essential hyperbolicity of the Einstein vacuum equations) by the simpler wave equation (and thus gravitational radiation by scalar radiation), replacing e.g. β with $\partial_v(r\phi)$. It is therefore not the behaviour of the curvature coefficients we are directly interested in, but the behaviour of the scalar field.

1.5 Case 2: Boundary data posed on a timelike hypersurface

In this section, we construct solutions for the setup with vanishing incoming radiation from past null infinity and with polynomially decaying *boundary data* (as opposed to characteristic data as considered in the previous section),

$$r\phi|_{\Gamma} \sim |t|^{-p+1}, \quad (1.5.1)$$

posed on a suitably regular timelike curve Γ , where t is time measured along that curve.

For instance, the reader can keep the example of a curve of constant $r = R$ in mind, with R being larger than what is the Schwarzschild radius in the linear case: $2M$. In general, however, r need not be constant and is also permitted to tend to infinity.

In contrast to the characteristic problem of section 1.4, this type of boundary problem does not permit *a priori estimates* of the same strength as those of section 1.4. Instead, we will need to develop the existence theory for these problems simultaneously to our estimates. The existence theory developed in the present section can then, *a fortiori*, be used to prove Proposition 1.4.1 of the previous section (i.e. to show the existence of solutions satisfying the assumptions of section 1.4.1).

1.5.1 Overview

The problem considered in this section differs from the previous problem in that there are two additional difficulties: In the null case, the two crucial observations were **a**) boundedness of the Hawking mass (see Prop. 1.4.3) and **b**) that $|u|$ is comparable to r along $\mathcal{C}_{\text{in}}$, which we used to propagate $|u|$ -decay from $\mathcal{C}_{\text{in}}$ outwards (see (1.4.26)). It is clear that **b**) will in general not be true in the timelike case (where $\mathcal{C}_{\text{in}}$ is replaced by Γ).

Regarding **a**), we recall that the boundedness of the Hawking mass ϖ was just a simple consequence of its monotonicity properties, which allowed us to essentially bound ϖ from above and below by its prescribed values on the initial ingoing or outgoing null ray, respectively. These values were in turn determined by the constraint equations for $\partial_u \varpi$ and $\partial_v \varpi$ (eqns. (1.3.18), (1.3.19)). However, when prescribing *boundary data* for (r, ϕ) on Γ , it is no longer possible to derive bounds for ϖ just in terms of the data.²⁷

Instead, we will therefore, inspired by the results for the null case, *bootstrap* both the boundedness of the Hawking mass and the $|u|$ -decay of $r\phi$. Unfortunately, the need to appeal to a bootstrap argument comes with a technical subtlety: To show the non-emptiness part of the bootstrap argument, we will need to exploit continuity in a compact region! This forces us to first consider *boundary data of compact support*, $\text{supp}(\phi_{\Gamma}) \cap (\{u \leq u_0\} \cap \Gamma) = \emptyset$ for some u_0 , which obey a polynomial decay bound that is independent of u_0 . Then, by the domain of

²⁷It is this fact which forms the main difficulty in proving local existence for this system.

dependence property²⁸, we can consider the finite problem where we set

$$r\phi = 0 (= \partial_v(r\phi) = \partial_v\varpi), \quad (1.5.2)$$

$$\varpi = M > 0 \quad (1.5.3)$$

on the outgoing null ray $\mathcal{C}_{u_0}$ of constant $u = u_0$ emanating from Γ . The goal is to show uniform bounds in u_0 for the solutions arising from this and, ultimately, to push u_0 to $-\infty$ using a limiting argument.

Structure

After showing local existence for this initial boundary value problem in section 1.5.4, we first make a couple of restrictive but severely simplifying assumptions (such as smallness of initial data). These allow us to prove a slightly weaker version of our final result, namely Thm. 1.5.3, in which we construct global solutions arising from *non-compactly supported* boundary data. This is done in the way outlined above: We first consider solutions arising from *compactly supported* boundary data and prove uniform bounds on ϖ and uniform decay of $r\phi$ for these in section 1.5.5 (both uniform in u_0). Subsequently, we send u_0 to $-\infty$ using a Grönwall-based limiting argument in section 1.5.6, hence removing the assumption of compact support.

Now, while the proof of Theorem 1.5.3 already exposes many of the main ideas, it does not show sharp decay for certain quantities. As a consequence, the theorem is not sufficient for showing that $\partial_v(r\phi) = Cr^{-3} \log r + \mathcal{O}(r^{-3})$ (unless the datum for r tends to infinity along Γ); instead, it only shows that $|\partial_v(r\phi)| \sim r^{-3} \log r$. We will overcome this issue by proving various refinements – the crucial ingredient to which is commuting with the generator of the timelike boundary Γ – which allow us to not only remove the aforementioned restrictive assumptions but also to show sharp decay on all quantities and, in particular, derive an asymptotic expression for $\partial_v(r\phi)$. This is done in section 1.5.7. The main results of this section, namely Theorems 1.5.6 and 1.5.7, are then proved in section 1.5.8. In particular, these theorems together prove Thm. 1.2.1 from the introduction. The confident reader may wish to skip to section 1.5.7 immediately after having finished reading section 1.5.5.

Since the construction of our final solution will span the next 40 pages, we feel that it may be helpful to immediately give a description of the final solutions of section 1.5.8. This is done in section 1.5.2. The sole purpose of this is for the reader to see already in the beginning what kind of solutions we will construct, it is in no way part of the mathematical argument.

Furthermore, in order for the limiting argument to become more concrete, we will work on an ambient background manifold with suitable coordinates. All solutions constructed in the present section will then be subsets of this manifold. This ambient manifold is introduced in section 1.5.3.

²⁸We can “extend” to the past, i.e. towards $\mathcal{I}^-$, by the Reissner–Nordström solution for $u \leq u_0$.

The Maxwell field

As we have seen in the previous section, the inclusion of the Maxwell field does not change the calculations in any notable way. We will thus, from now on, consider, in order to make the calculations less messy, the case $e^2 = 0$; however, all results of the present section can be recovered (with some minor adaptations) for $e^2 \neq 0$ as well.

1.5.2 Preliminary description of the final solution

In this section, we describe the solutions that we will ultimately construct in section 1.5.8.

Let U_0 be a negative number, $-\infty < U_0 < 0$, and define the set

$$\mathcal{D}_{U_0} := \{(u, v) \in \mathbb{R}^2 \mid -\infty < u \leq v < \infty \text{ and } u \leq U_0\}. \quad (1.5.4)$$

We denote, for $u \in (-\infty, U_0]$, the sets $\mathcal{C}_u := \{u\} \times [u, \infty)$ as *outgoing null rays* and, for $v \in [u, \infty)$, the sets $\mathcal{C}_v := (-\infty, U_0] \times \{v\}$ as *ingoing null rays*. We colloquially refer to $\{-\infty\} \times (-\infty, \infty)$ as $\mathcal{I}^-$ or *past null infinity*, to $(-\infty, U_0] \times \{\infty\}$ as $\mathcal{I}^+$ or *future null infinity*, to $\{-\infty\} \times \{-\infty\}$ as i^- or *past timelike infinity*, and to $\{-\infty\} \times \{\infty\}$ as i^0 or *spacelike infinity*. Furthermore, we denote the timelike part of the boundary of $\mathcal{D}_{U_0}$ by

$$\Gamma := \{(u, v) \in \mathcal{D}_{U_0} \mid u = v\}. \quad (1.5.5)$$

We denote the generator of Γ by $\mathbf{T} = \partial_u + \partial_v$. We will, in section 1.5.6, show the following statement:²⁹

Proposition. *Prescribe boundary data $r(u, u)$, $\phi(u, u)$ along Γ as follows: Let $M > 0$. Assume that $r(u, u) > 2M$ either tends to a finite limit $R > 2M$, or that it tends to an infinite limit, as $u \rightarrow -\infty$. In the case where it tends to a finite limit, we further assume that*

$$\mathbf{T}r(u, u) = \mathcal{O}(|u|^{-s}) \quad (1.5.6)$$

for some $s = 1 + \epsilon_r > 1$. In the case where it tends to an infinite limit, we instead assume that

$$-\mathbf{T}r(u, u) \sim |u|^{-s} \quad (1.5.7)$$

²⁹As we mentioned before, we will develop the existence theory at the same time as our estimates. The proposition below, however, extracts from this only the existence theory and the form of the final boundary/scattering data.

for some $s \in (0, 1]$. For the scalar field, we assume that

$$\mathbf{T}(r\phi)(u, u) = C_{\text{in}, \phi}^1 |u|^{-p} + \mathcal{O}(|u|^{-p-\epsilon_\phi}), \quad (1.5.8)$$

$$r\phi(u, u) = \frac{C_{\text{in}, \phi}^1}{p-1} |u|^{-p+1} + \mathcal{O}(|u|^{-p+1-\epsilon_\phi}) \quad (1.5.9)$$

for some constants $C_{\text{in}, \phi}^1 \neq 0$, $p > 1$ and $\epsilon_\phi \in (0, 1)$.

Then, if U_0 is chosen to be a sufficiently large negative number, there exists a unique triplet of C^2 -functions (r, ϕ, m) on $\mathcal{D}_{U_0}$ that solves the equations (1.3.18)–(1.3.22) pointwise throughout $\mathcal{D}_{U_0}$, restricts to the boundary data $r(u, u)$, $\phi(u, u)$ above, and that satisfies for any $v \in (-\infty, \infty)$:

$$\lim_{u \rightarrow -\infty} \partial_v(r\phi)(u, v) = 0, \quad \lim_{u \rightarrow -\infty} \partial_v r(u, v) = 1, \quad \lim_{u \rightarrow -\infty} m(u, u) = M > 0. \quad (1.5.10)$$

This solution moreover has the following properties: The area radius r tends to infinity along each of the ingoing and outgoing null rays, i.e., $\sup_{C_u} r(u, v) = \infty$ for all $u \in (-\infty, U_0]$, and $\sup_{C_v} r(u, v) = \infty$ for all $v \in (-\infty, \infty)$, and we have throughout $\mathcal{D}_{U_0}$ that $\nu < 0$ and $\lambda, \kappa > 0$. Furthermore, this solution satisfies for any $v \in (-\infty, \infty)$:

$$\lim_{u \rightarrow -\infty} m(u, v) - M = 0 = \lim_{u \rightarrow -\infty} r\phi(u, v). \quad (1.5.11)$$

In the above, uniqueness is understood with respect to the class of solutions with finite Hawking mass (see Remark 1.5.7).

The reader can again refer to the Penrose diagram below (Figure 1.6).

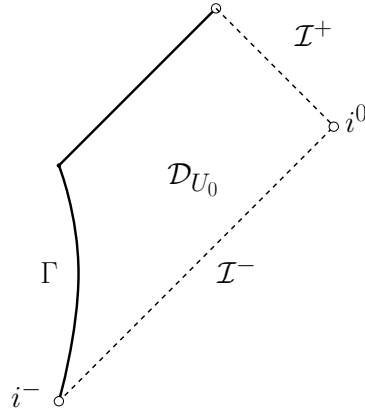


Figure 1.6 The Penrose diagram of $\mathcal{D}_{U_0}$. Note that, with our choice of coordinates (namely, $u = v$ on Γ), Γ should be a vertical line instead of a curved line. We depicted it as a curved line to avoid the possible confusion that Γ describes the centre of spacetime.

1.5.3 The ambient manifold

In this section, we introduce the ambient manifold and coordinate chart that shall provide us with the geometric background on which we shall be working.

Let $U < 0$, and let $\mathcal{D}_U$ be the manifold with boundary

$$\mathcal{D}_U := \{(u, v) \in \mathbb{R}^2 \mid -\infty < u \leq v < \infty \text{ and } u < U\}. \quad (1.5.12)$$

We can equip $\mathcal{D}_U$ with the Lorentzian metric $-du dv$. In the sequel, we shall prescribe on the boundary of $\mathcal{D}_U$ suitable boundary data $(\hat{r}, \hat{\phi})$ for the Einstein-Scalar field system (1.3.18)–(1.3.22). Together with suitable data on an outgoing null ray, these will, at least locally, lead to solutions (r, ϕ, m) , as will be demonstrated in the next section. These solutions correspond, according to section 1.3.1, to spherically symmetric spacetimes, whose quotient under the action of $SO(3)$ we will view as subsets of the ambient manifold $\mathcal{D}_U$.

Throughout this entire section, any causal-geometric concepts such as “null”, “timelike” or “future” will refer to the background manifold $(\mathcal{D}_U, -du dv)$.

1.5.4 A local existence result

We will start by proving a local existence result in the gauge specified above:

Proposition 1.5.1. *Let $\mathcal{D}_U$ be as described in section 1.5.3, let $u_0 \leq u_1 < U$ and $v_1 > u_0$, and let $\bar{\mathcal{C}} = \{u_0\} \times [u_0, v_1]$ be an outgoing null ray intersecting $\bar{\Gamma} = \{(u, u) \in \mathcal{D}_U \mid u_0 \leq u \leq u_1\}$ at a point $q = (u_0, u_0)$. Specify on $\bar{\mathcal{C}}$ two C^2 -functions $\bar{r}(v)$, $\bar{\phi}(v)$, and specify on $\bar{\Gamma}$ two C^2 -functions $\hat{r}(u)$, $\hat{\phi}(u)$. Moreover, specify a value $\bar{m}(q) < \bar{r}(u_0)/2$, and define on $\bar{\mathcal{C}}$ the function $\bar{m}(v)$ as the unique solution to the ODE*

$$\partial_v \bar{m} = \frac{1}{2} \left(1 - \frac{2\bar{m}}{\bar{r}} \right) \bar{r}^2 \frac{(\partial_v \bar{\phi})^2}{\partial_v \bar{r}} \quad (1.5.13)$$

with initial condition $\bar{m}(q)$. Finally, assume that the following data bounds are satisfied

$$\max_{\bar{\mathcal{C}}} \{ |\log \bar{r}|, |\log \partial_v \bar{r}|, |\bar{\phi}|, |\partial_v \bar{\phi}|, |\log(1 - 2\bar{m}(q)/\bar{r}(u_0))| \} \leq C, \quad (1.5.14)$$

$$\max_{\bar{\Gamma}} \{ |\log \hat{r}|, |\mathbf{T}\hat{r}|, |\hat{\phi}|, |\mathbf{T}\hat{\phi}| \} \leq C, \quad (1.5.15)$$

and assume the usual compatibility conditions at the corner q as well as³⁰

$$\mathbf{T}\hat{r}(u = v) - \partial_v \bar{r}(v) < 0. \quad (1.5.16)$$

Then, for ϵ sufficiently small and depending only on C , there is a region

$$\Delta_{u_0, \epsilon} := \{(u, v) \in \mathcal{D}_U \mid u_0 \leq u \leq v \leq u_0 + \epsilon\} \quad (1.5.17)$$

³⁰This should be thought of as implying that $\partial_u r < 0$ along Γ . Note that we cannot specify $\partial_u r$ as data on Γ .

in which a unique C^2 -solution to the spherically symmetric Einstein-Scalar field equations (1.3.18)–(1.3.23) that restricts correctly to the initial/boundary data exists. Moreover, higher regularity is propagated, that is to say: If the initial data are in C^k for $k > 2$, then the solution will also be in C^k .

Proof. The proof will follow a classical iteration argument: We will define a contraction map Φ on a complete metric space such that the fixed-point of this map will be a solution to the system of equations.

For an $\epsilon > 0$ to be specified later, define

$$Y(\Delta_{u_0, \epsilon}) := \{(r, \phi, m, \mu) \in C^1(\Delta_{u_0, \epsilon}) \times C^1(\Delta_{u_0, \epsilon}) \times C^0(\Delta_{u_0, \epsilon}) \times C^0(\Delta_{u_0, \epsilon})\} \quad (1.5.18)$$

and the corresponding subspace

$$Y_E(\Delta_{u_0, \epsilon}) := \{(r, \phi, m, \mu) \in Y(\Delta_{u_0, \epsilon}) \mid \max\{|\log r|, |\log \partial_v r|, |\log(-\partial_u r)|, \|\phi\|_{C^1(\Delta_{u_0, \epsilon})}, |m|, |\log(1 - \mu)|\} \leq E\}, \quad (1.5.19)$$

equipped with the metric

$$d((r_1, \phi_1, m_1, \mu_1), (r_2, \phi_2, m_2, \mu_2)) := \sup_{\Delta_{u_0, \epsilon}} \{|\log |r_1/r_2||, |\log |\partial_v r_1/\partial_v r_2||, |\log |\partial_u r_1/\partial_u r_2||, \|\phi_1 - \phi_2\|_{C^1(\Delta_{u_0, \epsilon})}, |m_1 - m_2|, |\log |1 - \mu_1|/|1 - \mu_2||\}.$$

For any element $(r, \phi, m, \mu) \in Y_E(\Delta_{u_0, \epsilon})$, we now define our candidate for the contraction map Φ via $(r', \phi', m', \mu') = \Phi((r, \phi, m, \mu))$, where the primed quantities r' , ϕ' are, for $(u, v) \in \Delta_{u_0, \epsilon}$, defined via

$$r'(u, v) = \hat{r}(u) + \bar{r}(v) - \bar{r}(u) - \int_u^v \int_{u_0}^u \frac{2m\nu\lambda}{r^2(1 - \mu)} d\bar{u} d\bar{v} \quad (1.5.20)$$

and (notice that these double integrals are simply integrals over rectangles)

$$\phi'(u, v) = \hat{\phi}(u) + \bar{\phi}(v) - \bar{\phi}(u) - \int_u^v \int_{u_0}^u \frac{1}{r} (\partial_u r \partial_v \phi + \partial_v r \partial_u \phi) d\bar{u} d\bar{v}, \quad (1.5.21)$$

and m' , μ' are defined as solutions to the ODE's

$$\partial_u \frac{\partial_v r'}{1 - \mu'} = r' \frac{(\partial_u \phi')^2}{\partial_u r'} \frac{\partial_v r'}{1 - \mu'}, \quad (1.5.22)$$

$$\partial_u m'(u, v) = \frac{1}{2} (1 - \mu') r'^2 \frac{(\partial_u \phi')^2}{\partial_u r'}, \quad (1.5.23)$$

with initial conditions $\mu'(u_0, v) = 2\bar{m}(v)/\bar{r}(v)$, $m'(u_0, v) = \bar{m}(v)$, respectively.

One now checks that **a)** Φ is a map from $Y_E(\Delta_{u_0, \epsilon})$ to itself, and that **b)** it is a contraction w.r.t. the associated metric d . Both these facts can easily be established by (after also integrating the equations for $\partial_v r'/(1 - \mu')$, m' , and solving the ODE for $\bar{m}$ for some suitably small interval) bounding in each case the integrand by a continuous function of E and then making the integrals sufficiently small by using the smallness in ϵ , whereas the initial data terms from integrating the equations can be bounded from above and below by continuous functions of C in case **a)**³¹, and they vanish in case **b)**.³²

Having established these two facts, we invoke the Banach fixed point theorem to obtain a unique fixed-point

$$(r, \phi, m, \mu) \in Y_E(\Delta_{u_0, \epsilon}),$$

which clearly solves the equations (1.3.18), (1.3.20)–(1.3.23) and restricts properly to the initial/boundary data. However, it is not yet clear that this fixed point has the desired regularity, that $\mu = 2m/r$, or that eq. (1.3.19) is satisfied.

To obtain the desired regularity, observe that the equations that the fixed point obeys immediately tell us that we have that $\partial_u \partial_v r$, $\partial_u \partial_v \phi$, $\partial_u m$ and $\partial_u \mu$ are, in fact, continuous. Moreover, we also have that $\mu = 2m/r$ everywhere since it holds initially (on $\bar{C}$) and we can differentiate in u to propagate equality inwards. (See also the argument below.) To infer higher regularity, consider now the equation that $\partial_u m$ satisfies (an ODE with coefficients that are continuously differentiable in v) to see that $\partial_v m$ is continuous; hence m is continuously differentiable. By a similar argument, we can then also infer that r and ϕ are in C^2 . This in turn implies that m is in C^2 , which allows us to propagate the constraint equation (1.5.13) inwards by differentiating both sides in u : Indeed, we have, by virtue of m being C^2 :

$$\begin{aligned} \partial_u \partial_v m &= \partial_v \left(\frac{1}{2} \left(1 - \frac{2m}{r} \right) r^2 \frac{(\partial_u \phi)^2}{\partial_u r} \right) \\ &= \left(\frac{-\partial_v m}{r} + \frac{m \partial_v r}{2r^2} \right) r^2 \frac{(\partial_u \phi)^2}{\partial_u r} + \left(1 - \frac{2m}{r} \right) r \frac{\partial_v r (\partial_u \phi)^2}{\partial_u r} \\ &\quad + \frac{1}{2} \left(1 - \frac{2m}{r} \right) r^2 \left(\frac{2 \partial_u \phi \partial_v \partial_u \phi}{\partial_u r} - \frac{(\partial_u \phi)^2 \partial_v \partial_u r}{(\partial_u r)^2} \right) \\ &= \left(-r \frac{(\partial_u \phi)^2}{\partial_u r} \right) \partial_v m + \left(1 - \frac{2m}{r} \right) r (\partial_u \phi \partial_v \phi), \end{aligned}$$

³¹This is the reason for the necessity of condition (1.5.16) – without it, we wouldn't be able to guarantee that the “initial value” term for $\partial_u r'$ on Γ would be negative.

³²Note that we introduced μ and m as a priori independent variables only to deduce more easily that $1 - \mu$ remains bounded away from zero.

where, in the last step, we used equations (1.3.13) and (1.3.23). On the other hand, we have:

$$\begin{aligned} & \partial_u \left(\frac{1}{2} \left(1 - \frac{2m}{r} \right) r^2 \frac{(\partial_v \phi)^2}{\partial_v r} \right) \\ &= \left(-r \frac{(\partial_u \phi)^2}{\partial_u r} \right) \left(\frac{1}{2} \left(1 - \frac{2m}{r} \right) r^2 \frac{(\partial_v \phi)^2}{\partial_v r} \right) + \left(1 - \frac{2m}{r} \right) r (\partial_u \phi \partial_v \phi), \end{aligned}$$

where we used eqns. (1.3.13) and (1.3.20) in the last step. Applying Grönwall's inequality to the two identities above shows that (1.3.19) holds everywhere in $\Delta_{u_0, \epsilon}$ and, thus, completes the main part of the proof.

In order to show that higher regularity is propagated if assumed initially, one first shows that if $\bar{r}$ is in C^3 , then $\partial_v^2 r$ is in C^1 by considering the equation satisfied by $\partial_u \partial_v^2 r$. One then shows a similar statement concerning $\partial_u^2 r$ by considering the equation for $\partial_v \partial_u^2 r$. One thus shows that r is in C^3 if the data for r are in C^3 . A similar argument gives that ϕ is in C^3 , from which it follows directly that m is in C^3 . One now proceeds inductively. $\square$

Remark 1.5.1. Comparing to the local existence proof for the characteristic initial value problem, the main difficulty here was that we couldn't treat the equation for $\partial_u m$ as a constraint equation that is prescribed initially (cf. 1.5.13), so we had to include it in the contraction map. The value of m along Γ is not known initially, but found dynamically via the fixed point theorem (by integrating $\partial_u m$ from $\bar{C}$). (We recall that, for the characteristic initial value problem, one can conveniently define the contraction map via the three wave equations for $\partial_u \partial_v \phi$, $\partial_u \partial_v r$ and $\partial_u \partial_v m$, and then propagate the equations for $\partial_u m$ and $\partial_v m$ "inwards" from initial data.)

1.5.5 The finite problem: Data on an outgoing null hypersurface $\mathcal{C}_{u_0}$

Now that we have established local existence for a small triangular region $\Delta_{u_0, \epsilon}$ as described above, we want to increase the region of existence. For this, we will first need to prove uniform bounds for r, m, ϕ (like those obtained in the null case) for initial/boundary data as described in section 1.5.2. As discussed in the introduction to this section, we will, in the present subsection, assume that the data for $r\phi$ are compactly supported on $\Gamma := \partial D_U$. We will remove this assumption of compact support in section 1.5.6.

Furthermore, as mentioned in the overview (sec. 1.5.1), we will from now on make an extra assumption in order to simplify the presentation, namely that the exponent p from the bound (1.5.8) be larger than $3/2$, that is, we will assume $p > 3/2$ instead of $p > 1$. This assumption will be removed in section 1.5.7. We will also introduce a lower bound on R (1.5.28) in order to more clearly expose the ideas. In reality, if we only want to show upper bounds, this bound can always be replaced by $R > 2M$, see Remark 1.5.3. We will only need a slightly stronger lower bound on R once we prove lower bounds for the radiation field in Theorem 1.5.3.

Theorem 1.5.1. *Let $\mathcal{D}_U$ be as described in section 1.5.3, and specify smooth functions $\hat{r}, \hat{\phi}$ on $\Gamma = \partial D_U = \{(u, u) \in \mathcal{D}_U\}$, with $\hat{\phi}$ having compact support. Let $\mathcal{C}_{u_0}$ denote the future-complete outgoing null ray emanating from a point $q = (u_0, u_0)$ on Γ that lies to the past of the support of $\hat{\phi}$. On $\mathcal{C}_{u_0}$, specify $\bar{m} \equiv M > 0$, $\bar{\phi} \equiv 0$ and an increasing smooth function $\bar{r}$ defined via $\bar{r}(v = u_0) = \hat{r}(u = u_0)$ and the ODE*

$$\partial_v \bar{r} = 1 - \frac{2M}{\bar{r}}. \quad (1.5.24)$$

Finally, assume that (denoting again the generator of Γ by $\mathbf{T} = \partial_u + \partial_v$) the following bounds hold on Γ :

$$|\mathbf{T}(\hat{r}\hat{\phi})(u)| \leq C_{\text{in},\phi}^1 |u|^{-p}, \quad (1.5.25)$$

$$|\mathbf{T}\hat{r}(u)| \leq C_{\text{in},r} |u|^{-s}, \quad (1.5.26)$$

$$\hat{r} \geq R > 2M, \quad (1.5.27)$$

with positive constants $p > 3/2$, $C_{\text{in},\phi}^1$, $C_{\text{in},r}$ and $s > 0$.

Let $\Delta_{u_0,\epsilon}$ denote the region of local existence (cf. (1.5.17)) of the solution (r, ϕ, m) arising from these initial/boundary data in the sense of Proposition 1.5.1, with ϵ only depending on the size of the data. Then we have, for sufficiently large negative values of U_0 (the choice of U_0 depending only on data), and if

$$R \geq \frac{2M}{1 - e^{-2+\delta(U_0)}} \quad (1.5.28)$$

for some function $\delta(u) \sim 1/|u|^{2p-3}$, that the following pointwise bounds hold throughout $\Delta_{u_0,\epsilon} \cap \{u \leq U_0\}$:

$$0 < \frac{M}{2} \leq m \leq M, \quad (1.5.29)$$

$$0 < 1 - \frac{2M}{r} \leq 1 - \mu \leq 1, \quad (1.5.30)$$

$$0 < 1 - \delta(u) = \kappa \leq 1, \quad (1.5.31)$$

$$0 < (1 - \delta(u)) \left(1 - \frac{2M}{r}\right) = \lambda \leq 1, \quad (1.5.32)$$

$$0 < d_\nu =: \left(1 - \frac{2M}{R}\right) e^{-\frac{2M}{R-2M}} \leq |\nu| \leq e^{\frac{2M}{R-2M}} =: C_\nu, \quad (1.5.33)$$

$$|r\phi| \leq \frac{C_{\text{in},\phi} |u|^{-p+1}}{1 - \frac{1}{2(1-\delta(U_0))} \log \frac{R}{R-2M}} =: C' |u|^{-p+1}, \quad (1.5.34)$$

$$|\partial_v(r\phi)| \leq MC' \frac{|u|^{-p+1}}{r^2}, \quad (1.5.35)$$

where $C_{\text{in},\phi} = C_{\text{in},\phi}^1/(p-1)$. In particular, all these bounds are independent of u_0 .

Proof. The proof will consist of a nested bootstrap argument. First, we will assume boundedness of the Hawking mass. This will essentially allow us to redo the calculations done in the proof of Prop. 1.4.4 to show boundedness of the geometric quantities λ, ν, μ . We will then assume $|u|$ -decay for the radiation field $r\phi$ and improve this decay by using the previously derived bounds on ν, λ, μ , the assumed bound on m , and by integrating the wave equation (1.3.24) in u and in v . We will then use the decay for $r\phi$ to get enough decay for ζ to also improve the bound on m by integrating $\partial_u m$. Indeed, it will turn out to slightly simplify things if we also introduce a bound on ζ as a bootstrap assumption.

Let us start the proof: It is easy to see that the assumptions of the theorem allow us to apply Proposition 1.5.1; in particular, a solution (r, ϕ, m) exists in $\Delta_{u_0, \epsilon}$ for sufficiently small ϵ . Next, notice that, by the monotonicity property of the Hawking mass³³, it is clear that $1 - \mu > 1 - \frac{2M}{R} =: d_\mu$. Having made these preliminary observations, we now initiate the bootstrap argument. Consider the set

$$\begin{aligned} \Delta &:= \{(u, v) \in \Delta_{u_0, \epsilon} \mid \text{such that, for all } (u', v') \in \Delta_{u_0, \epsilon} \text{ with } u' \leq u, v' \leq v : \\ &\quad |m(u', v')| \leq \eta \cdot M, \tag{BS(1)} \\ &\quad |\zeta(u', v')| \leq \tilde{C}|u'|^{-p+1}, \tag{BS(2)} \\ &\quad |r\phi(u', v')| \leq C'|u'|^{-p+1} \}, \tag{BS(3)} \end{aligned}$$

where $\eta, \tilde{C}$ and C' are positive constants with $\eta > 1$, $\tilde{C}$ sufficiently large, and $C' = \frac{\eta' C_{\text{in}, \phi}}{1 - \frac{1}{2(1 - \delta(U_0))} \log \frac{R}{R - 2M}} > 0$ (the positivity of this constant of course being precisely the condition that R be sufficiently large), where $\eta' > 1$ is arbitrary and $\delta(u) \sim |u|^{-2p+3}$ can be made arbitrarily small by choosing U_0 large enough.

Proposition 1.5.1 guarantees that Δ is non-empty by continuity; $\{q\} \subsetneq \Delta$ (q is trivially contained in Δ , but q alone wouldn't be enough, we instead need a small triangle in which we can integrate – this is where we need the assumption of compactness (and thus of compact support) to exploit continuity).

Furthermore, Δ is clearly closed, so it suffices to show that it is also open. We will essentially follow the same structure as we did in the null case for this: First, note that, in Δ , we again have the energy estimates (cf. (1.4.16), (1.4.17)) as a consequence of (BS(1))

$$\int_{v_1}^{v_2} \frac{1}{2} \frac{\theta^2}{\kappa}(u, v) dv \leq 2\eta M, \tag{1.5.36}$$

$$- \int_{u_1}^{u_2} \frac{1}{2} \frac{(1 - \mu)}{\nu} \zeta^2(u, v) du \leq 2\eta M. \tag{1.5.37}$$

We then obtain that

$$1 \geq \kappa \geq e^{-\frac{2\eta M}{R d_\mu}} =: d'_\kappa$$

³³By the above local existence result, we already know that $\nu < 0$ and $\lambda > 0$ in $\Delta_{u_0, \epsilon}$.

by integrating equation (1.3.20) for $\partial_u \kappa$ and applying the energy estimate as in the proof of Prop. 1.4.4. Later, we will want to show that the lower bound for κ can be improved beyond the estimate given by the energy estimate, which is why we have also introduced the bootstrap bound on ζ (BS(2)). But let us first derive bounds for λ and ν : It is clear that

$$d_\lambda := d'_\kappa d_\mu \leq \lambda \leq 1.$$

For $|\nu|$, we integrate eq. (1.3.23) from Γ :

$$|\nu(u, v)| = |\nu(u, u)| e^{2 \int_u^v \frac{\kappa}{r^2} m dv'}.$$

Using that

$$\left| \int_u^v \frac{\kappa}{r^2} m dv' \right| \leq \frac{\eta M}{d_\mu R},$$

as well as the fact that

$$\nu|_\Gamma = \mathbf{T}\hat{r} - \lambda|_\Gamma \leq -\frac{1}{2}\lambda|_\Gamma,$$

where we used that $\mathbf{T}\hat{r}$ satisfies $|\mathbf{T}\hat{r}| \leq |u|^{-s}$ and can thus be made small by choosing U_0 large enough, we thus get that

$$d_\nu \leq |\nu| \leq C_\nu,$$

with the constants d_ν, C_ν only depending on initial/boundary data.

Let us now invoke our second bootstrap assumption (BS(2)). Note that (BS(2)) directly implies that $|m| \leq M$. Indeed, integrating, as in the null case, the equation for $\partial_u m$ from $\mathcal{C}_{u_0}$ (see eq. (1.4.11)), we obtain

$$|m(u, v)| \leq M e^{\frac{\tilde{C}}{(2p-3)d_\nu R}} |u|^{-2p+3} + \frac{1}{2} \frac{\tilde{C}}{(2p-3)d_\nu} |u|^{-2p+3} e^{2 \frac{\tilde{C}}{(2p-3)d_\nu R}} |u|^{-2p+3}. \quad (1.5.38)$$

The above expression is strictly less than ηM for $|U_0|$ large enough; this improves the bootstrap assumption (BS(1)). Notice that the second term in the above expression is strictly smaller than the first one for large enough $|u|$. Therefore, by considering again (1.4.11), we also get that m is positive, say, $m > \frac{M}{2}$. By the monotonicity properties of m (namely, $\partial_u m \leq 0$), we thus conclude that $\frac{M}{2} \leq m \leq M$ (in fact, we have established that $m - M = \mathcal{O}(|u|^{-2p+3})$). Therefore, we shall henceforth assume that $|m| \leq M$.

We now use (BS(2)) to improve the lower bound on λ : Inserting (BS(2)) into the integration of eq. (1.3.20), we get

$$\kappa = 1 - \delta(u) > 0, \quad (1.5.39)$$

where $\delta(u) \sim 1 - e^{-\frac{1}{|u|^{2p-3}}}$ tends to 0 as $|u|$ tends to infinity. In particular, we now get the lower bound:

$$\lambda = (1 - \delta(u)) \left(1 - \frac{2M}{r}\right). \quad (1.5.40)$$

We finally invoke our bootstrap assumption (BS(3)) for the $|u|$ -decay for $r\phi$. From the wave equation $\partial_u \partial_v(r\phi) = 2m\nu\kappa \frac{r\phi}{r^3}$, we get, by integrating in u from $\mathcal{C}_{u_0}$:

$$|\partial_v(r\phi)(u, v)| \leq MC' \frac{|u|^{-p+1}}{r^2}. \quad (1.5.41)$$

In turn, integrating the above in v from Γ , we then get, plugging in the lower bound (1.5.40) to substitute v -integration with r -integration,

$$\begin{aligned} |r\phi(u, v)| &\leq \frac{C_{\text{in},\phi}}{|u|^{p-1}} + \frac{MC'}{|u|^{p-1}} \int_{r(u,u)}^{r(u,v)} \frac{1}{r(r-2M)(1-\delta(U_0))} dr \\ &\leq \frac{1}{|u|^{p-1}} \left(C_{\text{in},\phi} + \frac{C'}{2(1-\delta(U_0))} \log \left(\frac{r(u,u)}{r(u,v)} \cdot \frac{r(u,v)-2M}{r(u,u)-2M} \right) \right) \\ &\leq \frac{1}{|u|^{p-1}} \left(C_{\text{in},\phi} + \frac{C'}{2(1-\delta(U_0))} \log \left(\frac{R}{R-2M} \right) \right) < \frac{C'}{|u|^{p-1}}. \end{aligned} \quad (1.5.42)$$

The condition that this last term be less than $\frac{C'}{|u|^{p-1}}$ leads to the following lower bound on R :

$$1 - \frac{1}{2(1-\delta(U_0))} \log \frac{R}{R-2M} > 0 \implies R > \frac{2M}{1 - e^{-2(1-\delta(U_0))}}. \quad (1.5.43)$$

This closes the bootstrap assumption (BS(3)) for $r\phi$. Since $\eta' > 1$ was arbitrary, we can take the limit $\eta' \rightarrow 1$.

Finally, we use this decay in the scalar field to improve the bootstrap bound on ζ . This will essentially come from the wave equation: First, note that, on Γ ,

$$|\partial_u(r\phi)|(u, u) \leq |\partial_v(r\phi)|(u, u) + |\mathbf{T}(\hat{r}\hat{\phi})|(u) \leq |u|^{-p+1} \left(\frac{C_{\text{in},\phi}^1}{|u|} + \frac{MC'}{R^2} \right),$$

where the second inequality comes from the bound (1.5.41). We can now integrate the wave equation from Γ to obtain

$$|\partial_u(r\phi)(u, v)| \leq \left(\left(\frac{C_{\text{in},\phi}^1}{|u|} + \frac{MC'}{R^2} \right) + \frac{MC_\nu C'}{d_\mu R^2} \right) |u|^{-p+1}.$$

We thus get, for some constant C'' independent of $\tilde{C}$,

$$|\zeta(u, v)| \leq |\partial_u(r\phi)(u, v)| + |(\nu\phi)(u, v)| \leq C'' |u|^{-p+1},$$

improving (BS(2)) for large enough $\tilde{C}$, hence closing the bootstrap argument for ζ .

This shows that the set Δ is open and, thus, concludes the proof. $\square$

Remark 1.5.2. The reason why the above proof only works for $p > 3/2$ is that, with the method presented, we cannot show sharp decay for $\partial_u(r\phi)$ (just integrating the wave equation in v will always pick up the bad boundary term on Γ , and the decay shown for $\partial_v(r\phi)$ is sharp). This in turn means that we can only close the bootstrap assumption for m for $p > 3/2$, see the bound (1.5.38). We will explain how to deal with this issue later in section 1.5.7, where all the above bounds are made sharp. The reader interested in this may wish to skip to section 1.5.7 directly.

Remark 1.5.3. The lower bound on R is, in fact, wasteful, as one can already see from the fact that we did not explicitly use the decay for $r\phi$ inside the relevant integrals in (1.5.42). Alternatively, one can do a Grönwall argument as follows: We have

$$\begin{aligned} |r\phi|(u, v) &\leq \frac{C_{\text{in},\phi}}{|u|^{p-1}} + \int_u^v \int_{u_0}^u \left| 2m\nu\kappa \frac{r\phi}{r^3} \right| (u', v') \, du' \, dv' \\ &\leq \frac{C_{\text{in},\phi}}{|u|^{p-1}} + \int_{u_0}^u M \frac{\sup_{v' \in [u, v]} |r\phi|(u', v')}{(1 - \mathcal{O}(|u|^{-\epsilon}))r(r - 2M)(u', u)} \, du', \end{aligned} \quad (1.5.44)$$

where, in the second line, we applied Tonelli and then used that $\nu = 1 - 2M/r + \mathcal{O}(|u|^{-\epsilon})$ for some $\epsilon > 0$, which we will prove in section 1.5.7.1 (see (1.5.97)). Taking the supremum $\sup_{v' \in [u, v]}$ on the LHS of (1.5.44) and then applying Grönwall's inequality to this yields

$$\sup_{v' \in [u, v]} |r\phi|(u, v') \leq \frac{C_{\text{in},\phi}}{|u|^{p-1}} \left(\sqrt{\frac{R}{R - 2M}} + \mathcal{O}(|u|^{-\epsilon}) \right). \quad (1.5.45)$$

In other words, we can replace the bootstrap argument for (BS(3)) by a direct Grönwall argument, and this only requires the lower bound $R > 2M$. In particular, one can obtain an *a priori estimate* for $r\phi$ (without needing to assume compactness) provided that m is bounded. For now, however, we will continue working with the lower bound $R \geq \frac{2M}{1 - e^{-2+\delta(U_0)}}$.

The above theorem (when also taking into account the bound on $\partial_u(r\phi)$ shown in the proof) in particular allows us to increase the region of existence of the solution to

$$\mathcal{D}_{u_0, U_0} := \{(u, v) \in \mathcal{D}_U \mid u \in [u_0, U_0), v \in [u, \infty)\}, \quad (1.5.46)$$

i.e. a region extending towards a part of $\mathcal{I}^+$. In fact, we have:

Theorem 1.5.2. *Under the same assumptions as in Theorem 1.5.1, the resulting solution exists (and satisfies the bounds of Theorem 1.5.1) in all of $\mathcal{D}_U \cap \{u_0 \leq u \leq U_0\}$, and can be smoothly*

extended to all $u \leq u_0$ by the vacuum solution with mass M that satisfies $\partial_v r = 1 - \frac{2M}{r}$ on $\Gamma \cap \{u \leq u_0\}$.

Proof. The existence of the solution in the region $\mathcal{D}_{u_0, U_0}$ follows by continuously applying local existence theory (now also for the characteristic initial value problem) combined with the uniform bounds from Theorem 1.5.1. Moreover, in view of Birkhoff's theorem, we can smoothly extend to $u \leq u_0$ with the mass- M -Schwarzschild solution that satisfies $\partial_v r = 1 - \frac{2M}{r}$ on $\Gamma \cap \{u \leq u_0\}$ (see Figure 1.7).³⁴ To be more concrete, one can compute $\partial_u \left(\partial_v r - \left(1 - \frac{2M}{r}\right) \right) = \frac{2M\nu}{r^2(1-\mu)} \left(\partial_v r - \left(1 - \frac{2M}{r}\right) \right)$ and apply Grönwall's inequality combined with (1.5.24) to see that $\partial_v r = 1 - \frac{2M}{r}$ for all $v \geq u_0 \geq u$. Similarly, if one imposes that $\partial_v r = 1 - \frac{2M}{r}$ on $\Gamma \cap \{u \leq u_0\}$, one can apply Grönwall's inequality to $\partial_v \left(\partial_u r - \left(\mathbf{T}\hat{r} - \left(1 - \frac{2M}{r}\right)\right) \right)$ to show that $\partial_u r - \left(\mathbf{T}\hat{r} - \left(1 - \frac{2M}{r}\right)\right) = 0$ for all $u \leq u_0$. Combining these two facts uniquely determines the vacuum solution, which we shall henceforth refer to as $(r_0, 0, M)$. In particular, this solution is independent of u_0 in the sense that, for any two different values of u_0 , say $u_{0,a} < u_{0,b}$, the two arising solutions $(r_0, 0, M)$ are identical for $u \leq u_{0,a}$. $\square$

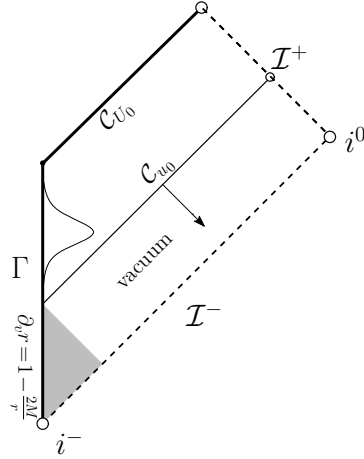


Figure 1.7 Extension of the finite solution towards $\mathcal{I}^-$ via the vacuum Schwarzschild solution. The finite solution arises from trivial data on $\mathcal{C}_{u_0}$ and boundary data on Γ that are compactly supported towards the future of $\mathcal{C}_{u_0}$ as depicted. We shaded the region that is uniquely determined by the specification of $\partial_v r = 1 - \frac{2M}{r}$ on $\Gamma \cap \{u \leq u_0\}$.

Remark 1.5.4. The resulting Schwarzschild solution $(r_0, 0, M)$ for $u \leq u_0$ can be related to the Schwarzschild solution $\mathcal{M}_M$ in double null Eddington–Finkelstein coordinates $(\tilde{u}, v)$ of section 1.3.2 by redefining the u -coordinate such that $u(\tilde{u}) = v$ on a past-complete timelike curve $\Gamma \subset \mathcal{M}_M$ on which the area radius function coincides with $\hat{r}$.

We have thus generated a family of semi-global solutions coming from compactly supported and decaying boundary data on a timelike curve Γ , equipped with bounds independent of the support of the data. We will remove the assumption of compact support in the next section by sending u_0 to $-\infty$.

³⁴This may seem a bit confusing at first: We are specifying the full data (both tangential and normal derivatives) on $\Gamma \cap \{u \leq u_0\}$, which is a timelike hypersurface! The reason that this works is that, in spherical symmetry, i.e. in $1+1$ dimensions, time and space are on the same footing. Of course, one does not have to exploit this and could, instead, “anchor” $\partial_v r$ on $\mathcal{I}^-$.

1.5.6 The limiting problem: Sending $\mathcal{C}_{u_0}$ to $\mathcal{I}^-$

In the previous proof, we crucially needed to exploit compactness to show the non-emptiness parts of the bootstrap arguments. We now remove this restriction to compact regions.

We shall follow a direct limiting argument: Given *non-compactly supported* boundary data, we construct a sequence of solutions with *compactly supported boundary data* that will tend to a unique limiting solution such that this limiting solution will restrict correctly to the initially prescribed non-compactly supported boundary data. The advantage of this approach over, say, a density argument, is that it will also provide us with a uniqueness statement as we will explain in Remark 1.5.7.

We will first state the “data assumptions” that the limiting solution is required to satisfy and, after that, write down the sequence of finite solutions. The remainder of the subsection will then contain the detailed analysis of the convergence of this sequence.

The “final” boundary data

Let $M > 0$. We restrict to sufficiently large negative values of $u \leq U_0 < 0$ and specify boundary data $(\hat{r}, \hat{\phi})$ on $\Gamma = \partial\mathcal{D}_U \cap \{u \leq U_0\}$ as follows: The datum $\hat{r} \in C^2(\Gamma)$ is to satisfy:³⁵

$$|\mathbf{T}\hat{r}(u)| \leq C_{\text{in},r}|u|^{-s}, \quad (1.5.47)$$

$$\hat{r} \geq R \geq \frac{2M}{1 - e^{-2+\delta(U_0)}} \quad (1.5.48)$$

for some positive constant $C_{\text{in},r}$, where $\delta(U_0) \sim |U_0|^{-2p+3}$ appeared before in Thm. 1.5.1. On the other hand, $\hat{\phi} \in C^2(\Gamma)$ is chosen to obey $\lim_{u \rightarrow -\infty} \hat{r}\hat{\phi}(u) = 0$ and

$$|\mathbf{T}(\hat{r}\hat{\phi})(u)| \leq C_{\text{in},\phi}^1 |u|^{-p} \quad (1.5.49)$$

with $p > 3/2$ and some constant $C_{\text{in},\phi}^1 > 0$.

The sequence of finite solutions (r_k, ϕ_k, m_k)

We finally prescribe a sequence of initial/boundary data as follows: In a slight abuse of notation, let, for $k \in \mathbb{N}$, $\mathcal{C}_k = \{u = -k, v \geq u\}$ denote an outgoing null ray emanating from Γ , and let $(\chi_k)_{k \in \mathbb{N}}$ denote a sequence of smooth cut-off functions on Γ (that are translates of each other),

$$\chi_k = \begin{cases} 1, & u \geq -k + 1, \\ 0, & u \leq -k. \end{cases} \quad (1.5.50)$$

³⁵We remind the reader that this lower bound on R can be replaced by $R > 2M$ in view of Remark 1.5.3.

On $\mathcal{C}_k$, we denote by $\bar{r}_k$ the solution to the ODE

$$\partial_v(\bar{r}_k) = 1 - \frac{2M}{\bar{r}_k} \quad (1.5.51)$$

with initial condition given by $\hat{r}(\Gamma \cap \mathcal{C}_k)$. Then our sequence of initial data is given by:

$$(I.D.)_k = \begin{cases} \hat{r}_k = \hat{r}, \hat{\phi}_k = \chi_k \hat{\phi} & \text{on } \Gamma, \\ \bar{r}_k, \bar{\phi}_k = 0, m = M & \text{on } \mathcal{C}_k. \end{cases} \quad (1.5.52)$$

This sequence of data leads, by Theorems 1.5.1 and 1.5.2 (for sufficiently large values of U_0), to a sequence of solutions (r_k, ϕ_k, m_k) , which we can smoothly extend (by Thm. 1.5.2) with the background mass- M -Schwarzschild solution $(r_0, 0, M)$ that satisfies $\partial_v r_0 = 1 - \frac{2M}{r_0}$ on Γ for $u \leq -k$. In particular, these solutions all obey the same bounds (uniformly in k) from Theorem 1.5.1 in the region where they are non-trivial – most notably, they obey the upper bound (1.5.34).

In the sequel, we will always mean this extended solution when referring to (r_k, ϕ_k, m_k) . We will now show that this sequence tends to a limiting solution, and that this limiting solution still obeys the bounds from Thm. 1.5.1 and, moreover, restricts correctly to the non-compactly supported “final” boundary data $\hat{r}, \hat{\phi}$ and vanishes on $\mathcal{I}^-$.

Before we state the theorem, let us recall the notation

$$\mathcal{D}_{U_0} := \{(u, v) \in \mathbb{R}^2 \mid u \in (-\infty, U_0], v \in [u, \infty)\}. \quad (1.5.53)$$

Theorem 1.5.3. *Let (r_k, ϕ_k, m_k) denote the sequence of solutions constructed above. Let $p \geq 2$, and let $U_0 < 0$ be sufficiently large. In the case $p = 2$, assume in addition that $C_{\text{in}, \phi}$ is sufficiently small.³⁶*

Then, as $k \rightarrow \infty$, the sequence (r_k, ϕ_k, m_k) uniformly converges to a limit (r, ϕ, m) ,

$$\|r_k \phi_k - r \phi\|_{C^1(\mathcal{D}_{U_0})} + \|r_k - r\|_{C^1(\mathcal{D}_{U_0})} + \|m_k - m\|_{C^1(\mathcal{D}_{U_0})} \rightarrow 0. \quad (1.5.54)$$

This limit is also a solution to the spherically symmetric Einstein-Scalar field equations. Moreover, (r, ϕ, m) restricts correctly to the boundary data $(\hat{r}, \hat{\phi})$ and satisfies, for all v ,

$$\lim_{u \rightarrow -\infty} r \phi(u, v) = \lim_{u \rightarrow -\infty} \partial_v m(u, v) = \lim_{u \rightarrow -\infty} \partial_v (r \phi)(u, v) = 0 \quad (1.5.55)$$

as well as

$$\lim_{u \rightarrow -\infty} m(u, v) = \lim_{u \rightarrow -\infty} m(u, u) = M. \quad (1.5.56)$$

³⁶The precise smallness conditions depends on both the ratios M/R and $C_{\text{in}, \phi}/R$. It is of no importance, however, since we will remove this assumption, as well as the assumption that $p \geq 2$, in Thm. 1.5.6. (These are the restrictive assumptions mentioned at the beginning of this section.)

Furthermore, (r, ϕ, m) , as well as the quantities λ, ν, κ , satisfy the same bounds as those derived in Theorem 1.5.1, i.e., we have throughout all of $\mathcal{D}_{U_0}$.³⁷

$$0 < \frac{M}{2} \leq m \leq M, \quad (1.5.57)$$

$$0 < 1 - \frac{2M}{r} = 1 - \mu \leq 1, \quad (1.5.58)$$

$$0 < 1 - \delta(u) \leq \kappa \leq 1, \quad (1.5.59)$$

$$0 < (1 - \delta(u)) \left(1 - \frac{2M}{r}\right) = \lambda \leq 1, \quad (1.5.60)$$

$$0 < d_\nu = (1 - 2M/R)e^{-\frac{2M}{R-2M}} \leq |\nu| \leq e^{\frac{2M}{R-2M}} = C_\nu, \quad (1.5.61)$$

$$|r\phi| \leq \frac{1}{p-1} \frac{C_{\text{in},\phi}^1 |u|^{-p+1}}{1 - \frac{1}{2(1-\delta(U_0))} \log \frac{R}{R-2M}} = C' |u|^{-p+1}, \quad (1.5.62)$$

$$|\partial_v(r\phi)| \leq MC' \frac{|u|^{-p+1}}{r^2}. \quad (1.5.63)$$

If we moreover assume that there exists a positive constant $d_{\text{in},\phi}^1 \leq C_{\text{in},\phi}^1$ such that

$$|\mathbf{T}(\hat{r}\hat{\phi})(u)| \geq d_{\text{in},\phi}^1 |u|^{-p}, \quad (1.5.64)$$

then, depending on the value of $C_{\text{in},\phi}^1 - d_{\text{in},\phi}^1$, and if R is large enough, there exists a positive constant d' depending only on data such that, throughout $\mathcal{D}_{U_0}$,

$$|r\phi| \geq d' |u|^{-p+1}, \quad (1.5.65)$$

and furthermore, C' can be replaced by $C_{\text{in},\phi}^1/(p-1)$ in the estimates (1.5.62), (1.5.63). In the special case where $d_{\text{in},\phi}^1 = C_{\text{in},\phi}^1$, the condition that R be large enough is given by

$$R > \frac{2M}{1 - e^{-1+\delta(U_0)}}. \quad (1.5.66)$$

Note that, from the proof of this theorem, one can, *a fortiori*, derive Proposition 1.4.1 from the previous section.

Proof of Theorem 1.5.3. We focus on the case $p = 2$; the cases $p > 2$ will follow *a fortiori*. We will show that the sequence $(r_k, r_k \phi_k, m_k)$ is a Cauchy sequence in the $C^1(\mathcal{D}_{U_0}) \times C^1(\mathcal{D}_{U_0}) \times C^0(\mathcal{D}_{U_0})$ -norm: Let $\epsilon > 0$. We want to show that there exists an $N(\epsilon)$ (to be specified later) such that

$$\|(r\phi)_n - (r\phi)_k\|_{C^1(\mathcal{D}_{U_0})} + \|r_n - r_k\|_{C^1(\mathcal{D}_{U_0})} + \|m_n - m_k\|_{C^0(\mathcal{D}_{U_0})} < \epsilon \quad (1.5.67)$$

³⁷Recall the definition of $\delta(u) \sim |u|^{-2p+3}$ from Theorem 1.5.1.

for all $n > k > N(\epsilon)$.

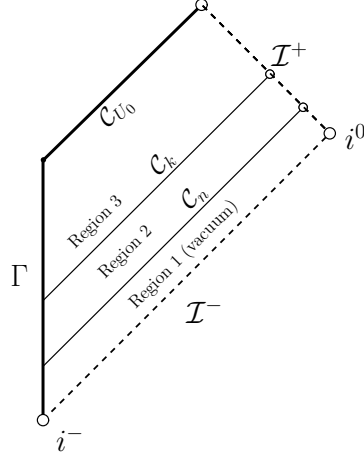


Figure 1.8 Depiction of the three regions in which we subdivide. In Region 1, both solutions are vacuum. In Region 2, both solutions can be shown to have small matter content for sufficiently large k . Estimating the r -difference in Region 2, and estimating all differences in Region 3, however, requires more work.

We will show this by splitting up into three regions (see Figure 1.8 above).

Note that, from now on, we will replace most uniform constants simply by C and adopt the usual algebra of constants ($C + D = CE = C = \dots$).

Region 1: For $u \leq -n$, the solutions (r_n, ϕ_n, m_n) , (r_k, ϕ_k, m_k) both agree with the vacuum solution $(r_0, 0, M)$, so the difference vanishes. See also the argument of Thm. 1.5.2.

Region 2: For $-n \leq u \leq -k$ (let's call this region $\mathcal{D}_{n,k}$), we obtain, in view of the decay of $|r\phi| \leq \frac{C}{|u|}$ and the related bounds for $\partial_u(r\phi)$ and $\partial_v(r\phi)$:

$$\|(r\phi)_n - (r\phi)_k\|_{C^1(\mathcal{D}_{n,k})} \leq \frac{C}{k}. \quad (1.5.68)$$

For the m -difference, recall that $|\zeta| \leq \frac{C}{|u|}$, so we can just integrate $\partial_u m \sim -\zeta^2$ from $\mathcal{C}_n$, where the difference $m_n - m_k$ is 0:

$$\|m_n - m_k\|_{C^0(\mathcal{D}_{n,k})} \leq \frac{C}{k}. \quad (1.5.69)$$

Controlling the r -difference, on the other hand, turns out to be more tricky: First, consider the κ -difference by integrating $\partial_u \kappa \sim -\frac{\zeta^2}{r} \kappa$ from $\mathcal{C}_n$ (where the difference $\kappa_n - \kappa_k$ is again 0):

$$\|\kappa_n - \kappa_k\|_{C^0(\mathcal{D}_{n,k})} \leq \frac{C}{k}. \quad (1.5.70)$$

For the other terms appearing in the r -difference, we will want to appeal to a Grönwall-type argument. Let $(u, v) \in \mathcal{D}_{n,k}$. Then we have, by the fundamental theorem of calculus:³⁸

$$\begin{aligned} & |r_n(u, v) - r_k(u, v)| + |\lambda_n(u, v) - \lambda_k(u, v)| + |\nu_n(u, v) - \nu_k(u, v)| \\ & \leq \left| \int_u^v \int_{-n}^u \partial_u \partial_v (r_n - r_k)(u', v') du' dv' \right| + \left| \int_u^v \partial_u \partial_v (r_n - r_k)(u, v') dv' \right| \\ & + \left| \int_{-n}^u \partial_u \partial_v (r_n - r_k)(u', v) du' \right| + \left| \int_{-n}^u \partial_u \partial_v (r_n - r_k)(u', u) du' \right|, \end{aligned} \quad (1.5.71)$$

where the fourth integral on the RHS comes from the difference $\nu_n - \nu_k$ on Γ , which we estimate by using that

$$|\nu_n(u, u) - \nu_m(u, u)| = |\lambda_n(u, u) - \lambda_m(u, u)|$$

and then integrating $\partial_u \lambda$ in u from $\mathcal{C}_n$ to estimate the right-hand side $\lambda_n(u, u) - \lambda_m(u, u)$. Otherwise, there are no boundary terms since r_n and r_k coincide on Γ and $\mathcal{C}_n$.

Our strategy shall now be to estimate the RHS of (1.5.71) against integrals over products of integrable functions and r -differences so that we can apply a Grönwall argument to it. Let's first focus on the double integral on the RHS of (1.5.71) (which controls $|r_n - r_k|$). By eq. (1.3.23), we have

$$\begin{aligned} \frac{1}{2} \partial_u \partial_v (r_n - r_k) &= (m_n - m_k) \left(\frac{\nu \kappa}{r^2} \right)_n + m_k \nu_k \frac{\kappa_n - \kappa_k}{r_n^2} \\ &+ m_k (\nu_n - \nu_k) \left(\frac{\kappa}{r^2} \right)_n + m_k \nu_k \frac{\kappa_k (r_n + r_k)}{r_n^2 r_k^2} (r_k - r_n). \end{aligned} \quad (1.5.72)$$

Notice already that, while the first two terms in the equation above are integrable, the $(\nu_n - \nu_k)$ -term in (1.5.72) comes with a non-integrable factor of $1/r^2$. We can, however, write³⁹

$$\begin{aligned} & \left| \int_u^v \int_{-n}^u \partial_u \partial_v (r_n - r_k)(u', v') du' dv' \right| \\ & \lesssim \left| \int_u^v \int_{-n}^u \frac{|m_n - m_k|}{r_n^2} + \frac{|\kappa_n - \kappa_k|}{r_n^2} du' dv' \right| + \left| \int_u^v \int_{-n}^u \frac{r_n + r_k}{r_n^2 r_k^2} |r_k - r_n| du' dv' \right| \\ & + \left| \int_u^v \int_{-n}^u \partial_u \left(\frac{r_n - r_k}{r_n^2} \kappa_n m_k \right) + \left(\frac{\partial_u r_n}{2r_n^3} \kappa_n m_k - \frac{\partial_u (\kappa_n m_k)}{r_n^2} \right) (r_n - r_k) du' dv' \right|, \end{aligned} \quad (1.5.73)$$

where we dealt with the bad $(\nu_n - \nu_k)$ -term by using the Leibniz rule.

For the first integral in this expression, we use the fact that⁴⁰ $\Delta \kappa + \Delta m \leq \frac{C}{|u|}$. In the region where $r \lesssim |u|$, i.e. away from $\mathcal{I}^+$, we can convert some of this $|u|$ -decay, say $|u|^{-\eta}$ for some $\eta > 0$, into r -decay so that the integral becomes integrable in r . Similarly, near null infinity, we

³⁸Strictly speaking, we already control the λ -difference, so we don't need to include it here.

³⁹Here, and in what follows, the constant hidden inside $\lesssim$ depends only on $M, C_{\text{in},\phi}^1$ and R .

⁴⁰We shall write differences $f_n - f_k$ as Δf .

can convert the r -decay into $|u|$ -decay. Thus, the first integral can be bounded by $Ck^{-1+\eta}$. It will play the role of the function multiplying the exponential in the Grönwall argument.

The second integral of (1.5.73) consists of an integrable function $\sim r^{-3}$ multiplied by Δr and, thus, is of the form we want (this term will appear inside the exponential term in the Grönwall lemma, along with the other remaining terms).

For the third integral of (1.5.73), the first term in it can be integrated using the fundamental theorem of calculus and gives, recalling that $\Delta r(u = -n) = 0$,

$$\left| \int_u^v \left(\frac{r_n - r_k}{r_n^2} \kappa_n m_k \right) (u, v') dv' \right| \lesssim \int_u^v \left(\frac{|r_n - r_k|}{r_n^2} \right) (u, v') dv', \quad (1.5.74)$$

so this integrand can also be written as a product of Δr and an integrable function. On the other hand, the second term in the third integral of (1.5.73) is, again, a product of Δr and an integrable function that goes like $r^{-3} + r^{-2}|u|^{-2}$ and, thus, can be dealt with in the same way as the second integral.

To summarise, we can estimate (1.5.73) (we also exchange the order of integration using Tonelli) via

$$\begin{aligned} |r_n - r_k|(u, v) &\leq \left| \int_u^v \int_{-n}^u \partial_u \partial_v (r_n - r_k)(u', v') du' dv' \right| \\ &\leq \frac{C}{k^{1-\eta}} + C \int_u^v \left(\frac{|r_n - r_k|}{r_n^2} \right) (u, v') dv' + C \int_{-n}^u \int_u^v f(u', v') |r_k - r_n|(u', v') dv' du' \\ &\leq \frac{C}{k^{1-\eta}} + C \int_{-n}^u r_n^{-2} |r_k - r_n|(u, v') dv' + C \int_{-n}^u \sup_{v' \in [u, v]} |r_k - r_n|(u', v') \int_u^v f(u', v') dv' du', \end{aligned} \quad (1.5.75)$$

where $f \geq 0$ is a positive, (doubly) integrable⁴¹ function obeying $f(u, v) \leq \frac{C}{r_n^3} + \frac{C}{r_n r_k^2} + \frac{C}{r_k r_n^2} + \frac{C}{r_n^2 |u|^2}$. Rewriting, we thus have

$$|\Delta r|(u, v) \leq \frac{C}{k^{1-\eta}} + C \int_u^v r_n^{-2} |\Delta r|(u, v') dv' + C \int_{-n}^u F(u') \sup_{v' \in [u, v]} |\Delta r|(u', v') du' \quad (1.5.76)$$

for some positive $F(u')$ which obeys, for all $u' \leq u$,

$$F(u') \leq \frac{C}{r_n^2(u', u)} + \frac{C}{r_n(u', u) r_k(u', u)} + \frac{C}{r_n(u', u) |u'|^2}.$$

Now, if we consider (1.5.76) for fixed u , and note that $\sup_{v' \in [u, v]} |r_k - r_n|(u', v')$ is non-decreasing in v , we obtain, by an application of Grönwall's inequality, that, for each u ,

$$|\Delta r|(u, v) \leq \left(\frac{C}{k^{1-\eta}} + C \int_{-n}^u F(u') \sup_{v' \in [u, v]} |\Delta r|(u', v') du' \right) \cdot e^{C \int_u^v r_n^{-2}(u, v') dv'}. \quad (1.5.77)$$

⁴¹Indeed, we have $\int \int \frac{1}{r_n r_k^2} dv du \lesssim \int \frac{1}{r_n r_k} du \leq \left(\int \frac{1}{r_n^2} du \right)^{\frac{1}{2}} \left(\int \frac{1}{r_k^2} du \right)^{\frac{1}{2}}$ by Cauchy-Schwarz.

The integral in the exponential can be bounded uniformly in u against $e^{C/R}$, and we thus obtain

$$|\Delta r|(u, v) \leq \frac{C}{k^{1-\eta}} + C \int_{-n}^u F(u') \sup_{v' \in [u, v]} |\Delta r|(u', v') du'. \quad (1.5.78)$$

Taking now the supremum in $v' \in [u, v]$ on the LHS, and applying another Grönwall estimate, we then obtain

$$\sup_{v' \in [u, v]} |\Delta r|(u, v') \leq \frac{C}{k^{1-\eta}}. \quad (1.5.79)$$

Notice that it was crucial that we took out the supremum in v in (1.5.75) – taking the supremum in u wouldn't work because the rectangle over which we integrate, namely $-n \leq u' \leq u \leq v' \leq v$, increases in v -length as u approaches $-n$.

Let's now move to the other three terms in (1.5.71). All of them can be dealt with in a simpler way than the first term. Consider, for example, the second term of (1.5.71):

$$\begin{aligned} & \left| \int_u^v \partial_u \partial_v (r_n - r_k)(u, v') dv' \right| \\ & \leq C \int_u^v \frac{|m_n - m_k| + |\kappa_n - \kappa_k|}{r_n^2} (u, v') dv' \\ & + C \int_u^v \left(\frac{1}{r_n^2 r_k} + \frac{1}{r_k r_n^2} \right) (|r_n - r_k| + |\nu_n - \nu_k|)(u, v') dv' \\ & \leq \frac{C}{k} + C \int_u^v \left(\frac{1}{r_n^2 r_k} + \frac{1}{r_k r_n^2} \right) (|r_n - r_k| + |\nu_n - \nu_k|)(u, v') dv'. \end{aligned} \quad (1.5.80)$$

The last two terms in (1.5.71) can be estimated similarly (after also taking the supremum $\sup_{v' \in [u, v]}$). One thus obtains the inequality (1.5.76), but with $|\Delta r|$ replaced by $|\Delta r| + |\Delta \nu| + |\Delta \lambda|$. From this, we conclude that

$$\|r_n - r_k\|_{C^1(\mathcal{D}_{n,k})} \leq \frac{C}{k^{1-\eta}} \leq \frac{\epsilon}{2} \quad (1.5.81)$$

for some $\eta > 0$ that can be chosen arbitrarily small. The last inequality holds if $N(\epsilon)$ is chosen accordingly. Similarly, we can make each of the RHS's of estimates (1.5.68) and (1.5.69) smaller than $\epsilon/4$ by choosing $N(\epsilon)$ sufficiently large. We thus conclude that

$$\|(r\phi)_n - (r\phi)_k\|_{C^1(\mathcal{D}_{n,k})} + \|r_n - r_k\|_{C^1(\mathcal{D}_{n,k})} + \|m_n - m_k\|_{C^0(\mathcal{D}_{n,k})} \leq \epsilon. \quad (1.5.82)$$

Region 3: Finally, we consider the region $-k \leq -u \leq -U_0$ (which we shall call $\mathcal{D}_k$). We will again want to perform a Grönwall argument. This time however, we will need to include the m - and the $r\phi$ -differences in the Grönwall estimate as well, since we can no longer appeal to smallness in $\frac{1}{k}$ to estimate them directly.

First, we write down estimates for the differences on the boundary Γ and on $\mathcal{C}_k$. On the boundary Γ , we have, schematically:

$$\begin{aligned}
|\partial_u r_n - \partial_u r_k|_{C^0(\Gamma)} &\leq |\partial_v r_n - \partial_v r_k|_{C^0(\mathcal{C}_k)} + \int \partial_u \partial_v (\Delta r) \, du, \\
|\partial_u (r\phi)_n - \partial_u (r\phi)_k|_{C^0(\Gamma)} &\leq |\mathbf{T}(r\phi)_n - \mathbf{T}(r\phi)_k|_{C^0(\Gamma)} \\
&\quad + |\partial_v (r\phi)_n - \partial_v (r\phi)_k|_{C^0(\mathcal{C}_k)} + \int \partial_u \partial_v (\Delta(r\phi)) \, du, \\
|(r\phi)_n - (r\phi)_k|_{C^1(\Gamma)} &\leq \frac{C}{k},
\end{aligned} \tag{1.5.83}$$

whereas, on $\mathcal{C}_k$, we have, by the previous result in region 2, that

$$\|(r\phi)_n - (r\phi)_k\|_{C^1(\mathcal{C}_k)} + \|r_n - r_k\|_{C^1(\mathcal{C}_k)} + \|m_n - m_k\|_{C^0(\mathcal{C}_k)} \leq \frac{C}{k^{1-\eta}}. \tag{1.5.84}$$

Moving now on to the Grönwall estimate, we write, using the above estimates for the initial/boundary differences,

$$\begin{aligned}
&(|(r\phi)_n - (r\phi)_k| + |\partial_u (r\phi)_n - \partial_u (r\phi)_k| + |\partial_v (r\phi)_n - \partial_v (r\phi)_k| \\
&\quad + |r_n - r_k| + |\partial_v r_n - \partial_v r_k| + |\partial_u r_n - \partial_u r_k| + \mathbf{D} \cdot |m_n - m_k|)(u, v) \\
&\leq \frac{C}{k^{1-\eta}} + \left| \int_{-m}^u \partial_u \partial_v ((r\phi)_n - (r\phi)_k)(u', u) \, du' \right| + \left| \int_{-m}^u \partial_u \partial_v (r_n - r_k)(u', u) \, du' \right| \\
&\quad + \underbrace{\left| \int_u^v \int_{-k}^u \partial_u \partial_v ((r\phi)_n - (r\phi)_k)(u', v') \, du' \, dv' \right|}_{(1)} \\
&\quad + \left| \int_u^v \partial_u \partial_v ((r\phi)_n - (r\phi)_k)(u, v') \, dv' \right| + \left| \int_{-k}^u \partial_u \partial_v ((r\phi)_n - (r\phi)_k)(u', v) \, du' \right| \\
&\quad + \underbrace{\left| \int_u^v \int_{-k}^u \partial_u \partial_v (r_n - r_k)(u', v') \, du' \, dv' \right|}_{(2)} \\
&\quad + \left| \int_u^v \partial_u \partial_v (r_n - r_k)(u, v') \, dv' \right| + \left| \int_{-k}^u \partial_u \partial_v (r_n - r_k)(u', v) \, du' \right| \\
&\quad + \underbrace{\mathbf{D} \left| \int_{-k}^u \partial_u (m_n - m_k)(u', v) \, du' \right|}_{(3)}.
\end{aligned} \tag{1.5.85}$$

Here, we introduced a positive constant $\mathbf{D} > 0$. Its relevance will become clear later. Note that, once we control the integrals (1), (2), we can, a fortiori, also control all the other integrals except for (3).

In order to estimate the underlined term (1), observe that, schematically (omitting the indices n, k),

$$\frac{1}{2}\Delta(\partial_u\partial_v(r\phi)) = \Delta\left(m\frac{\nu\lambda}{1-\mu}\right)\frac{r\phi}{r^3} + m\frac{\nu\lambda}{1-\mu}\frac{\Delta(r\phi)}{r^3} + m\frac{\nu\lambda}{1-\mu}\frac{r\phi}{r^4} \cdot 3\Delta r, \quad (1.5.86)$$

so (1) does not pose a problem (as all of the terms multiplying the Δ -differences are integrable).

Next, we look at the underlined term (2). Again, we have that, schematically,

$$\begin{aligned} \frac{1}{2}\Delta(\partial_u\partial_v r) &= \Delta\nu\frac{m\lambda}{1-\mu}\frac{1}{r^2} + \Delta\lambda\frac{m\nu}{1-\mu}\frac{1}{r^2} \\ &+ \frac{2\Delta r}{r^3}\frac{\nu m\lambda}{1-\mu} + \frac{\nu m\lambda}{(1-\mu)^2}\frac{1}{r^2}\left(\frac{2\Delta m}{r} + \frac{2m\Delta r}{r^2}\right) + \frac{\Delta(\mathbf{D}m)}{\mathbf{D}}\frac{\nu\lambda}{1-\mu}\frac{1}{r^2}. \end{aligned} \quad (1.5.87)$$

The first two terms on the RHS can be dealt with using an integration by parts as in (1.5.73) (after also interchanging the order of integration, using Fubini, for the $\Delta\lambda$ -term). The second two terms are, again, products of Δ -differences and integrable functions; they also pose no problem.

The last term, however, is borderline non-integrable and will lead to a log-divergence in the exponential of the Grönwall lemma. Indeed, we have (cf. Lemma 1.4.1)

$$\frac{1}{\mathbf{D}}\int_{-k}^u\int_u^v\frac{1}{r^2(u',v')}dv'du' \leq \frac{C}{\mathbf{D}}\int_{-k}^u\frac{1}{r(u',u)}du' \leq \frac{C}{\mathbf{D}}\log r(-k,u) \leq \frac{C}{\mathbf{D}}\log k.$$

However, we can deal with this divergence! The $\frac{C}{\mathbf{D}}\log k$ -term in the exponential of the Grönwall lemma will lead to a power $k^{C/\mathbf{D}}$. This is the reason for the inclusion of the constant $\mathbf{D}$: By making $\mathbf{D}$ large enough, we can ensure that the problematic term in (1.5.87) will only lead to a divergence of, say, $k^{\frac{1}{2}-\eta}$. In other words, we can ensure that it grows slower than the initial data difference ($\sim k^{-1+\eta}$) decays. This trick will come at a price, though, as we will need to absorb the largeness of $\mathbf{D}$ into the smallness of $C_{\text{in},\phi}^1$.⁴²

To see how this can be done, we now move on to the underlined term (3) in (1.5.85): We write

$$\partial_u m = \frac{1}{2}\frac{1-\mu}{\nu}\left((\partial_u(r\phi))^2 - 2\partial_u(r\phi)\frac{\nu r\phi}{r} + \left(\frac{\nu r\phi}{r}\right)^2\right). \quad (1.5.88)$$

Then, when considering the difference $\Delta(\partial_u m)$, there will again be precisely one borderline non-integrable term, with all other terms being easily controlled (recall that $\partial_u(r\phi)$, $r\phi$ are bounded by a term proportional to $\frac{C_{\text{in},\phi}^1}{|u|}$):

$$\mathbf{D} \cdot \Delta(\partial_u m) = \mathbf{D} \cdot \frac{1}{2}\frac{(1-\mu)\partial_u(r\phi)}{\nu}\Delta(\partial_u(r\phi)) + \dots \leq C\mathbf{D}\frac{C_{\text{in},\phi}^1}{|u|}\Delta(\partial_u(r\phi)) + \dots \quad (1.5.89)$$

⁴²A more careful investigation reveals that we can also absorb it into the smallness of M/R .

for some constant C , where the $\dots$ -terms denote products of integrable terms and Δ -differences. Hence, (1.5.89) will again contribute a logarithmic term to the exponential and, thus, a factor of $k^{\mathbf{CD}_{\text{in},\phi}^1}$ – but this factor can now be made small by choosing $C_{\text{in},\phi}^1$ sufficiently small.

In summary, we obtain a similar inequality to (1.5.76), with $|\Delta r|$ replaced by $|\Delta r| + |\Delta \nu| + |\Delta \lambda| + |\Delta r \phi| + |\Delta \partial_u(r \phi)| + |\Delta \partial_v(r \phi)| + |\Delta m|$, and can apply a similar Grönwall argument to finally obtain that

$$\|(r\phi)_n - (r\phi)_k\|_{C^1(\mathcal{D}_k)} + \|r_n - r_k\|_{C^1(\mathcal{D}_k)} + \|m_n - m_k\|_{C^0(\mathcal{D}_k)} \leq \frac{C}{\sqrt{k}} \leq \epsilon, \quad (1.5.90)$$

where the last inequality holds for $N(\epsilon) \geq \frac{C^2}{\epsilon^2}$.

Combining now the estimates (1.5.82) and (1.5.90) concludes the proof that the sequence (r_n, ϕ_n, m_n) is a Cauchy sequence. In particular, it converges to a limit (r, ϕ, m) in $\mathcal{D}_{U_0}$. Moreover, it not only converges in the $C^1 \times C^1 \times C^0$ -norm: By looking at e.g. the equation for $\partial_u \partial_v r$, we obtain that

$$\|\partial_u \partial_v r_n - \partial_u \partial_v r_k\|_{C^0(\mathcal{D}_{U_0})} \rightarrow 0$$

as well. Similarly, we get convergence in other higher differentiability norms – in particular, the limit is also a solution to the system of equations (1.3.18)–(1.3.22).

The uniform convergence then immediately tells us that all the pointwise bounds derived in Thm. 1.5.1 carry over to the limit.

It remains to show the lower bound (1.5.65). In order to derive it, we simply integrate $\partial_v(r\phi)$ from Γ and use the upper bound (1.5.63) for $\partial_v(r\phi)$:

$$\left| \int_u^v \partial_v(r\phi) \, dv' \right| \leq \frac{C_{\text{in},\phi}^1}{|u|} \frac{\log \frac{R}{R-2M}}{2(1-\delta(U_0))} \frac{1}{1 - \frac{1}{2(1-\delta(U_0))} \log \frac{R}{R-2M}}. \quad (1.5.91)$$

The condition that this be strictly smaller than $\frac{d_{\text{in},\phi}^1}{|u|}$ can always be satisfied for large enough R . In the case where $d_{\text{in},\phi}^1 = C_{\text{in},\phi}^1$, we obtain:⁴³

$$R > \frac{2M}{1 - e^{-1+\delta(U_0)}}.$$

Finally, if $r\phi$ has a sign, as implied by the lower bound above, then $\partial_v(r\phi)$ has the opposite sign, and, thus, C' can be replaced by $C_{\text{in},\phi}^1$ in the estimates (1.5.63), (1.5.62).

As the main difficulty was dealing with the bound for $\partial_u(r\phi) \leq |u|^{-p+1}$, which is borderline non-integrable for $p = 2$, the cases where $p > 2$ are strictly easier to deal with. This concludes the proof. $\square$

⁴³Note that if we inserted the better bound (1.5.45) here, we would obtain the marginally better lower bound $R > 2.95M$ instead. Again, this can be improved, but we do not present this here.

Remark 1.5.5. The proof above has an obvious shortcoming, as manifested in the estimate (1.5.89): Not only does it only work for $p \geq 2$ (as otherwise, the bound for $\partial_u(r\phi)$ wouldn't be integrable), but we even need to resort to some smallness condition for $C_{\text{in},\phi}^1/R$. Both of these difficulties can be traced back to the fact that we thus far have not shown sharp decay for $\partial_u(r\phi)$ (or for ζ). Conversely, if we knew that

$$|\partial_u(r\phi)| \lesssim |u|^{-p} + r^{-2}|u|^{-p+1}, \quad (1.5.92)$$

we could not only resolve the aforementioned difficulties, but the above proof would also work for all $p > 1$ (as this would also allow to show better bounds for (1.5.69), (1.5.70) and (1.5.89)). We will show this improved decay for $\partial_u(r\phi)$ in section 1.5.7, allowing us to remove both the smallness assumption *and* the restriction on p .

Remark 1.5.6. Even if one is happy with the smallness assumption (note that the smallness assumption is, in fact, not necessary in the uncoupled problem) and the restriction on p – after all, the case $p = 2$ is the one we are most interested in anyway – there is still one problem: Even though we already know at this point that $|u|r\phi(u, v)$ remains bounded from above and away from zero, we cannot yet show that it takes a limit on $\mathcal{I}^-$.⁴⁴ This means that, while we can show that $|\partial_v(r\phi)| \sim \log r/r^3$ near $\mathcal{I}^+$ by going through the same steps as in the proof of Thm. 1.4.2, we cannot write down precise asymptotics for $\partial_v(r\phi)$ yet, that is: We cannot yet say that $\partial_v(r\phi)(u, v) = C \log r/r^3 + \mathcal{O}(r^{-3})$ for some constant C . This problem will also be resolved in section 1.5.7.

Remark 1.5.7. Despite the shortcomings mentioned in Remarks 1.5.5, 1.5.6 (which we will fix anyway), the explicitness of the proof above allows one to directly extract a uniqueness statement (as claimed in Theorem 1.2.1) from it: Namely, one can extract that there exists a unique solution restricting correctly to the prescribed data on Γ and $\mathcal{I}^-$. The precise class with respect to which this uniqueness holds can be read off from the proof. Let us here only give a brief sketch of how this works:

Assume that there are two smooth solutions $(r, \phi, m)_i$, $i = 1, 2$, defined on $\mathcal{D}_U$ which satisfy, for $u \leq U_0$, and U_0 a sufficiently large negative number, the following: The corresponding geometric quantities $\lambda_i, \nu_i, 1 - \mu_i$ and m_i are uniformly comparable to 1, and the scalar fields satisfy $|\partial_u(r\phi_i)| \leq C_{\text{small}}|u|^{-1} + Cr^{-1}|u|^{-\varepsilon}$ for some $\varepsilon > 0$, some constant C , and a suitably small constant C_{small} (cf. (1.5.92)). Moreover, both solutions restrict correctly to $\hat{\phi}$, $\mathbf{T}\hat{\phi}$, $\hat{r}$ and $\mathbf{T}\hat{r}$ on Γ , and they satisfy $\lim_{\mathcal{I}^-} \partial_v r_i = 1$, as well as $\lim_{\mathcal{I}^-} m_i = M$ and $\lim_{\mathcal{I}^-} r\phi_i = \lim_{\mathcal{I}^-} \partial_v(r\phi_i) = 0$ on $\mathcal{I}^-$. Finally, $r_1 - r_2$ is a bounded quantity that tends to 0 as $u \rightarrow -\infty$.

With these assumptions, we then let $k \in \mathbb{N}$ be arbitrary and split $\mathcal{D}_{U_0}$ into the subsets $\mathcal{D}_{U_0} \cap \{u \leq -k\}$ and $\mathcal{D}_{U_0} \cap \{u > -k\}$. In the former set, we can treat the difference $\Delta(r, r\phi, m)$ of the two solutions as in the region $\mathcal{D}_{n,k}$ of the proof, leading to the estimate (1.5.82) (with ϵ

⁴⁴Unless, of course, $r|_{\Gamma} \rightarrow \infty$ as $t \rightarrow \infty$, in which case we can simply integrate $|u|\partial_v(r\phi)$ from Γ and use that $|u|r\phi(u, u)$ tends to a limit (which would have to be provided on data).

replaced by, say, $Ck^{-\varepsilon/2}$). In the latter set, we can then treat the difference as in the region $\mathcal{D}_k$ of the proof, leading to the estimate (1.5.90) (with ϵ replaced by, say, $Ck^{-\varepsilon/4}$). Taking $k \rightarrow \infty$ then shows that the two solutions agree.

In fact, if we also take into account the *a priori estimate* of Remark 1.5.3 (which, combined with the energy estimate (1.5.36), gives sharp decay for $r\phi$) as well as the corresponding *a priori* arguments that can be extracted from section 1.5.7 (cf. Remark 1.5.9), then all of the above global assumptions can be recovered from the assumptions that the m_i remain uniformly bounded. Thus, the solution constructed in Theorem 1.5.3 is the unique solution restricting correctly to the data on Γ and $\mathcal{I}^-$ that has a uniformly bounded Hawking mass.

1.5.7 Refinements

In this section, we will refine the above results and remove the unnecessary assumptions on p and on the smallness of $C_{\text{in},\phi}^1/R$ made thus far (see the Remarks 1.5.5, 1.5.6 in the previous section). More precisely, we will show sharp decay for $\partial_u(r\phi)$ and, thus, of ζ , hence allowing us to take $p > 1$ in the proofs of Thms. 1.5.1, 1.5.3 and to remove the smallness assumption on $C_{\text{in},\phi}/R$ in the latter. Finally, in order to compute the asymptotics of $\partial_v(r\phi)$, we will also show that the limit $\lim_{u \rightarrow -\infty} |u|^{p-1}r\phi(u, v)$ exists. These refinements will ultimately allow us to prove Theorems 1.5.6 and 1.5.7 in section 1.5.8.

Let us briefly sketch the ideas going into the following proofs. For simplicity, assume for the moment that Γ is a curve of constant $r = R$. The crucial observation is that, trivially, $\partial_u(r\phi) = \mathbf{T}(r\phi) - \partial_v(r\phi)$. Since we already know the sharp decay for $\partial_v(r\phi)$, it is thus left to show that the bound that $\mathbf{T}(r\phi)$ satisfies on Γ is propagated outwards. In the case of the linear wave equation on a fixed Schwarzschild background, this would be straight-forward; in fact, in that case, $\mathbf{T}$ commutes with the wave equation, and, hence, the bounds would propagate immediately by Thm. 1.5.1.

This approach cannot directly be used in the coupled problem. However, similarly to how we proved decay for $r\phi$ in Theorem 1.5.1 (using a bootstrap argument), we can hope to prove decay for $\mathbf{T}r$ since $\mathbf{T}r$ and $r\phi$ satisfy similar wave equations. Modulo technical difficulties arising from all the error terms coming from commuting with $\mathbf{T}$, this decay for $\mathbf{T}r$ then allows us to prove better decay for $\mathbf{T}(r\phi)$ and, thus, for $\partial_u(r\phi)$.

Notice that this argument needs to be slightly modified in the case where $r \rightarrow \infty$ along Γ . There, we will additionally use that we can convert some r -decay along Γ into $|u|$ -decay as we did when integrating the wave equation from $\mathcal{C}_{\text{in}}$ (see the proof of Thm. 1.4.1).

Finally, in order to show that $|u|^{p-1}r\phi$ attains a limit, we consider its derivative

$$\begin{aligned} \partial_u(|u|^{p-1}r\phi) &= |u|^{p-2}(-(p-1)r\phi + |u|\partial_u(r\phi)) \\ &= |u|^{p-2}(-(p-1)r\phi + |u|\mathbf{T}(r\phi)) - |u|^{p-1}\partial_v(r\phi). \end{aligned}$$

The goal is to show that the above is integrable, knowing already that the $\partial_v(r\phi)$ -term decays fast enough. To achieve this, we will compute the wave equation satisfied by the difference $-(p-1)r\phi + |u|\mathbf{T}(r\phi)$ and show that if this difference decays like $|u|^{-p+1-\epsilon}$ on Γ , then we can perform a similar bootstrap argument for it as we did for $r\phi$ in order to propagate this decay outwards.

The remainder of section 1.5 is structured as follows: In section 1.5.7.1, we will assume (as a bootstrap assumption) sharp decay on $\mathbf{T}(r\phi)$ and then prove decay of $\mathbf{T}r$ and $\mathbf{T}m$ as a consequence. In section 1.5.7.2, we will then recover the decay assumption on $\mathbf{T}(r\phi)$, using the decay of $\mathbf{T}r$ and $\mathbf{T}m$ proved in the preceding section. We will show decay for $-(p-1)r\phi + |u|\mathbf{T}(r\phi)$ in section 1.5.7.3. As all these proofs are based on bootstrap methods, we always need to work with compactly supported data. We will remove this assumption of compact support in section 1.5.8.1 by again performing a limiting argument as in the previous section. We finally derive the asymptotics of $\partial_v(r\phi)$ near $\mathcal{I}^+$ in section 1.5.8.2.

1.5.7.1 Proving decay for $\mathbf{T}r$ and $\mathbf{T}m$

Ultimately, we will want to make the following natural extra assumptions (in addition to $\hat{r} > 2M$) on the boundary data on $\Gamma = \partial\mathcal{D}_u$ in each of the two following cases:⁴⁵

Case 1: $\hat{r} \rightarrow R < \infty$: In this case, we assume that there exists a constant $C_{\text{in},r} > 0$ s.t.

$$|\mathbf{T}\hat{r}| \leq \frac{C_{\text{in},r}}{|u|^s}, \quad s = 1 + \epsilon_r > 1. \quad (1.5.93)$$

In fact, the upper bound could be replaced by any integrable function, but we here only write polynomial upper bounds in order to simplify notation. We do require the upper bound to be integrable, however.

Case 2: $\hat{r} \rightarrow \infty$: In this case, we assume that

$$-\mathbf{T}\hat{r} \sim \frac{1}{|u|^s}, \quad 1 \geq s > 0, \quad (1.5.94)$$

i.e., we assume both upper *and* lower bounds for $\mathbf{T}\hat{r}$. This means that either, for $s = 1$, $\hat{r} \sim \log|u|$, or, for $s < 1$, that $\hat{r} \sim |u|^\epsilon$ for $\epsilon = 1 - s$. The reason why we need the lower bound is that, in the case where $r|_\Gamma$ tends to infinity, we also need to convert some of the r -weights on Γ into $|u|$ -weights. Again, the above bounds can, in principle, be replaced by any non-integrable function.

⁴⁵In principle, we can also deal with cases where $\hat{r}$ oscillates, but, in order to simplify the presentation, and because these cases are also not physically interesting, we avoid discussing them here.

Remark 1.5.8. Note that if $s \leq \frac{1}{2}$, then the results of the following theorems follow trivially. Moreover, if $\hat{r} \sim |u|$, then, modulo the local existence part, we can apply the methods of section 1.4.

We will now prove decay for $\mathbf{T}r$ in each of these two cases. We will find that its decay is dictated by its initial decay on Γ and the decay of the scalar field. Thus, in order to capture the sharp decay of $\mathbf{T}r$, we will, in the theorem below, *assume* the sharp decay for $\mathbf{T}(r\phi)$. This assumption will be recovered in Thm. 1.5.5.

Theorem 1.5.4. *Let $\mathcal{D}_U$ be as described in section 1.5.3, and specify smooth functions $\hat{r}, \hat{\phi}$ on $\Gamma = \partial D_U = \{(u, u) \in \mathcal{D}_U\}$, with $\hat{\phi}$ having compact support. Let $\mathcal{C}_{u_0}$ denote the future-complete outgoing null ray emanating from a point $q = (u_0, u_0)$ on Γ that lies to the past of the support of $\hat{\phi}$. On $\mathcal{C}_{u_0}$, specify $\bar{m} \equiv M > 0$, $\bar{\phi} \equiv 0$, and an increasing smooth function $\bar{r}$ defined via $\bar{r}(v = u_0) = \hat{r}(u = u_0)$ and the ODE*

$$\partial_v \bar{r} = 1 - \frac{2M}{\bar{r}}.$$

Finally, assume that (denoting, again, the generator of Γ by $\mathbf{T} = \partial_u + \partial_v$) the following bounds hold on Γ :

$$|\mathbf{T}(\hat{r}\hat{\phi})(u)| \leq C_{\text{in},\phi}^1 |u|^{-p}, \quad (1.5.95)$$

$$|\mathbf{T}\hat{r}(u)| \leq C_{\text{in},r} |u|^{-s}, \quad (1.5.96)$$

with positive constants $p > 1$,⁴⁶ $C_{\text{in},\phi}^1$, $C_{\text{in},r}$, and $s > 0$; and assume that $\hat{r}$ tends to either an infinite (in the case $s \leq 1$) or a finite (in the case $s > 1$) limit $R \geq 4M$.

Let $\Delta_{u_0,\epsilon}$ denote the region of local existence from Proposition 1.5.1. If the assumption

$$|\mathbf{T}(r\phi)| \leq \frac{E}{|u|^p} \quad (\text{BS}(4))$$

is satisfied throughout $\Delta_{u_0,\epsilon}$ for some positive constant E , then we have, for sufficiently large negative values of U_0 (the choice of U_0 depending only on data), that, throughout $\Delta_{u_0,\epsilon} \cap \{u \leq U_0\}$ the estimates of Theorem 1.5.1 hold for the arising solution (r, ϕ, m) . Moreover, we have the additional bounds:⁴⁷

$$|\mathbf{T}r| \leq C'_r |u|^{-\min(s, 2p-1)}, \quad (1.5.97)$$

$$|\partial_v \mathbf{T}r| \leq D' M C'_r \frac{|u|^{-\min(s, 2p-1)}}{r^2}, \quad (1.5.98)$$

$$|\mathbf{T}m| \leq C_m \left(\frac{1}{|u|^{2p-1}r} + \frac{1}{|u|^{2p}} \right). \quad (1.5.99)$$

⁴⁶Note that this was $p > 3/2$ in Thm. 1.5.1.

⁴⁷Compare these bounds to the ones for $r\phi$ and $\partial_v(r\phi)$ in Thm. 1.5.1.

Here, C'_r and D' are constants which, for $2p - 1 > s$, only depend on the value of $C_{\text{in},r}$ and the ratio M/R and, for $2p - 1 \leq s$, also depend on E , whereas C_m always depends also on E . In particular, none of these constants depend on u_0 .

Remark 1.5.9. Again, the lower bound for R stated here is not necessary – one can prove the same result under the weaker assumption $R > 2M$ if one replaces the bootstrap arguments ((BS(5)), (BS(6))) below with Grönwall arguments as described in Remark 1.5.3. See footnote 50. The same applies to the subsequent results (Theorem 1.5.5 and Lemma 1.5.1).

Remark 1.5.10. In order to recover bootstrap assumption (BS(4)) in the next section, it is helpful to observe that C'_r and D' are independent of E for $2p - 1 > s$. The distinction between the cases $2p - 1 > s$ and $2p - 1 \leq s$ is best understood by looking at the estimate (1.5.97): If ϕ decays sufficiently slowly, then its decay dominates the decay of Tr .

Proof. Let's first draw our attention to the claim that p can be taken to be $p > 1$ instead of $p > 3/2$. The reason for this is that we assume $T(r\phi) \lesssim |u|^{-p}$. Recall that, in the proof of Thm. 1.5.1, the obstruction to taking $p > 1$ was that we were only able to show that ζ decayed as fast as $r\phi$, see Remark 1.5.2. However, if we assume ineq. (BS(4)), then we immediately get that

$$\partial_u(r\phi) = T(r\phi) + \partial_v(r\phi) \lesssim \frac{1}{|u|^p} + \frac{1}{|u|^{p-1}r^2},$$

which improves the bound on ζ to

$$|\zeta| \lesssim \frac{1}{|u|^{p-1}r} + \frac{1}{|u|^p},$$

so we can also prove Theorem 1.5.1 for $p > 1$. Therefore, we now assume the results of Thm. 1.5.1 to hold for $p > 1$. In fact, because of this improved bound for ζ , we can now take $\delta(u) \sim |u|^{2-2p}$ instead of $\delta(u) \sim |u|^{3-2p}$ (cf. (1.5.39)).

Furthermore, observe that, along $\mathcal{C}_{u_0}$, we have

$$\partial_v Tr = (\partial_v r + \partial_u r) \frac{2M}{r^2} = \frac{2M}{r^2} Tr \quad (1.5.100)$$

in view of $\partial_v \bar{r} = 1 - 2M/\bar{r}$ and the wave equation for r (1.3.23).

Outline of the main ideas: With these preliminary observations understood, we now give an outline of the heart of the proof: If we consider the set

$$\Delta := \{(u, v) \in \Delta_{u_0, \epsilon} \mid |Tr(u', v')| \leq \frac{C'_r}{|u'|^{\min(s, 2p-1)}} \forall (u', v') \in \Delta_{u_0, \epsilon} \text{ s.t. } u' \leq u, v' \leq v\}, \quad (1.5.101)$$

for some suitable $C'_r > C_{\text{in},r}$ (which will be specified later) and $\epsilon > 0$, then we immediately conclude that this is non-empty ($\{q\} \subsetneq \Delta$) by continuity and compactness. As in the proof of

Thm. 1.5.1, we want to show that Δ is open (it is clearly closed), that is, we want to improve the bound

$$|\mathbf{T}r| \leq C'_r |u|^{-\min(s, 2p-1)} \quad (\text{BS(5)})$$

inside Δ .

It turns out that we need to include another bootstrap assumption inside Δ in order for this to work, namely

$$|\partial_v \mathbf{T}r| \leq D \frac{|u|^{-\min(s, 2p-1)}}{r^2} \quad (\text{BS(6)})$$

for some suitable constant D . To see why, commute the wave equation for r , eq. (1.3.23), with the vector field $\mathbf{T}$:

$$\begin{aligned} \partial_u \partial_v (\mathbf{T}r) &= \mathbf{T}m \left(\frac{2\nu\kappa}{r^2} \right) + \mathbf{T}\mu \left(\frac{\kappa}{1-\mu} \frac{2m\nu}{r^2} \right) \\ &\quad + \mathbf{T}\lambda \left(\frac{2m\nu}{r^2(1-\mu)} \right) + \mathbf{T}\nu \left(\frac{2m\lambda}{r^2(1-\mu)} \right) - \mathbf{T}r \left(\frac{4m\nu\kappa}{r^3} \right). \end{aligned} \quad (1.5.102)$$

Looking at the $\mathbf{T}m$ -term, we will show below that the bootstrap assumptions (BS(4)) and (BS(5)) together imply that

$$\mathbf{T}m \lesssim \frac{1}{|u|^{2p-1}r} + \frac{1}{|u|^{2p}}, \quad (1.5.103)$$

which means that this term is in some sense independent of the bootstrap argument. In fact, it is this term which is responsible for the $\min(s, 2p-1)$ appearing in the bootstrap assumptions. Note already that ineq. (1.5.103) is an improvement over plugging in the naive bounds for ζ^2, θ^2 into $\mathbf{T}m$.

On the other hand, the $\mathbf{T}r$ -term in (1.5.102) comes with a factor $1/r^3$, so we expect to be able to treat it exactly like we treated $r\phi$ in Thm. 1.5.1. The $\mathbf{T}\mu$ -term can be written as a sum of faster decaying $\mathbf{T}m$ and $\mathbf{T}r$ -terms. As for the $\mathbf{T}\lambda$ -term, we expect to be able to bound it in the same way as we bounded $\partial_v(r\phi)$ in Thm. 1.5.1 (this explains the bootstrap assumption (BS(6))), so the non-integrable factor of $1/r^2$ multiplying $\mathbf{T}\lambda$ poses no problem.

However, for the $\mathbf{T}\nu$ -term, it seems like we're in trouble, as we cannot expect to show a similar bound for $\mathbf{T}\nu$ as for $\mathbf{T}\lambda$, for the same reason for which we weren't able to show a better bound for $\partial_u(r\phi)$ in Thm. 1.5.1. We deal with this problem by making the $\mathbf{T}\nu$ -term into a boundary term (using that $\mathbf{T}\nu = \partial_u \mathbf{T}r$), that is, we will write, inserting also the equations for $\partial_u \kappa$, $\partial_u \lambda$ and $\partial_u m$:

$$\begin{aligned} \partial_u \partial_v (\mathbf{T}r) &= \mathbf{T}m \left(\frac{2\nu\kappa}{r^2} + \frac{4m\nu\kappa}{r^3(1-\mu)} \right) \\ &\quad + \mathbf{T}\lambda \left(\frac{2m\nu}{r^2(1-\mu)} \right) + \partial_u \left(2m\kappa \frac{\mathbf{T}r}{r^2} \right) \\ &\quad + \mathbf{T}r \left(\frac{4m^2\nu\kappa}{r^4(1-\mu)} - \frac{2\kappa(1-\mu)\zeta^2}{r^2} \frac{1}{2\nu} - \frac{2m\kappa\zeta^2}{r^2} \frac{1}{r\nu} \right). \end{aligned} \quad (1.5.104)$$

The details: Having given a rough outline of how we will deal with each term in the above equation, we now give the details. First, we derive the estimate (1.5.103): Plugging in eqns. (1.3.18) and (1.3.19), we get

$$\begin{aligned} \mathbf{T}m &= \partial_u m + \partial_v m = \frac{1}{2}(1 - \mu) \left(\frac{\zeta^2}{\nu} + \frac{\theta^2}{\lambda} \right) \\ &= \frac{1}{2}(1 - \mu) \left(\frac{(\partial_v(r\phi) - \lambda\phi)^2}{\lambda} + \frac{(\mathbf{T}(r\phi) - \partial_v(r\phi) - \lambda\phi\frac{\nu}{\lambda})^2}{\lambda} \frac{\lambda}{\nu} \right). \end{aligned} \quad (1.5.105)$$

Now, by the bootstrap assumption (BS(5)), we have that

$$\frac{\nu}{\lambda} = \frac{\mathbf{T}r - \lambda}{\lambda} = -1 + \mathcal{O}(|u|^{-s}).$$

Upon inserting this back into equation (1.5.105), we get⁴⁸

$$\begin{aligned} \mathbf{T}m &= \frac{1 - \mu}{2\lambda} \left(-(\mathbf{T}(r\phi))^2 + 2\mathbf{T}(r\phi)(\partial_v(r\phi) - \lambda\phi + \mathcal{O}(|u|^{-s})) + \zeta^2 \mathcal{O}(|u|^{-s}) \right) \\ &\lesssim \frac{E^2}{|u|^{2p}} + \frac{E}{|u|^{2p-1}r} + \frac{C'_r E}{|u|^{2p-1}r} \frac{1}{|u|^{s-1}r}. \end{aligned} \quad (1.5.106)$$

Now, if $s > 1$, then the third term on the RHS of (1.5.106) has more $|u|$ -decay than the second term and can hence be ignored. If $s = 1$, the third term only has more r -decay, but in this case $r|_\Gamma \sim \log |u|$ by the lower bound in (1.5.94), so it can again be ignored. If $s < 1$, then, again by the lower bound in (1.5.94),

$$\frac{1}{|u|^{s-1}r} \leq 1,$$

so the third term decays just as fast as the second one, and it cannot be ignored. We conclude that

$$|\mathbf{T}m| \lesssim_{C'_r, E} \frac{1}{|u|^{2p-1}r} + \frac{1}{|u|^{2p}}. \quad (1.5.107)$$

It is important that the implicit constant in $\lesssim_{C'_r, E}$ only depends on C'_r if $r|_\Gamma \rightarrow \infty$. We shall return to this point later.

We now have all the tools to close the bootstrap argument, i.e. to improve assumptions (BS(5)) and (BS(6)). The idea is to integrate the wave equation (1.5.104) for $\mathbf{T}r$ twice along its characteristics. We consider the cases $2p - 1 > s$, $2p - 1 = s$ and $2p - 1 < s$ separately.

Case i): $2p - 1 > s$: Let us also assume, for simpler presentation, that $\mathbf{T}\hat{r}$ is compactly supported such that $\mathbf{T}\hat{r}(q) = 0$. This will just mean that we won't pick up a boundary term when integrating $\partial_u \partial_v \mathbf{T}r$ from $\mathcal{C}_{u_0}$, in view of (1.5.100). We will remove this assumption below.

⁴⁸Again, constants hidden inside $\lesssim$ only depend on initial/boundary data.

Integrating equation (1.5.104) from data, we thus obtain

$$|\partial_v \mathbf{T}r(u, v)| = \frac{2m\kappa \mathbf{T}r}{r^2} + \int_{u_0}^u \mathbf{T}m(\dots) + \mathbf{T}\lambda(\dots) + \mathbf{T}r(\dots) du'. \quad (1.5.108)$$

For the $\mathbf{T}m$ -term, we plug in the bound from (1.5.107), resulting in

$$\int_{u_0}^u \mathbf{T}m \left(\frac{2\nu\kappa}{r^2} + \frac{4m\nu\kappa}{r^3(1-\mu)} \right) du' \lesssim_E \frac{1}{r^2|u|^{2p-1}}.$$

For the other terms, we first use ν to turn the $|u|$ -integration into r -integration. We then plug in the bootstrap assumptions (BS(5)), (BS(6)), as well as all the bounds from Thm. 1.5.1; in particular, we use that $\kappa \leq 1$, $m \leq M$ and⁴⁹ $\mu - (1 - 2M/r) = \mathcal{O}(|u|^{-2p+1})$.⁵⁰ This yields

$$\begin{aligned} & |\partial_v \mathbf{T}r(u, v)| \\ & \leq \frac{2m\kappa C'_r}{r^2|u|^s} + \int_{r(u_0, v)}^{r(u, v)} \left| \mathbf{T}r \frac{4m^2\kappa}{r^3(r-2M)} \right| + \left| \mathbf{T}\lambda \frac{2m}{r(r-2M)} \right| dr + \mathcal{O} \left(\frac{1}{r^2|u|^{2p-1}} \right) \\ & \leq \frac{2MC'_r}{r^2|u|^s} + \frac{2MD + 4M^2C'_r}{|u|^s} \left(\frac{-\log(1 - \frac{2M}{r})}{(2M)^3} - \frac{r+M}{r^2(2M)^2} \right) + \mathcal{O} \left(\frac{1}{r^2|u|^{2p-1}} \right). \end{aligned} \quad (1.5.109)$$

In order to close the bootstrap argument for $\partial_v \mathbf{T}r$ (BS(6)), we require the RHS to be strictly smaller than $\frac{D}{r^2|u|^s}$. This leads to the following condition on D and C'_r :

$$2MC'_r \frac{(1 + \frac{1}{x^2} \log(\frac{1}{1-x}) - \frac{1}{x} - \frac{1}{2})}{(1 - \frac{1}{x^2} \log(\frac{1}{1-x}) + \frac{1}{x} + \frac{1}{2})} + \mathcal{O}(|u|^{s+1-2p}) < D, \quad (1.5.110)$$

where we wrote $x = \frac{2M}{r}$. The LHS is maximised when x is, so it is maximised for $x = \frac{2M}{r(u, u)}$. In the case where $r(u, u) \rightarrow \infty$ as $u \rightarrow -\infty$, (1.5.110) is trivially satisfied for large enough values of $|u|$.

On the other hand, if $r(u, u) \rightarrow R < \infty$, we have $r(u, u) \geq R - \mathcal{O}(|u|^{1-s})$. We can thus insert $x = \frac{2M}{R}$ into (1.5.110), resulting in another $o(1)$ -term. The estimate (1.5.110) then holds provided that $|U_0|$ is large enough, that $2M/R \lesssim 0.86$, and for

$$D = \eta 2MC'_r \cdot \frac{(1 + \frac{1}{x^2} \log(\frac{1}{1-x}) - \frac{1}{x} - \frac{1}{2})}{(1 - \frac{1}{x^2} \log(\frac{1}{1-x}) + \frac{1}{x} + \frac{1}{2})} =: \eta 2MC'_r \cdot A(x), \quad (1.5.111)$$

where $x = \frac{2M}{R}$ and $\eta > 1$ (where $\eta - 1 \rightarrow 0$ as $U_0 \rightarrow -\infty$).

⁴⁹The reader can instead just take the lower bound $1 - \frac{2M}{R} \leq \mu$. This will simplify the integrals below but lead to a slightly worse lower bound on R .

⁵⁰Similarly to the calculation in Remark 1.5.3, one can, instead of using bootstrap arguments and going through the calculations below, first apply a Grönwall estimate to $\partial_v \mathbf{T}r$ in the u -direction. From this, one can then obtain an inequality for $\mathbf{T}r$ similar to (1.5.76) and apply a similar double Grönwall estimate to it. This only requires the lower bound that $R > 2M$.

This improves the bootstrap assumption (BS(6)).

Next, we integrate the estimate (1.5.109) in v from Γ . In order to convert the v -integration into r -integration, we use the estimate $\lambda - (1 - 2M/r) = \mathcal{O}(|u|^{-2p+1})$. One obtains

$$\begin{aligned} |\mathbf{T}r(u, v)| &\leq \frac{C_{\text{in},r}}{|u|^s} + \frac{1}{|u|^s} \int_{r(u,u)}^{r(u,v)} \frac{2MC'_r}{r(r-2M)} dr \\ &\quad + \frac{\frac{D}{2M} + C'_r}{|u|^s} \int_{r(u,u)}^{r(u,v)} \frac{-\log(1 - \frac{2M}{r})}{2M(1 - \frac{2M}{r})} - \frac{r+M}{r(r-2M)} dr + \mathcal{O}(|u|^{1-2p}). \end{aligned} \quad (1.5.112)$$

The first integral has been computed before (cf. (1.5.42)). For the second integral, we substitute $x = 2M/r$, which brings it into the following form

$$\int_{2M/r(u,u)}^{2M/r(u,v)} \frac{\log(1-x)}{(1-x)x^2} + \frac{1}{x} \frac{1+\frac{1}{2}x}{1-x} dx. \quad (1.5.113)$$

This can now be computed using the dilogarithm $\text{Li}_2(x)$ (cf. (1.4.55)). Reinserting this back into (1.5.112), using also that $\text{Li}_2(1) = \pi^2/6$, we get the estimate:

$$\begin{aligned} |u|^s |\mathbf{T}r(u, v)| &\leq C_{\text{in},r} + C'_r \log\left(\frac{1}{1-x}\right) \\ &\quad + \underbrace{\left(\frac{D}{2M} + C'_r\right) \left(1 + \frac{\pi^2}{6} + \frac{1}{2} \log(1-x) \left(\frac{2}{x} + \log\left(\frac{1-x}{x^2}\right)\right) - \text{Li}_2(1-x)\right)}_{:=B(x)} + \mathcal{O}(|u|^{1-2p}) \end{aligned} \quad (1.5.114)$$

where we wrote $x = 2M/r(u, u)$: As before, we want this to be strictly smaller than $C'_r |u|^{-s}$ in order to close the bootstrap assumption (BS(5)). This can trivially be achieved for large enough $|U_0|$ in the case where $r(u, u) \rightarrow \infty$. In the case where $r(u, u) \rightarrow R$, we numerically find, plugging in (1.5.111), that the lower bound $2M/R \lesssim 0.516$ needs to hold. The constant C'_r can then be chosen to be

$$C'_r = \frac{\eta C_{\text{in},r}}{1 + \log(1 - 2M/R) - (1 + \eta A(2M/R))B(2M/R)}, \quad (1.5.115)$$

where $B(x)$ was defined in the estimate above and $A(x)$ was defined in (1.5.111). This bound is, in particular, independent of E .

This closes the bootstrap argument for $2p - 1 > s$ in the case of $\mathbf{T}\hat{r}$ being compactly supported.

If $\mathbf{T}\hat{r}$ is not compactly supported, the only difference is that we pick up a boundary term $\partial_v(\mathbf{T}r)(u_0, v)$ from integrating $\partial_u \partial_v \mathbf{T}r$. In view of equation (1.5.100), this boundary term can be bounded directly in terms of initial data (after applying a Grönwall inequality to bound $\mathbf{T}r$ on $\mathcal{C}_{u_0}$). This boundary term will slightly change the definitions of D and C'_r , but will not affect the lower bound on R in any way, precisely because it is bounded by data!

Case ii): $2p - 1 = s$: In this case, it seems like there is an additional difficulty since the $\mathcal{O}(r^{-2}|u|^{-2p+1})$ -term we treated as negligible in estimate (1.5.109) is now of the same order as the other terms: we therefore need to add a term $\lesssim_{C'_r, E} \frac{1}{r^2|u|^s}$ to the estimate (1.5.109). The E -dependence of the implicit constant in $\lesssim_{C'_r, E}$ then means that C'_r and D will depend on E . The C'_r -dependence in $\lesssim_{C'_r, E}$, on the other hand, seems like it would add a further restriction on the lower bound of R . This is where it is important that the implicit constant in $\lesssim_{C'_r, E}$ only depends on C'_r in the case where $r(u, u) \rightarrow \infty$ (see the remark below estimate (1.5.107)). Therefore, no new bound on R is introduced, and the bootstrap argument works in the same way, with D and C'_r now depending also on E .

Case iii): $2p - 1 < s$: In this case, the $\mathcal{O}(r^{-2}|u|^{-2p+1})$ -term we treated as negligible in estimate (1.5.109) now dominates all other terms and depends on E as well as on C'_r if $r(u, u) \rightarrow \infty$. By the same reasoning as above, the bootstrap argument then closes trivially, with C'_r and D again depending on E .

This concludes the proof. $\square$

We will now recover the bootstrap assumption (BS(4)):

1.5.7.2 Sharp decay for $T(r\phi)$ and $\partial_u(r\phi)$

In this section, we prove the next refinement to Thm. 1.5.1. This refinement shows sharp decay for $T(r\phi)$ and, thus, for $\partial_u(r\phi)$.

Theorem 1.5.5. *Let $\mathcal{D}_U$ be as described in section 1.5.3, and specify smooth functions $\hat{r}, \hat{\phi}$ on $\Gamma = \partial D_U = \{(u, u) \in \mathcal{D}_U\}$, with $\hat{\phi}$ having compact support. Let $\mathcal{C}_{u_0}$ denote the future-complete outgoing null ray emanating from a point $q = (u_0, u_0)$ on Γ that lies to the past of the support of $\hat{\phi}$. On $\mathcal{C}_{u_0}$, specify $\bar{m} \equiv M > 0$, $\bar{\phi} \equiv 0$, and an increasing smooth function $\bar{r}$ defined via $\bar{r}(v = u_0) = \hat{r}(u = u_0)$ and the ODE*

$$\partial_v \bar{r} = 1 - \frac{2M}{\bar{r}}.$$

Finally, assume that the following bounds hold on Γ :

$$|T(\hat{r}\hat{\phi})(u)| \leq C_{\text{in}, \phi}^1 |u|^{-p}, \quad (1.5.116)$$

$$|T\hat{r}(u)| \leq C_{\text{in}, r} |u|^{-s}, \quad (1.5.117)$$

with positive constants $p > 1$, $C_{\text{in}, \phi}^1$, $C_{\text{in}, r}$ and $s > 0$, and assume that $\hat{r}$ tends to either an infinite (in the case $s \leq 1$) or a finite (in the case $s > 1$) limit $R \geq 4M$. If $s \leq 1$, we moreover assume that there exists a positive constant $d_{\text{in}, r} < C_{\text{in}, r}$ such that

$$-T\hat{r}(u) \geq d_{\text{in}, r} |u|^{-s}. \quad (1.5.118)$$

Then we have, for sufficiently large negative values of U_0 (the choice of U_0 depending only on data), that, throughout $\Delta_{u_0, \epsilon} \cap \{u \leq U_0\}$, the estimates of Theorem 1.5.1 hold. Moreover, we have the following additional bounds⁵¹:

$$|\mathbf{T}(r\phi)| \leq C'_T |u|^{-p}, \quad (1.5.119)$$

$$|\partial_v \mathbf{T}(r\phi)| \leq D_T M C'_T \frac{|u|^{-p}}{r^2}. \quad (1.5.120)$$

Here, C'_T and D_T are constants which depend on the value of $C_{\text{in}, \phi}^1$ and the ratio M/R ; in particular, they do not depend on u_0 .

Finally, in view of (1.5.119), the estimates from Theorem 1.5.4 hold as well, with the constant E given by $E = C'_T$.

Proof. As in the previous proof, we will bootstrap the decay of $\mathbf{T}(r\phi)$, that is, we will assume

$$|\mathbf{T}(r\phi)(u, v)| \leq E |u|^{-p} \quad (\text{BS}(4))$$

for some suitable constant E , and we will subsequently improve this assumption. Note that, by the bootstrap assumption, the results of Thm. 1.5.4 hold.

We will distinguish between the cases $s > 1$, $s = 1$ and $s < 1$, i.e. between the cases where $r|_\Gamma$ tends to a finite or infinite limit.

Case i): $s > 1$: We start by commuting the wave equation for $r\phi$ with $\mathbf{T}$. As in the previous proof, we deal with the bad $\mathbf{T}\nu$ -term by converting it into a boundary term (cf. eq. (1.5.104)):

$$\begin{aligned} \partial_u \partial_v (\mathbf{T}(r\phi)) &= \mathbf{T}m \left(\frac{2\nu\kappa}{r^2} + \frac{4m\nu\kappa}{r^3(1-\mu)} \right) \frac{r\phi}{r} \\ &\quad + \mathbf{T}\lambda \left(\frac{2m\nu}{r^2(1-\mu)} \right) \frac{r\phi}{r} + \partial_u \left(2m\kappa \frac{\mathbf{T}r}{r^2} \frac{r\phi}{r} \right) \\ &\quad + \mathbf{T}r \left(\frac{4m^2\nu\kappa}{r^4(1-\mu)} - \frac{2\kappa(1-\mu)\zeta^2}{r^2} - \frac{2m\kappa\zeta^2}{r^2 r\nu} \right) \frac{r\phi}{r} + \mathbf{T}r \left(-\frac{2m\kappa}{r^2} \frac{\partial_u(r\phi)}{r} \right) \\ &\quad + \partial_u \partial_v r \cdot \frac{\mathbf{T}(r\phi)}{r}. \end{aligned} \quad (1.5.121)$$

The last term is exactly the same term that appears in $\partial_u \partial_v(r\phi)$, but with $r\phi$ replaced by $\mathbf{T}(r\phi)$. We will show that all other terms decay faster in $|u|$. More precisely, we will show that all other terms can be bounded by $\frac{1}{|u|^{p+\epsilon}r^3}$ for some $\epsilon > 0$.

For the $\mathbf{T}m$ -term, plugging in the bound (1.5.99) as well as $r\phi \lesssim |u|^{-p+1}$, we find that it is bounded by

$$\mathbf{T}m(\dots) \frac{r\phi}{r} \lesssim \frac{1}{|u|^{3p-2}r^4} + \frac{1}{|u|^{3p-1}r^3}.$$

⁵¹Compare these bounds to the ones for $r\phi$ and $\partial_v(r\phi)$ in Thm. 1.5.1.

The RHS can be bounded by $\frac{1}{|u|^{p+\epsilon}r^3} \left(\frac{1}{|u|} + \frac{1}{r} \right)$, where ϵ is given by $\epsilon = 2p - 2 > 0$.

The $\mathbf{T}\lambda$ - and the $\mathbf{T}r$ -terms can be dealt with similarly in view of the bounds (1.5.97) and (1.5.98) and since we assumed that $s > 1$ (and since $p > 1$ implies $2p - 1 > 1$).

For the boundary term in the second line, we find that, after integrating first in u and then in v , it can be bounded against $R^{-2}|u|^{-p+1-\min(s, 2p-1)}$.

In conclusion, we find that

$$\begin{aligned} |\mathbf{T}(r\phi)|(u, v) &\leq |\mathbf{T}(r\phi)|(u, u) + \left| \int_u^v dv' \int_{u_0}^u du' \partial_u \partial_v (\mathbf{T}r\phi) \right| \\ &\leq C_{\text{in}, \phi}^1 |u|^{-p} + \left| \int_u^v dv' \int_{u_0}^u du' \partial_u \partial_v r \frac{\mathbf{T}(r\phi)}{r^3} \right| + \mathcal{O}(|u|^{-p-\min(2p-2, s-1)}), \end{aligned} \quad (1.5.122)$$

so the bootstrap argument can be closed in the same manner as in the proof of Thm. 1.5.1 for $r\phi$. (Alternatively, one can perform a Grönwall argument as in Remark 1.5.3).

This concludes the proof in the case where $r(u, u)$ tends to a finite limit.

Case ii): $s=1$: In this case, the $\mathbf{T}r$ - and $\mathbf{T}\lambda$ -terms in eq. (1.5.121) are no longer subleading compared to the $\mathbf{T}(r\phi)$ -term (the $\mathbf{T}m$ -term remains unchanged). Nevertheless, since in this case $r(u, u) \sim \log |u|$ diverges, the bootstrap argument still closes.

Case iii): $s \leq 1$: Let us finally deal with the case where $r(u, u) \sim |u|^{1-s}$. Here, the $\mathbf{T}r$ - and $\mathbf{T}\lambda$ -terms in equation (1.5.121) exhibit less decay in u than the other terms, however, we can convert some of the extra r -decay present in these terms into u -decay according to

$$r^{-1} \lesssim |u|^{s-1}. \quad (1.5.123)$$

Using this, we have e.g. for the $\mathbf{T}\lambda$ -term in (1.5.121)

$$\left| \mathbf{T}\lambda \left(\frac{2m\nu}{r^2(1-\mu)} \right) \frac{r\phi}{r} \right| \lesssim \frac{1}{r^5|u|^s} \frac{1}{|u|^{p-1}} \lesssim \frac{1}{r^3} \frac{1}{|u|^{p+1-s}}. \quad (1.5.124)$$

The boundary term in the second line, as well as the third line of (1.5.121), can be dealt with in a similar fashion. For the term in the fourth line, we use the bootstrap assumption (BS(4)) as well as the bound for $\partial_v(r\phi)$ from (1.5.35) to conclude that

$$|\partial_u(r\phi)| \lesssim \frac{1}{|u|^p} + \frac{1}{|u|^{p-1}r^2}.$$

Plugging this bound back into the above, we see that

$$\mathbf{T}r \left(-\frac{2m\kappa}{r^2} \frac{\partial_u(r\phi)}{r} \right) \lesssim \frac{1}{r^3|u|^s} \left(\frac{1}{|u|^p} + \frac{1}{|u|^{p-1}r^2} \right) \lesssim \frac{1}{r^3|u|^{p+s}} + \frac{1}{r^3|u|^{p+1-s}},$$

where we again used (1.5.123). We conclude that the $\partial_u \partial_v r \frac{\mathbf{T}(r\phi)}{r}$ -term again dominates and that we can repeat the bootstrap argument as before.

This finishes the proof. $\square$

1.5.7.3 Convergence of $\lim_{u \rightarrow -\infty} |u|^{p-1} r \phi$

We now have all the tools at hand to reprove Theorem 1.5.3 without the smallness assumption on $C_{\text{in},\phi}$ and without the restriction on $p \geq 2$, see Remark 1.5.5. However, as explained in Remark 1.5.6, we still would not be able to conclude that $|u|^{p-1} r \phi(u, v)$ tends to a limit as u tends to $-\infty$. We now prove a lemma that allows us to do precisely this:

Lemma 1.5.1. *Under the same assumptions as in Theorem 1.5.5, assuming moreover that, on Γ ,*

$$\left| \hat{r} \hat{\phi} - \frac{|u|}{p-1} \mathbf{T}(\hat{r} \hat{\phi}) \right| \leq F |u|^{-p+1-\epsilon_\phi} \quad (1.5.125)$$

for some $1 > \epsilon_\phi > 0$ and a constant $F > 0$, we have that, for $s \neq 1$, for large enough values of $|U_0|$ and for $\hat{r} \geq R \geq 4M$:

$$\left| r \phi - \frac{|u|}{p-1} \mathbf{T}(r \phi) \right| \leq F' |u|^{-p+1-\epsilon'} \quad (1.5.126)$$

for a constant F' depending only on initial data and not on the value of u_0 . Here, ϵ' is given by

$$\epsilon' = \min(\epsilon_\phi, 2p-2, s, |s-1|) = \begin{cases} \min(\epsilon_\phi, 2p-2, s-1), & s > 1, \\ \min(\epsilon_\phi, 2p-2, s, 1-s), & s < 1. \end{cases} \quad (1.5.127)$$

If $s = 1$, then we instead have

$$\left| r \phi - \frac{|u|}{p-1} \mathbf{T}(r \phi) \right| \leq F' |u|^{-p+1} \frac{1}{\log^2 |u|}. \quad (1.5.128)$$

Proof. The proof will follow the same ideas as the previous proofs, and we will only sketch it. First, consider the case $s > 1$. We compute

$$\begin{aligned} \partial_u \partial_v \left(r \phi - \frac{|u|}{p-1} \mathbf{T}(r \phi) \right) &= \frac{\partial_u \partial_v r}{r} \left(r \phi - \frac{|u|}{p-1} \mathbf{T}(r \phi) \right) + \frac{1}{p-1} \partial_v \mathbf{T}(r \phi) \\ &\quad - \frac{|u|}{p-1} \left(\partial_u \partial_v \mathbf{T}(r \phi) - \partial_u \partial_v r \frac{\mathbf{T}(r \phi)}{r} \right). \end{aligned} \quad (1.5.129)$$

We will again assume (1.5.126) as a bootstrap assumption and improve it. The first term in the equation above is the usual one, and we can deal with it. It is left to show that the others decay faster:

For the second term, plugging in the bound (1.5.120) and converting some of the u -decay in it into r -decay and integrating in u and v does the job (this is where we need $\epsilon_\phi < 1$).

For the final term, we proceed exactly as in the proof of Theorem 1.5.5.

We proceed similarly in the cases $s = 1$ and $s \leq 1$. $\square$

Remark 1.5.11. Notice that one can replace the RHS of the assumption (1.5.125) with any function that is non-increasing in $|u|$ and recover the correspondingly adapted (1.5.126). In particular, we can add a constant to the RHS of (1.5.125). This will play a role later on because of the cut-off functions introduced below, see Remark 1.5.12

1.5.8 Proof of Thm. 1.2.1

1.5.8.1 Sending $\mathcal{C}_{u_0}$ to $\mathcal{I}^-$ (revisited)

We can now prove the refined version of Thm. 1.5.3. The setup will be the same as in section 1.5.6, with some minor modifications that we here point out:

The “final” boundary data

As in section 1.5.6, we let $M > 0$, and we restrict to sufficiently large negative values of $u \leq U_0 < 0$ and specify boundary data $(\hat{r}, \hat{\phi})$ on Γ as follows: The datum $\hat{r} \in C^2(\Gamma)$ is to satisfy $\hat{r} > 2M$ and either

$$\mathbf{T}\hat{r} \sim \frac{1}{|u|^s}, \quad 1 \geq s > 0, \quad \text{and} \quad \lim_{u \rightarrow -\infty} \hat{r} = \infty, \quad (1.5.130)$$

or⁵²

$$|\mathbf{T}\hat{r}| \lesssim \frac{1}{|u|^s}, \quad s > 1, \quad \text{and} \quad \lim_{u \rightarrow -\infty} \hat{r} = R > 2M. \quad (1.5.131)$$

On the other hand, $\hat{\phi} \in C^2(\Gamma)$ is chosen to obey $\lim_{u \rightarrow -\infty} \hat{r}\hat{\phi}(u) = 0$ and

$$\mathbf{T}(\hat{r}\hat{\phi}) = C_{\text{in},\phi}^1 |u|^{-p} + \mathcal{O}(|u|^{-p-\epsilon_\phi}) \quad (1.5.132)$$

for $p > 1$, $1 > \epsilon_\phi > 0$ and some constant $C_{\text{in},\phi}^1 > 0$.

The sequence of finite solutions (r_k, ϕ_k, m_k)

We finally prescribe a sequence of initial/boundary data as in section 1.5.6: We recall the notation that, for $k \in \mathbb{N}$, $\mathcal{C}_k = \{u = -k, v \geq u\}$, and we also recall the sequence of smooth cut-off functions $(\chi_k)_{k \in \mathbb{N}}$ on Γ from (1.5.50), which equal 1 for $u \geq -k + 1$, and which equal 0 for $u \leq -k$.

⁵²We now use the better lower bound on R , cf. Remarks 1.5.3 and 1.5.9.

Our sequence of initial data shall then be given⁵³ by:

$$(I.D.)_k = \begin{cases} \hat{r}_k = \hat{r}, \hat{r}\hat{\phi}_k(u) = \int_{-\infty}^u \chi_k T(\hat{r}\hat{\phi}) du' & \text{on } \Gamma, \\ \bar{r}_k = r_0, \bar{\phi}_k = 0, m = M & \text{on } \mathcal{C}_k. \end{cases} \quad (1.5.133)$$

These lead to a sequence of solutions (r_k, ϕ_k, m_k) , which we extend with the vacuum solution $(r_0, 0, M)$ for $u \leq -k$ (cf. Thm. 1.5.2) and which obey, uniformly in k , the bounds from Theorem 1.5.1 and also the refined bounds from Theorems 1.5.4 and 1.5.5 and Lemma 1.5.1.

Remark 1.5.12. There is one small technical subtlety here: The difference $\hat{r}_k \hat{\phi}_k - \frac{|u|}{p-1} \mathbf{T}(\hat{r}_k \hat{\phi}_k)$ has an error term coming from the cut-off function χ_k :

$$|\hat{r}_k \hat{\phi}_k - \frac{|u|}{p-1} \mathbf{T}(\hat{r}_k \hat{\phi}_k)| \leq \frac{F}{|u|^{p-1+\epsilon_\phi}} + \frac{C}{k^{p-1}}$$

for some positive constants F and C . As explained in Remark 1.5.11, Lemma 1.5.1 still applies to this. The error contribution Ck^{1-p} arising from the cut-off function then vanishes as $k \rightarrow \infty$.

Theorem 1.5.6. *Let $p > 1$ and $U_0 < 0$ be sufficiently large. Then, as $k \rightarrow \infty$, the sequence (r_k, ϕ_k, m_k) uniformly converges to a limit (r, ϕ, m) ,*

$$\|r_k \phi_k - r \phi\|_{C^1(\mathcal{D}_{U_0})} + \|r_k - r\|_{C^1(\mathcal{D}_{U_0})} + \|m_k - m\|_{C^1(\mathcal{D}_{U_0})} \rightarrow 0. \quad (1.5.134)$$

This limit is also a solution to the spherically symmetric Einstein-Scalar field equations. Moreover, (r, ϕ, m) restricts correctly to the boundary data $(\hat{r}, \hat{\phi})$ and satisfies, for all v ,

$$\lim_{u \rightarrow -\infty} r \phi(u, v) = \lim_{u \rightarrow -\infty} \partial_v m(u, v) = \lim_{u \rightarrow -\infty} \partial_v (r \phi)(u, v) = 0 \quad (1.5.135)$$

as well as

$$\lim_{u \rightarrow -\infty} m(u, v) = \lim_{u \rightarrow -\infty} m(u, u) = M. \quad (1.5.136)$$

Assume now that also $R > 2.95M$. Then the following sharp bounds hold throughout $\mathcal{D}_{U_0}$, for sufficiently large negative values of U_0 :

$$m - M = \mathcal{O}\left(\frac{1}{|u|^{2p-2}r} + \frac{1}{|u|^{2p-1}}\right), \quad (1.5.137)$$

$$\mathbf{T}m = \mathcal{O}\left(\frac{1}{|u|^{2p-1}r} + \frac{1}{|u|^{2p}}\right), \quad (1.5.138)$$

$$\kappa - 1 = \mathcal{O}\left(\frac{1}{|u|^{2p-2}r^2}\right), \quad (1.5.139)$$

$$\lambda - \left(1 - \frac{2M}{r}\right) = \mathcal{O}\left(\frac{1}{|u|^{2p-2}r}\right), \quad (1.5.140)$$

$$\nu + \lambda = \mathbf{T}r = \mathcal{O}(|u|^{-\min(s, 2p-1)}), \quad (1.5.141)$$

⁵³Notice that, for technical reasons, we here cut off $\mathbf{T}(\hat{r}\hat{\phi})$ rather than $\hat{r}\hat{\phi}$ itself.

$$|r\phi| \leq C_{\text{in},\phi}|u|^{-p+1}, \quad (1.5.142)$$

$$|\partial_v(r\phi)| \leq MC_{\text{in},\phi} \frac{|u|^{-p+1}}{r^2}, \quad (1.5.143)$$

$$|\partial_u(r\phi) + \partial_v(r\phi)| = |\mathbf{T}(r\phi)| \leq C_{\text{in},\phi}^1 |u|^{-p}, \quad (1.5.144)$$

$$|\partial_v \mathbf{T}(r\phi)| \leq \eta MC_{\text{in},\phi}^1 \frac{|u|^{-p}}{r^2}, \quad (1.5.145)$$

where $C_{\text{in},\phi} = C_{\text{in},\phi}^1/(p-1)$, $\eta > 1$ can be chosen arbitrarily close to 1 as $U_0 \rightarrow -\infty$ and all the constants implicit in $\mathcal{O}$ only depend on initial data.

Finally, the following limit exists and is non-zero:

$$\lim_{u \rightarrow -\infty} |u|^{p-1} r\phi(u, v) =: \Phi^- \neq 0. \quad (1.5.146)$$

More precisely, we have, for $s \neq 1$, that

$$r\phi(u, v) = \frac{\Phi^-}{|u|^{p-1}} + \mathcal{O}\left(\frac{1}{|u|^{p-1+\min(\epsilon, 2p-2, s, |s-1|)}} + \frac{1}{r|u|^{p-1}}\right), \quad (1.5.147)$$

and, for $s = 1$,

$$r\phi(u, v) = \frac{\Phi^-}{|u|^{p-1}} + \mathcal{O}\left(\frac{1}{|u|^{p-1} \log^2 |u|} + \frac{1}{r|u|^{p-1}}\right). \quad (1.5.148)$$

If $s \leq 1$, then the above limit (1.5.146) is given by $\Phi^- = C_{\text{in},\phi}/(p-1)$.

Remark 1.5.13. The lower bound $R > 2.95M$ is only necessary for the bounds (1.5.142)–(1.5.145), which otherwise still hold with slightly worse constants, and for the statement that $\Phi^- \neq 0$. In other words, it is only necessary for the proof of *lower* bounds, not of upper bounds. We expect that it can be improved.

Proof. First, notice that the reason that one can take the lower bound on R to be just $R > 2M$ is explained in Remarks 1.5.3 and 1.5.9. The first part of the theorem, namely that (r_k, ϕ_k, m_k) converges to a solution (r, ϕ, m) which restricts correctly to the boundary data, is then shown as in the proof of Thm. 1.5.3, now using the improved decay on $\mathbf{T}(r\phi)_k$ from Thm. 1.5.5. As discussed in Remark 1.5.5, the fact that we now have sharp decay for $\partial_u(r\phi)$ at our disposal removes the necessity to assume $p \geq 2$ as well as the smallness assumption on $C_{\text{in},\phi}$. Moreover, one can perform a similar argument to show convergence in higher derivative norms as well.

We thus obtain a limiting solution which, as before, satisfies all the bounds from Theorems 1.5.1, 1.5.4 and 1.5.5.

Furthermore, the improvements in the bounds (1.5.137), (1.5.139) and (1.5.140) can be obtained from redoing the proof of Theorem 1.5.1 with the improved bound on $\mathbf{T}(r\phi)$ from Thm. 1.5.5.

The bounds (1.5.138), (1.5.141), (1.5.142) and (1.5.143) come directly from Theorems 1.5.1 and 1.5.4, where, for the latter two bounds, we used that $r\phi$ also satisfies a lower bound provided that $R > 2.95$ (as shown in Thm. 1.5.3, estimate (1.5.91)), which allows us to improve C' to $C_{\text{in},\phi}$.

With similar reasoning as for $r\phi$ and $\partial_v(r\phi)$, one derives the improvements in the estimates (1.5.144) and (1.5.145) from Thm. 1.5.5 by showing that $\mathbf{T}(r\phi)$ also satisfies a lower bound and, in particular, has a sign. (This is done in the same way as for $r\phi$ in the proof of Thm. 1.5.3.)

To prove the final part of the theorem, we note that, if $s \leq 1$, it is trivial to show that $|u|^{p-1}r\phi$ attains a limit by looking at

$$|u|^{p-1}r\phi(u, v) = |u|^{p-1}r\phi(u, u) + |u|^{p-1} \int_u^v \partial_v(r\phi) \, dv'.$$

On the other hand, if $s \neq 1$, then we can show that $\partial_u(|u|^{p-1}r\phi)$ is integrable using the results of Lemma 1.5.1:

$$\begin{aligned} -\partial_u(|u|^{p-1}r\phi) &= (p-1)|u|^{p-2} \left(r\phi - \frac{|u|}{p-1} \partial_u(r\phi) \right) \\ &= (p-1)|u|^{p-2} \left(r\phi - \frac{|u|}{p-1} \mathbf{T}(r\phi) \right) + |u|^{p-1} \partial_v(r\phi). \end{aligned}$$

The second term in the second line is bounded by r^{-2} and, thus, is integrable. The first term in the second line, on the other hand, has been dealt with in Lemma 1.5.1, see (1.5.126) and Remark 1.5.12, and is also integrable.

This concludes the proof. $\square$

1.5.8.2 Asymptotics of $\partial_v(r\phi)$ near $\mathcal{I}^+$, i^0 and $\mathcal{I}^-$

We now state the asymptotics for the limiting solution (r, ϕ, m) in a neighbourhood of spatial infinity. By the above theorem, we have completely reduced the problem to the null case. We can therefore reproduce the proofs of section 1.4.4 to conclude the following:

Theorem 1.5.7. *Consider the solution (r, ϕ, m) constructed in Theorem 1.5.6, and let $p = 2$ in equation (1.5.132). Then, throughout $\mathcal{D}_{U_0} \cap \{v > 1\}$, for sufficiently large negative values of U_0 , we get the following asymptotic behaviour for $\partial_v(r\phi)$:*

$$|\partial_v(r\phi)| \sim \begin{cases} \frac{\log r - \log |u|}{r^3}, & u = \text{constant}, v \rightarrow \infty, \\ \frac{1}{r^3}, & v = \text{constant}, u \rightarrow -\infty, \\ \frac{1}{r^3}, & v + u = \text{constant}, v \rightarrow \infty. \end{cases} \quad (1.5.149)$$

More precisely, for fixed u , we have the following asymptotic expansion as $\mathcal{I}^+$ is approached:

$$\left| \partial_v(r\phi)(u, v) + 2M\Phi^- r^{-3} \left(\log r - \log(|u|) - \frac{3}{2} \right) \right| = \mathcal{O}(r^{-3} \log^{-2}(|u|) + r^{-4}|u|). \quad (1.5.150)$$

The $\log^{-2}|u|$ -term above can be replaced by $|u|^{-\epsilon'}$ for ϵ' as in (1.5.127) if $s \neq 1$.

Similarly, we can deal with higher-order asymptotics, that is with the cases $p = 3, 4, \dots$. See also Thm. 1.4.3.

Finally, in view of Remark 1.5.7, Theorems 1.5.6 and 1.5.7 combined prove Theorem 1.2.1 from the introduction.

1.6 An application: The scattering problem

In the previous sections 1.4 and 1.5, our motivation for the choice of initial (/boundary) data mainly came from Christodoulou's argument; in particular, the data were chosen so as to lead to solutions that satisfy the no incoming radiation condition and that agree with the prediction of the quadrupole approximation, that is, we chose initial data such that we would obtain the rate

$$\partial_u m(u, \infty) \sim -\frac{1}{|u|^4} \quad (1.6.1)$$

at future null infinity.

Alternatively, we could have motivated our choice of initial data by the observation that our data can be chosen to be conformally smooth near $\mathcal{I}^-$ for integer p and, nevertheless, lead to solutions that are not conformally smooth near $\mathcal{I}^+$.

In this section, we give yet another extremely natural motivation for our initial data of section 1.4. More precisely, we shall show in section 1.6.1 that the case $p = 3$ appears *generically* in evolutions of compactly supported scattering data on⁵⁴ $\mathcal{H}^-$ and $\mathcal{I}^-$. Our main theorem is Thm. 1.6.1, which contains Theorem 1.2.5 from the introduction.

We shall make further comments on *linear* scattering in sections 1.6.2 and 1.6.3, where we will, in particular, prove that the corresponding solutions are *never conformally smooth* (unless they vanish identically).

1.6.1 Non-linear scattering with a Schwarzschild or Minkowskian i^- (Proof of Thm. 1.2.5)

The Maxwell field As in the timelike case (section 1.5), we will ignore the Maxwell field, that is, we set $e^2 = 0$. However, all results of the present section can be recovered for $e^2 \neq 0$ as well.

The setup Let $M > 0$, $U < -2M$, and define the rectangle

$$\mathcal{E}_U := (-\infty, U] \times (-\infty, \infty) \subset \mathbb{R}^2. \quad (1.6.2)$$

We refer to the set $(-\infty, U] \times \{-\infty\}$ as $\mathcal{H}^-$ (to be thought of as the past event horizon of Schwarzschild), to the point $\{-\infty\} \times \{-\infty\}$ as i^- or past timelike infinity, and we otherwise keep the conventions from section 1.4.1.

We will now show that if we pose compactly supported scattering data for $r\phi$ on $\mathcal{I}^-$ (see Figure 1.9) and vanishing data on $\mathcal{H}^-$, then $\mathcal{E}_U$ generically contains as a subset a set $\mathcal{D}_{U_0}$ as defined in (1.4.8), in which the corresponding scattering solution satisfies the assumptions of

⁵⁴Since we always restrict to a region sufficiently close $\mathcal{I}^-$, that is to sufficiently large negative values of u , we may without loss of generality assume vanishing data on $\mathcal{H}^-$.

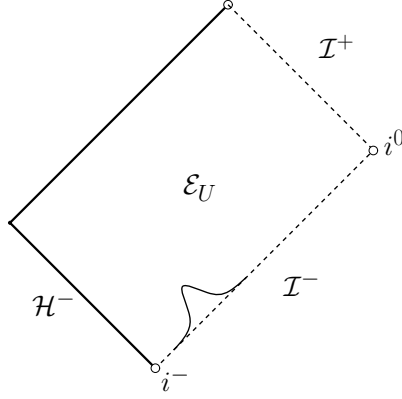


Figure 1.9 The Penrose diagram of $\mathcal{E}_U$. We pose compactly supported scattering data on $\mathcal{I}^-$ and $\mathcal{H}^-$. Since we are only interested in a region close to $\mathcal{I}^-$, we can, without loss of generality, set the data on $\mathcal{H}^-$ to be vanishing.

section 1.4.1 with $p = 3$, and hence, according to Theorem 1.4.3, has logarithmic terms at second highest order in the expansion of $\partial_v(r\phi)$ near $\mathcal{I}^+$.

Theorem 1.6.1. *Let $G(v)$ be a smooth compactly supported function, $\text{supp}(G) \subset (v_1, v_2)$. Then there exists a solution (r, ϕ, m) to the spherically symmetric Einstein-Scalar field system on $\mathcal{E}_U$ which satisfies $r|_{\mathcal{H}^-} = 2m|_{\mathcal{H}^-}$ on $\mathcal{H}^-$, and which satisfies $m(u, v) = M$ and $\phi(u, v) = 0$ for all $v \leq v_1$, and which finally satisfies $\lim_{u \rightarrow -\infty} r(u, v) = \infty$, $\partial_v r|_{\mathcal{I}^-}(v) = 1$, and $r\phi|_{\mathcal{I}^-}(v) = G(v)$ for all $v \in \mathbb{R}$. If we moreover fix $\partial_u r(u, v_2) = -1$ and $r(U, v_2) = -U$, then this solution is unique in the sense of Remark 1.5.7. We will call this solution the scattering solution.*

Furthermore, for sufficiently large negative values of U_0 , this scattering solution (r, ϕ, m) satisfies the following bounds throughout $\mathcal{E}_U \cap \{v \geq v_2\} \cap \{u \leq U_0\}$:

$$\left| r\phi(u, v) + \frac{I_0[G]}{u^2} \right| = \mathcal{O}(|u|^{-3}), \quad (1.6.3)$$

where $I_0[G]$ is a constant given by

$$I_0[G] := \int_{v_1}^{v_2} \left(M + \frac{1}{2} \int_{v_1}^v \left(\frac{dG}{dv} \right)^2 (v') dv' \right) G(v) dv. \quad (1.6.4)$$

In particular, by the results of section 1.4 (see Thm. 1.4.3), we have, for fixed values of u , the following asymptotic expression near $\mathcal{I}^+$ for $\partial_v(r\phi)$:

$$\left| \partial_v(r\phi)(u, v) - \frac{F(u)}{r^3} + 6\widetilde{M}I_0[G] \frac{\log(r) - \log|u|}{r^4} \right| = \mathcal{O}(r^{-4}), \quad (1.6.5)$$

where $F(u)$ is given by

$$F(u) = \int_{-\infty}^u \lim_{v \rightarrow \infty} (2m\nu r\phi)(u', v) du' = \frac{-2\widetilde{M}I_0[G]}{u} + \mathcal{O}(u^{-2}), \quad (1.6.6)$$

and where $\widetilde{M}$, the final value of the past Bondi mass, is given by

$$\widetilde{M} = \lim_{v \rightarrow \infty} m(-\infty, v) = M + \int_{v_1}^{v_2} \frac{1}{2} \left(\frac{dG}{dv} \right)^2 (v') dv' > M. \quad (1.6.7)$$

Finally, it is clear from its definition (1.6.4) that the constant $I_0[G]$ is generically non-zero (in an obvious sense).

Combined with Remark 1.6.1 below and the specialisation to the linear case described in section 1.3.3, this theorem proves Theorem 1.2.5 from the introduction.

Proof. We first restrict to $v < v_1$. There, by the domain of dependence property, the scattering solution exists and is identically Schwarzschild. The existence and uniqueness of the scattering solution for $v \geq v_1$ can then be obtained by combining the estimates of the present proof with the methods of section 1.5.6. (It is convenient to treat the regions $v \in [v_1, v_2]$ and $v > v_2$ separately.)

Let us now assume that we have already established the existence of the scattering solution. Then, we first note that the Hawking mass m on $\mathcal{I}^-$ is given by

$$m(-\infty, v) = M + \int_{v_1}^v \frac{1}{2} \left(\frac{dG}{dv} \right)^2 (v') dv', \quad (1.6.8)$$

which can be seen by integrating eq. (1.3.19) from i^- (and by standard limiting considerations, see the arguments below).

In the rest of the proof, we restrict to the region $v \in [v_1, v_2]$. We can then, using the monotonicity of the Hawking mass, redo the proofs of Propositions 1.4.3 and 1.4.4 to show that m , ν , λ and κ remain bounded from above, and away from zero, for $v \in [v_1, v_2]$. Moreover, we can apply the energy estimate as in the proof of Thm. 1.4.1 to show that $\sqrt{r}|\phi|$ is bounded from above as well, cf. (1.4.25).

In order to improve this bound on ϕ , we integrate the wave equation (1.3.24) from the ingoing null ray $v' = v_1$ (where ϕ vanishes), for $v \in [v_1, v_2]$:

$$|\partial_u(r\phi)(u, v)| \leq C \int_{v_1}^v r^{-\frac{5}{2}} dv' \leq \frac{C}{|u|^{\frac{5}{2}}} \quad (1.6.9)$$

for some positive constant C that depends only on initial data (in particular, C depends on $v_2 - v_1$) but which is allowed to change from line to line.

In turn, integrating estimate (1.6.9) from $\mathcal{I}^-$ implies that

$$|r\phi(u, v) - G(v)| \leq \frac{C}{|u|^{\frac{3}{2}}}.$$

Plugging this improved bound back into the wave equation and repeating the argument (1.6.9), we find that

$$|\partial_u(r\phi)(u, v)| \leq \frac{C}{|u|^3}$$

and, thus, by again integrating from $\mathcal{I}^-$,

$$|r\phi(u, v) - G(v)| \leq \frac{C}{u^2}.$$

With these decay rates for $r\phi$ and $\partial_u(r\phi)$, we can prove the analogue of Corollary 1.4.1; in particular, we can show that, for $v \in [v_1, v_2]$,

$$|\kappa(u, v) - 1| + |\nu(u, v) + 1| + |m(u, v) - m(-\infty, v)| = \mathcal{O}(u^{-2}).$$

To now obtain the asymptotic behaviour of $\partial_u(r\phi)$ along $v = v_2$, we calculate the v -derivative of $r^3\partial_u(r\phi)$: Using the above bounds, we find that

$$\begin{aligned} \partial_v(r^3\partial_u(r\phi)) \\ = 3r^2\lambda\partial_u(r\phi) + 2m\nu\kappa r\phi = -2 \left(M + \int_{v_1}^v \frac{1}{2} \left(\frac{dG}{dv} \right)^2 (v') dv' \right) G(v) + \mathcal{O}(|u|^{-1}). \end{aligned} \quad (1.6.10)$$

Integrating the estimate above from v_1 to v_2 yields

$$\left| r^3\partial_u(r\phi)(u, v_2) + \int_{v_1}^{v_2} 2 \left(M + \int_{v_1}^v \frac{1}{2} \left(\frac{dG}{dv} \right)^2 (v') dv' \right) G(v) dv \right| = \mathcal{O}(|u|^{-1}). \quad (1.6.11)$$

Since $\nu = -1$ on $v = v_2$, this puts us in precisely the setting of section 1.4.1 with $p = 3$: Indeed, integrating the equation (1.6.11) from $\mathcal{I}^-$ along $v = v_2$, we find

$$\left| r\phi(u, v_2) + \frac{1}{u^2} \int_{v_1}^{v_2} \left(M + \int_{v_1}^v \frac{1}{2} \left(\frac{dG}{dv} \right)^2 (v') dv' \right) G(v) dv \right| = \mathcal{O}(|u|^{-3}). \quad (1.6.12)$$

In fact, by Corollary 1.4.1, the same holds for any $v \geq v_2$. In particular, Thm. 1.4.3 applies with $p = 3$, with Φ^- given by

$$\Phi^- = - \int_{v_1}^{v_2} \left(M + \int_{v_1}^v \frac{1}{2} \left(\frac{dG}{dv} \right)^2 (v') dv' \right) G(v) dv, \quad (1.6.13)$$

and with M in (1.4.38) replaced by $\widetilde{M} = m(-\infty, v_2)$. This concludes the proof. $\square$

Remark 1.6.1 (Non-linear scattering for perturbations of Minkowski). In contrast to the setting in the previous section, we can now also have $M = 0$ and still see the logarithmic term. This is because the scattering data on $\mathcal{I}^-$ will always generate mass such that there will ultimately be

a mass term near i^0 . In particular, the results of Theorem 1.6.1 not only apply to scattering solutions with a Schwarzschildian i^- and compactly supported scattering data, but also to scattering solutions with a Minkowskian i^- (see the Penrose diagram below). This is because if one puts vanishing data on the center $r = 0$ and compactly supported data on $\mathcal{I}^-$, there will be a backwards null cone which is emanating from the center and on which $r\phi = 0$ by the domain of dependence property.

Moreover, we recall from Remark 1.2.4 that if the initial data on $\mathcal{I}^-$ are sufficiently small, then, according to [Chr93], the arising solution is causally geodesically complete and globally regular, it has a complete null infinity, and its Penrose diagram can be extended to a Minkowskian Penrose diagram as in Figure 1.10.

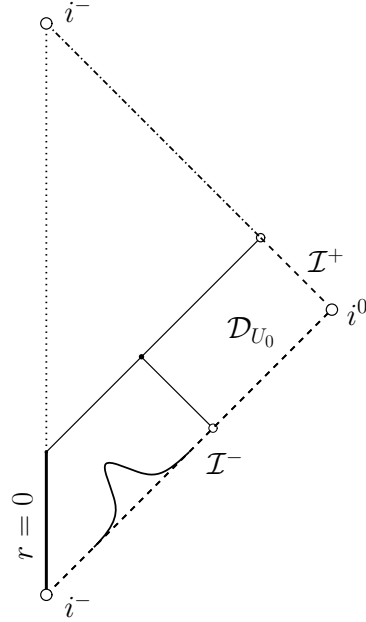


Figure 1.10 Scattering solution arising from compactly supported scattering data on $\mathcal{I}^-$ and a Minkowskian i^- . The solution fails to be conformally smooth near $\mathcal{I}^+$ by Theorem 1.6.1. Moreover, if the scattering data are suitably small, then the Penrose diagram can be extended to a Minkowskian Penrose diagram by the results of [Chr93].

1.6.2 Linear scattering on Schwarzschild

By the remarks in section 1.3.3, Theorem 1.6.1 also applies in the case of the linear wave equation on a fixed Schwarzschild background with mass $M > 0$ (see Figure 1.11 below). In the Eddington–Finkelstein double null coordinates⁵⁵ of section 1.3.2 (recall that $\partial_v r = -\partial_u r = 1 - \frac{2M}{r}$), the linear wave equation reads

$$\partial_u \partial_v (r\phi) = -2M \left(1 - \frac{2M}{r}\right) \frac{r\phi}{r^3}. \quad (1.6.14)$$

⁵⁵The gauge of the u -coordinate $\partial_u r = 1 - \frac{2M}{r}$ differs from our choice in the non-linear setting, where we set $\nu = -1$ on $v = v_2$. In these coordinates, the only difference to our results is then that the $\mathcal{O}(|u|^{-3})$ -term in (1.6.3) is replaced by an $\mathcal{O}(|u|^{-3} \log |u|)$ -term, and similarly for (1.6.6). This is completely inconsequential to any of our other results, however.

The only difference in the linear case is that $I_0[G]$ is now given by

$$I_0[G] := \int_{v_1}^{v_2} MG(v) dv, \quad (1.6.15)$$

since the scalar field no longer generates mass along past null infinity.

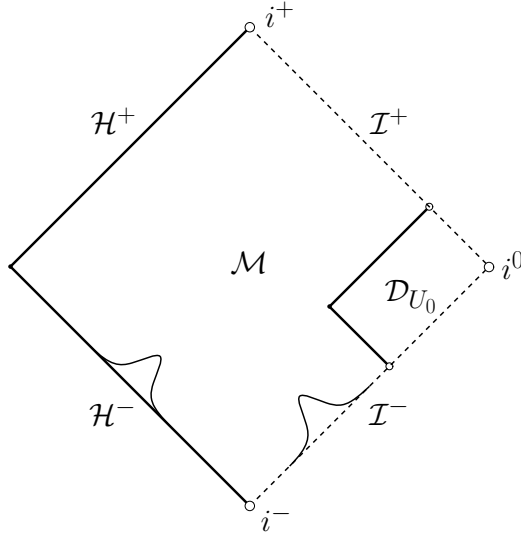


Figure 1.11 Smooth compactly supported scattering data for ϕ on a fixed Schwarzschild background. The solution generically contains a region $\mathcal{D}_{U_0}$ with $p = 3$ as in section 1.4. Moreover, the scalar field is *never* conformally smooth unless the scattering data vanish, see Theorem 1.6.2.

We now want to classify all spherically symmetric solutions to the linear wave equation arising from compactly supported scattering data in terms of their conformal smoothness near $\mathcal{I}^+$. We have already established that if

$$I_0[G] := \int_{v_1}^{v_2} MG(v) dv \neq 0,$$

then there will be a logarithmic term in the expansion of $\partial_v(r\phi)$ at order $\frac{\log r}{r^4}$. Let us now discuss the case when $I_0[G] = 0$. We prove the following theorem:

Theorem 1.6.2. *Prescribe compactly supported scattering data $G(v)$ on $\mathcal{I}^-$ and compactly supported scattering data on $\mathcal{H}^-$ for the spherically symmetric linear wave equation (1.6.14) on a fixed Schwarzschild background with mass $M > 0$. Then, by the results of [DRS18], there exists a unique smooth scattering solution ϕ attaining these data, with the uniqueness being understood in the class of finite-energy solutions.*

For $n \in \mathbb{N}_0$, define the scattering data constants $I^{(n)}[G]$ via

$$I^{(n)}[G] := M \int_{v_1}^{v_2} (-1)^n \frac{v^n}{n!} G(v) dv. \quad (1.6.16)$$

Let n denote the smallest natural number such that $I^{(n)}[G] \neq 0$.

Then the solution ϕ satisfies, for all $v \geq v_2$ and for all $u < U_0$, and for sufficiently large negative values of U_0 :

$$\left| r\phi(u, v) + \frac{I^{(n)}[G]}{|u|^{2+n}}(n+1)! \right| = \mathcal{O}(|u|^{-3-n} \log |u|). \quad (1.6.17)$$

Moreover, for fixed values of u , we have the following asymptotic expansion as $\mathcal{I}^+$ is approached:

$$\partial_v(r\phi) = \sum_{i=0}^n \frac{f_i^{(n)}(u)}{r^{3+i}} - (-1)^n(3+n)!I^{(n)}[G]M \frac{\log r - \log |u|}{r^{4+n}} + \mathcal{O}(r^{-4-n}) \quad (1.6.18)$$

for some smooth functions $f_i^{(n)}$.

This theorem shows, in particular, that the solution only remains conformally smooth near $\mathcal{I}^+$ if $G = 0$, that is to say, any smooth compactly supported linear scalar perturbation on $\mathcal{I}^-$ gives rise to a solution which is not conformally smooth.

Proof. The existence of the scattering solution ϕ follows by our previous methods or by the results of [DRS18]. The proof of the estimates (1.6.17) and (1.6.18) will be a proof via induction, with the base case having been established in Thm. 1.6.1. The crucial idea is to use *time integrals*.⁵⁶

To begin, let us state the following two basic facts: First, we have, for $v \in [v_1, v_2]$, and for any $n \in \mathbb{N}$:

$$\partial_u \partial_v (\partial_u^n(r\phi)) = - \left(2M \left(1 - \frac{2M}{r} \right) \frac{G(v)}{r^3} \right) \frac{(n+2)!}{2r^n} + \mathcal{O}(r^{-3-n-1}). \quad (1.6.19)$$

This can easily be established using the methods of the proof of Thm. 1.6.1. Secondly and similarly, we have that, for all $n \in \mathbb{N}$ and for all $v \geq v_2$,

$$\partial_u \partial_v (\partial_v^n(r\phi)) = - \left(2M \left(1 - \frac{2M}{r} \right) \frac{r\phi}{r^3} \right) (-1)^n \frac{(n+2)!}{2r^n} + \mathcal{O}(r^{-3-n-1}), \quad (1.6.20)$$

where this can easily be established from the asymptotics for $\partial_v(r\phi)$ proved in section 1.4.4 and an inductive argument. Note that both these facts also hold in the non-linear setting.

Let us now initiate the inductive step. We assume that (1.6.17) holds for some $n-1 \geq 0$, and we moreover assume that it commutes with ∂_u , that is to say, we assume that

$$\left| \partial_u^m \left(r\phi(u, v) + \frac{I^{(n-1)}[G]}{|u|^{2+n-1}}(n-1+1)! \right) \right| = \mathcal{O}(|u|^{-3-n+1-m} \log |u|) \quad (1.6.21)$$

⁵⁶These have been used in a similar context in [AAG18b].

for some $n - 1 \geq 0$, for all $m \in \mathbb{N}$, and for all $r\phi$ arising from compactly supported scattering data G such that $I^{(k)}[G] = 0$ for all $k < n - 1$. (That this holds in the base case $n = 1$ is an easy consequence of eqns. (1.6.12) and (1.6.19).)

Consider now compactly supported scattering data G such that $I^{(k)}[G] = 0$ for all $k < n$. These lead to a solution $r\phi$. The goal is to show that $r\phi$ can be written as $\mathbf{T}(r\phi^T)$, where $\mathbf{T} = \partial_u + \partial_v$, and where $r\phi^T$, the *time integral* of $r\phi$, is another solution coming from compactly supported data G^T such that $I^{(k)}[G^T] = 0$ for all $k < n - 1$. To achieve this, we take the obvious candidate for G^T :

$$G^T(v) = \int_{v_1}^v G(v') dv'. \quad (1.6.22)$$

Indeed, by the methods of the proof of Thm. 1.6.1, it is easy to see that the solution $r\phi^T$ arising from this satisfies

$$\mathbf{T}(r\phi^T)(-\infty, v) = \partial_v(r\phi^T)(-\infty, v) = (G^T)'(v) = G(v). \quad (1.6.23)$$

Therefore, since $\mathbf{T}$ also commutes with the wave equation⁵⁷, we indeed have $\mathbf{T}(r\phi^T) = r\phi$ by uniqueness.

It is left to show that $I^{(k)}[G^T] = 0$ for all $k < n - 1$. But this is an easy consequence of the fact that

$$\int_{v_1}^{v_2} \frac{v'^k}{k} G(v') dv' = - \int_{v_1}^{v_2} v'^{k-1} \int_{v_1}^{v'} G(v'') dv'' dv', \quad (1.6.24)$$

where we used that $I^{(k)}[G] = 0$ for all $k < n$. In particular, the above equation implies

$$I^{(n-1)}[G^T] = I^{(n)}[G], \quad (1.6.25)$$

and, similarly, that $I^{(k)}[G^T] = 0$ for all $k < n - 1$. From the induction assumption, it now follows that, for all $m \in \mathbb{N}$,

$$\left| \partial_u^m \left(r\phi^T(u, v) + \frac{I^{(n-1)}[G^T]}{|u|^{2+n-1}} n! \right) \right| = \mathcal{O}(|u|^{-3-n+1-m} \log |u|). \quad (1.6.26)$$

Finally, if we now write $r\phi(u, v) = \mathbf{T}(r\phi^T) = \partial_u(r\phi^T) + \partial_v(r\phi^T)$, then, as a consequence of the wave equation, the $\partial_v(r\phi^T)$ -term goes like $r\phi^T/|u|^2$ and is therefore sub-leading (for $v \geq v_2$). We thus obtain, for all $v \geq v_2$, that

$$\left| r\phi(u, v) + \frac{I^{(n)}[G]}{|u|^{2+n}} (n+1)! \right| = \mathcal{O}(|u|^{-3-n} \log |u|), \quad (1.6.27)$$

⁵⁷This is the only property used in this proof that fails to hold in the coupled case.

where we also used (1.6.25). One proceeds similarly for higher ∂_u -derivatives. (One can appeal to the wave equation (1.6.14) to deal with the arising $\partial_u \partial_v$ -terms.) This completes the inductive proof of eq. (1.6.17).

We proceed exactly in the same way for the proof of (1.6.18): We again make the inductive assumption that (1.6.18) holds for some n and moreover commutes with ∂_v^m for all m . By this, we mean the following: We assume that, for some fixed n and for all m ,

$$\partial_v^m(\partial_v(r\phi)) = \sum_{i=0}^n \frac{f_i^{(n,m)}(u)}{r^{3+i+m}} - (-1)^n(3+n)!I^{(n)}[G]M\partial_v^m\left(\frac{\log r}{r^{4+n}}\right) + \mathcal{O}(r^{-4-n-m}) \quad (1.6.28)$$

for all solutions $r\phi$ arising from compactly supported scattering data G that have $I^{(k)}[G] = 0$ for all $k < n$. Here, the $f_i^{(n,m)}$ are again some smooth functions. That this holds in the base case is a consequence of eq. (1.6.5) from Theorem 1.6.1 combined with eq. (1.6.20) and an inductive argument.

Then, in order to the inductive step, we assume that $r\phi$ is a solution arising from compactly supported scattering data G with $I^{(k)}[G] = 0$ for all $k < n$, and we anew write $r\phi$ as a time derivative, $r\phi = T(r\phi^T)$, and compute

$$\partial_v(r\phi)(u, v) = \partial_v^2(r\phi^T) + \partial_u \partial_v(r\phi^T) = \partial_v^2(r\phi^T) - 2M \left(1 - \frac{2M}{r}\right) \frac{r\phi^T}{r^3}. \quad (1.6.29)$$

It is then a simple exercise to write down the asymptotics for the second term by plugging in the asymptotics for $\partial_v(r\phi^T)$ into

$$r\phi^T(u, v) = r\phi^T(u, \infty) - \int_v^\infty \partial_v(r\phi^T)(u, v') dv'. \quad (1.6.30)$$

Leaving the details to the reader, one hence finds that the second term in (1.6.29) only produces log-terms at later orders than $\partial_v^2(r\phi^T)$ does, so the leading-order logarithmic contributions to the asymptotics of $\partial_v(r\phi)$ are determined by $\partial_v^2(r\phi^T)$. A similar argument works for higher derivatives.

This concludes the proof. $\square$

1.6.3 Linear scattering on extremal Reissner–Nordström

Finally, we remark that, by the “mirror symmetry” of the exterior of the extremal Reissner–Nordström spacetime [CT84] discussed in section 1.2.2.2 of this chapter (and the fact that all our results also apply when including a Maxwell field), we can state as an immediate corollary of our Thms. 1.6.1 and 1.6.2:

Corollary 1.6.1. *Consider the linear wave equation $\nabla^\mu \nabla_\mu \phi = 0$ on extremal Reissner–Nordström ($|e| = M$). Put smooth compactly supported spherically symmetric scattering data*

on $\mathcal{I}^-$ and on $\mathcal{H}^-$. Then, by the results of [AG20a], there exists a unique scattering solution attaining these data. This solution, in addition to not being conformally smooth near $\mathcal{I}^+$, fails to be smooth at the future event horizon $\mathcal{H}^+$ unless it vanishes identically, and one generically has that ϕ is not C^4 in the variable $r - r_+$.

This failure of the solution to remain smooth of course comes from the “mirrored” $\log(r - r_+)$ -terms of Theorem 1.6.1 that now appear in the ingoing derivative of ϕ instead of in $\partial_v(r\phi)$. Here, $r_+ = M$ is the r -value at $\mathcal{H}^+$.

Notice that this is in stark contrast to the Schwarzschild (or sub-extremal Reissner–Nordström) case, where the solution remains globally smooth in the exterior. This can be traced back to the existence of a bifurcation sphere in Schwarzschild, which does not exist in the extremal case. We refer the reader to section 1.2.2.2 for a more detailed discussion.

Appendix 1.A Useful curvature computations

In this appendix, we write down formulae for various curvature coefficients for the spherically symmetric Einstein-Scalar field system and, in particular, derive eq. (1.4.58). Recall that $g_{uv} = -\frac{1}{2}\Omega^2$ and $g_{AB} = r^2\gamma_{AB}$. Recall, moreover, that we use capital Latin letters to denote coordinates on the sphere.

We first compute the Christoffel symbols. The only non-vanishing ones are (apart from those related by symmetry and from Γ_{AB}^C):

$$\begin{aligned}\Gamma_{uu}^u &= \partial_u \log(\Omega^2), & \Gamma_{vv}^v &= \partial_v \log(\Omega^2), \\ \Gamma_{AB}^u &= -2\Omega^{-2}r\partial_u r\gamma_{AB}, & \Gamma_{AB}^v &= -2\Omega^{-2}r\partial_v r\gamma_{AB}, \\ \Gamma_{uB}^A &= \frac{\partial_u r}{r}\delta_B^A, & \Gamma_{vB}^A &= \frac{\partial_v r}{r}\delta_B^A.\end{aligned}\tag{1.A.1}$$

Next, we compute some of the Riemann tensor components, using the definition

$$R_{\nu\xi o}^\mu := \partial_\xi \Gamma_{\nu o}^\mu - \partial_o \Gamma_{\nu\xi}^\mu + \Gamma_{\xi\pi}^\mu \Gamma_{\nu o}^\pi - \Gamma_{o\pi}^\mu \Gamma_{\nu\xi}^\pi.\tag{1.A.2}$$

We have:

$$\begin{aligned}R_{AuBu} &= g_{AB}\frac{\zeta^2}{r^2}, & R_{AvBv} &= g_{AB}\frac{\theta^2}{r^2}, \\ R_{Avuv} &= R_{Auvu} = 0 = R_{ABuv}, \\ R_{3434} &= -\frac{\Omega^2}{2}\left(\Omega^2\frac{m}{r^3} - 2\partial_u\phi\partial_v\phi\right).\end{aligned}\tag{1.A.3}$$

Finally, we compute the Weyl curvature tensor

$$W_{\mu\nu\xi o} := R_{\mu\nu\xi o} - \frac{1}{2}(R_{\mu\xi}g_{\nu o} - R_{\mu o}g_{\nu\xi} + R_{\nu o}g_{\mu\xi} - R_{\nu\xi}g_{\mu o}) - \frac{R}{6}(g_{\mu\xi}g_{\nu o} - g_{\mu o}g_{\nu\xi})\tag{1.A.4}$$

by using the Einstein equations (here, T denotes the trace of $T_{\mu\nu}$, $T = g^{\mu\nu}T_{\mu\nu}$)

$$R_{\mu\nu} = 2T_{\mu\nu} - g_{\mu\nu}T\tag{1.A.5}$$

along with the fact that $T = \frac{4}{\Omega^2}\partial_u\phi\partial_v\phi$ to obtain that

$$W_{AuBu} = 0 = W_{AvBv} = W_{Auvu} = W_{Avuv} = W_{ABuv}\tag{1.A.6}$$

and

$$W_{uvuv} = -\frac{\Omega^4}{4}\frac{2m}{r^3} + \frac{8}{3}\Omega^2\partial_u\phi\partial_v\phi.\tag{1.A.7}$$

Appendix 1.B Constructing the time integral from characteristic initial data

In this appendix, we prove the analogue of Thm. 1.4.2 for general p , albeit only for the case of the linear wave equation. The crucial part of the proof is the construction of time integrals from characteristic initial data. Note that this has already been done in [AAG18c] and in an improved way in [AAG20b], Proposition 10.1. We will follow the approach of [AAG20b].

Theorem 1.B.1. *Consider the same setup as in section 1.4, but for the linear, uncoupled problem with Eddington–Finkelstein double null coordinates (u, v) (i.e. $\partial_v r = 1 - \frac{2M}{r} = -\partial_u r =: D$). Assume, moreover, that $2 \leq p \in \mathbb{Z}$. In view of Corollary 1.4.1, we then have, for all $v \geq 1$,*

$$r\phi(u, v) = \frac{\Phi_p^-}{|u|^{p-1}} + \mathcal{O}(|u|^{-p+1-\epsilon}). \quad (1.B.1)$$

Moreover, we have the following asymptotic expansion near $\mathcal{I}^+$:

$$\partial_v(r\phi) = \sum_{i=0}^{p-3} \frac{f_i^{(p)}(u)}{r^{3+i}} + (-1)^{p-1}(p-1)p \frac{\Phi_p^- M(\log r - \log |u|)}{r^{3+p-2}} + \mathcal{O}(r^{-3-p+2}) \quad (1.B.2)$$

for some smooth functions $f_i^{(p)}$.

Proof. Since we have already shown the result for $p = 2$, we will restrict to $p > 2$.

We want to use the time integral trick from the proof of Theorem 1.6.2. It is clear that the only ingredient missing in order to just repeat the proof from Thm. 1.6.2 is to show that we can write

$$r\phi = (\partial_u + \partial_v)(r\phi^T)$$

for some $r\phi^T$ that still solves the wave equation, which we shall call the time integral of $r\phi$.

In order to construct such a time integral, we note that if ϕ^T solves the wave equation (1.3.13), and if moreover $(\partial_u + \partial_v)\phi^T = \phi$, then we have

$$\begin{aligned} \partial_u(r^2\partial_u\phi^T) &= \partial_u(r^2\phi - r^2\partial_v\phi^T) = r\partial_u(r\phi) - Dr\phi - r^2\partial_u\partial_v\phi^T + 2Dr\partial_v\phi^T \\ &= r\partial_u(r\phi) - Dr\phi + rDT\phi^T = r\partial_u(r\phi). \end{aligned} \quad (1.B.3)$$

If we further impose that $r^2\partial_u\phi^T$ and ϕ^T vanish at $\mathcal{I}^-$, then it follows that

$$r\phi^T(u, v) = r \int_{-\infty}^u \frac{1}{r^2} \int_{-\infty}^{u'} r\partial_u(r\phi)(u'', v) du'' du'. \quad (1.B.4)$$

The rest of the proof then follows as in the proof of Theorem 1.6.2. $\square$

Chapter 2

A logarithmically modified Price's law

Abstract

In this chapter, we expand on the previous chapter's results by showing that the failure of “peeling” (and, thus, of smooth null infinity) at early times derived therein translates into logarithmic corrections at leading order to the well-known Price's law asymptotics near i^+ . This suggests that *the non-smoothness of $\mathcal{I}^+$ is physically measurable*.

More precisely, we consider the linear wave equation $\square_g \phi = 0$ on a fixed Schwarzschild background ($M > 0$), and we show the following: If one imposes initial data on an ingoing null hypersurface (extending to $\mathcal{H}^+$ and terminating at $\mathcal{I}^-$) that decay like $r\phi \sim |u|^{-1}$ as $u \rightarrow -\infty$, and vanishing data on $\mathcal{I}^-$ (this is the no incoming radiation condition), then the precise leading-order asymptotics of the solution ϕ are given by $r\phi|_{\mathcal{I}^+} = Cu^{-2} \log u + \mathcal{O}(u^{-2})$ along future null infinity, $\phi|_{r=R>2M} = 2C\tau^{-3} \log \tau + \mathcal{O}(\tau^{-3})$ along hypersurfaces of constant r , and $\phi|_{\mathcal{H}^+} = 2Cv^{-3} \log v + \mathcal{O}(v^{-3})$ along the event horizon. Moreover, the constant C is given by $C = 4MI_0^{(\text{past})}[\phi]$, where $I_0^{(\text{past})}[\phi] := \lim_{u \rightarrow -\infty} r^2 \partial_u (r\phi|_{\ell=0})$ is the *past Newman–Penrose constant* of ϕ on $\mathcal{I}^-$.

Thus, the precise late-time asymptotics of ϕ are completely determined by the early-time behaviour of the spherically symmetric part of ϕ near $\mathcal{I}^-$.

2.1	Introduction	116
2.1.1	The relation to Price's law	117
2.1.2	The main result (Theorem 2.1.1)	120
2.1.3	Structure of the chapter	122
2.2	The geometric setting	122
2.2.1	The Schwarzschild spacetime manifold	122
2.2.2	The spacelike-null foliation	123
2.2.3	Notational conventions	125

2.3 Preliminaries	125
2.3.1 The Cauchy problem for the wave equation	125
2.3.2 The modified Newman–Penrose constants $I_0[\phi]$, $I_0^{\log}[\phi]$ and $I_0^{\frac{\log r}{r^3}}[\phi]$	126
2.3.3 The main energy norms	126
2.3.4 The almost-sharp decay estimates	126
2.4 Asymptotics I: The case $I_0^{\log}[\phi] \neq 0$	127
2.4.1 The splitting of the spacetime and the region $\mathcal{B}_\alpha$	127
2.4.2 Asymptotics for $v^2\partial_v(r\phi)$ in the region $\mathcal{B}_\alpha$	128
2.4.3 Asymptotics for the radiation field $r\phi$ in $\mathcal{B}_\alpha$	129
2.4.4 Asymptotics for $T^k(r\phi)$ in the region $\mathcal{B}_{\alpha_k}$	131
2.4.5 Global asymptotics for the scalar field ϕ	134
2.4.6 Global asymptotics for $T^k\phi$	138
2.5 Time inversion for $I_0^{\log}[\phi] = 0$ and $I_0^{\frac{\log r}{r^3}}[\phi] \neq 0$	138
2.5.1 Construction of the time integral $\phi^{(1)}$	139
2.5.2 The time-inverted Newman–Penrose constant $I_0^{\log, (1)}[\phi]$	140
2.5.3 Initial energy norms for $\phi^{(1)}$	141
2.6 Asymptotics II: The case $I_0^{\log}[\phi] = 0$ and $I_0^{\frac{\log r}{r^3}}[\phi] \neq 0$	141
2.7 Proof of Theorem 2.1.1	143
2.8 Comments on higher-order asymptotics	145
2.9 Comments on the non-linear problem	147
Appendix 2.A Definitions of the main energy norms	149
2.A.1 The basic energy currents	149
2.A.2 Definition of the energy norms	149

2.1 Introduction

This chapter is concerned with the study of the precise late-time asymptotics of solutions to the wave equation

$$\square_g \phi = 0 \tag{2.1.1}$$

on the exterior of a fixed Schwarzschild (or a more general, spherically symmetric) background $(\mathcal{M}_M, g_M)$ with mass M under physically motivated assumptions on data. The most important of these assumptions is the *no incoming radiation condition* on $\mathcal{I}^-$, stating that the flux of the radiation field on past null infinity vanishes at late advanced times.

We initiated the study of such data in the previous chapter, where we constructed two classes of solutions¹ satisfying the no incoming radiation condition (as a condition on data

¹In fact, we also constructed solutions to the non-linear Einstein-Scalar field system in chapter 1.

on $\mathcal{I}^-$). The first class had polynomially decaying boundary data on a timelike boundary Γ terminating at i^- , whereas the second class had polynomially decaying characteristic initial data on an ingoing null hypersurface $\mathcal{C}_{\text{in}}$ terminating at $\mathcal{I}^-$. The choice for these data was in turn motivated by an argument due to D. Christodoulou [Chr02], which showed that the assumption of *Sachs peeling* and, thus, of (conformally) smooth null infinity, is incompatible with the no incoming radiation condition and the prediction of the quadrupole formula for N infalling masses from i^- . Indeed, we proved that the solutions from chapter 1 described above are not only in agreement with the quadrupole formula (which predicts that $\partial_u(r\phi) \sim |u|^{-2}$ near i^0), but also lead to logarithmic terms in the asymptotic expansion of $\partial_v(r\phi)$ as $\mathcal{I}^+$ is approached, thus contradicting the statement of Sachs peeling that such expansions can be expanded in powers of $1/r$. Roughly speaking, we obtained for the spherically symmetric mode ϕ_0 that if the limit

$$\lim_{\mathcal{C}_{\text{in}}, u \rightarrow -\infty} |u|r\phi_0 := \Phi^- \quad (2.1.2)$$

on initial data is non-zero (or if a similar condition on Γ holds), then, for sufficiently large negative values of u , one obtains on each outgoing null hypersurface of constant u the asymptotic expansion

$$\partial_v(r\phi_0)(u, v) = -2M\Phi^- \frac{\log r - \log |u|}{r^3} + \mathcal{O}(r^{-3}). \quad (2.1.3)$$

On the other hand, we will show in the next chapters 3 that higher ℓ -modes, under similar assumptions, decay slower, $\partial_v(r\phi_\ell) \sim r^{-2}$, with logarithmic terms appearing at order $r^{-3} \log r$ (or at a later order, depending on the setting).

The above results give a complete picture for the situation near $\mathcal{I}^-$. Naturally, one may then ask how the *early-time asymptotics* (2.1.3) translate into *late-time asymptotics* when one smoothly extends² the data on $\mathcal{C}_{\text{in}}$ (or Γ) all the way to the event horizon $\mathcal{H}^+$. In this work, we shall provide a detailed answer to this question. Let us already paraphrase the main statement (see also Figure 2.1.1):

Consider solutions ϕ to (2.1.1) which arise from the no incoming radiation condition and from smooth data on $\mathcal{C}_{\text{in}}$ that satisfy (2.1.2). Then their leading-order asymptotic behaviour towards i^+ is determined by the spherical mean ϕ_0 and contains logarithmic terms. Thus, the non-smoothness of null infinity near i^0 propagates and translates into logarithmic tails near i^+ .

2.1.1 The relation to Price's law

Showing a statement like the above is closely related to the task of proving Price's law. We recall that Price's law [Pri72, GPP94] roughly states that the evolutions of compactly supported Cauchy data under (2.1.1) satisfy the following asymptotics: $\phi|_{\mathcal{H}^+} \sim v^{-3}$ along the event

²It turns out that the leading-order asymptotics will not depend on the extension.

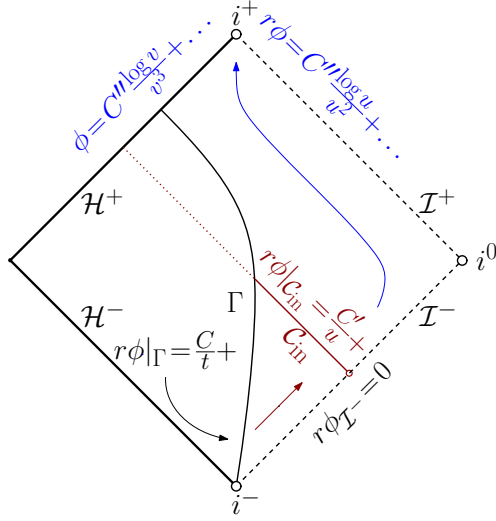


Figure 2.1.1 Schematic depiction of the results of the present chapter. We can either impose data $r\phi|_{\Gamma} = C/t + \dots$, which will, by our previous results from chapter 1, lead to behaviour on $\mathcal{C}_{\text{in}}$ as depicted, or we can directly impose data on $\mathcal{C}_{\text{in}}$. In both cases, we obtain logarithmic late-time asymptotics near i^+ provided the data are extended to $\mathcal{H}^+$.

horizon, $\phi|_{r=\text{constant}} \sim t^{-3}$ along hypersurfaces of constant r , and $r\phi|_{\mathcal{I}^+} \sim u^{-2}$ along future null infinity.

A rigorous proof of these asymptotics has only recently been obtained by Angelopoulos, Aretakis and Gajic [AAG18c, AAG18b] (see also the works [Hin22], [MZ22b] and [DSS11, DSS12], as well as [AAG19] for refined asymptotics). They, in fact, show that the leading-order asymptotics are determined by the $\ell = 0$ -mode, as higher ℓ -modes decay at least half a power faster. Their proof is split up into two parts. In the first [AAG18c], they derive *almost-sharp* decay estimates with an ϵ -loss, based on an extension of the r^p -method introduced in [DR10]. In the second part [AAG18b], they then use these almost-sharp estimates, together with certain conservation laws along null infinity $\mathcal{I}^+$ and a clever splitting into different spacetime regions, to obtain the precise leading-order asymptotics of the spherical mean. The upshot of this is that all the results in the first part [AAG18c] are, in some sense, blind to logarithmic corrections; the ϵ -loss in the almost-sharp decay estimates “swallows” the log-terms. Therefore, in order to find the late-time asymptotics of solutions coming from initial data satisfying (2.1.3), we only need to suitably adapt the second part of their proof [AAG18b] and can use the results of [AAG18c] as black box results.

Let us give some more detail on this second part: In a first step, they consider spherically symmetric initial data on a hyperboloidal slice Σ_0 (which extends to $\mathcal{H}^+$ and terminates at $\mathcal{I}^+$) and assume that the following limit exists and is non-vanishing:

$$\lim_{r \rightarrow \infty} r^2 \partial_r (r\phi_0)(u=0, r, \omega) =: I_0[\phi] < \infty. \quad (2.1.4)$$

Now, the crucial observation is that the quantity

$$\lim_{r \rightarrow \infty} r^2 \partial_r (r\phi_0)(u, r, \omega) =: I_0[\phi](u) \equiv I_0[\phi] \quad (2.1.5)$$

(called the *Newman–Penrose constant*) is, in fact, conserved along null infinity. It is this conservation law which is then exploited to derive the asymptotics of $r\phi_0$ in spacetime.

In a second step, they then consider spherically symmetric data for which $I_0[\phi] = 0$, and require that, in the spirit of peeling (i.e. smoothness in the conformal variable $s = 1/r$),

$$\lim_{r \rightarrow \infty} r^3 \partial_r(r\phi_0)(u = 0, r, \omega) < \infty. \quad (2.1.6)$$

There is no conservation law directly associated to this quantity. This difficulty is overcome by constructing the *time integral* $\phi^{(1)}$ of ϕ (which satisfies $T\phi^{(1)} = \phi$, where T is the stationary Killing field on Schwarzschild). It is shown that, generically, this time integral has a non-vanishing Newman–Penrose constant $I_0[\phi^{(1)}]$, which moreover can be computed from data for ϕ . The authors of [AAG18b] call this quantity the *time-inverted Newman–Penrose constant*:

$$I_0[\phi^{(1)}] =: I_0^{(1)}[\phi]. \quad (2.1.7)$$

If this quantity is non-vanishing (which it is, generically), then one can apply the methods from the first step to $\phi^{(1)}$ in order to find its precise asymptotics, and then convert these asymptotics of $\phi^{(1)}$ to asymptotics of ϕ by commuting with T . This then proves Price’s law. If, on the other hand, $I_0^{(1)}[\phi] = 0$, then one can construct the time integral of $\phi^{(1)}$ and proceed inductively to obtain faster decay.

Now, the conservation law (2.1.5) is, in fact, a special case of the more general statement that, under suitable assumptions,

$$\lim_{r \rightarrow \infty} f(r) \partial_r(r\phi_0)(u, r, \omega) =: I_0^{f(r)}[\phi](u) \quad (2.1.8)$$

is conserved along $\mathcal{I}^+$ if finite initially and if $f(r)/r^3 \rightarrow 0$ as $r \rightarrow \infty$.

We will be interested in the cases $f(r) = r^{-i} \log r$, $i = 2, 3$: Recall that the initial data we are interested in are to satisfy (2.1.3). The modified Newman–Penrose constant associated to (2.1.3) is given by $I_0^{\frac{\log r}{r^3}}[\phi] \equiv -2M\Phi^-$. Even though this quantity is itself conserved along null infinity, it turns out to be easier to work with the associated modified Newman–Penrose constant of the time integral instead. We have the following relation:

$$I_0^{\frac{\log r}{r^2}}[\phi^{(1)}] = -I_0^{\frac{\log r}{r^3}}[\phi]. \quad (2.1.9)$$

In the main body of this chapter, we will then present a modification of the argument in [AAG18b] that replaces (2.1.4) with the condition

$$0 \neq \lim_{r \rightarrow \infty} \frac{r^2}{\log r} \partial_r(r\phi_0)(u = 0, r, \omega) := I_0^{\frac{\log r}{r^2}}[\phi] := I_0^{\log}[\phi] < \infty. \quad (2.1.10)$$

This will allow us to show a *logarithmically modified Price's law* for the $\ell = 0$ -mode, see Thm. 2.1.1.

Remark 2.1.1 (Higher ℓ -modes). Recall from the above that it was shown in [AAG18c] that, in the setting of compactly supported Cauchy data, higher ℓ -modes generally decay at least half a power faster towards i^+ than the $\ell = 0$ -mode. However, the setting we are interested in (motivated by our results in chapter 1 and the next chapter 3) is such that, on Σ_0 , the $\ell > 0$ -modes decay to leading order like $\partial_v(r\phi_\ell) \sim r^{-2}$, whereas the $\ell = 0$ -mode decays like $\partial_v(r\phi_0) \sim r^{-3} \log r$ – more than half a power faster than the $\ell > 0$ -modes – so one might think that the $\ell = 0$ -mode does not determine the leading-order asymptotics in our setting. However, recent work by Angelopoulos, Aretakis and Gajic [AAG21] indicates that, even in this setting, one can still expect higher ℓ -modes to decay slightly faster. In particular, one can still expect the asymptotics of the $\ell > 0$ -modes to be subleading compared to the asymptotics of the $\ell = 0$ -mode obtained in this chapter. This will be discussed in detail in chapters 3 and 4, see also the remarks below Theorem 2.1.1. We will later on also see, however, that there are physically relevant scenarios where all ℓ -modes contribute at the same order to the decay at late times! For now, we restrict our presentation to the $\ell = 0$ -mode.

2.1.2 The main result

Let us now state a rough version of the main result of this chapter (see §2.2.1 for our choice of coordinates). The precise statement is written down in Theorems 2.6.1 and 2.7.1.

Theorem 2.1.1. *Let $\mathcal{C}_{\text{in}} = \{v = v_0\}$ be an ingoing null hypersurface starting from $\mathcal{I}^-$ and extending to $\mathcal{H}^+$, and let $\epsilon_\phi > 0$. Assume spherically symmetric initial data ϕ for (2.1.1) on a Schwarzschild background which satisfy³*

$$\frac{\partial_u(r\phi)}{\partial_u r}(u, v_0) = \frac{I_0^{(\text{past})}[\phi]}{r^2} + \mathcal{O}_4(r^{-2-\epsilon_\phi}) \quad (2.1.11)$$

for $u < 0$ and $I_0^{(\text{past})}[\phi] \neq 0$, and which also satisfy the no incoming radiation condition

$$\lim_{u \rightarrow -\infty} r\phi(u, v) = 0 \quad (2.1.12)$$

for all $v \geq v_0$. Assume further that the data smoothly extend to $\mathcal{H}^+$ (or that an appropriate energy norm of ϕ is finite) on $\{v = v_0\}$. Then, for all $u, v > 0$, the solution satisfies the following asymptotics near i^+ :

$$\left| \phi|_{\mathcal{H}^+}(v) + 8MI_0^{(\text{past})}[\phi] \frac{\log(1+v)}{(1+v)^3} \right| \leq C(v+1)^{-3}, \quad (2.1.13)$$

³If f and g are functions depending only on one variable x , we say $f = \mathcal{O}_k(g)$ if there exist uniform constants $C_j > 0$ such that $|\partial_x^j f| \leq C_j |\partial_x^j g|$ for $j = 0, \dots, k$.

$$\left| \phi|_{r=\text{constant}}(\tau) + 8MI_0^{(\text{past})}[\phi] \frac{\log(1+\tau)}{(1+\tau)^3} \right| \leq C(\tau+1)^{-3}, \quad (2.1.14)$$

$$\left| r\phi|_{\mathcal{I}^+}(u) + 4MI_0^{(\text{past})}[\phi] \frac{\log(u+1)}{(u+1)^2} \right| \leq C(u+1)^{-2}, \quad (2.1.15)$$

where $C > 0$ is a constant completely determined by data. Moreover, we have for all $u < \infty$ that

$$\lim_{v \rightarrow \infty} \frac{r^3}{\log r} \frac{\partial_v(r\phi)}{\partial_v r}(u, v) = -2MI_0^{(\text{past})}[\phi]. \quad (2.1.16)$$

We believe that a few remarks are in order:

- The appearance of logarithmic terms in *higher-order asymptotics* is well-known (see, for instance, [AAG19, GWS94, BVW18]). Similarly, modifications to Price's law have also been derived for spacetimes with different asymptotics than Schwarzschild (see, e.g., [CLSY95, MW21]). In contrast, the statement of Theorem 2.1.1 is that, *under physically motivated assumptions* (rather than assuming compact support or conformal smoothness on a Cauchy hypersurface), there are logarithmic corrections to Price's law *at leading order*.
- The above theorem is obtained for the wave equation on a fixed Schwarzschild background. However, it is easy to see that the proof generalises to other spherically symmetric spacetimes, most notably the subextremal Reissner–Nordström spacetimes. Moreover, the methods presented in this chapter can easily be applied to [AAG20b] to also obtain similar results for *extremal* Reissner–Nordström spacetimes. In this case, however, the asymptotics would depend crucially on the extension of the data to $\mathcal{H}^+$, in view of the *Aretakis constant* along $\mathcal{H}^+$. See also [AAG18a]. The generalisation to Kerr, on the other hand, will be the subject of future work (see also the recent [AAG23] and [Hin22]).
- The above theorem is formulated for initial data on an ingoing null hypersurface $\mathcal{C}_{\text{in}}$, however, by the results of chapter 1, an entirely analogous statement holds for boundary data on a past-complete timelike hypersurface Γ as considered in chapter 1 (see section 1.5.8.1 therein) which are suitably extended to $\mathcal{H}^+$.
- The above theorem is obtained for spherically symmetric solutions ϕ . However, as was mentioned before, the results of [AAG21] indicate that, even without symmetry assumptions, Theorem 2.1.1 gives the precise asymptotics since the higher ℓ -modes can be expected to decay faster. We will discuss the precise early- and late-time asymptotics of higher ℓ -modes in detail in chapters 3 and 4. In fact, we will find various different scenarios in chapter 3: In the case of polynomially decaying boundary data on a timelike hypersurface Γ , one can expect to recover a logarithmically modified Price's law ($r\phi_\ell|_{\mathcal{I}^+} \sim u^{-2-\ell} \log u$) for each ℓ -mode. In the case of polynomially decaying data on an ingoing null hypersurface $\mathcal{C}_{\text{in}}$, however, we will find that all higher ℓ -modes decay like $r\phi_\ell|_{\mathcal{I}^+} \sim u^{-2}$ along null infinity, i.e. one logarithm faster than the $\ell = 0$ -mode. Finally, in the case of smooth compactly

supported scattering data on $\mathcal{I}^-$ and $\mathcal{H}^-$, we will find that $r\phi_\ell|_{\mathcal{I}^+} \sim u^{-2}$ for all ℓ ! While the usual belief that higher ℓ -modes decay faster towards i^+ is hence violated on $\mathcal{I}^+$ in these settings, it still holds true away from $\mathcal{I}^+$, e.g. on hypersurfaces of constant r . See chapters 3 and 4 for details.

- The above theorem, in principle, gives a tool to *directly measure* the non-smoothness of future null infinity. (See also [AAG18a, Kro01] in this context.)

2.1.3 Structure of the chapter

This chapter is structured as follows: In §2.2, we shall introduce the geometry of the Schwarzschild spacetime and write down useful foliations of it. In §2.3, we then import the necessary theory for the wave equation and, in particular, the almost-sharp decay results of [AAG18c]. In §2.4, we shall derive the precise late-time asymptotics for ϕ in the case $I_0^{\log}[\phi] \neq 0$. We then derive the time inversion theory for the case $I_0^{\log}[\phi] = 0$ and $I_0^{\frac{\log r}{r^3}}[\phi] \neq 0$ in §2.5. Combining §2.4 and §2.5 then allows us to derive the precise late-time asymptotics for ϕ in the case $I_0^{\log}[\phi] = 0$ and $I_0^{\frac{\log r}{r^3}}[\phi] \neq 0$ in §2.6. We finally connect the results of §2.6 to our results obtained in chapter 1 and, thus, prove Theorem 2.1.1 in §2.7. We conclude the discussion of the linear wave equation by discussing higher-order asymptotics in §2.8.

Note that all the results of this chapter are obtained for the *linear* wave equation on Schwarzschild, despite the results of chapter 1 having been obtained for the coupled Einstein-Scalar field system as well. We therefore give two brief comments on potential extensions of the results of the present chapter to the coupled Einstein-Scalar field system in §2.9.

2.2 The geometric setting

2.2.1 The Schwarzschild spacetime manifold

We closely follow [AAG18c], with some minor adaptations:

The Schwarzschild family of spacetimes $(\mathcal{M}_M, g_M)$, $M > 0$, is given by the family of manifolds with boundary

$$\mathcal{M}_M = \mathbb{R} \times [2M, \infty) \times \mathbb{S}^2,$$

covered by the coordinate chart (v, r, θ, φ) with $v \in \mathbb{R}$, $r \in [2M, \infty)$, $\theta \in (0, \pi)$ and $\varphi \in (0, 2\pi)$, where (θ, φ) denote the standard spherical coordinates on $\mathbb{S}^2$, and by the family of metrics

$$g_M = -D(r) dv^2 + 2 dv dr + r^2(d\theta^2 + \sin^2 \theta d\varphi^2), \quad (2.2.1)$$

where

$$D(r) = 1 - \frac{2M}{r}. \quad (2.2.2)$$

Note that the vector field $T = \partial_v$ is a Killing vector field. We denote the boundary $\{r = 2M\} = \partial\mathcal{M}_M =: \mathcal{H}^+$ as the future event horizon.

Next, we introduce the *tortoise coordinate* r^* as

$$r^*(r) := R + \int_R^r D^{-1}(r') \, dr' \quad (2.2.3)$$

for some $R > 2M$ and define

$$u := v - 2r^*. \quad (2.2.4)$$

This gives rise to a covering (u, v, θ, φ) of $\mathcal{M}_M \setminus \mathcal{H}^+$ with $u \in (-\infty, \infty)$, $v \in (-\infty, \infty)$. The horizon is then “at $u = \infty$ ”. The metric in these double-null coordinates reads

$$g_M = -D(r) \, du \, dv + r^2 (d\theta^2 + \sin^2 \theta \, d\varphi^2). \quad (2.2.5)$$

We will drop the subscript M from now on.

With respect to the (u, v) -chart, we define the null vector fields

$$\underline{L} := \partial_u, \quad L := \partial_v.$$

We then have that

$$T = L + \underline{L}$$

and, in (v, r) -coordinates,

$$\underline{L} = -\frac{D}{2} \partial_r, \quad L = \frac{D}{2} \partial_r + \partial_v.$$

Remark 2.2.1. We can generalise our results to spacetimes which, instead of (2.2.2), have $D(r) = 1 - \frac{2M}{r} + \mathcal{O}_k(r^{-1-\gamma})$ for $\gamma > 0$ and for sufficiently large values of k , subject to the condition that these spacetimes satisfy certain Morawetz (integrated local energy decay) estimates (see sections 2.4.1 and 2.4.2 in [AAG18b]). Note that the sub-extremal Reissner–Nordström spacetime is such a spacetime, so the results of the present chapter also apply to sub-extremal Reissner–Nordström spacetimes.

2.2.2 The spacelike-null foliation

Let $h : [2M, \infty) \rightarrow \mathbb{R}_{\geq 0}$ be a non-negative, piecewise smooth function satisfying

$$0 \leq \frac{2}{D(r)} - h(r) = \mathcal{O}(r^{-1-\eta}) \quad (2.2.6)$$

for some constant $\eta > 0$. Let further $v_0 > 0$, and define $v_{\Sigma_0}(r)$ as well as the spherically symmetric hypersurface Σ_0 via

$$v_{\Sigma_0}(r) := v_0 + \int_{2M}^r h(r') dr', \quad \Sigma_0 := \{(v, r, \theta, \varphi) \mid v = v_{\Sigma_0}(r)\}. \quad (2.2.7)$$

By construction, Σ_0 is a spacelike-null hypersurface which crosses the event horizon and terminates at future null infinity (condition (2.2.6) ensures that v (or u) is monotonically increasing (or decreasing) in r along Σ_0).

For the sake of simpler notation, we will from now on restrict to examples of Σ_0 which moreover satisfy the following condition: There exist $2M < r_{\mathcal{H}} < 4M < r_{\mathcal{I}}$ and $v_0, u_0 > 0$ such that

$$\begin{aligned} \Sigma_0 \cap \{r \leq r_{\mathcal{H}}\} &= N_0^{\mathcal{H}} := \{v = v_0\} \cap \{r \leq r_{\mathcal{H}}\}, \\ \Sigma_0 \cap \{r \geq r_{\mathcal{I}}\} &= N_0^{\mathcal{I}} := \{u = u_0\} \cap \{r \geq r_{\mathcal{I}}\}, \end{aligned}$$

and such that, moreover, the part $\Sigma_0 \cap \{r_{\mathcal{H}} < r < r_{\mathcal{I}}\}$ is strictly spacelike. Furthermore, after potentially redefining u and r^* from eq. (2.2.3), we can choose $u_0 = 0$ and $r_{\mathcal{I}} = R$.

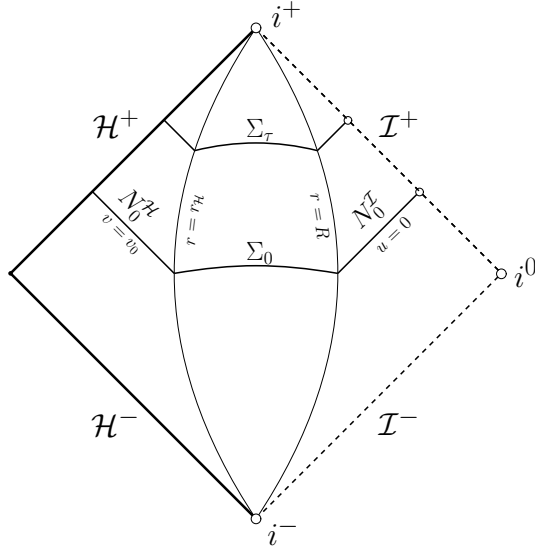


Figure 2.2.1 Depiction of the spacelike-null foliation of the Schwarzschild manifold $\mathcal{M}_M$ by the hypersurfaces Σ_τ .

Now, given Σ_0 , we define a time function $\tau : J^+(\Sigma_0) \rightarrow \mathbb{R}_{\geq 0}$ via the flow of the stationary Killing field as follows:

$$\tau|_{\Sigma_0} = 0, \quad T(\tau) = 1.$$

Let F_τ denote the flow of T , and define $\Sigma_\tau := F_\tau(\Sigma_0)$. This gives rise to a spacelike-null foliation of $J^+(\Sigma_0)$ (see Figure 2.2.1). Adapted to this foliation, we can cover $J^+(\Sigma_0)$ with

coordinates⁴ $(\tau, \rho, \theta, \varphi)$ with $\rho|_{\Sigma_0} = r|_{\Sigma_0}$ and ρ being constant along integral curves of T . In these coordinates, we have $T = \partial_\tau$, and the spherically symmetric vector field Y tangent to Σ_τ is given by

$$Y = \partial_\rho = \partial_r + h\partial_v = -\frac{2}{D}\underline{L} + hT.$$

We can then define the *red-shift vector field* N as follows:

$$N := T - Y \quad \text{in } \{2M \leq r \leq r_{\mathcal{H}}\}, \quad N := T \quad \text{in } \{r \geq r_{\mathcal{I}}\},$$

with the additional requirement that the smooth matching in $r_{\mathcal{H}} \leq r \leq r_{\mathcal{I}}$ is such that N remains time-invariant and strictly timelike.

2.2.3 Notational conventions

We use the notation $d\mu_{\Sigma_\tau}$ for the natural volume form on Σ_τ with respect to the induced metric, where, on the null parts of Σ_τ , this volume form is chosen to be $r^2 d\omega du$, $r^2 d\omega dv$, respectively, with $d\omega = \sin\theta d\theta d\varphi$. Similarly, we denote the normal to Σ_τ by n_{Σ_τ} , where we take the normals on the null parts to be $\underline{L}$, L , respectively.

We also say $f \sim g$ (or $f \lesssim g$) if there exists a uniform constant $C > 0$ such that $C^{-1}g \leq f \leq Cg$ (or $f \leq Cg$), and we use the usual algebra of constants ($C + D = C = CD \dots$).

2.3 Preliminaries

In this section, we recall the almost-sharp decay results obtained in [AAG18c, AAG18b]. We will first need to import some language.

2.3.1 The Cauchy problem for the wave equation

We recall the following standard result:

Proposition 2.3.1. *Let $\Phi \in C^\infty(\Sigma_0)$, $\Phi' \in C^\infty(\Sigma_0 \cap \{r_{\mathcal{H}} < r < r_{\mathcal{I}}\})$. Then there exists a unique smooth function $\phi : J^+(\Sigma_0) \rightarrow \mathbb{R}$ satisfying*

$$\phi|_{\Sigma_0} = \Phi, \quad n_{\Sigma_0}(\phi)|_{\Sigma_0 \cap \{r_{\mathcal{H}} < r < r_{\mathcal{I}}\}} = \Phi',$$

and

$$\square_g \phi = 0.$$

⁴Note that, for $\tau \geq 1$, we have $\tau \sim v$ for $r \leq r_{\mathcal{H}}$, $\tau \sim v \sim u$ for $r_{\mathcal{H}} \leq r \leq r_{\mathcal{I}}$, and $\tau \sim u$ for $r \geq r_{\mathcal{I}}$.

2.3.2 The modified Newman–Penrose constants $I_0[\phi]$, $I_0^{\log}[\phi]$ and $I_0^{\frac{\log r}{r^3}}[\phi]$

Let ϕ be a solution to the wave equation in the sense of Proposition 2.3.1. Let moreover f be a smooth function such that $\lim_{r \rightarrow \infty} r^{-3}f(r) = 0$. Then we define

$$I_0^f[\phi](u) := \frac{1}{4\pi} \lim_{r \rightarrow \infty} \int_{\mathbb{S}^2} f(r) \partial_r(r\phi)(u, r, \omega) d\omega. \quad (2.3.1)$$

It is shown e.g. in⁵ [AAG18c] that this quantity, if finite initially, is, in fact, independent of u . In this case, we can write

$$I_0^f[\phi](u) = I_0^f[\phi](u_0) =: I_0^f[\phi]. \quad (2.3.2)$$

We moreover introduce the following notation:

$$I_0^{\frac{\log r}{r^2}}[\phi] := I_0^{\log}[\phi], \quad I_0^{\frac{1}{r^2}}[\phi] := I_0[\phi].$$

The past Newman–Penrose constant We finally define the past analogue of the Newman–Penrose constant I_0 for scalar fields ϕ which solve $\square_g \phi = 0$ on all of $\mathcal{M}$:

$$I_0^{(\text{past})}[\phi](v) := \frac{1}{4\pi} \lim_{r \rightarrow \infty} \int_{\mathbb{S}^2} r^2 \partial_r(r\phi)(v, r, \omega) d\omega. \quad (2.3.3)$$

2.3.3 The main energy norms

In the sequel, we will refer to several initial data energy norms $E_k^N[\phi]$, $E_{0, I_0^{\log} \neq 0; k}^\epsilon[\phi]$, $\tilde{E}_{0, I_0^{\log} \neq 0; k}^\epsilon[\phi]$, $E_{0, I_0^{\log} = 0; k}^\epsilon[\phi]$ etc. These energy norms, which are defined on Σ_0 , measure the almost-sharp decay (with an ϵ -loss) and the regularity of the initial data on Σ_0 , see already Propositions 2.3.2 and 2.3.3. Since they are only used for the black box results of §2.3.4, their definitions are deferred to appendix 2.A. We remark already that, in the context of the scattering data we are ultimately interested in (which satisfy (2.1.3)), these energy norms will always be finite if enough regularity is assumed.

2.3.4 The almost-sharp decay estimates

We have now introduced all the necessary baggage to finally quote the following two black box results (these correspond to Proposition 5.2 and Corollary 7.6 from [AAG18b], respectively):

Proposition 2.3.2. *Let ϕ be a spherically symmetric solution of (2.1.1) in the sense of Proposition 2.3.1, let $k \in \mathbb{N}_0$, and assume that $E_{0, I_0^{\log} \neq 0; k+1}^\epsilon[\phi] < \infty$ for some $\epsilon \in (0, 1)$. Then*

⁵The proof there is only written for $f = r^2$, but it works for any smooth f as specified above.

there exists a constant $C(R, k, \epsilon)$ such that, for all $\tau \geq 0$:

$$|T^k \phi|(\tau, \rho) \leq C \sqrt{E_{0, I_0^{\log} \neq 0; k+1}^\epsilon[\phi]} (1 + \tau)^{-2-k+\epsilon}, \quad (2.3.4)$$

$$\sqrt{\rho + 1} \cdot |T^k \phi|(\tau, \rho) \leq C \sqrt{E_{0, I_0^{\log} \neq 0; k}^\epsilon[\phi]} (1 + \tau)^{-\frac{3}{2}-k+\epsilon}, \quad (2.3.5)$$

$$\rho \cdot |T^k \phi|(\tau, \rho) \leq C \sqrt{E_{0, I_0^{\log} \neq 0; k}^\epsilon[\phi]} (1 + \tau)^{-1-k+\epsilon}. \quad (2.3.6)$$

Proposition 2.3.3. *Let ϕ be a spherically symmetric solution of (2.1.1) in the sense of Proposition 2.3.1, let $k \in \mathbb{N}_0$, and assume that $\tilde{E}_{0, I_0^{\log} \neq 0; k+1}^\epsilon[\phi] < \infty$ for some $\epsilon \in (0, 1)$. Then there exists a constant $C(R, k, \epsilon)$ such that, for all $\tau \geq 0$:*

$$\sqrt{\rho + 1} (|NT^k(r\phi)| + |YT^k(r\phi)|)(\tau, \rho) \leq C \sqrt{\tilde{E}_{0, I_0^{\log} \neq 0; k+1}^\epsilon[\phi]} (1 + \tau)^{-\frac{5}{2}+\epsilon}. \quad (2.3.7)$$

2.4 Asymptotics I: The case $I_0^{\log}[\phi] \neq 0$

In this section, we derive the precise late-time asymptotics for spherically symmetric solutions ϕ to (2.1.1), evolving from initial data as in Proposition 2.3.1, which have finite $I_0^{\log}[\phi] \neq 0$. Let us from now on denote the causal future of Σ_0 as $\mathcal{R}$, $J^+(\Sigma_0) =: \mathcal{R}$.

We follow very closely section 8 of [AAG18b]. Even though the methods are essentially identical, all of the proofs in [AAG18b] require some adjustments in order to work in the case $I_0^{\log}[\phi] \neq 0$ (remember that in [AAG18b], it is assumed that $I_0[\phi] < \infty$). Since those parts which do not require adjustments usually make up for just a few lines, we here opt for a mostly self-contained presentation rather than frequently referring to [AAG18b]. Nevertheless, certain parts of our proofs will have a more detailed explanation in [AAG18b], in which case the reader will be informed of the precise reference.

2.4.1 The splitting of the spacetime and the region $\mathcal{B}_\alpha$

We define, for $\alpha \in (0, 1)$, the following subsets of $\mathcal{M}$:

$$\mathcal{B}_\alpha := \{r \geq R\} \cap \{0 \leq u \leq v - v^\alpha\}.$$

We moreover denote

$$\gamma_\alpha := \{v - u = v^\alpha\} \cap \{u \geq 0\};$$

this is a timelike hypersurface which contains part of the boundary of $\mathcal{B}_\alpha$. Without loss of generality, we assume that $v_{\gamma_\alpha}(u) \geq v_{r=R}(u)$ for all $u \geq 0$, where $v_{\gamma_\alpha}(u)$ is the unique v such that $(u, v) \in \gamma_\alpha$, and we similarly define $v_{r=R}(u)$ and $u_{\gamma_\alpha}(v)$.

In the sequel, we will split up $\mathcal{R}$ into the regions $\mathcal{B}_\alpha$ for some suitable α , $\mathcal{R} \cap \{r \geq R\} \setminus \mathcal{B}_\alpha$, and $\mathcal{R} \cap \{r \leq R\}$. See Figure 2.4.1 below. For the reader's convenience, we here collect a few relations between u, v and r which will frequently be used in the following: We have, throughout $\mathcal{B}_\alpha$, for sufficiently large R :

$$r \gtrsim v - u \geq v_{\gamma_\alpha}^\alpha(u) \gtrsim (u + 1)^\alpha, \quad (2.4.1)$$

$$r \gtrsim v - u \geq v - u_{\gamma_\alpha}(v) = v^\alpha, \quad (2.4.2)$$

$$v \geq u + 1 \geq \frac{v_{\gamma_\alpha}(u)}{2} \geq \frac{u + 1}{2}. \quad (2.4.3)$$

Moreover, we have throughout all of $\mathcal{R} \cap \{r \geq R\}$ that $\tau \sim u$ and that:

$$|(v - u - 1) - 2r| \lesssim \log r \lesssim \log v, \quad (2.4.4)$$

and thus, in particular, $r \sim v - u$. These relations can easily be checked using the definition of $\mathcal{B}_\alpha$ and eq. (2.2.3). The implicit constants in $\sim$ and $\lesssim$ depend only on M and R . Since $R > M$, they can, in fact, be chosen to depend only on R .

2.4.2 Asymptotics for $v^2 \partial_v(r\phi)$ in the region $\mathcal{B}_\alpha$

Throughout the rest of this section, we assume that ϕ is a smooth, spherically symmetric solution arising from initial data on Σ_0 . In addition to assuming that $I_0^{\log}[\phi] < \infty$, it will be convenient to also assume that the following limit is finite on initial data:

$$\lim_{r \rightarrow \infty} r^2 \left(\partial_r(r\phi)(u_0, r) - I_0^{\log}[\phi] \frac{\log r - \log 2}{r^2} \right) := I'_0[\phi]. \quad (2.4.5)$$

Let us then introduce the following L^∞ -norm on Σ_0 for $\beta > 0$:

$$P_{I_0^{\log}, I'_0, \beta}[\phi] := \left\| v^{2+\beta} \left(\partial_v(r\phi) - 2 \frac{I_0^{\log}[\phi] \log v}{v^2} - 2 \frac{I'_0[\phi]}{v^2} \right) \right\|_{L^\infty(\Sigma_0)}. \quad (2.4.6)$$

Our first proposition then concerns the asymptotics of $\partial_v(r\phi)$ in $\mathcal{B}_\alpha$:

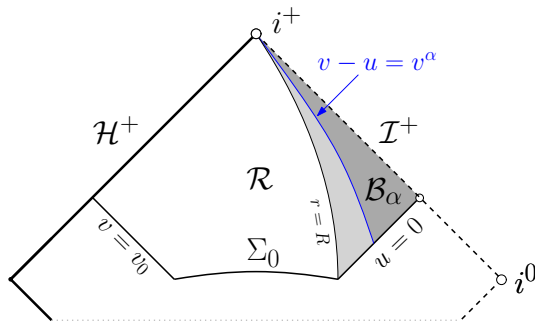


Figure 2.4.1 Depiction of $\mathcal{R} := J^+(\Sigma_0)$ and its subsets $\mathcal{B}_\alpha$ (to the right of the blue curve), $\mathcal{R} \cap \{r \geq R\} \setminus \mathcal{B}_\alpha$, and $\mathcal{R} \cap \{r \leq R\}$. The blue curve, in turn, corresponds to γ_α .

Proposition 2.4.1. *Let $\alpha \in (\frac{2}{3}, 1)$, $\epsilon \in (0, \frac{3\alpha-2}{2})$, and assume that $E_{0, I_0^{\log} \neq 0; 0}^\epsilon[\phi] < \infty$. If there exists $\beta > 0$ such that*

$$P_{I_0^{\log}, I_0', \beta}[\phi] < \infty, \quad (2.4.7)$$

then we have for all $(u, v) \in \mathcal{B}_\alpha$ that there exists a constant $C(R, \alpha, \epsilon) > 0$ such that

$$|v^2 \partial_v(r\phi)(u, v) - 2I_0^{\log}[\phi] \log v - 2I_0'[\phi]| \leq C \sqrt{E_{0, I_0^{\log} \neq 0; 0}^\epsilon[\phi]} \frac{1}{v^{3\alpha-2-2\epsilon}} + P_{I_0^{\log}, I_0', \beta}[\phi] v^{-\beta}. \quad (2.4.8)$$

Proof. The proof follows by integrating the wave equation for $r\phi$,

$$\partial_u \partial_v(r\phi) = -\frac{DD'}{4r} r\phi \left(\sim -\frac{r\phi}{r^3} \right) \quad (2.4.9)$$

(which is implied by (2.1.1) and where $'$ denotes r -differentiation), in u from initial data. This gives

$$\begin{aligned} |v^2 \partial_v(r\phi)(u, v) - v^2 \partial_v(r\phi)(0, v)| &\leq C v^{-(3\alpha-2-2\epsilon)} \int_0^u r^{-3} v^{3\alpha-2\epsilon} |r\phi|(u', v) du' \\ &\leq C \sqrt{E_{0, I_0^{\log} \neq 0; 0}^\epsilon[\phi]} v^{-(3\alpha-2-2\epsilon)} \int_0^u (u' + 1)^{-1-\epsilon} du', \end{aligned}$$

where we used the estimates (2.4.1), (2.4.2) and the almost-sharp decay estimate (2.3.6) for $r\phi$ with $k = 0$ (recall that $\tau \sim u$ in $\mathcal{B}_\alpha$). (Compare with the proof of Proposition 8.1 in [AAG18b].) $\square$

2.4.3 Asymptotics for the radiation field $r\phi$ in $\mathcal{B}_\alpha$

We will now use the asymptotics for $\partial_v(r\phi)$ obtained above to obtain decay for $r\phi$:

Proposition 2.4.2. *Under the assumptions of Proposition 2.4.1, with additionally $\alpha \in [\frac{5}{7}, 1)$ and $\epsilon \in (0, \frac{1}{6}(1 - \alpha))$, we have for all $(u, v) \in \mathcal{B}_\alpha$ that*

$$\begin{aligned} &\left| r\phi(u, v) - 2I_0^{\log}[\phi] \left(\frac{\log(u+1)+1}{u+1} - \frac{\log(v)+1}{v} \right) - 2I_0'[\phi] \left(\frac{1}{u+1} - \frac{1}{v} \right) \right| \\ &\leq C \left(\sqrt{E_{0, I_0^{\log} \neq 0; 0}^\epsilon[\phi]} + I_0^{\log}[\phi] + I_0'[\phi] \right) (u+1)^{\frac{\alpha}{2} - \frac{3}{2} + 2\epsilon} + C P_{I_0^{\log}, I_0', \beta}[\phi] (u+1)^{-1-\beta}, \end{aligned} \quad (2.4.10)$$

where $C = C(R, \epsilon, \alpha) > 0$ is a constant. In fact, if we further impose $\frac{1-\alpha}{2} < \beta + 2\epsilon$, then the estimate above provides asymptotics for $r\phi$ in the region $\mathcal{B}_\delta \subset \mathcal{B}_\alpha$, where δ is chosen such that $1 > \delta > \frac{\alpha+1}{2} + 2\epsilon > \alpha + 2\epsilon$.

In particular, setting $v = 0$, the estimate (2.4.10) provides us with asymptotics for $r\phi$ along $\mathcal{I}^+$.

Proof. Using the fundamental theorem of calculus, we write

$$r\phi(u, v) = r\phi(u, v_{\gamma_\alpha}(u)) + \int_{v_{\gamma_\alpha}(u)}^v \partial_v(r\phi)(u, v') dv'. \quad (2.4.11)$$

The boundary term can be bounded by writing $r\phi = r^{\frac{1}{2}}r^{\frac{1}{2}}\phi$, writing $r^{\frac{1}{2}}|_{\gamma_\alpha} \sim v_{\gamma_\alpha}^{\frac{\alpha}{2}} \sim (u+1)^{\frac{\alpha}{2}}$ by virtue of (2.4.4), (2.4.3) and the definition of γ_α , and finally using the almost-sharp decay estimate (2.3.5) with $k = 0$. We thus obtain:

$$|r\phi(u, v_{\gamma_\alpha})| \leq C \sqrt{E_{0, I_0^{\log} \neq 0; 0}^\epsilon[\phi]} (u+1)^{\frac{\alpha}{2} - \frac{3}{2} + \epsilon}.$$

In order to estimate the integral term, we plug in the result from the previous Proposition 2.4.1, resulting in the estimate:

$$\begin{aligned} & \left| \int_{v_{\gamma_\alpha}}^v \partial_v(r\phi)(u, v') dv' - 2I_0^{\log}[\phi] \left(\frac{\log v_{\gamma_\alpha}}{v_{\gamma_\alpha}} - \frac{\log v}{v} \right) - 2(I_0^{\log}[\phi] + I_0'[\phi]) \left(\frac{1}{v_{\gamma_\alpha}} - \frac{1}{v} \right) \right| \\ & \leq C \sqrt{E_{0, I_0^{\log} \neq 0; 0}^\epsilon[\phi]} (v_{\gamma_\alpha}^{-3\alpha+1+2\epsilon} - v^{-3\alpha+1+2\epsilon}) + CP_{I_0^{\log}, I_0', \beta}[\phi] (v_{\gamma_\alpha}^{-1-\beta} - v^{-1-\beta}). \end{aligned} \quad (2.4.12)$$

We first bound the terms from the LHS above. We write

$$v_{\gamma_\alpha}(u)^{-1} - v^{-1} = (v_{\gamma_\alpha}(u)^{-1} - (u+1)^{-1}) + ((u+1)^{-1} - v^{-1})$$

and estimate, using (2.4.1),

$$|v_{\gamma_\alpha}(u)^{-1} - (u+1)^{-1}| \leq (u+1)^{-1} v_{\gamma_\alpha}(u)^{\alpha-1} \leq C(u+1)^{-2+\alpha}.$$

Similarly, we write

$$\frac{\log v_{\gamma_\alpha}}{v_{\gamma_\alpha}} - \frac{\log v}{v} = \left(\frac{\log v_{\gamma_\alpha}(u)}{v_{\gamma_\alpha}(u)} - \frac{\log(u+1)}{u+1} \right) + \left(\frac{\log(u+1)}{u+1} - \frac{\log v}{v} \right)$$

and estimate

$$\begin{aligned} \frac{\log v_{\gamma_\alpha}(u)}{v_{\gamma_\alpha}(u)} - \frac{\log(u+1)}{u+1} &= \log v_{\gamma_\alpha}(u) \left(\frac{1}{v_{\gamma_\alpha}} - \frac{1}{u+1} \right) + \frac{1}{u+1} \log \frac{v_{\gamma_\alpha}(u)}{u+1} \\ &\leq C \frac{\log(u+1)}{(u+1)^{2-\alpha}} + \frac{1}{u+1} \log \left(1 + \frac{v_{\gamma_\alpha} - u - 1}{u+1} \right) \leq C \frac{\log(u+1)}{(u+1)^{2-\alpha}}, \end{aligned}$$

where, in order to obtain the last inequality, we used

$$\log \left(1 + \frac{v_{\gamma_\alpha} - u - 1}{u+1} \right) \leq \frac{v_{\gamma_\alpha} - u - 1}{u+1} \leq \frac{v_{\gamma_\alpha}^\alpha}{u+1} \leq C \frac{(u+1)^\alpha}{u+1}.$$

On the other hand, we estimate the terms on the RHS of (2.4.12) via

$$v_{\gamma_\alpha}(u)^{-1-\beta} - v^{-1-\beta} \leq v_{\gamma_\alpha}(u)^{-1-\beta} \leq (u+1)^{-1-\beta},$$

and identically for the $(v_{\gamma_\alpha}^{-3\alpha+1+2\epsilon} - v^{-3\alpha+1+2\epsilon})$ -term.

Finally, we can insert the estimates above back into (2.4.12) to find

$$\begin{aligned} & \left| \int_{v_{\gamma_\alpha}}^v \partial_v(r\phi)(u, v') \, dv' - 2I_0^{\log}[\phi] \left(\frac{\log(u+1)+1}{u+1} - \frac{\log(v)+1}{v} \right) - 2I'_0[\phi] \left(\frac{1}{u+1} - \frac{1}{v} \right) \right| \\ & \leq C \left(\sqrt{E_{0, I_0^{\log} \neq 0, 0}^\epsilon[\phi]} + I_0^{\log}[\phi] + I'_0[\phi] \right) (u+1)^{\frac{\alpha}{2} - \frac{3}{2} + 2\epsilon} + CP_{I_0^{\log}, I'_0, \beta}[\phi] (u+1)^{-1-\beta}, \end{aligned}$$

where we used that, for $\alpha \geq \frac{5}{7}$, we have

$$\max(-2 + \alpha, 1 - 3\alpha + 2\epsilon) \leq \frac{\alpha}{2} - \frac{3}{2} + 2\epsilon.$$

This concludes the proof of the first statement (2.4.10).

To see that (2.4.10) indeed gives the asymptotic behaviour in the region $\mathcal{B}_\delta$, we observe that, if $\epsilon \in (0, \frac{1}{6}(1 - \alpha))$, we have $\frac{\alpha}{2} - \frac{3}{2} + 2\epsilon < -1 - \epsilon$. Furthermore, we have in the region $\mathcal{B}_\delta$ that

$$\left| \frac{1}{u+1} - \frac{1}{v} \right| \geq \left| \frac{1}{u+1} - \frac{1}{v_{\gamma_\delta}} \right| = \left| \frac{v_{\gamma_\delta}^\delta}{v_{\gamma_\delta}(u+1)} \right| \sim (1+u)^{-2+\delta}.$$

Thus, if $1 > \delta > \frac{\alpha+1}{2} + 2\epsilon > \alpha + 2\epsilon$, and if moreover $\frac{1-\alpha}{2} < \beta + 2\epsilon$, we have $-2 + \delta > -1 - \beta$, and (2.4.10) indeed gives the asymptotic behaviour in $\mathcal{B}_\delta$. $\square$

2.4.4 Asymptotics for $T^k(r\phi)$ in the region $\mathcal{B}_{\alpha_k}$

We will now derive the asymptotics for $T^k(r\phi)$, $k > 0$. This will be crucial later on when going back from the time integral of a solution to the original solution.

In order to obtain the asymptotics for $T^k(r\phi)$ (Prop. 2.4.5), we again first derive the asymptotics for $\partial_v(T^k(r\phi))$ (Prop. 2.4.4). In turn, to derive the asymptotics for $\partial_v(T^k(r\phi))$, we will write $\partial_v T^k(r\phi) = \partial_v^{k+1}(r\phi) + \dots$, where the $\dots$ -terms denote terms which, by the wave equation, decay faster. We will therefore first derive the asymptotics for $\partial_v^{k+1}(r\phi)$ in Proposition 2.4.3.

In analogy to (2.4.6), we define the following higher-order analogues of the norm $P_{I_0^{\log}, I'_0, \beta}$ for $k \geq 0$:

$$P_{I_0^{\log}, I'_0, \beta; k}[\phi] := \max_{0 \leq j \leq k} \left\| v^{2+j+\beta} \partial_v^j \left(\partial_v(r\phi) - 2 \frac{I_0^{\log}[\phi] \log v}{v^2} - 2 \frac{I'_0[\phi]}{v^2} \right) \right\|_{L^\infty(\Sigma_0)}. \quad (2.4.13)$$

We then have

Proposition 2.4.3. *Let $k \in \mathbb{N}_0$, $\alpha_k \in (\frac{k+2}{k+3}, 1)$, let $\epsilon \in (0, \frac{1}{2}(k+3)\alpha - \frac{1}{2}(k+2))$, and assume that $E_{0, I_0^{\log} \neq 0; 0}^\epsilon[\phi] < \infty$. If moreover there exists $\beta > 0$ such that $P_{I_0^{\log}, I_0', \beta; k}[\phi] < \infty$, then we have for all $(u, v) \in \mathcal{B}_{\alpha_k}$:*

$$\begin{aligned} & \left| \partial_v^k \left(\partial_v(r\phi)(u, v) - 2I_0^{\log}[\phi] \frac{\log v}{v^2} + 2I_0'[\phi] \frac{1}{v^2} \right) \right| \\ & \leq CP_{I_0^{\log}, I_0', \beta; k}[\phi] v^{-2-k-\beta} + C \left(\sqrt{E_{0, I_0^{\log} \neq 0; 0}^\epsilon[\phi]} + I_0^{\log}[\phi] + I_0'[\phi] \right) v^{-(k+3)\alpha+2\epsilon}, \end{aligned} \quad (2.4.14)$$

where $C = C(R, \epsilon, \alpha_k, k) > 0$ is a constant.

Proof. This proof proceeds in the same way as the proof of Proposition 2.4.1, with the only difference being that we now inductively commute the wave equation (2.4.9) k times with ∂_v , multiply it with v^{k+2} , and only then integrate in u . See the proof of Proposition 8.3 of [AAG18b] for details. $\square$

Proposition 2.4.4. *Fix $k \in \mathbb{N}$. Under the assumptions of Proposition 2.4.3 and the additional assumption that $E_{0, I_0^{\log} \neq 0; k}^\epsilon[\phi] < \infty$, we have that*

$$\begin{aligned} & \left| \partial_v T^k(r\phi)(u, v) - \partial_v^k \left(2I_0^{\log}[\phi] \frac{\log v}{v^2} + 2I_0'[\phi] \frac{1}{v^2} \right) \right| \\ & \leq CP_{I_0^{\log}, I_0', \beta; k}[\phi] v^{-2-k-\beta} + C \left(\sqrt{E_{0, I_0^{\log} \neq 0; 0}^\epsilon[\phi]} + I_0^{\log}[\phi] + I_0'[\phi] \right) v^{-(k+3)\alpha+2\epsilon} \\ & + C \left(\sqrt{E_{0, I_0^{\log} \neq 0; k}^\epsilon[\phi]} + I_0^{\log}[\phi] + I_0'[\phi] \right) \sum_{l=0}^{k-1} r^{-3-l} (u+1)^{-k+l+\epsilon} \end{aligned} \quad (2.4.15)$$

for all $(u, v) \in \mathcal{B}_{\alpha_k}$, where $C = C(R, \epsilon, \alpha_k, k) > 0$ is a constant.

Proof. This proof is a consequence of the fact that

$$\partial_v T^k(r\phi) = \partial_v^{k+1}(r\phi) + \sum_{s=0}^{k-1} \sum_{\substack{l, m \geq 0; \\ l+m=k-1-s}} \mathcal{O}(r^{-2-s}) \partial_v^l T^m(r\phi),$$

combined with the results of the previous Proposition 2.4.3 and the estimate (2.3.5) from Proposition 2.3.2. See the proof of Proposition 8.4 in [AAG18c] for details. $\square$

Before we move on to the next proposition, we define a set of constants c_k via the relations

$$\partial_v^{k-1} \left(\frac{\log v}{v^2} \right) =: (-1)^k k! \frac{c_k + \log v}{v^{k+1}} \quad (2.4.16)$$

for $k \geq 1$ and set $c_0 := 1$. Note that $c_1 = 0$.

Proposition 2.4.5. Fix $k \in \mathbb{N}$. Under the assumptions of Proposition 2.4.4 and the additional assumptions that $\alpha_k \in [\frac{2k+5}{2k+7}, 1)$ and $\epsilon \in (0, \frac{1}{6}(1 - \alpha_k))$, we have that, for all $(u, v) \in \mathcal{B}_{\alpha_k}$,

$$\begin{aligned} & \left| T^k(r\phi)(u, v) - 2(-1)^k k! \left(I_0^{\log}[\phi] \left(\frac{\log(u+1) + c_k}{(u+1)^{k+1}} - \frac{\log(v) + c_k}{v^{k+1}} \right) \right. \right. \\ & \quad \left. \left. + I'_0[\phi] \left(\frac{1}{(u+1)^{k+1}} - \frac{1}{v^{k+1}} \right) \right) \right| \\ & \leq C \left(\sqrt{E_{0, I_0^{\log} \neq 0; k}^\epsilon[\phi]} + I_0^{\log}[\phi] + I'_0[\phi] \right) (u+1)^{-\frac{3}{2}-k+\frac{\alpha_k}{2}+2\epsilon} \\ & \quad + CP_{I_0^{\log}, I'_0, \beta; k}[\phi] (u+1)^{-1-k-\beta}, \end{aligned} \quad (2.4.17)$$

where $C = C(R, \epsilon, \alpha_k, k) > 0$ is a constant and c_k is defined in (2.4.16).

In fact, if we further impose $\frac{1-\alpha_k}{2} < \beta + 2\epsilon$, then the estimate above provides the asymptotics for $r\phi$ in the region $\mathcal{B}_{\delta_k} \subset \mathcal{B}_{\alpha_k}$ for $1 > \delta_k > \frac{\alpha_k+1}{2} + 2\epsilon > \alpha_k + 2\epsilon$.

In particular, we obtain the following asymptotics along $\mathcal{I}^+$:

$$\begin{aligned} & \left| T^k(r\phi)(u, \infty) - 2(-1)^k k! \left(I_0^{\log}[\phi] \frac{\log(u+1) + c_k}{(u+1)^{k+1}} + I'_0[\phi] \frac{1}{(u+1)^{k+1}} \right) \right| \\ & \leq C \left(\sqrt{E_{0, I_0^{\log} \neq 0; k}^\epsilon[\phi]} + I_0^{\log}[\phi] + I'_0[\phi] \right) (u+1)^{-1-k-\epsilon} \\ & \quad + CP_{I_0^{\log}, I'_0, \beta; k}[\phi] (u+1)^{-1-k-\beta}. \end{aligned} \quad (2.4.18)$$

Proof. The proof proceeds similarly to the proof of Proposition 2.4.2. We already proved the case $k = 0$ and can therefore restrict this proof to $k \geq 1$. We again apply the fundamental theorem of calculus in the v -direction, integrating from γ_{α_k} :

$$T^k(r\phi)(u, v) = T^k(r\phi)(u, v_{\gamma_{\alpha_k}}(u)) + \int_{v_{\gamma_{\alpha_k}}(u)}^v \partial_v T^k(r\phi)(u, v') dv. \quad (2.4.19)$$

We use the result of Proposition 2.4.4 to estimate the integral term:

$$\begin{aligned} & \left| \int_{v_{\gamma_{\alpha_k}}(u)}^v \partial_v T^k(r\phi)(u, v') dv' - 2 \left(\partial_v^{k-1} \left(\frac{I_0^{\log}[\phi] \log v}{v^2} + \frac{I'_0[\phi]}{v^2} \right) \right) \right|_{v_{\gamma_{\alpha_k}}(u)} \\ & \leq C \left(\sqrt{E_{0, I_0^{\log} \neq 0; k}^\epsilon[\phi]} + I_0^{\log}[\phi] + I'_0[\phi] \right) (v_{\gamma_{\alpha_k}}^{1-(k+3)\alpha_k+2\epsilon} - v^{1-(k+3)\alpha_k+2\epsilon}) \\ & \quad + CP_{I_0^{\log}, I'_0, \beta; k}(v_{\gamma_{\alpha_k}}^{-1-k-\beta} - v^{-1-k-\beta}) \\ & \quad + C \left(\sqrt{E_{0, I_0^{\log} \neq 0; k}^\epsilon[\phi]} + I_0^{\log}[\phi] + I'_0[\phi] \right) \sum_{l=0}^{k-1} \int_{v_{\gamma_{\alpha_k}}}^v r^{-3-l} (u+1)^{-k+l-\epsilon} dv'. \end{aligned}$$

We deal with the terms arising from the LHS by appealing to (2.4.16) and using

$$|v_{\gamma_{\alpha_k}}(u)^{-k-1} - (u+1)^{-k-1}| \leq C(u+1)^{-2-k+\alpha_k}$$

as well as

$$|v_{\gamma_{\alpha_k}}(u)^{-k-1} \log(u) - (u+1)^{-k-1} \log(u+1)| \leq C(u+1)^{-2-k+\alpha_k} \log(u+1).$$

Estimating the first and the second term on the RHS is done as in the proof of Proposition 2.4.2; we are hence left with the third term: Recalling (2.4.1), we find

$$\sum_{l=0}^{k-1} \int_{v_{\gamma_{\alpha_k}}}^v r^{-3-l} (u+1)^{-k+l+\epsilon} dv' \leq \sum_{l=0}^{k-1} \frac{C}{(u+1)^{k-l-\epsilon+(2+l)\alpha_k}} \leq C(u+1)^{-1+\epsilon-(k+1)\alpha_k}.$$

On the other hand, to estimate the boundary term in (2.4.19), we appeal to (2.3.5). This shows that the boundary term is bounded by (cf. (2.4.11))

$$|T^k(r\phi)(u, v_{\gamma_{\alpha_k}})| \leq C \sqrt{E_{0, I_0^{\log \neq 0; k}}^\epsilon[\psi]} (u+1)^{-\frac{3}{2}-k+\frac{\alpha_k}{2}+\epsilon}.$$

The first statement of the proposition, (2.4.17), now follows since, in view of $\alpha_k \geq \frac{2k+5}{2k+7}$ and $\epsilon \in (0, \frac{1}{2}(k+3)\alpha - \frac{1}{2}(k+2))$, all the relevant exponents of $(u+1)$ appearing above are dominated by $-\frac{3}{2}-k+\frac{\alpha_k}{2}+2\epsilon$.

On the other hand, to see that (2.4.17) provides asymptotics for $r\phi$ in the region $\mathcal{B}_{\delta_k}$ as specified, one proceeds as in the proof of Proposition 2.4.2. Compare with the proof of Proposition 8.5 in [AAG18b]. $\square$

2.4.5 Global asymptotics for the scalar field ϕ

In this section, we propagate the asymptotics obtained for $r\phi$ in $\mathcal{B}_\alpha$ into all of $\mathcal{R}$ and, in particular, into the region where $r \leq R$. In the region where r is large, this requires another splitting into different spacetime regions. On the other hand, in the region where r is small, we exploit that $\partial_\rho \phi$ exhibits good decay properties.

Proposition 2.4.6. *Let $\epsilon \in (0, \min(\frac{1}{98}, \beta))$, and assume that ϕ satisfies*

$$P_{I_0^{\log}, I_0', \beta}[\phi] < \infty$$

as well as

$$\tilde{E}_{0, I_0^{\log \neq 0; 1}}^\epsilon[\phi] < \infty.$$

Then we have for all $(u, v) \in \mathcal{R} \cap \{r \geq R\}$:

$$\begin{aligned} & \left| \phi(u, v) - \frac{4I_0^{\log}[\phi]}{v-u-1} \left(\frac{\log(u+1)}{u+1} - \frac{\log v}{v} \right) - 4 \frac{I_0^{\log}[\phi] + I'_0[\phi]}{(u+1)v} \right| \\ & \leq C \left(\sqrt{\tilde{E}_{0, I_0^{\log} \neq 0; 1}^\epsilon[\phi]} + I_0^{\log}[\phi] + I'_0[\phi] + P_{I_0^{\log}, I'_0, \beta}[\phi] \right) (u+1)^{-1-\epsilon} v^{-1}, \end{aligned} \quad (2.4.20)$$

where $C = C(R, \epsilon) > 0$ is a constant. On the other hand, we have, for another constant $C = C(R, \epsilon) > 0$, in all of $\mathcal{R} \cap \{r \leq R\}$:

$$\begin{aligned} & \left| \phi(\tau, \rho) - 4I_0^{\log}[\phi] \frac{\log(\tau+1)}{(\tau+1)^2} - 4I'_0[\phi] \frac{1}{(\tau+1)^2} \right| \\ & \leq C \left(\sqrt{\tilde{E}_{0, I_0^{\log} \neq 0; 1}^\epsilon[\phi]} + I_0^{\log}[\phi] + I'_0[\phi] + P_{I_0^{\log}, I'_0, \beta}[\phi] \right) (\tau+1)^{-2-\epsilon}. \end{aligned} \quad (2.4.21)$$

Proof. Let us first look at the case $(u, v) \in \mathcal{B}_\alpha$, with

$$\frac{5}{7} < \alpha < 1 - 6\epsilon \quad (2.4.22)$$

for some $\epsilon \in (0, \min(\frac{1}{21}, \beta))$ (which is four times the ϵ in the proposition). We essentially want to divide the estimate (2.4.10) (which is not an asymptotic estimate in all of $\mathcal{B}_\alpha$!) by r . Recalling (2.4.4), we have

$$\left| \frac{1}{2r} - \frac{1}{v-u-1} \right| \leq C \frac{\log v}{(v-u-1)^2}$$

and, using also (2.4.3),

$$\left| \frac{1}{r} \left(\frac{1}{u+1} - \frac{1}{v} \right) - \frac{2}{(u+1)v} \right| \leq C \frac{\log v}{r(u+1)v} \leq \frac{C}{(v-u-1)(u+1)^{3/2}}.$$

Dividing now (2.4.10) by r and making use of the two estimates above and also the fact that $\beta > \epsilon$, we obtain that

$$\begin{aligned} & \left| \phi(u, v) - \frac{4I_0^{\log}[\phi]}{v-u-1} \left(\frac{\log(u+1)}{u+1} - \frac{\log v}{v} \right) - 4 \frac{I_0^{\log}[\phi] + I'_0[\phi]}{(u+1)v} \right| \\ & \leq C \left(\sqrt{E_{0, I_0^{\log} \neq 0; 0}^\epsilon[\phi]} + I_0^{\log}[\phi] + I'_0[\phi] + P_{I_0^{\log}, I'_0, \beta}[\phi] \right) \frac{(u+1)^{\frac{\alpha-3}{2}+2\epsilon}}{v-u-1} \\ & \quad + C I_0^{\log}[\phi] \frac{\log v}{(v-u-1)^2} \left(\frac{\log(u+1)}{u+1} - \frac{\log v}{v} \right). \end{aligned} \quad (2.4.23)$$

In order to estimate the RHS, we need to restrict the region under consideration, namely $\mathcal{B}_\alpha$, to a smaller one, namely $\mathcal{B}_{\alpha+6\epsilon}$, and, moreover, partition this smaller region $\mathcal{B}_{\alpha+6\epsilon}$ into a region where v is large and one where $v \sim u+1$:

Asymptotics in $\mathcal{B}_{\alpha+6\epsilon} \cap \{v-u-1 > \frac{v}{2}\}$: In the region $\mathcal{B}_{\alpha+6\epsilon} \cap \{v-u-1 > \frac{v}{2}\}$, we estimate the first term on the RHS of (2.4.23) as follows:

$$\frac{(u+1)^{\frac{\alpha-3}{2}+2\epsilon}}{v-u-1} \leq 2 \frac{(u+1)^{\frac{\alpha-3}{2}+2\epsilon}}{v} < 2v^{-1}(1+u)^{-1-\epsilon}.$$

Here, we used (2.4.22) in the last estimate. Similarly, we estimate the second term of the RHS in (2.4.23) via

$$\frac{\log v}{(v-u-1)^2} \left(\frac{\log(u+1)}{u+1} - \frac{\log v}{v} \right) \leq C \frac{\log v}{v^2} \frac{\log(u+1)}{u+1} \leq Cv^{-1}(1+u)^{-1-\epsilon},$$

where we converted some v -decay into u -decay in the last estimate. This proves (2.4.20) in the region $\mathcal{B}_{\alpha+6\epsilon} \cap \{v-u-1 > \frac{v}{2}\}$.

Asymptotics in $\mathcal{B}_{\alpha+6\epsilon} \cap \{v-u-1 \leq \frac{v}{2}\}$: In the region $\mathcal{B}_{\alpha+6\epsilon} \cap \{v-u-1 \leq \frac{v}{2}\}$, we have, in particular, that $v \sim u+1$. Thus, using also that $v-u-1 \gtrsim v^{\alpha+6\epsilon}$ by definition of $\mathcal{B}_{\alpha+6\epsilon}$, we can estimate the first term on the RHS of (2.4.23) according to

$$\frac{(u+1)^{\frac{\alpha-3}{2}+2\epsilon}}{v-u-1} \leq Cv^{-1}v^{1-\alpha-6\epsilon}(u+1)^{\frac{\alpha-3}{2}+2\epsilon} \leq Cv^{-1}(u+1)^{-\frac{1+\alpha}{2}-4\epsilon}.$$

If we, in addition to (2.4.22), also require that $1-\alpha \leq 7\epsilon$, then we in fact have⁶

$$\frac{(u+1)^{\frac{\alpha-3}{2}+2\epsilon}}{v-u-1} \leq Cv^{-1}(u+1)^{-1-\frac{\epsilon}{2}}.$$

As for the second term on the RHS of (2.4.23), we simply write

$$\frac{\log v}{(v-u-1)^2} \left(\frac{\log(u+1)}{u+1} - \frac{\log v}{v} \right) \leq C \frac{1}{v^{2\alpha+12\epsilon}} \frac{\log^2(u+1)}{u+1} \leq Cv^{-1}(1+u)^{-1-\epsilon}.$$

This proves (2.4.20) in the region $\mathcal{B}_{\alpha+6\epsilon} \cap \{v-u-1 \leq \frac{v}{2}\}$.

Asymptotics in $\mathcal{R} \cap \{r \geq R\} \setminus \mathcal{B}_{\alpha+6\epsilon}$: We will use the fundamental theorem of calculus, integrating *inwards* along $N_\tau := \{r \geq R\} \cap \{u = \tau\}$ from $\gamma_{\alpha+6\epsilon}$, recalling the almost-sharp decay estimate

$$r^{\frac{1}{2}} |\partial_v \phi| \leq C \sqrt{\tilde{E}'_{0, I_0^{\log \neq 0; 1}}} (1+u)^{-\frac{5}{2}+\epsilon'}$$

⁶Note that this calculation would not have worked in the region $\mathcal{B}_\alpha$, hence the restriction to $\mathcal{B}_{\alpha+6\epsilon}$.

for $\epsilon' = \epsilon/4$, which follows directly from (2.3.7). Fixing moreover now

$$\alpha = 1 - 7\epsilon,$$

we can then follow the exact same steps of the proof of Proposition 8.6, pp. 59-60 in [AAG18b] to show that

$$|\phi(u, v) - \phi(u, v_{\gamma_{\alpha+6\epsilon}})| \leq C \sqrt{\tilde{E}_{0, I_0^{\log} \neq 0; 1}^{\epsilon'}} v^{-1} (1+u)^{-1-\frac{\epsilon}{4}}.$$

Plugging in the asymptotics for $\phi(u, v_{\gamma_{\alpha+6\epsilon}})$, which we have obtained already, and also noting that $E_{0, I_0^{\log} \neq 0; 0}^{\epsilon}[\phi] \leq \tilde{E}_{0, I_0^{\log} \neq 0; 1}^{\epsilon'}[\phi]$ for $\epsilon > \epsilon'$, we conclude the proof of (2.4.20) (notice that the ϵ in the proposition corresponds to ϵ' in the proof).

Asymptotics in $\mathcal{R} \cap \{r \leq R\}$: We finally extend the asymptotics into the region where $r \leq R$. We first need to convert the u - and v -decay from (2.4.20) into τ -decay on $r = R$. By definition, we have on $r = R$ that $v - u = R$ and $\tau = u$. Therefore, we have on $r = R$:

$$\begin{aligned} \frac{1}{v - u - 1} \left(\frac{\log(u+1)}{u+1} - \frac{\log v}{v} \right) &= \log(u+1) \frac{1}{v(u+1)} - \frac{\log(1 + \frac{R-1}{u+1})}{v(v-u-1)} \\ &= \frac{\log(\tau+1)}{(\tau+1)^2} - \frac{1}{(\tau+1)^2} + \mathcal{O}\left(\frac{\log(1+\tau)}{(1+\tau)^3}\right), \end{aligned}$$

where we used a standard estimate for $\log(1+x)$ in the last line. By the asymptotic estimate (2.4.20), we thus have

$$\begin{aligned} \left| \phi|_{\rho=R}(\tau) - 4I_0^{\log}[\phi] \frac{\log(\tau+1)}{(\tau+1)^2} - 4 \left((1-1)I_0^{\log}[\phi] + I_0'[\phi] \right) \frac{1}{(\tau+1)^2} \right| \\ \leq C \left(\sqrt{\tilde{E}_{0, I_0^{\log} \neq 0; 1}^{\epsilon}}[\phi] + I_0^{\log}[\phi] + I_0'[\phi] + P_{I_0^{\log}, I_0, \beta}[\phi] \right) (\tau+1)^{-2-\epsilon}. \end{aligned} \quad (2.4.24)$$

Finally, we have, by Proposition 2.3.3, that

$$\rho^{\frac{1}{2}} |\partial_{\rho} \phi| \leq C \sqrt{\tilde{E}_{0, I_0^{\log} \neq 0; 1}^{\epsilon}} (1+\tau)^{-\frac{5}{2}+\epsilon}.$$

Therefore, integrating along $\Sigma_{\tau} \cap \{r \leq R\}$ (recall that we set $r_{\mathcal{I}} = R$), we find

$$\begin{aligned} |\phi(\tau, \rho) - \phi(\tau, R)| &= \left| \int_{\rho}^R \partial_{\rho} \phi(\tau, \rho') d\rho' \right| \\ &\leq \int_{\tau}^R \rho^{-\frac{1}{2}} \rho^{\frac{1}{2}} |\partial_{\rho} \phi(\tau, \rho')| d\rho' \leq C \sqrt{\tilde{E}_{0, I_0^{\log} \neq 0; 1}^{\epsilon}} (1+\tau)^{-\frac{5}{2}+\epsilon}. \end{aligned} \quad (2.4.25)$$

Combining (2.4.24) and (2.4.25) completes the proof of the proposition. $\square$

2.4.6 Global asymptotics for $T^k \phi$

In order to apply our results to time derivatives of time integrals, we once again need to commute the global asymptotics of Proposition 2.4.6 with T .

Proposition 2.4.7. *Let $k \in \mathbb{N}_0$. There exists an $\epsilon > 0$ suitably small such that, under the assumptions $\tilde{E}_{0, I_0^{\log} \neq 0; k+1}^\epsilon[\phi] < \infty$ and $P_{I_0^{\log}, I_0', \beta; k}[\phi] < \infty$ for some $\beta > \epsilon$, we have that, for all $(u, v) \in \mathcal{R} \cap \{r \geq R\}$:*

$$\begin{aligned} & \left| T^k \phi(u, v) - 4(-1)^k k! \left(\frac{I_0^{\log}[\phi]}{v - u - 1} \left(\frac{\log(u+1)}{(u+1)^{k+1}} - \frac{\log v}{v^{k+1}} \right) \right. \right. \\ & \quad \left. \left. + (c_k I_0^{\log}[\phi] + I_0'[\phi]) \frac{1}{(u+1)^{k+1} v} \left(1 + \sum_{j=1}^k \left(\frac{u+1}{v} \right)^j \right) \right) \right| \\ & \leq C \left(\sqrt{\tilde{E}_{0, I_0^{\log} \neq 0; k+1}^\epsilon[\phi]} + I_0^{\log}[\phi] + I_0'[\phi] + P_{I_0^{\log}, I_0', \beta; k}[\phi] \right) (u+1)^{-k-1-\epsilon} v^{-1}, \end{aligned} \quad (2.4.26)$$

where $C = C(R, k, \epsilon) > 0$ and $c_k = c_k(k)$ are the constants defined in (2.4.16). In particular, $c_1 = 0$.

On the other hand, we have throughout $\mathcal{R} \cap \{r \leq R\}$ the estimate

$$\begin{aligned} & \left| T^k \phi(\tau, \rho) - 4(-1)^k k! \left(I_0^{\log}[\phi] \frac{\log(\tau+1)}{(\tau+1)^{2+k}} + ((c_k - 1) I_0^{\log}[\phi] + I_0'[\phi]) \frac{k+1}{(\tau+1)^{2+k}} \right) \right| \\ & \leq C \left(\sqrt{\tilde{E}_{0, I_0^{\log} \neq 0; k+1}^\epsilon[\phi]} + I_0^{\log}[\phi] + I_0'[\phi] + P_{I_0^{\log}, I_0', \beta; k}[\phi] \right) (\tau+1)^{-2-k-\epsilon}, \end{aligned} \quad (2.4.27)$$

where $C = C(R, k, \epsilon) > 0$ is a constant.

Proof. The proof follows the same structure as the proof of Proposition 2.4.6, now based on Proposition 2.4.4 instead of Proposition 2.4.2. The only additional ingredient required is the identity

$$\frac{1}{(u+1)^{k+1}} - \frac{1}{v^{k+1}} = \frac{1}{(u+1)^{k+1} v^{k+1} (v - u - 1)} \left(1 + \sum_{j=0}^k (u+1)^j v^{k-j} \right). \quad (2.4.28)$$

We refer the reader to the proof of Proposition 8.7 in [AAG18b] for more details. $\square$

2.5 Time inversion for $I_0^{\log}[\phi] = 0$ and $I_0^{\frac{\log r}{r^3}}[\phi] \neq 0$

In the previous section, we have derived the precise late-time asymptotics for solutions with $I_0^{\log}[\phi] \neq 0$. We now want to consider solutions with $I_0^{\log}[\phi] = 0$ and $I_0^{\frac{\log r}{r^3}}[\phi] \neq 0$. As explained in the introduction, we can reduce to the case $I_0^{\log}[\phi] \neq 0$ by considering the time integral $\phi^{(1)}$

of ϕ . The purpose of this section is thus to extract the conditions needed on ϕ so that we can apply the results of the previous section to its time integral $\phi^{(1)}$.

2.5.1 Construction of the time integral $\phi^{(1)}$

The approach of [AAG18b] does not allow one to directly construct the time integral for data with $I_0^{\frac{\log r}{r^3}}[\phi] \neq 0$. Therefore, we follow the more elegant approach of [AAG20b]:

Definition 2.5.1. *Let ϕ be a smooth, spherically symmetric solution to (2.1.1) in the sense of Proposition 2.3.1, with $I_0^{\log}[\phi]$ a well-defined limit. We then define the time integral $\phi^{(1)}$ of ϕ to be the unique spherically symmetric function $\phi^{(1)} : J^+(\Sigma_0) \rightarrow \mathbb{R}$ s.t.*

$$\begin{aligned} T\phi^{(1)} &= \phi, & \square_g \phi^{(1)} &= 0, \\ \lim_{v \rightarrow \infty} \phi^{(1)}(u_0, v) &= 0, & \lim_{u \rightarrow \infty} \underline{L}\phi^{(1)}(u, v_0) &= 0. \end{aligned}$$

Proposition 2.5.1. *If ϕ is as in the definition above, then its time integral $\phi^{(1)}$ satisfies along $N_0^{\mathcal{I}}$ the following relation:*

$$2r^2 L\phi^{(1)}(u_0, v) = C'_0[\phi] + 2 \int_{N_0^{\mathcal{I}} \cap \{v' \leq v\}} r L(r\phi)(u_0, v') dv', \quad (2.5.1)$$

where the constant $C'_0[\phi]$ is given by (writing $\Sigma_0 \cap \{r_{\mathcal{H}} \leq r \leq r_{\mathcal{I}}\} = \Sigma_0^*$)

$$4\pi C'_0[\phi] := 2 \int_{N_0^{\mathcal{H}}} r \underline{L}(r\phi) du' + 2 \int_{\Sigma_0^*} n_{\Sigma_0}(\phi) d\mu_{\Sigma_0} + 4\pi(r\phi|_{\Sigma_0 \cap \{r=r_{\mathcal{H}}\}} + r\phi|_{\Sigma_0 \cap \{r=r_{\mathcal{I}}\}}). \quad (2.5.2)$$

Let us moreover assume that $I_0^{\frac{\log r}{r^3}}[\phi] = \lim_{r \rightarrow \infty} \frac{r^3}{\log r} \partial_r(r\phi) < \infty$, and define $4\pi C_0[\phi] := 4\pi C'_0[\phi] + 2 \int_{N_0^{\mathcal{I}}} r L(r\phi)(u, v') d\omega dv'$.

Then we obtain the following additional relations along $N_0^{\mathcal{I}}$:

$$\phi^{(1)}|_{N_0^{\mathcal{I}}}(r) = -C_0 \int_r^\infty \frac{1}{Dr'^2} dr' + 2 \int_r^\infty \frac{1}{Dr'^2} \int_{r'}^\infty r'' \partial_r(r\phi)(u_0, r'') dr'' dr', \quad (2.5.3)$$

$$\begin{aligned} \partial_r(r\phi^{(1)})|_{N_0^{\mathcal{I}}}(r) &= C_0 \left(\frac{1}{Dr} - \int_r^\infty \frac{1}{Dr'^2} dr' \right) \\ &+ 2 \int_r^\infty \frac{1}{Dr'^2} \int_{r'}^\infty r'' \partial_r(r\phi)(u_0, r'') dr'' dr' - \frac{2}{Dr} \int_r^\infty r' \partial_r(r\phi)(u_0, r') dr', \end{aligned} \quad (2.5.4)$$

$$\partial_r^2(r\phi^{(1)})|_{N_0^{\mathcal{I}}}(r) = -C_0 \frac{D'}{D^2 r} + 2 \frac{D'}{D^2 r} \int_r^\infty r' \partial_r(r\phi)(u_0, r') dr' + \frac{2}{D} \partial_r(r\phi_0)|_{N_0^{\mathcal{I}}}(r). \quad (2.5.5)$$

Proof. A proof of the first statement is provided in Proposition 10.1 of [AAG20b]. We rewrite it as

$$L\phi^{(1)}(u_0, v) = \frac{C_0[\phi]}{2r^2} - \frac{1}{r^2} \int_v^\infty rL(r\phi)(u_0, v') dv'.$$

We then switch to (u, r) -coordinates and integrate (recall that $L = \frac{D}{2}\partial_r$) the above equality from $\mathcal{I}^+$, where $\phi^{(1)}$ vanishes by definition, to obtain the second statement. The last two statements are then obtained by multiplying by r and acting with ∂_r , ∂_r^2 , respectively. (Recall that $D' = \frac{d}{dr}D(r)$ etc.) $\square$

2.5.2 The time-inverted Newman–Penrose constant $I_0^{\log, (1)}[\phi]$

The following proposition expresses the Newman–Penrose constant $I_0^{\log}[\phi^{(1)}]$ in terms of ϕ :

Proposition 2.5.2. *Let ϕ be a smooth, spherically symmetric solution in the sense of Proposition 2.3.1, and let $\beta \in (0, 1)$ and J be constants. Assume that, on initial data, ϕ satisfies for some constant P :*

$$\left| \partial_r(r\phi)|_{N_0^{\mathcal{I}}}(r) - I_0^{\frac{\log r}{r^3}}[\phi] \frac{\log r}{r^3} - \frac{J}{r^3} \right| \leq Pr^{-3-\beta}. \quad (2.5.6)$$

Then there is a constant $C(R, P, \beta)$ such that the time integral $\phi^{(1)}$ of ϕ satisfies

$$\left| \partial_r(r\phi^{(1)})|_{N_0^{\mathcal{I}}}(r) + I_0^{\frac{\log r}{r^3}}[\phi] \frac{\log r}{r^2} + \frac{J + \frac{1}{2}I_0^{\frac{\log r}{r^3}}[\phi] - MC_0[\phi]}{r^2} \right| \leq Cr^{-2-\beta}, \quad (2.5.7)$$

$$\left| \partial_r^2(r\phi^{(1)})|_{N_0^{\mathcal{I}}}(r) - 2I_0^{\frac{\log r}{r^3}}[\phi] \frac{\log r}{r^3} - \frac{2J - 2MC_0[\phi]}{r^3} \right| \leq Cr^{-3-\beta}. \quad (2.5.8)$$

In particular, we have the following identities (recall the definition (2.4.5)):

$$I_0^{\log, (1)}[\phi] := I_0^{\log}[\phi^{(1)}] = -I_0^{\frac{\log r}{r^3}}[\phi], \quad (2.5.9)$$

$$I_0'[\phi^{(1)}] + \log(2)I_0^{\log}[\phi^{(1)}] = MC_0[\phi] - J - \frac{1}{2}I_0^{\frac{\log r}{r^3}}[\phi]. \quad (2.5.10)$$

More generally, we can also show that if the asymptotics for $\partial_r(r\phi)$ above commute with ∂_r^{k-1} on data, then the asymptotics for $\partial_r(r\phi^{(1)})$ commute with ∂_r^k on data.

Proof. We obtain (2.5.7) by plugging the estimate (2.5.6) into identity (2.5.4) and using that

$$\frac{1}{Dr} - \int_r^\infty \frac{1}{Dr'^2} dr' = \frac{M}{r^2} + \mathcal{O}(r^{-3}).$$

Similarly, we obtain (2.5.8) by plugging (2.5.6) into identity (2.5.5), noting that $D' = 2Mr^{-2}$. $\square$

We thus have as a direct corollary:

Corollary 2.5.1. *Under the assumptions of Proposition 2.5.2, we have that*

$$P_{I_0^{\log}, I_0', \beta}[\phi^{(1)}] < \infty \quad (2.5.11)$$

for $I_0^{\log}[\phi^{(1)}]$ and $I_0'[\phi^{(1)}]$ as in (2.5.9), (2.5.10). Moreover, we have

$$P_{I_0^{\log}, I_0', \beta, k}[\phi^{(1)}] < \infty \quad (2.5.12)$$

for $k = 1$, where the norms $P_{I_0^{\log}, I_0', \beta}$, $P_{I_0^{\log}, I_0', \beta, k}$ have been defined in eqns. (2.4.6) and (2.4.13), respectively.

2.5.3 Initial energy norms for $\phi^{(1)}$

Finally, in order to apply the results from section 2.4 to time integrals $\phi^{(1)}$ of initial data with $I_0^{\frac{\log r}{r^3}}[\phi] < \infty$, we need to estimate the relevant energy norms (namely $E_{0, I_0^{\log} \neq 0; k}^\epsilon$ and $\tilde{E}_{0, I_0^{\log} \neq 0; k}^\epsilon$) of $\phi^{(1)}$ in terms of initial data energy norms for ϕ . As these energy norms are blind to logarithmic corrections, we can simply quote the following result from Proposition 9.6 in [AAG18b]:

Proposition 2.5.3. *Let $k \in \mathbb{N}_0$, and let ϕ be a solution to the wave equation such that*

$$\tilde{E}_{0, I_0^{\log} = 0; k}^\epsilon[\phi] + \int_{\Sigma_0} J^N[N^2\phi] \cdot n_{\Sigma_0} d\mu_{\Sigma_0} < \infty \quad (2.5.13)$$

for some $\epsilon > 0$. Then there exist a constant $C = C(R, \Sigma_0, \epsilon, k) > 0$ such that the time integral $\phi^{(1)}$ of ϕ satisfies

$$E_{0, I_0^{\log} \neq 0; k+1}^\epsilon[\phi^{(1)}] \leq C \cdot \left(E_{0, I_0^{\log} = 0; k}^\epsilon[\phi] + \int_{\Sigma_0} J^N[N\phi] \cdot n_{\Sigma_0} d\mu_{\Sigma_0} \right), \quad (2.5.14)$$

$$\tilde{E}_{0, I_0^{\log} \neq 0; k+1}^\epsilon[\phi^{(1)}] \leq C \cdot \left(\tilde{E}_{0, I_0^{\log} = 0; k}^\epsilon[\phi] + \int_{\Sigma_0} J^N[N^2\phi] \cdot n_{\Sigma_0} d\mu_{\Sigma_0} \right). \quad (2.5.15)$$

2.6 Asymptotics II: The case $I_0^{\log}[\phi] = 0$ and $I_0^{\frac{\log r}{r^3}}[\phi] \neq 0$

We can now combine the results of sections 2.4 and 2.5 to derive the precise late-time asymptotics for solutions coming from smooth spherically symmetric initial data with $I_0^{\frac{\log r}{r^3}}[\phi] < \infty$. This is done by simply writing ϕ as a time derivative of its time integral $\phi^{(1)}$,

$$r\phi = T(r\phi^{(1)}),$$

for which then Propositions 2.4.4, 2.4.5 and 2.4.7 hold, assuming that the relevant energy and L^∞ -norms ($P_{I_0^{\log}, I_0', \beta}$ etc.) of $\phi^{(1)}$ are finite. This latter assumption can in turn be shown to hold using Proposition 2.5.3 and Corollary 2.5.1, respectively.

We summarise our findings in

Theorem 2.6.1. *Let $\epsilon > 0$ be sufficiently small. Let ϕ be a spherically symmetric solution to the wave equation arising from smooth initial data on Σ_0 (in the sense of Proposition 2.3.1) with $I_0^{\frac{\log r}{r^3}}[\phi] < \infty$, and assume that*

$$\tilde{E}_{0, I_0^{\log}=0;1}^\epsilon[\phi] + \int_{\Sigma_0} J^N[N^2\phi] \cdot n_{\Sigma_0} d\mu_{\Sigma_0} < \infty. \quad (2.6.1)$$

Assume moreover that there exist constants J , P and $\beta \in (\epsilon, 1)$ such that on initial data, for all $r \geq R$:

$$\left| \partial_r(r\phi)|_{N_0^T}(r) - I_0^{\frac{\log r}{r^3}}[\phi] \frac{\log r}{r^3} - \frac{J}{r^3} \right| \leq Pr^{-3-\beta}. \quad (2.6.2)$$

Then the time integral $\phi^{(1)}$ satisfies

$$I_0^{\log}[\phi^{(1)}] = -I_0^{\frac{\log r}{r^3}}[\phi], \quad (2.6.3)$$

$$I_0'[\phi^{(1)}] = MC_0[\phi] - J - \left(\frac{1}{2} + \log 2\right) I_0^{\frac{\log r}{r^3}}[\phi], \quad (2.6.4)$$

and

$$P_{I_0^{\log}, I_0', \beta}[\phi^{(1)}] + P_{I_0^{\log}, I_0', \beta;1}[\phi^{(1)}] + E_{0, I_0^{\log} \neq 0;2}^\epsilon[\phi] + \tilde{E}_{0, I_0^{\log} \neq 0;2}^\epsilon[\phi^{(1)}] < \infty, \quad (2.6.5)$$

where the P -norms have been defined in (2.4.6) and (2.4.13), respectively.

Moreover, we have the following asymptotic estimates for ϕ : For all $(u, v) \in \mathcal{R} \cap \{r \leq R\}$, we have:

$$\begin{aligned} & \left| \phi(\tau, \rho) + 4 \left(I_0^{\log}[\phi^{(1)}] \frac{\log(\tau+1)}{(\tau+1)^3} + (I_0'[\phi^{(1)}] - I_0^{\log}[\phi^{(1)}]) \frac{2}{(\tau+1)^3} \right) \right| \\ & \leq C \left(\sqrt{\tilde{E}_{0, I_0^{\log} \neq 0;2}^\epsilon[\phi^{(1)}]} + I_0^{\log}[\phi^{(1)}] + I_0'[\phi^{(1)}] + P_{I_0^{\log}, I_0', \beta;1}[\phi^{(1)}] \right) (\tau+1)^{-3-\epsilon}. \end{aligned} \quad (2.6.6)$$

On the other hand, we have that, for all $(u, v) \in \mathcal{R} \cap \{r \geq R\}$:

$$\begin{aligned} & \left| \phi(u, v) + \frac{4I_0^{\log}[\phi^{(1)}]}{v-u-1} \left(\frac{\log(u+1)}{(u+1)^2} - \frac{\log v}{v^2} \right) + \frac{4I_0'[\phi^{(1)}]}{(u+1)^2v} \left(1 + \frac{u+1}{v} \right) \right| \\ & \leq C \left(\sqrt{\tilde{E}_{0, I_0^{\log} \neq 0;2}^\epsilon[\phi^{(1)}]} + I_0^{\log}[\phi^{(1)}] + I_0'[\phi^{(1)}] + P_{I_0^{\log}, I_0', \beta;1}[\phi^{(1)}] \right) (u+1)^{-2-\epsilon} v^{-1}. \end{aligned} \quad (2.6.7)$$

In particular, the following asymptotics hold along $\mathcal{I}^+$:

$$\begin{aligned} & \left| r\phi(u, \infty) + 2 \left(I_0^{\log}[\phi^{(1)}] \frac{\log(u+1)}{(u+1)^2} + I'_0[\phi^{(1)}] \frac{1}{(u+1)^2} \right) \right| \\ & \leq C \left(\sqrt{E_{0, I_0^{\log} \neq 0; 1}^\epsilon} + I_0^{\log}[\phi^{(1)}] + I'_0[\phi^{(1)}] + P_{I_0^{\log}, I'_0, \beta; 1}[\phi^{(1)}] \right) (u+1)^{-2-\epsilon}. \end{aligned} \quad (2.6.8)$$

In each case, $C = C(R, \epsilon) > 0$ is a constant.

Proof. By Proposition 2.5.3 and Corollary 2.5.1, the statements (2.6.3), (2.6.4) and (2.6.5) for the time integral $\phi^{(1)}$ of ϕ follow. The remaining statements then follow by applying Propositions 2.4.5 and 2.4.7 to $\phi^{(1)}$ and $k = 1$, recalling that $c_k = 0$ for $k = 1$. $\square$

2.7 Proof of Theorem 2.1.1

We finally apply our results to the data of chapter 1 and, thus, prove the main theorem (Thm. 2.1.1):

Theorem 2.7.1. *Consider smooth, spherically symmetric initial/scattering data for (2.1.1) as in chapter 1, that is, assume that, on $\mathcal{C}_{\text{in}} := \{v = v_0\}$,*

$$\partial_u(r\phi)(u, v_0) = \frac{2I_0^{(\text{past})}[\phi]}{(1+u)^2} + \mathcal{O}_4((1+u)^{-2-\epsilon_\phi}) \quad (2.7.1)$$

for $u < 0$ and some $\epsilon_\phi > 0$, and that

$$\lim_{u \rightarrow -\infty} r\phi(u, v) = 0 \quad (2.7.2)$$

for all $v \geq v_0$. Moreover, assume that the data on $\{v = v_0\}$ extend smoothly to the event horizon $\mathcal{H}^+$.

Then there exist constants $J, P > 0$ and $\beta \in (0, \epsilon_\phi)$ such that the arising scattering solution satisfies

$$\left| \partial_r(r\phi)|_{N_0^{\mathcal{I}}}(r) - I_0^{\frac{\log r}{r^3}}[\phi] \frac{\log r}{r^3} - \frac{J}{r^3} \right| \leq Pr^{-3-\beta}, \quad (2.7.3)$$

where $I_0^{\frac{\log r}{r^3}}[\phi]$ is given by

$$I_0^{\frac{\log r}{r^3}}[\phi] = -2MI_0^{(\text{past})}[\phi]. \quad (2.7.4)$$

Moreover, we have on Σ_0 that

$$\tilde{E}_{0, I_0^{\log=0; 1}}^\epsilon[\phi] + \int_{\Sigma_0} J^N[N^2\phi] \cdot n_{\Sigma_0} d\mu_{\Sigma_0} < \infty. \quad (2.7.5)$$

In particular, Theorem 2.6.1 applies to ϕ .

Remark 2.7.1. An almost identical statement holds for the boundary value problem with polynomially decaying data on a spherically symmetric past-complete timelike hypersurface Γ as considered in chapter 1, see Theorem 1.2.1 therein. Moreover, a more detailed analysis shows that β can be chosen to equal ϵ_ϕ if $\epsilon_\phi < 1$, and that the RHS of (2.7.3) can be replaced by $Pr^{-4} \log r$ if $\epsilon_\phi \geq 1$. Lastly, it suffices to assume finite regularity for the data on $v = v_0$ and to assume that the energy expression

$$\sum_{i \leq 8, j \leq 1} \int_{\{v=v_0\}} \left(J^N[T^i \phi] + J^N[NT^j \phi] + J^N[N^2 \phi] \right) \cdot \underline{L} \, du$$

remains locally finite in a neighbourhood of $\mathcal{H}^+$.

Proof. We will, in this proof, change coordinates $(\bar{u}, \bar{v}) = (\frac{u}{2}, \frac{v}{2})$. Moreover, to match with the notation of chapter 1, we will also write $\Phi^- := I_0^{(\text{past})}[\phi]$. By the results of chapter 1 (Theorem 1.2.4 or, more precisely, Theorems 1.4.1 and 1.4.2 therein), a unique smooth solution assuming the initial/scattering data exists up to Σ_0 , and we have, for sufficiently large negative values of u , that

$$\left| r\phi(u, v) - \frac{\Phi^-}{|\bar{u}|} \right| \leq C|\bar{u}|^{-1-\epsilon_\phi} \quad (2.7.6)$$

for some uniform constant C .

In order to prove (2.7.3), we need to re-examine the steps of the proof of Theorem 1.4.2. By inserting the above estimate into the wave equation (2.4.9) and integrating the latter from past null infinity, we obtain that

$$\left| \partial_{\bar{v}}(r\phi)(\bar{u}, \bar{v}) + 2M \int_{-\infty}^{\bar{u}} D \frac{\Phi^-}{r^3 |\bar{u}'|} d\bar{u}' \right| \leq Cr^{-3} |\bar{u}|^{-\epsilon_\phi}$$

for sufficiently large negative values of u . Consider now the expression

$$\begin{aligned} & \partial_{\bar{u}} \left(r^3 \partial_{\bar{v}}(r\phi) + 2Mr^3 \int_{-\infty}^{\bar{u}} D \frac{\Phi^-}{r^3 |\bar{u}'|} d\bar{u}' \right) \\ &= \underbrace{2MD \left(\frac{\Phi^-}{|\bar{u}|} - r\phi \right)}_{\leq C|u|^{-1-\epsilon_\phi}} - \underbrace{3Dr^2 \left(\partial_{\bar{v}}(r\phi) + 2M \int_{-\infty}^{\bar{u}} D \frac{\Phi^-}{r^3 |\bar{u}'|} d\bar{u}' \right)}_{\leq Cr^{-1}|u|^{-\epsilon_\phi}}, \end{aligned} \quad (2.7.7)$$

and, after estimating, say, $r^{-1+\beta}$ against $|u|^{-1+\beta}$ for some $\beta < \epsilon_\phi$, integrate it from $-\infty$ to some fixed value $\bar{u}$ to obtain the more precise asymptotic expression:

$$\left| r^3 \partial_{\bar{v}}(r\phi)(\bar{u}, \bar{v}) + 2Mr^3 \int_{-\infty}^{\bar{u}} D \frac{\Phi^-}{r^3 |\bar{u}'|} d\bar{u}' - 2M \int_{-\infty}^{\bar{u}} \left(\frac{\Phi^-}{|\bar{u}'|} - r\phi(\bar{u}', \bar{v}) \right) d\bar{u}' \right| \leq \frac{C}{r^\beta}. \quad (2.7.8)$$

As in the proof of Theorem 1.4.2 in chapter 1, we evaluate the integral on the LHS by using $|\bar{v} - \bar{u} - r| \lesssim \log r$ and decomposing into partial fractions. We then have, for fixed $\bar{u}$,

$$\int_{-\infty}^{\bar{u}} \frac{D}{r^3 |\bar{u}'|} d\bar{u}' = \int_{-\infty}^{\bar{u}} \frac{1}{(\bar{v} - \bar{u}')^3 |\bar{u}'|} d\bar{u}' + \mathcal{O}\left(\frac{1}{r^{3+\beta}}\right) = \frac{\log r}{r^3} + \frac{J(\bar{u})}{r^3} + \mathcal{O}\left(\frac{1}{r^{3+\beta}}\right)$$

for some function $J(\bar{u})$.

Inserting this estimate back into (2.7.8) gives the asymptotics for $\partial_v(r\phi)$ for fixed, sufficiently large negative values of $\bar{u}$. However, in view of the wave equation for the radiation field (2.4.9), we can, in fact, propagate them for any finite u -distance; in particular, we can propagate them up to Σ_0 . This proves (2.7.3).

It is left to prove (2.7.5). Proving the finiteness of the J^N -based energies contained in the definition of $\tilde{E}_{0, I_0^{\log=0;1}}^\epsilon[\phi]$ is standard. On the other hand, to show the finiteness of terms like e.g.

$$\int_{N_0^\mathbb{I}} r^{5+2k-\epsilon} (\partial_r^{1+k}(r\phi))^2 dr$$

for $k = 1$, we use the asymptotic expression (2.7.3) as well as the fact that the asymptotics obtained in Theorem 1.4.2 of chapter 1 commute with ∂_v : For instance, we have that $\partial_v^2(r\phi) \sim r^{-4} \log r$ (see Remark 1.4.5 in chapter 1). Lastly, we similarly deal with terms such as

$$\int_{N_0^\mathbb{I}} r^{4-\epsilon} (\partial_r^1 T^3(r\phi))^2 dr$$

by writing $T = \partial_u + \partial_v$ and using the wave equation (2.4.9) for $\partial_u \partial_v$ -terms and the improved estimates mentioned above for $\partial_v \partial_v$ -terms etc. $\square$

2.8 Comments on higher-order asymptotics

In this section, we shall briefly discuss the issue of deriving higher-order asymptotics for ϕ .

In the settings where the solution is conformally smooth on $u = 0$, i.e.,

$$\partial_r(r\phi)(0, r) = \frac{I_0}{r^2} + \mathcal{O}(r^{-3}) \quad (2.8.1)$$

or

$$\partial_r(r\phi)(0, r) = \frac{J_0}{r^3} + \mathcal{O}(r^{-4}) \quad (2.8.2)$$

for some constants I_0, J_0 , the next-to-leading-order asymptotics for ϕ have been derived in [AAG19]. It was found there that these next-to-leading-order asymptotics contain logarithmic terms which, on the one hand, have contributions from the leading-order asymptotics of the solution itself. On the other hand, they have contributions from the simple fact that, on Schwarzschild, assuming smoothness in the variable $s = 1/r$ is incompatible with assuming

smoothness in the variable $s' = 1/v$. (This is because $r + 2M \log r - (v - u)/2 = \mathcal{O}(1)$.) In other words, the above initial data assumptions imply

$$\partial_v(r\phi)(0, r) = \frac{2I_0}{v^2} + \frac{16MI_0 \log v}{v^3} + \mathcal{O}(v^{-3}) \quad (2.8.3)$$

for (2.8.1) and, similarly, for (2.8.2):

$$\partial_v(r\phi)(0, r) = \frac{4J_0}{v^3} + \frac{48MJ_0 \log v}{v^4} + \mathcal{O}(v^{-4}). \quad (2.8.4)$$

These logarithmic higher-order asymptotics on initial data then lead (together with the contribution coming from the leading-order asymptotics of the solution itself) to logarithmic higher-order corrections in the asymptotic expansion of ϕ near i^+ .

Now, from the viewpoint of chapter 1, the asymptotics (2.8.1), (2.8.2) are of course inappropriate since they assume conformal smoothness or compact support: Instead, the results of chapter 1 motivate the following asymptotics on $u = 0$, where J_0 , J'_0 and J''_0 are constants:

$$\partial_r(r\phi)(0, r) = \frac{I_0 \frac{\log r}{r^3} [\phi] \log r}{r^3} + \frac{J_0}{r^3} + \mathcal{O}(\max(r^{-3-\epsilon_\phi}, r^{-4} \log r)) \quad (2.8.5)$$

or

$$\partial_r(r\phi)(0, r) = \frac{J'_0}{r^3} + \frac{J''_0 \log r}{r^4} + \mathcal{O}(r^{-4}), \quad (2.8.6)$$

where the latter expansion was obtained by e.g. considering the scattering problem on Schwarzschild with smooth compactly supported scattering data on $\mathcal{I}^-$ and $\mathcal{H}^-$ (and is also the generic case if the past Newman–Penrose constant vanishes and the data on $v = 1$ are conformally smooth), see Theorems 1.2.5 or 1.6.2 in chapter 1.

In the latter case (2.8.6), the situation is similar to (2.8.2), with the difference being that, now, there are two contributions to the logarithmic corrections on data, so, in principle, there could be cancellations in the higher-order late-time asymptotics (this would depend on the extension of the data to $\mathcal{H}^+$):

$$\partial_v(r\phi)(0, r) = \frac{4J'_0}{v^3} + \frac{(8J''_0 + 48MJ'_0) \log v}{v^4} + \mathcal{O}(v^{-4}). \quad (2.8.7)$$

On the other hand, in the former case (2.8.5), we have

$$\partial_v(r\phi)(0, r) = \frac{4I_0 \frac{\log r}{r^3} [\phi] (\log v - \log 2)}{v^3} + \frac{4J_0}{v^3} + \frac{48MI_0 \frac{\log r}{r^3} \log^2 v}{v^4} + \mathcal{O}(\max(v^{-3-\epsilon_\phi}, v^{-4} \log v)). \quad (2.8.8)$$

From this, one already sees that, if one were to suitably adapt the methods of [AAG19], and if one assumes that $\epsilon_\phi \geq 1$, then one would find corrections to the asymptotics of ϕ near i^+ which contain $\log^2$ -terms. (Again, there would also be a contribution coming from the leading-order

asymptotics themselves.) For instance, for the radiation field along future null infinity, we would obtain a correction at order $u^{-3} \log^2 u$. We leave the details to the reader.

2.9 Comments on the non-linear problem

We have shown in this chapter that, in the case of the linear wave equation on a fixed Schwarzschild background, the logarithmic terms obtained near spacelike infinity in chapter 1 can be translated into leading order logarithmic asymptotics for the radiation field near i^+ , $r\phi|_{\mathcal{I}^+} = Cu^{-2} \log u + \mathcal{O}(u^{-2})$. Now, the results of chapter 1 were, in fact, derived for the non-linear Einstein-Scalar field system and then obtained for the linear problem as a corollary. It is therefore interesting to ask if one can show analogous results to the ones obtained here for the non-linear case. We here only make two brief comments on two works which are related to this problem.

The black hole case If we consider the Einstein-Scalar field system under spherical symmetry, with assumptions as in [DR05] (in particular, we assume that an event horizon exists), and the additional assumption that, on some outgoing null hypersurface,

$$|\partial_v(r\phi)| \leq Cr^{-3} \log r \quad (2.9.1)$$

then we are in the realm of the non-linear proof of (almost) Price's law by Dafermos–Rodnianski [DR05] (again with the exception of the logarithmic terms). Carefully following their arguments, one finds the following results (with a choice of coordinates as in [DR05], see their Thm. 1.1): Along the event horizon, one obtains

$$|\phi| + |\partial_v \phi| \leq C_\epsilon v^{-3+\epsilon}, \quad (2.9.2)$$

whereas, along null infinity, one has

$$|r\phi| \leq Cu^{-2} \log u, \quad |\partial_u(r\phi)| \leq C_\epsilon u^{-3+\epsilon}, \quad (2.9.3)$$

for any $\epsilon > 0$ and a constant $C > 0$. C_ϵ is a constant that blows up as $\epsilon \rightarrow 0$. Note that these are only upper bounds. In particular, we expect that one can replace the ϵ -growth in (2.9.2), (2.9.3) with a logarithm.

The dispersive case If we consider the Einstein-Scalar field system under spherical symmetry and, instead of assuming that a black hole forms, assume that the solution disperses (and has a regular centre $r = 0$), then, under the assumptions of [LO15] and the additional assumption that

$$\partial_v(r\phi) \sim C(r+1)^{-3} \log(r+1) \quad (2.9.4)$$

on an outgoing null ray, we are in the setting of the proof of Luk–Oh [LO15] (except for the log-term). A very slight adaptation of their methods then gives the following *sharp* rates near timelike infinity: There exist positive constants A, B such that, near i^+ :

$$A \min\{u^{-3} \log u, r^{-1} u^{-2} \log u\} \leq \phi \leq B \min\{u^{-3} \log u, r^{-1} u^{-2} \log u\}, \quad (2.9.5)$$

$$A \min\{u^{-3} \log u, r^{-3} \log r\} \leq -\partial_v(r\phi) \leq B \min\{u^{-3} \log u, r^{-3} \log r\}, \quad (2.9.6)$$

$$Au^{-3} \log u \leq \partial_u(r\phi) \leq Bu^{-3} \log u. \quad (2.9.7)$$

See also Thms. 3.1 and 3.16 in [LO15]. To obtain the corresponding results for the logarithmic terms, one needs to slightly change their proof of Lemma 6.6 as well as the proof in section 10.

Appendix 2.A Definitions of the main energy norms

2.A.1 The basic energy currents

We define, with respect to any coordinate basis, the following energy momentum tensor for any scalar field ϕ :

$$\mathbf{T}_{\mu\nu}[\phi] := \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} \partial^\xi \phi \partial_\xi \phi.$$

Note that if ϕ is a solution to the wave equation, then $\mathbf{T}[\phi]$ is divergence-free.

For any vector field V , we further define the energy current $J^V[\phi]$ according to

$$J^V[\phi](\cdot) := \mathbf{T}[\phi](V, \cdot).$$

2.A.2 Definition of the energy norms

We work in (u, r, θ, φ) -coordinates, where $\partial_r = \frac{D}{2} L$.

Let ϕ be a spherically symmetric solution to $\square_g \phi = 0$ in the sense of Proposition 2.3.1, and let $\epsilon > 0$. Then we define the following initial data energy norms on Σ_0 , where the subscript “ $_0$ ” of the energy norms below denotes the fact that these are the energy norms for the $\ell = 0$ -mode.

$$E_k^N[\phi] := \sum_{i \leq k} \int_{\Sigma_0} J^N[T^i \phi] \cdot n_{\Sigma_0} \, d\mu_{\Sigma_0}, \quad (2.A.1)$$

$$\begin{aligned} E_{0, I_0 \neq 0; k}^\epsilon[\phi] &:= E_{3+3k}^N[\phi] + \int_{N_0^\mathcal{I}} r^{3+2k-\epsilon} (\partial_r^{1+k}(r\phi))^2 \, dr \\ &\quad + \sum_{i \leq 2k} \int_{N_0^\mathcal{I}} r^{3-\epsilon} (\partial_r T^i(r\phi))^2 + r^2 (\partial_r T^{1+i}(r\phi))^2 + r (\partial_r T^{2+i}(r\phi))^2 \, dr \\ &\quad + \sum_{\substack{m \leq k; \\ i \leq 2k-2m+\min(k,1)}} \int_{N_0^\mathcal{I}} r^{2+2m-\epsilon} (\partial_r^{1+m} T^i(r\phi))^2 \, dr, \end{aligned} \quad (2.A.2)$$

$$\begin{aligned} E_{0, I_0=0; k}^\epsilon[\phi] &:= E_{5+3k}^N[\phi] + \int_{N_0^\mathcal{I}} r^{5+2k-\epsilon} (\partial_r^{1+k}(r\phi))^2 \, dr \\ &\quad + \sum_{i \leq 2k} \int_{N_0^\mathcal{I}} r^{5-\epsilon} (\partial_r T^i(r\phi))^2 + r^{4-\epsilon} (\partial_r T^{1+i}(r\phi))^2 + r^{3-\epsilon} (\partial_r T^{2+i}(r\phi))^2 \\ &\quad \quad + r^2 (\partial_r T^{3+i}(r\phi))^2 + r (\partial_r T^{4+i}(r\phi))^2 \, dr \\ &\quad + \sum_{\substack{m \leq k; \\ i \leq 2k-2m+\min(k,1)}} \int_{N_0^\mathcal{I}} r^{4+2m-\epsilon} (\partial_r^{1+m} T^i(r\phi))^2 \, dr. \end{aligned} \quad (2.A.3)$$

We further define

$$\tilde{E}_{0,I_0 \neq 0;k}^\epsilon[\phi] := E_{0,I_0 \neq 0;k}^\epsilon[\phi] + \sum_{i \leq k} \int_{\Sigma_0} J^N[NT^i \phi] \cdot n_{\Sigma_0} \, d\mu_{\Sigma_0} \quad (2.A.4)$$

and

$$\tilde{E}_{0,I_0=0;k}^\epsilon[\phi] := E_{0,I_0=0;k}^\epsilon[\phi] + \sum_{i \leq k} \int_{\Sigma_0} J^N[NT^i \phi] \cdot n_{\Sigma_0} \, d\mu_{\Sigma_0}. \quad (2.A.5)$$

In an abuse of notation, we finally define

$$E_{0,I_0^{\log} \neq 0;k}^\epsilon[\phi] := E_{0,I_0 \neq 0;k}^\epsilon[\phi], \quad \tilde{E}_{0,I_0^{\log} \neq 0;k}^\epsilon[\phi] := \tilde{E}_{0,I_0 \neq 0;k}^\epsilon[\phi] \quad (2.A.6)$$

and

$$E_{0,I_0^{\log}=0;k}^\epsilon[\phi] := E_{0,I_0=0;k}^\epsilon[\phi], \quad \tilde{E}_{0,I_0^{\log}=0;k}^\epsilon[\phi] := \tilde{E}_{0,I_0=0;k}^\epsilon[\phi]. \quad (2.A.7)$$

This notation reflects the fact that the energy norms above “are blind” to logarithmic terms.

Chapter 3

Early-time asymptotics for higher ℓ -modes of linear waves on Schwarzschild

Abstract

In this chapter, we derive the early-time asymptotics for fixed-frequency solutions ϕ_ℓ to the wave equation $\square_g \phi_\ell = 0$ on a fixed Schwarzschild background ($M > 0$) arising from the no incoming radiation condition on $\mathcal{I}^-$ and polynomially decaying data, $r\phi_\ell \sim t^{-1}$ as $t \rightarrow -\infty$, on either a timelike boundary of constant area radius $r > 2M$ **(I)** or an ingoing null hypersurface **(II)**. In case **(I)**, we show that the asymptotic expansion of $\partial_v(r\phi_\ell)$ along outgoing null hypersurfaces near spacelike infinity i^0 contains logarithmic terms at order $r^{-3-\ell} \log r$. In contrast, in case **(II)**, we obtain that the asymptotic expansion of $\partial_v(r\phi_\ell)$ near spacelike infinity i^0 contains logarithmic terms already at order $r^{-3} \log r$ (unless $\ell = 1$).

These results suggest an alternative approach to the study of late-time asymptotics near future timelike infinity i^+ that does not assume conformally smooth or compactly supported Cauchy data: In case **(I)**, our results indicate a *logarithmically modified Price's law* for each ℓ -mode. On the other hand, the data of case **(II)** lead to much stronger deviations from Price's law. In particular, we conjecture that compactly supported scattering data on $\mathcal{H}^-$ and $\mathcal{I}^-$ lead to solutions that exhibit the same late-time asymptotics on $\mathcal{I}^+$ for each ℓ : $r\phi_\ell|_{\mathcal{I}^+} \sim u^{-2}$ as $u \rightarrow \infty$.

3.1	Introduction	153
3.1.1	Background and Motivation	153
3.1.2	Summary of the main results	157
3.1.3	Future applications: Late-time asymptotics and the role of the modified Newman–Penrose constants	163

3.1.4	Structure and guide to reading the chapter	170
3.2	Geometric preliminaries	172
3.2.1	The Schwarzschild spacetime manifold	172
3.2.2	The divergence theorem	173
3.3	Generalities on the wave equation	174
3.3.1	Existence and uniqueness	174
3.3.2	The basic energy currents	175
3.3.3	Decomposition into spherical harmonics	176
3.3.4	The commuted wave equations and the higher-order Newman–Penrose constants . .	176
3.3.5	Notational conventions	178
Part I:	The case $\ell = 1$.	179
3.4	Data on an ingoing null hypersurface $\mathcal{C}_{v=1}$	179
3.4.1	Initial data assumptions and the main theorem (Theorem 3.4.1)	179
3.4.2	Asymptotics for $r\phi_1$	180
3.4.3	Asymptotics for $\partial_v(r\phi_1)$ and proof of Thm. 3.4.1	184
3.4.4	Comments	186
3.5	Boundary data on a timelike hypersurface Γ_R	189
3.5.1	Overview of the ideas and structure of the section	189
3.5.2	The setup	190
3.5.3	Estimates for $\partial_v T^n(r\phi_1)$ and $T^n(r\phi_1)$	192
3.5.4	Estimates for $\partial_v T^n(r\phi_1 - u T(r\phi_1))$	201
3.5.5	Proof of Thm. 3.5.1	205
3.5.6	A comment on the stationary solution	209
3.6	Boundary data on a timelike hypersurface Γ_f	211
3.6.1	The setup	211
3.6.2	The case where $r _{\Gamma_f}$ attains a finite limit ($s > 1$):	211
3.6.3	The case where $r _{\Gamma_f}$ diverges ($s \leq 1$):	213
Part II:	Generalising to all $\ell \geq 0$	218
3.7	The higher-order Newman–Penrose constants	218
3.7.1	Generalising the approximate conservation law (3.3.13)	218
3.7.2	Generalising the approximate conservation law (3.3.15)	220
3.8	Boundary data on a timelike hypersurface Γ_R	221
3.8.1	Initial/boundary data assumptions	221
3.8.2	The main theorem (Theorem 3.8.1)	222
3.8.3	Overview of the proof	222
3.8.4	Proof of Theorem 3.8.1	224

3.9 Data on an ingoing null hypersurface $\mathcal{C}_{v=1}$ I	236
3.9.1 Initial data assumptions	236
3.9.2 The main theorem (Theorem 3.9.1)	236
3.9.3 Asymptotics for $r\phi_L$	238
3.9.4 Asymptotics for $\partial_v(r\phi_L)$ and proof of Theorem 3.9.1	240
3.9.5 Comments	242
3.10 Data on an ingoing null hypersurface $\mathcal{C}_{v=1}$ II	244
3.10.1 Initial data assumptions	244
3.10.2 The main theorem (Theorem 3.10.1)	244
3.10.3 Overview of the proof	246
3.10.4 Computing transversal derivatives on data	246
3.10.5 Precise leading-order behaviour of $[r^2\partial_v]^L(r\phi_L)$	247
3.10.6 Precise leading-order behaviour of $[r^2\partial_v]^{L-i}(r\phi_L)$	251
3.10.7 Cancellations in the initial data contributions	253
3.10.8 Summary and proof of Theorem 3.10.1	256
3.10.9 Comments: More severe modifications to Price's law	259
Appendix 3.A Proofs omitted from the chapter's main body	260
3.A.1 Proofs of Propositions 3.7.1 and 3.7.2	260
3.A.2 Proof of Lemma 3.8.3	262
3.A.3 Proof of Lemma 3.9.1	263
3.A.4 Proof of Lemma 3.10.2	263
3.A.5 Proof of Lemma 3.10.3	263

3.1 Introduction

3.1.1 Background and Motivation

In this chapter, we study the early-time asymptotics, i.e. asymptotics near spatial infinity i^0 , of solutions, localised on a single angular frequency $\ell = L$, to the wave equation

$$\square_{g_M} \phi_{\ell=L} = 0 \tag{3.1.1}$$

on the exterior of a fixed Schwarzschild (or a more general spherically symmetric) background $(\mathcal{M}_M, g_M)$ under certain assumptions on data near past infinity. The most important of these assumptions is the *no incoming radiation condition* on $\mathcal{I}^-$, stating that the flux of the radiation field on past null infinity vanishes at late advanced times. In addition, we will assume polynomially decaying (boundary) data on *either* a past-complete timelike hypersurface, *or* a past-complete null hypersurface.

3.1.1.1 The spherically symmetric mode

We initiated the study of such data in chapter 1, where we constructed *spherically symmetric* solutions arising from the no incoming radiation condition, as a condition on data on $\mathcal{I}^-$, and polynomially decaying boundary data on a timelike hypersurface Γ terminating at i^- (or polynomially decaying characteristic initial data on an ingoing null hypersurface $\mathcal{C}_{\text{in}}$ terminating at $\mathcal{I}^-$).

The choice for these data, in turn, was motivated by an argument due to D. Christodoulou [Chr02] (based on the monumental proof of the stability of the Minkowski spacetime [CK93]), which showed that the assumption of *Sachs peeling* [Sac61, Sac62b] and, thus, of (*conformally*) *smooth null infinity* [Pen65] is incompatible with the no incoming radiation condition and the prediction of the quadrupole formula for N infalling masses from i^- . The latter predicts that the rate of change of gravitational energy along $\mathcal{I}^+$ is given by $\sim -1/|u|^4$ near i^0 . Indeed, modelling gravitational radiation by scalar radiation, we showed in chapter 1 that the data described above lead to solutions which not only agree with the prediction of the quadrupole approximation (namely that $r^2(\partial_u \phi)^2|_{\mathcal{I}^+} \sim |u|^{-4}$ near i^0), but also have logarithmic terms in the asymptotic expansion of the *spherically symmetric mode* $\partial_v(r\phi_0)$ as $\mathcal{I}^+$ is approached, thus contradicting the statement of Sachs peeling that such expansions are analytic in $1/r$. More precisely, we obtained for the spherically symmetric mode ϕ_0 that if the limit

$$\lim_{\mathcal{C}_{\text{in}}, u \rightarrow -\infty} |u|r\phi_0 := \Phi^- \quad (3.1.2)$$

on initial data is non-zero (or, in the timelike case, if a similar condition on Γ holds), then it is, in fact, a conserved quantity along $\mathcal{I}^-$, and, for sufficiently large negative values of u , one obtains on each outgoing null hypersurface of constant u the asymptotic expansion

$$\partial_v(r\phi_0)(u, v) = -2M\Phi^- \frac{\log r - \log |u|}{r^3} + \mathcal{O}(r^{-3}). \quad (3.1.3)$$

In wide parts of the literature, it has been (and still is) assumed that physically relevant spacetimes do possess a smooth null infinity and that, therefore, logarithms as in (3.1.3) do not appear. The result of chapter 1, in line with [CK93], thus further puts this assumption in doubt. Furthermore, we showed in chapter 2 that the failure of peeling manifested by the *early-time* asymptotics (3.1.3) translates into logarithmic *late-time* asymptotics near i^+ , providing evidence for the physical measurability of the failure of null infinity to be smooth. We will return to the discussion of late-time asymptotics in section 3.1.3.

Finally, we note that the results from chapter 1 were, in fact, obtained for the non-linear Einstein-Scalar field system ($G_{\mu\nu}[g] = T_{\mu\nu}^{sf}[\phi_0]$) under spherical symmetry and then, *a fortiori*, carried over to the linear case ($G_{\mu\nu}[g] = 0, \square_g \phi_0 = 0$).

3.1.1.2 Higher ℓ -modes

Ultimately, we would like to develop an understanding of the situation for the *Einstein vacuum equations* without symmetry assumptions (for which the spherically symmetric Einstein-Scalar field system only served as a toy model) in order to close the circle to Christodoulou's original argument [Chr02], which was an argument pertaining to gravitational, not scalar, radiation. In particular, we would like to understand the prediction of the quadrupole approximation, namely that the rate of gravitational energy loss along $\mathcal{I}^+$ is given by $-1/|u|^4$ as $u \rightarrow -\infty$, *dynamically*, i.e. arising from suitable scattering data, rather than imposing it on $\mathcal{I}^+$ as was done in [Chr02]. In view of the multipole structure of gravitational radiation, it thus seems to be necessary to first understand the answer to the following question:

What are the early-time asymptotics for higher ℓ -modes of solutions to the wave equation $\square_g \phi = 0$ on a fixed Schwarzschild background, arising from the no incoming radiation condition, i.e., what is the analogue of (3.1.3) for $\ell > 0$?

We shall provide a detailed answer to this question in this chapter. Let us already paraphrase two special cases of the main statements (which are summarised in section 3.1.2). Statement 1) below corresponds to Theorems 3.1.3, 3.1.4, and Statement 2) corresponds to Theorem 3.1.5.

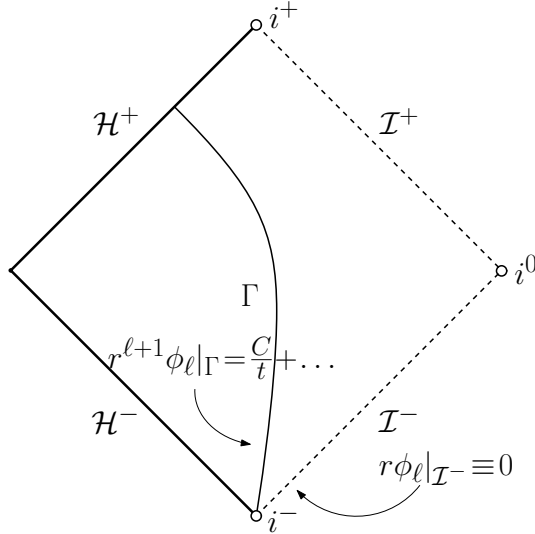


Figure 3.1.1 Schematic depiction of the data setup considered in 1): We consider polynomially decaying data on a spherically symmetric timelike hypersurface Γ , and vanishing data on $\mathcal{I}^-$. The latter condition is to be thought of as the no incoming radiation condition.

1) Consider solutions ϕ_ℓ to (3.1.1) arising from polynomially decaying data $r^{\ell+1}\phi_\ell \sim |t|^{-1}$ as $t \rightarrow -\infty$ on a spherically symmetric timelike hypersurface Γ and the no incoming radiation condition on $\mathcal{I}^-$. (See Figure 3.1.1.) Then, schematically, $r\phi_\ell|_{\mathcal{I}^+} \sim |u|^{-\ell-1}$ along $\mathcal{I}^+$ as $u \rightarrow -\infty$, and the asymptotic expansion of $\partial_v(r\phi_\ell)$ along outgoing null hypersurfaces of constant u near spacelike infinity i^0 reads:

$$\partial_v(r\phi_\ell) = \frac{f_0(u)}{r^2} + \dots + \frac{f_\ell(u)}{r^{2+\ell}} + C \frac{\log r}{r^{3+\ell}} + \dots, \quad (3.1.4)$$

where C is a non-vanishing constant.

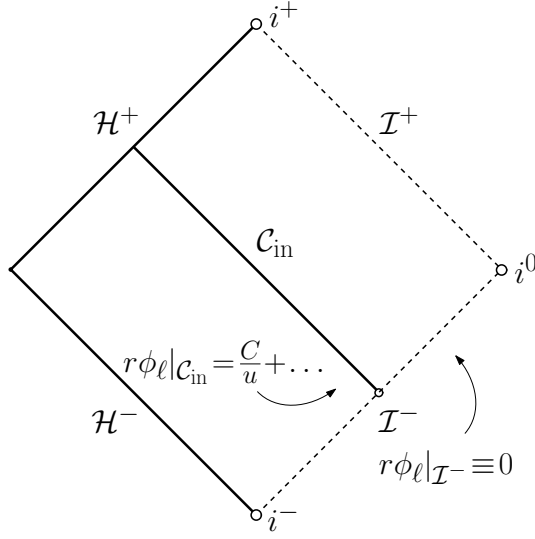


Figure 3.1.2 Schematic depiction of the data setup considered in 2): We consider polynomially decaying data on a spherically symmetric ingoing null hypersurface C_{in} , and vanishing data on the part of I^- that lies to the future of C_{in} . The latter condition is to be thought of as the no incoming radiation condition.

2) Alternatively, consider solutions ϕ_ℓ to (3.1.1) arising from polynomially decaying data $r\phi_\ell \sim |u|^{-1}$ as $u \rightarrow -\infty$ on a null hypersurface C_{in} and the no incoming radiation condition. (See Figure 3.1.2.) Then, schematically, $r\phi_\ell|_{I^+} \sim |u|^{-\min(\ell+1,2)}$ as $u \rightarrow -\infty$, and the asymptotic expansion of $\partial_v(r\phi_\ell)$ along outgoing null hypersurfaces of constant u near spacelike infinity i^0 reads:

$$\partial_v(r\phi_\ell) = \frac{f_0(u)}{r^2} + C \frac{\log r}{r^3} + \dots, \quad (3.1.5)$$

unless $\ell = 1$, in which case we instead have that

$$\partial_v(r\phi_\ell) = \frac{f_0(u)}{r^2} + \frac{f_1(u)}{r^3} + C \frac{\log r}{r^4} + \dots \quad (3.1.6)$$

In both cases, C is a generically non-vanishing constant.

By incorporating an r^ℓ -weight into the boundary data assumption (namely $r^{\ell+1}\phi_\ell|_\Gamma \sim |t|^{-1}$), we phrased statement 1) in such a way as to be independent of the behaviour of the area radius r on Γ : Independently of whether r is constant along Γ or divergent (e.g. $r|_\Gamma \sim |t|$), the $|t|^{-1}$ -decay of $r^{\ell+1}\phi$ on Γ translates into $|u|^{-\ell-1}$ decay of $r\phi_\ell$ near I^- , causing the logarithmic term in (3.1.4) to appear ℓ orders later than in (3.1.5).

The difference between (3.1.5) and (3.1.6), on the other hand, is a manifestation of certain cancellations that happen if $r\phi_\ell \sim |u|^{-\ell}$ on C_{in} . Similar cancellations are responsible for $r\phi_\ell$ decaying faster on I^+ than on C_{in} in case 2). These cancellations, together with the precise and more general versions of the above statements, will be discussed in detail in section 3.1.2 below, see also Remark 3.1.4.

Let us finally remark that, even though higher ℓ -modes thus decay slower than the spherically symmetric mode near spacelike infinity, we still expect the leading-order asymptotics near *future timelike infinity* i^+ to be dominated by the spherically symmetric mode in the two data setups

described above, see also chapter 2 and [AAG21]. However, in the case of smooth compactly supported scattering data on $\mathcal{I}^-$ and the past event horizon $\mathcal{H}^-$, it turns out that all ℓ -modes can be expected to have the same decay along $\mathcal{I}^+$ as i^+ is approached. We will discuss this in detail in section 3.1.3, see already Figures 3.1.3–3.1.5.

3.1.2 Summary of the main results

We now give a summary of the main theorems obtained in this chapter. They are all stated with respect to Eddington–Finkelstein double null coordinates (u, v) ($\partial_v r = 1 - \frac{2M}{r} = -\partial_u r$). Let's first focus on solutions to (3.1.1) supported on a single $\ell = 1$ -frequency.

3.1.2.1 The case $\ell = 1$

Let $\Gamma \subset \mathcal{M}_M$ be a spherically symmetric, past-complete timelike hypersurface of constant area radius function $r = R > 2M$.¹ Let $\ell = 1$ and $|m| \leq 1$, and prescribe on Γ smooth boundary data for $\phi_{\ell=1} = \phi_1 \cdot Y_{1m}$ that satisfy, as $u \rightarrow -\infty$,

$$\left| r^2 \phi_1|_{\Gamma} - \frac{C_{\Gamma}}{|u|} \right| = \mathcal{O}_5(|u|^{-1-\epsilon}) \quad (3.1.7)$$

for some constant C_{Γ} and for some $\epsilon \in (0, 1)$. Moreover, prescribe in a limiting sense that

$$\lim_{u \rightarrow -\infty} \partial_v^n (r \phi_1)(u, v) = 0, \quad n = 0, \dots, 5 \quad (3.1.8)$$

for all $v \in \mathbb{R}$. We interpret this as the condition of no incoming radiation from $\mathcal{I}^-$. We then prove the following theorem, in its rough form (see Theorem 3.5.1 for the precise version):

Theorem 3.1.1. *Given smooth boundary data satisfying (3.1.7), there exists a unique smooth (finite-energy) solution to (3.1.1) (restricted to the $(1, m)$ -angular frequency) in the domain of dependence of $\Gamma \cup \mathcal{I}^-$ that restricts correctly to these data and satisfies (3.1.8). Moreover, this solution satisfies along any spherically symmetric ingoing null hypersurface:*

$$\lim_{u \rightarrow -\infty} r^2 \partial_u (r \phi_1)(u, v) = 0, \quad (3.1.9)$$

$$\lim_{u \rightarrow -\infty} r^2 \partial_u (r^2 \partial_u (r \phi_1))(u, v) \equiv I_{\ell=1}^{\text{past}}[\phi], \quad (3.1.10)$$

where $I_{\ell=1}^{\text{past}}[\phi]$ is a constant which is non-vanishing as long as C_{Γ} is non-vanishing and $R/2M$ is sufficiently large, and we further have that

$$\left| r^2 \partial_u (r^2 \partial_u (r \phi_1))(u, v) - I_{\ell=1}^{\text{past}}[\phi_1] \right| = \mathcal{O}(\max(r^{-1}, |u|^{-\epsilon})). \quad (3.1.11)$$

¹In fact, the theorem below also applies to spherically symmetric hypersurfaces Γ on which r is allowed to vary and, in particular, tend to infinity. We will show this in the main body of the chapter.

In particular, $r\phi_1$ decays like u^{-2} towards $\mathcal{I}^-$.

The next theorem translates these results into logarithmic asymptotics along outgoing null hypersurfaces in a neighbourhood of spacelike infinity. Let $\mathcal{C}_{\text{in}}$ be a spherically symmetric, past-complete ingoing null hypersurface (e.g. $v = 1$). Prescribe on $\mathcal{C}_{\text{in}}$ smooth data for $\phi_{\ell=1} = \phi_1 \cdot Y_{1m}$ that satisfy

$$\lim_{u \rightarrow -\infty} r^2 \partial_u(r\phi_1) = C_{\text{in}}^{(1)}, \quad (3.1.12)$$

$$\left| r^2 \partial_u(r^2 \partial_u(r\phi_1)) - C_{\text{in}}^{(2)} \right| = \mathcal{O}(|u|^{-\epsilon}), \quad (3.1.13)$$

for some constants $C_{\text{in}}^{(1)}$, $C_{\text{in}}^{(2)}$ and for some $\epsilon \in (0, 1)$. Moreover, prescribe equation (3.1.8) to hold in a limiting sense to the future of $\mathcal{C}_{\text{in}}$ for $n = 0, 1, 2$. Then we have (see Theorem 3.4.1 for the precise version):

Theorem 3.1.2. *Given smooth data satisfying (3.1.12) and (3.1.13), there exists a unique smooth solution to (3.1.1) (restricted to the $(1, m)$ -angular frequency) in the domain of dependence of $\mathcal{C}_{\text{in}} \cup \mathcal{I}^-$ that restricts correctly to these data and satisfies (3.1.8). Moreover, this solution satisfies, for sufficiently large negative values of u , the following asymptotics as $\mathcal{I}^+$ is approached along any outgoing spherically symmetric null hypersurface:*

$$\begin{aligned} r^2 \partial_v(r\phi_1)(u, v) = & -C_{\text{in}}^{(1)} - 2 \int_{-\infty}^u F(u') du' - \frac{2MC_{\text{in}}^{(1)} - 2M \int_{-\infty}^u F(u') du'}{r} \\ & - 2M(C_{\text{in}}^{(2)} - 2MC_{\text{in}}^{(1)}) \frac{\log r - \log |u|}{r^2} + \mathcal{O}(r^{-2}), \end{aligned} \quad (3.1.14)$$

where $F(u)$ is given by the limit of the radiation field $r\phi_1$ on $\mathcal{I}^+$,

$$F(u) := \lim_{v \rightarrow \infty} r\phi_1(u, v) = \frac{C_{\text{in}}^{(2)} - 2MC_{\text{in}}^{(1)}}{6|u|^2} + \mathcal{O}(|u|^{-2-\epsilon}). \quad (3.1.15)$$

The asymptotics of $r\phi_1$ near $\mathcal{I}^+$ can be obtained by integrating $\partial_v(r\phi_1)$ from $\mathcal{I}^+$ and combining (3.1.14) and (3.1.15). In particular, if $M(C_{\text{in}}^{(2)} - 2MC_{\text{in}}^{(1)}) \neq 0$, then peeling fails at future null infinity.

Theorem 3.1.2 applies to the solution of Theorem 3.1.1, with $C_{\text{in}}^{(1)} = 0$ and $C_{\text{in}}^{(2)} = I_{\ell=1}^{\text{past}}[\phi]$.

Remark 3.1.1. Let us already make the following observation: We recall from section 3.1.1.1 that, in the spherically symmetric case ($\ell = 0$) studied in chapter 1, the initial u -decay of $r\phi_0$ was transported all the way to $\mathcal{I}^+$, that is, we had that $\lim_{v \rightarrow \infty} r\phi_0(u, v) \sim |u|^{-1}$. This fact was closely related to the approximate conservation law satisfied by $\partial_u(r\phi_0)$.² For $\ell = 1$, we see that this is no longer the case: The initial $|u|^{-1}$ -decay of $r\phi_1$ translates into $|u|^{-2}$ -decay

²Recall that $\partial_v \partial_u(r\phi_0) = -2M(1 - \frac{2M}{r}) \cdot \frac{r\phi_0}{r^3}$.

on $\mathcal{I}^+$. This improvement in the u -decay on $\mathcal{I}^+$ can be traced back to certain cancellations that happen if the $|u|$ -decay of the data comes with a specific power: In fact, notice from (3.1.15) that if $C_{\text{in}}^{(1)} = 0$, the u -decay of $r\phi_1$ on $\mathcal{I}^+$ sees *no improvement* over its initial decay. We will understand these cancellations in more generality in the theorems below, see already equation (3.1.27) of Theorem 3.1.5 and the Remark 3.1.4. See also §3.4.4.3 for a schematic explanation of these cancellations.

3.1.2.2 The case of general $\ell \geq 0$

Let $\ell = L \in \mathbb{N}_0$ and $|m| \leq L$, let Γ be as in §3.1.2.1, and prescribe on Γ smooth boundary data for $\phi_{\ell=L} = \phi_L \cdot Y_{Lm}$ that satisfy, as u tends to $-\infty$,

$$\left| r^{L+1} \phi_L|_{\Gamma} - \frac{C_{\Gamma}}{|u|} \right| = \mathcal{O}_{L+4}(|u|^{-1-\epsilon}) \quad (3.1.16)$$

for some constant C_{Γ} and some $\epsilon \in (0, 1)$, and prescribe again, in a limiting sense, that for all $v \in \mathbb{R}$:

$$\lim_{u \rightarrow -\infty} \partial_v^n (r\phi_L(u, v)) = 0, \quad n = 0, \dots, L+4. \quad (3.1.17)$$

Then we have (see Theorem 3.8.1 for the precise version):

Theorem 3.1.3. *Given smooth boundary data satisfying (3.1.16), there exists a unique smooth solution to (3.1.1) (restricted to the (L, m) -angular frequency) in the domain of dependence of $\Gamma \cup \mathcal{I}^-$ that restricts correctly to these data and satisfies (3.1.17). Moreover, this solution satisfies along any spherically symmetric ingoing null hypersurface:*

$$\lim_{u \rightarrow -\infty} (r^2 \partial_u)^{L-j} (r\phi_L)(u, v) = 0, \quad j = 0, \dots, L, \quad (3.1.18)$$

$$\lim_{u \rightarrow -\infty} (r^2 \partial_u)^{L+1} (r\phi_L)(u, v) \equiv I_{\ell=L}^{\text{past}}[\phi], \quad (3.1.19)$$

where $I_{\ell=L}^{\text{past}}[\phi]$ is a constant which is non-vanishing as long as C_{Γ} is non-vanishing and $R/2M$ is sufficiently large, and we further have that

$$\left| (r^2 \partial_u)^{L+1} (r\phi_1)(u, v) - I_{\ell=L}^{\text{past}}[\phi] \right| = \mathcal{O}(\max(r^{-1}, |u|^{-\epsilon})). \quad (3.1.20)$$

In particular, $r\phi_{\ell}$ decays like $|u|^{-\ell-1}$ towards $\mathcal{I}^-$.

Remark 3.1.2. Theorem 3.1.3 also applies to boundary data on more general spherically symmetric timelike hypersurfaces on which r is allowed to tend to infinity. See also Theorems 3.6.1 and 3.6.2.

Moreover, the proof can also be applied to any inverse polynomial rate for the $|u|$ -decay of the boundary data. In fact, if $r|_{\Gamma} \rightarrow \infty$, one can more generally apply it to growing polynomial rates, $r^{L+1} \phi_L|_{\Gamma} \sim |u|^{-p}$ for some $p < 0$, so long as the quantity $r\phi_L|_{\Gamma}$ itself is

decaying. This leads to some obvious changes in equations (3.1.19), (3.1.20). (Schematically, if $r^{L+1}\phi_L|_\Gamma \sim |u|^{-p}$ along Γ , then $r\phi_L \sim |u|^{-L-p}$ along hypersurfaces of constant v .)

Remark 3.1.3. Notice that the regularity required for the boundary data (eq. (3.1.16)), when restricted to $L = 0$, is higher than that of chapter 1. This is because, for general $L \geq 0$, we need to work with certain energy estimates in order to obtain the sharp decay for transversal derivatives on Γ , which is not necessary for the $\ell = 0$ -mode.

As before, the results of Theorem 3.1.3 translate into logarithmic asymptotics near spacelike infinity: Prescribe on $\mathcal{C}_{\text{in}}$ smooth data for $\phi_{\ell=L} = \phi_L \cdot Y_{Lm}$ that satisfy

$$\lim_{u \rightarrow -\infty} (r^2 \partial_u)^{L-j} (r\phi_L)(u, v) = 0, \quad j = 0, \dots, L, \quad (3.1.21)$$

$$\left| (r^2 \partial_u)^{L+1} (r\phi_1)(u, v) - C_{\text{in}}^{(L,0)} \right| = \mathcal{O}(|u|^{-\epsilon}) \quad (3.1.22)$$

for some constant $C_{\text{in}}^{(L,0)}$ and some $\epsilon \in (0, 1)$, and further prescribe equation (3.1.17) to hold in the future of $\mathcal{C}_{\text{in}}$ for $n = 0, \dots, L+1$. We prove the following theorem in its rough form (see Theorem 3.9.1 for the precise version):

Theorem 3.1.4. *Given smooth data satisfying (3.1.21) and (3.1.22), there exists a unique smooth solution to (3.1.1) (restricted to the (L, m) -angular frequency) in the domain of dependence of $\mathcal{C}_{\text{in}} \cup \mathcal{I}^-$ that restricts correctly to these data and satisfies (3.1.17). Moreover, this solution satisfies, for sufficiently large negative values of u , the following asymptotics as $\mathcal{I}^+$ is approached along any outgoing spherically symmetric null hypersurface:*

$$r^2 \partial_v (r\phi_L)(u, v) = \sum_{i=0}^L \frac{f_i^{(L)}(u)}{r^i} + \frac{I_{\ell=L}^{\text{future}, \frac{\log r}{r^3}}[\phi](\log r - \log |u|)}{r^{L+1}} + \mathcal{O}(r^{-L-1}) \quad (3.1.23)$$

where the $f_i^{(L)}$ are smooth functions of u which satisfy $f_i(u) = \frac{\beta_i^{(L)}}{|u|^{L-i}} + \mathcal{O}(|u|^{-L+i-\epsilon})$ for some explicit numerical constants $\beta_i^{(L)}$, and $I_{\ell=L}^{\text{future}, \frac{\log r}{r^3}}[\phi]$ is an explicit constant which can be expressed as a non-vanishing numerical multiple of M and $C_{\text{in}}^{(L,0)}$. In addition, we have that

$$\lim_{v \rightarrow \infty} r\phi_L(u, v) = \frac{L! C_{\text{in}}^{(L,0)}}{(2L+1)! |u|^{L+1}} + \mathcal{O}(|u|^{-L-1-\epsilon}). \quad (3.1.24)$$

The asymptotics of $r\phi_L$ near $\mathcal{I}^+$ can again be obtained by integrating $\partial_v(r\phi_L)$ from $\mathcal{I}^+$ and combining the above two estimates.

Now, while Theorem 3.1.3 generalises Theorem 3.1.1 in every sense, Theorem 3.1.4 does not fully generalise Theorem 3.1.2 since it excludes initial data that satisfy

$$\lim_{u \rightarrow -\infty} [r^2 \partial_u]^{L-j} (r\phi_L) = C_{\text{in}}^{(L, j+1)} \quad (3.1.25)$$

for $j = 0, \dots, L$ and *non-vanishing* constants $C_{\text{in}}^{(L,j)}$. If only $C_{\text{in}}^{(L,1)}$ is non-vanishing, then, in fact, the above theorem remains valid, albeit with some modifications to the $f_i(u)$ and to the constant $I_{\ell=L}^{\text{future}, \frac{\log r}{r^3}}[\phi]$. More generally, however, we have the following:

Instead of (3.1.21), (3.1.22), prescribe on C_{in} that

$$\left| r\phi_L(u, 1) - \frac{C_{\text{in}}}{r^p} \right| = \mathcal{O}_1(r^{-p-\epsilon}) \quad (3.1.26)$$

for some $\epsilon \in (0, 1]$, a constant $C_{\text{in}} \neq 0$, and for some $p \in \mathbb{N}_0$ ($p = 0$ is permitted). Moreover, assume the no incoming radiation condition (3.1.17) to hold for $n = 1, \dots, L+1$. Then we have (see Theorem 3.10.1 for the precise version):

Theorem 3.1.5. *Given smooth data satisfying (3.1.26), there exists a unique smooth solution to (3.1.1) (restricted to the (L, m) -angular frequency) in the domain of dependence of $C_{\text{in}} \cup \mathcal{I}^-$ that restricts correctly to these data and satisfies (3.1.17). Define $r_0 := |u| - 2M \log |u|$. Then the limit of the radiation field satisfies*

$$\lim_{v \rightarrow \infty} r\phi_L(u, v) = F(u) = \begin{cases} \mathcal{O}(r_0^{-p-\epsilon}), & \text{if } p \leq L \text{ and } p \neq 0, \\ C(L, p) \cdot C_{\text{in}} r_0^{-p} + \mathcal{O}(r_0^{-p-\epsilon}), & \text{if } p > L \text{ or } p = 0, \end{cases} \quad (3.1.27)$$

for some smooth function $F(u)$ and some non-vanishing numerical constant $C(L, p)$.

Moreover, if $p < L$, this solution satisfies, for sufficiently large negative values of u , the following asymptotics as $\mathcal{I}^+$ is approached along any outgoing spherically symmetric null hypersurface:

$$r^2 \partial_v(r\phi_L)(u, v) = \sum_{i=0}^{p-1} \frac{f_i^{(L,p)}(u)}{r^i} + \frac{I_{\ell=L}^{\text{future}, r^{2+p-L}}[\phi](\log r - \log |u|)}{r^p} + \mathcal{O}\left(\frac{|u|}{r^p}\right), \quad (3.1.28)$$

where the $f_i^{(L,p)}$ are smooth functions which satisfy $f_i^{(L,p)} = \mathcal{O}(r_0^{-p+i+1-\epsilon})$ if $i < p-1$, and $f_i^{(L,p)} = \beta_i^{(L,p)} + \mathcal{O}(r_0^{-\epsilon})$ for some constant $\beta_i^{(L,p)}$ if $i = p-1$. $I_{\ell=L}^{\text{future}, r^{2+p-L}}[\phi]$ is a non-vanishing constant which depends on p, L, C_{in} and M .

On the other hand, if $p \geq L$, then

$$r^2 \partial_v(r\phi_L)(u, v) = \sum_{i=0}^{\max(L, p-1)} \frac{f_i^{(L,p)}(u)}{r^i} + \mathcal{O}\left(\frac{\log r}{r^{\max(L+1, p)}}\right), \quad (3.1.29)$$

where the $f_i^{(L,p)}$ are smooth functions which satisfy $f_i^{(L,p)} = \mathcal{O}(r_0^{-p+i+1-\epsilon})$ if $p = L$ and $i < L-1$, and which satisfy $f_i^{(L,p)} = \beta_i^{(L,p)} r_0^{-p+i+1} + \mathcal{O}(r_0^{-p+i-\epsilon})$ for some constants $\beta_i^{(L,p)}$ otherwise (i.e. if $p = L = i$, $p = L = i+1$, or if $p > L$).

The asymptotics of $r\phi_L$ near $\mathcal{I}^+$ can again be obtained by integrating $\partial_v(r\phi_L)$ from $\mathcal{I}^+$ and combining the above estimates.

Some remarks are in order.

Remark 3.1.4. Notice the different behaviour in the cases $p < L$, $p = L$ and $p > L$ in Theorem 3.1.5. We want to direct the reader's attention to the following points:

- Equation (3.1.27) shows that if $0 \neq p \leq L$, then there is a *cancellation* and $\lim_{v \rightarrow \infty} (r\phi_L)(u, v)$ decays faster in u than $r\phi_L(u, 1)$. See §3.4.4.3 for a schematic explanation of these cancellations. Such cancellations *do not* happen if $p = 0$ or $p > L$. Moreover, they can be viewed as Minkowskian behaviour, i.e., they can already be seen if $M = 0$. In fact, in the course of the proof of Theorem 3.1.5, we will derive simple and effective expressions for solutions of $\square_g \phi_L = 0$ on Minkowski arising from the no incoming radiation condition and initial data $r\phi_L(u, 1) = C/r^p$ (see Proposition 3.10.4).
- In view of (3.1.29), we see that “the first logarithmic term” in the expansion of $r^2 \partial_v (r\phi_L)$ appears at order $r^{-p-1} \log r$ unless $p = L$, in which case it appears one order later. In particular, it never appears at order $r^{-L-1} \log r$.

Remark 3.1.5. The proof of Theorem 3.1.5 can be generalised to positive non-integer p in (3.1.26) (and even to certain negative p). However, if $p \notin \{1, \dots, L\}$, we expect no cancellations of the type above to occur. On the other hand, if we assume, for instance, that $r\phi_L(u, 1) \sim r^{-p} \log r$ initially, then the same cancellations occur in the range $p \in \{1, \dots, L\}$, and one will obtain that $r\phi_L|_{\mathcal{I}^+} \sim |u|^{-p}$ if $p \in \{1, \dots, L\}$ and $r\phi_L|_{\mathcal{I}^+} \sim |u|^{-p} \log |u|$ otherwise. This observation will be of relevance in future work.

Remark 3.1.6. All of the above theorems make crucial use of certain approximate conservation laws. These are generalisations of the Minkowskian identities

$$\partial_u \left(r^{-2\ell-2} (r^2 \partial_v)^{\ell+1} (r\phi_\ell) \right) = 0, \quad \partial_v \left(r^{-2\ell-2} (r^2 \partial_u)^{\ell+1} (r\phi_\ell) \right) = 0,$$

and have been used in a very similar context in the recent [AAG21], see also [MZ22b]. See already section 3.3.4 and section 3.7 for a discussion and derivation of these in the cases $\ell \leq 1$, $\ell \geq 0$, respectively. The reason why we stated Theorems 3.1.4 and 3.1.5 separately is that the former can be proved in a rather simple way using the second conservation law, i.e. by propagating the initial decay for $(r^2 \partial_u)^{\ell+1} (r\phi_\ell)$ in v , whereas, in order to prove Theorem 3.1.5, we will need to use the conservation law in the u -direction.

Remark 3.1.7. The constants $I_{\ell=L}^{\text{future},f}[\phi]$ appearing in the above theorems are modified Newman–Penrose constants. These are closely related to the approximate conservation laws mentioned before. We will discuss this further in the next section.

Remark 3.1.8. One can generalise all of the above theorems to hold on more general spherically symmetric spacetimes such as the Reissner–Nordström spacetimes in the full physical range of charge parameters $|e| \leq M$. In the extremal case $|e| = M$, one can moreover apply the

well-known conformal “mirror” isometry to obtain results on the asymptotics near the future event horizon $\mathcal{H}^+$, see §1.2.2.2 of chapter 1.

3.1.3 Future applications: Late-time asymptotics and the role of the modified Newman–Penrose constants

The approximate conservation laws mentioned in Remarks 3.1.6, 3.1.7 are closely related to the ℓ -th order Newman–Penrose constants $I_\ell[\phi]$ defined on future and past null infinity, respectively (see also the original [NP65, NP68], and, more tailored to our context, [AAG18b, AAG21] and section 3.7 of the present chapter). In fact, these ℓ -th order Newman–Penrose constants play an important role in the study of both *early-time asymptotics* (near i^0) and *late-time asymptotics* (near i^+) of fixed- ℓ solutions to the wave equation on Schwarzschild.³

While the question of early-time asymptotics has not been investigated much elsewhere, the study of late-time asymptotics has been an active field for decades. The most prominent result in this line of research is the so-called *Price’s law* [Pri72, GPP94], see also [Lea86]. Price’s law states that smooth, compactly supported data on a Cauchy hypersurface (i.e. data with trivial early-time asymptotics) for fixed angular frequency solutions $\phi_{\ell=L} = Y_{Lm}\phi_{Lm}$ to the wave equation (3.1.1) generically lead to the following asymptotics near future timelike infinity i^+ (we suppress the m -index in the following):

$$r\phi_L|_{\mathcal{I}^+} \sim u^{-2-L}, \quad \phi_L|_{r=\text{constant}} \sim \tau^{-2L-3}, \quad \phi_L|_{\mathcal{H}^+} \sim v^{-2L-3} \quad (3.1.30)$$

along future null infinity, hypersurfaces of constant r , and the event horizon $\mathcal{H}^+$, respectively. This statement has been satisfactorily proved in the recent works [AAG18c, AAG18b, AAG21], see also [Hin22] and [MZ22b]. (For earlier rigorous works on *pointwise upper bounds* (not asymptotics), see [DSS11, DSS12] as well as [MTT12].) We also refer the reader to these papers for more general background and motivation for the study of late-time asymptotics.

The question of late-time asymptotics for *compactly supported Cauchy data* has thus been completely understood. Similar results have been obtained for non-compactly supported data, but in that case, it has typically been assumed that the data are conformally smooth. However, if one’s motivation for studying late-time asymptotics comes from gravitational wave astronomy (i.e. the hope that some devices will eventually be able to measure these asymptotics), then the assumption of smooth compactly supported (or conformally smooth) data on a Cauchy hypersurface becomes questionable – as long as one accepts the general framework of an *isolated system*. For, if one assumes that the gravitational waves emitting system under consideration has existed for all times, then it will certainly have radiated for all times: Thus, a spacetime describing this system cannot be expected to contain Cauchy hypersurfaces with compact radiation content. On the other hand, the data considered in chapter 1 and the present chapter

³This is discussed in much more detail in chapter 4.

have a clear physical motivation⁴ and, thus, seem like a more reasonable starting point for the question of physically relevant late-time asymptotics.

Motivated by this, we shall now discuss consequences that our results from section 3.1.2 have on late-time asymptotics. It turns out that one can gain a simple, intuitive understanding of these in terms of the aforementioned Newman–Penrose constants.

3.1.3.1 The timelike case: A logarithmically modified Price’s law for all ℓ

Let’s assume that we have a spherically symmetric timelike hypersurface Γ that has *constant* area radius near i^- and terminates at $\mathcal{H}^+$. (Note that, if we chose Γ to terminate at i^+ , then we would have to essentially *prescribe* the late-time asymptotics as boundary data on Γ . On the other hand, if we choose Γ to terminate at $\mathcal{H}^+$, then it will turn out that the leading-order late-time asymptotics are completely determined by the data’s behaviour near i^- . In particular, they do not depend on the extension of the data towards $\mathcal{H}^+$.) Consider first the spherically symmetric mode, and prescribe smooth data for it which, near past timelike infinity i^- , behave like $r\phi_0 = C|u|^{-1} + \mathcal{O}(|u|^{-1-\epsilon})$, and which smoothly extend to the future event horizon $\mathcal{H}^+$; and impose the no incoming radiation condition on $\mathcal{I}^-$. Then the results of chapter 1 show⁵ that the past Newman–Penrose constant exists and is conserved along $\mathcal{I}^-$:

$$I_{\ell=0}^{\text{past}}[\phi] := \lim_{u \rightarrow -\infty} r^2 \partial_u(r\phi_0) \neq 0. \quad (3.1.31)$$

Moreover, we showed that the finiteness of the past N–P constant, together with the no incoming radiation condition, implies that, even though the future Newman–Penrose constant vanishes ($\lim_{v \rightarrow \infty} r^2 \partial_v(r\phi_0) = 0$), a *logarithmically modified* future Newman–Penrose constant exists and is conserved along $\mathcal{I}^+$:

$$I_{\ell=0}^{\text{future}, \frac{\log r}{r^3}}[\phi] := \lim_{v \rightarrow \infty} \frac{r^3}{\log r} \partial_v(r\phi_0) = -2MI_{\ell=0}^{\text{past}}[\phi] \neq 0. \quad (3.1.32)$$

In chapter 2, we then applied slight adaptations of the methods of [AAG18b] to show that this logarithmically modified Newman–Penrose constant completely determines the leading-order late-time asymptotics near i^+ :

$$r\phi_0(u, \infty) = \frac{1}{2} I_{\ell=0}^{\text{future}, \frac{\log r}{r^3}}[\phi] \frac{\log u}{u^2} + \mathcal{O}(u^{-2}), \quad (3.1.33)$$

$$\phi_0(u, v_R(u)) = \frac{1}{2} I_{\ell=0}^{\text{future}, \frac{\log r}{r^3}}[\phi] \frac{\log \tau}{\tau^3} + \mathcal{O}(\tau^{-3}), \quad (3.1.34)$$

⁴In addition to the remarks at the beginning of this chapter, see also chapter 0 or §sec:introduction and §sec:introduction:scalar of chapter 1 for a more detailed discussion of the physical motivation.

⁵In fact, we only showed in chapter 1 that $\lim_{u \rightarrow -\infty} |u|r\phi_0$ is finite. However, it follows directly from Lemma lemma:trick therein that $\lim_{u \rightarrow -\infty} |u|r\phi_0 = \lim_{u \rightarrow -\infty} |u|^2 T(r\phi_0) = \lim_{u \rightarrow -\infty} r^2 \partial_u(r\phi_0)$, since $\partial_v(r\phi_0) \lesssim r^{-2}|u|^{-1}$. Alternatively, one can also refer to Thm. 3.1.3 of the present chapter.

$$\phi_0(\infty, v) = \frac{1}{2} I_{\ell=0}^{\text{future}, \frac{\log r}{r^3}} [\phi] \frac{\log v}{v^3} + \mathcal{O}(v^{-3}), \quad (3.1.35)$$

along $\mathcal{I}^+$, hypersurfaces of constant R , and $\mathcal{H}^+$, respectively.⁶ In particular, the leading-order late-time behaviour is independent of the extension of the data towards $\mathcal{H}^+$ and only depends on the behaviour of the data near i^- . We called this a *logarithmically modified Price's law* for the $\ell = 0$ -mode.

Consider now the $\ell = 1$ -case. If we assume data as in Theorem 3.1.1 and smoothly extend them to $\mathcal{H}^+$, then we obtain that the past N–P constant of order $\ell = 1$ exists and is conserved along $\mathcal{I}^-$:

$$I_{\ell=1}^{\text{past}, f} [\phi] := \lim_{u \rightarrow -\infty} r^2 \partial_u (r^2 \partial_u (r \phi_1) + M r \phi_1) = \lim_{u \rightarrow -\infty} r^2 \partial_u (r^2 \partial_u (r \phi_1)) \neq 0. \quad (3.1.36)$$

It then follows from Theorem 3.1.2 that the decay encoded in (3.1.36) (namely, $r \phi_1 \sim u^{-2}$), along with the no incoming radiation condition, implies that the future N–P constant vanishes, but that a *logarithmically modified future Newman–Penrose constant of order $\ell = 1$* exists and is conserved along $\mathcal{I}^+$ (see also Theorem 3.4.2 in §3.4.4):

$$I_{\ell=1}^{\text{future}, \frac{\log r}{r^3}} [\phi] := \lim_{v \rightarrow \infty} \frac{r^3}{\log r} \partial_v (r^2 \partial_v (r \phi_1) - M r \phi_1) = 4M I_{\ell=1}^{\text{past}} [\phi]. \quad (3.1.37)$$

One should then be able to combine the results above with those of Angelopoulos–Aretakis–Gajic [AAG21], with adaptations exactly as in chapter 2 (which combined the results of chapter 1 and [AAG18c, AAG18b]), in order to obtain near i^+ :

$$r \phi_1(u, \infty) = C_1 \frac{\log u}{u^3} + \mathcal{O}(u^{-3}), \quad (3.1.38)$$

$$\phi_1(u, v_R(u)) = C_2 \frac{\log \tau}{\tau^5} + \mathcal{O}(\tau^{-5}), \quad (3.1.39)$$

$$\phi_1(\infty, v) = C_2 \frac{\log v}{v^5} + \mathcal{O}(v^{-5}), \quad (3.1.40)$$

where C_1, C_2 are given by numerical multiples of $I_{\ell=1}^{\log, \text{future}} [\phi]$. In particular, these constants C_1, C_2 should be independent of the extension of the data towards $\mathcal{H}^+$. Thus, we would obtain a logarithmically modified Price's law for $\ell = 1$ (cf. §3.4.4.2).

In fact, we expect the same structure to hold in the case of general $\ell = L$. The data of Theorem 3.1.3 lead to solutions which have finite L -th order past N–P constant and, by Theorem 3.1.4 (see also Theorem 3.9.1), have a finite *logarithmically modified L -th order future* N–P constant $I_{\ell=L}^{\text{future}, \frac{\log r}{r^3}} [\phi]$, see §3.7 for the definition of these. In other words, Theorems 3.1.3 and 3.1.4 prove the precise analogues of (3.1.36) and (3.1.37) for general $\ell = L$. In view of the

⁶Notice that the (u, v) -gauge used in this chapter is related to that of chapter 2 by $(u, v) = (u'/2, v'/2)$.

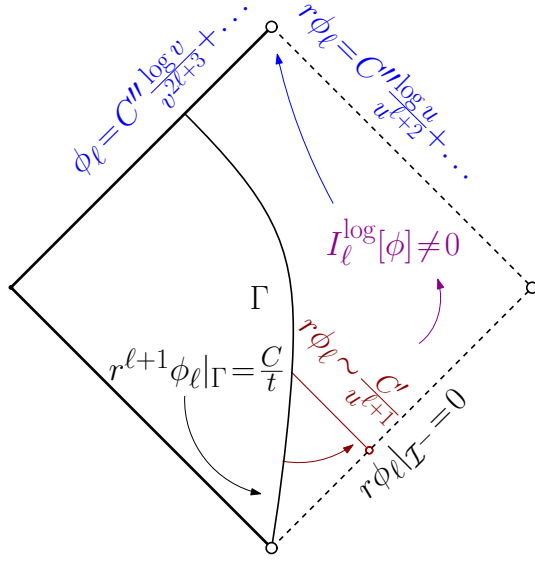


Figure 3.1.3 Schematic depiction of the situation of §3.1.3.1: Given smooth data for $r\phi_\ell$ on Γ which decay like $1/t$ near i^- , the solution decays like $u^{-\ell-1}$ near $\mathcal{I}^-$ by Thm. 3.1.3 and has finite logarithmically modified N-P constant on $\mathcal{I}^+$ by Thm. 3.1.4. The depicted late-time behaviour near i^+ should follow from the methods of [AAG21] and should be independent of the data's extension towards $\mathcal{H}^+$.

remarks above, one should then be able to recover a *logarithmically modified Price's law* for each ℓ from this. See Figure 3.1.3.

What would be more difficult, however, is to show such a statement for *fixed, finite regularity* of ϕ_L instead of assuming smoothness or regularity that is dependent on L . We therefore make the following conjecture:

Conjecture 3.1.1. *Prescribe data for ϕ on Γ that have sufficient but fixed, finite regularity and which satisfy $r^\ell \phi_\ell \sim t^{-1}$ as $t \rightarrow -\infty$ for all ℓ . Moreover, prescribe the no incoming radiation condition on $\mathcal{I}^-$. Then there exists an $\ell_0 \in \mathbb{N}$, increasing with the prescribed regularity of the data, such that, for all $\ell \leq \ell_0$, the ℓ -modes ϕ_ℓ of the corresponding solution will exhibit the following late-time asymptotics near i^+ :*

$$r\phi_\ell|_{\mathcal{I}^+} \sim u^{-2-\ell} \log u, \quad \phi_\ell|_{r=\text{constant}} \sim \tau^{-2\ell-3} \log \tau, \quad \phi_\ell|_{\mathcal{H}^+} \sim v^{-2\ell-3} \log v. \quad (3.1.41)$$

Moreover, the projection onto higher $\ell > \ell_0$ -modes $\phi_{\ell > \ell_0}$ satisfies the upper bounds

$$r\phi_{\ell > \ell_0}|_{\mathcal{I}^+} = \mathcal{O}(u^{-2-\ell_0-\epsilon}), \quad \phi_{\ell > \ell_0}|_{r=\text{constant}} = \mathcal{O}(\tau^{-2\ell_0-3-\epsilon}), \quad \phi_{\ell > \ell_0}|_{\mathcal{H}^+} = \mathcal{O}(v^{-2\ell_0-3-\epsilon}) \quad (3.1.42)$$

for some $\epsilon > 0$. If the data are chosen to be smooth, then ℓ_0 can be chosen to be ∞ .

See also the comments in §3.9.5.

A proof of the above conjecture would require revisiting the proof of Thm. 3.1.3 since, as stated, Thm. 3.1.3 requires boundary data regularity increasing in angular frequency L . However, if one imposes fixed, finite regularity, it should still be possible to extract weaker decay (compared to that of Thm. 3.1.3) from the methods of the proof that is consistent with (3.1.42).

On the other hand, once these modifications are understood, one should be able to directly apply the methods of [AAG21], with modifications as in chapter 2, to prove the conjecture.

It would also be interesting to find a definitive answer to the question whether or not the rate (3.1.42) can be improved *without assuming additional regularity*.

We finally note that, on the one hand, if the $1/t$ -decay on initial data is replaced by any *integrable* decay rate, then the logarithms in (3.1.41) would disappear and we would expect the usual Price's law tails. *On the other hand, if one considers a timelike hypersurface Γ on which $r|_\Gamma \rightarrow \infty$ as $u \rightarrow -\infty$, say, $r|_\Gamma(u) \sim |u|$, and only imposes $r\phi_\ell|_\Gamma \sim |t|^{-1}$, then the expected modifications to Price's law are much more severe and exactly as in the null case with $p = 1$. We will discuss this latter case now.*

3.1.3.2 The null case: More severe deviations from Price's law

In contrast to the timelike case, it turns out that the data considered in Theorem 3.1.2, which were posed for the $\ell = 1$ -mode on an ingoing null hypersurface ($r\phi_1|_{\mathcal{C}_{\text{in}}} \sim C_{\text{in}}^{(1)}|u|^{-1}$), generally lead to a non-vanishing future Newman–Penrose constant

$$I_{\ell=1}^{\text{future}}[\phi] := \lim_{v \rightarrow \infty} r^2 \partial_v (r^2 \partial_v (r\phi_1) - Mr\phi_1) \neq 0 \quad (3.1.43)$$

if $C_{\text{in}}^{(1)} \neq 0$, cf. Theorem 3.4.2. In this case, one recovers the following late-time asymptotics (provided that one smoothly extends the data to $\mathcal{H}^+$):

$$r\phi_1|_{\mathcal{I}^+} \sim u^{-2}, \quad \phi_1|_{r=R} \sim \tau^{-4}, \quad \phi_1|_{\mathcal{H}^+} \sim v^{-4}, \quad (3.1.44)$$

with the leading-order asymptotics only depending on $I_{\ell=1}^{\text{future}}[\phi]$. These late-time asymptotics are one power worse than the Price's law decay (3.1.30) for compactly supported data and have also been derived in [AAG21].⁷

In the case of general $\ell \geq 1$, however, the situation is more subtle: The data considered in Theorem 3.1.5, i.e. $r\phi_L(u, 1) \sim |u|^{-p}$, lead, for $p \leq L \neq 0$, to solutions where the usual Newman–Penrose constant is infinite, $I_{\ell=L}^{\text{future}}[\phi] := \lim r^2 \partial_v \Phi_L = \infty$, where Φ_L is defined in section 3.7, eq. (3.7.8). Instead, the following $(L - p)$ -modified Newman–Penrose constant remains finite and conserved along null infinity (see also Thm. 3.10.1):

$$I_{\ell=L}^{\text{future}, r^{2-L+p}}[\phi] := \lim_{v \rightarrow \infty} r^{2-L+p} \partial_v \Phi_L \neq 0. \quad (3.1.45)$$

With the decay encoded in (3.1.45), which is $L - p$ powers worse than in the case of finite unmodified N–P constant, we expect that one can further modify the methods of [AAG21] to

⁷ In fact, the authors of [AAG21] first derived late-time asymptotics for data with $I_{\ell=1}^{\text{future}}[\phi] \neq 0$, and then showed that solutions ϕ arising from smooth compactly supported data can generically be written as time derivatives of solutions ϕ^T that satisfy $I_{\ell=1}^{\text{future}}[\phi^T] \neq 0$. They then showed that time derivatives decay one power faster, which proved Price's law.

then derive late-time asymptotics near i^+ which are $L - p + 1$ powers slower than the Price's law decay (3.1.30) and which do not depend on the data's extension towards $\mathcal{H}^+$ (see Figure 3.1.4), provided that the solution is smooth. Cf. the comments in §3.10.9.

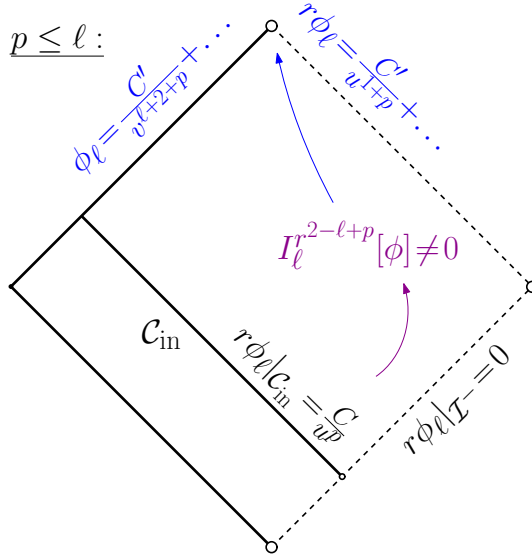


Figure 3.1.4 Schematic depiction of the situation of §3.1.3.2: Given data for $r\phi_\ell$ on $\mathcal{C}_{\text{in}}$ which decay like $1/u^p$ near $\mathcal{I}^-$, the solution has finite $(\ell - p)$ -modified N-P constant (see (3.1.45)) on $\mathcal{I}^+$ by Thm. 3.1.5, provided that $p \leq \ell$. We also depicted the conjectured late-time behaviour near i^+ .

Analogously to Conjecture 3.1.1, we also make the following conjecture for the finite regularity problem:

Conjecture 3.1.2. *Let $1 \leq p \in \mathbb{N}$, and prescribe data for ϕ on $\mathcal{C}_{\text{in}}$ that have sufficient but fixed, finite regularity and which satisfy $r\phi_\ell \sim |u|^{-p}$ as $u \rightarrow -\infty$ for all ℓ . Moreover, prescribe the no incoming radiation condition on $\mathcal{I}^-$. Then, for all $\ell \geq p > 0$,⁸ the ℓ -modes ϕ_ℓ of the corresponding solution will exhibit the following late-time asymptotics near i^+ along $\mathcal{I}^+$:*

$$r\phi_\ell|_{\mathcal{I}^+} \sim u^{-p-1}. \quad (3.1.46)$$

Moreover, there exists an $\ell_0 \in \mathbb{N}$, increasing with the prescribed regularity of the data, such that, away from $\mathcal{I}^+$, and for some $\epsilon > 0$,

$$\phi_\ell|_{r=\text{constant}} \sim \tau^{-\ell-p-2}, \quad \phi_\ell|_{\mathcal{H}^+} \sim v^{-\ell-p-2}, \quad \text{for all } \ell \in \{p, \dots, \ell_0\}, \quad (3.1.47)$$

$$\phi_{\ell > \ell_0}|_{r=\text{constant}} = \mathcal{O}(\tau^{-\ell_0-p-2-\epsilon}), \quad \phi_{\ell > \ell_0}|_{\mathcal{H}^+} = \mathcal{O}(v^{-\ell_0-p-2-\epsilon}). \quad (3.1.48)$$

If the data are chosen to be smooth, then ℓ_0 can be chosen to be ∞ .

Remarkably, if one takes $p = 1$ in (3.1.26), then the asymptotics (3.1.46), (3.1.47) for $\ell \geq 1$ would still be a logarithm faster than the ones for $\ell = 0$, (3.1.33)–(3.1.35), despite the decay of $\partial_v(r\phi_\ell)$ towards spatial infinity being slower for $\ell > 0$ than for $\ell = 0$.

⁸For $\ell = p - 1$, one would obtain a logarithmically modified Price's law (3.1.41), and, for $\ell < p - 1$, one would generically obtain the usual Price's law behaviour (3.1.30).

We note that even if one is willing to assume smoothness, then the modifications to [AAG21] needed to prove (3.1.47) are different than those of chapter 2, as one now has to deal with a difference in *integer powers* in decay. (In chapter 2, we treated non-integer modifications in decay.) We expect that one should be able to use time *derivatives*, rather than time integrals, of our solutions to reduce to the cases treated in [AAG21], and then integrate the asymptotics of these time derivatives from i^+ to obtain the asymptotics of the original solution. (This would be the opposite procedure of that described in footnote 7.)

In contrast, the fixed, finite regularity problem, i.e. a proof of Conjecture 3.1.2, would require much more elaborate modifications. In fact, since we conjecture the precise late-time asymptotics *for all* ℓ in (3.1.46), one would now also have to modify the methods of [AAG21] since the procedure outlined in the previous paragraph would, again, require regularity that is increasing in ℓ . We also want to point out that, since the conjectured asymptotics (3.1.46) along $\mathcal{I}^+$ are independent of ℓ , an understanding of the fixed, finite regularity problem would be all the more important for applications!

3.1.3.3 Compactly supported scattering data on $\mathcal{H}^-$ and $\mathcal{I}^-$

One final natural configuration of data we want to consider is the case of smooth, compactly supported scattering data on $\mathcal{I}^-$ and the past event horizon $\mathcal{H}^-$. In order to apply our results, we can, without loss of generality, assume that the data on $\mathcal{H}^-$ are vanishing (this can be achieved by restricting to sufficiently large negative values of u). Similarly, we can assume, without loss of generality, that the data on $\mathcal{I}^-$ are supported in $v_1 \leq v \leq 1$. If we then integrate the wave equation satisfied by the radiation field $r\phi_L$, namely

$$\partial_v \partial_u (r\phi_L) = \left(1 - \frac{2M}{r}\right) \left(\frac{-L(L+1)}{r^2} r\phi_L - \frac{2M}{r^3} r\phi_L\right), \quad (3.1.49)$$

from $v = v_1$ to $v = 1$, we obtain that, generically, $\partial_u(r\phi_0)(u, 1) \sim r^{-3}$ if $L = 0$ and $\partial_u(r\phi_L)(u, 1) \sim r^{-2}$ if $L \geq 1$. More precisely, one can derive from (3.1.49) that if the data on $\mathcal{I}^-$ are given by $r\phi_L(-\infty, v) =: G(v)$, then

$$\begin{aligned} r^2 \cdot \partial_u(r\phi_L)(u, 1) &= -L(L+1) \int_{v_1}^1 G(v) dv + \mathcal{O}(r^{-1}), & \text{if } L > 0, \\ r^3 \cdot \partial_u(r\phi_0)(u, 1) &= -2M \int_{v_1}^1 G(v) dv + \mathcal{O}(r^{-1}), & \text{if } L = 0, \end{aligned}$$

See also §1.2.2 and §1.6.1 of chapter 1 for a detailed discussion of this restricted to the spherically symmetric mode. Thus, since the integrals above are non-vanishing for generic scattering data G , one can show that Theorem 3.1.5 applies, with (generically) $p = 2$ if $L = 0$ and with $p = 1$ if $L \geq 1$.

The results of Theorem 3.1.5 then show that if $L = 0$, then $\lim_{v \rightarrow \infty} r^3 \partial_v(r\phi_0) < \infty$, whereas if $L \geq 1$, then, generically, $I_{\ell=L}^{\text{future}, r^{2-L+1}}[\phi]$ is finite and non-vanishing. Therefore, if $L = 0$, one obtains the following late-time asymptotics near i^+ [AAG21]:

$$r\phi_0|_{\mathcal{I}^+} \sim u^{-2}, \quad \phi_0|_{r=R} \sim \tau^{-3}, \quad \phi_0|_{\mathcal{H}^+} \sim v^{-3}. \quad (3.1.50)$$

On the other hand, if $L > 0$, then, since $p = 1$, Conjecture 3.1.2 would imply that, generically,

$$r\phi_L|_{\mathcal{I}^+} \sim u^{-2}, \quad \phi_L|_{r=R} \sim \tau^{-L-3}, \quad \phi_L|_{\mathcal{H}^+} \sim v^{-L-3}. \quad (3.1.51)$$

We are thus led to a third conjecture (see Figure 3.1.5):

Conjecture 3.1.3. *Consider compactly supported scattering data on $\mathcal{H}^-$ and $\mathcal{I}^-$ for (3.1.1), supported on all angular frequencies, with sufficient but finite regularity. Then there exists an $\ell_0 \in \mathbb{N}$, increasing with the prescribed regularity of the data, such that, away from $\mathcal{I}^+$, and for some $\epsilon > 0$,*

$$\phi_\ell|_{r=\text{constant}} \sim \tau^{-\ell-p-2}, \quad \phi_\ell|_{\mathcal{H}^+} \sim v^{-\ell-p-2}, \quad \text{for all } \ell \in \{0, \dots, \ell_0\}, \quad (3.1.52)$$

$$\phi_{\ell > \ell_0}|_{r=\text{constant}} = \mathcal{O}(\tau^{-\ell_0-p-2-\epsilon}), \quad \phi_{\ell > \ell_0}|_{\mathcal{H}^+} = \mathcal{O}(v^{-\ell_0-p-2-\epsilon}). \quad (3.1.53)$$

On the other hand, along future null infinity $\mathcal{I}^+$, we have the asymptotic expression

$$r\phi|_{\mathcal{I}^+} = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} C_{\ell m} \cdot Y_{\ell m}(\theta, \phi) u^{-2} + o(u^{-2}) \quad (3.1.54)$$

for some constants $C_{\ell m}$ which can be computed explicitly from the scattering data on $\mathcal{I}^-$ and which are generically non-zero.

The asymptotics (3.1.54) would be in stark contrast to the usual expectation that the asymptotic behaviour on $\mathcal{I}^+$, i.e. the *physically measurable* behaviour, is dominated by low frequencies. It would therefore also be interesting to find the precise form of the constants $C_{\ell m}$ to see how much each frequency contributes. We leave this, as well as the resolution of Conjectures 3.1.1–3.1.3, to future work.

3.1.4 Structure and guide to reading the chapter

This chapter is structured as follows: We first recall the family of Schwarzschild spacetimes and recall some geometric preliminaries in §3.2. We then recall relevant results on the wave equation on a Schwarzschild background in §3.3.

The rest of the chapter is divided into two parts: In part I, which comprises sections 3.4–3.6, we focus solely on the $\ell = 1$ -case. This part is written with an emphasis on being instructive

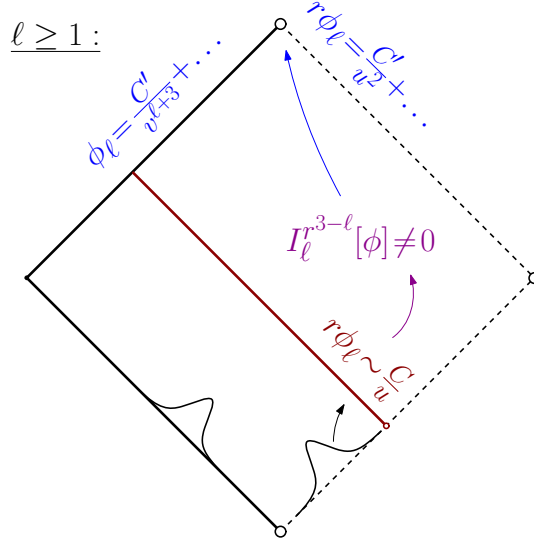


Figure 3.1.5 Given compactly supported scattering data for $r\phi_\ell$ on $\mathcal{H}^-$ and $\mathcal{I}^-$, the solution generically decays like $1/u$ near $\mathcal{I}^-$ (away from the support of the data) unless $\ell = 0$. It thus has finite $(\ell - 1)$ -modified N-P constant (see (3.1.45)) on $\mathcal{I}^+$ by Thm. 3.1.5. We also depicted the conjectured late-time asymptotics for all $\ell \geq 0$.

and providing intuition for the main results and some (but not all) of the methods used to prove them, which might otherwise be camouflaged by the large amount of inductions in the case of general ℓ . In part II, which comprises sections 3.7–3.10, we then develop a more systematic approach for the case of general ℓ .

Part I is structured as follows: In §3.4, we treat the case of data on an ingoing null hypersurface and prove Theorem 3.1.2. In §3.5, we then treat the case of boundary data on a timelike hypersurface Γ of constant area radius and prove Theorem 3.1.1. We shall explain how to lift the restriction to constant area radii and treat boundary data on more general spherically symmetric hypersurfaces in §3.6.

Part II is structured as follows: In §3.7, we derive the higher-order approximate conservation laws for general ℓ -modes and the associated higher-order Newman–Penrose constants. Equipped with these, we then consider the case of boundary data on a hypersurface Γ of constant area radius and prove Theorem 3.1.3 in §3.8. The generalisation to more general Γ proceeds similarly to the one in §3.6 and is left to the reader. The last two sections, §3.9 and §3.10, again concern data on a null hypersurface. In §3.9, we consider the fast initial decay implied by Thm. 3.1.3 and prove Theorem 3.1.4. Section 3.10 then generalises these results to slowly decaying data (using different methods) and contains the proof of Theorem 3.1.5. Various inductive proofs of statements made in part II are deferred to the appendix 3.A.

Depending on the reader’s taste, she can either begin with a thorough reading of part I and then skim through §3.7–§3.9 of part II and carefully read §3.10, which introduces an approach not presented in part I.

Alternatively, she can skip directly to part II and occasionally refer back to part I for details, e.g. on the treatment of boundary data on a timelike hypersurface of varying area radius in §3.6.

In any case, an effort was made to make each section of the chapter as self-contained as possible.

3.2 Geometric preliminaries

3.2.1 The Schwarzschild spacetime manifold

The (exterior of the) Schwarzschild family of spacetimes $(\mathcal{M}_M, g_M)$, $M > 0$,⁹ is given by the family of manifolds

$$\mathcal{M}_M = \mathbb{R} \times (2M, \infty) \times \mathbb{S}^2,$$

covered by the coordinate chart (v, r, θ, φ) , with $v \in \mathbb{R}$, $r \in (2M, \infty)$, $\theta \in (0, \pi)$ and $\varphi \in (0, 2\pi)$, where (θ, φ) denote the standard spherical coordinates on $\mathbb{S}^2$, and by the family of metrics

$$g_M = -D(r) dv^2 + 2 dv dr + r^2(d\theta^2 + \sin^2 \theta d\varphi^2), \quad (3.2.1)$$

where

$$D(r) = 1 - \frac{2M}{r}. \quad (3.2.2)$$

Upon introducing the *tortoise coordinate* r^* as

$$r^*(r) := R + \int_R^r D^{-1}(r') dr' \quad (3.2.3)$$

for some $R > 2M$, and defining

$$u := v - r^*, \quad (3.2.4)$$

one obtains a double null covering (u, v, θ, φ) of $\mathcal{M}_M$, with $u \in (-\infty, \infty)$, $v \in (-\infty, \infty)$. In these coordinates, the metric takes the form

$$g_M = -4D(r) du dv + r^2(d\theta^2 + \sin^2 \theta d\varphi^2). \quad (3.2.5)$$

Throughout the remainder of this chapter, we will always work within this (u, v) -coordinate system.

From the definitions (3.2.3), (3.2.4), it follows that $\partial_v r = -\partial_u r = D$ and that, for sufficiently large values of r :

$$|r - (v - u) + 2M \log r| = \mathcal{O}(1). \quad (3.2.6)$$

The estimate (3.2.6) will be used frequently throughout the chapter.

The vector field $T = \partial_u + \partial_v$ is a Killing vector field, the *static* Killing vector field of the Schwarzschild spacetime, which equips $(\mathcal{M}_M, g_M)$ with a time orientation.

⁹For $M = 0$, one recovers the Minkowski spacetime, to which all the results of the chapter apply as well.

While the metric (3.2.5) in double null coordinates (u, v) becomes singular near $u = \infty$, we see from the form (3.2.1) that one can smoothly extend $(\mathcal{M}_M, g_M)$ to and beyond “ $u = \infty$ ” in (v, r) -coordinates. The set “ $u = \infty$ ” is referred to as $\mathcal{H}^+$, or *future event horizon*, and the region beyond it as the black hole region of the Schwarzschild spacetime. Similarly, one can extend $(\mathcal{M}_M, g_M)$ to and beyond “ $v = -\infty$ ” (denoted $\mathcal{H}^-$) by working in coordinates (u, r) .

On the other hand, we will often consider functions $f \in C^\infty(\mathcal{M}_M)$ such that e.g. the limit $\lim_{v \rightarrow \infty} f(u, v, \theta, \varphi)$ exists and is continuous in u, θ and φ . In these cases, we will interpret the limit as living on the abstract set $\{u, v = \infty, \theta, \varphi\}$, which we will refer to as *future null infinity* or $\mathcal{I}^+$. Similarly, *past null infinity* $\mathcal{I}^-$ corresponds to the set of points $\{u = -\infty, v, \theta, \varphi\}$. One can think of these sets as being attached to $\mathcal{M}$ as boundaries, but the differentiable structure of this extension plays no role in this chapter. See Figure 3.2.1.

We introduce two null foliations of $\mathcal{M}_M$: A foliation by ingoing null hypersurfaces

$$\mathcal{C}_{v=V} = \mathcal{M}_M \cap \{v = V\}, \quad (3.2.7)$$

and a foliation by outgoing null hypersurfaces

$$\mathcal{C}_{u=U} = \mathcal{M}_M \cap \{u = U\}. \quad (3.2.8)$$

We will often just write $\mathcal{C}_U$ instead of $\mathcal{C}_{u=U}$, and, similarly, $\mathcal{C}_V$ instead of $\mathcal{C}_{v=V}$. It will always be clear from the context whether we refer to ingoing or outgoing null hypersurfaces. Moreover, if $f : \mathbb{R} \rightarrow (2M, \infty)$ is a smooth function of u , we shall denote by Γ_f the following timelike hypersurface:

$$\Gamma_f = \mathcal{M}_M \cap \{v = v_{\Gamma_f}(u)\}, \quad (3.2.9)$$

where $v_{\Gamma_f}(u)$ is defined via

$$v_{\Gamma_f}(u) - u = 2r^*(f(u)).$$

In the special case where $f(u) = R > 2M$ is a constant, we simply write $\Gamma_f = \Gamma_R$ and $v_{\Gamma_f}(u) = v_R(u)$.

In the sequel, we will drop the subscript M in $\mathcal{M}_M$ and g_M , and we will frequently quotient out the spheres for a given spherically symmetric subset of $\mathcal{M}$ without writing it (for instance, we will denote the set of all points (u, v) s.t. $(u, v, \theta, \varphi) \in \Gamma_R$ by Γ_R , too).

3.2.2 The divergence theorem

Let $\mathcal{D}$ be any simply connected subset of $\mathcal{M}$ with piecewise smooth boundary $\partial\mathcal{D}$. If J is a smooth 1-form, then we have by the divergence theorem:

$$\int_{\mathcal{D}} \operatorname{div} J = \int_{\partial\mathcal{D}} J \cdot n_{\partial\mathcal{D}} \quad (3.2.10)$$

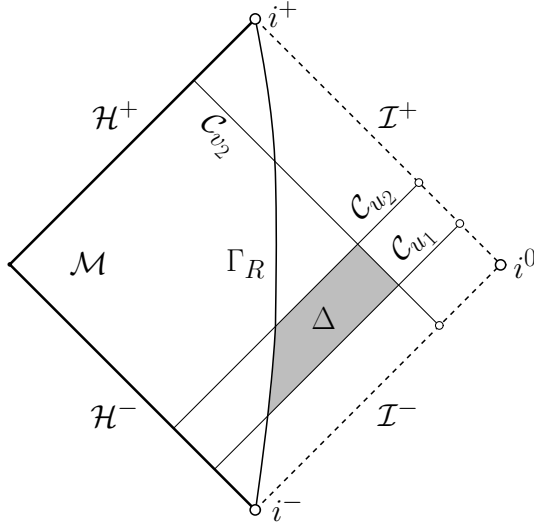


Figure 3.2.1 Depiction of the Schwarzschild manifold $\mathcal{M}$. Also depicted is the region Δ in which we apply the divergence theorem (3.2.11).

Here, $n_{\partial\mathcal{D}}$ is the normal to $\partial\mathcal{D}$, and integration over the canonical volume form is implied. If $\partial\mathcal{D}$ contains null pieces, then there is no canonical choice of volume form or normal on these. In this case, we shall choose the product of volume form and normal in such a way that the divergence theorem (3.2.10) applies (using Stokes' Theorem). For instance, if Δ is the region bounded by $\Gamma_R \cap \{u_1 \leq u \leq u_2\}$, $\mathcal{C}_{u_1} \cap \{v_R(u_1) \leq v \leq v_2\}$, $\mathcal{C}_{u_2} \cap \{v_R(u_2) \leq v \leq v_2\}$ and $\mathcal{C}_{v_2} \cap \{u_1 \leq u \leq u_2\}$, then we have

$$\begin{aligned} & \int_{\Gamma_R \cap \{u_1 \leq u \leq u_2\}} r^2 d(u+v) d\Omega J \cdot (\partial_u - \partial_v) + \int_{\mathcal{C}_{u_1} \cap \{v_R(u_1) \leq v \leq v_2\}} r^2 dv d\Omega J \cdot \partial_v \\ &= \int_{\mathcal{C}_{v_2} \cap \{u_1 \leq u \leq u_2\}} r^2 du d\Omega J \cdot \partial_u + \int_{\mathcal{C}_{u_2} \cap \{v_R(u_2) \leq v \leq v_2\}} r^2 dv d\Omega J \cdot \partial_v - \int_{\Delta} 2Dr^2 du dv d\Omega \operatorname{div} J, \end{aligned} \quad (3.2.11)$$

where $d\Omega = \sin\theta d\theta d\varphi$ is the volume form of the unit sphere. See Figure 3.2.1 for a depiction of this region.

3.3 Generalities on the wave equation

In this section, we collect some important facts about the wave equation

$$\square_g \phi := \nabla^\mu \nabla_\mu \phi = 0 \quad (3.3.1)$$

on a Schwarzschild background, where ∇ denotes the Levi-Civita connection of g .

3.3.1 Existence and uniqueness

We recall the following two standard existence results:

Proposition 3.3.1 (Existence for characteristic initial data). *Let $f \in C^\infty(\mathcal{C}_{v_1} \cap \{u_1 \leq u \leq u_2\})$ and $h \in C^\infty(\mathcal{C}_{u_1} \cap \{v_1 \leq v \leq v_2\})$ be two smooth functions satisfying the usual corner condition. Then there exists a unique smooth function $\phi : \mathcal{M} \cap \{v_1 \leq v \leq v_2, u_1 \leq u \leq u_2\} \rightarrow \mathbb{R}$ such that*

$$\phi|_{\mathcal{C}_{v_1} \cap \{u_1 \leq u \leq u_2\}} = f, \quad \phi|_{\mathcal{C}_{u_1} \cap \{v_1 \leq v \leq v_2\}} = h,$$

and

$$\square_g \phi = 0.$$

Proposition 3.3.2 (Existence for mixed characteristic/boundary data). *Let $f \in C^\infty(\Gamma_R \cap \{u_1 \leq u \leq u_2\})$ and $h \in C^\infty(\mathcal{C}_{u_1} \cap \{v_R(u_1) \leq v \leq v_2\})$ be two smooth functions satisfying the usual corner condition. Then there exists a unique smooth function $\phi : \mathcal{M} \cap \{u_1 \leq u \leq u_2, v_R(u) \leq v \leq v_2\} \rightarrow \mathbb{R}$ such that*

$$\phi|_{\Gamma_R \cap \{u_1 \leq u \leq u_2\}} = f, \quad \phi|_{\mathcal{C}_{u_1} \cap \{v_R(u_1) \leq v \leq v_2\}} = h,$$

and

$$\square_g \phi = 0.$$

3.3.2 The basic energy currents

We define, with respect to any coordinate basis, and for any smooth scalar field $f \in C^\infty(\mathcal{M})$, the following *energy momentum tensor*:

$$\mathbf{T}_{\mu\nu}[f] := \partial_\mu f \partial_\nu f - \frac{1}{2} g_{\mu\nu} \partial^\xi f \partial_\xi f$$

Moreover, if V is any smooth vector field on $\mathcal{M}$, we define the energy current $J^V[f]$ according to

$$J^V[\phi](\cdot) := \mathbf{T}[\phi](V, \cdot).$$

With the divergence theorem (3.2.10) in mind, we compute

$$\operatorname{div} J^V[f] = K^V[f] + \mathcal{E}^V[f], \quad (3.3.2)$$

where

$$K^V[f] := \mathbf{T}^{\mu\nu} \nabla_\mu V_\nu, \quad (3.3.3)$$

$$\mathcal{E}^V[f] := V(f) \square_g f. \quad (3.3.4)$$

Note that $K^V[f]$ vanishes if V is Killing (in view of the symmetry of $\mathbf{T}$), whereas $\mathcal{E}^V[f]$ vanishes if f is a solution to the wave equation. Thus, $K^V[f]$ measures the failure of V to be Killing and $\mathcal{E}^V[f]$ measures the failure of f to solve the wave equation.

3.3.3 Decomposition into spherical harmonics

One can decompose any smooth function $f : \mathcal{M} \rightarrow \mathbb{R}$ into its projections onto spherical harmonics,

$$f = \sum_{\ell'=0}^{\infty} f_{\ell'=\ell'},$$

such that

$$f_{\ell=\ell'}(u, v, \theta, \varphi) = \sum_{m=-\ell'}^{m=\ell'} f_{\ell'm}(u, v) Y_{\ell'm}(\theta, \varphi),$$

where the $Y_{\ell'm}$ are the spherical harmonics. These form a complete basis on $L^2(\mathbb{S}^2)$ of orthogonal eigenfunctions to the spherical Laplacian $\Delta_{\mathbb{S}^2}$, with eigenvalues $-\ell'(\ell' + 1)$. In particular, in view of the spherical symmetry of the Schwarzschild spacetime, if ϕ solves $\square_g \phi = 0$, so does $\phi_{\ell=L}$:

$$\square_g \phi = 0 \implies \square_g \phi_{\ell=L} = 0$$

for any $L \geq 0$. In the sequel, we will frequently suppress the m -index of $\phi_{\ell m}(u, v)$ and just write ϕ_ℓ instead.

Finally, we recall the Poincaré inequality on the sphere:

Lemma 3.3.1. *Let $L > 0$, and let $f_{\ell \geq L} \in C^2(\mathbb{S}^2)$ be supported only on ℓ -modes with $\ell \geq L$. Then*

$$\int_{\mathbb{S}^2} f_{\ell \geq L}^2 d\Omega \leq -\frac{1}{L(L+1)} \int_{\mathbb{S}^2} f_{\ell \geq L} \cdot \Delta_{\mathbb{S}^2} f_{\ell \geq L} d\Omega = \frac{1}{L(L+1)} \int_{\mathbb{S}^2} |\nabla_{\mathbb{S}^2} f_{\ell \geq L}|^2 d\Omega. \quad (3.3.5)$$

3.3.4 The commuted wave equations and the higher-order Newman–Penrose constants

In the double null coordinates (3.2.5), the wave operator $\square_g$ acting on any scalar function f takes the form

$$\square_g f = -\frac{\partial_u \partial_v f}{D} + \frac{1}{r} \partial_v f - \frac{1}{r} \partial_u f + \frac{1}{r^2} \Delta_{\mathbb{S}^2} f. \quad (3.3.6)$$

Hence, if ϕ solves the wave equation $\square_g \phi = 0$, then we obtain the following wave equation for the radiation field $r\phi$ (recall that $\partial_v r = D = -\partial_u r$):

$$\partial_u \partial_v (r\phi) = \frac{D}{r^2} \Delta_{\mathbb{S}^2} (r\phi) - \frac{2MD}{r^3} r\phi. \quad (3.3.7)$$

Notice that if we restrict to the spherically symmetric mode $r\phi_{\ell=0}$, this gives rise to the approximate conservation law

$$\partial_u \partial_v (r\phi_0) = -\frac{2MD}{r^3} r\phi_0. \quad (3.3.8)$$

This equation (3.3.8) is closely related to the existence of conserved quantities along null infinity, the so-called the Newman–Penrose constants

$$I_{\ell=0}^{\text{future}}[\phi](u) := \lim_{v \rightarrow \infty} r^2 \partial_v (r\phi_0)(u, v), \quad (3.3.9)$$

$$I_{\ell=0}^{\text{past}}[\phi](v) := \lim_{u \rightarrow -\infty} r^2 \partial_u (r\phi_0)(u, v), \quad (3.3.10)$$

which, under suitable assumptions on ϕ , remain conserved along $\mathcal{I}^+$, $\mathcal{I}^-$, respectively. Equation (3.3.8) (or rather, the non-linear analogue thereof) played a crucial role in proving our results from chapter 1 and is, in fact, ubiquitous in the studies of asymptotics for the wave equation on Schwarzschild backgrounds, see e.g. [DR05], [AAG18b].

However, for higher ℓ -modes, the approximate conservation law (3.3.8) is no longer available, and the RHS of $\partial_u \partial_v (r\phi_{\ell=L})$ has a bad r^{-2} -weight. This difficulty appears already in the Minkowski spacetime, i.e. for $M = 0$. There, it can be resolved by commuting with $(r^2 \partial_v)^\ell$, $(r^2 \partial_u)^\ell$, respectively. Indeed, if $M = 0$, one has the following precise conservation laws:

$$\partial_u (r^{-2L-2} (r^2 \partial_v)^{(L+1)} (r\phi_L)) = 0,$$

$$\partial_v (r^{-2L-2} (r^2 \partial_u)^{(L+1)} (r\phi_L)) = 0.$$

One can find generalisations of these conservation laws in Schwarzschild. This is done in §3.7 of the chapter. For now, we believe it to be more instructive to only explain what happens to the $\ell = 1$ -modes. If we naively commute the wave equation for $\ell = 1$, namely

$$\partial_u \partial_v (r\phi_1) = -\frac{2D}{r^2} r\phi_1 \left(1 + \frac{M}{r}\right), \quad (3.3.11)$$

with $r^2 \partial_v$, then we find

$$\partial_u (r^{-2} \partial_v (r^2 \partial_v (r\phi_1))) = -10MD \frac{r^2 \partial_v (r\phi_1)}{r^5} - 2MD \frac{r\phi_1}{r^4} \left(1 + \frac{4M}{r}\right). \quad (3.3.12)$$

We see that the top-order term in (3.3.12) comes with a good r^{-5} -weight. Moreover, the problematic r^{-4} -weight multiplying $r\phi_1$ can be removed by subtracting $Mr\phi_1$ in the following way:

$$\partial_u (r^{-2} \partial_v (r^2 \partial_v (r\phi_1) - Mr\phi_1)) = -12MD \frac{r^2 \partial_v (r\phi_1)}{r^5} - 6M^2 D \frac{r\phi_1}{r^5}. \quad (3.3.13)$$

Similarly, for u and v interchanged, we obtain

$$\partial_v(r^{-2}\partial_u(r^2\partial_u(r\phi_1))) = -10MD\frac{r^2\partial_u(r\phi_1)}{r^5} + 2MD\frac{r\phi_1}{r^4} \left(1 + \frac{4M}{r}\right) \quad (3.3.14)$$

and

$$\partial_v(r^{-2}\partial_u(r^2\partial_u(r\phi_1) + Mr\phi_1)) = -12MD\frac{r^2\partial_u(r\phi_1)}{r^5} + 6M^2D\frac{r\phi_1}{r^5}. \quad (3.3.15)$$

The *approximate conservation laws* (3.3.13), (3.3.15) give rise to the following higher-order Newman–Penrose constants:

$$I_{\ell=1}^{\text{future}}[\phi](u) := \lim_{v \rightarrow \infty} r^2\partial_v(r^2\partial_v(r\phi_1) - Mr\phi_1)(u, v), \quad (3.3.16)$$

$$I_{\ell=1}^{\text{past}}[\phi](v) := \lim_{u \rightarrow -\infty} r^2\partial_u(r^2\partial_u(r\phi_1) + Mr\phi_1)(u, v), \quad (3.3.17)$$

which, under suitable assumptions on ϕ , remain conserved along $\mathcal{I}^+$, $\mathcal{I}^-$, respectively. Equations (3.3.13) and (3.3.15) will play a similar role in the asymptotic analysis of the $\ell = 1$ -mode as equation (3.3.8) did in the analysis of chapter 1.

3.3.5 Notational conventions

We use the notation that $f \sim g$ (or $f \lesssim g$) if there exists a uniform constant $C > 0$ such that $C^{-1}g \leq f \leq Cg$ (or $f \leq Cg$). Similarly, we use the convention that $f = \mathcal{O}(g)$ if there exists a uniform constant $C > 0$ such that $|f| \leq Cg$. If f and g are functions depending on a single variable x , and if $k \in \mathbb{N}$, we also say that $f = \mathcal{O}_k(g)$ if there exist uniform constants $C_j > 0$ such that $|\partial_x^j f| \leq C_j |\partial_x^j g|$ for all $j \leq k$. Finally, we use the usual algebra of constants ($C + D = C = CD \dots$).

Part I:

The case $\ell = 1$.

In this part of the chapter, we focus solely on the analysis of the $\ell = 1$ -modes. The aim of this part is to give some intuition for the decay rates and the methods used to prove them. The confident reader may wish to skip directly to the discussion of general ℓ in Part II.

We first treat the case of data on an ingoing null hypersurface and prove Theorem 3.1.2 in §3.4. We then treat the case of boundary data on a timelike hypersurface of constant area radius r and prove Theorem 3.1.1 in §3.5. Finally, we explain how to generalise to the case of boundary data on timelike hypersurfaces on which r is allowed to vary in §3.6.

Throughout Part I, ϕ will always denote a solution to $\square_g \phi = 0$ which is localised on an (ℓ, m) -frequency with $\ell = 1$, $|m| \leq 1$ fixed. We use the notation from §3.3.3, that is, we write $\phi = \phi_{\ell=1} = \phi_1(u, v) \cdot Y_{1m}(\theta, \varphi)$.

3.4 Data on an ingoing null hypersurface $\mathcal{C}_{v=1}$

In this section, we consider solutions ϕ arising from polynomially decaying data on an ingoing null hypersurface $\mathcal{C}_{v=1}$ and from vanishing data on $\mathcal{I}^-$, and we show asymptotic estimates near spatial infinity for these. In particular, this section contains the proof of Theorem 3.1.2 from the introduction.

3.4.1 Initial data assumptions and the main theorem (Theorem 3.4.1)

Prescribe smooth characteristic/scattering data for the wave equation (3.1.1) restricted to $(1, m)$ which satisfy on $\mathcal{C}_{v=1}$

$$r^2 \partial_u (r \phi_1)(u, 1) = C_{\text{in}}^{(1)} + \mathcal{O}(r^{-1}), \quad (3.4.1)$$

$$r^2 \partial_u (r^2 \partial_u (r \phi_1))(u, 1) = C_{\text{in}}^{(2)} + \mathcal{O}(r^{-\eta}) \quad (3.4.2)$$

for some $\eta \in (0, 1)$, and which moreover satisfy for all $v \geq 1$:

$$\lim_{u \rightarrow -\infty} \partial_v^n (r \phi_1)(u, v) = 0 \quad (3.4.3)$$

for $n = 0, 1, 2$. We interpret this latter assumption as the no incoming radiation condition.

The main result of this section then is:

Theorem 3.4.1. *By standard scattering theory [DRS18], there exists a unique smooth scattering solution $\phi_1 \cdot Y_{1m}$ in $\mathcal{M} \cap \{v \geq 1\}$ attaining these data. Let U_0 be a sufficiently large negative number. Then, for all $(u, v) \in \mathcal{D} := (-\infty, U_0] \times [1, \infty)$, the outgoing derivative of $r \phi_1$ satisfies,*

for fixed values of u , the following asymptotic expansion as $\mathcal{I}^+$ is approached:

$$\begin{aligned} r^2 \partial_v(r\phi_1)(u, v) = & -C_{\text{in}}^{(1)} - 2 \int_{-\infty}^u F(u') du' - \frac{2MC_{\text{in}}^{(1)} - 2M \int_{-\infty}^u F(u') du'}{r} \\ & - 2M(C_{\text{in}}^{(2)} - 2MC_{\text{in}}^{(1)}) \frac{\log r - \log |u|}{r^2} + \mathcal{O}(r^{-2}), \end{aligned} \quad (3.4.4)$$

where $F(u)$ is given by the limit of the radiation field $r\phi_1$ on $\mathcal{I}^+$

$$F(u) := \lim_{v \rightarrow \infty} r\phi_1(u, v) = \frac{C_{\text{in}}^{(2)} - 2MC_{\text{in}}^{(1)}}{6|u|^2} + \mathcal{O}(|u|^{-2-\eta}). \quad (3.4.5)$$

In particular, if $M(C_{\text{in}}^{(2)} - 2MC_{\text{in}}^{(1)}) \neq 0$, then peeling fails at future null infinity.

Remark 3.4.1. The methods of our proof can also directly be applied to data which only have

$$r^2 \partial_u(r\phi_1)(u, 1) = C_{\text{in}}^{(1)} + \mathcal{O}_1(r^{-\eta})$$

for $\eta \in (0, 1)$. In that case, one would, schematically, obtain $\partial_v(r\phi_1) = \frac{f_1(u)}{r^2} + \frac{f_2(u)}{r^3} + \mathcal{O}(r^{-3-\eta})$.

In order to prove the theorem, we shall first establish the asymptotics of $r\phi_1$, using equations (3.3.11) and (3.3.15), in §3.4.2, and then establish the asymptotics of $\partial_v(r\phi_1)$, using (3.3.11) and (3.3.13), in §3.4.3. We shall make some important comments in §3.4.4.

3.4.2 Asymptotics for $r\phi_1$

We recall from §3.3.4 the two wave equations

$$\partial_u \partial_v(r\phi_1) = -\frac{2D}{r^2} r\phi_1 \left(1 + \frac{M}{r}\right) \quad (3.4.6)$$

and

$$\partial_v(r^{-2} \partial_u(r^2 \partial_u(r\phi_1))) = -10MD \frac{r^2 \partial_u(r\phi_1)}{r^5} + 2MD \frac{r\phi_1}{r^4} \left(1 + \frac{4M}{r}\right). \quad (3.4.7)$$

The reason that we here choose to work with (3.4.7) rather than (3.3.15) is that, in view of the no incoming radiation condition, the bad r^{-4} -weight multiplying $r\phi_1$ in (3.4.7) is not a problem (since $r\phi_1$ itself will decay).

Throughout the rest of §3.4, U_0 will be a sufficiently large negative number (the largeness depending only on data), and $\mathcal{D}$ will be as in Thm. 3.4.1.

3.4.2.1 A weighted energy estimate and almost-sharp decay for $r\phi_1$

We first prove almost-sharp decay using an energy estimate:

Proposition 3.4.1. *Define the following energies:*

$$\begin{aligned} E_q^{[u_1, u_2]}(v) &:= \int_{u_1}^{u_2} |u|^q \left((\partial_u(r\phi_1))^2 + (r\phi_1)^2 \frac{2D}{r^2} \left(1 + \frac{M}{r} \right) \right) (u, v) \, du, \\ E_q^{[v_1, v_2]}(u) &:= \int_{v_1}^{v_2} |u|^q \left((\partial_v(r\phi_1))^2 + (r\phi_1)^2 \frac{2D}{r^2} \left(1 + \frac{M}{r} \right) \right) (u, v) \, dv. \end{aligned}$$

Then the following energy inequality holds for all $v_2 > v_1 \geq 1$, $q \geq 0$ and for $0 > U_0 \geq u_2 > u_1$:

$$E_q^{[u_1, u_2]}(v_2) + E_q^{[v_1, v_2]}(u_2) \leq E_q^{[u_1, u_2]}(v_1) + E_q^{[v_1, v_2]}(u_1). \quad (3.4.8)$$

Proof. Multiply the wave equation (3.4.6) with $2T(r\phi_1)$ (recall that $T = \partial_u + \partial_v$) to obtain:

$$0 = \partial_u \left((\partial_v(r\phi_1))^2 \right) + \partial_v \left((\partial_u(r\phi_1))^2 \right) + T \left(\frac{2D(r\phi_1)^2}{r^2} \left(1 + \frac{M}{r} \right) \right).$$

This would already lead to the standard energy estimate, but we can exploit a certain monotonicity to obtain a weighted energy estimate: For this, we multiply the above expression with $|u|^q$ and recall that $u < 0$:

$$\begin{aligned} 0 &= \partial_v \left(|u|^q (\partial_u(r\phi_1))^2 + \frac{2D|u|^q(r\phi_1)^2}{r^2} \left(1 + \frac{M}{r} \right) \right) \\ &\quad + \partial_u \left(|u|^q (\partial_v(r\phi_1))^2 + \frac{2D|u|^q(r\phi_1)^2}{r^2} \left(1 + \frac{M}{r} \right) \right) \\ &\quad + q|u|^{q-1} \left((\partial_v(r\phi_1))^2 + \frac{2D(r\phi_1)^2}{r^2} \left(1 + \frac{M}{r} \right) \right). \end{aligned}$$

Finally, integrating this in u and v using the fundamental theorem of calculus gives

$$\begin{aligned} E_q^{[u_1, u_2]}(v_2) + E_q^{[v_1, v_2]}(u_2) &= E_q^{[u_1, u_2]}(v_1) + E_q^{[v_1, v_2]}(u_1) \\ &\quad - \int_{v_1}^{v_2} \int_{u_1}^{u_2} q|u|^{q-1} \left((\partial_v(r\phi_1))^2 + \frac{2D(r\phi_1)^2}{r^2} \left(1 + \frac{M}{r} \right) \right) \, du \, dv. \end{aligned} \quad (3.4.9)$$

□

Remark 3.4.2. A similar result holds for any fixed angular frequency solution. Moreover, in view of Lemma 3.3.5, the above proof also works for any ϕ supported on angular frequencies $\ell \geq L$, for some $L \geq 1$.

From this weighted L^2 -estimate, we can already derive almost-sharp pointwise decay:

Corollary 3.4.1. *There is a constant C depending only on data such that, throughout $\mathcal{D}$:*

$$|r\phi_1(u, v)| \leq \frac{C}{|u|}, \quad |\partial_u(r\phi_1)| \leq \frac{C}{|u|^2}. \quad (3.4.10)$$

Moreover, we have that, for all $v \geq 1$:

$$\lim_{u \rightarrow -\infty} r^2 \partial_u(r\phi_1)(u, v) \equiv C_{\text{in}}^{(1)}. \quad (3.4.11)$$

Proof. We consider the energy estimate above with $q = 2$ and let $(u, v) \in \mathcal{D}$. Then

$$\begin{aligned} r\phi_1(u, v) &= \int_{-\infty}^u \partial_u(r\phi_1)(u', v) \, du' \\ &\leq \left(\int_{-\infty}^u |u'|^{-2} \, du' \right)^{\frac{1}{2}} \left(\int_{-\infty}^u |u'|^2 (\partial_u(r\phi_1))^2(u, v) \, du' \right)^{\frac{1}{2}} \\ &\leq \left(\int_{-\infty}^u |u'|^{-2} \, du' \right)^{\frac{1}{2}} \left(E_2^{[-\infty, u]}(1) + \lim_{u' \rightarrow -\infty} E_2^{[1, v]}(u') \right)^{\frac{1}{2}} \leq \frac{C}{|u|} \end{aligned} \quad (3.4.12)$$

for some constant C solely determined by initial data. Here, we used the no incoming radiation condition (3.4.3) in the first step, Cauchy–Schwarz in the second step, and the energy estimate in the third step. In the last estimate, we then inserted the initial data assumptions¹⁰ (3.4.1) and used that $\lim_{u' \rightarrow -\infty} E_2^{[1, v]}(u') = 0$. To show this latter statement, consider first the energy estimate with $q = 0$ to obtain a bound of the form $\phi_1 \lesssim r^{-\frac{1}{2}}$. Then, insert this bound into (3.4.6) to obtain $\partial_v(r\phi_1) \lesssim r^{-\frac{1}{2}}$, and repeat the argument with, say, $q = 1/2$, and iterate.

Plugging the bound (3.4.12) into the wave equation (3.4.6) and integrating from initial data $v = 1$, we moreover obtain that

$$|\partial_u(r\phi_1)| \leq \frac{C}{u^2},$$

and that, in fact, the limit of $|u|^2 \partial_u(r\phi_1)$ remains constant along $\mathcal{I}^-$. $\square$

3.4.2.2 Asymptotics for $\partial_u(r\phi_1)$ and $r\phi_1$

We now make the decay from Corollary 3.4.1 sharp:

Proposition 3.4.2. *There is a constant C depending only on data such that $r\phi_1$ satisfies the following asymptotic expansion throughout $\mathcal{D}$:*

$$\left| r\phi_1(u, v) - \frac{C_{\text{in}}^{(1)}}{r} - \frac{C_{\text{in}}^{(2)} - 2MC_{\text{in}}^{(1)}}{6|u|^2} \right| \leq \frac{C}{|u|^{2+\eta}} + \frac{C}{r|u|}. \quad (3.4.13)$$

In particular, we have

$$\lim_{v \rightarrow \infty} r\phi_1(u, v) = \frac{C_{\text{in}}^{(2)} - 2MC_{\text{in}}^{(1)}}{6|u|^2} + \mathcal{O}(|u|^{-2-\eta}). \quad (3.4.14)$$

¹⁰Recall that, in view of (3.2.6), $r(1, u) = |u| + \mathcal{O}(\log |u|)$.

Proof. We integrate the approximate conservation law (3.4.7) from $v = 1$:

$$\begin{aligned} r^{-2}\partial_u(r^2\partial_u(r\phi_1))(u, v) &= r^{-2}\partial_u(r^2\partial_u(r\phi_1))(u, 1) \\ &+ \int_1^v \left(\frac{-10MD\partial_u(r\phi_1)}{r^3} + \frac{2MDr\phi_1}{r^4} \left(1 + \frac{4M}{r} \right) \right) (u, v') dv'. \end{aligned} \quad (3.4.15)$$

Using that $\partial_v r = D$ and plugging in the initial data assumption (3.4.2) as well as the almost sharp bounds obtained in Corollary 3.4.1, we obtain

$$\partial_u(r^2\partial_u(r\phi_1))(u, v) \lesssim \frac{r^2}{|u|^4}, \quad (3.4.16)$$

from which, in turn, we obtain via integrating that

$$\left| r^2\partial_u(r\phi_1) - \lim_{u \rightarrow -\infty} r^2\partial_u(r\phi_1) \right| \lesssim \int_{-\infty}^u \frac{r^2}{|u'|^4} du' \lesssim \frac{r^2}{|u|^3}, \quad (3.4.17)$$

where the last inequality can be seen by recalling that $r \sim v - u$, or by an integration by parts, see also eq. (3.4.20) below. Now, by Corollary 3.4.1, we have $\lim_{u \rightarrow -\infty} r^2\partial_u(r\phi_1) = C_{\text{in}}^{(1)}$. Thus, integrating once more in u and using that $r\phi_1$ vanishes on $\mathcal{I}^-$, we obtain that

$$\left| r\phi_1 - C_{\text{in}}^{(1)} r^{-1} \right| \lesssim |u|^{-2}.$$

This estimate provides us with the leading-order behaviour of $r\phi_1$ in r . To also understand the leading-order u -decay of $r\phi_1$, we insert our improved bounds back into equation (3.4.15):

$$r^{-2}\partial_u(r^2\partial_u(r\phi_1))(u, v) = \frac{C_{\text{in}}^{(2)}}{|u|^4} + \int_1^v \frac{-10MC_{\text{in}}^{(1)}}{r^5} + \frac{2MC_{\text{in}}^{(1)}}{r^5} dv' + \mathcal{O}(|u|^{-4-\eta}). \quad (3.4.18)$$

Hence, by again converting the v -integration into r -integration using $\partial_v r = D$,

$$\partial_u(r^2\partial_u(r\phi_1)) = r^2 \left(\frac{C_{\text{in}}^{(2)}}{|u|^4} - \frac{2MC_{\text{in}}^{(1)}}{|u|^4} + \frac{2MC_{\text{in}}^{(1)}}{|r|^4} \right) + \mathcal{O}(r^2|u|^{-4-\eta}). \quad (3.4.19)$$

Integrating this from past null infinity, we again encounter the integral $\int_{-\infty}^u \frac{r^2}{|u'|^4} du'$. We compute this as follows:

$$\begin{aligned} \int \frac{r^2}{|u|^4} du &= \int \partial_u \left(\frac{r^2}{3|u|^3} \right) + \frac{2Dr}{3|u|^3} du \\ &= \frac{r^2}{3|u|^3} + \int \partial_u \left(\frac{r}{3|u|^2} \right) + \frac{D}{3|u|^2} - \frac{4M}{3|u|^3} du = \frac{1}{3} \sum_{k=0}^2 \frac{r^k}{|u|^{k+1}} + \mathcal{O}(|u|^{-2}). \end{aligned} \quad (3.4.20)$$

We therefore obtain the following estimate for $\partial_u(r\phi_1)$:

$$\partial_u(r\phi_1)(u, v) = \frac{C_{\text{in}}^{(1)}}{r^2} + \frac{C_{\text{in}}^{(2)} - 2MC_{\text{in}}^{(1)}}{3} \left(\frac{1}{|u|^3} + \frac{1}{|u|^2 r} + \frac{1}{|u| r^2} \right) + \mathcal{O}(r^{-3} + |u|^{-3-\eta}). \quad (3.4.21)$$

In particular, we thus get that

$$\lim_{v \rightarrow \infty} \partial_u(r\phi_1)(u, v) = \frac{C_{\text{in}}^{(2)} - 2MC_{\text{in}}^{(1)}}{3|u|^3} + \mathcal{O}(|u|^{-3-\eta}). \quad (3.4.22)$$

Integrating once more in u finishes the proof of the proposition. $\square$

3.4.3 Asymptotics for $\partial_v(r\phi_1)$ and proof of Thm. 3.4.1

Equipped with an asymptotic expression for $r\phi_1$, we can now compute the asymptotics of $\partial_v(r\phi_1)$. We first derive the leading-order asymptotics of $\partial_v(r\phi_1)$ up to order $\mathcal{O}(r^{-3})$, using only the wave equation (3.4.6), and then determine the next-to-leading-order asymptotics up to $\mathcal{O}(r^{-4} \log r)$ using the commuted equation (3.3.12).

3.4.3.1 Leading-order asymptotics of $\partial_v(r\phi_1)$

Plugging the asymptotics (3.4.13) of $r\phi_1$ into the wave equation (3.4.6) and integrating the latter from past null infinity, we obtain

$$\partial_v(r\phi_1)(u, v) = -\frac{C_{\text{in}}^{(1)}}{r^2} + \mathcal{O}(r^{-2}|u|^{-1}). \quad (3.4.23)$$

In order to find the $\mathcal{O}(r^{-2}|u|^{-1})$ -term, we commute the wave equation with r^2 ,

$$\partial_u(r^2 \partial_v(r\phi_1)) = -2Dr \partial_v(r\phi_1) - 2D \left(r\phi_1 + \frac{M}{r} r\phi_1 \right), \quad (3.4.24)$$

to find, upon integrating, that

$$r^2 \partial_v(r\phi_1)(u, v) = -C_{\text{in}}^{(1)} - \frac{C_{\text{in}}^{(2)} - 2MC_{\text{in}}^{(1)}}{3|u|} + \mathcal{O} \left(\frac{\log(v-u) - \log|u|}{v} + \frac{1}{|u|^{1+\eta}} + \frac{1}{r} \right), \quad (3.4.25)$$

where we used eq. (3.2.6) and the fact that

$$\int_{-\infty}^u \frac{1}{r(u', v)|u'|} du' \sim \int_{-\infty}^u \frac{1}{(v-u')|u'|} du' = \frac{\log(v-u) - \log|u|}{v}. \quad (3.4.26)$$

In fact, the $\mathcal{O}(\log r)$ -terms in (3.4.25) do not appear: By writing $r\phi_1$ as¹¹ $r\phi_1 = \lim_{\mathcal{I}^+} r\phi_1 - \int_v^\infty \partial_v(r\phi_1)$ in eq. (3.4.24), we can improve the asymptotic estimate (3.4.25) to

$$r^2 \partial_v(r\phi_1)(u, v) = -C_{\text{in}}^{(1)} - 2 \int_{-\infty}^u \lim_{v \rightarrow \infty} r\phi_1(u', v) du' + \mathcal{O}(r^{-1}).$$

This cancellation is related to the one that gives rise to the approximate conservation law (3.3.13). In the above, we used (see also eq. (1.4.56) of chapter 1) that

$$\int_{-\infty}^u \frac{\log(v - u') - \log|u'|}{vr} du' \lesssim \int_{-\infty}^u \frac{\log(v - u') - \log|u'|}{v(v - u')} du' \leq \frac{\pi^2}{6} \frac{1}{v - u} \lesssim \frac{1}{r}. \quad (3.4.27)$$

We summarise our findings in

Proposition 3.4.3. *We have the following asymptotics throughout $\mathcal{D}$:*

$$\partial_v(r\phi_1)(u, v) = \frac{\lim_{\mathcal{I}^+} r^2 \partial_v(r\phi_1)(u)}{r^2} + \mathcal{O}(r^{-3}), \quad (3.4.28)$$

$$r\phi_1(u, v) = \lim_{\mathcal{I}^+} r\phi_1(u) - \frac{\lim_{\mathcal{I}^+} r^2 \partial_v(r\phi_1)(u)}{r} + \mathcal{O}(r^{-2}), \quad (3.4.29)$$

where $\lim_{\mathcal{I}^+} r\phi_1(u)$ is given by (3.4.14), and where

$$\lim_{\mathcal{I}^+} r^2 \partial_v(r\phi_1)(u) = -C_{\text{in}}^{(1)} - 2 \int_{-\infty}^u \lim_{\mathcal{I}^+} r\phi_1(u') = -C_{\text{in}}^{(1)} - \frac{C_{\text{in}}^{(2)} - 2MC_{\text{in}}^{(1)}}{3|u|} + \mathcal{O}(|u|^{-1-\eta}). \quad (3.4.30)$$

3.4.3.2 Next-to-leading-order asymptotics for $\partial_v(r\phi_1)$ (Proof of Thm. 3.4.1)

Proof of Theorem 3.4.1. Equipped with the leading-order asymptotics for $\partial_v(r\phi_1)$ and $r\phi_1$, we now find the asymptotic behaviour of $\partial_v(r^2 \partial_v(r\phi_1))$ using the commuted wave equation

$$\partial_u(r^{-2} \partial_v(r^2 \partial_v(r\phi_1))) = -10MD \frac{r^2 \partial_v(r\phi_1)}{r^5} - 2MD \frac{r\phi_1}{r^4} \left(1 + \frac{4M}{r}\right). \quad (3.4.31)$$

By the no incoming radiation condition (3.4.3) and the fundamental theorem of calculus, we have

$$r^{-2} \partial_v(r^2 \partial_v(r\phi_1))(u, v) = \int_{-\infty}^u \frac{-10MD r^2 \partial_v(r\phi_1)}{r^5} - \frac{2MD r\phi_1}{r^4} \left(1 + \frac{4M}{r}\right) du'. \quad (3.4.32)$$

Plugging the asymptotics from Prop. 3.4.3 into the above, we obtain that

$$\begin{aligned} & r^{-2} \partial_v(r^2 \partial_v(r\phi_1))(u, v) \\ &= \int_{-\infty}^u -\frac{8MD \lim_{\mathcal{I}^+} r^2 \partial_v(r\phi_1)(u')}{r^5} - \frac{2MD \lim_{\mathcal{I}^+} r\phi_1(u')}{r^4} du' + \mathcal{O}(r^{-5}). \end{aligned} \quad (3.4.33)$$

¹¹We write $(\lim_{\mathcal{I}^+} f)(u) = \lim_{v \rightarrow \infty} f(u, v)$.

Evaluating the integrals in a similar way to (3.4.26), we thus find

$$r^2 \partial_v (r^2 \partial_v (r \phi_1)) = 2MC_{\text{in}}^{(1)} - M \frac{C_{\text{in}}^{(2)} - 2MC_{\text{in}}^{(1)}}{3|u|} + \mathcal{O}\left(\frac{\log(1-v/u)}{v} + \frac{1}{|u|^{1+\eta}} + \frac{1}{r}\right). \quad (3.4.34)$$

Notice that the $\mathcal{O}$ -terms in (3.4.34) all integrate to $\mathcal{O}(1/r)$ when multiplied by $1/r$ (cf. (3.4.27)).

To find the next-to-leading-order logarithmic terms, we commute the approximate conservation law (3.4.31) with r^4 :

$$\partial_u (r^2 \partial_v (r^2 \partial_v (r \phi_1))) = -\frac{4D}{r} r^2 \partial_v (r^2 \partial_v (r \phi_1)) - \frac{10MD}{r} r^2 \partial_v (r \phi_1) - 2MDr \phi_1 \left(1 + \frac{4M}{r}\right).$$

Integrating this from past null infinity and plugging in (as in (3.4.33)) the asymptotics for $r^2 \partial_v (r^2 \partial_v (r \phi_1))$, $r^2 \partial_v (r \phi_1)$ and $r \phi_1$ from (3.4.34) and Prop. 3.4.3, respectively, we find:

$$\begin{aligned} & r^2 \partial_v (r^2 \partial_v (r \phi_1))(u, v) \\ &= 2MC_{\text{in}}^{(1)} + \int_{-\infty}^u \frac{12MD}{r} \frac{C_{\text{in}}^{(2)} - 2MC_{\text{in}}^{(1)}}{3|u'|} - 2M \lim_{\mathcal{I}^+} r \phi_1(u') du' + \mathcal{O}(r^{-1}) \\ &= 2MC_{\text{in}}^{(1)} - 2M \int_{-\infty}^u \lim_{\mathcal{I}^+} r \phi_1 du' + 4M(C_{\text{in}}^{(2)} - 2MC_{\text{in}}^{(1)}) \frac{\log(v-u) - \log|u|}{v} + \mathcal{O}(r^{-1}). \end{aligned} \quad (3.4.35)$$

We can now fix u and integrate the above in v from $\mathcal{I}^+$ to obtain for $\partial_v (r \phi_1)$:

$$\begin{aligned} r^2 \partial_v (r \phi_1)(u, v) &= \lim_{\mathcal{I}^+} r^2 \partial_v (r \phi_1)(u) - \frac{\lim_{\mathcal{I}^+} r^2 \partial_v (r^2 \partial_v (r \phi_1))(u)}{r} \\ &\quad - 2M(C_{\text{in}}^{(2)} - 2MC_{\text{in}}^{(1)}) \frac{\log(v-u) - \log|u|}{r^2} + \mathcal{O}(r^{-2}), \end{aligned} \quad (3.4.36)$$

where

$$\lim_{\mathcal{I}^+} r^2 \partial_v (r^2 \partial_v (r \phi_1))(u) = 2MC_{\text{in}}^{(1)} - 2M \int_{-\infty}^u \lim_{\mathcal{I}^+} r \phi_1 du'.$$

This concludes the proof of Theorem 3.4.1. $\square$

3.4.4 Comments

3.4.4.1 The Newman–Penrose constant $I_{\ell=1}^{\text{future}}[\phi]$

It is instructive to also write down the asymptotics of the quantity related to the higher-order Newman–Penrose constant $I_{\ell=1}^{\text{future}}[\phi]$ (recall the definition (3.3.16)):

Theorem 3.4.2. *Let U_0 be a sufficiently large negative number. Then, throughout $\mathcal{D} = (-\infty, U_0] \times [1, \infty)$, the outgoing derivative of the combination $r^2 \partial_v (r \phi_1) - Mr \phi_1$ satisfies, for*

fixed values of u , the following asymptotic expansion as $\mathcal{I}^+$ is approached:

$$r^2 \partial_v (r^2 \partial_v (r \phi_1) - M r \phi_1) = 3M C_{\text{in}}^{(1)} + 4M (C_{\text{in}}^{(2)} - 2M C_{\text{in}}^{(1)}) \frac{\log r - \log |u|}{r} + \mathcal{O}(r^{-1}). \quad (3.4.37)$$

In particular, $I_{\ell=1}^{\text{future}}[\phi](u) \equiv 3M C_{\text{in}}^{(1)}$ is conserved along $\mathcal{I}^+$.

3.4.4.2 The case $C_{\text{in}}^{(1)} = 0$: A logarithmically modified Price's law

Notice that if $C_{\text{in}}^{(1)} = 0$, then $I_{\ell=1}^{\text{future}}[\phi] = 0$. However, one can still define a conserved quantity along future null infinity in this case, namely

$$I_{\ell=1}^{\text{future}, \frac{\log r}{r^3}}[\phi](u) := \lim_{v \rightarrow \infty} \frac{r^3}{\log r} \partial_v (r^2 \partial_v (r \phi_1) - M r \phi_1)(u, v), \quad (3.4.38)$$

which, in our case, is given by $4M C_{\text{in}}^{(2)}$. In particular, by using similar methods to the ones from chapter 2, which combined the results of chapter 1 and [AAG18b], one can thus obtain that the late time asymptotics of the $\ell = 1$ -mode, if one smoothly extends the data to $\mathcal{H}^+$, have *logarithmic corrections at leading order*. In particular, one can obtain that $r \phi_1(u, \infty) = C u^{-3} \log u + \mathcal{O}(u^{-3})$ along $\mathcal{I}^+$, and that $\partial_v \phi_1(\infty, v) = C' v^{-5} \log v + \mathcal{O}(v^{-5})$ along the event horizon $\mathcal{H}^+$, where the constants C and C' can be expressed explicitly in terms of $C_{\text{in}}^{(2)}$.

In order to show this, one needs to combine the results of the present chapter with those of the recent [AAG21] and make modifications to [AAG21] similar to those in chapter 2, see the discussion of §3.1.3.

3.4.4.3 Discussion of the cancellations of Remark 3.1.4 and the case of general ℓ

Recall the cancellations discussed for general ℓ in Remark 3.1.4. Let us here give some intuition for them, restricting, of course, to the case $\ell = 1$.

Theorem 3.4.1 shows that, if $r \phi_1 \sim 1/|u|$ initially, this translates to $r \phi|_{\mathcal{I}^+} \sim u^{-2}$ on null infinity. We found this “cancellation” somewhat tacitly, namely by transporting decay for the commuted quantity $r^2 \partial_u (r^2 \partial_u (r \phi_1))$ along $\mathcal{I}^-$. It is maybe easiest to explain why this approach produces no cancellations for $p \in (0, 1)$ or $p \in (1, 2)$: If $p \in (1, 2)$, then the estimate (3.4.16) becomes *worse*, not better, since the initial data term of (3.4.15) now decays slower. On the other hand, if $p \in (0, 1)$, then (3.4.17) fails, as the limit $\lim r^2 \partial_u (r \phi_1)$ diverges. In fact, this shows that the proof of the present section fails for $p < 1$.

There also is a more direct way of understanding the cancellation for $p = 1$: In view of the estimate (3.4.23), we have that, schematically,

$$r \phi(u, v) = r \phi(u, 1) + \int_1^v \partial_v (r \phi) dv' = r \phi(u, 1) + \int_1^v \frac{-C_{\text{in}}^{(1)}}{r^2(u, v')} dv' = \frac{C_{\text{in}}^{(1)}}{|u|} + \frac{C_{\text{in}}^{(1)}}{r} - \frac{C_{\text{in}}^{(1)}}{|u|} = \frac{C_{\text{in}}^{(1)}}{r},$$

where we used that $r(u, 1) \sim |u|$. From this point of view, it is clear that such cancellations only happen if $r\phi_1 \sim 1/|u|^p$ for $p = 1$. Our more systematic approach of §3.10, in which we analyse general ℓ -modes, will understand the cancellations of Remark 3.1.4 in a generalised form of the above computation. Indeed, in §3.10, we will avoid using the conservation law in the v -direction entirely, and instead only use the conservation law in the u -direction: Instead of propagating decay for $(r^2\partial_u)^{\ell+1}(r\phi_\ell)$ in v and then integrating this $\ell + 1$ times from $\mathcal{I}^-$, we will directly obtain an estimate for $(r^2\partial_v)^{\ell+1}(r\phi_\ell)$ by integrating from $\mathcal{I}^-$ in u , and then integrate this estimate ℓ times from $v = 1$, carefully analysing at each step the initial data contributions. In particular, this approach will also allow for slower decay in the initial data. See already §3.10.3 for a more detailed overview of the approach for general ℓ .

3.5 Boundary data on a timelike hypersurface Γ_R

Having obtained asymptotic estimates for solutions arising from polynomially decaying initial data on an ingoing null hypersurface in the previous section, we now want to obtain similar estimates for solutions arising from polynomially decaying *boundary data on a timelike hypersurface* Γ_R . The main result of this section is the proof of Theorem 3.1.1.

In contrast to the previous section, we here need to construct our solutions at the same time as we prove estimates on them.

We use the notation from §3.3.3, that is, we write $\phi = \phi_{\ell=1} = \phi_1(u, v) \cdot Y_{1m}(\theta, \varphi)$.

3.5.1 Overview of the ideas and structure of the section

Let us briefly recall the approach that we followed in our treatment of the $\ell = 0$ -mode in chapter 1: Given polynomially decaying boundary data on Γ_R , we first considered a sequence of compactly supported boundary data that would approach the original boundary data. This allowed us to use the method of continuity, i.e. bootstrap arguments. We then *assumed* decay for $r\phi_0$, and improved it by essentially integrating the wave equation (3.3.8) first in u and then in v (from Γ_R) and exploiting $2M/R$ as a “small” parameter. In fact, we also showed that one can avoid exploiting smallness in $2M/R$ using a Grönwall argument.

If we want to follow a similar approach for $\ell = 1$, it is not sufficient to consider the uncommuted wave equation (3.3.11) in view of its non-integrable r^{-2} -weight. Instead, it seems more appropriate to use the approximate conservation law (3.3.13) and bootstrap decay on the combination

$$\Phi := r^2 \partial_v(r\phi_1) - Mr\phi_1.$$

The first and main difficulty then becomes apparent: $\Phi|_{\Gamma_R}$ is not given by boundary data (we prescribe boundary data *tangent* to Γ). One way of overcoming this difficulty is to exploit certain cancellations in the wave equation; this however requires one to have knowledge on the T -derivative of $r\phi_1$. Alternatively, one can estimate $r^2 \partial_v(r\phi_1)|_{\Gamma_R}$ using an energy estimate which only uses “a square root” of the bootstrapped decay of $r^2 \partial_v(r\phi_1)$. We will make use of both of these approaches, the former for lower-order derivatives $r^2 \partial_v T^n(r\phi_1)$ (where we have room to make assumptions on $T^{n+1}(r\phi_1)$), and the latter for the top-order derivative $r^2 \partial_v T^N(r\phi_1)$, $n < N$. In fact, using only the latter approach is sufficient, but we find it instructive to also include the former as it since it highlights the importance of commuting with T . In the more systematic approach of the discussion of general ℓ in §3.8, we will, however, exclusively use the latter approach.

Equipped with a boundary estimate on Φ , we can then hope to close the bootstrap argument by simply integrating (3.3.13) first in u and then in v , and exploiting $2M/R$ as a small parameter. In fact, as in the $\ell = 0$ -case, one can avoid this smallness assumption. The only additional subtlety here is that, in order to estimate the RHS of (3.3.13), we need to control $r\phi_1$ and

$\partial_v(r\phi_1)$, which is not directly provided by a bootstrap assumption on the combination Φ . We will deal with this by estimating $r\phi_1$ against the integral over $\partial_v(r\phi_1)$ from Γ_R , and either just exploiting smallness in $2M/R$ or using a more elaborate Grönwall argument.

Structure We first state our initial boundary data assumptions for ϕ_1 , as well as the main theorem, in §3.5.2.1. Then, in order to gain access to the method of continuity, we smoothly cut-off the boundary data in §3.5.2.2. These will lead to finite solutions $\phi_1^{(k)}$ in the sense of Proposition 3.3.2. Using bootstrap methods as outlined above, we can then estimate $r^2\partial_v T^n(r\phi_1^{(k)})$ and $T^n(r\phi_1^{(k)})$ in §3.5.3.

In order to later show that Theorem 3.4.1 can be applied (i.e. to show that the limit $\lim_{u \rightarrow -\infty} (r^2\partial_u)^2(r\phi_1)(u, v)$ exists), we will also need to show some auxiliary estimates on the differences $r^2\partial_v T^n(r\phi_1^{(k)} - |u|T(r\phi_1^{(k)}))$. This is done in §3.5.4.

In §3.5.5, we finally show that the finite solutions $\phi_1^{(k)}$ tend to a limiting solution and show that Theorem 3.4.1 can be applied to it, thus proving Theorem 3.1.1. We make some closing comments in §3.5.6.

3.5.2 The setup

3.5.2.1 The initial/boundary data and the main theorem (Theorem 3.5.1)

Throughout the rest of this section, we shall assume that $R > 2M$ is a constant. In particular, $T = \partial_u + \partial_v$ will be tangent to Γ_R . We then prescribe smooth boundary data $\hat{\phi}_1$ on $\Gamma_R = \mathcal{M}_M \cap \{v = v_R(u)\}$ that satisfy, for $u \leq U_0 < 0$ and $|U_0|$ sufficiently large, the upper bounds

$$|T^n(r\hat{\phi}_1)| \leq \frac{n!C_{\text{in}}^\Gamma}{R|u|^{n+1}}, \quad n = 0, 1, \dots, N+1, \quad (3.5.1)$$

$$|T^n(r\hat{\phi}_1 - |u|T(r\hat{\phi}_1))| \leq \frac{C_{\text{in},\epsilon}^\Gamma}{R|u|^{n+1+\epsilon}}, \quad n = 0, \dots, N'+1 \quad (3.5.2)$$

for some positive constants C_{in}^Γ , $C_{\text{in},\epsilon}^\Gamma$, $\epsilon \in (0, 1)$ and $N, N' \geq 0$ integers, and which also satisfy the following lower bound:

$$|T(r\hat{\phi}_1)| \geq \frac{C_{\text{in}}^\Gamma}{2R|u|^2} > 0. \quad (3.5.3)$$

Moreover, we demand, in a limiting sense, that, for all v ,

$$\lim_{u \rightarrow -\infty} \partial_v^n(r\phi_1)(u, v) = 0, \quad n = 1, \dots, N+1. \quad (3.5.4)$$

Then the main result of this section is

Theorem 3.5.1. *Let $R > 2M$ be a constant. Then there exists a unique solution ϕ_1 to eq. (3.3.11) in $\mathcal{D}_{\Gamma_R} := \mathcal{M} \cap \{v \geq v_R(u)\}$ that restricts correctly to $\hat{\phi}_1$ on Γ_R , $\phi_1|_{\Gamma_R} = \hat{\phi}_1$, and that satisfies (3.5.4). Moreover, if U_0 is a sufficiently large negative number, then there exists a*

constant $C = C(2M/R, C_{\text{in}}^\Gamma)$, depending only on data, such that ϕ_1 obeys the following bounds throughout $\mathcal{D}_{\Gamma_R} \cap \{u \leq U_0\}$:

$$\left| r^2 \partial_v T^n(r\phi_1)(u, v) \right| \leq \frac{C}{|u|^{n+1}}, \quad n = 0, \dots, N, \quad (3.5.5)$$

$$|T^n(r\phi_1)(u, v)| \leq \frac{C}{|u|^{n+1}} \max(r^{-1}, |u|^{-1}), \quad n = 0, \dots, N-1. \quad (3.5.6)$$

Finally, if $N \geq 4$ and $N' \geq 2$, then we have, along any ingoing null hypersurface $\mathcal{C}_v$, that

$$r^2 \partial_u(r\phi_1)(u, v) = \mathcal{O}(r^{-1}), \quad (3.5.7)$$

$$r^2 \partial_u(r^2 \partial_u(r\phi_1))(u, v) = \tilde{C} + \mathcal{O}(r^{-1} + |u|^{-\epsilon}), \quad (3.5.8)$$

where $\tilde{C}$ is a constant that is non-vanishing if $R/2M$ is sufficiently large. In particular, ϕ_1 satisfies the assumptions of Theorem 3.4.1 with $C_{\text{in}}^{(1)} = 0$, $C_{\text{in}}^{(2)} = \tilde{C}$ and $\epsilon = \eta$.

Remark 3.5.1. Let us already draw the reader's attention to the fact that the data above lead to solutions with $C_{\text{in}}^{(1)} = 0$ (cf. (3.4.1)). In view of the comments in §3.4.4.2, this suggests that the data considered here lead to a logarithmically modified Price's law near i^+ .

Remark 3.5.2. Instead of considering data with $\hat{\phi}_1 \sim |u|^{-1}$, we can also consider data with $\hat{\phi}_1 \sim |u|^{-p}$ for $p > 0$ and derive a similar result with some obvious modifications.

Remark 3.5.3. It may be instructive for the reader to keep the following solution to (3.3.11) in the case $M = 0$ in mind:

$$\partial_u \partial_v \left(\frac{1}{2|u|^2} + \frac{1}{|u|r} \right) = -\frac{2}{r^2} \left(\frac{1}{2|u|^2} + \frac{1}{|u|r} \right). \quad (3.5.9)$$

3.5.2.2 Cutting of the data and replacing $\mathcal{I}^-$ with $\mathcal{C}_{u=-k}$

As mentioned before, in order to appeal to bootstrap arguments, we need to work in compact regions. We therefore need to cut the boundary data off and then recover the original data using a limiting argument. Let $(\chi_k(u))_{k \in \mathbb{N}}$ be a sequence of positive smooth cut-off functions such that

$$\chi_k = \begin{cases} 1, & u \geq -k+1, \\ 0, & u \leq -k, \end{cases}$$

and cut off the highest-order derivative: $\chi_k \cdot T^{N+1} \hat{\phi}$. We then have

$$\int_{-\infty}^u \chi_k T^{N+1} \hat{\phi}_1 = \chi_k T^N \hat{\phi}_1 - \int_{-\infty}^u (T\chi_k)(T^N \hat{\phi}_1) = \chi_k T^N \hat{\phi}_1 + \theta_k \cdot \mathcal{O}(k^{-N-1}),$$

where θ_k equals 1 on $\{u \geq -k\}$ and 0 elsewhere. Similarly, we obtain inductively that

$$\underbrace{\int \cdots \int}_{n \text{ times}} \chi_k T^{N+1} \hat{\phi}_1 = \chi_k T^{N+1-n} \hat{\phi}_1 + \theta_k \cdot \mathcal{O}(k^{-N-2+n}).$$

In particular, if we denote $\overbrace{\int \cdots \int}^{N+1 \text{ times}} \chi_k T^{N+1} \hat{\phi}_1$ as $\hat{\phi}_1^{(k)}$, then the bounds (3.5.1), (3.5.2) imply, for sufficiently large negative values of u and for some constant $C'_{\text{in}} = C'_{\text{in}}(N, N')$:

$$|T^n(r\hat{\phi}_1^{(k)})| \leq \frac{n!C_{\text{in}}^\Gamma}{R|u|^{n+1}}, \quad n = 0, 1, \dots, N+1, \quad (3.5.10)$$

$$|T^n(r\hat{\phi}_1^{(k)} - |u|T(r\hat{\phi}_1^{(k)}))| \leq \frac{C_{\text{in},\epsilon}^\Gamma}{R|u|^{n+1+\epsilon}} + C'_{\text{in}}\theta_k \cdot \frac{C_{\text{in}}^\Gamma}{Rk^{n+1}}, \quad n = 0, 1, \dots, N'+1 \quad (3.5.11)$$

Notice that, in the second line above, we lose some decay due to the θ_k -term arising from the cut-off. Since we will take the limit $k \rightarrow \infty$ in the end, this only poses a minor difficulty.

Throughout the next two sections (§3.5.3 and §3.5.4), we shall assume initial/boundary data satisfying the estimates (3.5.10) and (3.5.11) and moreover satisfying

$$\phi_1(u = -k, v) = 0 \quad (3.5.12)$$

for all $v \geq v_R(-k)$. We shall denote the unique solutions to these initial/boundary value problems as $\phi_1^{(k)}$. For the next two sections, we shall drop the superscript (k) , only to reinstate it in §3.5.5, where we will show that the solutions $\phi_1^{(k)}$ tend towards a limiting solution.

3.5.3 Estimates for $\partial_v T^n(r\phi_1)$ and $T^n(r\phi_1)$

Let U_0 be a sufficiently large negative number, and let $\hat{\phi}_1$ be smooth data on Γ_R , supported on $\Gamma_R \cap \{-k < u\}$ and satisfying (3.5.10). By Prop. 3.3.2, there exists a unique smooth solution ϕ_1 throughout $\mathcal{D}_{\Gamma_R} \cap \{-k \leq u\}$ such that $\phi_1(-k, v) = 0$ for all $v \geq v_R(-k)$ and such that $\phi_1|_{\Gamma_R} = \hat{\phi}_1$. We will now derive the following uniform-in- k estimates on this solution ϕ_1 :

Proposition 3.5.1. *Let ϕ_1 be the solution as described above, and let $N \geq 1$. Then, if $|U_0|$ is sufficiently large, there exists a constant $C = C(2M/R, C_{\text{in}}^\Gamma)$ (in particular, this constant does not depend on k), which can be chosen to be independent of $R/2M$ for large enough $R/2M$, such that the following estimates hold throughout $\mathcal{D}_{\Gamma_R} \cap \{-k \leq u \leq U_0\}$:*

$$|r^2 \partial_v T^n(r\phi_1)(u, v)| \leq \frac{C}{|u|^{n+1}}, \quad n = 0, 1, \dots, N, \quad (3.5.13)$$

$$|T^n(r\phi_1)(u, v)| \leq \frac{C}{|u|^{n+1}} \max(r^{-1}, |u|^{-1}), \quad n = 0, 1, \dots, N-1. \quad (3.5.14)$$

Proof. The proof is divided into the sections §3.5.3.1–§3.5.3.5. In §3.5.3.1–§3.5.3.4, we present a bootstrap argument and exploit $2M/R$ as a small parameter to improve the bootstrap assumptions. An overview over this bootstrap argument will be given in §3.5.3.1.

We will then explain how to lift the smallness assumption on $2M/R$ by partially replacing the bootstrap argument with a Grönwall-type argument in §3.5.3.5.

3.5.3.1 The bootstrap assumptions

Let $\{C_{\text{BS}}^{(n)}, n = 1, \dots, N\}$ and $\{C_{\text{BS},\phi}^{(m)}, m = 0, \dots, N-1\}$ be two sets of sufficiently large positive constants. We shall make the following bootstrap assumptions on ϕ_1 :

$$\left| r^2 \partial_v T^n(r\phi_1)(u, v) \right| \leq \frac{C_{\text{BS}}^{(n)}}{|u|^{n+1}} \quad (\text{BS}(\mathbf{n}))$$

for $n = 1, \dots, N$, and

$$|T^m(r\phi_1)(u, v)| \leq \frac{C_{\text{BS},\phi}^{(m)}}{|u|^{m+1}} \max(r^{-1}, |u|^{-1}) \quad (\text{BS}'(\mathbf{m}))$$

for $m = 0, \dots, N-1$.

We now define Δ to be the subset of all $(u, v) \in X := \{(u, v) \mid -k < u \leq U_0, v_R(u) < v\}$ such that, for all $(u', v') \in X$ with $u' \leq u$ and $v' \leq v$, $(\text{BS}(\mathbf{n}))$ and $(\text{BS}'(\mathbf{m}))$ hold for all $n = 1, \dots, N$, $m = 0, \dots, N-1$, respectively.

By compactness and continuity, Δ is non-empty if the constants $C_{\text{BS}}^{(n)}, C_{\text{BS},\phi}^{(m)}$ are chosen sufficiently large. Moreover, Δ is trivially closed in X . We shall show that Δ is also open by improving each of the bootstrap assumptions within Δ .

We shall first improve the bootstrap assumptions for the lower-order T -derivatives ($n \leq N-2$) by explicitly exploiting the precise behaviour for higher T -derivatives in §3.5.3.2 in order for the reader to get a clear intuition for the origin of the assumed rates. In §3.5.3.3, we will then improve the bootstrap assumption away from the top-order derivative ($n \leq N-1$), where we no longer have the sharp decay for $T^N(r\phi_1)$ available. Finally, in §3.5.3.4, we will improve the bootstrap assumptions for the top-order derivatives.

Since the approach of §3.5.3.4 applies to derivatives of any order, the reader can in principle skip §3.5.3.2–§3.5.3.3, which are included for pedagogical reasons, and go directly to §3.5.3.4. In fact, §3.5.3.4 only requires the bootstrap assumptions $(\text{BS}(\mathbf{n}))$ (and not $(\text{BS}'(\mathbf{m}))$). In particular, when going through §3.5.3.2–§3.5.3.3, the reader can focus on the arguments without having to pay close attention to the bootstrap constants.

3.5.3.2 Closing away from the top-order derivatives $j \leq N - 2$

The idea is to exploit the fact that, for $M = 0$, $\phi_1 = 1/r^2$ is a stationary solution. In particular, we expect $\partial_v(r^2\phi_1)$ to have some cancellations (see (3.5.17)), and $r^2\partial_v(r\phi_1)$ to remain approximately conserved in u and v (see (3.5.19)). (We remind the reader of the example solution (3.5.9).)

Proposition 3.5.2. *Let $0 \leq j \leq N - 2$. Then, for sufficiently large values of $R/2M$ and $|U_0|$, and if the ratios $C_{\text{BS},\phi}^{(j)}/C_{\text{BS},\phi}^{(j+1)}$, $C_{\text{BS}}^{(j)}/C_{\text{BS},\phi}^{(j+1)}$ are chosen large enough, we have throughout Δ that, in fact,*

$$\left| r^2 \partial_v T^j(r\phi_1)(u, v) \right| \leq \frac{1}{2} \frac{C_{\text{BS}}^{(j)}}{|u|^{j+1}}, \quad (3.5.15)$$

$$\left| T^j(r\phi_1)(u, v) \right| \leq \frac{1}{2} \frac{C_{\text{BS},\phi}^{(j)}}{|u|^{j+1}} \max(r^{-1}, |u|^{-1}). \quad (3.5.16)$$

Proof. Fix $j \leq N - 2$ and assume (BS'(m)) for $m = j, j + 1$. Motivated by the comment above, we compute

$$\partial_u(r^{-2}\partial_v(r^2\phi_1)) = \frac{DT(r\phi_1)}{r^2} - \frac{8MD}{r^4}r\phi_1. \quad (3.5.17)$$

Commuting with T^j , plugging in the bootstrap assumptions, and integrating (3.5.17) from $u = -k$, we find (recall that $\partial_u r = -D$):

$$\left| r^{-2} T^j \partial_v(r^2\phi_1)(u, v) \right| \leq \int_{r_v(-k)}^{r_v(u)} \frac{C_{\text{BS},\phi}^{(j+1)}}{r^3 |u|^{j+2}} + \frac{8MC_{\text{BS},\phi}^{(j)}}{r^5 |u|^{j+1}} dr \leq \frac{C_{\text{BS},\phi}^{(j+1)}}{2r^2 |u|^{j+2}} + \frac{2MC_{\text{BS},\phi}^{(j)}}{r^4 |u|^{j+1}}. \quad (3.5.18)$$

Here, we denoted $r_v(u)$ as the unique r such that $r^*(r) = v - u$. Now, we similarly compute

$$\partial_u(r^2\partial_v(r\phi_1)) = -2D\partial_v(r^2\phi_1) - 6DM\frac{r\phi_1}{r}. \quad (3.5.19)$$

Commuting again with T^j , plugging in the bound (3.5.18) for $T^j\partial_v(r^2\phi_1)$ from above, and integrating (3.5.19) in u , we then find:

$$\begin{aligned} \left| r^2 \partial_v T^j(r\phi_1)(u, v) \right| &\leq \int_{r_v(-k)}^{r_v(u)} \frac{C_{\text{BS},\phi}^{(j+1)}}{|u|^{j+2}} du + \int_{r_v(-k)}^{r_v(u)} \frac{4MC_{\text{BS},\phi}^{(j)}}{r^2 |u|^{j+1}} + \frac{6MC_{\text{BS},\phi}^{(j)}}{r^2 |u|^{j+1}} dr \\ &\leq \frac{C_{\text{BS},\phi}^{(j+1)}}{(j+1)|u|^{j+1}} + \frac{10MC_{\text{BS},\phi}^{(j)}}{r|u|^{j+1}}. \end{aligned} \quad (3.5.20)$$

For large enough R and $C_{\text{BS}}^{(j)}/C_{\text{BS},\phi}^{(j+1)}$, this proves the first part of the proposition.

Moreover, inserting (3.5.20) back into (3.5.18) and writing $\partial_v(r^2\phi_1) = r\partial_v(r\phi_1) + Dr\phi_1$, we obtain

$$\begin{aligned} D|T^j(r\phi_1)| &\leq |T^j\partial_v(r^2\phi_1)| + \frac{1}{r}|r^2T^j\partial_v(r\phi_1)| \\ &\leq \frac{C_{\text{BS},\phi}^{(j+1)}}{2|u|^{j+2}} + \frac{2MC_{\text{BS},\phi}^{(j)}}{r^2|u|^{j+1}} + \frac{C_{\text{BS},\phi}^{(j+1)}}{(j+1)r|u|^{j+1}} + \frac{10MC_{\text{BS},\phi}^{(j)}}{r^2|u|^{j+1}}. \end{aligned} \quad (3.5.21)$$

This proves the second part of the proposition for large enough R and $C_{\text{BS},\phi}^{(j)}/C_{\text{BS},\phi}^{(j+1)}$. $\square$

3.5.3.3 Closing away from the top-order derivatives $j = N - 1$

In the previous proof, we crucially needed the sharp decay of $T^{j+1}(r\phi_1)$, which we no longer have access to if $j + 1 = N$. We therefore proceed differently now. We will use the approximate conservation law (3.3.13). In fact, since we still have sharp decay for $T^j(r\phi_1)$, it will suffice to consider (the T -commuted)

$$\partial_u(r^{-2}\partial_v(r^2\partial_v(r\phi_1))) = -10MD\frac{r^2\partial_v(r\phi_1)}{r^5} - 2MD\frac{r\phi_1}{r^4}\left(1 + \frac{4M}{r}\right), \quad (3.5.22)$$

since, as long as we have the extra r -decay of $T^j(r\phi_1)$, the bad r^{-4} -weight multiplying $T^j(r\phi_1)$ poses no problem.

Proposition 3.5.3. *Let $0 \leq j \leq N - 1$. Then, for sufficiently large values of $R/2M$ and $|U_0|$, and if $C_{\text{BS}}^{(j)}$ and $C_{\text{BS},\phi}^{(j)}$ are chosen large enough, we have throughout Δ that, in fact,*

$$|r^2\partial_v T^j(r\phi_1)(u, v)| \leq \frac{1}{2} \frac{C_{\text{BS}}^{(j)}}{|u|^{j+1}}, \quad (3.5.23)$$

$$|T^j(r\phi_1)(u, v)| \leq \frac{1}{2} \frac{C_{\text{BS},\phi}^{(j)}}{r|u|^{j+1}} \max(r^{-1}, |u|^{-1}). \quad (3.5.24)$$

Proof. Fix $j \leq N - 1$ and assume (BS(n)) for $n = j, j + 1$ and (BS'(m)) for $m = j$. The idea is to integrate (3.5.22) twice, first from $u = -k$ and then from Γ_R . In doing so, we will pick up the boundary term $r^2T^j(r\phi_1)|_{\Gamma_R}$, which is not given by data. We will therefore estimate this boundary term by using the (T -commuted) eq. (3.5.17): First, note that, by integrating the bound (BS(n)) for $n = j + 1$ from Γ_R , we have

$$|T^{j+1}(r\phi_1)(u, v)| \leq \frac{(j+1)!C_{\text{in}}^\Gamma}{R|u|^{j+1}} + \frac{C_{\text{BS}}^{(j+1)}}{R|u|^{j+1}}. \quad (3.5.25)$$

Hence, by integrating equation (3.5.17) from $u = -k$, we obtain

$$\left| r^{-2} T^j \partial_v (r^2 \phi_1)(u, v) \right| \leq \int_{r_v(-k)}^{r_v(u)} \frac{(j+1)! C_{\text{in}}^\Gamma + C_{\text{BS}}^{(j+1)}}{r^2 R |u|^{j+2}} + \frac{8MC_{\text{BS},\phi}^{(j)}}{r^5 |u|^{j+1}} dr,$$

and, consequentially,

$$\left| r^2 T^j \partial_v (r \phi_1) + Dr \cdot T^j (r \phi_1) \right| \leq r^2 \frac{(j+1)! C_{\text{in}}^\Gamma + C_{\text{BS}}^{(j+1)}}{R |u|^{j+2}} + \frac{2MC_{\text{BS},\phi}^{(j)}}{r |u|^{j+1}}.$$

Evaluating this bound on Γ_R and applying the triangle inequality gives

$$\left| r^2 T^j \partial_v (r \phi_1) \right|_{\Gamma_R} \leq \left(1 - \frac{2M}{R} \right) \frac{j! C_{\text{in}}^\Gamma}{|u|^{j+1}} + \frac{2MC_{\text{BS},\phi}^{(j)}}{R |u|^{j+1}} + R \frac{(j+1)! C_{\text{in}}^\Gamma + C_{\text{BS}}^{(j+1)}}{|u|^{j+2}}. \quad (3.5.26)$$

Notice that, for sufficiently large $|U_0|$, the last term becomes subleading. Moreover, if $2M/R$ is suitably small, the first term, in fact, dominates.

Equipped with this estimate for the boundary term, we can now integrate (the T^j -commuted) approximate conservation law (3.5.22), first from $u = -k$:

$$\begin{aligned} & r^{-2} \partial_v (r^2 \partial_v T^j (r \phi_1))(u, v) \\ &= \int_{-k}^u -10MD \frac{r^2 \partial_v T^j (r \phi_1)}{r^5} - 2MD \frac{T^j (r \phi_1)}{r^4} \left(1 + \frac{4M}{r} \right) du' \\ &\leq \int_{-r_v(k)}^{r_v(u)} \frac{10MC_{\text{BS}}^{(j)}}{r^5 |u|^{j+1}} + \frac{2MC_{\text{BS},\phi}^{(j)}}{r^5 |u|^{j+1}} \left(1 + \frac{4M}{r} \right) dr. \end{aligned}$$

We thus obtain:

$$\left| \partial_v (r^2 \partial_v T^j (r \phi_1)) \right| \leq \frac{10MC_{\text{BS}}^{(j)}}{4r^2 |u|^{j+1}} + \frac{MC_{\text{BS},\phi}^{(j)}}{2r^2 |u|^{j+1}} \left(1 + \frac{16M}{5R} \right). \quad (3.5.27)$$

Finally, integrating (3.5.27) from Γ_R , and estimating the boundary term via (3.5.26), we obtain

$$\begin{aligned} \left| r^2 \partial_v T^j (r \phi_1) \right| &\leq D(R) \frac{j! C_{\text{in}}^\Gamma}{|u|^{j+1}} + \frac{2MC_{\text{BS},\phi}^{(j)}}{R |u|^{j+1}} + R \frac{(j+1)! C_{\text{in}}^\Gamma + C_{\text{BS}}^{(j+1)}}{|u|^{j+2}} \\ &\quad + D^{-1}(R) \left(\frac{10MC_{\text{BS}}^{(j)}}{4R |u|^{j+1}} + \frac{MC_{\text{BS},\phi}^{(j)}}{2R |u|^{j+1}} \left(1 + \frac{16M}{5R} \right) \right). \end{aligned} \quad (3.5.28)$$

(The factor D^{-1} comes from substituting dv with dr in the integral.) If U_0 and $R/2M$ are sufficiently large, and if $C_{\text{BS}}^{(j)}$ is chosen suitably large relative to $j! C_{\text{in}}^\Gamma$, then the RHS can be shown to be smaller than $\frac{C_{\text{BS}}^{(j)}}{2|u|^{j+1}}$. We thus recover the first statement of the proposition.

In order to show the second statement, we exploit the fact that we have also obtained an estimate on $r^2\partial_v(r^2\partial_v T^j(r\phi_1))$ and use the following identity (which follows directly from the wave equation (3.3.11)):

$$-\frac{2D}{r^2}T^j(r\phi_1)\left(1+\frac{M}{r}\right)=T^j\partial_v\partial_u(r\phi_1)=-\partial_v^2T^j(r\phi_1)+\partial_vT^{j+1}(r\phi_1). \quad (3.5.29)$$

The last term of the equation above is controlled by the bootstrap assumption (BS(n)) for $n = j + 1$. For the other term, we can write:

$$\partial_v^2T^j(r\phi_1)=\frac{1}{r^2}\partial_v\left(r^2\partial_vT^j(r\phi_1)\right)-\frac{2D}{r}\partial_vT^j(r\phi_1).$$

We therefore can estimate $T^j(r\phi_1)$ as follows:

$$2D\left(1+\frac{M}{r}\right)|T^j(r\phi_1)|\leq\left|\partial_v\left(r^2\partial_vT^j(r\phi_1)\right)\right|+\frac{2D}{r}\left|r^2\partial_vT^j(r\phi_1)\right|+\left|r^2\partial_vT^{j+1}(r\phi_1)\right|. \quad (3.5.30)$$

Finally, plugging in the estimates (3.5.27), (3.5.28), as well as the assumption (BS(n)) for $n = j + 1$, into the estimate (3.5.30) shows that if U_0 and $R/2M$ are sufficiently large, and if $C_{\text{BS},\phi}^{(j)}$ is chosen suitably large relative to $j!C_{\text{in}}^\Gamma/2$, then the RHS is smaller than $\frac{C_{\text{BS},\phi}^{(j)}}{2r|u|^{j+1}}$, thus proving the proposition.¹² $\square$

3.5.3.4 Closing the top-order derivative $j = N$

In the previous proof, we used the sharp u -decay of $T^{j+1}(r\phi_1)$ (see (3.5.26)), combined with equation (3.5.17), to estimate the boundary term $r^2\partial_vT^j(r\phi_1)|_{\Gamma_R}$ on Γ_R . At the highest order in derivatives, we can no longer do this. Instead, we will estimate the boundary term using an energy estimate. This energy estimate will be wasteful in terms of r -decay, but sharp in terms of u -decay and, therefore, useful on Γ_R . Moreover, it only requires a “square root” of the bootstrap estimate on $r^2\partial_vT^j(r\phi_1)$ and, thus, allows for improvement.

Another difference to the previous section will be that, since we can no longer assume the sharp decay of $T^{j+1}(r\phi_1)$, we will have to work with the approximate conservation law (3.3.13) instead of (3.5.22). (Recall that the former has a better r -weight multiplying $T^j(r\phi_1)$.) This will give us an estimate on $T^j\Phi = r^2\partial_vT^j(r\phi_1) - MT^j(r\phi_1)$. As mentioned in the introduction to this section, we will simply exploit the largeness in $R/2M$ to estimate $r^2\partial_vT^j(r\phi_1)$ in terms of $T^j\Phi$.

¹²Notice that the last term on the RHS of (3.5.30) is the reason why we need the $\max(r^{-1}, |u|^{-1})$ in (BS'(m)) rather than just r^{-1} . This only becomes relevant near $\mathcal{I}^+$.

Proposition 3.5.4. *Let $0 \leq j \leq N$. Then, for sufficiently large values of $R/2M$ and $|U_0|$, and if $C_{\text{BS}}^{(j)}$ is chosen large enough, we have throughout Δ that, in fact,*

$$\left| r^2 \partial_v T^j(r\phi_1)(u, v) \right| \leq \frac{1}{2} \frac{C_{\text{BS}}^{(j)}}{|u|^{j+1}}. \quad (3.5.31)$$

Proof. Fix $j \leq N$ and assume (BS(n)) for $n = j$.

Recall the definition (3.3.2). Since T is Killing and $T^j\phi_{\ell=1}$ solves the wave equation, we have

$$\text{div} J^T[T^j\phi_{\ell=1}] = 0. \quad (3.5.32)$$

We want to apply the divergence theorem to this identity. We recall the notation $\phi_{\ell=1} = \phi_1 \cdot Y_{1m}$, and denote by $\nabla_{\mathbb{S}^2}$ the covariant derivative on the unit sphere. We compute

$$\begin{aligned} J^T[T^j\phi_{\ell=1}] \cdot \partial_u &= \mathbf{T}[T^j\phi_{\ell=1}](T, \partial_u) = (\partial_u T^j\phi_{\ell=1})^2 + \frac{D}{r^2} \left| \nabla_{\mathbb{S}^2} T^j\phi_{\ell=1} \right|^2, \\ J^T[T^j\phi_{\ell=1}] \cdot \partial_v &= \mathbf{T}[T^j\phi_{\ell=1}](T, \partial_v) = (\partial_v T^j\phi_{\ell=1})^2 + \frac{D}{r^2} \left| \nabla_{\mathbb{S}^2} T^j\phi_{\ell=1} \right|^2, \\ J^T[T^j\phi_{\ell=1}] \cdot (\partial_u - \partial_v) &= \mathbf{T}[T^j\phi_{\ell=1}](T, \partial_u - \partial_v) = T^{j+1}\phi_{\ell=1} \cdot (\partial_u - \partial_v) T^j\phi_{\ell=1}. \end{aligned}$$

Let now $(u, v) \in \Delta$. Then, applying the divergence theorem as in (3.2.11) to (3.5.32), we obtain

$$\begin{aligned} & \int_{\mathcal{C}_v \cap \{-k \leq u' \leq u\}} r^2 du' d\Omega J^T[T^j\phi_{\ell=1}] \cdot \partial_u \\ & \leq \int_{\Gamma_R \cap \{-k \leq u' \leq u\}} r^2 d(u' + v') d\Omega J^T[T^j\phi_{\ell=1}] \cdot (\partial_u - \partial_v). \end{aligned} \quad (3.5.33)$$

Doing the integrals over the sphere, using that $u + v = 2u + r^*(R)$ along Γ_R , and plugging in the expressions for the fluxes from above, we obtain

$$\begin{aligned} & \int_{-k}^u \left(r^2 (\partial_u T^j\phi_1)^2 + D |T^j\phi_1|^2 \right) (u', v) du' \\ & \leq \int_{\Gamma_R \cap \{-k \leq u' \leq u\}} \left(2r^2 T^{j+1}\phi_1 \cdot (T - 2\partial_v) T^j\phi_1 \right) (u', u' + r^*(R)) du'. \end{aligned} \quad (3.5.34)$$

Observe that we can estimate the right-hand side of (3.5.34) by using the boundary data assumption (3.5.10) for $T^{j+1}\phi_1$ and the bootstrap assumption (BS(n)) with $n = j$ for $\partial_v T^j\phi_1$.

On the other hand, the left-hand side of (3.5.34) controls $\sqrt{r} T^j\phi$, as can be seen by applying first the fundamental theorem of calculus and then the Cauchy–Schwarz inequality:

$$\left| T^j\phi_1(u, v) \right| \leq \left(\int_{-k}^u \frac{1}{r^2(u', v)} du' \right)^{\frac{1}{2}} \left(\int_{-k}^u r^2 (\partial_u T^j\phi_1)^2 (u', v) du' \right)^{\frac{1}{2}}.$$

Applying the energy identity (3.5.34) to the above estimate gives

$$\begin{aligned} Dr(T^j \phi_1)^2 &\leq \int_{\Gamma_R} 2|T^{j+1}(r\phi_1)| \left(|T^{j+1}(r\phi_1)| + 2|\partial_v T^j(r\phi_1)| + \frac{2D}{r}|T^j(r\phi_1)| \right) du' \\ &\leq \int_{\Gamma_R} \frac{2(j+1)!C_{\text{in}}^\Gamma}{R|u'|^{j+2}} \left(\frac{2(j+1)!C_{\text{in}}^\Gamma}{R|u'|^{j+2}} + \frac{2C_{\text{BS}}^{(j)}}{R^2|u'|^{j+1}} + \frac{2D}{R} \frac{2j!C_{\text{in}}^\Gamma}{R|u'|^{j+1}} \right) du' \\ &\leq \frac{1}{R^3|u|^{2j+2}} \underbrace{\left(4D(j!)^2(C_{\text{in}}^\Gamma)^2 + 2j!C_{\text{in}}^\Gamma C_{\text{BS}}^{(j)} \right)}_{:=A} + \mathcal{O}\left(\frac{1}{|u|^{2j+3}R^2}\right). \end{aligned}$$

Plugging this bound¹³ into the wave equation (3.3.11) and integrating (3.3.11) from $u = -k$ results in the following bound on the boundary term $\partial_v T^j(r\phi_1)|_{\Gamma_R}$:

$$\left| \partial_v T^j(r\phi_1)|_{\Gamma_R} \right| \leq \int \frac{2\sqrt{AD}}{R^{\frac{3}{2}}|u|^{j+1}} \frac{1}{r^{3/2}} \left(1 + \frac{M}{r} \right) du \leq \frac{4\sqrt{A}}{R^2|u|^{j+1}} \frac{1 + \frac{M}{3R}}{\sqrt{1 - \frac{2M}{R}}}. \quad (3.5.35)$$

In fact, we see that the estimate on the boundary term closes by itself!

Having obtained a bound on the boundary term, we can proceed as in the previous proof. We insert the bootstrap estimate (BS(n)) for $n = j$ and the estimate

$$|T^j(r\phi_1)| \leq \frac{j!C_{\text{in}}^\Gamma}{R|u|^{j+1}} + \frac{C_{\text{BS}}^{(j)}}{R|u|^{j+1}} \quad (3.5.36)$$

implied by it into the approximate conservation law (3.3.13) to find (recall $\Phi = r^2\partial_v(r\phi_1) - Mr\phi_1$):

$$\begin{aligned} |r^{-2}\partial_v T^j \Phi(u, v)| &\leq \int_{-k}^u \frac{6M^2 D}{r^5} \frac{j!C_{\text{in}}^\Gamma + C_{\text{BS}}^{(j)}}{R|u'|^{j+1}} + \frac{12MD}{r^5} \frac{C_{\text{BS}}^{(j)}}{|u'|^{j+1}} du' \\ &\leq \frac{3M^2}{2r^4} \frac{j!C_{\text{in}}^\Gamma + C_{\text{BS}}^{(j)}}{R|u|^{j+1}} + \frac{3MC_{\text{BS}}^{(j)}}{r^4|u|^{j+1}}. \end{aligned} \quad (3.5.37)$$

Multiplying the above estimate by r^2 , integrating from Γ_R and using the bound (3.5.35) to estimate the boundary term, we thus obtain

$$|T^j \Phi| \leq \frac{4\sqrt{A}}{|u|^{j+1}} \frac{1 + \frac{M}{3R}}{\sqrt{1 - \frac{2M}{R}}} + \frac{Mj!C_{\text{in}}^\Gamma}{R|u|^{j+1}} + \frac{1}{1 - \frac{2M}{R}} \left(\frac{3M^2}{2} \frac{j!C_{\text{in}}^\Gamma + C_{\text{BS}}^{(j)}}{R^2|u|^{j+1}} + \frac{3MC_{\text{BS}}^{(j)}}{R|u|^{j+1}} \right). \quad (3.5.38)$$

Importantly, $C_{\text{BS}}^{(j)}$ in the above estimate is either multiplied by decaying R -weights, or appears sublinearly inside a square root (in A).

¹³We from now on ignore the $\mathcal{O}(|u|^{-2j-3})$ -term, for it can be easily absorbed into the slightly wasteful estimates we make at each step by choosing U_0 large enough.

We can now combine (3.5.38) with (3.5.36) and write

$$|r^2 \partial_v T^j(r\phi_1)| \leq |T^j \Phi| + M |T^j(r\phi_1)| \quad (3.5.39)$$

to close the bootstrap assumption, provided that R and $C_{\text{BS}}^{(j)}$ are chosen large enough. $\square$

In order to close the entire bootstrap argument, one can now first apply Proposition 3.5.4 to $j = N$, then apply Proposition 3.5.3 to $j = N - 1$ and, finally, apply Proposition 3.5.2 to all $j \leq N - 2$. One thus obtains that Δ is open and hence closes the bootstrap argument. In particular, we have established the proof of Proposition 3.5.1.

More systematically, one could instead make only the bootstrap assumptions (BS(n)) (without assuming (BS'(m))), apply Proposition 3.5.4 to all $j \leq N$ in order to close the bootstrap argument, and then use the identity (3.5.29) in order to obtain the remaining estimates for $T^j(r\phi_1)$. This will be the approach followed in section 3.8. $\square$

3.5.3.5 Removing the smallness assumption on $2M/R$.

In the proofs of the previous sections §3.5.3.2–§3.5.3.4, we exclusively followed continuity methods, which required us to exploit $2M/R$ as a small parameter at various steps. It turns out that one can partially replace the continuity argument with a Grönwall argument to remove all smallness assumptions on $2M/R$. Let us briefly sketch how this works.

Let ϕ denote the finite solution as described in the beginning of §3.5.3. First, we remark that the proof of Proposition 3.5.4 shows that one can obtain an estimate of the form

$$|\partial_v T^j(r\phi_1)|_{\Gamma_R} \leq \frac{C}{\sqrt{R}} |u|^{-j-1} \quad (3.5.40)$$

without requiring largeness in R (this can be obtained by assuming a bootstrap estimate on $r^2 \partial_v T^j(r\phi_1)|_{\Gamma_R}$ and improving it using the energy estimate, cf. (3.5.35)).

Equipped with this boundary term estimate, one can then obtain an estimate on $r^2 \partial_v T^j(r\phi)$ throughout $\mathcal{D}_{\Gamma_R} \cap \{u \geq -k\}$ as follows: Let (u, v) in $\mathcal{D}_{\Gamma_R} \cap \{u \geq -k\}$. For simpler notation, set $j = 0$. Then, by the fundamental theorem of calculus,

$$|\Phi(u, v) - \Phi(u, v_R(u))| \leq \int_{v_R(u)}^v r^2(u, v') \int_{-k}^u \left| 12MD \frac{r^2 \partial_v(r\phi_1)}{r^5} - 6M^2 D \frac{r\phi_1}{r^5} \right| (u', v') du' dv'. \quad (3.5.41)$$

Recalling the definition of $\Phi = r^2 \partial_v(r\phi_1) - Mr\phi_1$, estimating the $Mr\phi_1$ -term against the integral over $\partial_v(r\phi_1)$ from Γ , and applying Tonelli in the inequality above, we obtain:

$$|r^2 \partial_v(r\phi_1)(u, v)| \leq \left| r^2 \partial_v(r\phi_1)(u, v_R(u)) \right| + \int_{v_R(u)}^v M \frac{|r^2 \partial_v(r\phi_1)|}{r^2}(u, v') dv'$$

$$+ \int_{-k}^u \int_{v_R(u)}^v r^2(u, v') \left| 12MD \frac{r^2 \partial_v(r\phi_1)}{r^5} - 6M^2 D \frac{r\phi_1}{r^5} \right| (u', v') dv' du'. \quad (3.5.42)$$

We now also write the $r\phi_1$ -term in the double integral as an integral over $\partial_v(r\phi_1)$ from Γ and pull out the relevant suprema:

$$\begin{aligned} |r^2 \partial_v(r\phi_1)(u, v)| &\leq \left| r^2 \partial_v(r\phi_1)(u, v_R(u)) \right| + \int_{v_R(u)}^v M \frac{|r^2 \partial_v(r\phi_1)|}{r^2}(u, v') dv' \\ &+ C \int_{-k}^u \frac{\sup_{v' \in [v_R(u), v]} |r^2 \partial_v(r\phi_1)(u', v')|}{r^2(u', v_{\Gamma_R}(u))} + \frac{\sup_{v' \in [v_R(u'), v]} |r^2 \partial_v(r\phi_1)(u', v')|}{r^2(u', v_R(u))} du'. \end{aligned} \quad (3.5.43)$$

We already control the boundary term on the RHS. If we now fix u and just regard the last two integrals on the RHS as some monotonically increasing function of v , we can apply Grönwall's inequality in the v -direction to obtain that

$$|r^2 \partial_v(r\phi_1)(u, v)| \leq C \left(\left| r^2 \partial_v(r\phi_1)(u, v_R(u)) \right| + \int_{-k}^u \frac{\sup_{v' \in [v_R(u'), v]} |r^2 \partial_v(r\phi_1)(u', v')|}{r^2(u', v_{\Gamma_R}(u))} du' \right). \quad (3.5.44)$$

Finally, we take the supremum in v , $\sup_{v' \in [v_R(u), v]}$, on the RHS and apply another Grönwall inequality, this time in u . This then shows that

$$\sup_{v' \in [v_R(u), v]} |r^2 \partial_v(r\phi_1)(u, v')| \leq C \left| r^2 \partial_v(r\phi_1)(u, v_R(u)) \right|, \quad (3.5.45)$$

and thus shows (3.5.5) for $n = 0$.

Clearly, this approach requires no smallness assumption on $2M/R$ other than $R > 2M$. Nevertheless, in hopes of simplifying the presentation, we will keep exploiting $2M/R$ as a small parameter throughout the remainder of the chapter (i.e. §3.5.4 and §3.8). However, as the argument above shows, these smallness assumptions can always be lifted if one only wants to show *upper bounds*. The only times where we really need $2M/R$ as a small parameter is when we show *lower bounds* on $r\phi_1$ etc., see already §3.5.5.2.

3.5.4 Estimates for $\partial_v T^n(r\phi_1 - |u|T(r\phi_1))$

The results obtained thus far are sufficient to show the first two estimates of Theorem 3.5.1. In fact, not much modification is needed to also show certain lower bounds. However, something different needs to be done in order to establish the existence of the limit $\lim_{u \rightarrow -\infty} r^2 \partial_u(r^2 \partial_u(r\phi_1))$ (i.e. to prove eq. (3.5.8)). A crucial ingredient for this is to prove decay estimates for the differences $T^j(\partial_v(r\phi_1) - |u|\partial_v T(r\phi_1))$ (the reader may wish to already have a look at §3.5.5.2 to understand the role played by these quantities).

Therefore, let from now on ϕ_1 be as described in the beginning of §3.5.3, but with the additional assumption that also the lower bound (3.5.11) holds on the boundary data. We will now establish the following uniform decay estimates:

Proposition 3.5.5. *Let ϕ_1 be the solution described above, and let $1 \leq N' \leq N + 2$. Then, if $|U_0|$ is sufficiently large, there exists a constant $C = C(2M/R, C_{\text{in}}^\Gamma, C_{\text{in},\epsilon}^\Gamma)$ (in particular, this constant does not depend on k), which can be chosen independent of R for large enough R , such that the following estimates hold throughout $\mathcal{D}_{\Gamma_R} \cap \{-k \leq u \leq U_0\}$:*

$$\left| r^2 T^n \partial_v (r\phi_1 - |u|T(r\phi_1)) \right| \leq \frac{C}{|u|^{n+1+\epsilon}} + \theta_k \cdot \frac{C}{k^{n+1}}, \quad n = 0, \dots, N'. \quad (3.5.46)$$

Proof. The proof will be very similar to the proof of Proposition 3.5.1. We will again treat $2M/R$ as a small parameter, keeping in mind that this restriction can be lifted as in §3.5.3.5.

3.5.4.1 The bootstrap assumptions

Let $\{C_{\text{BS},\epsilon}^{(n)}, 0 = 1, \dots, N'\}$ be a set of sufficiently large positive constants, and let Δ be defined as in 3.5.3.1, with the additional requirement that also

$$\left| r^2 T^n \partial_v (r\phi_1 - |u|T(r\phi_1)) \right| \leq \frac{C_{\text{BS},\epsilon}^{(n)}}{|u|^{n+1+\epsilon}} + \theta_k \cdot \frac{C_{\text{BS},\epsilon}^{(n)}}{k^{n+1}} \quad (\text{BS}''(n))$$

holds for $n = 0, \dots, N'$. We shall improve these estimates in the following. Note that we only assume estimates on the ∂_v -derivatives, so we can just use the method of the proof of Proposition 3.5.4 with some adaptations. The crucial observation is that, while the differences $T^n(\phi_1 - |u|T\phi_1)$ do not solve the wave equation, the error term is of the form $\partial_v T^{n+1}(r\phi_1)$, over which we already have sharp control by Proposition 3.5.1.

3.5.4.2 Improving the bootstrap assumptions

Proposition 3.5.6. *Let $j \in \{0, \dots, N'\}$ for $N' \leq N + 2$. Then, for sufficiently large values of $R/2M$ and $|U_0|$, and if $C_{\text{BS},\epsilon}^{(j)}$ is chosen large enough, we have throughout Δ that, in fact,*

$$\left| r^2 T^j \partial_v (r\phi_1 - |u|T(r\phi_1)) \right| \leq \frac{1}{2} \frac{C_{\text{BS},\epsilon}^{(j)}}{|u|^{j+1+\epsilon}} + \frac{1}{2} \theta_k \cdot \frac{C_{\text{BS},\epsilon}^{(j)}}{k^{n+1}}. \quad (3.5.47)$$

Proof. We shall only need to assume (BS''(n)) for $n = j$ and, in addition, the results of Proposition 3.5.1. We shall follow the structure of the proof of Proposition 3.5.4.

First, we require an estimate of $r^2 T^j \partial_v (r\phi_1 - |u|T(r\phi_1))$ on the boundary Γ_R . Recall that, in the previous proof, we obtained such an estimate by using an energy estimate to obtain a bound on $\sqrt{r} T^j \phi_1$ with sharp decay in u , and by then using the wave equation to convert this into a bound for $\partial_v T^j(r\phi_1)$ on Γ_R . Proceeding along the same lines for the differences under

consideration, we are led to consider the current $J^T[T^j(\phi_1 - |u|T\phi_1)]$. The divergence of this current is no longer vanishing. Instead, we have¹⁴

$$\begin{aligned} \operatorname{div} J^T [T^j(\phi_1 - |u|T\phi_1)] &= \square_g (T^j(\phi_1 - |u|T\phi_1)) \cdot T (T^j(\phi_1 - |u|T\phi_1)) \\ &= -\frac{1}{Dr} \partial_v T^{j+1}(r\phi_1) \cdot \frac{1}{r} T^{j+1}(r\phi_1 - |u|T(r\phi_1)), \end{aligned} \quad (3.5.48)$$

where we used the formula (3.3.6) for $\square_g$. Using the estimates from Proposition 3.5.1 and the fact that $j+2 \leq N'+2 \leq N$, we can thus bound $\operatorname{div} J^T$ as follows:

$$\left| \operatorname{div} J^T [T^j(\phi_1 - |u|T\phi_1)] \right| \leq 2 \cdot \frac{C^2}{R \cdot r^4 |u|^{2j+4}}. \quad (3.5.49)$$

Applying the divergence theorem, in the form of (3.2.11), to the current J^T , and doing the integrals over the sphere, we then arrive at (compare to eq. (3.5.34))

$$\begin{aligned} &\int_{-k}^u r^2 \left(\partial_u T^j(\phi_1 - |u'|T\phi_1) \right)^2 du' \\ &\leq \int_{\Gamma_R \cap \{-k \leq u' \leq u\}} 2r^2 T^{j+1}(\phi_1 - |u'|T\phi_1) \cdot (2\partial_v - T) T^j(\phi_1 - |u'|T\phi_1) du' \\ &+ \int_{-k}^u \int_{v_R(u')}^v r^2 \left| \operatorname{div} J^T [T^j(\phi_1 - |u'|T\phi_1)] \right| du' dv'. \end{aligned} \quad (3.5.50)$$

We have already estimated the terms inside the bulk term in (3.5.49). Indeed, we can see that the contribution to the RHS of the estimate (3.5.50) above is subleading:

$$\int_{-k}^u \int_{v_R(u)}^v r^2 \left| \operatorname{div} J^T [T^j(\phi_1 - |u'|T\phi_1)] \right| du' dv' \leq \frac{\tilde{C}}{R^2 |u|^{2j+3}} \quad (3.5.51)$$

for some constant $\tilde{C}$. On the other hand, we can estimate the boundary term in (3.5.50) by plugging in the boundary data assumptions (3.5.11) for $n = j, j+1$ and the bootstrap assumption (BS''(n)) for $n = j$. This gives:

$$\begin{aligned} &\int_{\Gamma_R \cap \{-k \leq u' \leq u\}} 2r^2 T^{j+1}(\phi_1 - |u'|T\phi_1) \cdot (2\partial_v - T) T^j(\phi_1 - |u'|T\phi_1) du' \\ &\leq \int_{\Gamma_R \cap \{-k \leq u' \leq u\}} 2 \left(\frac{C_{\text{in},\epsilon}^\Gamma}{R |u'|^{2+j+\epsilon}} + \frac{C_{\text{in}}^\Gamma}{R k^{j+2}} \right) \\ &\quad \cdot \left(\frac{2}{R^2} \left(\frac{C_{\text{BS},\epsilon}^{(j)}}{|u'|^{j+1+\epsilon}} + \frac{C_{\text{BS},\epsilon}^{(j)}}{k^{j+1}} \right) + \left(\frac{2D}{R} + \frac{1}{|u'|} \right) \left(\frac{C_{\text{in},\epsilon}^\Gamma}{R |u'|^{1+j+\epsilon}} + \frac{C_{\text{in}}^\Gamma}{R k^{j+1}} \right) \right) du' \\ &\leq \frac{8}{R^3} \max(C_{\text{in},\epsilon}^\Gamma, C_{\text{in}}^\Gamma) \left(C_{\text{BS},\epsilon}^{(j)} + \max(C_{\text{in},\epsilon}^\Gamma, C_{\text{in}}^\Gamma) \right) \left(\frac{1}{k^{j+1}} + \frac{1}{|u|^{j+1+\epsilon}} \right)^2 \cdot (1 + \mathcal{O}(|u|^{-1})). \end{aligned}$$

¹⁴We abuse notation and write $\phi_{\ell=1} = \phi_1$.

We thus find, using the fundamental theorem of calculus, Cauchy–Schwarz and the energy estimate (3.5.50) above, that

$$\sqrt{Dr}|T^j(\phi_1 - |u|T\phi_1)| \leq \frac{\sqrt{A'}}{R^{\frac{3}{2}}} \left(\frac{1}{k^{j+1}} + \frac{1}{|u|^{j+1+\epsilon}} \right) (1 + \mathcal{O}(|u|^{-\frac{1}{2}})), \quad (3.5.52)$$

where A' is a constant which, importantly, only depends *linearly* on $C_{\text{BS},\epsilon}^{(j)}$. We now use the wave equation (3.3.11) in order to derive an estimate for the ∂_v -derivative on the boundary. We compute from (3.3.11) that

$$\partial_u \partial_v (T^j(r\phi_1 - |u|T(r\phi_1))) = -\frac{2D}{r^2} \left(1 + \frac{M}{r} \right) (T^j(r\phi_1 - |u|T(r\phi_1))) + \partial_v T^{j+1}(r\phi_1). \quad (3.5.53)$$

Note that we control the error term $\partial_v T^{j+1}(r\phi_1)$ by Proposition 3.5.1; in fact, it is subleading in terms of u -decay. Integrating (3.5.53) from $u = -k$, and plugging in the estimate (3.5.52), we find that (see also (3.5.35))

$$|\partial_v T^j(r\phi_1 - |u|T(r\phi_1))| \leq \frac{4\sqrt{A'}}{R^2} \frac{1 + \frac{M}{3R}}{\sqrt{1 - \frac{2M}{R}}} \left(\frac{1}{k^{j+1}} + \frac{1}{|u|^{j+1+\epsilon}} \right) (1 + \mathcal{O}(|u|^{-\frac{1}{2}})). \quad (3.5.54)$$

We have thus established an estimate on the boundary term. Now, in order to improve the bootstrap assumption, we want to appeal to the approximate conservation law (3.3.13). We compute that

$$\begin{aligned} \partial_u \left(r^{-2} \partial_v T^j(\Phi - |u|T\Phi) \right) &= r^{-2} \partial_v T^{j+1}\Phi \\ &\quad - \frac{6M^2D}{r^5} (T^j(r\phi_1 - |u|T(r\phi_1))) - \frac{12MD}{r^5} r^2 \partial_v T^j(r\phi_1 - |u|T(r\phi_1)). \end{aligned} \quad (3.5.55)$$

Again, we control the error term $r^{-2} \partial_v T^{j+1}\Phi$ by Proposition 3.5.1; in fact, it has more u -decay than the other terms:

$$|r^{-2} \partial_v T^{j+1}\Phi| \leq \frac{C}{r^4 |u|^{j+2}}.$$

Converting some of the additional $|u|$ -decay present in $\partial_v T^{j+1}\Phi$ into r -decay,

$$|r^{-2} \partial_v T^{j+1}\Phi| \leq \frac{C}{r^{5-\delta} |u|^{j+1+\delta}},$$

for some suitable $1 > \delta > \epsilon$, and repeating the computations leading to (3.5.38), we thus find

$$|T^j(\Phi - |u|T\Phi)| \leq \frac{4\sqrt{A'}}{R^2} \frac{1 + \frac{M}{3R}}{\sqrt{1 - \frac{2M}{R}}} \left(\frac{1}{k^{j+1}} + \frac{1}{|u|^{j+1+\epsilon}} \right) + \frac{MC_{\text{in},\epsilon}^\Gamma}{R|u|^{j+1+\epsilon}} + \frac{MC_{\text{in}}^\Gamma}{Rk^{j+1}}$$

$$+ \frac{1}{1 - \frac{2M}{R}} \left(\frac{3M^2}{2} \frac{C_{\text{in},\epsilon}^\Gamma + C_{\text{BS},\epsilon}^{(j)}}{R^2} + \frac{3MC_{\text{BS},\epsilon}^{(j)}}{R} \right) \left(\frac{1}{k^{j+1}} + \frac{1}{|u|^{j+1+\epsilon}} \right) + \mathcal{O}(|u|^{-j-1-\delta}). \quad (3.5.56)$$

Importantly, $C_{\text{BS},\epsilon}^{(j)}$ in the above estimate appears either multiplied by decaying R -weights or sublinearly inside a square root (in A'). Therefore, if $R/2M$ and $C_{\text{BS},\epsilon}^{(j)}$ are sufficiently large, we can improve the bootstrap assumption (and thus prove the proposition) by again writing

$$\left| r^2 T^j \partial_v (r\phi_1 - |u|T(r\phi_1)) \right| \leq \left| T^j (\Phi - |u|T\Phi) \right| + M \left| T^j (r\phi_1 - |u|T(r\phi_1)) \right|. \quad (3.5.57)$$

□

This concludes the proof of Proposition 3.5.5. □

3.5.5 Proof of Thm. 3.5.1

Recall from §3.5.2.2 the definition of the sequence of solutions $\phi_1^{(k)}$, each arising from data satisfying (3.5.10), (3.5.11) and (3.5.12). We have shown sharp, uniform-in- k decay for these solutions in Propositions 3.5.1 and 3.5.5.

We will now smoothly extend these solutions to the zero solution for $u \leq -k$ and show that they converge uniformly to a pointwise limit ϕ_1 as $k \rightarrow -\infty$, which therefore still satisfies the uniform bounds of Propositions 3.5.1 and 3.5.5.

3.5.5.1 Sending $\mathcal{C}_{u=-k} \rightarrow \mathcal{I}^-$

Proposition 3.5.7. *Let $\{\phi_1^{(k)}\}_{k \in \mathbb{N}}$ be the sequence of solutions described in §3.5.2.2 extended with the zero solution for $u \leq -k$. This sequence $\{\phi_1^{(k)}\}_{k \in \mathbb{N}}$ tends to a uniform limit ϕ_1 as $k \rightarrow \infty$,*

$$\lim_{k \rightarrow \infty} \|\phi_1^{(k)} - \phi_1\|_{C^N(\mathcal{D}_{\Gamma_R})} = 0. \quad (3.5.58)$$

In fact, this limiting solution is the unique smooth solution that restricts correctly to the data of §3.5.2.1, and it satisfies, throughout $\mathcal{D}_{\Gamma_R} \cap \{u \leq U_0\}$, and for sufficiently large negative values of U_0 , the following bounds for some constant $C = C(2M/R, C_{\text{in}}^\Gamma, C_{\text{in},\epsilon}^\Gamma)$ which can be chosen independent of R for large enough R :

$$\left| r^2 \partial_v T^n (r\phi_1)(u, v) \right| \leq \frac{C}{|u|^{n+1}}, \quad n = 0, 1, \dots, N, \quad (3.5.59)$$

$$|T^n (r\phi_1)(u, v)| \leq \frac{C}{|u|^{n+1}} \max \left(r^{-1}, |u|^{-1} \right), \quad n = 0, 1, \dots, N-1. \quad (3.5.60)$$

Moreover, if $N' \leq N+2$, we also have

$$\left| r^2 T^n \partial_v (r\phi_1 - |u|T(r\phi_1)) \right| \leq \frac{C}{|u|^{n+1+\epsilon}}, \quad n = 0, 1, \dots, N'. \quad (3.5.61)$$

Proof. We show that the sequence is Cauchy. Let $\delta > 0$ arbitrary. We need to show that there exists $K \in \mathbb{N}$ such that

$$\|\phi_1^{(n)} - \phi_1^{(k)}\|_{C^N(\mathcal{D}_{\Gamma_R})} < \delta \quad (3.5.62)$$

for all $n, k > K$. This is done by splitting $\mathcal{D}_{\Gamma_R}$ into three regions: $u \leq -n$, $-n \leq u \leq -k+1$ and $-k+1 \leq u$, where we assumed without loss of generality that $n > k$.

In the first region, $u \leq -n$, both solutions vanish, so there is nothing to show.

Notice that, by linearity, the difference $\Delta\phi_1 := \phi_1^{(n)} - \phi_1^{(k)}$ is itself a solution to the wave equation (3.3.11), with vanishing data on $u = -n$ and compactly supported boundary data on $\Gamma_R \cap \{-n \leq u \leq -k+1\}$. We can therefore simply apply the results of Proposition 3.5.1 to $\Delta\phi_1$ in the second region, $-n \leq u \leq -k+1$, and obtain, for some constant C_1 , that

$$\|\phi_1^{(n)} - \phi_1^{(k)}\|_{C^N(\mathcal{D}_{\Gamma_R} \cap \{-n \leq u \leq -k+1\})} \leq \frac{C_1}{k}. \quad (3.5.63)$$

In the third region¹⁵, $-k+1 \leq u$, we apply the energy estimate (3.5.34) to the difference $\Delta\phi_1$:

$$\begin{aligned} & \int_{-n}^u r^2 (\partial_u T^j \Delta\phi_1)^2 + D |T^j \Delta\phi_1|^2 du' \\ & \leq \int_{\Gamma_R \cap \{-n \leq u' \leq -k+1\}} 2r^2 T^{j+1} \Delta\phi_1 \cdot (2\partial_v - T) T^j \Delta\phi_1 du'. \end{aligned}$$

Here, we used that the boundary data for $\Delta\phi_1$ are compactly supported in $u \leq -k+1$. We can now estimate the integral over Γ_R by plugging in the boundary data assumptions for the $T^j \Delta\phi_1$ -terms and by plugging in the previously obtained estimate (3.5.63) for the terms $\partial_v T^j \Delta\phi_1$. We thus find that

$$|T^j \Delta\phi(u, v)| \leq \left(\int_{-n}^u \frac{1}{r^2} du' \right)^{\frac{1}{2}} \left(\int_{-n}^u r^2 (\partial_u T^j \Delta\phi_1(u, v))^2 du' \right)^{\frac{1}{2}} \leq \frac{1}{\sqrt{r}} \frac{C}{k^{j+1}}$$

for some constant $C > 0$. From these estimates on $T^j \Delta\phi_1$, we can obtain estimates on $\partial_v T^j \Delta\phi_1$ by simply integrating (3.3.11) from $u = -k+1$ (where $\partial_v T^j \Delta\phi_1 \lesssim k^{-j-1}$). This shows that there exists a constant C_2 such that

$$\|\phi_1^{(n)} - \phi_1^{(k)}\|_{C^N(\mathcal{D}_{\Gamma_R} \cap \{u \geq -k+1\})} \leq \frac{C_2}{k}. \quad (3.5.64)$$

Combining (3.5.63) and (3.5.64) shows that (3.5.62) holds for all $n > k > K$ provided that $K > \frac{C_1+C_2}{2\delta}$.

We have thus established the uniform convergence of the sequence $\{\phi_1^{(k)}\}$. In view of the uniformity of the convergence, the bounds from Propositions 3.5.1 and 3.5.5 carry over to the limiting solution, thus proving the estimates (3.5.59)–(3.5.61). Moreover, the methods of the

¹⁵In fact, the approach for the third region can be used in all of $\mathcal{D}$.

proof show that this is the unique solution that has vanishing energy flux on $\mathcal{I}^-$ and satisfies the assumptions of §3.5.2.1. This concludes the proof. $\square$

3.5.5.2 The limit $\lim_{u \rightarrow -\infty, v=\text{constant}} r^2 \partial_u (r^2 \partial_u (r\phi))$

Finally, we establish that the limiting solution constructed above satisfies (3.5.8). For this, we will also need to assume the lower bound (3.5.3) on data.

Proposition 3.5.8. *Consider the solution of Proposition 3.5.7, and assume in addition that $N \geq 4$ and $N' \geq 2$, as well as the lower bound on data (3.5.3). Then the following limit exists, is independent of v , and is non-vanishing so long as C_{in}^Γ is non-vanishing and $R/2M$ is sufficiently large:*

$$\lim_{u \rightarrow -\infty} r^2 \partial_u (r^2 \partial_u (r\phi_1(u, v))) = \tilde{C} \neq 0. \quad (3.5.65)$$

Moreover, we have that $\lim_{u \rightarrow -\infty} r^2 \partial_u (r\phi_1)(u, v) = 0$ and

$$r^2 \partial_u (r^2 \partial_u (r\phi_1(u, v))) - \lim_{u \rightarrow -\infty} r^2 \partial_u (r^2 \partial_u (r\phi_1(u, v))) = \mathcal{O}(\max(|u|^{-\epsilon}, r^{-1})). \quad (3.5.66)$$

Proof. We first establish the existence of the limit

$$\lim_{u \rightarrow -\infty} |u|^2 r^2 \partial_v T(r\phi_1)(u, v) =: \mathcal{L}(v) \quad (3.5.67)$$

by computing

$$\partial_u (|u|^2 r^2 \partial_v T(r\phi_1)) = \underbrace{-2|u| r^2 \partial_v T(r\phi_1) + |u|^2 r^2 \partial_v T^2(r\phi_1)}_{=\mathcal{O}(|u|^{-1-\epsilon})} - \underbrace{|u|^2 \partial_v (r^2 \partial_v T(r\phi_1))}_{=\mathcal{O}(r^{-2})}, \quad (3.5.68)$$

where we bounded the first two terms using (3.5.61), and the third term by plugging in the bounds (3.5.59), (3.5.60) into the approximate conservation law (3.3.13) and integrating in u (see also (3.5.27)). In fact, the bound on the last term also shows that $\mathcal{L}(v)$ is independent of v , $\mathcal{L}(v) \equiv \mathcal{L}$. We have thus shown that

$$|u|^2 r^2 \partial_v T(r\phi_1) - \lim_{u \rightarrow -\infty} |u|^2 r^2 \partial_v T(r\phi_1) = \mathcal{O}(\max(|u|^{-\epsilon}, r^{-1})). \quad (3.5.69)$$

We now show that $\mathcal{L}$ is non-vanishing by using the lower bound (3.5.3): We have from (the T -commuted) equation (3.5.17) that

$$r^{-2} \partial_v T(r^2 \phi_1) = \int \frac{DT^2(r\phi_1)}{r^2} - \frac{8MD}{r^4} T(r\phi_1) du \leq \frac{MC}{r^4 |u|^2} + \mathcal{O}(|u|^{-3} r^{-2}).$$

Evaluating the above on Γ_R gives

$$\left| R^2 \partial_v T(r\phi_1) + DR^2 T\phi_1|_{\Gamma_R} \right| \leq \frac{MC}{R|u|^2} + \mathcal{O}(|u|^{-3} R^{-2}),$$

which, if R is chosen large enough (recall that the constant C in the estimates above can be chosen independently of R if R is large enough), can be chosen to be less than $\frac{C_{\text{in}}^\Gamma}{4|u|^2}$. This results in the following lower bound on Γ_R :

$$\left| r^2 \partial_v T(r\phi_1) \right|_{\Gamma_R} \geq \frac{C_{\text{in}}^\Gamma}{4|u|^2}. \quad (3.5.70)$$

Using once more the estimate on $|u|^2 \partial_v(r^2 \partial_v T(r\phi_1)) \sim r^{-2}$ and integrating it from Γ_R shows that, if R is chosen suitably large, we in fact have

$$\left| r^2 \partial_v T(r\phi_1) \right|_{\Gamma_R} \geq \frac{C_{\text{in}}^\Gamma}{8|u|^2}. \quad (3.5.71)$$

In summary, we have thus established that $\lim_{u \rightarrow -\infty} |u|^2 r^2 \partial_v T(r\phi_1)(u, v) = \mathcal{L}(v) \equiv \mathcal{L} \neq 0$.

Finally, in order to relate $\mathcal{L}$ to the limit of $r^2 \partial_u(r^2 \partial_u(r\phi_1))$ in question, we write

$$\begin{aligned} r^2 \partial_u(r^2 \partial_u(r\phi_1)) &= r^4 T^2(r\phi_1) - r^2 \partial_v(r^2 T(r\phi_1)) - r^2 \partial_u(r^2 \partial_v(r\phi_1)) \\ &= r^4 T^2(r\phi_1) - r^4 \partial_v T(r\phi_1) - 2Dr^3 T(r\phi_1) + 2Dr^2 \partial_v(r^2 \phi_1) + 6MDr^2 \phi_1, \end{aligned} \quad (3.5.72)$$

where we used eq. (3.5.19) in the last line. Notice that the last term decays like $|u|^{-1}$, whereas the other terms do not decay. We now express the limits of each of the other terms in terms of $\mathcal{L}$.

Computing $\lim_{u \rightarrow -\infty} |u|^{j+1} r T^j(r\phi_1)$. Observe that (3.5.61) implies that

$$\begin{aligned} \lim_{u \rightarrow -\infty} |u|^2 r^2 \partial_v T(r\phi_1) &= \frac{1}{2} \lim_{u \rightarrow -\infty} |u|^3 r^2 \partial_v T^2(r\phi_1) \\ &= \frac{1}{6} \lim_{u \rightarrow -\infty} |u|^4 r^2 \partial_v T^3(r\phi_1) = \dots = \frac{1}{(j+1)!} \lim_{u \rightarrow -\infty} |u|^{2+j} r^2 \partial_v T^{j+1}(r\phi_1) \end{aligned} \quad (3.5.73)$$

for all $j \leq N'$. Now, use the wave equation (3.3.11) to write

$$\begin{aligned} -2DT^j(r\phi_1) \left(1 + \frac{M}{r} \right) &= r^2 \partial_v T^{j+1}(r\phi_1) - r^2 \partial_v^2 T^j(r\phi_1) \\ &= r^2 \partial_v T^{j+1}(r\phi_1) + 2Dr \partial_v T^j(r\phi_1) - \partial_v(r^2 \partial_v T^j(r\phi_1)). \end{aligned}$$

The last term decays faster than the others, and we conclude that

$$-2 \lim_{u \rightarrow -\infty} |u|^{j+1} r T^j(r\phi_1) = \lim_{u \rightarrow -\infty} |u|^{j+2} r^2 \partial_v T^{j+1}(r\phi_1) + 2 \lim_{u \rightarrow -\infty} |u|^{j+1} r^2 \partial_v T^j(r\phi_1). \quad (3.5.74)$$

Plugging (3.5.73) into the expression above, and setting $j = 1, 2$, respectively, we thus obtain

$$\lim_{u \rightarrow -\infty} |u|^2 r T(r\phi_1) = -\frac{1}{2} (2\mathcal{L} + 2\mathcal{L}) = -2\mathcal{L}, \quad (3.5.75)$$

$$\lim_{u \rightarrow -\infty} |u|^3 r T^2(r\phi_1) = -\frac{1}{2}(6\mathcal{L} + 2 \cdot 2\mathcal{L}) = -5\mathcal{L}. \quad (3.5.76)$$

Computing $\lim_{u \rightarrow -\infty} r^2 \partial_v(r^2 \phi_1)$. In order to compute the limit of $r^2 \partial_v(r^2 \phi_1)$, we use equation (3.5.17) to write

$$\begin{aligned} r^2 \partial_v(r^2 \phi_1) &= r^4 \int \frac{DT(r\phi_1)}{r^2} - \frac{8MD(r\phi_1)}{r^4} du \\ &= r^4 \int \frac{\lim_{u \rightarrow -\infty} |u|^2 r T(r\phi_1)}{r^3 |u|^2} du + \mathcal{O}(|u|^{-1} + r|u|^{-1-\epsilon}), \end{aligned}$$

from which we conclude, using (3.2.6), that

$$\lim_{u \rightarrow -\infty} r^2 \partial_v(r^2 \phi_1) = \frac{1}{4} \lim_{u \rightarrow -\infty} |u|^2 r T(r\phi_1) = -\frac{1}{2}\mathcal{L}. \quad (3.5.77)$$

Finally, inserting the identities (3.5.75), (3.5.76) and (3.5.77) back into (3.5.72), we find that

$$\lim_{u \rightarrow -\infty} r^2 \partial_u(r^2 \partial_u(r\phi_1)) = -5\mathcal{L} - \mathcal{L} + 2 \cdot 2\mathcal{L} - \frac{2}{2}\mathcal{L} = -3\mathcal{L}. \quad (3.5.78)$$

This proves equation (3.5.65). Estimate (3.5.66) as well as the vanishing of $\lim_{u \rightarrow -\infty} r^2 \partial_u(r\phi)$ follow similarly. $\square$

Combining the previous two propositions, Propositions 3.5.7 and 3.5.8, proves Theorem 3.5.1.

3.5.6 A comment on the stationary solution

We have already remarked in §3.4.4 that we expect the vanishing of $\lim_{u \rightarrow -\infty} r^2 \partial_u(r\phi_1)$ to lead to late-time asymptotics with logarithmic terms appearing at leading order if the data on Γ_R are smoothly extended to $\mathcal{H}^+$ (for instance, we would have $r\phi_1|_{\mathcal{I}^+}(u) \sim \frac{\log u}{u^3}$ as $u \rightarrow \infty$). In other words, we expect our choice of polynomially decaying boundary data to lead to a logarithmically modified Price's law for $\ell = 1$.

Note that the limit $\lim_{u \rightarrow -\infty} r^2 \partial_u(r\phi_1)$ would not vanish if we included the stationary solution, that is to say: if we added a constant to our initial data. Using the structure of equations¹⁶ (3.5.17), (3.5.19) presented in the proof of Proposition 3.5.2, or other methods, it is indeed not difficult to see that the stationary solution behaves like $\partial_v(r\phi_1) \sim -\partial_u(r\phi_1) \sim r^{-2}$.

On the other hand, we see that if we prescribe *decaying* data on Γ_R , then the solution will behave, roughly speaking, like the stationary solution multiplied by that decay. Now, since

¹⁶For the sake of completeness, we mention here that these equations are special cases of

$$\partial_u(r^{-2L} \partial_v(r^L \cdot r\phi_L)) = \frac{LD}{r^{L+1}} T(r\phi_L) - \frac{2MD(L+1)^2}{r^{L+3}} r\phi_L, \quad (3.5.79)$$

$$\partial_u(r^{L+1} \partial_v(r\phi_L)) = -(L+1)D\partial_v(r^L \cdot r\phi_L) - (1+L(L+1))2MDr^{L-2}r\phi_L. \quad (3.5.80)$$

the stationary solution for higher ℓ -modes will decay faster in ℓ , $r\phi_\ell \sim r^{-\ell}$ (see eqns. (3.5.79), (3.5.80)), we thus expect that if we prescribe decaying boundary data for higher ℓ -modes, then the corresponding solution will decay increasingly faster towards $\mathcal{I}^-$, and $(r^2\partial_u)^j(r\phi_\ell)$ will vanish to higher and higher orders. We will build on this intuition and make it precise in §3.8.

3.6 Boundary data on a timelike hypersurface Γ_f

In the previous section, we showed how to construct solutions and prove sharp decay in the case of polynomially decaying boundary data on hypersurfaces of constant $r = R$. We now outline how to generalise to spherically symmetric hypersurfaces on which r is allowed to vary. In fact, not much modification will be needed.

3.6.1 The setup

For the sake of notational simplicity, we restrict our attention to spherically symmetric hypersurfaces $\Gamma_f \subset \mathcal{M}$ that have timelike generators that are given by¹⁷

$$T^{(s)} = \partial_u + \frac{1}{1 + |u|^{-s}} \partial_v = T - \frac{|u|^{-s}}{1 + |u|^{-s}} \partial_v, \quad s > 0. \quad (3.6.1)$$

Notice that we normalised $T^{(s)}$ such that $T^{(s)}u = 1$.

Since the cases $s > 1$ and $s \leq 1$ are quite different, we shall treat them separately. Let's first consider the case $s > 1$:

3.6.2 The case where $r|_{\Gamma_f}$ attains a finite limit ($s > 1$):

3.6.2.1 Initial/boundary data assumptions and the main theorem

Let Γ_f be as described above, and let $s > 1$. We then prescribe smooth boundary data $\hat{\phi}_1$ for ϕ_1 on Γ_f which satisfy, for $u \leq U_0 < 0$ and $|U_0|$ sufficiently large:

$$\left| (T^{(s)})^n (r\hat{\phi}_1) \right| \leq \frac{n! C_{\text{in}}^\Gamma}{r|u|^{n+1}}, \quad n = 0, 1, \dots, N+1, \quad (3.6.2)$$

$$\left| (T^{(s)})^n (r\hat{\phi}_1 - |u|T^{(s)}(r\hat{\phi}_1)) \right| \leq \frac{C_{\text{in},\epsilon}^\Gamma}{r|u|^{n+1+\epsilon}}, \quad n = 0 \dots, N'+1 \quad (3.6.3)$$

for some positive constants C_{in}^Γ , $C_{\text{in},\epsilon}^\Gamma$, $\epsilon \in (0, 1)$ and for $N, N' > 1$ positive integers. Moreover, we demand, in a limiting sense, that, for all v

$$\lim_{u \rightarrow -\infty} \partial_v^n (r\phi_1)(u, v) = 0, \quad n = 1, \dots, N+1. \quad (3.6.4)$$

Then we obtain, as in the previous section:

Theorem 3.6.1. *Let $N \geq 4$ and $N' \geq 2$. Then there exists a unique solution ϕ_1 to eq. (3.3.11) in $\mathcal{D}_{\Gamma_f} = \mathcal{M} \cap \{v \geq v_{\Gamma_f}(u)\}$ which restricts correctly to $\hat{\phi}_1$ on Γ_f , $\phi_1|_{\Gamma_f} = \hat{\phi}_1$, and which satisfies (3.6.4).*

¹⁷The proofs presented below also directly apply to slightly more general spherically symmetric timelike generators, e.g. $T^{(s)} \sim T - f_s \partial_v$, with $f_s \sim |u|^{-s}$, or $T^{(s)} \sim T - f_s \log |u| \partial_v$ etc.

Moreover, the estimates from Theorem 3.5.1 apply to this solution, with $\tilde{C} \neq 0$ being non-zero provided that a lower bound on data is specified and that $R/2M$ is sufficiently large.

3.6.2.2 Outline of the proof

As the proof only requires small modifications to the proof of Theorem 3.5.1, we will only give an outline. There are two closely related ways one can go about this: One can either work with the generators of Γ_f , i.e. replace all T 's from the proof of Theorem 3.5.1 with $T^{(s)}$'s, and estimate the resulting error terms (which would always exhibit faster decay than the other terms) – this was the approach of chapter 1. Alternatively, one can continue working with T and exploit the fact that the difference of, say, $T(r\phi_1) - T^{(s)}(r\phi_1) = \frac{|u|^{-s}}{1+|u|^{-s}}\partial_v(r\phi_1)$ decays faster than either of the terms on the left-hand side as long as $s > 1$. Thus, an estimate on $T(r\phi_1)$ immediately gives control on $T^{(s)}(r\phi_1)$ and vice versa.¹⁸ We shall follow the second approach:

Proof of Theorem 3.6.1. First, we cut off the data as in section 3.5.2.2. We then introduce the set of bootstrap assumptions as in section 3.5.3.1 (with the only modification that the set X defined below eq. (BS'(m)) now contains all $v \geq v_{\Gamma_f}(u)$). The proof of Proposition 3.5.2 remains unchanged. The proof of Proposition 3.5.3 requires the modification that, now, it isn't $T^j(r\phi_1)$ which on Γ_f is given by data, but $(T^{(s)})^j(r\phi_1)$. However, this can easily be dealt with by writing

$$T(r\phi_1) = T^{(s)}(r\phi_1) + \frac{|u|^{-s}}{1+|u|^{-s}}\partial_v(r\phi_1) \quad (3.6.5)$$

(and similarly for T^j), and then plugging in the bootstrap assumption for $\partial_v(r\phi_1)$, using the fact that, because $s > 1$, the $\partial_v(r\phi_1)$ -term has more u -decay than the $T^{(s)}(r\phi_1)$ -term. It can thus be absorbed into the latter for large enough $|U_0|$.

Let's now move to the proof of Proposition 3.5.4. Applying the divergence theorem gives (we denote the induced volume element on Γ_f by $r^2 dt_{\Gamma_f} d\Omega$)

$$\begin{aligned} & \int_{\mathcal{C}_v \cap \{-k \leq u' \leq u\}} r^2 du' d\Omega J^T[T^j \phi_{\ell=1}] \cdot \partial_u \\ & \leq \int_{\Gamma_f \cap \{-k \leq u' \leq u\}} r^2 dt_{\Gamma_f} d\Omega J^T[T^j \phi_{\ell=1}] \cdot \left(\partial_u - \partial_v + \frac{|u'|^{-s}}{1+|u'|^{-s}}\partial_v \right), \end{aligned} \quad (3.6.6)$$

¹⁸Note that this is no longer true if $s \leq 1$.

which implies (cf. (3.5.34))

$$\begin{aligned} & \int_{-k}^u r^2 (\partial_u T^j \phi_1)^2 + D |T^j \phi_1|^2 du' \\ & \lesssim \int_{\Gamma_f \cap \{-k \leq u' \leq u\}} r^2 \left(T^{j+1} \phi_1 \cdot (T - 2\partial_v) T^j \phi_1 + \frac{|u'|^{-s}}{1 + |u'|^{-s}} \left((T^j \partial_v \phi_1)^2 + \frac{2D}{r^2} (T^j \phi_1)^2 \right) \right) du'. \end{aligned} \quad (3.6.7)$$

As before, we can now write $T^j(r\phi_1)|_{\Gamma_f} = (T^{(s)})^j(r\phi_1)|_{\Gamma_f} + \mathcal{O}(|u|^{-j-s})$ to find that

$$\int_{-k}^u r^2 (\partial_u T^j \phi_1)^2 + D |T^j \phi_1|^2 du' \lesssim \int_{\Gamma_f \cap \{-k \leq u' \leq u\}} -2r^2 \left((T^{(s)})^{j+1} \phi_1 \cdot 2\partial_v T \phi_1 \right) + \mathcal{O}(|u'|^{-2j-3-s}) du'.$$

From here, we arrive at the analogue of the estimate (3.5.38). We can thus prove the analogue of Proposition 3.5.4.

In a similar fashion, one can then follow the steps of sections 3.5.4 and 3.5.5 to conclude the proof of Theorem 3.6.1. $\square$

3.6.3 The case where $r|_{\Gamma_f}$ diverges ($s \leq 1$):

There are two main differences in the case $s \leq 1$. On the one hand, if we write, as above,

$$T(r\phi_1) = \underbrace{T^{(s)}(r\phi_1)}_{\sim |u|^{-2}r^{-1}} + \frac{|u|^{-s}}{1 + |u|^{-s}} \underbrace{\partial_v(r\phi_1)}_{\sim |u|^{-1}r^{-2}}, \quad (3.6.8)$$

then we immediately see that, on Γ_f , where $r \sim |u|^{1-s}$ if $s \neq 1$, both terms on the RHS can be expected to have the same decay.¹⁹ This means that we have to be more careful in estimating the boundary terms in the energy estimate. On the other hand, since now $r|_{\Gamma}$ tends to infinity, it will be much more straight-forward to show the existence of the limit $\lim_{u \rightarrow -\infty} |u|^2 r^2 \partial_v T(r\phi_1)$.

3.6.3.1 Initial/boundary data assumptions and the main theorem

Let Γ_f be as described in section 3.6.1, and let $s \leq 1$. We prescribe smooth boundary data $\hat{\phi}_1$ for ϕ_1 on Γ_f which satisfy, for $u \leq U_0 < 0$ and $|U_0|$ sufficiently large:

$$\left| (T^{(s)})^n (r^2 \hat{\phi}_1) - \frac{n! C_{\text{in}}^\Gamma}{|u|^{n+1}} \right| = \mathcal{O}(|u|^{-n-1-\epsilon}), \quad n = 0, \dots, 5 \quad (3.6.9)$$

for some positive constant C_{in}^Γ . Moreover, we demand, in a limiting sense, that, for all v

$$\lim_{u \rightarrow -\infty} \partial_v^n (r\phi_1)(u, v) = 0 \quad n = 1, \dots, 5. \quad (3.6.10)$$

¹⁹Notice that if $s = 1$, the second term on the RHS decays faster by $\log^{-2} |u|$.

Then we obtain, as in the previous section:

Theorem 3.6.2. *There exists a unique solution ϕ_1 to eq. (3.3.11) in $\mathcal{D}_{\Gamma_f} = \mathcal{M} \cap \{v \geq v_{\Gamma_f}(u)\}$ which restricts correctly to $\hat{\phi}_1$ on Γ_f , $\phi_1|_{\Gamma_f} = \hat{\phi}_1$, and which satisfies (3.6.10).*

Moreover, if U_0 is a sufficiently large negative number, then there exists a constant $C = C(C_{\text{in}}^\Gamma)$ (depending only on data) such that ϕ_1 obeys the following bounds throughout $\mathcal{D}_{\Gamma_f} \cap \{u \leq U_0\}$:

$$|r^2 \partial_v T^n(r\phi_1)(u, v)| \leq \frac{C}{|u|^{n+1}}, \quad n = 0, \dots, 4, \quad (3.6.11)$$

$$|T^n(r\phi_1)(u, v)| \leq \frac{C}{|u|^{n+1}} \max(r^{-1}, |u|^{-1}), \quad n = 0, \dots, 3. \quad (3.6.12)$$

Finally, along any ingoing null hypersurface $\mathcal{C}_v$, we have

$$r^2 \partial_u(r\phi_1)(u, v) = \mathcal{O}(|u|^{-1}), \quad (3.6.13)$$

$$r^2 \partial_u(r^2 \partial_u(r\phi_1))(u, v) = \begin{cases} \tilde{C} + \mathcal{O}(|u|^{-\epsilon'}), & \text{if } s < 1, \\ \tilde{C} + \mathcal{O}(\log^{-1} |u|), & \text{if } s = 1, \end{cases} \quad (3.6.14)$$

where $\tilde{C} = 3C_{\text{in}}^\Gamma$ is determined explicitly by initial data, and $\epsilon' = \min(\epsilon, s, 1 - s)$.

Remark 3.6.1. We remark that, in the case $s = 1$, the fact that the $\mathcal{O}(\log^{-1} |u|)$ -term in (3.6.14) is non-integrable means that Theorem 3.4.1 cannot be applied directly. Since this is a very specific issue, we make no attempts to fix it in this presentation.

3.6.3.2 Outline of the proof

Proof. As before, only a sketch of the proof will be provided.

We cut the data off as before. Let us first show (3.6.11) for $n = 0$:

Proof of (3.6.11) for $n = 0$: We follow the proof of Proposition 3.5.4. We first need to acquire an estimate for $\partial_v(r\phi_1)$ on Γ_f . We assume as a bootstrap assumption that

$$|r^2 \partial_v(r\phi_1)| \leq \frac{C_{\text{BS}}}{|u|} \quad (3.6.15)$$

for a suitable constant C_{BS} . We recall from the energy estimate (3.6.7):

$$\begin{aligned} & \int_{-k}^u r^2 (\partial_u \phi_1)^2 + D |\phi_1|^2 du' \\ & \lesssim \int_{\Gamma_f \cap \{-k \leq u' \leq u\}} r^2 \left(T \phi_1 \cdot (T - 2\partial_v) \phi_1 + \frac{|u'|^{-s}}{1 + |u'|^{-s}} \left((\partial_v \phi_1)^2 + \frac{2D}{r^2} \phi_1^2 \right) \right) du' \end{aligned} \quad (3.6.16)$$

Note that the $(\partial_v \phi_1)^2$ -terms in the above are potentially dangerous since they could lead to a C_{BS}^2 -term, which would make it impossible to improve the bootstrap assumption. However, upon writing again

$$T\phi_1 = T^{(s)}\phi_1 + \frac{|u|^{-s}}{1 + |u|^{-s}}\partial_v \phi_1, \quad (3.6.17)$$

we find that the $(\partial_v \phi_1)^2$ -terms, in fact, appear with a benign sign:

$$\begin{aligned} & \int_{\Gamma_f} r^2 \left(T\phi_1 \cdot (T - 2\partial_v)\phi_1 + \frac{|u'|^{-s}}{1 + |u'|^{-s}} \left((\partial_v \phi_1)^2 + \frac{2D}{r^2} \phi_1^2 \right) \right) du' \\ &= \int_{\Gamma_f} r^2 \left((T\phi_1)^2 - \frac{|u'|^{-s}}{1 + |u'|^{-s}} (\partial_v \phi_1)^2 - 2\partial_v \phi_1 T^{(s)}\phi_1 + \frac{|u'|^{-s}}{1 + |u'|^{-s}} \frac{2D}{r^2} \phi_1^2 \right) du' \\ &\leq \int_{\Gamma_f} r^2 \left((T^{(s)}\phi_1)^2 + 2|\partial_v \phi_1| |T^{(s)}\phi_1| \frac{1 + 2|u'|^{-s}}{1 + |u'|^{-s}} + \frac{|u'|^{-s}}{1 + |u'|^{-s}} \frac{2D}{r^2} \phi_1^2 \right) + r^2 \frac{|u'|^{-2s}}{(1 + |u'|^{-s})^2} (\partial_v \phi_1)^2 du' \\ &\lesssim \int_{\Gamma_f} r^2 \left(\frac{(C_{\text{in}}^\Gamma)^2}{r^4 |u'|^4} + \frac{(C_{\text{in}}^\Gamma)^2}{r^6 |u'|^{2+2s}} + \frac{C_{\text{in}}^\Gamma (C_{\text{in}}^\Gamma + C_{\text{BS}})}{r^5 |u'|^2} \left(\frac{1}{|u'|} + \frac{1}{r|u'|^s} \right) + \frac{(C_{\text{in}}^\Gamma)^2}{r^6 |u'|^{2+s}} \right) + r^2 \frac{(C_{\text{in}}^\Gamma + C_{\text{BS}})^2}{r^6 |u'|^{2+2s}} du', \end{aligned}$$

where we used the boundary data assumption and the bootstrap assumption in the last estimate. Using now the fact that $|u|^{1-s} \lesssim r$ if $s \neq 1$ (or $\log |u| \lesssim r$ if $s = 1$), as well as the fundamental theorem of calculus and the Cauchy–Schwarz inequality, combined with the energy estimate (3.6.16), we obtain that

$$r\phi_1^2 \lesssim \frac{C_{\text{in}}^\Gamma (C_{\text{in}}^\Gamma + C_{\text{BS}})}{r^3 |\Gamma_f| |u|^2} \quad (3.6.18)$$

Importantly, C_{BS} does not appear quadratically in the above estimate. Plugging this bound into the wave equation (3.3.11) and integrating in u , we obtain that

$$\left| r^2 \partial_v (r\phi_1) \right|_{\Gamma_f} \lesssim \frac{\sqrt{C_{\text{BS}}^2 + C_{\text{BS}} C_{\text{in}}^\Gamma}}{|u|^2}. \quad (3.6.19)$$

Having obtained a bound for the boundary term, we can now, as in the proof of Proposition 3.5.4, use the approximate conservation law (3.3.13) to close the bootstrap argument for $\partial_v (r\phi_1)$. Indeed, we can obtain a bound for Φ (similarly to how we obtained (3.5.38)) and then use the fact that, by integrating the bootstrap assumption from Γ_f , we have

$$|r\phi_1| \leq \frac{C_{\text{in}}^\Gamma + C_{\text{BS}}}{r |\Gamma_f| |u|}. \quad (3.6.20)$$

In view of $\log |u| \lesssim r |\Gamma_f|$, this decays faster than $r^2 \partial_v (r\phi_1)$. Therefore, the bound for Φ immediately translates into a bound for $r^2 \partial_v (r\phi_1)$. This closes the bootstrap argument.

Proof of (3.6.11) for $n > 0$: Having proved (3.6.11) for $n = 0$, we now outline the proof for $n > 0$. In fact, the only thing that changes is that, in the energy equality (3.6.16), we now need to express $T^j \phi_1$ in terms of $(T^{(s)})^j \phi_1$ for $j > 1$, which leads to more “error” terms. For instance, we have

$$T^2 \phi_1 = (T^{(s)})^2 \phi_1 + 2 \frac{|u|^{-s}}{1 + |u|^{-s}} \partial_v T \phi_1 + T \left(\frac{|u|^{-s}}{1 + |u|^{-s}} \right) \partial_v \phi_1 - \frac{|u|^{-2s}}{(1 + |u|^{-s})^2} \partial_v^2 \phi_1. \quad (3.6.21)$$

We have already obtained estimates for the last two terms. Moreover, we can estimate the first term above from the boundary data assumptions, and the second one via a bootstrap assumption on $\partial_v T(r\phi_1)$. Plugging these estimates into (3.6.7) for $j = 2$ then improves the bootstrap assumption.

We leave the cases $j > 2$ to the reader. (Notice that when e.g. expressing $T^4 \phi_1$ in terms of $(T^{(s)})^4 \phi_1$, there will also be, for instance, a term containing $\partial_v^4 \phi_1$. We can estimate this by simply commuting the wave equation twice with ∂_v and appealing to the proof of (3.6.11) for $n = 0$. The other terms can be dealt with similarly.)

Proof of (3.6.12): We can obtain the estimates (3.6.12) for $n \leq 3$ by using the wave equation as in (3.5.29) and the already obtained bounds (3.6.11).

Proof of (3.6.13) The proof of (3.6.13) is straight-forward. We simply write:

$$r^2 \partial_u(r\phi_1) = r^2 T(r\phi_1) - r^2 \partial_v(r\phi_1). \quad (3.6.22)$$

Proof of (3.6.14): Finally, we prove (3.6.14). As in the proof of Proposition 3.5.8, we will first compute the limit $\lim_{u \rightarrow -\infty} |u|^2 r^2 \partial_v T(r\phi_1)$.

In view of the approximate conservation law (3.3.13) and the fact that $r|_{\Gamma_f}$ tends to infinity, we have that

$$\lim_{u \rightarrow -\infty} |u|^2 r^2 \partial_v T(r\phi_1)(u, v) = \lim_{u \rightarrow -\infty} |u|^2 r^2 \partial_v T(r\phi_1)(u, v_{\Gamma_f}(u)). \quad (3.6.23)$$

We estimate $\partial_v T(r\phi_1)|_{\Gamma_f}$ as follows. Integrating the T -commuted (3.5.17), we obtain that

$$r^2 \partial_v T(r\phi_1) + r^2 T \phi_1 = \mathcal{O} \left(\frac{r}{|u|^3} + \frac{1}{r|u|^2} \right), \quad (3.6.24)$$

from which we read off that

$$|u|^2 r^2 \partial_v T(r\phi_1)(u, v) = -C_{\text{in}}^\Gamma + \begin{cases} \mathcal{O} \left(\frac{1}{\log |u|} \right), & s = 1, \\ \mathcal{O} \left(\frac{1}{|u|^s} + \frac{1}{|u|^{1-s}} + \frac{1}{|u|^\epsilon} \right), & s < 1, \end{cases} \quad (3.6.25)$$

and, in particular, that

$$\mathcal{L} := \lim_{u \rightarrow -\infty} |u|^2 r^2 \partial_v T(r\phi_1)(u, v) = -C_{\text{in}}^\Gamma. \quad (3.6.26)$$

Here, we used that

$$r^2 T\phi_1 = T(r^2\phi_1) = T^{(s)}(r^2\phi_1) + \frac{|u|^{-s}}{1 + |u|^{-s}} \partial_v(r^2\phi_1)$$

and the fact that, in view of (3.5.17), the second term above decays faster than the first.

Similarly, we find that

$$\lim_{u \rightarrow -\infty} |u|^{j+1} r^2 \partial_v T^j(r\phi_1)(u, v) = -j! C_{\text{in}}^\Gamma \quad (3.6.27)$$

for $j \leq 3$. We can now compute, exactly as in the proof of Proposition 3.5.8, the expressions (3.5.75), (3.5.76) and (3.5.77), from which it follows, using the identity (3.5.72), that

$$\lim_{u \rightarrow -\infty} r^2 \partial_u (r^2 \partial_u (r\phi_1))(u, v) = -3\mathcal{L} = 3C_{\text{in}}^\Gamma. \quad (3.6.28)$$

This concludes the proof. □

Part II:

Generalising to all $\ell \geq 0$.

Having understood the case $\ell = 1$ in detail in the previous sections §3.4–§3.6, we now want to analyse the general case. As explained in §3.1.4, this second part of the chapter can be understood mostly independently of part I.

As a zeroth step, we need to establish higher ℓ -analogues of the approximate conservation laws (3.3.13), (3.3.15). This is achieved in §3.7. We then treat the case of timelike boundary data in §3.8, restricting the presentation however to cases of hypersurfaces of constant area radius. Then, we treat the case of characteristic initial data in §3.9 and §3.10.

The sections §3.8 and §3.9 are similar in spirit to §3.5 and §3.4, respectively. (The reasons for the reversed order of the sections are solely of presentational, not of mathematical nature.) On the other hand, §3.10 follows a different mathematical structure than §3.4: While the methods of §3.4 and §3.9 can only treat data that decay like $r\phi_\ell \lesssim |u|^{-\ell}$, the approach of §3.10 allows us to also treat slowly decaying (and even growing) initial data.

3.7 The higher-order Newman–Penrose constants

In this section, we derive higher-order conservation laws and define the Newman–Penrose constants associated with them.

3.7.1 Generalising the approximate conservation law (3.3.13)

In order to generalise the approximate conservation law in u (3.3.13), we first require a general formula for commutations of the wave equation with $[r^2\partial_v]^N$:

Proposition 3.7.1. *Let ϕ be a smooth solution to $\square_g\phi = 0$, and let $N \geq 0$. Then ϕ satisfies*

$$\begin{aligned} \partial_u\partial_v[r^2\partial_v]^N(r\phi) &= -\frac{2DN}{r}\partial_v[r^2\partial_v]^N(r\phi) \\ &\quad + \sum_{j=0}^N \frac{D}{r^2}(2M)^j \left(a_j^N + b_j^N \Delta_{\mathbb{S}^2} - c_j^N \cdot \frac{2M}{r} \right) [r^2\partial_v]^{N-j}(r\phi), \end{aligned} \quad (3.7.1)$$

where the constants a_j^N, b_j^N and c_j^N are given explicitly by

$$a_j^N = (2^j - 1) \binom{N}{j} + (2^{j+2} - 2) \binom{N+1}{j+2}, \quad (3.7.2)$$

$$b_j^N = \binom{N}{j}, \quad (3.7.3)$$

$$c_j^N = 2^j \binom{N}{j} + 2^{j+2} \binom{N+1}{j+2}, \quad (3.7.4)$$

and we use the convention that $\binom{N}{j} = 0$ if $j > N$.

Proof. A proof is given in the appendix 3.A.1. $\square$

Notice that, in particular, $a_0^N = N(N+1)$ and $b_0^N = 1$. Hence, if we restrict to solutions supported on the $\ell = L$ -angular frequencies, and consider (3.7.1) for $N = L$, there will be a cancellation in the highest-order derivatives. One can then iteratively subtract (3.7.1) for $N < L$, multiplied with a suitable constant, to obtain an approximate conservation law. This is done in

Corollary 3.7.1. *Let $\phi = \sum_{|m| \leq L} \phi_{Lm} \cdot Y_{Lm}$ be a smooth solution to $\square_g \phi = 0$ supported on the angular frequencies $\ell = L \geq 0$. In what follows, we shall suppress the m -index. Let $N \geq 0$, and define, for $j \leq N$,*

$$\tilde{a}_j^{N,L} := a_j^N - b_j^N \cdot L(L+1), \quad (3.7.5)$$

and let $\{x_i^{(L)}\}_{0 \leq i \leq L}$ be a set of constants with $x_0^{(L)} = 1$. Then ϕ satisfies

$$\begin{aligned} \partial_u \partial_v \left([r^2 \partial_v]^L (r \phi_L) + \sum_{i=1}^L (2M)^i x_i^{(L)} [r^2 \partial_v]^{L-i} (r \phi_L) \right) \\ = -\frac{2LD}{r} \partial_v [r^2 \partial_v]^L (r \phi_L) - \sum_{i=1}^L (2M)^i x_i^{(L)} \frac{2(L-i)D}{r} \partial_v [r^2 \partial_v]^{L-i} (r \phi_L) \\ + \sum_{j=0}^L \frac{D}{r^2} (2M)^j [r^2 \partial_v]^{L-j} (r \phi_L) \sum_{i=0}^j \left(x_i^{(L)} \tilde{a}_{j-i}^{L-i,L} - x_i^{(L)} c_{j-i}^{L-i} \cdot \frac{2M}{r} \right). \end{aligned} \quad (3.7.6)$$

Proof. This is a straight-forward computation. $\square$

Definition 3.7.1 (The generalised higher-order future Newman–Penrose constant). *Let ϕ be as in Corollary 3.7.1, and define the constants $x_i^{(L)}$ for $1 \leq i \leq L$ as follows:*

$$x_i^{(L)} := -\frac{1}{\tilde{a}_0^{L-i,L}} \sum_{k=0}^{i-1} \tilde{a}_{i-k}^{L-k,L} x_k^{(L)}. \quad (3.7.7)$$

This is well-defined since $\tilde{a}_0^{L-i,L} \neq 0$ for $i > 0$ and since $x_0^{(L)} = 1$. We further denote

$$\Phi_L := [r^2 \partial_v]^L (r \phi_L) + \sum_{i=1}^L (2M)^i x_i^{(L)} [r^2 \partial_v]^{L-i} \phi_L, \quad (3.7.8)$$

and define, for any smooth function $f(r) = o(r^3)$, the generalised higher-order Newman–Penrose constant according to

$$I_{\ell=L}^{\text{future},f}[\phi](v) := \lim_{v \rightarrow \infty} f \partial_v \Phi_L(u, v). \quad (3.7.9)$$

Corollary 3.7.2. *The quantity Φ_L defined above satisfies the following approximate conservation law:*

$$\partial_u(r^{-2L}\partial_v\Phi_L) = \sum_{j=0}^L \frac{D}{r^{3+2L}}(2M)^{j+1}[r^2\partial_v]^{L-j}(r\phi_L) \left(2(j+1)x_{j+1}^{(L)} - \sum_{i=0}^j x_i^{(L)}c_{j-i}^{L-i} \right). \quad (3.7.10)$$

Here, we used the notation that $x_i^{(L)} = 0$ for $i \geq L$.

In particular, under suitable assumptions on ϕ , the generalised higher-order N–P constant defined above is conserved along $\mathcal{I}^+$:

$$I_{\ell=L}^{\text{future},f}[\phi](v) \equiv I_{\ell=L}^{\text{future},f}[\phi]. \quad (3.7.11)$$

Remark 3.7.1. It is helpful to keep in mind that the quantity $\partial_v\Phi_L$ can always be written as

$$\partial_v\Phi_L = \partial_v(r^2p_1(r^2p_2\partial_v(\dots r^2p_L\partial_v(r\phi_L)\dots))), \quad (3.7.12)$$

where the p_i are polynomials in $1/r$. Intuitively, this indicates that one should typically be able to recover an estimate for $r\phi_L$ from Φ_L by simply integrating L times. However, we will never need to make use of the form (3.7.12) in this chapter.

3.7.2 Generalising the approximate conservation law (3.3.15)

We follow a similar procedure to derive approximate conservation laws in the ∂_v -direction. We have

Proposition 3.7.2. *Let ϕ be a smooth solution to $\square_g\phi = 0$, and let $N \geq 0$. Then ϕ satisfies*

$$\begin{aligned} \partial_v\partial_u[r^2\partial_u]^N(r\phi) &= \frac{2DN}{r}\partial_u[r^2\partial_u]^N(r\phi) \\ &\quad + \sum_{j=0}^N \frac{D}{r^2}(2M)^j \left(\underline{a}_j^N + \underline{b}_j^N \Delta_{\mathbb{S}^2} - \underline{c}_j^N \cdot \frac{2M}{r} \right) [r^2\partial_u]^{N-j}(r\phi), \end{aligned} \quad (3.7.13)$$

where $\underline{a}_j^N = (-1)^j a_j^N$, $\underline{b}_j^N = (-1)^j b_j^N$ and $\underline{c}_j^N = (-1)^j c_j^N$.

Proof. The proof follows along the same steps as the one of Proposition 3.7.1. See the appendix 3.A.1 for details. $\square$

Definition 3.7.2 (The generalised higher-order past Newman–Penrose constant). *Let ϕ be as in Corollary 3.7.1, let $\tilde{a}_j^{N,L} := (-1)^j \tilde{a}_j^{N,L}$, let $x_0^{(L)} = 1$, and define, for $1 \leq i \leq L$,*

$$\underline{x}_i^{(L)} = -\frac{1}{\tilde{a}_0^{L-i,L}} \sum_{k=0}^{i-1} \tilde{a}_{i-k}^{L-k,L} \underline{x}_k^{(L)}. \quad (3.7.14)$$

We then denote

$$\underline{\Phi}_L := [r^2 \partial_u]^L (r \phi_L) + \sum_{i=1}^L (2M)^i \underline{x}_i^{(L)} [r^2 \partial_u]^{L-i} (r \phi_L), \quad (3.7.15)$$

and, moreover, define for any smooth function $f(r) = o(r^3)$ the generalised higher-order Newman–Penrose constant according to

$$I_{\ell=L}^{\text{past},f}[\phi](u) := \lim_{u \rightarrow -\infty} f \partial_u \underline{\Phi}_L(u, v). \quad (3.7.16)$$

Corollary 3.7.3. *The quantity $\underline{\Phi}_L$ defined above satisfies the following approximate conservation law:*

$$\partial_v (r^{-2L} \partial_u \underline{\Phi}_L) = \sum_{j=0}^L \frac{D}{r^{3+2L}} (2M)^{j+1} [r^2 \partial_u]^{L-j} (r \phi_L) \left(-2(j+1) \underline{x}_{j+1}^{(L)} - \sum_{i=0}^j \underline{x}_i^{(L)} \underline{c}_{j-i}^{L-i} \right). \quad (3.7.17)$$

Here, we used the notation that $\underline{x}_i^{(L)} = 0$ for $i \geq L$.

In particular, under suitable assumptions on $\underline{\Phi}$, the generalised higher-order N–P constant defined above is conserved along $\mathcal{I}^-$:

$$I_{\ell=L}^{\text{past},f}[\phi](u) \equiv I_{\ell=L}^{\text{past},f}[\phi]. \quad (3.7.18)$$

3.8 Boundary data on a timelike hypersurface Γ_R

Equipped with the approximate conservation laws (3.7.10), (3.7.17), we now generalise the results of §3.5. More precisely, we construct higher ℓ -mode solutions to (3.1.1) (and derive estimates for them) that arise from polynomially decaying boundary data on a timelike hypersurface Γ_R of constant area radius $r = R$ and the no incoming radiation condition. In particular, the present section contains the proof of Theorem 3.1.3. The generalisation to boundary data on hypersurfaces on which r is allowed to vary then proceeds as in §3.6 and is left to the reader.

Throughout the rest of this section, we shall assume that $R > 2M$ is a constant and that ϕ is a solution to (3.1.1) supported on a single angular frequency (L, m) , with $|m| \leq L$ and $L \geq 0$. In the usual abuse of notation of §3.3.3, we omit the m -index, that is, we write $\phi = \phi_{Lm} Y_{Lm} = \phi_L Y_{Lm}$.

3.8.1 Initial/boundary data assumptions

We prescribe smooth boundary data $\hat{\phi}_L$ on $\Gamma_R = \mathcal{M}_M \cap \{v = v_R(u)\}$ that satisfy, for $u \leq U_0 < 0$ and $|U_0|$ sufficiently large, the following upper bounds:

$$\left| T^n(r \hat{\phi}_L) \right| \leq \frac{n! C_{\text{in}}^\Gamma}{R^L |u|^{n+1}}, \quad n = 0, 1, \dots, N+1, \quad (3.8.1)$$

$$\left| T^n(r\hat{\phi}_L - |u|T(r\hat{\phi}_L)) \right| \leq \frac{C_{\text{in},\epsilon}^\Gamma}{R^L|u|^{n+1+\epsilon}}, \quad n = 0, \dots, N' + 1 \quad (3.8.2)$$

for some positive constants C_{in}^Γ , $C_{\text{in},\epsilon}^\Gamma$, $\epsilon \in (0, 1)$ and $N, N' \geq 0$ integers, and which also satisfy the lower bound

$$\left| T(r\hat{\phi}_L) \right| \geq \frac{C_{\text{in}}^\Gamma}{2R^L|u|^{n+1}} > 0. \quad (3.8.3)$$

Moreover, we demand, in a limiting sense, that, for all v ,

$$\lim_{u \rightarrow -\infty} \partial_v^n(r\phi_L)(u, v) = 0, \quad n = 1, \dots, N + 1. \quad (3.8.4)$$

This latter condition is to be thought of as the no incoming radiation condition.

3.8.2 The main theorem (Theorem 3.8.1)

The main result of this section is

Theorem 3.8.1. *Let $R > 2M$. Then there exists a unique solution $\phi_L \cdot Y_{Lm}$ to eq. (3.1.1) in $\mathcal{D}_{\Gamma_R} = \mathcal{M} \cap \{v \geq v_R(u)\}$ that restricts correctly to $\hat{\phi}_L \cdot Y_{Lm}$ on Γ_R , $\phi_L|_{\Gamma_R} = \hat{\phi}_L$, and that satisfies (3.8.4).*

Moreover, if U_0 is a sufficiently large negative number, then there exists a constant $C = C(2M/R, C_{\text{in}}^\Gamma, L)$, depending only on data, such that ϕ_L obeys the following bounds throughout $\mathcal{D}_{\Gamma_R} \cap \{u \leq U_0\}$:

$$\left| [r^2 \partial_v]^{L-j} T^i(r\phi_L) \right| \leq \frac{C}{|u|^{i+1}} \min(r, |u|)^{-\min(\tilde{j}, N-i)} \quad (3.8.5)$$

for all $j = -1, \dots, L$ and for all $i = 0, \dots, N$, and $\tilde{j} := \max(j, 0)$.

Finally, if $N - 2 \geq N' \geq L + 1$, then we have along any ingoing null hypersurface $\mathcal{C}_v$:

$$[r^2 \partial_u]^{L-j}(r\phi_L)(u, v) = \mathcal{O}(r^{-1-j}), \quad j = 0, \dots, L, \quad (3.8.6)$$

$$[r^2 \partial_u]^{L+1}(r\phi_L)(u, v) = \tilde{C} + \mathcal{O}(r^{-1} + |u|^{-\epsilon}) \quad (3.8.7)$$

for some constant $\tilde{C}$ which can be shown to be non-vanishing if $R/2M$ is sufficiently large.

Remark 3.8.1. A similar result holds true for more general timelike hypersurfaces Γ_f (on which, in particular, r is allowed to tend to infinity) as discussed in §3.6. We leave the proof to the interested reader.

3.8.3 Overview of the proof

We shall first give an overview over the proof of Theorem 3.8.1.

- I In a first step, we construct a sequence of smooth compactly supported data $\hat{\phi}_L^{(k)}$ as in §3.5.2.2, which lead to solutions $\phi_L^{(k)}$ in the sense of Prop. 3.3.2. The purpose of this is that we will then be able to use the method of continuity (i.e. bootstrap arguments) on these finite solutions $\phi_L^{(k)}$.
- II We then assume (in the form of a bootstrap assumption) that the estimate $\left| r^2 \partial_v (r \phi_L^{(k)})|_{\Gamma_R} \right| \leq \frac{C_{\text{BS}}}{R^{L-1}|u|}$ holds on Γ_R . An application of an energy estimate will imply that $\left| r^2 \partial_v (r \phi_L^{(k)}) \right| \leq C'(\text{data}) \cdot \frac{\sqrt{C_{\text{BS}}}}{R^L|u|}$ and, thus, improve this assumption. From this, we then inductively integrate equation (3.7.1) to obtain estimates for the boundary terms $\left| [r^2 \partial_v]^{L-j} (r \phi_L^{(k)})|_{\Gamma_R} \right|$, $j = 0, \dots, L$. The same estimates hold upon commuting with T^i .
- III In a third step, we assume decay on $[r^2 \partial_v]^L (r \phi_L^{(k)})$ and integrate the approximate conservation law (3.7.10) in u and in v (the integration in v from Γ_R outwards is why we need the estimates on the boundary terms from step II) to improve this decay, exploiting $2M/R$ as a small parameter. (We recall from §3.5.3.5 that any smallness assumptions on $2M/R$ can be recovered by replacing the bootstrap argument with a Grönwall argument.) Integrating this estimate for $[r^2 \partial_v]^L (r \phi_L^{(k)})$ then j times from Γ_R and also commuting with T establishes the following estimates:

$$\left| [r^2 \partial_v]^{L-j} T^i (r \phi_L^{(k)}) \right| \leq \frac{C}{|u|^{i+1}} R^{-\max(j,0)} \quad (3.8.8)$$

for $i = 0, \dots, N$, $j = -1, 0, \dots, L$, and for $C \neq C(k)$ a constant.

- IV In a fourth step, we adapt the methods of steps II and III as in §3.5.4 to obtain estimates on the boundary terms $[r^2 \partial_v]^{L-j} T^i (r \phi_L^{(k)} - |u| T(r \phi_L^{(k)}))|_{\Gamma_R}$ and, from these, establish the auxiliary estimates (modulo corrections arising from the cut-off terms, cf. (3.8.31)):

$$\left| [r^2 \partial_v]^{L-j} T^i (r \phi_L^{(k)} - |u| T(r \phi_L^{(k)})) \right| \leq \frac{C}{|u|^{i+1+\epsilon}} R^{-\max(j,0)} \quad (3.8.9)$$

for $i = 0, \dots, N'$ and $j = -1, 0, \dots, L$.

- V In a fifth step, we show, as in §3.5.5, that the solutions $\phi_L^{(k)}$ tend uniformly to a limiting solution ϕ_L , which still satisfies the estimates (3.8.8) and (3.8.9) above.
- VI In a sixth step, we use the estimate (3.8.8), together with the identities

$$\begin{aligned} & \sum_{j=0}^N \left(a_j^N - L(L+1)b_j^N - c_j^N \frac{2M}{r} \right) [r^2 \partial_v]^{N-j} T^i (r \phi_L) \\ &= [r^2 \partial_v]^{N+1} T^{i+1} (r \phi_L) + \frac{2D(N+1)}{r} [r^2 \partial_v]^{N+1} T^i (r \phi_L) - \frac{1}{r^2} [r^2 \partial_v]^{N+2} T^i (r \phi_L) \end{aligned} \quad (3.8.10)$$

implied by (3.7.1), to obtain the improved estimates (3.8.5), i.e. to convert the R -weights of (3.8.8) into r -weights, using an “upwards-downwards induction”.

VII In a seventh step, we use equation (3.8.10) to obtain a *lower bound* for $[r^2\partial_v]^L T(r\phi_L)|_\Gamma$ on Γ , provided that the lower bound (3.8.3) for $T(r\phi_L)|_\Gamma$ on data is specified. Using the estimate (3.8.8) with $j = -1$, we can then obtain a global lower bound for $[r^2\partial_v]^L T(r\phi_L)$, provided that $R/2M$ is sufficiently large.

Furthermore, and independently of this lower bound, we can use the auxiliary estimate (3.8.9) to show that the following limit exists and is independent of v :

$$\lim_{u \rightarrow -\infty} |u|^2 [r^2\partial_v]^L T(r\phi_L)(u, v) =: \mathcal{L}. \quad (3.8.11)$$

By the lower bounds obtained before, this limit is non-vanishing.

VIII Finally, we prove (3.8.6) and (3.8.7) by writing

$$[r^2\partial_u]^{L+1-j}(r\phi_L) = [r^2T - r^2\partial_v]^{L+1-j}(r\phi_L), \quad (3.8.12)$$

and by expressing each term in the expansion of the above expression in terms of $\mathcal{L}$, using the relations (3.8.9) and (3.8.10).

3.8.4 Proof of Theorem 3.8.1

We now prove Theorem 3.8.1, following the structure outlined above. The proof will be self-contained, with the exceptions of steps IV and V, for which we will refer to §3.5 for details.

Proof. Throughout this proof, C shall denote a constant that depends only on $C_{\text{in}}^\Gamma, C_{\text{in},\epsilon}^\Gamma, 2M/R, M, N, N', L$ (and, in particular, not on k) and can be bounded independently of R for sufficiently large R . Moreover, C is allowed to vary from line to line. We will also assume U_0 to be sufficiently large, where this largeness, again, only depends on data, i.e. on C .

Step I: Cutting off the data

We let $(\chi_k(u))_{k \in \mathbb{N}}$ be a sequence of smooth cut-off functions such that

$$\chi_k(u) = \begin{cases} 1, & u \geq -k + 1, \\ 0, & u \leq -k, \end{cases}$$

and cut-off the the highest-order derivative: $\chi_k T^{N+1} \hat{\phi}_L^{(k)}$. We then define, as in §3.5.2.2, $\hat{\phi}_L^{(k)}$ to be the $N+1$ -th T -integral of $\chi_k T^{N+1} \hat{\phi}_L$ from $-\infty$. Then $\hat{\phi}_L^{(k)}$ satisfies the following bounds:

$$\left| T^n(r\hat{\phi}_L^{(k)}) \right| \leq \frac{n! C_{\text{in}}^\Gamma}{R^L |u|^{n+1}}, \quad n = 0, 1, \dots, N+1, \quad (3.8.13)$$

$$\left| T^n \left(r\hat{\phi}_L^{(j)} - |u| T(r\hat{\phi}_L^{(k)}) \right) \right| \leq \frac{C_{\text{in},\epsilon}^\Gamma}{R^L |u|^{n+1+\epsilon}} + C\theta_k \cdot \frac{C_{\text{in}}^\Gamma}{Rk^{n+1}}, \quad n = 0, 1, \dots, N'+1, \quad (3.8.14)$$

where θ_k equals 1 if $u \geq -k$, and 0 otherwise.

The boundary data $\hat{\phi}_L^{(k)}$, combined with the no incoming radiation condition (3.8.4), lead to unique solutions $\phi_L^{(k)}$, which vanish identically for $u \leq -k$, and which solve the finite initial/boundary value problem (in the sense of Prop. 3.3.2) where $\hat{\phi}_L^{(k)}$ is specified on Γ_R , and where $r\phi_L^{(k)} = 0$ on $\{u = -k\}$.

In steps II–IV below, we will show uniform-in- k estimates on these solutions $\phi_L^{(k)}$, temporarily dropping the superscript (k) and denoting them simply by ϕ_L . We will re-instate this superscript in step V, where we will show that the solutions $\phi_L^{(k)}$ tend to a limiting solution as $k \rightarrow -\infty$.

Step II: Estimates on the boundary terms

Claim 1. *Let U_0 be a sufficiently large negative number. Then there exist constants $B^{(i)}$ such that*

$$\left| r^2 \partial_v T^i(r\phi_L) \right| (u, v_{\Gamma_R}(u)) \leq \frac{B^{(i)}}{R^{L-1} |u|^{i+1}} \quad (3.8.15)$$

for $i = 0, \dots, N$ and for all $u \leq U_0$.

Proof. We fix $i \leq N$, and assume (3.8.15) with $B^{(i)}$ sufficiently large as a bootstrap assumption.

Recall the definition of the energy current (3.3.2). We apply the divergence theorem in the form (3.2.11) to the identity (we abuse notation and omit the Y_{Lm})

$$\text{div} J^T[T^i \phi_L] = 0 \quad (3.8.16)$$

as in equation (3.5.34) in order to obtain

$$\begin{aligned} & \int_{-k}^u r^2 \left(\partial_u T^i \phi_1 \right)^2 (u', v) \, du' \\ & \leq \int_{\Gamma_R \cap \{-k \leq u' \leq u\}} 2r^2 \left(\left| T^{i+1} \phi_L \right|^2 + 2 \left| T^{i+1} \phi_L \cdot \partial_v T^i \phi_L \right| \right) (u', v_{\Gamma_R}(u')) \, du'. \end{aligned} \quad (3.8.17)$$

We estimate, on Γ_R :

$$\left| r \partial_v T^i \phi_L \right| = \left| \partial_v T^i(r\phi_L) - D T^i \phi_L \right| \leq \frac{B^{(i)}}{R^{2+L-1} |u|^{i+1}} + \frac{(i+1)! C_{\text{in}}^\Gamma}{R^{L+1} |u|^{i+1}}.$$

(In the above, we used the bootstrap assumption (3.8.15) and the boundary data assumption (3.8.13).) We thus obtain that

$$\begin{aligned} & \int_{-k}^u r^2 \left(\partial_u T^j \phi_1 \right)^2 (u', v) du' \\ & \leq \int_{\Gamma_R \cap \{-k \leq u' \leq u\}} 2R^2 \cdot \left(\left(\frac{(i+2)! C_{\text{in}}^\Gamma}{R^{L+1} |u|^{i+2}} \right)^2 + 2 \frac{(i+2)! C_{\text{in}}^\Gamma (B^{(i)} + (i+1)! C_{\text{in}}^\Gamma)}{R^{2L+1} |u|^{2i+3}} \right) (u', v_{\Gamma_R}(u')) du' \\ & \leq C \cdot \frac{B^{(i)}}{R^{2L+1} |u|^{2i+2}} \end{aligned}$$

for some constant C as described in the beginning of the proof. We now apply the fundamental theorem of calculus and the Cauchy–Schwarz inequality to obtain

$$T^i \phi_L(u, v) = \int_{-k}^u \partial_u T^i \phi_L(u', v) du' \leq \frac{1}{\sqrt{Dr}} \cdot \frac{\sqrt{CB^{(i)}}}{R^{L+\frac{1}{2}} |u|^{i+1}}.$$

Inserting this bound into (3.7.1) with $N = 0$, and integrating the latter from $u = -k$, we find that

$$\left| \partial_v T^i(r\phi_L) \right| (u, v_{\Gamma_R}(u)) \leq \int_{-k}^u \frac{\sqrt{CB^{(i)}}}{R^{L+\frac{1}{2}} |u'|^{i+1}} \left(L(L+1) + \frac{2M}{r} \right) \frac{D}{r^{\frac{3}{2}}} du' \leq \frac{C\sqrt{B^{(i)}}}{R^{L+1} |u|^{i+1}}. \quad (3.8.18)$$

This improves the bootstrap assumption (3.8.15), provided that $B^{(i)}$ is chosen sufficiently large. $\square$

Claim 2. *Let U_0 be a sufficiently large negative number. Then there exists a constant C such that*

$$\left| [r^2 \partial_v]^{L-j}(r\phi_L) \right| (u, v_{\Gamma_R}(u)) \leq \frac{C}{R^j |u|^{i+1}} \quad (3.8.19)$$

for $i = 0, \dots, N$ and $j = 0, \dots, L-1$, and for all $u \leq U_0$.

Proof. In the proof of the previous claim, we have in fact shown that (cf. (3.8.18))

$$\left| \partial_v T^i(r\phi_L) \right| (u, v) \leq \frac{C}{R^{L+\frac{1}{2}} \sqrt{r} |u|^{i+1}} \quad (3.8.20)$$

for all $v \geq v_{\Gamma_R}(u)$. Let us assume inductively that

$$\left| \partial_v [r^2 \partial_v]^n T^i(r\phi_L) \right| (u, v) \leq \frac{C r^{-\frac{1}{2}+n}}{R^{L+\frac{1}{2}} |u|^{i+1}} \quad (3.8.21)$$

for some fixed $n < \max(L-2, 1)$ and for all $u \leq U_0, v \geq v_{\Gamma_R}(u)$, noting that we have already established the case $n = 0$. We then insert this inductive assumption into (3.7.1) with $N = n+1$

and integrate the latter in u to find

$$\begin{aligned} & \left| r^{-2(n+1)} \partial_v [r^2 \partial_v]^{n+1} T^i(r\phi_L) \right| (u, v) \\ & \leq \int_{-k}^u \sum_{j=0}^{n+1} \frac{D}{r^{2(n+1)+2}} \cdot \frac{C r^{\frac{3}{2}+n}}{R^{L+\frac{1}{2}} |u'|^{i+1}} du' \leq \frac{1}{r^{2(n+1)}} \cdot \frac{C r^{\frac{1}{2}+n}}{R^{L+\frac{1}{2}} |u|^{i+1}}; \end{aligned}$$

so (3.8.21) holds for $n+1$ as well. Evaluating on Γ_R completes the proof. $\square$

Step III: The main estimates

Claim 3. *Let U_0 be a sufficiently large negative number. There exists a constant C such that*

$$\left| [r^2 \partial_v]^{L-j} T^i(r\phi_L) \right| (u, v) \leq \frac{C}{|u|^{i+1}} R^{-\max(j,0)} \quad (3.8.22)$$

for $i = 0, \dots, N$, $j = -1, 0, \dots, L$, and for all $u \leq U_0$, $v \geq v_{\Gamma_R}(u)$.

Proof. In order to simplify the presentation, we will additionally assume that $R/2M$ is sufficiently large. This largeness assumption can be lifted by replacing bootstrap argument below by a Grönwall argument as in §3.5.3.5.

Let us fix $i \leq N$. We make the following bootstrap assumption:

$$\left| [r^2 \partial_v]^L T^i(r\phi_L) \right| (u, v) \leq \frac{C_{\text{BS}}^{(i)}}{|u|^{i+1}}, \quad (3.8.23)$$

where $C_{\text{BS}}^{(i)}$ is a constant to be specified later. Notice that, by integrating this up to L times from Γ , estimating at each step the boundary term by (3.8.19), this implies

$$\left| [r^2 \partial_v]^{L-j} T^i(r\phi_L) \right| (u, v) \leq \frac{C + C_{\text{BS}}^{(i)}}{R^j |u|^{i+1}} \quad (3.8.24)$$

for $j = 0, \dots, L$. In particular, if R and $C_{\text{BS}}^{(i)}$ are chosen sufficiently large, then we have

$$\left| [r^2 \partial_v]^L T^i(r\phi_L) \right| (u, v) \leq 2 \cdot |\Phi_L|(u, v), \quad (3.8.25)$$

where we recall the definition (3.7.8) of Φ_L . We now plug the bounds (3.8.24) into the approximate conservation law (3.7.10) and integrate the latter in u from $u = -k$ to obtain that

$$\left| r^{-2L} \partial_v T^i \Phi_L \right| (u, v) \leq \frac{C \cdot C_{\text{BS}}^{(i)}}{r^{2L+2} |u|^{i+1}}. \quad (3.8.26)$$

We then integrate this bound in v from Γ , estimating the boundary term $T^i \Phi_L|_{\Gamma_R}$ using (3.8.15), to obtain that

$$|T^i \Phi_L|(u, v) \leq \frac{C}{|u|^{i+1}} + \frac{C \cdot C_{\text{BS}}^{(i)}}{R|u|^{i+1}}. \quad (3.8.27)$$

Finally, we can choose R and $C_{\text{BS}}^{(i)}$ large enough such that

$$|[r^2 \partial_v]^L T^i(r\phi_L)| \leq |2T^i \Phi_L| \leq \frac{C_{\text{BS}}^{(i)}}{2|u|^{i+1}}. \quad (3.8.28)$$

This improves the bootstrap assumption (3.8.23) and thus proves (3.8.22) for $j = 0$. The result for $j > 0$ then follows in view of the estimates (3.8.24), and the result for $j = -1$ follows from (3.8.26). $\square$

Step IV: The auxiliary estimates

As in the $\ell = 1$ -case (cf. §3.5.4), we will need some auxiliary estimates in order to later be able to show that certain quantities attain limits on $\mathcal{I}^-$. These auxiliary estimates will be estimates on the differences $[r^2 \partial_v]^L T^i(r\phi_L^{(k)} - |u|T(r\phi_L^{(k)}))$. As in step II, we first need estimates on the boundary terms:

Claim 4. *Let U_0 be a sufficiently large negative number, and let $N' \leq N - 2$. Then there exists a constant C such that*

$$|[r^2 \partial_v]^{L-j} T^i(r\phi_L - |u|T(r\phi_L))|(u, v_{\Gamma_R}(u)) \leq \frac{C}{R^j |u|^{i+1+\epsilon}} + \theta_k \cdot \frac{C}{R^j k^{i+1}} \quad (3.8.29)$$

for $i = 0, \dots, N'$ and $j = 0, \dots, L$, and for all $u \leq U_0$.

Proof. The proof for the case $j = L - 1$ is similar to that of Claim 1: We first assume (3.8.29) (with $j = L - 1$) as a bootstrap assumption. The main difference to the proof of Claim 1 is that we then apply the divergence theorem to

$$\begin{aligned} \operatorname{div} J^T[T^i(\phi_L - |u|T\phi_L)] &= \square_g(T^i(\phi_L - |u|T\phi_L)) \cdot T(T^i(\phi_L - |u|T\phi_L)) \\ &= -\frac{1}{Dr} \partial_v T^{i+1}(r\phi_L) \cdot \frac{1}{r} T^{i+1}(r\phi_L - |u|T(r\phi_L)), \end{aligned}$$

rather than to $\operatorname{div} J^T[T^i \phi_L] = 0$. (Here, we used the expression (3.3.6) for $\square_g$.) This gives rise to a non-trivial bulk term. However, the estimates we established in Claim 3 provide sufficient bounds for this term.

Using the fundamental theorem of calculus and the Cauchy–Schwarz inequality, one then obtains an estimate on (cf. (3.5.52))

$$\sqrt{r} T^i(\phi_L - |u|T\phi_L) \leq \frac{C}{R^{\frac{2L+1}{2}}} \left(\frac{1}{|u|^{i+1+\epsilon}} + \frac{1}{k^{i+1}} \right).$$

In order to translate this into an estimate on $\partial_v T^i(r\phi_L - |u|T(r\phi_L))$, we consider the wave equation satisfied by $\partial_u \partial_v T^i(r\phi_L - |u|T(r\phi_L))$,

$$\partial_u \partial_v T^i(r\phi_L - |u|T(r\phi_L)) = -\frac{D}{r^2} \left(L(L+1) + \frac{2M}{r} \right) T^i(r\phi_L - |u|T(r\phi_L)) + \overbrace{\partial_v T^{i+1}(r\phi_L)}^{\leq Cr^{-L-1}|u|^{-i-2}}, \quad (3.8.30)$$

where we again note that the error term $\partial_v T^{i+1}(r\phi_L)$ can be bounded by the previous estimates (Claim 3), and integrate in u . This improves the bootstrap assumption.

The general case $j \leq L-1$ then follows as in the proof of Claim 2, noting that, when considering the wave equations for $\partial_u \partial_v [r^2 \partial_v]^j T^i(r\phi_L - |u|T(r\phi_L))$, the error terms compared to (3.7.1) will always be given by $\partial_v [r^2 \partial_v]^j T^{i+1}(r\phi_L)$, which we already control by Claim 3. See also the proof of Proposition 3.5.5 for more details. $\square$

Having obtained estimates on the boundary terms, we can now prove:

Claim 5. *Let U_0 be a sufficiently large negative number, and let $N' \leq N-2$. Then there exists a constant C such that*

$$\left| [r^2 \partial_v]^{L-j} T^i(r\phi_L^{(k)} - |u|T(r\phi_L^{(k)})) \right| (u, v) \leq \frac{C}{R^{\max(j,0)}} \left(\frac{1}{|u|^{i+1+\epsilon}} + \frac{1}{k^{i+1}} \right) \quad (3.8.31)$$

for $i = 0, \dots, N'$, $j = -1, 0, \dots, L$, and for all $u \leq U_0$, $v \geq v_{\Gamma_R}(u)$.

Proof. The proof is similar to that of Claim 3, with the main modifications being that we now use Claim 4 in order to estimate the boundary terms. Furthermore, instead of the approximate conservation law (3.7.10), we now consider the equations

$$\begin{aligned} \partial_u (r^{-2L} \partial_v T^i(\Phi_L - |u|T\Phi_L)) &= \overbrace{r^{-2L} \partial_v T^{i+1}\Phi_L}^{\leq Cr^{-2L-2}|u|^{-i-2}} \\ &+ \sum_{j=0}^L \frac{D(2M)^{j+1}}{r^{2L+3}} [r^2 \partial_v]^{L-j} T^i(r\phi_L - |u|T(r\phi_L)) \left(2(j+1)x_{j+1}^{(L)} - \sum_{i=0}^j x_i^{(L)} c_{j-i}^{L-i} \right), \end{aligned} \quad (3.8.32)$$

in which we again control the error terms $r^{-2L} \partial_v T^{i+1}\Phi_L$ by Claim 3. Indeed, they decay faster near Γ_R (that is, they have more u -decay).²⁰ See the proof of Proposition 3.5.5 for more details. $\square$

²⁰Notice, however, that they do *not* decay faster near $\mathcal{I}^-$. It is for this reason that the R -weights in (3.8.31) cannot directly be upgraded to r -weights.

Step V: Taking the limit $k \rightarrow \infty$

So far, we have proved uniform-in- k estimates on the sequence of solutions $\phi_L^{(k)}$ (whose elements vanish on $u \leq -k$) constructed in Step I. We now show that these solutions converge uniformly to another solution ϕ_L :

Claim 6. *The sequence $\{\phi_L^{(k)}\}_{k \in \mathbb{N}}$ tends to a uniform limit ϕ_L as $k \rightarrow \infty$,*

$$\lim_{k \rightarrow \infty} \|\phi_L^{(k)} - \phi_L\|_{C^N(\mathcal{D}_{\Gamma_R})} = 0. \quad (3.8.33)$$

In fact, this limiting solution is the unique smooth solution that restricts correctly to the data of §3.8.1, and it satisfies for all $u \leq U_0$ and $v \geq v_{\Gamma_R}(u)$, for sufficiently large negative values of U_0 , the following bounds for some constant C :

$$\left| [r^2 \partial_v]^{L-j} T^i(r \phi_L^{(k)}) \right| (u, v) \leq \frac{C}{|u|^{i+1}} R^{-\max(j,0)} \quad (3.8.34)$$

for $i = 0, \dots, N$ and $j = -1, 0, \dots, L$. Furthermore, if $N' \leq N - 2$, then we also have

$$\left| [r^2 \partial_v]^{L-j} T^i(r \phi_L^{(k)} - |u| T(r \phi_L^{(k)})) \right| (u, v) \leq \frac{C}{|u|^{i+1+\epsilon}} R^{-\max(j,0)}. \quad (3.8.35)$$

for $i = 0, \dots, N'$ and $j = -1, 0, \dots, L$.

Proof. The proof proceeds, *mutatis mutandis*, as the proof of Proposition 3.5.7. $\square$

Step VI: Proving sharp decay for $[r^2 \partial_v]^{L-j} T^i(r \phi_L)$ (Proof of (3.8.5))

Claim 7. *Let U_0 be a sufficiently large negative number. There exists a constant C such that the solution ϕ_L from Claim 6 satisfies*

$$\left| [r^2 \partial_v]^{L-j} T^i(r \phi_L) \right| (u, v) \leq \frac{C}{|u|^{i+1}} \min(r, |u|)^{-\min(\tilde{j}, N-i)} \quad (3.8.36)$$

for all $j = -1, \dots, L$, $i = 0, \dots, N$, and for all $u \leq U_0$, $v \geq v_{\Gamma_R}(u)$. Here, $\tilde{j} := \max(j, 0)$.

Proof. We will prove this inductively by showing the following lemma:

Lemma 3.8.1. *Let $n < L$. Then there exists a constant C such that*

$$\left| [r^2 \partial_v]^{n-j} T^i(r \phi_L) \right| (u, v) \leq \frac{C}{|u|^{i+1}} \min(r, |u|)^{-\min(j+1, N-i)} \quad (3.8.37)$$

for all $j = 0, \dots, n$, for all $i = 0, \dots, N$, and for all $u \leq U_0$, $v \geq v_{\Gamma_R}(u)$.

Indeed, once this lemma is shown for $n = L - 1$, then (3.8.36) follows in view of Claim 3 (which already provides the sharp estimates for $n = L, L + 1$). $\square$

Proof of Lemma 3.8.1. We first show (3.8.37) for $n = 0$. We derive from (3.7.1) with $N = 0$ that

$$\begin{aligned} -\frac{D}{r^2}T^i(r\phi_L)\left(L(L+1)+\frac{2M}{r}\right) &= -\partial_v^2T^i(r\phi_L)+\partial_vT^{i+1}(r\phi_L) \\ &= -\frac{1}{r^2}\partial_v(r^2\partial_vT^i(r\phi_L))+\frac{2D}{r}\partial_vT^i(r\phi_L)+\partial_vT^{i+1}(r\phi_L). \end{aligned} \quad (3.8.38)$$

We can assume, without loss of generality, that $i < N$, as (3.8.37) follows directly from (3.8.22) if $i = N$. If $i < N$, then we can insert the estimates from (3.8.22) into (3.8.38) to find that

$$|T^i(r\phi_L)| \leq \frac{C}{r^2|u|^{i+1}} + \frac{C}{r|u|^{i+1}} + \frac{C}{|u|^{i+2}}. \quad (3.8.39)$$

This establishes (3.8.37) for $n = 0$.

Let us now assume that (3.8.37) holds for some fixed $n < L - 1$. We shall show that it then also holds for $n + 1$. We derive from (3.7.1) the following generalisation of (3.8.38):

$$\begin{aligned} D\left(a_0^{n+1}-b_0^{n+1}L(L+1)-c_0^{n+1}\frac{2M}{r}\right)[r^2\partial_v]^{n+1}T^i(r\phi_L) \\ = -\sum_{j=0}^n(2M)^{j+1}D\left(a_{j+1}^{n+1}-b_{j+1}^{n+1}L(L+1)-c_{j+1}^{n+1}\frac{2M}{r}\right)[r^2\partial_v]^{n-j}T^i(r\phi_L) \\ + [r^2\partial_v]^{n+2}T^{i+1}(r\phi_L) + \frac{2D(n+2)}{r}[r^2\partial_v]^{n+2}T^i(r\phi_L) - \frac{1}{r^2}[r^2\partial_v]^{n+3}T^i(r\phi_L). \end{aligned} \quad (3.8.40)$$

Notice that, since $n + 1 < L$, the difference $a_0^{n+1} - b_0^{n+1}L(L + 1)$ is non-zero. Therefore, estimating the terms in the second line of (3.8.40) using the induction assumption, and the terms in the third line of (3.8.40) using (3.8.22) (keeping in mind that $n + 3 \leq L + 1$ and assuming as before that $i < N$), we obtain

$$\begin{aligned} \left|[r^2\partial_v]^{n+1}T^i(r\phi_L)\right| &\leq \frac{C}{|u|^{i+1}}\min(r,|u|)^{-\min(1,N-i)} \\ &\quad + \frac{C}{|u|^{i+2}} + \frac{C}{r|u|^{i+1}} + \frac{C}{r^2|u|^{i+1}} \leq \frac{C}{|u|^{i+1}}\min(r,|u|)^{-\min(1,N-i)}. \end{aligned} \quad (3.8.41)$$

This establishes (3.8.37) for $n + 1$, *restricted to $j = 0$* . For $j > 0$, we use another induction, this time going down in derivatives:

Subclaim 1. There exists a constant C such that

$$\left|[r^2\partial_v]^{n+1-j}T^i(r\phi_L)\right|(u,v) \leq \frac{C}{|u|^{i+1}}\min(r,|u|)^{-\min(1+j,N-i)} \quad (3.8.42)$$

for all $j = 0, \dots, n + 1$, and for all $u \leq U_0$, $v \geq v_{\Gamma_R}(u)$.

Proof of Sublemma 1. We have already established (3.8.42) for $j = 0$. Let us now assume that it holds for $j \leq n$ fixed. We shall show that (3.8.42) then also holds for $j + 1$. We can restrict to $i < N - (j + 2)$ since the result would be trivial otherwise. Using (3.8.40), we obtain the estimate (writing $n + 1 - (j + 1) = n - j$):

$$\begin{aligned} |[r^2 \partial_v]^{n-j} T^i(r\phi_L)| &\leq \sum_{k=0}^{n-j-1} \frac{C}{|u|^{i+1}} \min(r, |u|)^{-\min(k+2+j, N-i)} \\ &+ \frac{C}{|u|^{i+2}} \min(r, |u|)^{-\min(j+1, N-(i+1))} + \frac{C}{|u|^{i+1}r} \min(r, |u|)^{-\min(j+1, N-i)}, \end{aligned} \quad (3.8.43)$$

where we used the induction assumption (3.8.37) to estimate the first term on the RHS. This establishes (3.8.42) for $j + 1$ and, thus, proves the sublemma. $\square$

Sublemma 1 proves (3.8.37) for $n + 1$ and, hence, completes the proof of Lemma 3.8.1. $\square$

Step VII: The limit $\mathcal{L} = \lim_{u \rightarrow -\infty} |u|^2 [r^2 \partial_v]^L T(r\phi_L)$

Throughout the rest of the proof, we shall assume that $N - 2 \geq N' \geq L + 1$, and that ϕ_L denotes the solution from Claim 6.

Claim 8. *The limit $\mathcal{L}(v) := \lim_{u \rightarrow -\infty} |u|^2 [r^2 \partial_v]^L T(r\phi_L)(u, v)$ exists and is independent of v . Moreover, along any ingoing null hypersurfaces of constant v ,*

$$|u|^2 [r^2 \partial_v]^L T(r\phi_L)(u, v) - \mathcal{L} = \mathcal{O}(r^{-1} + |u|^{-\epsilon}). \quad (3.8.44)$$

In fact, if the lower bound (3.8.3) is assumed, and if $R/2M$ is chosen large enough, then $\mathcal{L} \neq 0$.

Proof. We show that the limit exists by computing

$$\begin{aligned} \partial_u(|u|^2 [r^2 \partial_v]^L T(r\phi_L)) &= \underbrace{-2|u| [r^2 \partial_v]^L T(r\phi_L) + |u|^2 [r^2 \partial_v]^L T^2(r\phi_L)}_{\leq C|u|^{-1-\epsilon}} - \underbrace{|u|^2 \partial_v [r^2 \partial_v]^L T(r\phi_L)}_{\leq Cr^{-2}}. \end{aligned} \quad (3.8.45)$$

The first two terms together can be bounded by $|u|^{-1-\epsilon}$ in view of estimate (3.8.35) from Claim 6. The third term can be bounded by r^{-2} in view of estimate (3.8.36) from Claim 7. This establishes the existence of the limit $\mathcal{L}(v)$. The independence of v follows directly from the bound on the third term.

It is left to show that this limit is not zero. For this, we will also need to establish a lower bound for $|u|^2 [r^2 \partial_v]^L T(r\phi_L)$ on Γ_R . First, we observe that, in view of the identity (3.8.38), we have, on Γ_R , that

$$\left| R^L \cdot T(r\phi_L)(L(L+1)) + 2R^{L-1} [r^2 \partial_v] T(r\phi_L) \right| \leq \frac{C}{|u|^3} + \frac{C}{R|u|^2}. \quad (3.8.46)$$

Thus, if R and $|U_0|$ are chosen sufficiently large, we have, as a consequence of the lower bound (3.8.3):

$$\left| r^2 \partial_v T(r\phi_L) \right| (u, v_{\Gamma_R}(u)) \geq \frac{C_{\text{in}}^\Gamma}{4R^{L-1}|u|^2} \quad (3.8.47)$$

Similarly, we can now show inductively, using (3.8.40) instead of (3.8.38), that, say,

$$\left| [r^2 \partial_v]^{L-j} T(r\phi_L) \right| (u, v_{\Gamma_R}(u)) \geq \frac{C_{\text{in}}^\Gamma}{2^{L-j+1} R^j |u|^2}. \quad (3.8.48)$$

for $j = 0, \dots, L-1$, provided that R is chosen sufficiently large.²¹

Using the lower bound above for $j = 0$ and then integrating the bound (3.8.36) for $j = -1$ from Γ_R , one obtains, provided that R is chosen sufficiently large,

$$\left| [r^2 \partial_v]^L T(r\phi_L) \right| (u, v) \geq \frac{C_{\text{in}}^\Gamma}{2^{L+2}|u|^2} \quad (3.8.49)$$

for all $v \geq v_{\Gamma_R}(u)$, $u \leq U_0$. This shows that $\mathcal{L} \neq 0$ and thus completes the proof. (Notice that this approach only works for the highest-order v -derivative (i.e. for $j = 0$).) $\square$

We now show that various different limits can be computed from this limit.

Lemma 3.8.2. *The following limits exist and satisfy the relations*

$$\lim_{u \rightarrow -\infty} |u|^{1+i} r^j [r^2 \partial_v]^{L-j} T^i(r\phi_L) = \mathcal{L}^{(i,j)} \quad (3.8.50)$$

for $i = 0, \dots, N' + 1$ and $j = 0, \dots, L$, provided that $i + j \leq N' + 1$, where $\mathcal{L}^{(i,j)}$ are rational multiples of $\mathcal{L}$.

Proof. We will first show (3.8.50) for $j = 0$. Indeed, we have $\mathcal{L}^{(0,0)} = \mathcal{L}^{(1,0)} = \mathcal{L}$, and, in view of the estimates (3.8.35) from Claim 6, we have the relations

$$\begin{aligned} \lim_{u \rightarrow -\infty} |u|^2 [r^2 \partial_v]^L T(r\phi_L) &= \frac{1}{2} \lim_{u \rightarrow -\infty} |u|^3 [r^2 \partial_v]^L T^2(r\phi_L) \\ &= \frac{1}{6} \lim_{u \rightarrow -\infty} |u|^4 [r^2 \partial_v]^L T^3(r\phi_L) = \dots = \frac{1}{i!} \lim_{u \rightarrow -\infty} |u|^{1+i} [r^2 \partial_v]^L T^i(r\phi_L), \end{aligned} \quad (3.8.51)$$

provided that $i \leq N' + 1$. We thus have established that, for all $i \leq N' + 1$:

$$\mathcal{L}^{(i,0)} = i! \mathcal{L}. \quad (3.8.52)$$

We now prove (3.8.50) inductively in j . Without loss of generality, we may assume that $L > 0$. We assume that we have already established (3.8.50) for (i, j) , for some fixed $j < L$

²¹Notice that this leads to an extremely wasteful lower bound on R . One can improve this using a different approach, or just not show the lower bound. We do not analyse this issue any further.

and for all $i \leq N' + 1 - j$. We then show that (3.8.50) also holds for $(i, j + 1)$, provided that $i + j + 1 \leq N' + 1$: Indeed, if $i + j \leq N'$, we can appeal to equation (3.8.40) (with $n + 1$ replaced by $L - j - 1$) to obtain

$$\begin{aligned} & (a_0^{L-j-1} - b_0^{L-j-1}L(L+1)) \lim_{u \rightarrow -\infty} |u|^{2+i} r^{j+1} [r^2 \partial_v]^{L-j-1} T^i(r\phi_L) \\ &= \lim_{u \rightarrow -\infty} |u|^{2+i+1} r^j [r^2 \partial_v]^{L-j} T^{i+1}(r\phi_L) + 2(L-j) \lim_{u \rightarrow -\infty} |u|^{2+i} r^j [r^2 \partial_v]^{L-j} T^i(r\phi_L), \end{aligned} \quad (3.8.53)$$

where we used (3.8.36) to estimate the lower-order derivatives $[r^2 \partial_v]^{L-j-2-k}$ for $k \geq 0$. We have thus established that

$$\mathcal{L}^{(i,j+1)} = \frac{1}{a_0^{L-j-1} - b_0^{L-j-1}L(L+1)} \cdot \left(\mathcal{L}^{(i+1,j)} + 2(L-j)\mathcal{L}^{(i,j)} \right) \quad (3.8.54)$$

for all $0 \leq i \leq N' + 1 - (j + 1)$, $0 \leq j < L$. (Notice that, in this range of indices, $a_0^{L-j-1} - b_0^{L-j-1}L(L+1) \neq 0$.) We leave it to the reader to derive explicit expressions for $\mathcal{L}^{(i,j)}$ from the recurrence relations above. $\square$

Step VIII: The limit $\lim_{u \rightarrow -\infty} [r^2 \partial_u]^{L+1}(r\phi_L)$ (Proof of (3.8.6) and (3.8.7))

Claim 9. *The limit $\lim_{u \rightarrow -\infty} [r^2 \partial_u]^{L+1}(r\phi_L) = \tilde{C} \neq 0$ exists and can be computed explicitly in terms of the constants $\mathcal{L}^{(i,j)}$. Moreover, we have along any ingoing null hypersurface of constant v :*

$$[r^2 \partial_u]^{L-j}(r\phi_L)(u, v) = \mathcal{O}(\min(r, |u|)^{-1-j}), \quad j = 0, \dots, L, \quad (3.8.55)$$

$$[r^2 \partial_u]^{L+1}(r\phi_L)(u, v) = \tilde{C} + \mathcal{O}(r^{-1} + |u|^{-\epsilon}). \quad (3.8.56)$$

Proof. We will prove this by writing $\partial_u = T - \partial_v$ and using the following lemma:

Lemma 3.8.3. *Let f be a smooth function, and let $n \in \mathbb{N}$. Then*

$$(r^2 \partial_u)^n f = (r^2 T - r^2 \partial_v)^n f = \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} \left(\sum_{i=0}^{k-1} \alpha_i^{(n,k)} (r + \mathcal{O}(1))^i (r^2 T)^{k-i} \right) [r^2 \partial_v]^{n-k} f \quad (3.8.57)$$

for some constants $\alpha_i^{(n,k)}$.

Proof. A proof is provided in the appendix 3.A.2. $\square$

Indeed, applying this lemma with $n = L - j$ for $j \geq 0$ immediately proves (3.8.55) upon inserting the bounds (3.8.36) from Claim 7. On the other hand, applying the lemma with

$n = L + 1$ and recalling Lemma 3.8.2, we find

$$\lim_{u \rightarrow -\infty} [r^2 \partial_u]^{L+1} (r \phi_L) = \sum_{k=0}^{L+1} (-1)^{L+1-k} \binom{L+1}{k} \sum_{i=0}^{k-1} \alpha_i^{(L+1,k)} \mathcal{L}^{(k-i, L+1-k)} = \tilde{C} \quad (3.8.58)$$

since we assumed that $N' \geq L + 1$. Equation (3.8.56) then follows similarly.

Finally, we need to show that $\tilde{C} \neq 0$. Instead of explicitly computing the sum above, we proceed by contradiction: Suppose that (3.8.56) holds with $\tilde{C} = 0$. Then, in view of (3.8.55), we have that, say, on $v = 1$, $r \phi_L(u, 1) = \mathcal{O}(|u|^{-L-1-\epsilon})$. Inductively inserting this estimate into (3.7.10) and integrating in u (cf. Proposition 3.10.1), this implies that $[r^2 \partial_v]^L (r \phi_L)(u, v) = \mathcal{O}(|u|^{-1-\epsilon})$, which would imply that $\mathcal{L} = 0$, a contradiction. $\square$

This concludes the proof of Theorem 3.8.1. $\square$

3.9 Data on an ingoing null hypersurface $\mathcal{C}_{v=1}$ I

Having obtained an understanding of solutions arising from timelike boundary data in the previous section, we now aim to understand solutions arising from polynomially decaying initial data on an ingoing null hypersurface. We will, in the present section, focus on initial data which decay as predicted by Theorem 3.8.1 of the previous section. While the present section completely generalises the methods of §3.4 and contains a proof of Theorem 3.1.4, it requires fast initial decay on the data (depending on ℓ). The case of more slowly decaying data will thus need to be treated differently and is discussed in §3.10.

Throughout this section, we shall again assume that ϕ is a solution to (3.1.1), supported on a single angular frequency (L, m) with $|m| \leq L$. In the usual abuse of notation, we omit the m -index, that is, we write $\phi = \phi_{Lm} \cdot Y_{Lm} = \phi_L \cdot Y_{Lm}$.

3.9.1 Initial data assumptions

Prescribe smooth characteristic/scattering data for (3.1.1), restricted to the angular frequency (L, m) , that satisfy on $\mathcal{C}_{v=1}$

$$\lim_{u \rightarrow -\infty} [r^2 \partial_u]^{L+1-i} (r\phi_L)(u, 1) = C_{\text{in}}^{(L,i)}, \quad i = 1, \dots, L, \quad (3.9.1)$$

$$[r^2 \partial_u]^{L+1} (r\phi_L)(u, 1) = C_{\text{in}}^{(L,0)} + \mathcal{O}(r^{-\epsilon}) \quad (3.9.2)$$

for some $\epsilon \in (0, 1)$, where the $C_{\text{in}}^{(L,i)}$ are constants, and which moreover satisfy for all $v \geq 1$:

$$\lim_{u \rightarrow -\infty} \partial_v^n (r\phi_L)(u, v) = 0 \quad (3.9.3)$$

for $n = 0, \dots, L+1$. We interpret this latter assumption as the no incoming radiation condition.

3.9.2 The main theorem (Theorem 3.9.1)

Motivated by the previous Theorem 3.8.1, we will only consider the case where $C_{\text{in}}^{(L,i)} = 0$ for $i > 0$ for now. The other cases, and further generalisations that do not require conformal regularity on the initial data, will be treated in §3.10, as they have to be dealt with in a different way. Let us mention, however, that the proof of the present section still works if additionally $C_{\text{in}}^{(L,1)} \neq 0$.

Theorem 3.9.1. *By standard scattering theory [DRS18], there exists a unique smooth scattering solution $\phi_L \cdot Y_{Lm}$ in $\mathcal{M} \cap \{v \geq 1\}$ attaining the data of §3.9.1.*

Let U_0 be a sufficiently large negative number. Assume moreover that $C_{\text{in}}^{(L,i)} = 0$ for all $i = 1, \dots, L$ and that $C_{\text{in}}^{(L,0)} \neq 0$. Then, for all $u \leq U_0$, the limit of the radiation field on future

null infinity is given by

$$\lim_{v \rightarrow \infty} (r\phi_L)(u, v) = \frac{L!C_{\text{in}}^{(L,0)}}{(2L+1)!|u|^{L+1}} + \mathcal{O}(|u|^{-L-1-\epsilon}), \quad (3.9.4)$$

and, throughout $\mathcal{D} = (-\infty, U_0] \times [1, \infty)$, the outgoing derivative of $r\phi_L$ satisfies, for fixed values of u , the following asymptotic expansion as $\mathcal{I}^+$ is approached:

$$\partial_v(r\phi_L)(u, v) = \sum_{i=0}^L \frac{f_i^{(L)}(u)}{r^{2+i}} + \frac{(-1)^L M \cdot B^* \log r - \log |u|}{(L+1)! r^{3+L}} + \mathcal{O}(r^{-3-L}). \quad (3.9.5)$$

Here, the $f_i^{(L)}(u)$ are smooth functions of u which satisfy $f_i^{(L)}(u) = \frac{(-1)^i \beta_{L-1-i}^{(L)}}{i!|u|^{L-i}} + \mathcal{O}(|u|^{-L+i-\epsilon})$

for $i < L$ and $f_i^{(L)}(u) = \frac{2M(-1)^L \tilde{a}_1^{L,L} \beta_1^{(L)}}{L!|u|} + \mathcal{O}(|u|^{-1-\epsilon})$ for $i = L$, and $B^* = 2(2x_1^{(L)} - c_0^L) \beta_0^{(L)}$.

The constants $\beta_i^{(L)}$ are given explicitly by the formulae²²

$$\beta_i^{(L)} = \frac{L!C_{\text{in}}^{(L,0)}}{(2L+1)!} \cdot \frac{i!}{L!} \prod_{k=0}^{L-i-1} (a_0^k - b_0^k L(L+1)) = (-1)^{L-i} \frac{(2L-i)!C_{\text{in}}^{(L,0)}}{(2L+1)!}. \quad (3.9.6)$$

Moreover, the quantity Φ_L defined in eq. (3.7.8) has the expansion

$$\partial_v \Phi_L = \frac{M \cdot B^* (\log r - \log |u|)}{r^3} + \mathcal{O}(r^{-3}), \quad (3.9.7)$$

and, in particular, the logarithmically modified Newman–Penrose constant is finite and conserved:

$$I_{\ell=L}^{\text{future}, \frac{\log r}{r^3}}[\phi](u) := \lim_{v \rightarrow \infty} \frac{r^3}{\log r} \partial_v \Phi_L(u, v) = M \cdot B^* \neq 0. \quad (3.9.8)$$

Remark 3.9.1. The first two statements of the above theorem, (3.9.4) and (3.9.5), still apply if one lifts the restriction $C_{\text{in}}^{(L,1)} = 0$, albeit with different constants and with different $f_i^{(L)}(u)$. See also Remark 3.9.2. In particular, we again have a cancellation if $r\phi_L \sim 1/|u|^L$ initially: The initial $|u|^{-L}$ -decay translates into $|u|^{-L-1}$ -decay on $\mathcal{I}^+$. Equations (3.9.7) and (3.9.8), on the other hand change, change: The leading-order decay behaviour of $\partial_v \Phi_L$ is now given by $\sim \frac{1}{r^2}$, and, in particular, the usual Newman–Penrose constant

$$I_{\ell=L}^{\text{future}}[\phi] := \lim_{v \rightarrow \infty} r^2 \partial_v \Phi_L(u, v) \quad (3.9.9)$$

²²For the readers convenience, we recall that $\tilde{a}_1^{L,L} = -L^2$, that $2x_1^{(L)} = -L$, and that $c_0^L = 1 + 2L(L+1)$. Finally, we recall that $a_0^k - b_0^k L(L+1) = k(k+1) - L(L+1)$. All these constants have been defined in §3.7.

will be finite, generically non-vanishing, and conserved along future null infinity. If one also allows $C_{\text{in}}^{(L,i)} \neq 0$ for $i > 1$, then the modifications to Theorem 3.9.1 will be more severe, see already Theorem 3.10.1.

Overview of the proof

We will prove the theorem in two steps. First, we will obtain an asymptotic estimate for $r\phi_L$ (which will, in particular, imply (3.9.4)) by integrating (3.7.13) in v from data and then integrating the result $L + 1$ times in u from $\mathcal{I}^-$.

Then, we will use this estimate to get the leading-order decay of $\partial_v(r\phi_L)$ by integrating equation (3.7.1) with $N = 0$. Once this is achieved, we will inductively obtain leading-order asymptotics for $(r^2\partial_v)^n(r\phi_L)$ using the corresponding (3.7.1), from which we can, in turn, deduce higher-order asymptotics for $\partial_v(r\phi_L)$. This will prove (3.9.5). Equation (3.9.7) then follows in a similar fashion from the approximate conservation law (3.7.10).

3.9.3 Asymptotics for $r\phi_L$

Proposition 3.9.1. *There exists a constant C depending only on data such that $r\phi_L$ satisfies the following asymptotic expansion throughout $\mathcal{D}$:*

$$\left| r\phi_L(u, v) - \frac{L!C_{\text{in}}^{(L,0)}}{(2L+1)!|u|^{L+1}} \right| \leq \frac{C}{|u|^{L+1+\epsilon}} + \frac{C}{r|u|^L}. \quad (3.9.10)$$

In particular, we have

$$\lim_{v \rightarrow \infty} r\phi_L(u, v) = \frac{L!}{(2L+1)!|u|^{L+1}} + \mathcal{O}(|u|^{-L-1-\epsilon}). \quad (3.9.11)$$

Proof. By applying the weighted energy estimate of Proposition 3.4.1 (whose proof still works for higher ℓ -modes), we obtain the decay estimate (cf. Corollary 3.4.1):

$$|r\phi_L(u, v)| \leq C|u|^{-L-1} \quad (3.9.12)$$

for some constant C depending only on initial data.

By inserting this estimate into eq. (3.7.13) with $N = 0$ and integrating the latter from $v = 1$, we then obtain that $|\partial_u(r\phi_L)| \leq C|u|^{-L-2}$. Similarly, by inductively integrating eq. (3.7.13) for higher $N \leq L$ from $v = 1$, we find that

$$|(r^2\partial_u)^n(r\phi)| \leq C \frac{r^{2n}}{|u|^{L+1+n}}. \quad (3.9.13)$$

We can plug these estimates into (3.7.13) with $N = L$,

$$\begin{aligned}
r^{2L} \partial_v (r^{-2L} \partial_u [r^2 \partial_u]^L (r\phi_L)) &= -c_0^L \frac{2MD}{r^3} \overbrace{[r^2 \partial_u]^L (r\phi_L)}^{\lesssim r^{2L} |u|^{-2L-1}} \\
&\quad + \sum_{j=1}^N \frac{D}{r^2} (2M)^j \underbrace{[r^2 \partial_u]^{L-j} (r\phi_L)}_{\lesssim r^{2L-2} |u|^{-2L}} \cdot \left(\underline{a}_j^L - \underline{b}_j^L - \underline{c}_j^L \frac{2M}{r} \right), \quad (3.9.14)
\end{aligned}$$

and integrate in v from $v = 1$ to find that

$$\left| \partial_u [r^2 \partial_u]^L (r\phi_L)(u, v) - C_{\text{in}}^{(L,0)} \frac{r^{2L}}{|u|^{2L+2}} \right| \leq C \frac{r^{2L}}{|u|^{2L+2+\epsilon}}. \quad (3.9.15)$$

Essentially, we can now integrate (3.9.15) $L + 1$ times from $\mathcal{I}^-$ to improve the bootstrap assumption. For this, we invoke the following

Lemma 3.9.1. *Let $N, N' \in \mathbb{N}$ with $N > N' + 1$. Then*

$$(N-1) \int_{-\infty}^u \frac{r^{N'}}{|u'|^N} du' = \frac{r^{N'}}{|u|^{N-1}} + \sum_{k=1}^{N'} \frac{r^{N'-k}}{|u|^{N-1-k}} \prod_{j=1}^k \frac{N'+1-j}{N-1-j} \left(1 + \mathcal{O}(r^{-1}) \right). \quad (3.9.16)$$

Proof. The proof is straight-forward, but nevertheless provided in the appendix 3.A.3. $\square$

We now apply Lemma 3.9.1 with $N' = 2L$, $N = 2L + 2$ to (3.9.15) (and divide by r^2) to obtain that

$$\left| \partial_u [r^2 \partial_u]^{L-1} (r\phi_L)(u, v) - \frac{C_{\text{in}}^{(L,0)}}{2L+1} \left(\frac{r^{2(L-1)}}{|u|^{2L+1}} + \sum_{k=1}^{2L} \frac{r^{2(L-1)-k}}{|u|^{2L+1-k}} \right) \right| \leq C \frac{r^{2(L-1)}}{|u|^{2L+1+\epsilon}}. \quad (3.9.17)$$

Here, we used that the boundary term vanishes,²³

$$\lim_{u \rightarrow -\infty} [r^2 \partial_u]^L (r\phi_L)(u, v) = 0.$$

Notice that the terms inside the sum $\sum_{k=1}^{2L}$ decay faster near $\mathcal{I}^+$ than the $\frac{r^{2(L-1)-k}}{|u|^{2L+1-k}}$ -term inside (3.9.17). Therefore, inductively applying the above procedure L more times²⁴, one obtains

$$\left| r\phi_L(u, v) - \frac{1}{(2L+1) \cdots (L+1)} \frac{C_{\text{in}}^{(L,0)}}{|u|^{L+1}} \right| \leq \frac{C}{|u|^{L+1+\epsilon}} + \frac{C}{r|u|^L}. \quad (3.9.18)$$

This proves the proposition. $\square$

Remark 3.9.2. We stated in Remark 3.9.1 that parts of Theorem 3.9.1 still apply if one assumes that also $C_{\text{in}}^{(L,1)} = \lim_{u \rightarrow -\infty} [r^2 \partial_u]^L (r\phi_L) \neq 0$. Let us explain the modifications to the proof

²³In the general case, $C_{\text{in}}^{(L,i)} \neq 0$, these boundary terms would of course not vanish.

²⁴Notice that the $\sum_{k=1}^{2L}$ -sum in (3.9.17) also contains the terms $\frac{1}{|u|^3}$, $\frac{1}{r|u|^2}$ and $\frac{1}{r^2|u|}$. The latter two are not of the form of Lemma 3.9.1, but can simply be estimated against the former.

above needed to see this: First, one needs to replace the $|u|^{-L-1}$ -decay in (3.9.12) with $|u|^{-L}$ -decay. This then leads to the RHS's of (3.9.13) and (3.9.14) to decay one power in u slower. However, this still produces the same leading-order decay of $\partial_u[r^2\partial_u]^L(r\phi_L)$ in (3.9.15). Upon integrating this from $\mathcal{I}^-$, one picks up the limit $C_{\text{in}}^{(L,1)} = \lim_{u \rightarrow -\infty} [r^2\partial_u]^L(r\phi_L)$ and obtains an asymptotic estimate for $[r^2\partial_u]^L(r\phi_L)$ near $\mathcal{I}^-$. One can then, as was done in §3.4, insert this asymptotic estimate for $[r^2\partial_u]^L(r\phi_L)$ back into (3.9.14) and proceed with the rest of the proof as above. In view of the boundary term coming from $\lim[r^2\partial_u]^L(r\phi_L)$, one then obtains, instead of (3.9.10),

$$\left| r\phi_L(u, v) - \frac{L!C_{\text{in}}^{(L,0)} + C'(C_{\text{in}}^{(L,1)})}{(2L+1)!|u|^{L+1}} - \frac{C''(C_{\text{in}}^{(L,1)})}{r^L} \right| \leq \frac{C}{|u|^{L+1+\epsilon}} \quad (3.9.19)$$

for some $C'(C_{\text{in}}^{(L,1)})$, $C''(C_{\text{in}}^{(L,1)})$ which depend only on M , L and $C_{\text{in}}^{(L,1)}$.

The proof of the asymptotics for $\partial_v(r\phi_L)$, presented in the next section, remains largely unchanged. See also §3.4 for details.

3.9.4 Asymptotics for $\partial_v(r\phi_L)$ and proof of Theorem 3.9.1

Proof of Theorem 3.9.1. Having obtained the asymptotics for $r\phi_L$ along $\mathcal{I}^+$, we can now compute the asymptotics of $\partial_v(r\phi_L)$. For the sake of notational simplicity, we restrict to $L \neq 0$ for now, the case $L = 0$ is recovered in (3.9.32).

We first compute the leading-order asymptotics by integrating the wave equation (3.7.1) with $N = 0$,

$$\partial_u\partial_v(r\phi_L) = -\frac{D}{r^2} \left(L(L+1) + \frac{2M}{r} \right) r\phi_L \quad (3.9.20)$$

from past null infinity (where $\partial_v(r\phi_L)$ vanishes by assumption (3.9.3)) and by plugging in the estimate (3.9.10). This yields, after also commuting (3.9.20) with r^2 ,

$$\left| r^2\partial_v(r\phi_L)(u, v) + \frac{(L+1)!}{(2L+1)!} \frac{C_{\text{in}}^{(L,0)}}{|u|^L} \right| \leq \frac{C}{|u|^{L+\epsilon}} + \frac{C}{r|u|^{L-1}}, \quad (3.9.21)$$

where, from now on, C will be a constant which depends only on data and which is allowed to vary from line to line. More precisely, by writing $r\phi_L(u, v) = r\phi_L(u, \infty) - \int_v^\infty \partial_v(r\phi)(u, v) dv'$ in (3.9.20), we can write

$$\left| r^2\partial_v(r\phi_L)(u, v) + L(L+1) \int_{-\infty}^u \lim_{v \rightarrow \infty} (r\phi)(u', v) du' \right| \leq \frac{C}{r|u|^{L-1}}. \quad (3.9.22)$$

Let us now make the following induction assumption. Let $n \geq 1$. Then we assume that for all $L \geq i \geq n$:

$$\left| [r^2\partial_v]^{L-i}(r\phi_L) - \frac{\beta_i^{(L)}}{|u|^{i+1}} \right| \leq \frac{C}{|u|^{i+1+\epsilon}} + \frac{C}{r|u|^i} \quad (3.9.23)$$

for some non-vanishing constants $\beta_i^{(L)} \in \mathbb{Q}$. Since we have already established that this holds true for $n = L$ with $\beta_L^{(L)} = \frac{L!C_{\text{in}}^{(L,0)}}{(2L+1)!}$, it suffices to show that (3.9.23) also holds for $n - 1 \geq 1$, provided that it holds for n . Therefore, we now consider (3.7.1) with $N = L - n \geq 0$:

$$\begin{aligned} & \partial_u \left(r^{-2(L-n)} \partial_v [r^2 \partial_v]^{L-n} (r\phi_L) \right) \\ &= \frac{1}{r^{2(L-n)}} \sum_{i=n}^L \frac{D(2M)^{i-n}}{r^2} [r^2 \partial_v]^{L-i} (r\phi_L) \left(a_{i-n}^{L-n} - b_{i-n}^{L-n} L(L+1) - c_{i-n}^{L-n} \frac{2M}{r} \right). \end{aligned} \quad (3.9.24)$$

Plugging in the induction assumption (3.9.23) for the terms on the RHS and then integrating (3.9.24) gives that $\partial_v [r^2 \partial_v]^{L-n} (r\phi_L)$ is of order $\mathcal{O}(r^{-2}|u|^{-n})$. Moreover, commuting now (3.9.24) with $r^{2(L-(n-1))}$, we obtain that

$$\begin{aligned} \partial_u (r^2 \partial_v [r^2 \partial_v]^{L-n} (r\phi_L)) &= - \overbrace{\frac{2(L-(n-1))D}{r}}^{\lesssim r^{-1}|u|^{-n}} \cdot r^2 \partial_v [r^2 \partial_v]^{L-n} (r\phi_L) \\ &+ \sum_{i=n}^L D(2M)^{i-n} \underbrace{[r^2 \partial_v]^{L-i} (r\phi_L)}_{\lesssim |u|^{-n-1}} \left(a_{i-n}^{L-n} - b_{i-n}^{L-n} L(L+1) - c_{i-n}^{L-n} \frac{2M}{r} \right), \end{aligned} \quad (3.9.25)$$

from which, in turn, we recover, by again integrating from $\mathcal{I}^-$, that

$$\left| [r^2 \partial_v]^{L-(n-1)} (r\phi_L) - \frac{\beta_{n-1}^{(L)}}{|u|^n} \right| \leq \frac{C}{|u|^{n+\epsilon}} + \frac{C}{r|u|^{n-1}}, \quad (3.9.26)$$

with $\beta_{n-1}^{(L)}$ given by

$$n\beta_{n-1}^{(L)} = \beta_n^{(L)} \left(a_0^{L-n} - b_0^{L-n} L(L+1) \right) \neq 0. \quad (3.9.27)$$

This proves (3.9.23) for all $n \geq 1$ and, thus, that (3.9.26) holds for all $n \geq 2$. In fact, it is easy to see that (3.9.26) also holds for $n = 1$, with the $r^{-1}|u|^{-n+1}$ -term on the RHS replaced by $\log(1 - v/u)/v$ (cf. (3.4.26)).

In order to get a similar estimate to (3.9.26) for $n = 0$, we recall the crucial cancellation in (3.7.10) for $N = L$ (namely $a_0^L - b_0^L L(L+1) = 0$). We are thus led to consider, in a very similar fashion to the above, the equation

$$\begin{aligned} \partial_u (r^{-2L} \partial_v [r^2 \partial_v]^L (r\phi_L)) &= - \frac{1}{r^{2L}} \frac{D}{r^2} \cdot c_0^L \frac{2M}{r} [r^2 \partial_v]^L (r\phi_L) \\ &+ \frac{1}{r^{2L}} \sum_{i=1}^L \frac{D(2M)^i}{r^2} [r^2 \partial_v]^{L-i} (r\phi_L) \left(a_i^L - b_i^L L(L+1) - c_i^L \frac{2M}{r} \right). \end{aligned} \quad (3.9.28)$$

The first term on the RHS is bounded by $Cr^{-2L-3}|u|^{-1}$, whereas the other terms in the sum are bounded by $Cr^{-2L-2}|u|^{-2}$. More precisely, we have

$$\begin{aligned} \partial_u(r^{-2L}\partial_v[r^2\partial_v]^L(r\phi_L)) &= -\frac{2MD}{r^{2L+3}|u|}c_0^L\beta_0^{(L)} \\ &+ \frac{2MD}{r^{2L+2}|u|^2}\left(a_1^L - b_1^L L(L+1)\right)\beta_1^{(L)} + \mathcal{O}\left(r^{-2L-2}|u|^{-2-\epsilon} + r^{2L-3}\frac{\log 1 - \frac{v}{u}}{v}\right). \end{aligned} \quad (3.9.29)$$

Integrating this from $\mathcal{I}^-$ then yields that

$$[r^2\partial_v]^{L+1}(r\phi_L)(u, v) = \frac{2M}{|u|}\left(a_1^L - b_1^L L(L+1)\right)\beta_1^{(L)} + \mathcal{O}\left(|u|^{-1-\epsilon} + \frac{1}{r} + \frac{\log 1 - \frac{v}{u}}{v}\right), \quad (3.9.30)$$

where, in the two asymptotic equalities above, we made use of the integral estimates (3.4.26) and (3.4.27). In order to find the logarithmic next-to leading order asymptotics, we insert the estimates above into the approximate conservation law (3.7.10):

$$\partial_u(r^2\partial_v\Phi_L) = (2L+2)Dr\partial_v\Phi_L + \sum_{j=0}^L \frac{D}{r}(2M)^{j+1}[r^2\partial_v]^{L-j}(r\phi_L) \left(2(j+1)x_{j+1}^{(L)} - \sum_{i=0}^j x_i^{(L)}c_{j-i}^{L-i}\right). \quad (3.9.31)$$

From this, we then obtain, in a similar way to how we proved the estimates above, that

$$r^2\partial_v\Phi_L(u, v) = 2M(2x_1^{(L)} - c_0^L)\beta_0^{(L)}\frac{\log(v-u) - \log|u|}{v} + \mathcal{O}(r^{-1}). \quad (3.9.32)$$

Notice that the difference $2x_1^{(L)} - c_0^L$ is non-vanishing for all $L(\geq 0)$ since $x_1^{(L)} = -\frac{L}{2}$ and $c_0^L = 1 + 2L(L+1)$.

Comparing equations (3.9.30) and (3.9.32) then gives us the next-to-leading-order asymptotics for $[r^2\partial_v]^{L+1}(r\phi_L)$ since, for $i > 0$, the terms $[r^2\partial_v]^{L-i}(r\phi_L)$ contained in Φ_L do not contain logarithmic terms at next-to-leading order, which can be seen by integrating (3.9.30) i times from $\mathcal{I}^+$.

Finally, the statement (3.9.5) follows by simply integrating these asymptotics L times from $\mathcal{I}^+$ and using (3.9.23) for the arising boundary terms on $\mathcal{I}^+$.

This concludes the proof of Theorem 3.9.1. $\square$

3.9.5 Comments

3.9.5.1 A logarithmically modified Price's law at all orders

We expect Theorem 3.9.1 (and, in particular, eq. (3.9.8)) to imply a logarithmically modified Price's law for the $\ell = L$ -mode (see also the remarks in §3.4.4). However, Theorem 3.9.1 only applies if $C_{\text{in}}^{(L,i)} = 0$ for all $i > 0$. This assumption, in turn, is motivated by the results of the previous §3.8 (eq. (3.8.6) from Theorem 3.8.1). Therefore, although the general situation

($C_{\text{in}}^{(L,i)} \neq 0$) might (and will) be quite different, we can expect that the data considered in §3.8, i.e. data on a timelike boundary data which decay like $r\phi_\ell \sim |t|^{-1}$ near i^- and which are smoothly extended to $\mathcal{H}^+$, lead to a logarithmically modified Price's law, for each ℓ . To be concrete, the expected decay rates would then be

$$r\phi_L|_{\mathcal{I}^+} \sim u^{-L-2} \log u, \quad \phi_L|_{\mathcal{H}^+} \sim v^{-2L-3} \log v \quad (3.9.33)$$

near i^+ . Moreover, the leading-order asymptotics should be independent of the extension of the data towards $\mathcal{H}^+$. As has been discussed in §3.1.3, the proof of the above expectation should follow by combining the results of this chapter with those of [AAG21], similarly to how chapter 2 combined the results of chapter 1 with those of [AAG18c, AAG18b], so long as sufficient regularity (depending on L) is assumed. The fixed-regularity problem, on the other hand, seems much more difficult, cf. Conjecture 3.1.1.

3.9.5.2 The case $C_{\text{in}}^{(L,i)} \neq 0$

Notably, the proof presented in this section cannot be directly applied to the case $C_{\text{in}}^{(L,i)} \neq 0$ for $i > 1$, since one would encounter several difficulties related to the quantities $(r^2 \partial_u)^i (r\phi_L)$. (Notice already that the limits $\lim_{u \rightarrow -\infty} [r^2 \partial_u]^i (r\phi_L)$ grow like v^{i-1} for $i = 1, \dots, L$, and like v^{L-1} for $i = L+1$.) Furthermore, working with the quantity $[r^2 \partial_u]^{L+1} (r\phi_L)$ requires strong conformal regularity assumptions. In the next section, we shall therefore obtain asymptotics for much more general data by working only with the quantities $[r^2 \partial_v]^i (r\phi_L)$ and not using the approximate conservation law in v (3.7.17) at all.

3.10 Data on an ingoing null hypersurface $\mathcal{C}_{v=1}$ II

In this final section, we present a different approach towards obtaining the early-time asymptotics of $\partial_v(r\phi_L)$ of solutions ϕ_L arising from polynomially decaying initial data on a null hypersurface $\mathcal{C}_{v=1}$, without requiring any conformal regularity and/or fast decay on initial data. In particular, this section contains the proof of Theorem 3.1.5 from the introduction and can also treat the cases $C_{\text{in}}^{(L,i)} \neq 0$ from the previous section.

Throughout this section, we shall again assume that ϕ is a solution to (3.1.1), supported on a single angular frequency (L, m) with $|m| \leq L$. In the usual abuse of notation, we omit the m -index, that is, we write $\phi = \phi_{Lm} \cdot Y_{Lm} = \phi_L \cdot Y_{Lm}$.

3.10.1 Initial data assumptions

Prescribe smooth characteristic/scattering data for (3.1.1), restricted to the angular frequency (L, m) , that satisfy on $\mathcal{C}_{v=1}$

$$\left| r\phi_L(u, 1) - \frac{C_{\text{in}}}{r^p} \right| = \mathcal{O}_1(r^{-p-\epsilon}) \quad (3.10.1)$$

for some $\epsilon \in (0, 1]$, a constant $C_{\text{in}} > 0$ and for some $p \in \mathbb{N}_0$, and which moreover satisfy

$$\lim_{u \rightarrow -\infty} \partial_v^n(r\phi_L)(u, v) = 0 \quad (3.10.2)$$

for all $v \geq 1$ and for all $n = 1, \dots, L+1$.

Notice that if $p = 1$ in (3.10.1), then this includes the cases $C_{\text{in}}^{(L,i)} \neq 0$ from §3.9.

3.10.2 The main theorem (Theorem 3.10.1)

Theorem 3.10.1. *By standard scattering theory [DRS18], there exists a unique smooth scattering solution $\phi_L \cdot Y_{Lm}$ in $\mathcal{M} \cap \{v \geq 1\}$ attaining the data of §3.10.1.*

Let U_0 be a sufficiently large negative number, let $\mathcal{D} = (-\infty, U_0] \times [1, \infty)$, and let $r_0 := r(u, 1) = |u| - 2M \log |u| + \mathcal{O}(1)$. Then the following statements hold for all $(u, v) \in \mathcal{D}$:

a) We have that:

$$\lim_{v \rightarrow \infty} r\phi_L(u, v) = F(u) = \begin{cases} \mathcal{O}(r_0^{-p-\epsilon}), & \text{if } p \leq L \text{ and } p \neq 0 \\ C_0 \cdot C_{\text{in}} r_0^{-p} + \mathcal{O}(r_0^{-p-\epsilon}), & \text{if } p > L \text{ or } p = 0, \end{cases} \quad (3.10.3)$$

for some smooth function $F(u)$ and some explicit, non-vanishing constant $C_0 = C_0(L, p)$.

b) Moreover, the outgoing derivative of the radiation field $\partial_v(r\phi_L)$ satisfies the following asymptotic expansion if $p < L$:

$$r^2 \partial_v(r\phi_L)(u, v) = \sum_{i=0}^{p-1} \frac{f_i^{(L,p)}(u)}{r^i} + \frac{M \cdot C_1(\log r - \log |u|) + C_2 r_0}{r^p} + \mathcal{O}\left(\frac{|u|^{1-\epsilon}}{r^p}\right), \quad (3.10.4)$$

where the $f_i^{(L,p)}$ are smooth functions of order $f_i^{(L,p)} = \mathcal{O}(r_0^{-p+i+1-\epsilon})$ if $i < p-1$, or of order $f_i^{(L,p)} = C_3^{L,p,i} + \mathcal{O}(r_0^{-\epsilon})$ if $i = p-1$.

On the other hand, if $p \geq L$, then

$$r^2 \partial_v(r\phi_L)(u, v) = \sum_{i=0}^L \frac{f_i^{(L,p)}(u)}{r^i} + \mathcal{O}\left(\frac{\log r}{r^{L+1}}\right), \quad (3.10.5)$$

where the $f_i^{(L,p)}$ are smooth functions of order $f_i^{(L,p)} = \mathcal{O}(r_0^{-p+i+1-\epsilon})$ if $p = L$ and $i < L-1$, and which are given by $f_i^{(L,p)} = C_3^{L,p,i} r_0^{-p+i+1} + \mathcal{O}(r_0^{-p+i-\epsilon})$ otherwise (i.e. if $p = L = i$, if $p = L = i+1$, or if $p > L$).

In each case, we have explicit, non-vanishing expressions for the constants $C_1, C_2, C_3^{L,p,i}$ that depend only on L, p, i, C_{in} (and not on M).

c) Finally, if $p \leq L$, then the following limit exists, is non-vanishing, and is independent of u :

$$\lim_{v \rightarrow \infty} r^{2+p-L} \partial_v \Phi_L(u, v) = I_{\ell=L}^{\text{future}, r^{2+p-L}}[\phi] \neq 0. \quad (3.10.6)$$

If $p = L+1$, then the following limit exists, is non-vanishing, and is independent of u :

$$\lim_{v \rightarrow \infty} \frac{r^3}{\log r} \partial_v \Phi_L(u, v) = I_{\ell=L}^{\text{future}, \frac{\log r}{r^3}}[\phi] \neq 0. \quad (3.10.7)$$

If $p > L+1$, then $\partial_v \Phi_L = \mathcal{O}(r^{-3})$, and all modified Newman–Penrose constants vanish on $\mathcal{I}^+$.

In each case, we have explicit expressions for the constants $I_{\ell=L}^{\text{future}, r^{2+L-p}}[\phi], I_{\ell=L}^{\text{future}, \frac{\log r}{r^3}}[\phi]$. These depend only on L, p, M, C_{in} .

All explicit expressions for constants are listed in the proof of this theorem on page 258.

Remark 3.10.1. Notice that we often expressed u -decay in terms of r_0 rather than u in order to compactly express logarithmic contributions: For instance, if $\epsilon = 1$, and if we express decay directly in terms of u , then we have an additional $\mathcal{O}(|u|^{-p-1} \log |u|)$ -contribution in the second line of (3.10.3).

Remark 3.10.2. The faster decay in (3.10.3) for $p \leq L$ can be traced back to certain cancellations. These already happen for $M = 0$. In fact, we will, in the course of the proof, derive effective expressions for exact solutions to the wave equation on Minkowski which have data $r\phi_L(u, 1) = C_{\text{in}}/r^p$ and satisfy the no incoming radiation condition (3.10.2). (See Proposition 3.10.4.)

Remark 3.10.3. If also the next-to-leading-order behaviour on initial data is specified in (3.10.1), we can upgrade the $\mathcal{O}$ -symbols in (3.10.3) etc. to precise asymptotics, see Corollary 3.10.1.

Remark 3.10.4. With a bit more effort, one can extend the analysis of the proof (using for instance time integrals as was done in chapter 1) to show that (3.10.5) can be improved to

$$r^2 \partial_v(r\phi_L)(u, v) = \sum_{i=0}^{\max(L, p-1)} \frac{f_i^{(L, p)}(u)}{r^i} + \mathcal{O}\left(\frac{\log r}{r^{\max(L+1, p)}}\right).$$

Notice that “the first logarithmic term” of the expansion of $r^2 \partial_v(r\phi_L)$ never appears at order $r^{-L} \log r$: It either appears at order $r^{-L-i} \log r$ or at order $r^{-L+i} \log r$, with $i > 0$. In this sense, there is a cancellation happening at $L = p$. In particular, if $p = 1$, then the expansion of $\partial_v(r\phi_L)$ contains a logarithmic term at order $r^{-3} \log r$ for all $L \neq 1$ (including $L = 0$), whereas the first logarithmic term for $L = 1$ only appears at order $r^{-4} \log r$.

Remark 3.10.5. Using the methods of the proof, one can show a very similar result if one assumes more generally that $0 \leq p \in \mathbb{R}$. (In fact, one should also be able to consider a certain range of positive p !) In this case, however, the cancellation (3.10.3) in general no longer appears; it seems to be a special property of $p \in \{1, \dots, L\}$.

3.10.3 Overview of the proof

In contrast to the proof of §3.9, we will, in this section, only use the approximate conservation law (3.7.10) and obtain an asymptotic estimate for $[r^2 \partial_v]^i(r\phi_L)$ directly from data. For this, we will first need to compute $[r^2 \partial_v]^N(r\phi_L)$ for $N \leq L$ on data, i.e. on $v = 1$, by inductively integrating the relevant equation (3.7.1). This is done in Proposition 3.10.1 in §3.10.4. We then make a bootstrap assumption on the decay of $[r^2 \partial_v]^L(r\phi_L)$ and improve it using (3.7.10). Once we have obtained a sharp estimate on $[r^2 \partial_v]^L(r\phi_L)$ in this way (Proposition 3.10.2 in §3.10.5), we can then inductively integrate from $v = 1$ to obtain a sharp estimate for $[r^2 \partial_v]^{L-i}(r\phi_L)$ (Proposition 3.10.3 in §3.10.6). In doing so, we pick up an “initial data term” with each integration. These data terms will all be of the same order, so there might be cancellations. We will understand these cancellations in Proposition 3.10.4 in §3.10.7. The results are then summarised in Corollary 3.10.1. Finally, the proof of Theorem 3.10.1 is given in §3.10.8.

The disadvantage of this more direct approach to the asymptotics of ∂_v -derivatives of $r\phi_L$ is that we gain no direct information on the asymptotics of ∂_u -derivatives. On the other hand, this should also be seen as an advantage since this approach requires no assumption on the conformal regularity of the initial data on $v = 1$.

3.10.4 Computing transversal derivatives on data

Inserting the initial data bound (3.10.1) into the wave equation (3.7.1) with $N = 0$, and integrating from $u = -\infty$, where $\partial_v(r\phi_L)$ vanishes by the no incoming radiation condition (3.10.2),

we obtain that on $v = 1$

$$\left| \partial_v(r\phi_L) + \frac{L(L+1)C_{\text{in}}}{(2+p-1)r^{1+p}} \right| \leq Cr^{-1-p-\epsilon} \quad (3.10.8)$$

for some constant C . In turn, inserting this estimate into (3.7.1) with $N = 1$, one obtains an estimate for $[r^2\partial_v]^2(r\phi_L)$. Proceeding inductively, one obtains the following

Proposition 3.10.1. *Let ϕ_L be as in Theorem 3.10.1. Then we have on $v = 1$ that*

$$[r^2\partial_v]^{N+1}(r\phi_L) = C_{[r^2\partial_v]^{N+1}}r^{N+1-p} + \mathcal{O}(r^{N+1-p-\epsilon}), \quad \text{for } N = 0, \dots, L-1, \quad (3.10.9)$$

$$[r^2\partial_v]^{L+1}(r\phi_L) = C_{[r^2\partial_v]^{L+1}}r^{L-p} + \mathcal{O}(r^{L-p-\epsilon}). \quad (3.10.10)$$

Here, we defined the constants

$$C_{[r^2\partial_v]^{N+1}} := \frac{p!C_{\text{in}}}{(N+1+p)!} \cdot \prod_{i=0}^N (a_0^i - b_0^i L(L+1)), \quad N = 0, \dots, L-1, \quad (3.10.11)$$

$$C_{[r^2\partial_v]^{L+1}} := \frac{2M}{L+2+p} \left(-c_0^L C_{[r^2\partial_v]^L} + (a_1^L - b_1^L L(L+1) C_{[r^2\partial_v]^{L-1}}) \right). \quad (3.10.12)$$

Proof. Inductively integrate equation (3.7.1). $\square$

3.10.5 Precise leading-order behaviour of $[r^2\partial_v]^L(r\phi_L)$

Equipped with the initial data estimates (3.10.9), we now prove

Proposition 3.10.2. *Let ϕ_L be as in Theorem 3.10.1. Then we have*

$$\begin{aligned} [r^2\partial_v]^L(r\phi_L)(u, v) &= [r^2\partial_v]^L(r\phi_L)(u, 1) \\ &+ 2MC_{[r^2\partial_v]^L} (2x_1^{(L)} - c_0^L) \frac{(L - \min(p, L))!(L+1+p)!}{(2L+2)!} \cdot \begin{cases} \frac{r^{L-1-p}}{(L-1-p)} (1 + \mathcal{O}(\frac{|u|}{r} + \frac{1}{r^\epsilon})), & L > p+1, \\ \log r - \log |u| + \mathcal{O}(1), & L = p+1, \\ \mathcal{O}(|u|^{L-1-p}), & L < p+1. \end{cases} \end{aligned} \quad (3.10.13)$$

If $L = p$, then we can write more precisely:

$$[r^2\partial_v]^L(r\phi_L)(u, v) = [r^2\partial_v]^L(r\phi_L)(u, 1) + 2MC_{[r^2\partial_v]^L} \left(\frac{(2x_1^{(L)} - c_0^L)}{2L+2} - x_1^{(L)} \right) r_0^{-1} + \mathcal{O}(|u|^{-1-\epsilon}). \quad (3.10.14)$$

Notice that if $L < p+1$, then the second line of (3.10.13) decays faster than the first line, whereas if $L \geq p+1$, the second line determines the leading-order r -behaviour.

In principle, we can also compute the case $L < p$ more precisely, but since it has already been dealt with in §3.9, we choose not to. Suffice it to say that if $L < p$, then there will also be

a logarithmic term at order $(\log r - \log |u|)/r^{p+1-L}$, and if $L = p$, there will be a logarithmic term at order $(\log r - \log |u|)/r^{p+2-L}$. Cf. Remark 3.10.4.

Proof. We will prove the proposition by first making a bootstrap assumption to obtain a preliminary estimate on $[r^2\partial_v]^L(r\phi_L)$ (see (3.10.22)), and then using this preliminary estimate to obtain the sharp leading-order decay.

3.10.5.1 A preliminary estimate

We can deduce from the energy estimate from Proposition 3.4.1 that $|r\phi_L| \leq C|u|^{-p}$ for some constant C solely determined by initial data. Cf. Corollary 3.4.1. (This is the reason why we also assumed decay on the first derivative in (3.10.1).) By repeating the calculations done in §3.10.4, we can then derive from $|r\phi_L| \leq C|u|^{-p}$ the estimates

$$[r^2\partial_v]^{N+1}(r\phi_L)(u, v) \leq Cr^{N+1}|u|^{-p} + \mathcal{O}(r^{N+1-p-\epsilon}) \quad (3.10.15)$$

for $N = 0, \dots, L-1$ and another constant C .

Consider now the set X of all $V > 1$ such that the bootstrap assumption

$$\left| [r^2\partial_v]^L(r\phi_L) \right| (u, v) \leq C_{\text{BS}} \max(r^{L-p}, |u|^{L-p}) \quad (3.10.16)$$

holds for all $1 \leq v \leq V$ and for some suitably chosen constant C_{BS} . The max above distinguishes between the cases of growth ($L-p > 0$) and decay ($L-p < 0$). *For easier readability, we will suppose for the next few lines that $L \geq p$. This assumption will be removed in (3.10.22).* In view of the estimate (3.10.15), this set is non-empty provided that C_{BS} is sufficiently large, and it suffices to improve the assumption (3.10.16) within X to deduce that $X = (1, \infty)$. Indeed, if we assume that $\sup_X v$ is finite and improve estimate (3.10.16) within X , then (3.10.15) shows that $\sup_X v + \delta$ would still be in X for sufficiently small δ by continuity. (Here, we used that the RHS of (3.10.15) can be written as $C(V) \cdot \max(r^{N+1-p}, |u|^{N+1-p})$ for some continuous function $C(V)$.)

Let us therefore improve the bound (3.10.16) inside X : First, note that (3.10.16) implies that

$$\left| [r^2\partial_v]^N(r\phi_L) \right| (u, v) \lesssim C_{\text{BS}} \max(r^{N-p}, |u|^{N-p}). \quad (3.10.17)$$

for $N = 0, \dots, L$, where we also used (3.10.9). Recall now equation (3.7.10):

$$\partial_u(r^{-2L}\partial_v\Phi_L) = \sum_{j=0}^L \frac{D}{r^{3+2L}} (2M)^{j+1} [r^2\partial_v]^{L-j}(r\phi_L) \left(2(j+1)x_{j+1}^{(L)} - \sum_{i=0}^j x_i^{(L)} c_{j-i}^{L-i} \right). \quad (3.10.18)$$

As a consequence of (3.10.17) and the bootstrap assumption (3.10.16), the RHS is bounded by r^{-L-3-p} :

$$\left| \partial_u(r^{-2L}\partial_v\Phi_L) \right| \lesssim \frac{2MD}{r^{L+3+p}} \cdot C_{\text{BS}} + \mathcal{O}(r^{-L-4-p}) \lesssim \frac{C_{\text{BS}}}{r^{L+3+p}}. \quad (3.10.19)$$

We integrate this bound in u from $\mathcal{I}^-$, where $r^{-2L}\partial_v\Phi_L = 0$ by the no incoming radiation condition (3.10.2). This yields

$$\left| r^{-2L}\partial_v\Phi_L \right| \lesssim \frac{C_{\text{BS}}}{r^{L+2+p}}. \quad (3.10.20)$$

We now recall the definition of Φ_L from (3.7.8) and estimate the difference $\partial_v\Phi_L - \partial_v[r^2\partial_v]^L(r\phi_L)$ using once more the bootstrap assumption, resulting in the bound

$$\left| \partial_v[r^2\partial_v]^L(r\phi_L) \right| \lesssim C_{\text{BS}}r^{L-2-p}. \quad (3.10.21)$$

Finally, we integrate the bound above from $v = 1$,

$$\left| [r^2\partial_v]^L(r\phi_L)(u, v) - [r^2\partial_v]^L(r\phi_L)(u, 1) \right| \lesssim \begin{cases} C_{\text{BS}}r^{L-1-p}, & L > 1 + p, \\ C_{\text{BS}}(\log r - \log |u|), & L = 1 + p, \\ C_{\text{BS}}|u|^{L-1-p}, & L < 1 + p, \end{cases} \quad (3.10.22)$$

which, combined with the initial data bound (3.10.9), improves the bootstrap assumption. (The third case in (3.10.22) follows from considerations similar to the above.) However, we can already read off from (3.10.22) that, unless $L < 1 + p$, it is actually the RHS of (3.10.22) that determines the leading-order r -behaviour of $[r^2\partial_v]^L(r\phi_L)$, whereas the data term on the LHS only determines the leading-order u -behaviour. We will understand the precise behaviour of the RHS in the next section.

3.10.5.2 Precise leading-order behaviour of $[r^2\partial_v]^L(r\phi_L)$

We again restrict to $p \leq L$ for simpler notation, the only major difference if $p > L$ is explained in Remark 3.10.6. We will also assume for simplicity that $\epsilon < 1$, leaving the case $\epsilon = 1$ to the reader.

In order to find the precise leading-order behaviour of $[r^2\partial_v]^L(r\phi_L)$, we repeat the previous steps, with the difference that we now use the improved estimate²⁵

$$\left| [r^2\partial_v]^L(r\phi_L) - C_{[r^2\partial_v]^L}|u|^{L-p} \right| \leq Cr^{L-1-p}(\delta_{L,p+1}(\log r - \log |u|) + 1) + C|u|^{L-p-\epsilon}, \quad (3.10.23)$$

²⁵The $|u|^{L-p-\epsilon}$ term in the RHS above needs to be replaced by $|u|^{L-1-p} \log |u|$ if $\epsilon = 1$ because $r \sim |u| - 2M \log |u|$ on $v = 1$. This can be fixed by replacing $|u|^{L-p}$ with r_0^{L-p} .

implied by (3.10.22) and (3.10.9), instead of the preliminary estimate (3.10.16). Similarly, we improve the estimate (3.10.17) to

$$\left| [r^2 \partial_v]^N (r \phi_L) \right| \leq C |u|^{N-p} + C r^{N-1-p} (\log r - \log |u| + 1) \quad (3.10.24)$$

for all $N = 0, \dots, L-1$. Inserting these two bounds into (3.7.10), we obtain

$$\begin{aligned} \partial_u (r^{-2L} \partial_v \Phi_L) &= \frac{2MD}{r^{2L+3}} (2x_1^{(L)} - x_0^{(L)} c_0^L) \cdot C_{[r^2 \partial_v]^L} |u|^{L-p} \\ &\quad + \mathcal{O} \left(r^{-L-4-p} (\log r - \log |u| + 1) + r^{-2L-3} |u|^{L-p-\epsilon} \right). \end{aligned} \quad (3.10.25)$$

Integrating the above in u gives

$$r^{-2L} \partial_v \Phi_L = \int_{-\infty}^u \frac{2MD}{r^{2L+3}} (2x_1^{(L)} - x_0^{(L)} c_0^L) \cdot C_{[r^2 \partial_v]^L} |u'|^{L-p} du' + \mathcal{O}(r^{L-2-p-\epsilon}). \quad (3.10.26)$$

On the LHS of (3.10.26), we have

$$r^{-2L} \partial_v \Phi_L = r^{-2L} \partial_v [r^2 \partial_v]^L (r \phi_L) + 2M x_1^{(L)} \frac{C_{[r^2 \partial_v]^L} |u|^{L-p}}{r^{2L+2}} + \mathcal{O} \left(\frac{r^{L-1-p} + \log r + |u|^{L-p-\epsilon}}{r^{2L+2}} \right). \quad (3.10.27)$$

In order to estimate the RHS of (3.10.26), we recall that $x_0^{(L)} = 1$, and compute the integral using the following

Lemma 3.10.1. *Let $N, N' \in \mathbb{N}$ with $N > N' + 1$. Then*

$$(N-1) \int_{-\infty}^u \frac{|u'|^{N'}}{|r|^N} du' = \sum_{k=0}^{N'} \frac{|u|^{N'-k}}{|r|^{N-1-k}} \prod_{j=1}^k \frac{N'+1-j}{N-1-j} + \mathcal{O}(r^{N'-N}). \quad (3.10.28)$$

Proof. The proof proceeds almost identically to the proof of Lemma 3.9.1. Alternatively, one can also compute the integral directly by writing $|u'| = r + v + \mathcal{O}(\log r)$. This latter approach is also useful for $N' \notin \mathbb{N}$. $\square$

Remark 3.10.6. When considering the case $p > L$, then the lemma above slightly changes (i.e. for $N' < 0$). While it is trivial to obtain the $|u|^{L-1-p}$ -decay claimed in (3.10.13), one can also obtain a more precise statement: In fact, if $N' < 0$, then the above integral is precisely the one that gave rise to the logarithmic terms in the previous sections §3.4 and §3.9 (see, for instance, eq. (3.4.26)). In particular, if $N > 0 > N'$, the integrals of (3.10.28) will lead to logarithmic terms at order $(\log r - \log |u|)/r^{N-N'-1}$.

Applying Lemma 3.10.1 (with $N' = L - p$ and $N = 2L - 3$) to (3.10.26), we obtain

$$r^{-2L} \partial_v \Phi_L = 2MC_{[r^2 \partial_v]^L} (2x_1^{(L)} - c_0^L) \frac{(L-p)!}{(2L+2) \cdots (L+2+p)} \cdot \frac{1}{r^{L+2+p}} \left(1 + \mathcal{O} \left(\frac{|u|}{r} + \frac{1}{r^\epsilon} \right) \right) \quad (3.10.29)$$

where we used that $|u|^q/r^q \leq 1$ for any $q > 0$. Finally, using (3.10.27) to write $\partial_v \Phi_L \sim \partial_v [r^2 \partial_v]^L (r\phi_L)$, and integrating from $v = 1$, we obtain, if $L > 1 + p$,

$$\begin{aligned} [r^2 \partial_v]^L (r\phi_L)(u, v) &= [r^2 \partial_v]^L (r\phi_L)(u, 1) \\ &+ 2MC_{[r^2 \partial_v]^L} (2x_1^{(L)} - c_0^L) \frac{(L-p)!(L+1+p)!}{(L-1-p)(2L+2)!} \cdot r^{L-1-p} \left(1 + \mathcal{O} \left(\frac{|u|}{r} + \frac{1}{r^\epsilon} \right) \right). \end{aligned} \quad (3.10.30)$$

On the other hand, if $L = 1 + p$, then we obtain

$$\begin{aligned} [r^2 \partial_v]^L (r\phi_L)(u, v) &= [r^2 \partial_v]^L (r\phi_L)(u, 1) \\ &+ 2MC_{[r^2 \partial_v]^L} (2x_1^{(L)} - c_0^L) \frac{(L-p)!(L+1+p)!}{(2L+2)!} \cdot (\log r - \log |u|) + \mathcal{O}(1) \end{aligned} \quad (3.10.31)$$

where we used that $\log r(u, 1) = \log |u| + \mathcal{O}(|u|^{-1})$.

The case $L < 1 + p$ follows in much the same way. The only difference is that one now also needs to take the second term on the RHS of (3.10.27) into account since it will give a contribution of the same order as the $r^{-2L} \partial_v \Phi_L$ -term. For the latter, one can derive an estimate similar to (3.10.29). This concludes the proof of Proposition 3.10.2. $\square$

3.10.6 Precise leading-order behaviour of $[r^2 \partial_v]^{L-i} (r\phi_L)$

Proposition 3.10.3. *Let ϕ_L be as in Theorem 3.10.1, and let $0 \leq j \leq L$. Then we have for $j < L - 1 - p$:*

$$\begin{aligned} [r^2 \partial_v]^{L-j} (r\phi_L)(u, v) &= \text{data}_{L-j} \\ &+ 2MC_{[r^2 \partial_v]^L} (2x_1^{(L)} - c_0^L) \frac{(L-2-p-j)!(L-p)(L+1+p)!}{(2L+2)!} \cdot r^{L-1-p-j} \left(1 + \mathcal{O} \left(\frac{|u|}{r} + \frac{1}{r^\epsilon} \right) \right) \end{aligned} \quad (3.10.32)$$

On the other hand, if $j = L - 1 - p$, we have

$$\begin{aligned} [r^2 \partial_v]^{p+1} (r\phi_L)(u, v) &= \text{data}_{p+1} \\ &+ 2MC_{[r^2 \partial_v]^L} (2x_1^{(L)} - c_0^L) \frac{(L-p)(L+1+p)!}{(2L+2)!} \cdot (\log r - \log |u|) + \mathcal{O}(1). \end{aligned} \quad (3.10.33)$$

Finally, if $j > L - 1 - p$, we have

$$[r^2 \partial_v]^{L-j}(r\phi_L)(u, v) = \text{data}_{L-j} + \mathcal{O}(|u|^{-j+(L-1-p)}). \quad (3.10.34)$$

Moreover, if also $p \leq L - 1$, then $[r^2 \partial_v]^{L-j}(r\phi_L)$ possesses an asymptotic expansion in powers of $1/r$ up to r^{-j+L-p} , with a logarithmic term appearing at order

$$2MC_{[r^2 \partial_v]^L}(2x_1^{(L)} - c_0^L) \frac{(L-p)!(L+p+1)!}{(L-p-1)!(2L+2)!} \frac{(-1)^{j-(L-1-p)}}{(j-(L-1-p))!} \cdot \frac{\log r - \log |u|}{r^{j-(L-1-p)}}. \quad (3.10.35)$$

In the above, the expression data_{L-j} is shorthand for

$$\begin{aligned} \text{data}_{L-j} := & [r^2 \partial_v]^{L-j}(r\phi_L)(u, 1) \\ & + \underbrace{\sum_{i=1}^j \int_{r(u,1)}^{r(u,v)} \frac{1}{Dr_{(i)}^2} \cdots \int_{r(u,1)}^{r(2)} \frac{1}{Dr_{(1)}^2} dr_{(1)} \cdots dr_{(i)}}_{i \text{ integrals}} \cdot [r^2 \partial_v]^{L-j+i}(r\phi_L)(u, 1). \end{aligned} \quad (3.10.36)$$

Remark 3.10.7. Notice that (3.10.35) only holds for $p \leq L - 1$. Indeed, we already know from the results of §3.9 (or §3.4 for $L = 1$) that if $p = L$, then the first logarithmic term in the expansion of $[r^2 \partial_v]^{L-j}(r\phi_L)$ will appear at order $\frac{\log r - \log |u|}{r^{j-(L-2-p)}}$. For $p > L$, in contrast, one can show the first logarithmic term will appear at $\frac{\log r - \log |u|}{r^{j-(L-1-p)}}$, although we won't show this here. (Again, we have in fact already shown this for $p = L + 1$ in §3.9.) In this sense, there is a cancellation happening at $p = L$.

Proof. We simply need to integrate (3.10.30) (or, in general, the bound (3.10.13)) j times from $v = 1$, using at each step the initial data bounds (3.10.9). If $j < L - 1 - p$, one obtains inductively that

$$\begin{aligned} [r^2 \partial_v]^{L-j}(r\phi_L)(u, v) &= [r^2 \partial_v]^{L-j}(r\phi_L)(u, 1) \\ &+ \sum_{i=1}^j \int_{r(u,1)}^{r(u,v)} \frac{1}{Dr_{(i)}^2} \int_{r(u,1)}^{r(i)} \frac{1}{Dr_{(i-1)}^2} \cdots \int_{r(u,1)}^{r(2)} \frac{1}{Dr_{(1)}^2} dr_{(1)} \cdots dr_{i-1} dr_{(i)} [r^2 \partial_v]^{L-j+i}(r\phi_L)(u, 1) \\ &+ 2MC_{[r^2 \partial_v]^L}(2x_1^{(L)} - c_0^L) \frac{(L-p)!(L+1+p)! \cdot r^{L-1-p-j}}{(L-1-p) \cdots (L-1-p-j) \cdot (2L+2)!} \cdot \left(1 + \mathcal{O}\left(\frac{|u|}{r} + \frac{1}{r^\epsilon}\right)\right). \end{aligned} \quad (3.10.37)$$

Here, the terms in the second line come from integrating the initial data contributions (divided by r^2) in the estimates for $[r^2 \partial_v]^{L-j+i}(r\phi_L)$. Notice that all these terms have the same leading-order u -decay, so there might be cancellations between them. We will return to this in §3.10.7. For now, we simply leave them as they are and write them as data_{L-j} .

From (3.10.37), one deduces that if $j = L - 1 - p$:

$$\begin{aligned} [r^2 \partial_v]^{p+1}(r\phi_L)(u, v) &= \text{data}_{p+1} \\ &+ 2MC_{[r^2 \partial_v]^L}(2x_1^{(L)} - c_0^L) \frac{(L-p)!(L+1+p)!}{(L-1-p)!(2L+2)!} \cdot (\log r - \log |u|) + \mathcal{O}(1). \end{aligned} \quad (3.10.38)$$

Assuming that $L - 1 - p \geq 0$, we finally integrate (3.10.38) again from $v = 1$ (and write $j' := j - (L - 1 - p)$) to obtain that, for all $L \geq j \geq L - 1 - p$:

$$\begin{aligned} [r^2 \partial_v]^{L-j}(r\phi_L)(u, v) &= \text{data}_{L-j} \\ &+ 2MC_{[r^2 \partial_v]^L}(2x_1^{(L)} - c_0^L) \frac{(L-p)!(L+p+1)!}{(L-p-1)!(2L+2)!} \frac{(-1)^{j'}}{j'!} \cdot \left(\frac{\log r - \log |u|}{r^{j'}} \right) + \mathcal{O}(|u|^{-j'}), \end{aligned} \quad (3.10.39)$$

where we inductively used that, for any $q > 0$,

$$(q-1) \int_{r(u,1)}^{r(u,v)} \frac{\log r - \log |u|}{r^q} dr = -\frac{\log r - \log |u|}{r^{q-1}} + \frac{1}{|u|^{q-1}} - \frac{1}{r^{q-1}}.$$

Notice that, for $j > L - 1 - p$, in contrast to (3.10.37) and (3.10.38), the leading-order r -decay of $[r^2 \partial_v]^{p+1-j}(r\phi_L)$ is no longer determined by the second line of (3.10.39), but by the first line, namely the initial data terms. (If $j = L - p$, the second line still provides the next-to-leading-order behaviour in r .) To nevertheless prove the fourth claim (3.10.35) of the proposition, one can simply obtain an analogue of (3.10.39) by integrating the estimate (3.10.38) j times from *future null infinity*, rather than from $v = 1$. This concludes the proof. $\square$

Setting $j = L$ in the above, we get

$$\begin{aligned} r\phi_L(u, v) &= r\phi_L(u, 1) + \sum_{i=1}^L \int_{r(u,1)}^{r(u,v)} \frac{1}{Dr_{(i)}^2} \cdots \int_{r(u,1)}^{r_{(2)}} \frac{1}{Dr_{(1)}^2} dr_{(1)} \cdots dr_{(i)} [r^2 \partial_v]^i (r\phi_L)(u, 1) \\ &+ \mathcal{O}\left(\frac{\log r - \log |u|}{r^{p+1}}\right) + \mathcal{O}\left(\frac{1}{|u|^{p+1}}\right). \end{aligned} \quad (3.10.40)$$

In view of (3.10.9), the above estimate shows that $r\phi_L = C|u|^{-p} + \mathcal{O}(r^{-1}|u|^{-p+1})$ for some constant C , however, this constant C might potentially be zero. Indeed, we already know that this is what happens in the case $L = 1 = p$ (discussed in §3.4), to which, in fact, (3.10.40) applies. (Recall that we showed that $r\phi_1 \sim 1/r + 1/|u|^2$ if $r\phi_1 \sim 1/|u|$ initially.) We discuss these potential cancellations now.

3.10.7 Cancellations in the initial data contributions

We now analyse the $v = 1$ -contributions data_{L-j} in the first (and second) line(s) of (3.10.37)–(3.10.39) in more detail. Define $r(u, 1) = r_0(u)$. We will prove the following

Proposition 3.10.4. *Let $0 \leq j \leq L$. The expression data_{L-j} defined in (3.10.36) evaluates to²⁶*

$$\text{data}_{L-j} = C_{\text{in}} r_0^{L-p-j} \sum_{n=0}^j S_{L,p,j,n} \left(\frac{r(u,1)}{r(u,v)} \right)^n + \mathcal{O}(r_0^{L-p-j-\epsilon}), \quad (3.10.41)$$

where the $S_{L,p,j,n}$ are constants that are computed explicitly in eq. (3.10.53). They never vanish if $p > L$. However, if $p \leq L$, then they vanish if and only if $L - p + n + 1 \leq j \leq L$.

Remark 3.10.8. The computations required for the proof of the above are completely Minkowskian. This is to be understood in the sense that the M -dependence of (3.10.41) is entirely contained in the $\mathcal{O}(r_0^{L-p-j-\epsilon})$ term. In fact, the above proposition provides us with exact solutions to the linear wave equation on Minkowski that arise from initial data $r\phi_L(u, 1) = C_{\text{in}} r_0^{-p}$ and the no incoming radiation condition (3.10.2).

Proof. We first require an expression for the integrals in data_{L-j} . For this, we prove

Lemma 3.10.2. *Let $k \in \mathbb{N}$. Then*

$$\underbrace{\int_{r(u,1)}^{r(u,v)} \frac{1}{r_{(k)}^2} \cdots \int_{r(u,1)}^{r_{(2)}} \frac{1}{r_{(1)}^2} dr_{(1)} \cdots dr_{(k)}}_{k \text{ integrals}} = \frac{1}{k!} \left(\frac{1}{r(u,1)} - \frac{1}{r(u,v)} \right)^k. \quad (3.10.42)$$

Proof. The proof is deferred to the appendix 3.A.4. □

Equipped with this lemma, we can write the data contributions in the estimates of Proposition 3.10.3, namely

$$\text{data}_{L-j} = \sum_{i=0}^j \int_{r(u,1)}^{r(u,v)} \frac{1}{Dr_{(i)}^2} \cdots \int_{r(u,1)}^{r_{(2)}} \frac{1}{Dr_{(1)}^2} dr_{(1)} \cdots dr_{(i)} [r^2 \partial_v]^{L-j+i} (r\phi_L)(u, 1), \quad (3.10.43)$$

as follows, writing from now on $r = r(u, v)$ and $r_0 = r(u, 1)$ and estimating the D^{-1} -terms against $1 + \mathcal{O}(r_0^{-1})$:

$$\text{data}_{L-j} = \sum_{i=0}^j \frac{1}{i!} \left(\frac{1}{r_0} - \frac{1}{r} \right)^i [r^2 \partial_v]^{L-j+i} (r\phi_L)(u, 1) \cdot \left(1 + \mathcal{O}(r_0^{-1}) \right). \quad (3.10.44)$$

We now insert the estimates (3.10.9) to write this as

$$\text{data}_{L-j} = \sum_{i=0}^j \frac{p! \prod_{k=0}^{L-j+i-1} (a_0^k - L(L+1))}{i! (L+p-j+i)!} \left(\frac{1}{r_0} - \frac{1}{r} \right)^i \cdot C_{\text{in}} r_0^{L-p-j+i} \left(1 + \mathcal{O}(r_0^{-\epsilon}) \right), \quad (3.10.45)$$

²⁶Note that since we express u -decay in terms of r_0 , (3.10.41) holds for all $\epsilon \leq 1$. Cf. footnote 25.

where we used that $b_0^k = 1$. By noting that

$$a_0^k - L(L+1) = k(k+1) - L(L+1) = -(L+k+1)(L-k),$$

we can further express the product as

$$\prod_{k=0}^{L-j+i-1} (a_0^k - L(L+1)) = (-1)^{L-j+i} \frac{(2L-j+i)!L!}{L!(j-i)!} = (-1)^{L-j+i} \frac{(2L-j+i)!}{(j-i)!}. \quad (3.10.46)$$

This yields

$$\text{data}_{L-j} = \sum_{i=0}^j \frac{(-1)^{L-j+i} p! (2L-j+i)!}{i! (L+p-j+i)! (j-i)!} \left(\frac{1}{r_0} - \frac{1}{r} \right)^i \cdot C_{\text{in}} r_0^{L-p-j+i} \left(1 + \mathcal{O}(r_0^{-\epsilon}) \right). \quad (3.10.47)$$

A cancellation at leading-order, i.e. at order r_0^{L-p-j} , takes place if the sum

$$\sum_{i=0}^j (-1)^{L-j+i} \frac{p! (2L-j+i)!}{i! (L+p-j+i)! (j-i)!} =: (-1)^{L-j} p! \cdot \text{sum}(L, p, j) \quad (3.10.48)$$

vanishes. To understand when this happens, we prove the following

Lemma 3.10.3. *Let $0 \leq j \leq L$. If $p > L$ or $p = 0$, then $\text{sum}(L, p, j)$ never vanishes. If $0 < p \leq L$, then $\text{sum}(L, p, j)$ vanishes if and only if $j \in \{L-p+1, \dots, L\}$.*

More precisely, if $p > L$, then

$$\text{sum}(L, p, j) = \overbrace{\int_0^1 \int_0^{x_{p-L}} \dots \int_0^{x_2}}^{p-L \text{ integrals}} x_1^{2L-j} \frac{(1-x_1)^j}{j!} dx_1 \dots dx_{p-L-1} dx_{p-L}, \quad (3.10.49)$$

which is manifestly positive. On the other hand, if $p \leq L$, then

$$\text{sum}(L, p, j) = (-1)^j \binom{L-p}{L-p-j} \frac{(2L-j)!}{(L+p)!}, \quad (3.10.50)$$

where we use the convention that $\binom{n}{k} = 0$ if $k < 0 \leq n$.

In fact, equation (3.10.50) also applies to $p > L$ if we define in the standard way $\binom{L-p}{L-p-j} := (-1)^j \binom{p-L+j-1}{j}$.

Proof. The proof is deferred to the appendix 3.A.5. Notice, however, that one can make certain soft statements without having to do any computations. For instance, if we consider the case $p = 1$ and suppose there are no cancellations for $j = L$, then we would obtain from (3.10.40) an estimate of the form $r\phi_L = C/|u| + \mathcal{O}(1/r + 1/|u|^2)$. Inserting this into the wave equation (3.7.1) with $N = 0$ would then give that $\partial_v(r\phi_L) \sim \log r/r^2$, a contradiction to the estimate (3.10.39)

for $j = L - 1$. Thus, there has to be a cancellation at $j = 1$; in other words, $\text{sum}(L, 1, 1) = 0$. However, we here choose to calculate the sums explicitly. $\square$

Lemma 3.10.3 provides us with an understanding of cancellations at leading-order, i.e. at order r_0^{L-p-j} . Similarly, we can understand cancellations at higher order in (3.10.47), say at order $r_0^{L-p-j+n}r^{-n}$, by considering the corresponding sum

$$\sum_{i=0}^j (-1)^n \binom{i}{n} (-1)^{L-j+i} \frac{p!(2L-j+i)!}{i!(L+p-j+i)!(j-i)!} =: (-1)^{L-j} p! \cdot \text{sum}(L, p, j, n). \quad (3.10.51)$$

Understanding this sum is straight-forward: We have

$$\begin{aligned} \text{sum}(L, p, j, n) &= \sum_{i=0}^j (-1)^n \binom{i}{n} (-1)^i \frac{(2L-j+i)!}{i!(L+p-j+i)!(j-i)!} \\ &= \frac{1}{n!} \sum_{i=n}^j (-1)^{i-n} \frac{(2L-j+i)!}{(i-n)!(L+p-j+i)!(j-i)!} \\ &= \frac{1}{n!} \sum_{i=0}^{j-n} (-1)^i \frac{(2L-j+i+n)!}{i!(L+p-j+i+n)!(j-i-n)!}, \end{aligned}$$

and thus

$$\text{sum}(L, p, j, n) = \frac{1}{n!} \text{sum}(L, p, j-n), \quad (3.10.52)$$

where we set $\text{sum}(L, p, j) = 0$ if $j < 0$. In particular, in view of Lemma 3.10.3 above, we obtain that if $j \geq n$ and $p > L$, no cancellations occur. On the other hand, if $j \geq n$ and $p \leq L$, then cancellations occur if and only if $L-p+n+1 \leq j \leq L$ and $j-n \geq 1$.

This concludes the proof of Proposition 3.10.4, with the constants $S_{L,p,j,n}$ being given by

$$S_{L,p,j,n} = (-1)^{L-j} \frac{p!}{n!} \text{sum}(L, p, j-n), \quad (3.10.53)$$

where $\text{sum}(L, p, j-n)$ is computed explicitly in Lemma 3.10.3. $\square$

3.10.8 Summary and proof of Theorem 3.10.1

We can roughly (and schematically) summarise the results obtained so far as

$$[r^2 \partial_v]^{L-j} (r \phi_L)(u, v) \sim \begin{cases} |u|^{L-p-j}, & \text{if } L < p \text{ or } j = L-p \geq 0, \\ |u|^{L-p-j} (|u|^{-\epsilon} + |u| r^{-1}), & \text{if } L \geq p \text{ and } j \geq L-p+1, \\ \log r - \log |u| + |u|, & \text{if } L > p \text{ and } j = L-p-1, \\ r^{L-1-p-j} + |u|^{L-p-j}, & \text{if } L > p \text{ and } 0 \leq j < L-p-1. \end{cases} \quad (3.10.54)$$

The first case follows from estimate (3.10.34) from Proposition 3.10.3 and the fact that *there are no* cancellations in the data term data_{L-j} in view of Proposition 3.10.4. The leading-order behaviour is thus entirely determined by the data.

In contrast, the second case follows from (3.10.34) from Proposition 3.10.3 and the fact that *there are* cancellations in the data term data_{L-j} in view of Proposition 3.10.4. Notice moreover that if $\epsilon < 1$, then the leading-order behaviour will only have contributions from the data. (To see this, one needs to repeat the calculations of Proposition 3.10.1, taking into account also the subleading terms.) If $\epsilon = 1$, then there will, in addition, be contributions from the $\mathcal{O}$ -terms in (3.10.34). Note that, if desired, all of these contributions can be computed explicitly by following the steps above but without discarding the subleading terms.

The third case follows from (3.10.33) from Proposition 3.10.3, with the $|u|$ -term coming again from the data contribution data_{L-j} , which contains no cancellations in view of Proposition 3.10.4.

The fourth case follows in the same way from (3.10.32) from Proposition 3.10.3, with the $|u|^{L-p-j}$ -term coming again from the data contribution data_{L-j} , which contains no cancellations in view of Proposition 3.10.4.

More precisely, we have the following

Corollary 3.10.1. *Let ϕ_L and $\mathcal{D}$ be as in Theorem 3.10.1, and recall that $r_0 := r(u, 1) = |u| - 2M \log |u| + \mathcal{O}(1)$, as well as the constants $S_{L,p,j,n}$ defined in (3.10.53).*

1.) *If $L < p$ and $j \geq 0$, or if $j = L - p \geq 0$, then we have throughout $\mathcal{D}$:*

$$[r^2 \partial_v]^{L-j}(r\phi_L) = C_{\text{in}} S_{L,p,j,0} \cdot r_0^{L-p-j} (1 + \mathcal{O}(|u|r^{-1} + |u|^{-\epsilon})). \quad (3.10.55)$$

2.) *If $L \geq p$ and $j \geq L - p + 1$, then we have throughout $\mathcal{D}$:*

$$[r^2 \partial_v]^{L-j}(r\phi_L) = \mathcal{O}(r_0^{L-p-j-\epsilon}) + \mathcal{O}(r^{-1}|u|^{L-p-j+1}). \quad (3.10.56)$$

Indeed, if we suppose instead of (3.10.1) that

$$|r\phi - C_{\text{in}} r^{-p} + C_{\text{in},\epsilon} r^{-p-\epsilon}| \leq C r^{-p-\epsilon'} \quad (3.10.57)$$

for some constants $C, C_{\text{in},\epsilon}$ and for some $0 < \epsilon \leq 1 < \epsilon'$, then we have

$$[r^2 \partial_v]^{L-j}(r\phi_L) = \tilde{C} \cdot r_0^{L-p-j-\epsilon} + \mathcal{O}(|u|^{L-p-j-1}) + \mathcal{O}(r^{-1}|u|^{L-p-j+1}) \quad (3.10.58)$$

for some constant $\tilde{C} = \tilde{C}(L, p, j, \epsilon, M, C_{\text{in}}, C_{\text{in},\epsilon})$ which we can compute explicitly.

3.) *If $L \geq p$ and $j = L - p - 1$, then we have throughout $\mathcal{D}$:*

$$[r^2 \partial_v]^{p+1}(r\phi_L) = C'_1 \cdot (\log r - \log |u|) + C_{\text{in}} S_{L,p,L-p-1,0} \cdot r_0 + \mathcal{O}(|u|^{1-\epsilon}), \quad (3.10.59)$$

with the constant C'_1 being given by

$$C'_1 = (-1)^L \cdot 2M(2x_1^{(L)} - c_0^L) \cdot \frac{p!(L-p)(L+1+p)}{(2L+2)(2L+1)} \cdot C_{\text{in}}. \quad (3.10.60)$$

4.) If $L > p$ and $0 \leq j \leq L - p - 2$, then we have throughout $\mathcal{D}$:

$$[r^2 \partial_v]^{L-j}(r\phi_L) = C'_2 \cdot r^{L-1-p-j} + C_{\text{in}} S_{L,p,j,0} \cdot r_0^{L-p-j} + \mathcal{O}(|u|^{L-p-j-\epsilon}) + \mathcal{O}(r^{L-2-p-j}|u|), \quad (3.10.61)$$

with the constant C'_2 being given by $C'_2 = (L-2-p-j)! \cdot C'_1$.

Proof. The proof is obtained by combining the results of Propositions 3.10.1–3.10.4 in the manner described above (below (3.10.54)). Notice that we expressed the constant $C_{[r^2 \partial_v]^L}$ appearing in (3.10.32) and (3.10.33) (and defined in (3.10.11)) as $C_{[r^2 \partial_v]^L} = (-1)^L \frac{p!(2L)!}{(L+p)!} \cdot C_{\text{in}}$, which follows from (3.10.46). $\square$

Proof of Theorem 3.10.1

Proof of Theorem 3.10.1. Part a) of the theorem follows directly from Corollary 3.10.1, with C_0 given by $C_0 = S_{L,p,L,0}$.

The first part of b) follows by dividing (3.10.59) by r^2 , integrating from $\mathcal{I}^+$, and repeating the procedure $p-1$ terms. The boundary terms $\lim_{v \rightarrow \infty} [r^2 \partial_v]^{L-p-i}(u, v)$ we pick up with each integration are estimated via (3.10.55) or (3.10.56), thus giving rise to the functions $f_i^{(L,p)}$ and their leading-order behaviour. This proves (3.10.4), with the constants C_1, C_2 given by

$$MC_1 = \frac{(-1)^p}{p!} C'_1, \quad C_2 = \frac{(-1)^p}{p!} S_{L,p,L-p-1,0} \cdot C_{\text{in}}. \quad (3.10.62)$$

The second part of b) follows similarly: We take (3.10.55) with $j=0$ and integrate $L-1$ times from $\mathcal{I}^+$, using at each step either (3.10.55) or (3.10.56) to estimate the boundary terms on $\mathcal{I}^+$. This alone only gives an expansion of $r^2 \partial_v(r\phi_L)$ up to r^{1-L} . The higher-order behaviour can be obtained by also taking into account the estimate for $\partial_v[r^2 \partial_v]^L(r\phi_L)$ implied by the estimates (3.10.27) and (3.10.29). One can obtain expressions for the constants $C_3^{L,p,i}$ in much the same way as for C_1, C_2 , using also (3.10.27) and (3.10.29) for $i=L$.

Finally, part c) of Theorem 3.10.1 follows by (in the case $p=L+1$ a slightly modified version of) (3.10.29). We have for $p \leq L$ (recall $C_{[r^2 \partial_v]^L} = (-1)^L \frac{p!(2L)!}{(L+p)!} \cdot C_{\text{in}}$):

$$I_{\ell=L}^{\text{future}, r^{2+p-L}}[\phi] = 2MC_{[r^2 \partial_v]^L}(2x_1^{(L)} - c_0^L) \frac{(L-p)!}{(2L+2) \cdots (L+2+p)}, \quad (3.10.63)$$

and for $p=L+1$:

$$I_{\ell=L}^{\text{future}, \frac{\log r}{r^3}}[\phi] = 2MC_{[r^2 \partial_v]^L}(2x_1^{(L)} - c_0^L). \quad (3.10.64)$$

This concludes the proof of Theorem 3.10.1. $\square$

3.10.9 Comments: More severe modifications to Price's law

We have already discussed in detail in §3.9.5 that we expect to obtain a *logarithmically modified Price's law* for each ℓ provided that one smoothly extends the data to the event horizon and that $p = L + 1$ in (3.10.1), which, in turn, is the decay predicted by the results of §3.8. On the other hand, in view of equation (3.10.6), one can expect that the modification to Price's law is much more severe for $p \leq L > 0$: Indeed, we expect that if $p \leq L$, then one obtains asymptotics near i^+ that are $L - p + 1$ powers worse than in the case of smooth compactly supported data, i.e., we expect that

$$r\phi_L|_{\mathcal{I}^+} \sim u^{-L-2+(L-p+1)} = u^{-1-p}, \quad \phi_L|_{\mathcal{H}^+} \sim v^{-2L-3+(L-p+1)} = v^{-L-p-2} \quad (3.10.65)$$

near i^+ . The reader should compare this to the behaviour of the $\ell = 0$ -mode for $p = 1$,

$$r\phi_0|_{\mathcal{I}^+} \sim u^{-2} \log u, \quad \phi_0|_{\mathcal{H}^+} \sim v^{-3} \log v, \quad (3.10.66)$$

which was proved in chapter 2.

We again refer the reader to §3.1.3 and Conjecture 3.1.2 therein for a more detailed discussion.

Appendix 3.A Proofs omitted from the chapter's main body

This appendix contains various proofs which have been omitted in the main body of the chapter.

3.A.1 Proofs of Propositions 3.7.1 and 3.7.2

Proof of Proposition 3.7.1. We prove (3.7.1) by induction, noting that it is true for $N = 0$ with $a_0^0 = 0$, $b_0^0 = 1 = c_0^0$. Assume now that (3.7.1) holds for a fixed N . We have

$$\begin{aligned} \partial_v \partial_u [r^2 \partial_v]^{N+1} (r\phi) &= \partial_v \left(r^2 \partial_u \partial_v [r^2 \partial_v]^N (r\phi) \right) + \partial_v \left(-2r^2 \cdot \frac{D}{r} \partial_v [r^2 \partial_v]^N (r\phi) \right) \\ &= \partial_v \left(r^2 \partial_u \partial_v [r^2 \partial_v]^N (r\phi) \right) - \frac{2D}{r} \partial_v [r^2 \partial_v]^{N+1} (r\phi) + \left(1 - \frac{4M}{r} \right) \frac{2D}{r^2} [r^2 \partial_v]^{N+1} (r\phi). \end{aligned} \quad (3.A.1)$$

Using the induction hypothesis, we compute the first term on the RHS according to

$$\begin{aligned} &\partial_v \left(r^2 \partial_u \partial_v [r^2 \partial_v]^N (r\phi) \right) \\ &= -2N \partial_v \left(r^2 \frac{D}{r} \partial_v [r^2 \partial_v]^N (r\phi) \right) + \partial_v \left(\sum_{j=0}^N D(2M)^j \left((a_j^N + b_j^N \Delta_{\mathbb{S}^2} - c_j^N \cdot \frac{2M}{r}) [r^2 \partial_v]^{N-j} (r\phi) \right) \right) \\ &= -\frac{2ND}{r} \partial_v [r^2 \partial_v]^{N+1} (r\phi) + \frac{2ND}{r^2} \left(1 - \frac{4M}{r} \right) \cdot [r^2 \partial_v]^{N+1} (r\phi) \\ &\quad + \sum_{j=0}^N \frac{D(2M)^j}{r^2} \left(a_j^N + b_j^N \Delta_{\mathbb{S}^2} - c_j^N \cdot \frac{2M}{r} \right) [r^2 \partial_v]^{N-j+1} (r\phi) \\ &\quad + \sum_{j=0}^N \frac{D(2M)^{j+1}}{r^2} \left(a_j^N + b_j^N \Delta_{\mathbb{S}^2} - c_j^N \cdot \frac{2M}{r} + c_j^N - \frac{2M}{r} c_j^N \right) [r^2 \partial_v]^{N-j} (r\phi) \\ &= -\frac{2ND}{r} \partial_v [r^2 \partial_v]^{N+1} (r\phi) \\ &\quad + \sum_{j=1}^{N+1} \frac{D(2M)^j}{r^2} \left((a_{j-1}^N + c_{j-1}^N + a_j^N) + (b_{j-1}^N + b_j^N) \Delta_{\mathbb{S}^2} - (c_j^N + 2c_{j-1}^N) \cdot \frac{2M}{r} \right) [r^2 \partial_v]^{N+1-j} (r\phi) \\ &\quad + \frac{D}{r^2} \left((a_0^N + 2N) - b_0^N \Delta_{\mathbb{S}^2} - (c_0^N + 4N) \cdot \frac{2M}{r} \right) [r^2 \partial_v]^{N+1} (r\phi), \end{aligned} \quad (3.A.2)$$

where we defined $a_j^N, b_j^N, c_j^N := 0$ for $j > N$.

Plugging the above equation back into (3.A.1), we thus obtain

$$\begin{aligned} \partial_u \partial_v [r^2 \partial_v]^{N+1} (r\phi) &= -\frac{2D(N+1)}{r} \partial_v [r^2 \partial_v]^{N+1} (r\phi) \\ &\quad + \sum_{j=0}^{N+1} \frac{D}{r^2} (2M)^j \left(a_j^{N+1} + b_j^{N+1} \Delta_{\mathbb{S}^2} - c_j^{N+1} \cdot \frac{2M}{r} \right) [r^2 \partial_v]^{N+1-j} (r\phi), \end{aligned} \quad (3.A.3)$$

with

$$a_0^{N+1} = a_0^N + 2(N+1), \quad b_0^{N+1} = b_0^N, \quad c_0^{N+1} = c_0^N + 4(N+1), \quad (3.A.4)$$

and the additional relations

$$a_j^{N+1} = a_j^N + a_{j-1}^N + c_{j-1}^N, \quad (3.A.5)$$

$$b_j^{N+1} = b_j^N + b_{j-1}^N, \quad (3.A.6)$$

$$c_j^{N+1} = c_j^N + 2c_{j-1}^N. \quad (3.A.7)$$

This proves equation (3.7.1).

To find the explicit expressions for a_j^N, b_j^N and c_j^N , we need to solve the recurrence relations above. From (3.A.4), we read off that

$$a_0^N = N(N+1), \quad b_0^N = 1, \quad c_0^N = 1 + 2N(N+1)$$

for all $N \geq 0$. From this and (3.A.6), we can then read off that $b_j^N = \binom{N}{j}$ for all $N \geq 0$ and $0 \leq j \leq N$. Similarly, by writing $\tilde{c}_j^N = 2^{-j}c_j^N$, we find from (3.A.7) that

$$\tilde{c}_j^N = \binom{N}{j} + 4\binom{N+1}{j+2}.$$

Finally, plugging in the expression for c_j^N into (3.A.5), one finds

$$a_j^N = (2^j - 1)\binom{N}{j} + (2^{j+2} - 2)\binom{N+1}{j+2}.$$

This completes the proof of Proposition 3.7.1. $\square$

Proof of Proposition 3.7.2. The proof follows along the same lines as the previous one, with the difference that one now obtains the linear system

$$\underline{a}_0^{N+1} = \underline{a}_0^N + 2(N+1), \quad \underline{b}_0^{N+1} = \underline{b}_0^N, \quad \underline{c}_0^{N+1} = \underline{c}_0^N + 4(N+1), \quad (3.A.8)$$

and the additional relations

$$\underline{a}_j^{N+1} = \underline{a}_j^N - \underline{a}_{j-1}^N - \underline{c}_{j-1}^N, \quad \underline{b}_j^{N+1} = \underline{b}_j^N - \underline{b}_{j-1}^N, \quad \underline{c}_j^{N+1} = \underline{c}_j^N - 2\underline{c}_{j-1}^N. \quad (3.A.9)$$

One can relate this to the previous system (3.A.4)–(3.A.7) by writing $\tilde{b}_j^N = (-1)^j \underline{b}_j^N$ etc. $\square$

3.A.2 Proof of Lemma 3.8.3

Proof. Equation (3.8.57) clearly holds for $n = 0$. Assume that it holds for a fixed n . We shall show that it also holds for $n + 1$. Letting $[\cdot, \cdot]$ denote the usual commutator, we have

$$\begin{aligned}
(r^2 \partial_u)^{n+1} f &= (r^2 T - r^2 \partial_v)(r^2 T - r^2 \partial_v)^n f \\
&= \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} \left(\sum_{i=0}^{k-1} \alpha_i^{(n,k)} (r + \mathcal{O}(1))^i (r^2 T)^{k+1-i} \right) [r^2 \partial_v]^{n-k} f \\
&\quad + \sum_{k=0}^n (-1)^{n+1-k} \binom{n}{k} \left(\sum_{i=0}^{k-1} \alpha_i^{(n,k)} (r + \mathcal{O}(1))^i (r^2 T)^{k-i} \right) [r^2 \partial_v]^{n+1-k} f \\
&\quad - \sum_{k=0}^n (-1)^{n+1-k} \binom{n}{k} \left(\sum_{i=0}^{k-1} \alpha_i^{(n,k)} [r^2 \partial_v, (r + \mathcal{O}(1))^i (r^2 T)^{k-i}] \right) [r^2 \partial_v]^{n-k} f
\end{aligned} \tag{3.A.10}$$

by commuting $r^2 \partial_v$ past the other terms. We now write

$$\begin{aligned}
&[r^2 \partial_v, (r + \mathcal{O}(1))^i (r^2 T)^{k-i}] f \\
&= [r^2 \partial_v, (r + \mathcal{O}(1))^i] (r^2 T)^{k-i} f + (r + \mathcal{O}(1))^i [r^2 \partial_v, (r^2 T)^{k-i}] f,
\end{aligned} \tag{3.A.11}$$

and compute

$$[r^2 \partial_v, (r + \mathcal{O}(1))^i] f = i D r^{i+1} f + \mathcal{O}(r^i) f, \tag{3.A.12}$$

as well as

$$[r^2 \partial_v, (r^2 T)^{k-i}] f = (k-i) [r^2 \partial_v, r^2 T] (r^2 T)^{k-i-1} f = (k-i) 2 D r (r^2 T)^{k-i} f, \tag{3.A.13}$$

where we used that $[V_1, V_2^n] = n[V_1, V_2]V_2^{n-1}$, which holds true if $[[V_1, V_2], V_2] = 0$. Plugging the above identities back into (3.A.10), one then recovers (3.8.57) for $n + 1$ in a standard way. This gives rise to recurrence relations for the $\alpha_i^{(n,k)}$, which can be solved explicitly. We leave this to the interested reader.

Alternatively, we could have used the non-commutative binomial theorem to write

$$(r^2 T - r^2 \partial_v)^n f = \sum_{k=0}^n \binom{n}{k} (r^2 T + [r^2 \partial_v, \cdot])^k (-r^2 \partial_v)^{n-k} f, \tag{3.A.14}$$

and computed the commutators directly. $\square$

3.A.3 Proof of Lemma 3.9.1

Proof. The result clearly holds $N = 2$. Let us now assume that (3.9.16) holds for a fixed N and for all $N' < N - 1$. Then it also holds for $N + 1$ and for any $N' < N$. Indeed,

$$N \int_{-\infty}^u \frac{r^{N'}}{|u'|^{N+1}} du' = \int_{-\infty}^u \partial_u \left(\frac{r^{N'}}{|u'|^N} \right) + \frac{N' r^{N'-1}}{|u'|^N} \left(1 - \frac{2M}{r} \right) du', \quad (3.A.15)$$

and we can apply the induction assumption to the second term on the RHS to obtain the result (after some standard shifting of indices). $\square$

3.A.4 Proof of Lemma 3.10.2

Proof. Let us inductively assume that

$$\int_{r_0}^r \frac{1}{r_{(k)}^2} \cdots \int_{r_0}^{r_{(2)}} \frac{1}{r_{(1)}^2} dr_{(1)} \cdots dr_{(k)} = \frac{1}{k!} \left(\frac{1}{r_0} - \frac{1}{r} \right)^k \quad (3.A.16)$$

for any $r \geq r_0 > 0$. This holds true for $k = 1$. Going from k to $k + 1$, we have

$$\int_{r_0}^r \frac{1}{r'^2} \int_{r_0}^{r'} \frac{1}{r_{(k)}^2} \cdots \int_{r_0}^{r_{(2)}} \frac{1}{r_{(1)}^2} dr_{(1)} \cdots dr_{(k)} dr' = \int_{r_0}^r \frac{1}{r'^2} \frac{1}{k!} \left(\frac{1}{r_0} - \frac{1}{r'} \right)^k dr'. \quad (3.A.17)$$

We write $x = 1/r$ and $x_0 = 1/r_0$ in the integral above to obtain

$$\int_{r_0}^r \frac{1}{r'^2} \frac{1}{k!} \left(\frac{1}{r_0} - \frac{1}{r'} \right)^k dr' = (-1)^{k+1} \int_{x_0}^x \frac{1}{k!} (x' - x_0)^k dx' = \frac{1}{(k+1)!} (x_0 - x)^{k+1} \quad (3.A.18)$$

for any $r \geq r_0 > 0$. This concludes the proof. $\square$

3.A.5 Proof of Lemma 3.10.3

Proof. For convenience, we recall the definition

$$\text{sum}(L, p, j) := \sum_{i=0}^j (-1)^i \frac{(2L - j + i)!}{i!(L + p - j + i)!(j - i)!}. \quad (3.A.19)$$

We begin with the crucial observation that if $p = L$ and $j \geq 1$, then

$$\text{sum}(L, L, j) = \sum_{i=0}^j (-1)^i \frac{1}{i!(j - i)!} = \sum_{i=0}^j \frac{1}{j!} \binom{j}{i} (-1)^i \cdot 1^j = \frac{1}{j!} (1 - 1)^j = 0. \quad (3.A.20)$$

The idea is to interpret the above sum as a function of some variable x evaluated at $x = 1$: Consider, for instance, the case where $p = L - 1$. Then

$$\begin{aligned} \text{sum}(L, L - 1, j) &= \sum_{i=0}^j (-1)^i \frac{2L + i - j}{i!(j - i)!} = \frac{d}{dx} \Big|_{x=1} \sum_{i=0}^j (-1)^i \frac{1}{i!(j - i)!} x^{2L+i-j} \\ &= (-1)^j \frac{d}{dx} \Big|_{x=1} \left(x^{2L-j} \sum_{i=0}^j (-1)^{j-i} x^i \frac{1}{i!(j - i)!} \right) \quad (3.A.21) \\ &= (-1)^j \frac{d}{dx} \Big|_{x=1} \left(x^{2L-j} \frac{(x - 1)^j}{j!} \right). \end{aligned}$$

Similarly, if $p = L - k$ for some $k \geq 0$, then we obtain inductively that

$$\text{sum}(L, L - k, j) = (-1)^j \frac{d^k}{dx^k} \Big|_{x=1} \left(x^{2L-j} \frac{(x - 1)^j}{j!} \right). \quad (3.A.22)$$

We then compute the k -th derivative above using the Leibniz rule:

$$\frac{d^k}{dx^k} \left(x^{2L-j} \frac{(x - 1)^j}{j!} \right) = \sum_{n=0}^k \binom{k}{n} \frac{(2L - j)! x^{2L-j-n}}{(2L - j - n)!} \cdot \frac{(x - 1)^{j-(k-n)}}{(j - (k - n))!}, \quad (3.A.23)$$

where we use the convention that $\frac{1}{(-n)!} = 0$ for all $n \in \mathbb{N}^+$. In particular, upon evaluating the expression (3.A.23) at $x = 1$, we get

$$\frac{d^k}{dx^k} \Big|_{x=1} \left(x^{2L-j} \frac{(x - 1)^j}{j!} \right) = \sum_{n=0}^k \binom{k}{n} \frac{(2L - j)!}{(2L - j - n)!} \cdot \delta_{k-n,j} = \binom{k}{k - j} \frac{(2L - j)!}{(2L - k)!}, \quad (3.A.24)$$

which proves the second formula (3.10.50) of the proposition.

On the other hand, if $p = L + 1$, we *integrate* the corresponding sum (instead of differentiating):

$$\begin{aligned} \text{sum}(L, L + 1, j) &= \sum_{i=0}^j (-1)^i \frac{1}{i!(j - i)!(2L + 1 - j + i)} \\ &= \int_0^1 \sum_{i=0}^j (-1)^i \frac{1}{i!(j - i)!} x^{2L+i-j} dx = \int_0^1 x^{2L-j} \frac{(1 - x)^j}{j!} dx, \end{aligned} \quad (3.A.25)$$

which is manifestly positive. Equation (3.10.49) then follows inductively. $\square$

Chapter 4

A dictionary for asymptotics near $\mathcal{I}^-$, $\mathcal{I}^+$ and i^+ for all ℓ -modes

Abstract

In this chapter, we distil some of the key ideas from recent works concerning the asymptotic structure of gravitational radiation in dynamical, astrophysical spacetimes, and we assemble them in a new way in order to make them more accessible to the wider general relativity community. In the process, we also discuss new physical findings.

First, we introduce the conserved $f(r)$ -modified Newman–Penrose charges on asymptotically flat spacetimes, and we show that these charges provide a dictionary that relates asymptotics of massless, general spin fields in different regions: Asymptotic behaviour near i^+ (“late-time tails”) can be read off from asymptotic behaviour towards $\mathcal{I}^+$, and, similarly, asymptotic behaviour towards $\mathcal{I}^+$ can be read off from asymptotic behaviour near i^- or $\mathcal{I}^-$.

Using this dictionary, we then explain how: **(I)** the quadrupole approximation for a system of N infalling masses from i^- causes the “peeling property towards $\mathcal{I}^+$ ” to be violated, and **(II)** this failure of peeling results in deviations from the usual predictions for tails in the late-time behaviour of gravitational radiation: Instead of the Price’s law rate $r\Psi^{[4]}|_{\mathcal{I}^+} \sim u^{-6}$ as $u \rightarrow \infty$, we predict that $r\Psi^{[4]}|_{\mathcal{I}^+} \sim u^{-3}$, with the coefficient of this latter decay rate being a multiple of the monopole and quadrupole moments of the matter distribution in the infinite past.

4.1	Introduction	266
4.1.1	Motivation and background	267
4.1.2	The main result	269
4.1.3	Structure	270
4.2	Coordinates, foliations and conventions	270
4.3	The $f(r)$-modified Newman–Penrose charges	272
4.4	Deriving asymptotics towards i^+ from asymptotics towards $\mathcal{I}^+$	274

4.4.1	The case of nonvanishing N-P charge $I_\ell^f[\psi]$	275
4.4.2	Vanishing $I_\ell^f[\psi]$	279
4.4.3	Summing over ℓ	281
4.5	Deriving asymptotics towards $\mathcal{I}^+$ from physical data near i^-	281
4.5.1	The data setup	282
4.5.2	Analysis of the spherically symmetric part ϕ_0	283
4.5.3	Higher ℓ -modes ψ_ℓ	285
4.6	Completing the dictionary and extending to gravitational perturbations	287
4.7	Further directions	289
4.8	Conclusion	291

4.1 Introduction

Over the last few years, there has been significant progress in the mathematical study of general relativity concerning our understanding of how certain asymptotic conservation laws—related to the *Newman–Penrose charges* [NP65, NP68]—can serve as a mechanism for deriving statements about asymptotics of gravitational waves on black hole spacetimes. This mechanism was first explored in [AAG18b] and has since lead to new, potentially physically measurable predictions ([AAG18a, BKS19] and chapter 2 of this thesis). The present chapter aims to distil and generalise some of the main ideas behind this mechanism, and to provide a physical and more accessible interpretation thereof. We also preview several upcoming results.

The ideas presented in this note generally pertain to the system of linearised equations of gravity around a Kerr black hole background $(\mathcal{M}_{M,a}, g_{M,a})$. This system of equations both contains, and is, at a fundamental level, governed¹ by the spin- ± 2 Teukolsky equations² [Teu73]:

$$\mathcal{T}_{g_{M,a}}^{[s]} \Psi^{[|s|-s]} = 0, \quad s = \pm 2, \quad (4.1.1)$$

with $\mathcal{T}_{g_{M,a}}^{[s]}$ a differential operator similar to the wave operator $\square_{g_{M,a}}$ on Kerr, and with $\Psi^{[0]}$ and $\Psi^{[4]}$ the gauge-invariant, extremal components of the perturbed Weyl tensor in the Newman–Penrose formalism [NP62].

To keep the presentation as clear as possible, however, we will restrict most of the discussion to the simpler case of $s = 0$, in which (4.1.1) in fact equals the scalar wave equation

$$\square_{g_{M,a}} \psi = 0. \quad (4.1.2)$$

¹In particular, any admissible perturbation that has $\Psi^{[0]} = 0 = \Psi^{[4]}$ must be the sum of a pure gauge solution and a linearised Kerr solution [Wal73].

²In fact, the ideas discussed in the present chapter apply to the Teukolsky equations for any spin $s \in \frac{1}{2}\mathbb{N}_0$.

We will moreover restrict to the subcase with specific angular momentum $a = 0$, where $g_{M,a}$ reduces to the Schwarzschild metric g_M . Extensions to non-zero s and a will be discussed at the end of the chapter.

This chapter will answer the following two questions (see Fig. 4.1.1):

- A) How can we read off late-time asymptotics of ψ near future timelike infinity i^+ —in particular, *along* future null infinity $\mathcal{I}^+$ and the event horizon $\mathcal{H}^+$ —from asymptotics *towards* $\mathcal{I}^+$?
- B) How can we derive asymptotics towards $\mathcal{I}^+$ from physically motivated scattering data assumptions modelling a system of N infalling masses from the infinite past i^- and excluding incoming radiation from past null infinity $\mathcal{I}^-$?

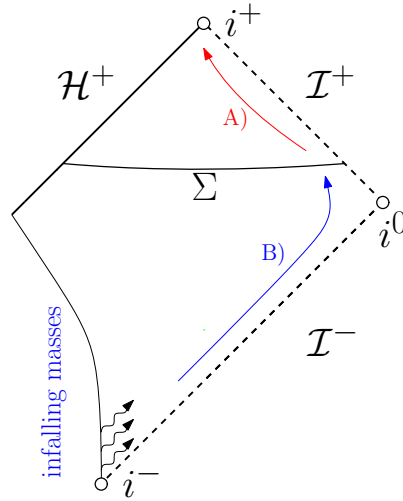


Figure 4.1.1 Depiction of the problems **A)** and **B)** described in the paragraph above.

4.1.1 Motivation and background

Let us first motivate problems **A)** and **B)** individually:

A) The study of the dynamics of (4.1.1) at late times is motivated by the ambitious final state conjecture, which in particular asserts that two inspiralling black holes will settle down to a Kerr solution outside the horizon [Pen69],³ and by the fact that, mathematically, a large part of the radiation emitted as dynamical spacetimes settle down to a Kerr spacetime is encoded precisely in the solutions $\Psi^{[4]}$ and $\Psi^{[0]}$ to eq. (4.1.1).

Hence, in the idealisation of an isolated gravitational system, where gravitational wave observatories operate at $\mathcal{I}^+$, the mathematical late-time behaviour (of appropriate rescalings)

³In the restricted case of spacetimes arising from small perturbations of subextremal Kerr initial data, this conjecture is known as the Kerr stability conjecture, see for example [DHRT21, KS21] and references therein for recent mathematical progress towards a proof thereof.

of the Teukolsky variables $\Psi^{[4]}$ and $\Psi^{[0]}$ along $\mathcal{I}^+$ offers predictions for the late-time part of signals measured by actual gravitational wave detectors; see for example the analysis of the late-time parts of gravitational signals coming from recent black hole mergers in [A⁺16, A⁺21]. This gives rise to several interesting points:

- 1) Suppose one manages to measure the decay rates and coefficients in the late-time asymptotics of the Teukolsky variables, and interprets these as asymptotics towards i^+ along $\mathcal{I}^+$. Given a good mathematical understanding of these late-time tails, one may then extract from the measurements information about both the nature of the final Kerr black hole state as well as the asymptotic properties of the initial state. Late-time tails therefore serve as important signatures of black holes in dynamical astrophysical processes.
- 2) In addition, the late-time behaviour along $\mathcal{I}^+$ is *mathematically strongly correlated* with the late-time behaviour of appropriate renormalisations of the Teukolsky variables along $\mathcal{H}^+$.
- 3) In turn, the late-time behaviour along $\mathcal{H}^+$ is related to the strength of the null singularity that is expected to exist generically inside dynamical black holes; see for example [Daf05a, LO19]. A sufficiently precise understanding of the behaviour of the Teukolsky variables along $\mathcal{H}^+$ is therefore necessary for resolving the *Strong Cosmic Censorship conjecture* (see [DL17] and references therein) in the setting of dynamical black hole spacetimes. Hence, via 2), the information measured along $\mathcal{I}^+$ would also relate to behaviour in the black hole interior!

Of course, the late-time asymptotics near i^+ will depend on the assumptions made on the initial state, i.e. the choice of initial data hypersurface and the prescribed initial data. For instance, if the initial data are posed on an asymptotically hyperboloidal hypersurface Σ (see Fig. 4.1.1), we will see that the leading-order asymptotics near i^+ are, in many cases, completely determined by how the data for ψ behave near $\mathcal{I}^+$, i.e. by the asymptotics of ψ along Σ towards $\mathcal{I}^+$. But what should these early-time asymptotics towards $\mathcal{I}^+$ be?

B) In large (but not all) parts of the literature, there have been two predominant data assumptions on the asymptotics towards $\mathcal{I}^+$ along Σ : It has been assumed (e.g. in the original heuristic work on late-time asymptotics [Pri72]) that these asymptotics are either trivial, i.e. that the data are of compact support along Σ and therefore vanish identically near $\mathcal{I}^+$, or—typically justified by Penrose’s concept of smooth conformal compactification of spacetime (a.k.a. smooth null infinity) [Pen65]—it has been assumed that the initial data satisfy “peeling” [Sac61, Sac62b], i.e. that they have an asymptotic expansion in powers of $1/r$ and exhibit certain leading-order decay towards $\mathcal{I}^+$ (e.g. $\Psi^{[0]} = O(r^{-5})$, $\Psi^{[4]} = O(r^{-1})$).

Now, we would argue that the former assumption is incompatible with the model of an isolated system, as any such system will have radiated *for all times* and, therefore, will not have hypersurfaces Σ of compact radiation content, see Fig. 4.1.1.

The assumption of peeling becomes similarly questionable for physically relevant systems, as will be explained in this chapter. Indeed, the approach we take here is to *not make assumptions* on the asymptotics along Σ towards $\mathcal{I}^+$, but to instead *derive them from physical principles*: We shall consider a *scattering data* setup as in chapter 1 that **a)** has no incoming radiation from $\mathcal{I}^-$ and that **b)** attempts to capture the gravitational radiation of N masses—approaching each other from infinitely far away in the infinite past—by imposing data on some null cone $\mathcal{C}$ (to be thought of as enclosing these masses) that are predicted by post-Newtonian arguments (such as the quadrupole approximation) [WW79b, Dam86, Chr02]. From this scattering setup, we will then *dynamically derive* the asymptotics towards $\mathcal{I}^+$, see Fig. 4.1.2.

4.1.2 The main result

Let us here already give an outline of the results we obtain from the scattering data setup described above, focussing first on the simpler case $s = 0$. See also Fig. 4.1.2.

- We start by assuming that, if ψ_0 denotes the spherically symmetric part of ψ , and if $\phi_0 := r\psi_0$, then $\phi_0|_{\mathcal{C}} \sim u^{-1}$ as $u \rightarrow -\infty$ along some ingoing null hypersurface $\mathcal{C}$. This assumption is motivated by post-Newtonian arguments for systems of N infalling masses from i^- , see §4.5.
- Combining this assumption with the condition of no incoming radiation from $\mathcal{I}^-$, we then prove that $\partial_v \phi_0 \sim r^{-3} \log r$ towards $\mathcal{I}^+$, so the peeling property fails. In fact, we prove that the limit $\lim_{\mathcal{I}^+} \frac{r^3}{\log r} \partial_v \phi_0$ is conserved along $\mathcal{I}^+$.
- Finally, if one smoothly, but arbitrarily, extends the data along $\mathcal{C}$ towards $\mathcal{H}^+$, then this failure of peeling will lead to the following late-time asymptotics *along* $\mathcal{I}^+$: $\phi_0|_{\mathcal{I}^+} \sim u^{-2} \log u$ as $u \rightarrow \infty$. This should be contrasted with the Price's law⁴ rate one obtains for Cauchy data of compact support: $\phi_0|_{\mathcal{I}^+} \sim u^{-2}$.

For the scalar field ($s = 0$), the *measurable* rate along $\mathcal{I}^+$ thus only differs by a logarithm from the Price's law rate. This difference is much less subtle in the case of gravitational perturbations ($|s| = 2$): **There, under the same setup, we expect the following: $\Psi^{[0]}$ violates the peeling property towards $\mathcal{I}^+$, $\Psi^{[0]} \sim \mathcal{O}(r^{-4})$, and this leads to $r\Psi^{[4]}|_{\mathcal{I}^+}$ decaying like u^{-3} along $\mathcal{I}^+$ as $u \rightarrow \infty$, as opposed to the u^{-6} -rate that one obtains in the case of compactly supported Cauchy data and the u^{-5} -rate that one obtains for data consistent with the peeling property!**

The proof of this expectation is work in progress (see the next two chapters 5 and 6), but §4.6 of the present chapter already contains a preview of the argument.

⁴Price's law is the statement that the following asymptotic behaviour should hold given sufficiently rapidly decaying data: $\psi_\ell \sim t^{-3-2\ell}$ along curves of constant r as $t \rightarrow \infty$, and $\phi_\ell|_{\mathcal{I}^+} \sim u^{-2-\ell}$ as $u \rightarrow \infty$. These asymptotics were first predicted from heuristic arguments in [Pri72] and [Lea86], respectively, and proved using mathematically rigorous arguments in [AAG18b, Hin22, AAG21].

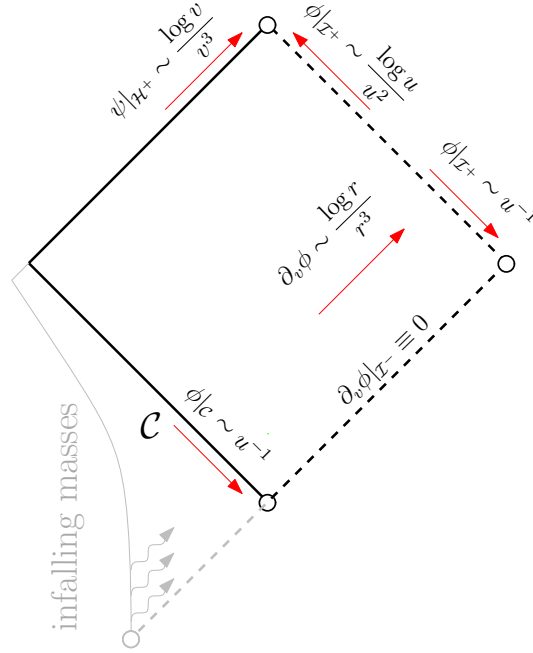


Figure 4.1.2 Summary of the main results of the chapter, focussing on the spherically symmetric ($\ell = 0$) part of the scalar field ($s = 0$).

4.1.3 Structure

The geometry, coordinates and foliations of the Schwarzschild spacetime are set up in §4.2. We then give the definitions of the $f(r)$ -modified Newman–Penrose charges and their associated conservation laws in §4.3. In §4.4, we use these conservation laws to explain how to translate various asymptotics *towards* $\mathcal{I}^+$ into late-time asymptotics *along* $\mathcal{I}^+$ and near i^+ . In §4.5, we then construct a simple model capturing a system of N infalling masses from i^- and discuss the asymptotics towards $\mathcal{I}^+$ exhibited by this model. In §4.6, we combine the results of §4.5 and §4.4 to obtain a complete dictionary translating asymptotics near $\mathcal{I}^-$ to asymptotics near i^+ . In particular, this gives predictions on the (in principle) measurable late-time asymptotics along $\mathcal{I}^+$.

We briefly touch upon some extensions of the methods to different settings in §4.7, and we conclude in §4.8.

4.2 Coordinates, foliations and conventions

We consider the Schwarzschild black hole exterior spacetimes, denoted $(\mathring{\mathcal{M}}_M, g_M)$, where $\mathring{\mathcal{M}}_M = \mathbb{R}_t \times (2M, \infty)_r \times \mathbb{S}_{(\theta, \varphi)}^2$, and where

$$g_M = -D dt^2 + D^{-1} dr^2 + r^2(d\theta^2 + \sin^2 \theta d\varphi^2), \quad D = 1 - \frac{2M}{r}.$$

Note that the constant- t slices foliating $\mathring{\mathcal{M}}_M$ are *asymptotically flat* and, in the Kruskal extension of the spacetime, approach the bifurcation sphere $\mathcal{B}$ (see Fig. 4.2.1). To capture radiative properties in the spacetime, it will be more convenient to introduce the following τ -slicing by *asymptotically hyperboloidal* hypersurfaces that penetrate the event horizon strictly to the future of $\mathcal{B}$: Consider the new time function

$$\tau = t + r_* - 2(r - 2M) - 4M \log\left(\frac{r}{2M}\right) \quad \text{with} \quad r_* = r + 2M \log\left(\frac{r-2M}{M}\right).$$

It may easily be verified that we have the following expression for g_M in $(\tau, r, \theta, \varphi)$ coordinates:

$$g_M = -\left(1 - \frac{2M}{r}\right) d\tau^2 - 2\left(1 - \frac{8M^2}{r^2}\right) d\tau dr + \frac{16M^2}{r^2} \left(1 + \frac{2M}{r}\right) dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2).$$

In fact, this metric is well-defined on the manifold-with-boundary $\mathcal{M}_M = \mathbb{R}_\tau \times [2M, \infty)_r \times \mathbb{S}_{(\theta, \varphi)}^2$, which may be thought of as an extension of $\mathring{\mathcal{M}}_M$ that includes the future event horizon $\mathcal{H}^+$ as the level set $\mathcal{H}^+ = \{r = 2M\}$. We denote with Σ_τ the constant- τ level sets:

$$\Sigma_{\tau_0} = \{\tau = \tau_0\}.$$

We will moreover make use of the double null functions

$$v = t + r_*, \quad u = t - r_*, \quad (4.2.1)$$

and we will consider double null coordinates (u, v, θ, φ) on $\mathring{\mathcal{M}}_M$. The level sets of constant u or v are null hypersurfaces, and we may formally⁵ represent future and past null infinity, $\mathcal{I}^+$ and $\mathcal{I}^-$, as the level sets

$$\mathcal{I}^+ = \{v = \infty\}, \quad \mathcal{I}^- = \{u = -\infty\}.$$

A final word on conventions: The letter C appearing in inequalities will always denote a positive constant whose precise value is not relevant and which is allowed to change from line to line. In other words, C obeys the “algebra of constants”: $C \cdot C = C + C = C \dots$. We will occasionally omit C and just write $f \lesssim g$ or $f \gtrsim g$ if the quantities f, g satisfy $f \leq Cg$ or $f \geq Cg$, respectively. If both $f \lesssim g$ and $f \gtrsim g$, we write $f \sim g$.

If we talk about constants whose specific values do matter, we will typically use either the letter A or B , the former in the case when A is just a constant and nonvanishing multiple of certain data quantities, the latter when B is a more complicated expression (that can in principle vanish).

⁵One can also view these sets as conformal boundaries of the spacetime.

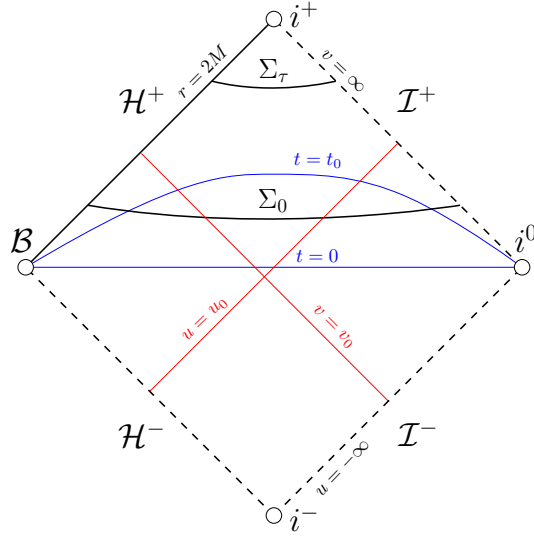


Figure 4.2.1 A Penrose diagrammatic depiction of $\mathcal{M}_M$ with the asymptotically hyperboloidal foliation by Σ_τ , constant- t level sets and u - and v -null hypersurfaces.

4.3 The $f(r)$ -modified Newman–Penrose charges

In double null coordinates (4.2.1), one can derive from (4.1.2) the following wave equation for the rescaled quantity $\phi := r\psi$:

$$4\partial_u\partial_v\phi = \frac{D}{r^2}\Delta_{\mathbb{S}^2}\phi - \frac{2MD}{r^3}\phi, \quad (4.3.1)$$

where $\Delta_{\mathbb{S}^2}$ is the spherical Laplacian with eigenvalues $-\ell(\ell+1)$, $\ell \in \mathbb{N}_0$.

To develop an intuition for the $f(r)$ -modified Newman–Penrose (N–P) charges, let us first consider **the case $M = 0$** : Projecting onto spherical harmonics $Y_{\ell m}$, $\phi_{\ell m} := \langle \phi, Y_{\ell m} \rangle_{L^2(\mathbb{S}^2)}$ (and suppressing the m -index), it is straightforward to derive from (4.3.1) the following infinite set of *exact conservation laws* for $N = \ell$:

$$\partial_u(r^{-2N}\partial_v(r^2\partial_v)^N\phi_\ell) = \frac{(N-\ell)(N+\ell+1)}{r^{2N+2}}(r^2\partial_v)^N\phi_\ell. \quad (4.3.2)$$

If $M \neq 0$, we no longer have global conservation laws, but we can still derive *asymptotic* conservation laws. **Focussing first on the $\ell = 0$ -mode ϕ_0** , the analogue of (4.3.2) reads

$$4\partial_u\partial_v\phi_0 = -\frac{2MD}{r^3}\phi_0. \quad (4.3.3)$$

Clearly, $\partial_v\phi_0$ is no longer globally conserved; however, the RHS exhibits a good r^{-3} -weight. Therefore, for any function f with

$$(f(r))^{-1} = o(r^3), \quad (4.3.4)$$

the $f(r)$ -modified Newman–Penrose charge

$$I_0^f[\psi](u) := \lim_{v \rightarrow \infty} (f(r))^{-1} \partial_v \phi_0(u, v) \quad (4.3.5)$$

will, under suitable assumptions and so long as it exists for some value of u , be conserved in u . This essentially follows from commuting (4.3.3) with $f(r)^{-1}$, see eq. (4.4.6) in §4.4.1 for a proof.

While the charges $I_0^f[\psi]$ have first been defined for $f(r) = r^{-2}$ [NP68, AAG21], we will see in the present chapter that other choices of f are equally important; see also [Kro00, Kro01] as well as the previous chapters 2 and 3.

Notice however the restriction $f(r)^{-1} = o(r^3)$ (also a consequence of the r^{-3} -weight in (4.3.3)). In particular, if $\partial_v \phi_0 = O(r^{-3})$ initially, then *all* ($\ell = 0$) $f(r)$ -modified N–P charges vanish, and one cannot *directly* associate a non-zero asymptotic conserved charge to (4.3.3).

Generalising (4.3.3) to higher ℓ : If we naively compute the RHS of (4.3.2) for $M \neq 0$ and $N = \ell$, then the highest-order term in derivatives will be adorned with a good $r^{-2\ell-3}$ -weight, whereas all lower-order derivatives will come with a bad $r^{-2\ell-2}$ -weight. This problem can be addressed by considering not $(r^2 \partial_v)^\ell \phi_\ell$, but a suitable combination of $(r^2 \partial_v)^\ell \phi_\ell$ and lower-order derivatives. Moreover, it is more natural to work with the rescaled null vector field $\hat{\mathbf{L}} := D^{-1} r^2 \partial_v$ rather than with $r^2 \partial_v$.⁶ To be precise, if we replace $(r^2 \partial_v)^\ell \phi_\ell$ in (4.3.2) with

$$\Phi_\ell := \hat{\mathbf{L}}^\ell \phi_\ell + \sum_{i=1}^{\ell} x_i^{(\ell)} \cdot M^i \cdot \hat{\mathbf{L}}^{\ell-i} \phi_\ell, \quad \text{where} \quad x_i^{(\ell)} = \frac{1}{i!} \frac{(2\ell-i)!}{(2\ell)!} \left(\frac{\ell!}{(\ell-i)!} \right)^3, \quad (4.3.6)$$

then a lengthy computation shows that

$$\partial_u (D^\ell r^{-2\ell} \partial_v \Phi_\ell) = \frac{D^{\ell+1} M}{r^{2\ell+3}} \sum_{i=0}^{\ell} \left(y_i^{(\ell)} + \frac{M}{r} z_i^{(\ell)} \right) \cdot M^i \cdot \hat{\mathbf{L}}^{\ell-i} \phi_\ell, \quad (4.3.7)$$

where $\{y_i^{(\ell)}, z_i^{(\ell)}\}$ is a set of numerical constants. As before, we can now associate, for any f with $(f(r))^{-1} = o(r^3)$, the following $f(r)$ -modified N–P charges to (4.3.7):

$$I_\ell^f[\psi](u) := \lim_{v \rightarrow \infty} f(r)^{-1} \partial_v \Phi_\ell(u, v). \quad (4.3.8)$$

Under suitable assumptions, these $I_\ell^f[\psi](u)$ will again be conserved if finite for some value of u , see §4.4.1.

Importantly, in addition to being conserved, the charges $I_\ell^f[\psi]$ also provide a measure of conformal regularity, i.e. regularity in the variable $x = 1/r$, of the field ψ , see Footnote 6. In order to illustrate this point, we consider the following example: Given data ψ on the

⁶Note that in coordinates $(u, x = 1/r, \theta, \varphi)$, we have $2\hat{\mathbf{L}} = -\partial_x$.

hyperboloidal hypersurface Σ_0 that only have a finite asymptotic expansion in powers of $1/r$, say,

$$r\psi_\ell = \sum_{i=0}^{[p_\ell]} \frac{C_i}{r^i} + C^* \frac{\log^{q_\ell} r}{r^{p_\ell+1}} + \dots \quad \text{for some } q_\ell \in \mathbb{N}, p_\ell \in \mathbb{R}_+,$$

then the $f(r)$ -modified N–P charges associated to ψ can be read off from Table 4.3.1 below:

Values of p_ℓ, q_ℓ	$(p_\ell < \ell, q_\ell > 0)$ or $(p_\ell = \ell, q_\ell > 0)$:	$(p_\ell = \ell, q_\ell = 0)$:	$(p_\ell > \ell, q_\ell \geq 0)$:
$\partial_v \Phi_\ell =$	$\frac{A \log^{q_\ell} r}{r^{2-(\ell-p_\ell)}} + \dots$	$\frac{B}{r^2} + \dots$	$\frac{B}{r^2} + \frac{A \log^{q_\ell} r}{r^{2-(\ell-p_\ell)}} + \dots$

Table 4.3.1 The letter A is a placeholder for a nonvanishing constant multiple of C^* , whereas B stands for a linear combination of the C_i and may therefore be vanishing. The second row of the table indicates for which f the $f(r)$ -modified N–P charge is finite and nonvanishing. For instance, if $p_\ell < \ell$, then we can take $f(r) = r^{\ell-p_\ell-2} \log^{q_\ell} r$ and $I_\ell^f[\psi] = A$. On the other hand, if the data satisfy peeling, then we always have $f(r) = r^{-2}$.

The $f(r)$ -modified N–P charges for higher spin s : In a similar fashion, one can define conserved charges for more general spin. For instance, if $s = \pm 2$ and $M = 0$, then the generalisation of the Minkowskian identities (4.3.2) is given by

$$\partial_u(r^{-2N-2s} \partial_v(r^2 \partial_v)^N(r^{|s|+s+1} \Psi_\ell^{[|s|-s]})) = \frac{(N+s-\ell)(N+s+\ell+1)}{r^{2N+2s+2}} (r^2 \partial_v)^N(r^{|s|+s+1} \Psi_\ell^{[|s|-s]}), \quad (4.3.9)$$

where $\Psi_\ell^{[|s|-s]}$ is the projection of $\Psi^{[|s|-s]}$ onto the spin- s weighted spherical harmonics ${}_s Y_{\ell m}$, which are defined for $\ell \geq |s|$ (here, we again suppressed the m -index). From (4.3.9), one can then derive an equation similar to (4.3.7) in order to derive the relevant $f(r)$ -modified N–P charges for $M \neq 0, |s| = 2$. Comparing (4.3.2) with (4.3.9), one thus finds that, roughly speaking, **the ℓ -th mode of $r^{|s|+s} \Psi^{[|s|-s]}$ behaves like the $(\ell - s)$ -mode of ψ would**, as the RHS of (4.3.9) vanishes for $N = \ell - s$.

4.4 From asymptotics towards $\mathcal{I}^+$ to asymptotics towards i^+

We will now sketch the derivation of the leading-order late-time asymptotics in time τ as $\tau \rightarrow \infty$ of solutions to the wave equation (4.1.2) **starting from initial data on the hyperboloidal initial hypersurface Σ_0** . The arguments in this section generalise arguments from [AAG18c, AAG18b, AAG21] and chapter 2.

We first consider data for which there exists f_ℓ such that $I_\ell^{f_\ell}[\psi] \neq 0$ in §4.4.1. As we will see, the late-time tails for ψ_ℓ are directly encoded in the value of $I_\ell^{f_\ell}[\psi] \neq 0$ in this case. In §4.4.2, we then consider data for which $I_\ell^{f_\ell}[\psi] = 0$ for any choice of f_ℓ , and reduce this case to that of §4.4.1. The analyses of §4.4.1 and §4.4.2 produce the late-time tails for fixed angular frequency solutions ψ_ℓ . We comment on general solutions in §4.4.3.

4.4.1 The case of nonvanishing N–P charge $I_\ell^f[\psi]$

We assume smooth initial data on Σ_0 and take f_ℓ to be a function of r with the following general form:

$$f_\ell(r) = r^{-p'_\ell}(\log r)^{q_\ell}, \quad (4.4.1)$$

where $1 - \ell < p'_\ell < 3$ and $q_\ell \geq 0$, $p'_\ell \in \mathbb{R}$, $q_\ell \in \mathbb{N}$. Given f_ℓ , we make the following assumptions on the r -asymptotics of the initial data on Σ_0 : The ℓ -th spherical harmonic mode ψ_ℓ satisfies

$$\begin{aligned} \partial_v \Phi_\ell|_{\Sigma_0} &= I_\ell^{f_\ell}[\psi] f_\ell(r) + O(r^{-p'_\ell}) \quad \text{if } q_\ell > 0, \\ \partial_v \Phi_\ell|_{\Sigma_0} &= I_\ell^{f_\ell}[\psi] f_\ell(r) + O(r^{-p'_\ell - \beta}) \quad \text{if } q_\ell = 0, \end{aligned} \quad (4.4.2)$$

with $I_\ell^{f_\ell}[\psi] \neq 0$, $\beta > 0$, and with Φ_ℓ defined in (4.3.6).

In Steps 0–3 below, we outline how we can translate the above initial data r -asymptotics to the following late-time τ -asymptotics and u -asymptotics:

$$\psi_\ell|_{r=r_0}(\tau) = A_\ell w_\ell(r_0) I_\ell^{f_\ell}[\psi] f_\ell(\tau) \tau^{-2\ell} + \dots \quad (\tau \rightarrow \infty), \quad (4.4.3)$$

$$r\psi_\ell|_{\mathcal{I}^+}(u) = A_\ell I_\ell^{f_\ell}[\psi] f_\ell(u) u^{1-\ell} + \dots \quad (u \rightarrow \infty), \quad (4.4.4)$$

with $A_\ell \in \mathbb{R}$ constants that depend only on p, q, ℓ , and $w_\ell(r_0)$ also depending on r_0 (see (4.4.12) for the precise r_0 -dependence), and where $\dots$ schematically denote terms that contribute as higher-order terms in τ^{-1} or u^{-1} .

Step 0: The mechanism for deriving late-time tails relies on the following type of *upper bound time-decay estimate*:

$$|r^{-\ell} \psi_\ell| \leq B_\epsilon (\tau + 1)^{1-p'_\ell - \ell + \epsilon} (\tau + r)^{-\ell - 1}, \quad (4.4.5)$$

with $\epsilon > 0$ arbitrarily small and $B_\epsilon > 0$ an appropriately large constant depending on L^2 -type initial data norms and diverging as $\epsilon \rightarrow 0$. In light of the expected time-decay that can be read off from (4.4.3) and (4.4.4), the estimate (4.4.5) can be thought of as an *almost sharp* time-decay estimate.

We will moreover make use of the fact that, in (τ, r) -coordinates, when acting with the vector fields ∂_τ , $r\partial_r$ and $r^2\partial_r$ on $r\psi_\ell$, the τ -decay rate in (4.4.5) changes according to Table 4.4.1:

Vector field	Change in power of τ -factor in (4.4.5)
∂_τ	-1
$r\partial_r$	+0
$r^2\partial_r$	+1

Table 4.4.1 Change in decay rate when acting with weighted vector fields on ψ_ℓ .

The estimate (4.4.5) and the properties of Table 4.4.1 are slight generalisations of what is derived in [AAG18c, AAG18b, AAG21]. The methods used to derive (4.4.5) build on the

vast literature on uniform boundedness and decay estimates for linear waves on black hole spacetime backgrounds and involve geometric properties of Schwarzschild like the trapping of null geodesics and the red-shift effect; see [DRSR16] for a comprehensive overview of this literature. A derivation of these almost-sharp estimates lies beyond the scope of the present chapter, so we will view them simply as *black box assumptions* in a self-contained derivation of late-time tails.

In Steps 1–3 below, we will outline the derivation of the precise leading-order late-time asymptotics and late-time tails for the spherically symmetric $\ell = 0$ -mode. We will briefly describe the generalization to $\ell \geq 1$ afterwards.

Step 1: Let $\alpha \in (0, 1)$. We first restrict to the region $\{v \geq v_\gamma(u)\}$. Here, γ is a timelike curve along which

$$r = u^\alpha + \dots = v^\alpha + \dots,$$

with $\dots$ denoting terms that are higher order in u^{-1} and v^{-1} , and $v_\gamma(u)$ denotes the unique v such that $(u, v) \in \gamma$. In this region, we apply (4.3.3) and (4.4.2) to obtain:

$$\partial_u \partial_v \phi_0 = O(r^{-3})\phi_0 + \dots, \quad \text{with } \partial_v \phi_0|_{\Sigma_0}(v) = I_0^{f_0}[\psi]f_0\left(\frac{v}{2}\right) + \dots,$$

with $\dots$ again denoting terms that contribute to sub-leading order in the argument. Here, we used moreover that $r = \frac{1}{2}(v - u) + \dots$ to leading order in $v - u$, so $r|_{\Sigma_0} = \frac{v}{2} + \dots$.

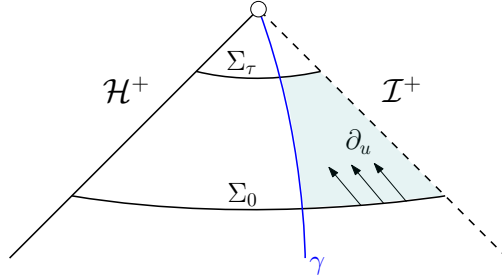


Figure 4.4.1 Step 1: integrating $\partial_u \partial_v \phi_0$ in u .

Since $\alpha > 0$, r grows along γ , and the factor r^{-3} on the right-hand side of $\partial_u \partial_v \phi_0$ decays in the region $\{v \geq v_\gamma(u)\}$. Therefore, we can integrate the equation for $\partial_u \partial_v \phi_0$ in the u -direction (see Fig. 4.4.1) and plug in the almost-sharp estimate (4.4.5) on the right-hand side to propagate the asymptotics of $\partial_v \phi_0$ from Σ_0 to the rest of the region $\{v \geq v_\gamma(u)\}$:

$$\partial_v \phi_0(u, v) = I_0^{f_0}[\psi]f_0\left(\frac{v}{2}\right) + \dots \quad (4.4.6)$$

Note that (4.4.6) in particular implies the following modified N–P conservation law along $\mathcal{I}^+$: The limit $\lim_{r \rightarrow \infty} \frac{1}{f_0(r)} \partial_v \phi_0(u, r)$ is conserved in u along $\mathcal{I}^+$ and is equal to $I_0^{f_0}[\psi]$.

Step 2: In this step, we integrate (4.4.6) in the v -direction in the region $\{v \geq v_\gamma(u)\}$, now starting from γ (see Fig. 4.4.2):

$$\phi_0(u, v) = \phi_0|_\gamma(u) + \int_{v_\gamma(u)}^v \partial_v \phi_0(u, v') dv'.$$

The curve γ is chosen such that $\phi_0|_\gamma$ only contributes to higher order. Indeed, using our definition of γ ($r_\gamma \sim u^\alpha$ along γ), we can split

$$|\phi_0|_\gamma| = r_\gamma |\psi_0|_\gamma| \sim u^\alpha |\psi_0|_\gamma|.$$

Since $\alpha < 1$, and since $u \sim \tau$ along γ , we can then apply the estimate (4.4.5) with $\epsilon > 0$

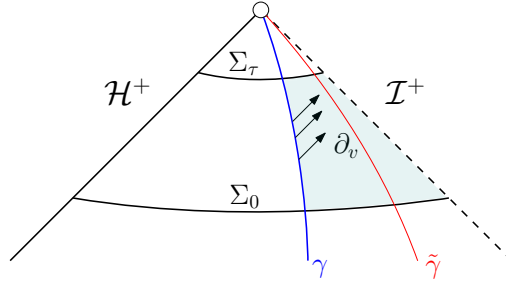


Figure 4.4.2 Step 2: integrating $\partial_v \phi_0$ in v from γ .

suitably small to conclude that $\phi_0|_\gamma$ in fact decays *faster* than $u^{-p'_0+1}$. Hence,

$$\phi_0(u, v) = \int_{v_\gamma(u)}^v \partial_v \phi_0(u, v') dv' + \dots = I_0^{f_0}[\psi] \int_{v_\gamma(u)}^v f_0\left(\frac{v'}{2}\right) dv' + \dots$$

Given the form of f_0 assumed in (4.4.1), we can evaluate the above integral to obtain:

$$\begin{aligned} \phi_0(u, v) &= I_0^{f_0}[\psi] \int_{v_\gamma(u)}^v \left(\frac{v'}{2}\right)^{-p'_0} \log^{q_0}\left(\frac{v'}{2}\right) dv' + \dots \\ &= \frac{2^{p'_0}}{p'_0-1} I_0^{f_0}[\psi] \cdot \left(\log^{q_0}\left(\frac{v_\gamma(u)}{2}\right) (v_\gamma(u)^{-p'_0+1} - v^{-p'_0+1}) + v^{-p'_0+1} \log^{q_0}\left(\frac{v_\gamma(u)}{v}\right) \right) \dots \end{aligned}$$

everywhere in $\{v \geq v_\gamma(u)\}$, where $\dots$ here denote terms that will contribute as higher order terms in u^{-1} or τ^{-1} below. In particular, restricting to a smaller region $\{v \geq v_{\tilde{\gamma}}(u)\}$, where $\tilde{\gamma}$ is a timelike curve along which $r = u^{\tilde{\alpha}} + \dots = v^{\tilde{\alpha}} + \dots$, for $\alpha < \tilde{\alpha} < 1$ with $\tilde{\alpha}$ suitably close to 1, we obtain:

$$\phi_0(u, v) = \frac{2^{p'_0}}{p'_0-1} I_0^{f_0}[\psi] (\log^{q_0}(u) (u^{-p'_0+1} - v^{-p'_0+1}) + \dots) \quad (4.4.7)$$

Taking the limit $v \rightarrow \infty$, we thus obtain

$$\phi_0|_{\mathcal{I}^+}(u) = \frac{2^{p'_0}}{p'_0-1} I_0^{f_0}[\psi] u^{-p'_0+1} \log^{q_0} u + \dots,$$

which proves (4.4.4) for $\ell = 0$. Moreover, by Taylor expanding the expression $\frac{2}{v_{\tilde{\gamma}}(u)-u}(u^{-p+1} - v_{\tilde{\gamma}}(u)^{-p+1})$ in $\frac{v_{\tilde{\gamma}}(u)-u}{u} = u^{\tilde{\alpha}-1}$ around 0, we can also obtain the late-time asymptotics of ψ_0 along $\tilde{\gamma}$:

$$\psi_0|_{\tilde{\gamma}}(u) = \frac{2\phi_0|_{\tilde{\gamma}}(u)}{v_{\tilde{\gamma}}(u)-u} + \dots = 2^{p'_0+1} I_0^{f_0}[\psi] u^{-p'_0} \log^{q_0} u + \dots \quad (4.4.8)$$

Notice, in particular, that the $\log^{q_0} u$ -term in (4.4.8) directly corresponds to the $\log^{q_0} r$ -term in (4.4.1).

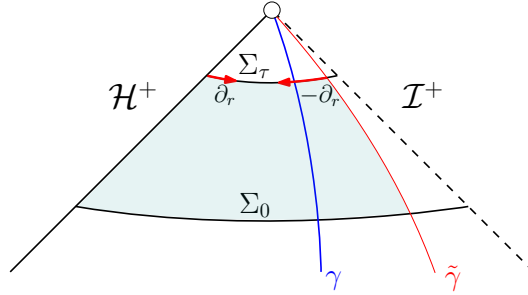


Figure 4.4.3 Step 3: integrating $\partial_r(Dr^2\partial_r\psi_0)$ along Σ_τ from $r = 2M$ and then $\partial_r\psi_0$ from $\tilde{\gamma}$.

Step 3: To also conclude (4.4.3), we *propagate* the late-time asymptotics (4.4.8) derived along the curve $\tilde{\gamma}$ in Step 2 all the way to the event horizon at $r = 2M$. We use that, in $(\tau, r, \theta, \varphi)$ coordinates, the wave equation for ψ_0 takes the following schematic form:

$$\partial_r((r^2 - 2Mr)\partial_r\psi_0) = O(r) \cdot (\partial_\tau\psi_0 + r\partial_r\partial_\tau\psi_0 + \partial_\tau^2\psi_0). \quad (4.4.9)$$

We integrate eq. (4.4.9) from $r = 2M$ (see Fig. 4.4.3) to obtain everywhere in $2M \leq r \leq r_{\tilde{\gamma}}(\tau)$:

$$(r^2 - 2Mr)|\partial_r\psi_0|(\tau, r) \lesssim \int_{2M}^r r \left(|\partial_\tau\psi_0| + |r\partial_r\partial_\tau\psi_0| + |\partial_\tau^2\psi_0| \right) dr'.$$

Now, we can exploit the fact that the right-hand side above involves terms with at least one ∂_τ -derivative: Indeed, the estimate (4.4.5) together with Table 4.4.1 tells us that the integrand on the RHS goes like $r \cdot \tau^{-p'_0-1+\epsilon}$. Doing the integral, we thus obtain, for all $r \leq r_{\tilde{\gamma}}(\tau)$:

$$|\partial_r\psi_0| \leq C(1 + \tau)^{-p'_0-1+\epsilon}. \quad (4.4.10)$$

Finally, for any $r \in [2M, r_{\tilde{\gamma}}(\tau)]$, we write $\psi_0(\tau, r) = \psi_0(\tau, r_{\tilde{\gamma}}(\tau)) + \int \partial_r\psi_0(\tau, r') dr'$, and combine estimates (4.4.8) and (4.4.10). Again, the integral term only contributes at higher order, so we obtain

$$\psi_0(\tau, r) = \psi_0(\tau, r_{\tilde{\gamma}}(\tau)) + \dots = 2^{p'_0+1} I_0^{f_0}[\psi] \cdot \tau^{-p'_0} (\log \tau)^{q_0} + \dots \quad (4.4.11)$$

everywhere in $2M \leq r \leq r_\gamma(\tau)$. This concludes the derivation of (4.4.3) in the case $\ell = 0$.

Treatment of $\ell \geq 1$ modes: The argument for higher ℓ -modes runs similarly, with the role of ϕ_0 now being played by Φ_ℓ . We start **Step 1** by taking (4.3.7), which schematically reads

$$\partial_u(D^\ell r^{-2\ell} \partial_v \Phi_\ell) = O(r^{-3-2\ell}) \Phi_\ell + \dots,$$

inserting the estimates (4.4.5) into the RHS, and finally integrating in u . This proves the analogue of (4.4.6), giving us a sharp estimate for $\partial_v \Phi_\ell$ and the conservation of $I_\ell^{f_\ell}[\psi]$. Then, in **Step 2**, we integrate $\partial_v \Phi_\ell$ in v to first obtain the late-time behaviour of Φ_ℓ , and we subsequently use that

$$\Phi_\ell(u, v) = (D^{-1} r^2 \partial_v)^\ell (r \psi_\ell) + \dots,$$

with $\dots$ terms that contribute as higher-order terms, to obtain the late-time behaviour of ϕ_ℓ after another ℓ integrations in v . Since each of these integrations picks up an extra $1/u$ -term from a suitably chosen γ , this thus proves (4.4.4), as well as the analogue of (4.4.8).

Finally, in **Step 3**, we consider the weighted quantity $w_\ell^{-1}(r) \psi_\ell$, with w_ℓ a smooth, non-zero radial weight function that satisfies

$$\square_g(w_\ell Y_{\ell m}) = 0 \quad \text{for all } |m| \leq \ell, \quad \text{and} \quad w_\ell(r) \sim r^\ell \quad \text{as } r \rightarrow \infty. \quad (4.4.12)$$

The product $w_\ell^{-1}(r) \psi_\ell$ then satisfies the equation:

$$\partial_r((r^2 - 2Mr) w_\ell^2 \partial_r(w_\ell^{-1} \psi_\ell)) = O(r^{\ell+1}) \cdot (\partial_\tau \psi_\ell + r \partial_r \partial_\tau \psi_\ell + \partial_\tau^2 \psi_\ell), \quad (4.4.13)$$

which involves only terms with ∂_τ -derivatives on the right-hand side. We integrate this equation to show that $\partial_r(w_\ell^{-1} \psi_\ell)$ is higher-order in τ^{-1} . One more integration in r then results in the global late-time asymptotics for $w_\ell^{-1} \psi_\ell$ and, in particular, proves (4.4.3).

4.4.2 Vanishing $I_\ell^f[\psi]$

When $I_\ell^{f_\ell}[\psi] = 0$ in the initial data assumptions (4.4.2), the steps outlined in Section 4.4.1 cannot be applied directly. We now explain how to proceed in this case. Let $f_\ell(r) = r^{-p'_\ell} (\log r)^{q_\ell}$ with $p'_\ell \geq 2$, and assume the following r -asymptotics on initial data:

$$\partial_v \Phi_\ell|_{\Sigma_0} = J_\ell r^{-1} f_\ell(r) + O(r^{-p'_\ell-1}) \quad \text{if } q_\ell > 0, \quad (4.4.14)$$

$$\partial_v \Phi_\ell|_{\Sigma_0} = J_\ell r^{-1} f_\ell(r) + O(r^{-p'_\ell-1-\beta}) \quad \text{if } q_\ell = 0, \quad (4.4.15)$$

for some $J_\ell \in \mathbb{R}$. Assumption (4.4.15) implies that $I_\ell^f[\psi] = 0$ for any choice of f . Furthermore, since we allow J_ℓ to be zero, these assumptions include initial data of compact support.

The object that is key to deriving late-time tails for these rapidly r -decaying initial data is the *time integral*:

$$\partial_\tau^{-1}\psi_\ell(\tau, r) := - \int_\tau^\infty \psi_\ell(\tau', r) d\tau'. \quad (4.4.16)$$

By the decay estimates (4.4.5) in Step 0, $\partial_\tau^{-1}\psi_\ell$ is well-defined and regular at the event horizon. Moreover, by definition, $\partial_\tau(\partial_\tau^{-1}\psi_\ell) = \psi_\ell$, and, by the symmetries of the Schwarzschild spacetime, $\partial_\tau^{-1}\psi_\ell$ still solves (4.1.2) projected to ℓ . Lastly, we can determine the value of $\partial_\tau^{-1}\psi_\ell|_{\Sigma_0}$ via simple integration of:

$$\partial_r((r^2 - 2Mr)w_\ell^2\partial_r(w_\ell^{-1}\partial_\tau^{-1}\psi_\ell))|_{\Sigma_0} = O(r^{\ell+1})[\psi_\ell + r\partial_r\psi_\ell + \partial_\tau\psi_\ell]|_{\Sigma_0}, \quad (4.4.17)$$

using that $(r^2 - 2Mr)w_\ell^2\partial_r(w_\ell^{-1}\partial_\tau^{-1}\psi_\ell)$ vanishes at $r = 2M$ and $w_\ell^{-1}\partial_\tau^{-1}\psi_\ell$ vanishes as $r \rightarrow \infty$.

It then follows that the asymptotics of $\partial_v\partial_\tau^{-1}\Phi_\ell$ can be grouped into two cases:

$$\begin{aligned} \partial_v\partial_\tau^{-1}\Phi_\ell|_{\Sigma_0} &= (I_\ell^{(1)})^{f_\ell}[\psi]f_\ell(r) + O(r^{-p'_\ell}) \quad \text{if } q_\ell > 0 \text{ and } p'_\ell = 2 \text{ and } J_\ell \neq 0, \\ \partial_v\partial_\tau^{-1}\Phi_\ell|_{\Sigma_0} &= (I_\ell^{(1)})^{r^{-2}}[\psi]r^{-2} + O(r^{-2-\beta'}) \quad \text{with } \beta' > 0, \text{ if } q_\ell = 0 \text{ or } p'_\ell > 2 \text{ or } J_\ell = 0, \end{aligned}$$

where $(I_\ell^{(1)})^f[\psi]$ is the $f(r)$ -modified N-P charge of $\partial_\tau^{-1}\psi$, which can be expressed either as a constant multiple of J_ℓ if $q_\ell > 0$, $p'_\ell = 2$ and $J_\ell \neq 0$, or in terms of an integral of initial data for ψ along Σ_0 otherwise. In the latter case, $(I_\ell^{(1)})^{r^{-2}}[\psi]$ is *generically nonvanishing* if $M \neq 0$.⁷

We can now apply the arguments of §4.4.1 to obtain the late-time asymptotics of $\partial_\tau^{-1}\psi_\ell$ (cf. (4.4.3), (4.4.4)) and finally take a ∂_τ -derivative to deduce the asymptotics for ψ itself. We obtain

$$\psi_\ell|_{r=r_0}(\tau, \theta, \varphi) = A_\ell w_\ell(r_0)(I_\ell^{(1)})^f[\psi] \frac{d}{d\tau}(f(\tau)\tau^{-2\ell}) + \dots \quad (\tau \rightarrow \infty), \quad (4.4.18)$$

$$r\psi_\ell|_{\mathcal{I}^+}(u, \theta, \varphi) = A_\ell (I_\ell^{(1)})^f[\psi] \frac{d}{du}(f(u)u^{1-\ell}) + \dots \quad (u \rightarrow \infty), \quad (4.4.19)$$

with $f(x)$ either given by $f_\ell(x)$ or by x^{-2} . Thus, for rapidly decaying initial data, **the τ -decay rates in the corresponding late-time tails are encoded in the r -decay of the initial data of the *time integral* $\partial_\tau^{-1}\psi$!**

The results of §4.4.1 and §4.4.2 are summarised in the table below, where $f_\ell = r^{-p'_\ell} \log^{q_\ell} r$ with $p'_\ell > 1 - \ell$ and $q_\ell \geq 0$:

⁷Here, “generic” can be given a precise meaning: $(I_\ell^{(1)})^f[\psi] = 0$ only for a codimension-1 subset of data in any suitably r -weighted function space. In fact, within this codimension-1 subset, we can consider *another* time integral, so we apply time integration twice. Using multiple time inversions, we can conclude that the set of data leading to solutions that do not behave inverse polynomially in time is therefore of infinite codimension.

	$p'_\ell < 3$ or $p'_\ell = 3$ and $q_\ell > 0$:	$p'_\ell > 3$ or $p'_\ell = 3$ and $q_\ell = 0$:
$\partial_v \Phi_\ell _{\Sigma_0}$ as $r \rightarrow \infty$:	$I_\ell^{f_\ell}[\psi] \cdot f_\ell(r) + \dots$	$J_\ell \cdot f_\ell(r) + \dots$
$\phi_\ell _{\mathcal{I}^+}$ as $u \rightarrow \infty$:	$A_\ell I_\ell^{f_\ell}[\psi] \cdot f_\ell(u) u^{1-\ell} + \dots$	$A_\ell \cdot (I_\ell^{(1)})^{r^{-2}}[\psi] u^{-2-\ell} + \dots$
$\psi_\ell _{r=r_0}$ as $\tau \rightarrow \infty$:	$A_\ell w_\ell(r_0) I_\ell^{f_\ell}[\psi] \cdot f_\ell(\tau) \tau^{-2\ell} + \dots$	$A_\ell w_\ell(r_0) \cdot (I_\ell^{(1)})^{r^{-2}}[\psi] \cdot \tau^{-3-2\ell} + \dots$

Table 4.4.2 Summary of the results of §4.4. The A_ℓ are placeholders for nonvanishing numerical constants. The function $w(r_0)$ is defined in (4.4.12). The rates in the second column are independent of the mass M , whereas the constant $(I_\ell^{(1)})^{r^{-2}}[\psi]$ appearing in the third column does depend on M .

4.4.3 Summing over ℓ

In §4.4.1 and §4.4.2, we considered the late-time asymptotics of spherical harmonic modes of fixed ℓ . When choosing initial data such that higher ℓ -modes have more regularity in $\frac{1}{r}$ as $r \rightarrow \infty$, and, in particular, when considering smooth, compactly supported data, higher modes will decay faster. In such cases, the estimate (4.4.5) can be extended to

$$|r^{-\ell} \psi_{\geq \ell}| \leq C \tau^{1-p'_\ell-\ell+\epsilon} (\tau + r)^{-\ell-1}. \quad (4.4.20)$$

Then, by (4.4.20), it follows that when splitting $\psi_{\geq \ell} = \psi_\ell + \psi_{\geq \ell+1}$, the $\psi_{\geq \ell+1}$ contributes at higher order in τ^{-1} and u^{-1} , so the late-time asymptotics of $\psi_{\geq \ell}$ agree with the late-time asymptotics of ψ_ℓ , which are given by (4.4.3) and (4.4.4), or (4.4.18) and (4.4.19). However, initial data on Σ_0 for which higher modes do not have higher regularity at infinity may arise naturally in scattering problems, see §3.1.3.3 of chapter 3. There, it is conjectured that compactly supported scattering data lead to solutions where *all modes* contribute to the late-time asymptotics at the same order. See already the fourth row, second column of Table 4.6.1.

4.5 Deriving asymptotics towards $\mathcal{I}^+$ from physical data near i^-

In §4.4, we derived late-time asymptotics near i^+ from given asymptotics along some hyperboloidal initial data hypersurface Σ_0 , i.e. from asymptotics *towards* $\mathcal{I}^+$. Naturally, to decide what the “correct” predictions for late-time asymptotics are, one thus needs a way to decide what the “correct” asymptotics towards $\mathcal{I}^+$ are. As explained in the introduction, instead of *assuming*, say, vanishing or “peeling” asymptotics near $\mathcal{I}^+$, we will *dynamically derive* the asymptotics towards $\mathcal{I}^+$ from a scattering data setup that **a**) has no incoming radiation from $\mathcal{I}^-$ and **b**) resembles—in some sense—a system of N infalling masses following unbound Keplerian orbits near the infinite past. This section follows the methods of the previous chapters 1–3.

4.5.1 The data setup

In the context of the scalar wave equation on a fixed Schwarzschild background, one simple model with data that realise **a)** and **b)** is depicted in Fig. 4.5.1: In order to satisfy **a)** on $\mathcal{I}^-$, we demand $\partial_v(r\psi)|_{\mathcal{I}^-} \equiv 0$ to be vanishing identically. This corresponds to a vanishing energy flux along $\mathcal{I}^-$. Realising **b)**, on the other hand, is less straightforward. The idea is as follows: While it may, for now, be too ambitious to try and analytically treat a system of N infalling masses, we can instead, for sufficiently large negative retarded times u , consider an ingoing null cone $\mathcal{C}$ from $\mathcal{I}^-$, to be thought of as enclosing the N infalling masses, and impose data on $\mathcal{C}$ that capture the structure of the radiation emitted by these N infalling masses, see Fig. 4.5.1. The

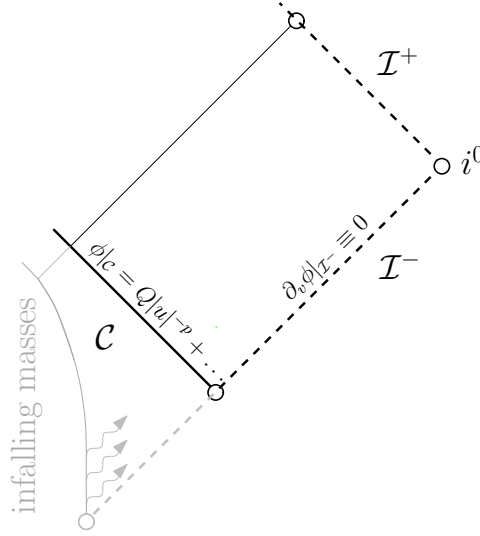


Figure 4.5.1 Depiction of the data setup: We aim to capture the radiation of the infalling masses by imposing the quadrupole approximation on $\mathcal{C}$. The depicted spacetime can be thought of as part of the spacetime depicted in Fig. 1.

heuristic tool that allows us to take this step is the *quadrupole approximation*, which predicts that (gravitational) radiation decays polynomially along $\mathcal{C}$ with a certain rate.

Identifying this radiation with scalar radiation ψ , we now give a brief sketch of where this polynomial decay comes from: The quadrupole approximation for N infalling masses predicts that the loss of gravitational energy along $\mathcal{I}^+$ is given by

$$\frac{dE_{\text{grav}}}{dt} \sim -\ddot{\mathbf{Q}}_{ij}^{TT} \ddot{\mathbf{Q}}_{ij}^{TT},$$

the $\mathbf{Q}_{ij}$ denoting the quadrupole moment of the mass distribution. In the case of hyperbolic orbits⁸ (i.e. if the relative velocities of the masses tend to nonzero constants in the infinite

⁸We could do a similar analysis for parabolic orbits, which would give the exponent 10/3 instead of 4 in (4.5.1).

past), one thus gets that (see [Chr02] for a derivation)

$$\frac{dE_{\text{grav}}}{dt} \sim -C|u|^{-4} + \dots \quad \text{as } u \rightarrow -\infty. \quad (4.5.1)$$

We can now identify this gravitational energy along $\mathcal{I}^+$ with the scalar field energy, namely the flux of the Noether current associated to $T = \partial_t$:

$$\frac{dE_{\text{grav}}}{dt} \sim \frac{dE_{\text{scalar}}}{dt} = - \int_{\mathbb{S}^2} (\partial_u \phi|_{\mathcal{I}^+})^2 d\sigma_{\mathbb{S}^2}. \quad (4.5.2)$$

In this sense, the quadrupole approximation “predicts” that $|\partial_u \phi|_{\mathcal{I}^+}| \sim |u|^{-2}$, and hence $|\phi|_{\mathcal{I}^+}| \sim |u|^{-1}$.

One can make a more sophisticated argument at the level of gravitational (i.e. not scalar) perturbations ($s = \pm 2$) that also allows one to obtain such a rate on an ingoing null hypersurface at a *finite* v -distance $\mathcal{C}$ as opposed to $\mathcal{I}^+$ [WW79b, Dam86]. This justifies the assumption that $\phi|_{\mathcal{C}} = Q|u|^{-p} + \dots$ for $p = 1$, $Q \neq 0$.

In fact, a more detailed perturbative analysis would result in different predictions on the exponent p for each angular mode ℓ . For the purpose of this expository note, we will simply make the assumption that

$$\phi_\ell|_{\mathcal{C}} = Q_\ell|u|^{-p_\ell} + \dots \quad (4.5.3)$$

for some general $p_\ell \in \mathbb{R}_+$, $Q_\ell \in \mathbb{R} \setminus \{0\}$, keeping in mind that for $\ell = 0$, $p_0 = 1$ is the “physically relevant” exponent.⁹

We will now complete our dictionary by explaining how these asymptotics along $\mathcal{C}$ translate into asymptotics towards $\mathcal{I}^+$, first for $\ell = 0$ in §4.5.2, then for general ℓ in §4.5.3. We will give an outlook on the case of $s = \pm 2$ in §4.6.

4.5.2 Analysis of the spherically symmetric part ϕ_0

We now explain how to treat solutions ψ_0 arising from a data setup as in Fig. 4.5.1, i.e. with data $\phi_0|_{\mathcal{C}}(u) = Q_0|u|^{-p_0} + \dots$ on $\mathcal{C}$ and $\partial_v \phi_0|_{\mathcal{I}^-} \equiv 0$ on $\mathcal{I}^-$. These data are to be understood as *scattering data*. Recently developed scattering theory [Nic16, DRS18, Mas22a] ensures the unique existence of a solution ψ_0 that attains the prescribed data along $\mathcal{C}$ and $\mathcal{I}^-$, and moreover provides us with the (far from optimal) global bound that (see also chapter 1)

$$|r\psi_0| \leq C\sqrt{r}. \quad (4.5.4)$$

⁹Of course, we can only speak of physical relevance in the case of gravitational radiation, i.e. for $s = \pm 2$, which we discuss in §4.6. However, since the quadrupole approximation gives a prediction for the lowest angular modes of $\Psi^{[0]}$ and $\Psi^{[4]}$, namely the $\ell = 2$ modes, and the lowest angular mode of ψ is given by $\ell = 0$, we here take $p_0 = 1$ to be physically relevant exponent for the scalar field. See also the arguments relating the analysis of the $s = 0$ case to the $s = \pm 2$ case in §4.6 (cf. Footnote 12).

To obtain asymptotic estimates, the analysis will roughly follow two steps:

- (I) We first insert the weak estimate $|\phi_0| \lesssim \sqrt{r}$ into (4.3.3) and integrate (4.3.3) from $\mathcal{I}^-$ to obtain a weak estimate on $\partial_v \phi_0$.
- (II) We can then integrate this estimate for $\partial_v \phi_0$ from $\mathcal{C}$ to obtain an estimate for ϕ_0 .

The crucial observation then is that, as a consequence of the strong r^{-3} -weight appearing on the right-hand side of (4.3.3), this new estimate on ϕ_0 will be improved compared to the original one. One can then iterate steps (I) and (II) until one obtains a sharp estimate. Now, the details:

(I): Inserting $|\phi_0| \leq \sqrt{r}$ into (4.3.3), and integrating (4.3.3) from $\mathcal{I}^-$, we find

$$\partial_v \phi_0(u, v) = \int_{-\infty}^u \partial_u \partial_v \phi_0 \, du' \leq \int_{-\infty}^u \frac{MCD}{2r^{5/2}} \, du' \leq Cr^{-3/2}. \quad (4.5.5)$$

Here, we used the no-incoming-radiation condition $\partial_v \phi_0|_{\mathcal{I}^-} \equiv 0$.

(II): Integrating (4.5.5) from $\mathcal{C}$ (where $2r|_{\mathcal{C}} \sim |u|$), we then find

$$|\phi_0(u, v) - \phi|_{\mathcal{C}}(u)| \leq C \int_1^v r^{-3/2} \, dv' \leq C|u|^{-1/2}. \quad (4.5.6)$$

This clearly improves the initial estimate $|\phi_0| \leq C\sqrt{r}$ to $|\phi_0| \leq C|u|^{-\min(p_0, 1/2)}$.

One can now insert this improved estimate back into (4.5.5) and iteratively repeat the procedure of steps (I) and (II), i.e. of (4.5.5), (4.5.6), to eventually obtain the estimate

$$|\phi_0(u, v) - \phi|_{\mathcal{C}}(u)| \leq C|u|^{-p_0-1}. \quad (4.5.7)$$

Since $\phi_0|_{\mathcal{C}}(u) = Q_0|u|^{-p_0} + \dots$, (4.5.7) implies that the leading-order term in the asymptotics of ϕ_0 as $u \rightarrow -\infty$ is $Q_0|u|^{-p_0}$.

Finally, in order to also obtain an asymptotic estimate for $\partial_v \phi_0$ as $v \rightarrow \infty$, we once again repeat the calculation (4.5.5), this time equipped with the asymptotic estimate (4.5.7). This gives

$$\partial_v \phi_0(u, v) = \int_{-\infty}^u \partial_u \partial_v \phi_0 \, du' = \int_{-\infty}^u -\frac{MDQ_0|u|^{-p_0}}{2r^3} \, du' + \dots \quad (4.5.8)$$

The integral on the RHS can be computed by writing $2r = v - u + \mathcal{O}(\log(v - u))$. Let us here give the concrete computation only in the case $p_0 = 1$:

$$\begin{aligned} \int_{-\infty}^u \frac{1}{(v - u')^3 u'} \, du' &= \int_{-\infty}^u \frac{1}{v^3} \left(\frac{1}{u'} + \frac{1}{v - u'} + \frac{v}{(v - u')^2} + \frac{v^2}{(v - u')^3} \right) \, du' \\ &= \frac{\log|u| - \log(v - u)}{v^3} + \frac{3v - 2u}{2v^2(v - u)^2}. \end{aligned} \quad (4.5.9)$$

In particular, we conclude that, along $u = \text{const}$ (where $v \sim r$), we have

$$\partial_v \phi_0(u, v) = -\frac{MQ_0}{2} \frac{\log r - \log |u|}{r^3} + \mathcal{O}(r^{-3}). \quad (4.5.10)$$

Similar computations can be done for general p_0 , see Table 4.5.1 below:

$\phi_0 _{\mathcal{C}} = Q_0 u ^{-p_0} + \dots$	$p_0 \in (0, 1):$	$p_0 = 1:$	$p_0 \in (1, 2):$	$p_0 = 2:$	$\dots$
$\partial_v \phi_0(u = \text{const}, v) =$	$\frac{AM}{r^{2+p_0}} +$	$\frac{AM \log r}{r^3} +$	$\frac{h(u)}{r^3} + \frac{AM}{r^{2+p_0}} +$	$\frac{h(u)}{r^3} + \frac{AM \log r}{r^4} +$	$\dots$

Table 4.5.1 Relation between decay of ϕ_0 as $u \rightarrow -\infty$ and asymptotics for $\partial_v \phi_0$ as $v \rightarrow \infty$. Here, $h(u)$ is a function decaying in u as $u \rightarrow -\infty$, and A stands for a constant multiple of Q_0 .

In particular, if $p_0 = 1$, then (4.5.10) implies that $I_0^{f_0}[\psi] = -MQ_0/2$ for $f_0 = r^{-3} \log r$. Thus, combining the findings of the present section and of §4.4 (cf. Table 4.4.2), we can in particular conclude the following (cf. chapters 1,2):

Theorem 4.5.1. *Consider data on $\mathcal{C}$ and $\mathcal{I}^-$ that satisfy $\phi_0|_{\mathcal{C}} = Q_0|u|^{-1} + \mathcal{O}(|u|^{-1-\epsilon})$ as $u \rightarrow -\infty$ and $\partial_v \phi_0|_{\mathcal{I}^-} \equiv 0$. Then, along any outgoing null hypersurface of constant u , $2\partial_v \phi_0 = -MQ_0 r^{-3} \log r + \dots$ as $v \rightarrow \infty$, so $I_0^{r^{-3} \log r}[\psi] \equiv -MQ_0/2$.*

Moreover, if one smoothly extends the data along $\mathcal{C}$ to $\mathcal{H}^+$ as in Fig. 4.5.2, one obtains the following late-time asymptotics near i^+ .¹⁰

$$\psi_0|_{\mathcal{H}^+} = -4MQ_0 v^{-3} \log v + \dots \quad \phi_0|_{\mathcal{I}^+} = -2MQ_0 u^{-2} \log u + \dots \quad (4.5.11)$$

Finally, we point out that, regardless of the value of p_0 , ϕ_0 will never fully satisfy peeling towards $\mathcal{I}^+$ under this setup, as indicated by Table 4.5.1.

4.5.3 Higher ℓ -modes ψ_ℓ

We now turn our attention to higher ℓ -modes: We consider data such that $r\psi_\ell|_{\mathcal{C}}(u) = Q_\ell|u|^{-p_\ell} + \dots$ on $\mathcal{C}$ and such that $\partial_v(r\psi_\ell)|_{\mathcal{I}^-} \equiv 0$ along $\mathcal{I}^-$.

The analysis of higher ℓ -modes will again be similar to the analysis of the $\ell = 0$ mode, with the relevant equation now being (4.3.7) instead of (4.3.3). For simplicity, let us work with a simplified version of (4.3.7), namely (recall $\hat{\mathbf{L}} := D^{-1}r^2\partial_v$):

$$\partial_u(D^\ell r^{-2\ell} \partial_v \hat{\mathbf{L}}^\ell \phi_\ell) = \frac{MD^{\ell+1}y_0^{(\ell)}}{r^{2\ell+3}} \hat{\mathbf{L}}^\ell \phi_\ell. \quad (4.5.12)$$

This simplification corresponds to setting $x_i^{(\ell)} = 0 = y_i^{(\ell)}$ for all $i > 0$ and $z_i^{(\ell)} = 0$ for all $i \geq 0$ in (4.3.7), but still allows us to capture the main ideas of the proof: Essentially, we can

¹⁰Note that, owing to the conservation of $I_0^{r^{-3} \log r}[\phi]$ along $\mathcal{I}^+$, the leading-order asymptotics (4.5.11) are independent of the extension of the data towards $\mathcal{H}^+$.

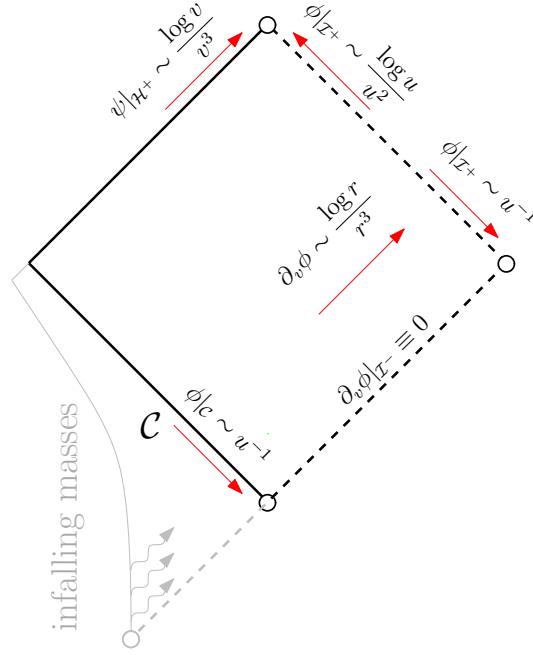


Figure 4.5.2 Depiction of the resulting asymptotics in various regions if $\phi|_{\mathcal{C}} \sim u^{-1}$ initially.

now repeat a procedure almost identical to steps (I) and (II) of §4.5.2, with ϕ_0 replaced by $\Phi_\ell := \widehat{\mathbf{L}}^\ell \phi_\ell$. The main difference is that, in order to get an estimate similar to (4.5.7), one first needs to compute the values of Φ_ℓ along $\mathcal{C}$. This is achieved by inductively integrating the equations satisfied by $\partial_u(r^{-2N}\partial_v\widehat{\mathbf{L}}^N\phi_\ell)$ for $N < \ell$ from $u = -\infty$, with these equations in turn being obtained by simply commuting the wave equation (4.3.1) with $\widehat{\mathbf{L}}^N$ (cf. (4.3.2)). This gives:

$$\widehat{\mathbf{L}}^N \phi_\ell|_{\mathcal{C}}(u) = A_N^{(\ell)} |u|^{-p_\ell+N} + \dots \quad (4.5.13)$$

for some constants $A_N^{(\ell)}$ that are nonvanishing multiples of Q_ℓ for $N \leq \ell$.

Equipped with this estimate for $\Phi_\ell|_{\mathcal{C}}$, we can now, similarly to how we showed (4.5.7), show that

$$|\Phi_\ell(u, v) - \Phi_\ell|_{\mathcal{C}}(u)| \leq C \begin{cases} r^{\ell-1-p_\ell} & \text{if } p_\ell < \ell - 1 \\ \log r - \log |u| & \text{if } p_\ell = \ell - 1 \\ |u|^{\ell-1-p_\ell} & \text{if } p_\ell > \ell - 1 \end{cases} \quad (4.5.14)$$

(Note that the first two cases in (4.5.14) did not appear for $\ell = 0$ since we assumed p_ℓ to be positive. Formally, however, the calculations here are valid for *any* value of $p_\ell \in \mathbb{R}$.) As with the $\ell = 0$ -case, we can now insert this estimate into (4.5.12) to obtain an asymptotic estimate

on $\partial_v \Phi_\ell$:

$$\partial_v \Phi_\ell(u, v) = r^{2\ell} \int_{-\infty}^u \frac{MDy_0^{(\ell)}}{r^{2\ell+3}} \Phi_\ell|_{\mathcal{C}} du' + \dots = \begin{cases} Ar^{\ell-p_\ell-2} + \dots & \text{if } p_\ell < \ell + 1 \\ Ar^{-3}(\log r - \log |u|) + \dots & \text{if } p_\ell = \ell + 1 \\ A|u|^{\ell+1-p_\ell} r^{-3} + \dots & \text{if } p_\ell > \ell + 1 \end{cases} \quad (4.5.15)$$

Notice that if $p_\ell \leq \ell + 1$, then the RHS of (4.5.15) gives the relevant $f(r)$ -modified N-P charge and shows that it is conserved as well. For instance, for $p_\ell < \ell + 1$ and $f_\ell = r^{\ell-p_\ell-2}$, we have $I_\ell^{f_\ell}[\psi] = A$. On the other hand, if $p_\ell > \ell + 1$, then all $f(r)$ -modified N-P charges vanish.

Finally, by integrating (4.5.15) $\ell + 1$ times from $\mathcal{C}$ (where $2r = |u| + \dots$), each time picking up a term on $\mathcal{C}$ that is given by (4.5.13), we can now also obtain an estimate for ϕ_ℓ itself.¹¹ A more detailed discussion of this can be found in chapter 3.

4.6 Completing the dictionary and extending to gravitational perturbations

Completing the dictionary: We can finally summarise our findings and combine the results of §4.5 with those of §4.4. Suppose that we have data for ψ_ℓ such that $\phi_\ell|_{\mathcal{C}} = Q_\ell |u|^{-p_\ell} + \dots$ near $\mathcal{I}^-$ and such that $\partial_v \phi_\ell|_{\mathcal{I}^-} \equiv 0$. We can then read off the relevant choice of $f(r)$ -modified N-P charge from (4.5.15). Thus, combining this with the results of §4.4, cf. Table 4.4.2 (and again smoothly but arbitrarily extending the data towards $\mathcal{H}^+$), we can directly connect the behaviour of ϕ_ℓ near $\mathcal{I}^-$ to its behaviour near i^+ . This is done in Table 4.6.1 below.

$\phi_\ell _{\mathcal{C}} = Q_\ell u ^{-p_\ell} + \dots$ as $u \rightarrow -\infty$:	$p_\ell < \ell + 1$:	$p_\ell = \ell + 1$:	$p_\ell > \ell + 1$:
$\phi_\ell _{\mathcal{I}^+}$ as $u \rightarrow -\infty$:	$A u ^{-p_\ell}$ unless $p_\ell \in \{1, \dots, \ell\}$		
$\partial_v \Phi_\ell$ as $u = \text{const}, v \rightarrow \infty$:	$AMr^{-2+\ell-p_\ell}$	$AMr^{-3} \log r$	Br^{-3}
$\phi_\ell _{\mathcal{I}^+}$ as $u \rightarrow \infty$:	AMu^{-1-p_ℓ}	$AMu^{-2-\ell} \log u$	$Bu^{-2-\ell}$
$\psi_\ell _{\mathcal{H}^+}$ as $v \rightarrow \infty$:	$AMv^{-2-\ell-p_\ell}$	$AMv^{-2\ell-3} \log v$	$Bv^{-2\ell-3}$

Table 4.6.1 The full thesaurus: The letter A is always a placeholder for a constant, nonvanishing multiple of Q_ℓ (that is different from cell to cell), whereas B stands for a constant that also depends on the extension of the data along $\mathcal{C}$ towards $\mathcal{H}^+$ and is only generically nonvanishing.

¹¹We resort to an example in order to schematically explain this. Consider the $\ell = 1$ -mode ψ_1 with $p_1 > 0$. From the estimate (4.5.14), we obtain that $r^2 \partial_v \phi_1(u, v) = r^2 \partial_v \phi_1|_{\mathcal{C}}(u) + \dots$. We further compute from (4.3.1) that $r^2 \partial_v \phi_1|_{\mathcal{C}}(u) = A_1^{(1)} \cdot |u|^{-p_1+1} + \dots$, with $A_1^{(1)} = -Q_1/(2(p_1 + 1))$ (cf. (4.5.13)). Therefore, we can schematically compute $\phi_1(u, v)$ via

$$\phi_1(u, v) = \phi_1|_{\mathcal{C}}(u) + \int \partial_v \phi_1(u, v) dv = \frac{Q_1}{|u|^{p_1}} + \frac{A_1^{(1)}}{|u|^{p_1-1}} \int r^{-2} dv + \dots = \frac{Q_1}{|u|^{p_1}} - \frac{Q_1}{(p_1 + 1)|u|^{p_1-1}} \left(\frac{1}{r|_{\mathcal{C}}} - \frac{1}{r} \right) + \dots$$

In particular, $\lim_{v \rightarrow \infty} \phi_1(u, v)$ decays faster in u than $\phi_1|_{\mathcal{C}}(u)$ does initially iff $p_1 = 1$.

More generally, if $\phi_\ell|_{\mathcal{C}}$ decays like $|u|^{-p_\ell}$ initially, then $\phi_\ell|_{\mathcal{I}^+}$ will decay faster than $|u|^{-p_\ell}$ if and only if $p_\ell \in \{1, \dots, \ell\}$.

Extending to gravitational perturbations ($s = \pm 2$): So far, we have focussed mostly on $s = 0$. The more realistic case of gravitational perturbations will be discussed in detail in chapters 5 and 6, but we shall already list the main points here: Post-Newtonian arguments [WW79b, Dam86] predict the following rates along $\mathcal{C}$ at the level of quadrupolar radiation (i.e. for $\ell = 2$): $\Psi^{[0]} \sim |u|^{-3}$ and $\Psi^{[4]} \sim |u|^{-4}$ as $u \rightarrow -\infty$.¹² Combining this with the no-incoming-radiation condition, a preliminary analysis then gives:

- α) In addition to $\Psi^{[4]}$ violating the peeling rate near $\mathcal{I}^-$ (which states that $\Psi^{[4]} = \mathcal{O}(r^{-5})$), the peeling rate of $\Psi^{[0]}$ near $\mathcal{I}^+$ is also violated: Instead of the peeling rate $\Psi^{[0]} = \mathcal{O}(r^{-5})$, one will obtain that $\Psi^{[0]} \sim r^{-4}$ towards $\mathcal{I}^+$. Motivations for this rate have appeared before in [Dam86, Chr02].
- β) While the radiation field for $\Psi^{[0]}$, $\lim_{\mathcal{I}^+} r^5 \Psi^{[0]}$, thus blows up, the radiation field for $\Psi^{[4]}$, $\lim_{\mathcal{I}^+} r \Psi^{[4]}$, is still defined, and we conjecture that the failure of peeling (i.e. of conformal regularity) in α) translates into the following late-time decay rate along $\mathcal{I}^+$: $r \Psi^{[4]}|_{\mathcal{I}^+} \sim u^{-3}$ as $u \rightarrow \infty$. This should be contrasted with the Price's law rate, which predicts that $r \Psi^{[4]}|_{\mathcal{I}^+} \sim u^{-6}$; see [MZ22b] for a derivation of the Price's law rate for $|s| = 2$.

In fact, at a heuristic level, the reader can already guess the rates in α) and β) by the observation from §4.3 that the following correspondences hold:

$$r^5 \Psi_{\ell=2}^{[0]} \leftrightarrow \phi_{\ell=0}, \quad r \Psi_{\ell=2}^{[4]} \leftrightarrow \phi_{\ell=4}.$$

Therefore, our rates on $\mathcal{C}$ read:

$$\begin{aligned} |u|^2 \sim r^5 \Psi_{\ell=2}^{[0]}|_{\mathcal{C}} \leftrightarrow \phi_{\ell=0}|_{\mathcal{C}} &\implies p_0 = -2, \\ |u|^{-3} \sim r \Psi_{\ell=2}^{[4]}|_{\mathcal{C}} \leftrightarrow \phi_{\ell=4}|_{\mathcal{C}} &\implies p'_4 = 3. \end{aligned}$$

However, the behaviour of $\Psi_{\ell=2}^{[4]}$ is actually governed by $p_4 = 2$, not $p'_4 = 3$. Leaving the details to chapters 5 and 6, we here only note that this is related to the fact that, as a consequence of the extra r^{-2s} -weight in (4.3.9), the $|u|^{-3}$ -decay of $r \Psi_{\ell=2}^{[4]}|_{\mathcal{C}}$ leads to a non-integrable RHS of (4.3.9) for $N = 0$ when trying to compute the transversal derivatives of $r \Psi_{\ell=2}^{[4]}$ along $\mathcal{C}$ as in (4.5.13): This leads to $\widehat{\mathcal{L}}(r \Psi^{[4]})$ decaying only like $|u|^{-2} \log |u|$ along $\mathcal{C}$, and to $\widehat{\mathcal{L}}^2(r \Psi^{[4]}) \sim 1$ not decaying along $\mathcal{C}$. So, compared to (4.5.13), higher-order transversal derivatives decay one power slower compared to the case $s = 0$.

¹²The reader could have loosely guessed these rates using the same simplistic heuristics as we presented for $s = 0$: The rate $r \Psi^{[4]}|_{\mathcal{C}} \sim |u|^{-3}$ as $u \rightarrow -\infty$ on data will remain the same on $\mathcal{I}^+$, where $r \Psi^{[4]} = \partial_u N$ gives the rate of change of the News function N . Furthermore, we have the Bondi mass loss formula [BVdBM62, Sac62b, CK93] along $\mathcal{I}^+$: $\frac{dE_{\text{grav}}}{dt} = -\frac{1}{4\pi} \int_{\mathbb{S}^2} |N|^2 d\sigma_{\mathbb{S}^2}$. The rate for $r \Psi^{[4]}$ thus comes from the quadrupole approximation prediction that $\frac{dE_{\text{grav}}}{dt} \sim -|u|^{-4}$. The rate for $\Psi^{[0]}$ is then enforced by the Teukolsky–Starobinsky identities [TP74] and the no-incoming-radiation condition.

By now applying Table 4.6.1 with these correspondences and values for p_ℓ , we obtain in particular:

$$\partial_v(r^5\Psi_{\ell=2}^{[0]})|_{u=\text{const}} \leftrightarrow \partial_v\phi_{\ell=0}|_{u=\text{const}} \stackrel{p_0=-2}{\sim} r^{-2+0-(-2)} = r^0 \quad \text{as } r \rightarrow \infty$$

$$\stackrel{\text{integrating}}{\implies} \Psi_{\ell=2}^{[0]}|_{u=\text{const}} \sim r^{-4} \quad \text{as } r \rightarrow \infty, \quad (4.6.1)$$

$$r\Psi_{\ell=2}^{[4]}|_{\mathcal{I}^+} \leftrightarrow \phi_{\ell=4}|_{\mathcal{I}^+} \stackrel{p_4=2}{\sim} u^{-1-2} = u^{-3} \quad \text{as } u \rightarrow \infty. \quad (4.6.2)$$

The numerology presented above is rooted in the assumption of hyperbolic Keplerian orbits in the infinite past and (4.5.1). Using similar arguments, the reader can also find the numerology in the case of *parabolic orbits*, cf. Footnote 8.

4.7 Further directions

We append the main body of the chapter with some remarks on extensions of the presented methods.

From Schwarzschild to subextremal Kerr: For solutions to the scalar wave equation on subextremal Kerr arising from conformally regular or compactly supported initial data, the leading-order late-time asymptotics feature the same rates as in Schwarzschild [Hin22, AAG23]. See also [SRTdC20, SRTdC23, MZ23] for related recent results in the setting of the Teukolsky equations and [BO99a, BO99b] for a heuristic analysis of late-time tails in Kerr spacetimes.

The effects of the non-zero angular momentum of Kerr do, however, affect the decay rates of higher angular modes. Since Kerr is not spherically symmetric, there is no *a priori* canonical definition of spherical harmonics and corresponding angular modes,¹³ and, whichever definition is chosen, obtaining late-time tails for each mode will involve the difficulty of *mode coupling*.

This difficulty was addressed and studied in [AAG23], where it was shown that the choice of spherical harmonics with respect to Boyer–Lindquist spheres at infinity allows for a modified analogue of Price’s law in Kerr. The topic of generalising Price’s law on Kerr spacetimes has a long history featuring various conflicting predictions. See [ZKB14, BK14] and references therein for an overview of this problem and for the latest numerical results, which are in alignment with the mathematically rigorous results derived in [AAG23].

We expect that the techniques outlined in the present chapter will also be applicable in combination with the methods developed in [AAG23] to study the effects that a violation of peeling has on late-time tails for the Teukolsky equations on subextremal Kerr backgrounds.

¹³Note however that in phase space, the Boyer–Lindquist time-frequency-dependent spheroidal harmonics form a natural choice of angular modes, since they are involved in the separability of the wave equation after taking a Fourier transform in time.

Extremal black holes: *Extremal* black holes feature several additional fascinating phenomena that have an effect on the rates of decay in late-time tails and are inherently connected to the degeneracy of extremal event horizons. For instance, the wave equation on extremal Reissner–Nordström black holes and the axisymmetric wave equation on extremal Kerr black holes possess additional conserved charges along the event horizon [Are15] that lead to different decay rates from the subextremal setting [AAG20b] and are connected to the presence of asymptotic instabilities known in the literature as the *Aretakis instabilities* [Are15]; see also earlier heuristics in [Ori13, Sel16]. In fact, in extremal Reissner–Nordström, these additional conservation laws can be related to the Newman–Penrose charges by applying the Couch–Torrence conformal isometry that maps null infinity to the event horizon [CT84, BF13, LMRT13].

In upcoming work (see [Gaj21, Gaj23]), it is shown that the late-time tails of non-axisymmetric solutions to the wave equation on extremal Kerr exhibit even stronger deviations from the subextremal case, as well as stronger instabilities, consistent with the heuristics in [GA01, CGZ16]. Due to the lack of conserved charges along the event horizon in the non-axisymmetric setting, we moreover need a different mechanism for deriving late-time tails from the one presented in the present chapter.¹⁴

Late-time tails in gravitational radiation are of particular observational relevance in the extremal setting since they have been predicted to form the dominant part of gravitational wave signals at much earlier stages in the ringdown process than in the subextremal setting [YZZ⁺13]. They therefore provide a promising observational signature of (near)-extremality of black holes. Since extremal late-time tails also decay much slower, and since they are a phenomenon associated to regularity at the future event horizon and not at future null infinity, *we do not expect the failure of peeling to have an effect on decay rates*, in contrast to the subextremal setting.

Moving beyond linear perturbations: The results of §4.5 have been extended to the coupled Einstein-scalar field system under the assumption of spherical symmetry in chapter 1. The effects of nonlinearities on Price’s law (i.e. including backreaction) have been investigated heuristically and numerically in [BCR09, BR10], where deviations to Price’s law have been predicted for higher spherical harmonic modes arising from compactly supported data. See also the recently announced results of Luk–Oh on late-time tails on fixed, but *dynamical* black hole spacetime backgrounds, where mathematically rigorous methods are applied to obtain similar deviations [Luk21]. The above works suggest that in the full nonlinear theory, compactly supported Cauchy data should lead to the following late-time tails along $\mathcal{I}^+$:

$$r\Psi^{[4]}|_{\mathcal{I}^+} \sim u^{-5}.$$

¹⁴A similar lack of conserved charges occurs also in the model problems of scalar fields with an inverse-square potential on Schwarzschild, and a new mechanism is developed in [Gaj22] to overcome this.

While this is slower than the u^{-6} -tail expected from linear theory (for compactly supported data), this late-time tail still decays *faster* than the u^{-3} -tail that we predicted above as a consequence of the failure of peeling. In light of this, we expect that, even in the fully nonlinear theory, the dominant late-time behaviour will be u^{-3} , provided that we consider the physically motivated scattering data of the present chapter.

Finally, in view of the impressive techniques that have been developed to study the dynamics of gravitational radiation in the setting of the full system of the *nonlinear Einstein vacuum equations* [CK93, DHRT21], we expect a mathematically rigorous investigation of the above prediction to be within reach.

4.8 Conclusion

We want to conclude with the following points:

- ◊ If one has initial data on some hyperboloidal hypersurface Σ_0 for which one can define nonvanishing $f(r)$ -modified N–P charges, then the late-time asymptotics towards i^+ can be read off from these N–P charges according to Table 4.4.2. The crucial question then is: What is the right choice of $f(r)$ -modified N–P charge?
- ◊ If one poses polynomially decaying data on some ingoing null hypersurface emanating from $\mathcal{I}^-$ and excludes radiation coming in from $\mathcal{I}^-$, then, because of the nonvanishing background mass M near spatial infinity i^0 , the backscatter of gravitational radiation at early times will lead to $\mathcal{I}^+$ not being smooth, and this failure of smoothness will determine the choice of $f(r)$ -modified N–P charge and, therefore, the late-time tails of gravitational radiation near i^+ , see Table 4.6.1. The smoothness of $\mathcal{I}^-$ plays no role here.
- ◊ The assumption of polynomial decay towards $\mathcal{I}^-$, in turn, comes from post-Newtonian arguments and the assumption that the system under consideration, e.g. two infalling masses, follows approximately hyperbolic Keplerian orbits in the infinite past.

We record again that it is frequently assumed throughout large parts of the literature that one has spatially compact support on Σ_0 , or that gravitational radiation has only started radiating at some fixed, finite time (which of course implies the former). However, in the context of an isolated system describing an astrophysical process, we believe the assumptions of the present chapter, i.e. that the system under consideration has radiated *for all times*, to be more natural.

Independently of the above considerations, we also hope to have convinced the reader that, even from a purely theoretical point of view, the assumption of smooth null infinity might be too rigid, and that by avoiding this assumption, one can perform many more general arguments that give new and deeper insights into the nature of general relativity!

Chapter 5

Early-time asymptotics for linearised gravity—An overview

Abstract

In this chapter, we finally move on to the system of linearised gravity around Schwarzschild. We provide an overview of the physical ideas involved in setting up the mathematical problem, the mathematical challenges that need to be overcome once the problem is posed, as well as the main new results we will obtain in chapter 6 and upcoming works.

From the physical perspective, this includes a discussion of how Post-Newtonian theory provides a prediction on the gravitational radiation emitted by N infalling masses from the infinite past *in the intermediate zone*, i.e. up to some finite advanced time.

From the mathematical perspective, we then take this prediction, together with the condition that there be no incoming radiation from $\mathcal{I}^-$, as a starting point to set up a scattering problem for the linearised Einstein vacuum equations around Schwarzschild and near spacelike infinity, and we outline how to solve this scattering problem and obtain the asymptotic properties of the scattering solution near i^0 and $\mathcal{I}^+$.

We also provide commentary on the consequences our results have.

5.1	Introduction	294
5.1.1	Structure of the remainder of the chapter	295
5.2	The physical setup for the generation problem	296
5.2.1	The physical picture that we want to describe	296
5.2.2	The multipolar Post-Minkowskian expansion and the Post-Newtonian prediction	296
5.2.3	Comments and conceptual problems	298
5.3	The mathematical setup for the propagation problem	299
5.3.1	The system of linearised gravity around Schwarzschild (LGS)	299
5.3.2	The pure gauge solutions and the linearised Kerr solutions of (LGS)	301

5.3.3	Scattering theory for (LGS) : The seed scattering data	301
5.3.4	Bondi normalisation of the seed scattering data	302
5.3.5	Seed scattering data describing the exterior of the N -body problem	302
5.3.6	The main result	303
5.4	Solving the scattering problem I: Constructing the unique solution	304
5.4.1	The Teukolsky equations and the Regge–Wheeler equation	304
5.4.2	Scattering theory for the Regge–Wheeler equation	305
5.4.3	Scattering theory for the remainder of (LGS) :	306
5.5	Solving the scattering problem II: Asymptotic analysis of solutions $\alpha^{[s]}$ to the Teukolsky equations	306
5.5.1	The induced scattering data for $\alpha^{[+2]}$ and $\alpha^{[-2]}$	306
5.5.2	A sketch of the problem for the lowest angular mode $\alpha_{\ell=2}^{[+2]}$	307
5.5.3	A sketch for higher angular modes $\alpha_{\ell}^{[+2]}$	308
5.5.4	A sketch for higher angular modes $\alpha_{\ell}^{[-2]}$	309
5.5.5	The summing of the ℓ -modes	310
5.6	Solving the scattering problem III: Asymptotic analysis of the remainder of (LGS)	311
5.6.1	Bondi normalising $\mathcal{I}^+$ in addition to $\mathcal{I}^-$	311
5.6.2	Corrections to the quadrupole formula: dE/du along $\mathcal{I}^+$	311
5.6.3	The antipodal matching condition	312
5.7	The question of late-time asymptotics	312
5.8	Conclusion	313

5.1 Introduction

In this chapter, we physically motivate and give an overview of mathematical constructions of solutions to the linearised Einstein vacuum equations around Schwarzschild with the following features:

- (i) They describe the far field region of a system of N infalling masses coming from the infinite past and following approximately hyperbolic Keplerian orbits.
- (ii) They have no incoming radiation from past null infinity.
- (iii) They violate peeling near past null infinity: Near $\mathcal{I}^-$, the extremal component of the Weyl tensor Ψ_4 (a.k.a. $\alpha^{[-2]}$ a.k.a. $\underline{\alpha}$) decays like r^{-4} rather than r^{-5} .

- (iv) They violate peeling near future null infinity: Near $\mathcal{I}^+$, the other extremal component of the Weyl tensor, Ψ_0 (a.k.a. $\alpha^{[+2]}$ a.k.a. α), decays like $r^{-4} + r^{-5} \log^2 r$ rather than r^{-5} . In particular, the constructed class of spacetimes does not admit a smooth null infinity/conformal compactification.
- (v) They decay slower towards spacelike infinity than assumed in most stability works. Both Ψ_0 and Ψ_4 decay like r^{-3} along $t = 0$, as opposed to the $o(r^{-7/2})$ -decay rate assumed in [CK93].

In the list above, the bullet points (ii) and (iii) play the role of *scattering data assumptions*, from which we will construct a (unique) *scattering solution*. These scattering data assumptions, in turn, are motivated by the physical considerations involved in (i). Finally, (iv) and (v) play the role of theorems that we prove as we analyse asymptotic properties of this scattering solution.

The detailed construction and asymptotic analysis of these solutions will be presented in the next chapter (chapter 6) and [KK23], the latter being work in progress. The purpose of the present chapter is to already give a rough outline of the main problems, ideas and results of the construction, as well as a discussion of the physical motivation for the construction.

For the reader who has not yet read the introduction to this thesis, we now strongly recommend to read §0.1 for historical background and context for the problem studied in this chapter as we will be directly referencing §0.1.

5.1.1 Structure of the remainder of the chapter

In §5.2, we give an outline of the Post-Newtonian analysis of the *generation problem* for a system of N infalling masses coming from the infinite past.

In §5.3, we describe how this Post-Newtonian analysis informs the mathematical setup of a scattering problem for the linearised Einstein vacuum equations around Schwarzschild, i.e. the mathematical setup for the *propagation problem*.

In §5.4 and §5.5, we describe how this scattering problem is solved, and we study the asymptotic properties of the Weyl curvature tensor of the scattering solution. In particular, we give a quick and intuitive sketch of why peeling fails in the way it does.

In §5.6, starting from the asymptotics of the Weyl tensor, we sketch how to derive the asymptotics of the remainder of the system and comment on various interesting properties that the resulting solutions have. We briefly discuss consequences of our results on late-time asymptotics in §5.7 and conclude in §5.8.

5.2 The physical setup for the generation problem

We here sketch the physical setup that will inform our mathematical construction. This section does not aim or claim to be rigorous, instead, we want to give a “quick and dirty” justification for the assumptions we will make for our mathematical scattering setup in §5.3. Some conceptual difficulties are mentioned in §5.2.3.

5.2.1 The physical picture that we want to describe...

...is that of a system of two infalling masses (with negligible internal structure) whose trajectories approach hyperbolic (or parabolic) Keplerian orbits in the infinite past. More generally, the contents of this section apply to systems of N infalling masses from the infinite past whose relative velocities approach constant, nonrelativistic values. In addition, we prohibit incoming radiation from $\mathcal{I}^-$. By virtue of these masses moving slowly and the separation between them becoming large, we expect perturbations around the Newtonian theory to provide us with a good prediction on the gravitational radiation generated by this system.

The plan is now as follows: We enclose the system by an ingoing null cone $\mathcal{C}$ from $\mathcal{I}^-$, truncated at some finite retarded time since we only care about the behaviour of the system in the distant past. Up until this null cone (we denote this region *the intermediate region*), we assume the validity of a suitable Post-Newtonian framework in order to understand the *generation problem*, in particular, we use this framework to get a prediction on the behaviour of gravitational radiation along $\mathcal{C}$ towards $\mathcal{I}^-$. See Fig. 5.2.1.

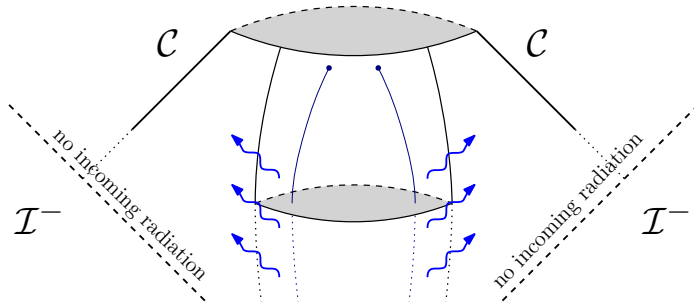


Figure 5.2.1 We apply perturbative methods to study the generation of gravitational waves up until some null cone $\mathcal{C}$. The region beyond $\mathcal{C}$ will later be studied rigorously by taking the behaviour along $\mathcal{C}$ as given.

5.2.2 The multipolar Post-Minkowskian expansion and the Post-Newtonian prediction

The generation of gravitational waves by Newtonian or Post-Newtonian sources has been studied in early references, see, for instance, [EW75, WW76] (and [Bla14] for a more recent review). The physical picture of two weakly interacting masses in the infinite past as described in §5.2.1 was first studied to leading order in the Post-Newtonian expansion parameter v/c by Walker and Will [WW79a, WW79b], v denoting the characteristic speed of the system and c the speed of light. This expansion corresponds to an expansion that only takes into account the

quadrupole moment of gravitational radiation, higher multipole moments being of higher order in v/c . The relevant results can be read off from eqns. (61), (62) in [WW79b]. In particular (the reason why we single out the following results will be made clear in §5.3), the following asymptotic behaviour was found along the incoming Minkowskian null cones: Ψ_0 was found to decay like r^{-3} towards $\mathcal{I}^-$, Ψ_4 was found to decay like r^{-4} (violating peeling near $\mathcal{I}^-$ (0.1.4)), and the ingoing shear along the null cones, λ , was found to decay like r^{-2} . In each case, the coefficient of the relevant decay rate is related to (time derivatives of) the quadrupole moment of the system, more precise expressions will be given below.

In order to obtain a more complete picture, we also want to understand the asymptotic behaviour of higher multipoles; for this, we consult Thorne’s work on multipole expansions in GR [Tho80] (see also the appendix of [Sac61] or Pirani’s lecture notes [TPB64]).

Let’s give a quick summary of the computations in [Tho80]: We will temporarily work in Cartesian Minkowskian coordinates $(x^0 = t, x^1, x^2, x^3)$, and we will take Latin indices to only range from 1 to 3. First, Thorne considers linearised perturbations $g_{\mu\nu}^1 = g_{\mu\nu} - \eta_{\mu\nu}$ around the Minkowski metric $\eta_{\mu\nu}$. By imposing de Donder gauge, the linearised Einstein vacuum equations then reduce to Minkowskian wave equations for the Cartesian components of the trace-reversed metric perturbation $g_{\mu\nu}^1 - \frac{1}{2}\eta_{\mu\nu}\eta^{\alpha\beta}g_{\alpha\beta}^1$, and the general *outgoing wave* solution for g^1 , assuming the coordinates to be mass-centered, is then given in equations (8.13) of [Tho80] in terms of an expansion into multipoles $\mathcal{J}_{A_\ell}$ and $\mathcal{S}_{A_\ell}$, the radiation field corresponding to this solution being given by eq. (4.8) of [Tho80].

Let now (t, r, θ, φ) denote the usual spherical polar coordinates, introduce the double null coordinates $u = t - r$, $v = t + r$, define the null tetrad $l = \partial_t + \partial_r = 2\partial_v$, $n = \partial_t - \partial_r = 2\partial_u$, $m_{\pm 2} = \frac{1}{\sqrt{2}r}(\partial_\theta \pm i\frac{\partial_\varphi}{\sin\theta})$, and then define the Newman–Penrose scalars

$$\Psi_0 = W_{\mu\nu\rho\sigma}^1 l^\mu m^\nu l^\rho m^\sigma, \quad \Psi_4 = W_{\mu\nu\rho\sigma}^1 n^\mu \bar{m}^\nu n^\rho \bar{m}^\sigma, \quad (5.2.1)$$

where W^1 denotes the linearised Weyl tensor of g . From the form of the metric (8.13), we then compute (this generalises [JN65, Lam66]), for $s = \pm 2$

$$\Psi_{2-s} = -2^s \sum_{\ell=2}^{\infty} \sum_{m=-\ell}^{\ell} \frac{(\ell+s)!}{(\ell-2)!} r^{\ell-|s|} \partial_u^{\ell-s} \left(\frac{I_{\ell m}(u) T_{ij}^{\text{E}2, \ell m} - \frac{s}{|s|} S_{\ell m}(u) T_{ij}^{\text{B}2, \ell m}}{r^{1+\ell+s}} \right) m_s^i m_s^j \quad (5.2.2)$$

where the $I_{\ell m}$ and $S_{\ell m}$ are related to $\mathcal{J}_{A_\ell}$ and $\mathcal{S}_{A_\ell}$ via Thorne’s (4.7) and $T_{ij}^{\text{E}2, \ell m}$ and $T_{ij}^{\text{B}2, \ell m}$ denote the electric and magnetic transverse traceless tensor spherical harmonics defined in Thorne’s (2.30). (See Thorne’s eq. (2.38) for the relation to spin-weighted harmonics.)

By now considering slow-motion, weak internal gravity sources in the “near zone” of spacetime, by using the outgoing Minkowskian Green’s function and by then matching it to the vacuum expression for the metric (8.13), it is derived that, to leading order in v/c , the moments $I_{\ell m}$ and $S_{\ell m}$ are given by the Newtonian mass and current multipole moments (eq. (5.27)

of [Tho80]):

$$\begin{aligned} I_{\ell m} &= \frac{16\pi}{(2\ell+1)!!} \sqrt{\frac{(\ell+1)(\ell+2)}{2(\ell-1)\ell}} \int \rho \cdot r^\ell Y^{\ell m*} d^3x, \\ S_{\ell m} &= \frac{32\pi}{(2\ell+1)!!} \sqrt{\frac{(\ell+2)}{2(\ell-1)}} \int (\epsilon_{jpq} x_p \cdot \rho v_q) \cdot r^{\ell-1} Y_j^{B,\ell m*} d^3x. \end{aligned} \quad (5.2.3)$$

In the formulae above, ϵ denotes the Levi-Civita symbol, ρ and ρv_j denote the Newtonian mass and momentum density, and $Y^{B,\ell m}$ is the magnetic vector spherical harmonic (defined in Thorne's (2.18)).

With these computations at hand, we can now extend the results of [WW79b] for the setting of two pointlike particles moving along approximately hyperbolic Keplerian orbits beyond the mass quadrupole approximation. For such orbits, the leading order behaviour of the relative position vector $x_{\text{rel}}(t)$ of the particles is given by $x_{\text{rel}}^i(t) = v^i t + a^i \log |t| + b^i + \dots$ as $t \rightarrow -\infty$ (see [Ede83] for a similar statement for N particles). Here, the coefficient v corresponds to the asymptotic relative velocity of the orbit, and $|a| \sim (m_1 + m_2)/v^2$ by Newton's equations of motion.

If we insert this asymptotic behaviour for $x_{\text{rel}}(t)$ into the multipole moments (5.2.3), we find that $I_{\ell m}(u) = I_{\ell m}^0 u^\ell + I_{\ell m}^1 u^{\ell-1} \log |u| + \dots$, and $S_{\ell m}(u) = S_{\ell m}^0 u^\ell + S_{\ell m}^1 u^{\ell-1} \log |u| + \dots$, where $|I_{\ell m}^0|$ is of magnitude $(v/c)^\ell$, and where $|S_{\ell m}^0| \sim (v/c)^{\ell+1}$. We conclude from this and (5.2.2) that, as we approach $\mathcal{I}^-$, all ℓ -modes $\Psi_{2\pm 2, \ell} \sim r^{\ell-2} \partial_u^{\ell\pm 2} (\frac{u^\ell + u^{\ell-1} \log |u|}{r^{\ell+1\mp 2}})$ of the Newman–Penrose scalars $\Psi_{2\pm 2}$ decay like:

$$\Psi_{0, \ell} = \frac{\Psi_{0, \ell}^0}{r^3} + \frac{\Psi_{0, \ell}^1 \log r}{r^4} + \frac{\Psi_{0, \ell}^2}{r^4} + \dots, \quad \Psi_{4, \ell} = \frac{\Psi_{4, \ell}^0}{r^4} + \dots, \quad (5.2.4)$$

where $\Psi_{0, \ell}^0$ is computed from $I_{\ell m}^0$ (and $S_{\ell m}^0$), $\Psi_{0, \ell}^1$ is computed from $I_{\ell m}^1$ (and $S_{\ell m}^1$) and, up to complex conjugation and ℓ -dependent multiple, $\Psi_{0, \ell}^1 \sim \Psi_{4, \ell}^0$. In the above, we used that $|u| \sim r$ near $\mathcal{I}^-$. It is also straight-forward to compute that all higher ℓ -modes of the ingoing shear decay as for $\ell = 2$ (cf. eq. (63) of [WW79b]), $\lambda_\ell \sim r^{-2}$.

A similar story can be told for parabolic orbits, where $x_{\text{rel}}(t) \sim t^{\frac{2}{3}}$. The behaviour of the $\ell = 2$ modes of Ψ_0 and Ψ_4 near $\mathcal{I}^-$ is then given by $r^{-\frac{11}{3}}$. In contrast to hyperbolic orbits, since parabolic orbits do not grow linearly in time, higher ℓ -modes will now decay faster towards $\mathcal{I}^-$: $\Psi_{2\pm 2, \ell} \lesssim r^{-3-\ell+\frac{2\ell}{3}}$.

5.2.3 Comments and conceptual problems

We conclude with some comments. First, notice that (5.2.2) implies that $\Psi_0 = \mathcal{O}(r^{-5})$ near $\mathcal{I}^+$, independently of the structure of the multipole moments. Historically, this fact was used as justification for the peeling property (0.1.3), e.g. in [NP62]. Similarly, the fact that outgoing wave solutions to the linearised field equations around Minkowski have an expansion in powers of $1/r$ (see Thorne's (8.13)) motivated the setup of Bondi coordinates.

What we will see in the present note, however, is that Thorne's expression for the metric perturbation (8.13) breaks down at late advanced times, i.e. near $\mathcal{I}^+$, due to the effects of the Schwarzschild term M/r . In fact, taking into account this Schwarzschild term will also lead to corrections in Einstein's classical quadrupole formula ((4.18) of [Tho80]). Of course, one might then wonder whether it is already too much to assume the validity of the framework all the way until the null cone $\mathcal{C}$, rather than, say, just assuming it to hold up until some timelike cylinder $\mathcal{T}$ along which $|t| \sim r$ as in [Dam86], see Fig. 5.2.1. Let us here just mention that the propagation of radiation from such a timelike cylinder to the null cone $\mathcal{C}$ was studied at the level of scalar waves on Schwarzschild in chapters 1 and 3, the result of these studies is that the leading order decay towards $u \rightarrow -\infty$ is the same along the cylinder and along $\mathcal{C}$.

Secondly, the approximations described in [Tho80] assume a compact system, and it would be nice to clearly formulate them for the system considered here, namely a system that becomes unbound in the infinite past. A particular feature of the system that we consider, growing linearly in time, is that all higher-order corrections within the Post-Newtonian expansion (see §V.D of [Tho80]), even though they are of smaller magnitude in v/c , feature the same decay towards $\mathcal{I}^-$, in just the same way as all angular modes of Ψ_0 or Ψ_4 feature the same decay towards $\mathcal{I}^-$. Of course, one could interpret the approximate statements (5.2.3) as the leading order result for metric perturbations on fixed angular frequency. If we really wanted to do consistent perturbation theory, however, then writing down the leading-order expressions (5.2.3) for the ℓ -th electric multipole moment means that we should also consider higher-order Post-Newtonian expansions of all lower electric multipoles (i.e. expand the $(\ell - n)$ th multipole up to n orders) etc.

5.3 The mathematical setup for the propagation problem

In the previous section, we gave a rough sketch of the *generation problem* of gravitational radiation in a Post-Newtonian framework. We now describe how we convert the information from §5.2, which was perturbatively obtained for linearised gravity around Minkowski, into a mathematically formulated scattering setup for linearised gravity around Schwarzschild.

Studying linearised gravity around Schwarzschild will on the one hand provide us with a clear understanding of the failure of peeling due to the presence of mass near spatial infinity. On the other hand, we will set up the problem in such a way as to serve as a strong foundation for the study of the actual, nonlinear Einstein vacuum equations.

5.3.1 The system of linearised gravity around Schwarzschild (*LGS*)

Consider the Schwarzschild metric in the familiar form $g_M = -(1 - \frac{2M}{r}) dt^2 + (1 - \frac{2M}{r})^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)$. We introduce the double null coordinates $u = (t - r^*)/2$, $v = (t + r^*)/2$,

where $r^* = r + 2M \log(r/2M)$, so that the metric takes the form

$$g_M = -4\Omega^2 du dv + \mathfrak{g}_{AB} d\theta^A d\theta^B. \quad (5.3.1)$$

Here, $\Omega^2 = 1 - \frac{2M}{r}$, and $\mathfrak{g}$ denotes the standard metric on $\mathbb{S}_r^2$, with $(\theta^1, \theta^2) = (\theta, \varphi)$. (In the following, capital Latin indices will always refer to indices on the sphere and range from 1 to 2.) We will restrict our attention to the very exterior of spacetime, that is to say, we will work on the manifold $\mathcal{M}_M = (-\infty, -1]_u \times [1, \infty)_v \times \mathbb{S}^2$ (see Fig. 5.3.1); and we will work with the ON frame

$$(e_1 = \frac{1}{r}\partial_\theta, e_2 = \frac{1}{r\sin\theta}\partial_\varphi, e_3 = \frac{1}{\Omega}\partial_u, e_4 = \frac{1}{\Omega}\partial_v). \quad (5.3.2)$$

The (double null) system of linearised gravity around Schwarzschild is obtained by considering a general 1-parameter family of Einstein metrics $\mathbf{g}(\varepsilon)$ on $\mathcal{M}_M$ such that $\mathbf{g}(0) = g_M$:

$$\mathbf{g}(\varepsilon) = -4\Omega^2(\varepsilon) du dv + \mathfrak{g}_{AB}(\varepsilon)(d\theta^A - \mathbf{b}^A(\varepsilon) dv)(d\theta^B - \mathbf{b}^B(\varepsilon) dv); \quad (5.3.3)$$

and by then linearising the arising Einstein and Bianchi equations in ε (i.e. by writing $\Omega = \sqrt{1 - \frac{2M}{r}} + \varepsilon^{(1)}\Omega$ etc. and only keeping terms of order ε). See [DHR19b] for details.

The result is a coupled system of 10 hyperbolic equations for the linearised curvature coefficients—the linearised Bianchi equations—together with a system of around 30 transport and elliptic equations for the linearised metric and connection coefficients. We will refer to this system as **(LGS)**.

Examples for linearised metric components are the linearised lapse $\Omega^{(1)}$ and the linearised metric on the spheres $\mathfrak{g}^{(1)}$, which is split up into its trace and its tracefree part, $\mathfrak{g}^{(1)} = \hat{\mathfrak{g}}^{(1)} + \frac{1}{2}\text{tr}\mathfrak{g}^{(1)} \cdot \mathfrak{g}$.

Examples for linearised connection coefficients are the linearisation of the ingoing and outgoing null expansions, denoted $(\Omega\text{tr}\underline{\chi})^{(1)}$ and $(\Omega\text{tr}\chi)^{(1)}$, respectively, and of the ingoing and outgoing null shears: $\hat{\underline{\chi}}^{(1)}$ and $\hat{\chi}^{(1)}$. The work [DHR19b] employs the Christodoulou–Klainerman formalism. For those familiar with the very closely related Newman–Penrose formalism, the ingoing and outgoing null expansions (or shears) are called μ and ρ (or λ and σ), respectively.

Examples for linearised curvature coefficients are the linearised Gauss curvature $\hat{K}^{(1)}$ on the spheres, as well as the extremal curvature components $\alpha^{[-2]}$ and $\alpha^{[+2]}$:

$$\alpha^{[-2]}(e_A, e_B) = \Omega^2 \hat{W}^{(1)}(e_3, e_A, e_3, e_B), \quad \alpha^{[+2]}(e_A, e_B) = \Omega^{-2} \hat{W}^{(1)}(e_4, e_A, e_4, e_B). \quad (5.3.4)$$

These are the real-valued symmetric tracefree two-tensor analogues to $\Psi_4^{(1)}$ and $\Psi_0^{(1)}$: Identifying $l = \Omega e_4$, $n = \Omega^{-1} e_3$, $\sqrt{2}m_{\pm 2} = e_1 \pm ie_2$, we have $\hat{\Psi}_{2\mp 2}^{(1)} = \alpha^{[\pm 2]}(m_{\pm 2}, m_{\pm 2})$.

Examples for equations of **(LGS)** are given by (writing $\hat{\chi}_{AB}^{(1)} = \hat{\chi}(e_A, e_B)$ etc.)

$$\partial_u(\hat{\mathfrak{g}}_{AB}^{(1)}) = 2\Omega \hat{\underline{\chi}}_{AB}^{(1)}, \quad \partial_u(\Omega^{-1} r^2 \hat{\underline{\chi}}_{AB}^{(1)}) = -\Omega^{-2} r^2 \alpha_{AB}^{[-2]}, \quad \partial_v(\Omega^{-1} r^2 \hat{\chi}_{AB}^{(1)}) = -\Omega^2 r^2 \alpha_{AB}^{[2]}. \quad (5.3.5)$$

5.3.2 The pure gauge solutions and the linearised Kerr solutions of (LGS)

Central to the understanding of (LGS) is the existence of two classes of explicit solutions to it. The first class of these consists of the **linearised Kerr solutions**: These are produced by linearising a nearby Kerr metric in double null coordinates around Schwarzschild, or by linearising a Schwarzschild solution with nearby mass around the original Schwarzschild.

The other class of solutions is given by the **pure gauge solutions**. These are generated by linearising nonlinear coordinate transformations of the double null coordinates $(u, v, \theta^1, \theta^2)$ that preserve the double null form of the metric (5.3.3).

It turns out that the linearised Kerr solutions are entirely supported on spherical harmonics with $\ell < 2$. Conversely, any solution supported on $\ell \leq 1$ is given by a linear combination of such linearised Kerr and pure gauge solutions. In other words, the $\ell \leq 1$ -part of (LGS) doesn't carry any dynamics. In what follows, we will therefore exclusively discuss solutions supported on $\ell \geq 2$.

In view of the existence of these solutions, it is very helpful to extract quantities from the system (LGS) that are invariant under addition of pure gauge or Kerr solutions. Examples for such *gauge invariant* quantities are $\alpha^{[+2]}$ and $\alpha^{[-2]}$ (i.e. $\overset{(i)}{\Psi}_0$ and $\overset{(i)}{\Psi}_4$). In fact, any solution that has both $\alpha^{[-2]}$ and $\alpha^{[+2]}$ (or $\overset{(i)}{\Psi}_4$ and $\overset{(i)}{\Psi}_0$) vanishing is given by a combination of linearised Kerr and pure gauge solutions. See [Wal73, DHR19b] for details.

5.3.3 Scattering theory for (LGS) : The seed scattering data

The quantitative stability of (LGS) was first shown in [DHR19b] (and provided the central ingredient to the more recent nonlinear stability of Schwarzschild proof [DHRT21]), and a global scattering theory to (LGS) was written down in [Mas22b].

In the present chapter, we will be concerned with the *semi-global scattering theory* for (LGS) : Given appropriate data on the truncated ingoing null cone $\mathcal{C} = \mathcal{M}_M \cap \{v = 1\}$ (whose future end sphere we will denote as $\mathcal{S}_1$ and whose past limiting sphere we will denote as $\mathcal{S}_\infty$, see Fig. 5.3.1) coming from $\mathcal{I}^-$ together with data along $\mathcal{I}^-$ to the future of this cone, we want to construct the unique solution to (LGS) “restricting” to these data in the limit.

We now describe what scattering data consist of, and which parts of these scattering data carry physical information. As the $\ell < 2$ -part (LGS) is non-dynamical, we will only present the discussion for data and solutions supported on $\ell \geq 2$.

Definition 5.3.1. *An $\ell \geq 2$ seed scattering data set consists of the following prescribed quantities: Along $\mathcal{C}$, prescribe $\overset{(i)}{\hat{\chi}}$ and $\partial_u \overset{(i)}{\Omega}$. Along $\mathcal{I}^-$ (to the future of $\mathcal{S}_\infty$), prescribe $\overset{(i)}{\hat{\phi}}$, $\overset{(i)}{\Omega}$ and the radiation field of the outgoing null shear $r \cdot \overset{(i)}{\hat{\chi}}$ (a.k.a. the News tensor). Finally, prescribe on $\mathcal{S}_1$ the values of $\partial_v \alpha^{[-2]}$ as well as the ingoing null expansion $(\Omega \text{tr} \overset{(i)}{\chi})$, and prescribe on $\mathcal{S}_\infty$ the weighted outgoing null expansion $r \cdot (\Omega \text{tr} \overset{(i)}{\chi})$ together with the Gauss curvature $r^2 \overset{(i)}{K}$.*

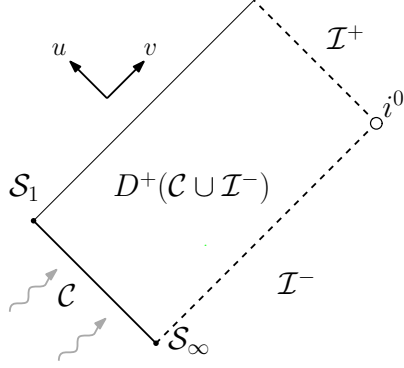


Figure 5.3.1 We rigorously study the propagation problem to the future of the null cone $\mathcal{C}$ by formulating a scattering problem for **(LGS)** in $D^+(\mathcal{C} \cup \mathcal{I}^-)$. We always think of the scattering data along $\mathcal{C}$ as capturing the radiation of some physical system, cf. Fig. 5.2.1.

From these seed scattering data, one can construct all other quantities (e.g. $\alpha^{[-2]}$ or $\alpha^{[+2]}$) along $\mathcal{C}$ and $\mathcal{I}^-$, and then construct the unique solution to **(LGS)** in the entire domain of dependence $D^+(\mathcal{C} \cup \mathcal{I}^-)$ that restricts to these data (in the limit). A sketch of this is given in §5.4.

5.3.4 Bondi normalisation of the seed scattering data

We now discuss which part of the seed data carry physical meaning. In chapter 6, it is shown that, upon adding pure gauge solutions, any seed scattering data set can be Bondi normalised at $\mathcal{I}^-$; this means:

- $\hat{\mathcal{G}}^{(1)}$ and $\hat{\Omega}^{(1)}$ along $\mathcal{I}^-$ can be set to zero, and
- $r^2 \hat{K}^{(1)}$ and $r(\Omega \text{tr} \chi)^{(1)}$ on $\mathcal{S}_\infty$ can be set to zero.

Together, the two bullet points imply that $r^2 \hat{K}^{(1)}$ and $r(\Omega \text{tr} \chi)^{(1)}$ vanish along all of $\mathcal{I}^-$, meaning that the spheres at $\mathcal{I}^-$ are the standard round spheres. This normalisation captures the spirit of the original Bondi coordinates, except that it does not make any assumptions on $1/r$ -expansions. Henceforth, we will always assume our seed data to be Bondi normalised.

The Bondi normalisation as described above still leaves us with some remaining gauge freedom which, for example, contains the group of BMS transformations on $\mathcal{I}^-$, see [Mas22b] for details. The remaining gauge freedom can also be used to set $\partial_u \hat{\Omega}^{(1)}$ along $\mathcal{C}$ and $(\Omega \text{tr} \underline{\chi})^{(1)}|_{\mathcal{S}_1}$ to 0. Doing this, we see that the physical content of the seed scattering data is carried by $r \hat{\chi}^{(1)}|_{\mathcal{I}^-}$, $\hat{\chi}^{(1)}|_{\mathcal{C}}$ as well as $\partial_v \alpha^{[-2]}|_{\mathcal{S}_1}$.

5.3.5 Seed scattering data describing the exterior of the N -body problem

Having understood what constitutes scattering data for **(LGS)**, we now want to pose scattering data as motivated by the considerations in §5.2. What we would like to do is to just take the statements (5.2.4) and demand that our seed scattering data are such that the predicted asymptotics for $\alpha^{[-2]}(=\Psi_4)$ and $\alpha^{[+2]}(=\Psi_0)$ near $\mathcal{I}^-$ are satisfied. But of course, the results

of §5.2 concern linearised gravity around Minkowski and, as such, do not translate directly into results for linearised gravity around Schwarzschild. Complications in relating the two include the logarithmic divergence of the null cones in Schwarzschild compared to Minkowski, metric perturbations around Schwarzschild being different objects from metric perturbations around Minkowski, as well as changes to the treatment of the near zone generating the relationship to the Newtonian theory. As these complications are quite severe, we will not attempt to treat them, but instead just hope that all of these changes are subleading in terms of decay towards $\mathcal{I}^-$. On the basis of this hope, we formulate the following definition:

Definition 5.3.2. *A Bondi normalised seed scattering data set is said to describe the exterior of a system of N infalling masses following approximately hyperbolic orbits if the following conditions are satisfied for all $\ell \geq 2$:*

- (I) $\alpha^{[-2]}$ along $\mathcal{C}$ satisfies $\alpha_\ell^{[-2]} = \mathcal{B}_\ell r^{-4} + \mathcal{O}(r^{-4-\epsilon})$ for some $\mathcal{B}_\ell \neq 0$ independent of u , and the limit $\lim_{u \rightarrow -\infty} r^2 \overset{(i)}{\chi}_\ell$ along $\mathcal{C}$ is finite and nonvanishing.
- (II) The limit $\lim_{u \rightarrow -\infty} r^3 \alpha_\ell^{[+2]} = \mathcal{A}_\ell$ is finite and nonvanishing.

Finally, we say that such data also satisfy the no incoming radiation condition from $\mathcal{I}^-$ if additionally the News tensor $r \overset{(i)}{\chi}$ vanishes along $\mathcal{I}^-$.

In the above definition, $\alpha_\ell^{[+2]}$ denotes the projection of $\alpha^{[+2]}$ onto the electric and magnetic tensor harmonics $T^{\text{E}2,\ell m}$ and $T^{\text{B}2,\ell m}$ introduced below (5.2.2). We will occasionally use an overline to denote magnetic conjugation: Given $\mathcal{A} = \mathcal{A}^{\text{E}2} + \mathcal{A}^{\text{B}2}$, $\overline{\mathcal{A}}$ denotes $\overline{\mathcal{A}} = \mathcal{A}^{\text{E}2} - \mathcal{A}^{\text{B}2}$.

The assumptions (I) and (II) are directly motivated by eq. (5.2.4), whereas the last condition of Def. 5.3.2 realises the physical requirement of excluding incoming radiation from $\mathcal{I}^-$, it is the linearised version of demanding that the Bondi mass along $\mathcal{I}^-$ be constant.

It is, of course, possible to formulate the conditions of Def. 5.3.2 directly in terms of the seed data—see chapter 6 (in fact, for (I), the reader can see this from (5.3.5)).

5.3.6 The main result

We already formulate a rough version of one of the main results of chapter 6:

Theorem 5.3.1. *There exists a unique solution to (LGS) attaining the scattering data of Def. 5.3.2. This solution satisfies, throughout $D^+(\mathcal{C} \cap \mathcal{I}^-)$:*

$$\begin{aligned} \alpha_\ell^{[+2]} = & \frac{1}{r^5} \sum_{n=0}^{\ell-2} f_{n,\ell}(u) \left(\frac{|u|}{r} \right)^n \\ & + \frac{2M(-1)^{\ell+1}}{(\ell-1)(\ell+2)} \cdot \frac{\mathcal{A}_\ell}{r^4} + \frac{M(-1)^{\ell+1}(\ell-1)(\ell+2)}{2} \frac{\overline{\mathcal{B}}_\ell \log^2 r}{r^5} + \mathcal{O}(Mr^{-5} \log r) \end{aligned} \quad (5.3.6)$$

Here, $f_n(u) = C_{n,\ell} \mathcal{A}_\ell u^2 + \mathcal{O}(|u| \log |u|)$ for some algebraic constants $C_{n,\ell}$.

Moreover, the solution satisfies, as $u \rightarrow -\infty$,

$$\lim_{u=\text{const}, v \rightarrow \infty} r\alpha_\ell^{[-2]} = (-1)^{\ell+1} \cdot 2|u|^{-3} \left(\frac{\mathcal{B}_\ell}{\ell(\ell+1)} + \frac{2M\overline{\mathcal{A}}_\ell(\ell-2)!}{(\ell+2)!} \right) + \mathcal{O}(|u|^{-3-\epsilon}). \quad (5.3.7)$$

The proof of this result will be sketched in §§5.4–5.5, and further commentary on the result will be provided in the sections afterwards.

5.4 Solving the scattering problem I: Constructing the unique solution

In §5.3, we outlined the mathematical setup of the scattering problem for **(LGS)** and related this setup to the generation of gravitational waves discussed in §5.2 in §5.3.2. We will now sketch the general construction of scattering solutions to **(LGS)**.

5.4.1 The Teukolsky equations and the Regge–Wheeler equation

The system **(LGS)** may at first seem quite untractable, on the one hand owing to its size (it consists of 37 equations for 19 unknowns), on the other hand due to the large space of pure gauge solutions. A key realisation in unlocking **(LGS)**, bypassing both difficulties, is that the gauge invariant quantities $\alpha^{[+2]}$ and $\alpha^{[-2]}$ each satisfy a decoupled wave equation known as the Teukolsky equation, which, in components ($\alpha_{AB}^{[s]} = \alpha^{[s]}(e_A, e_B)$), reads:

$$\begin{aligned} & \partial_u(r^{-2s}\Omega^{2s}\partial_v(r^{|s|+s+1}\alpha_{AB}^{[s]})) \\ &= \frac{\Omega^{2s+2}}{r^{2s+2}}r^{|s|+s+1}((\overset{\circ}{\Delta} + s)\alpha^{[s]})_{AB} - \frac{2M\Omega^{2s+2}}{r^{2s+3}}(1+s)(1+2s)r^{|s|+s+1}\alpha_{AB}^{[s]}, \quad s = \pm 2. \end{aligned} \quad (5.4.1)$$

Here, $\overset{\circ}{\Delta}$ denotes the Laplacian on $\mathbb{S}^2$ acting on a two-tensor, its eigenvalues are $-\ell(\ell+1)+4$, $\ell \in \mathbb{N}_{\geq 2}$. Henceforth, we will by $\partial_v\alpha^{[s]}$ or $\partial_u\alpha^{[s]}$ always mean the derivative of the components of $\alpha^{[+2]}$ (i.e. $\partial_v(\alpha^{[s]}(e_A, e_B))$), whereas, by $\overset{\circ}{\Delta}\alpha^{[s]}$, we will mean the components of $\overset{\circ}{\Delta}$ acting on a two-tensor (i.e. $(\overset{\circ}{\Delta}\alpha^{[s]})(e_A, e_B)$).

Despite the large body of literature on linear wave equations on black hole backgrounds, a robust analysis of the Teukolsky equation for a long time seemed inaccessible due to the first order terms that appear when writing (5.4.1) in standard wave form ($\partial_u\partial_v\alpha^{[s]} = \dots$). Triumph over this difficulty was achieved in [DHR19b] by exploiting the following observation: The first order terms in (5.4.1) disappear for certain weighted higher-order derivatives of $\alpha^{[s]}$. Indeed, if $\overset{(i)}{\Phi}$ denotes either $(\Omega^{-2}r^2\partial_u)^2(r\Omega^4\alpha^{[+2]})$ or $(\Omega^{-2}r^2\partial_v)^2(r\alpha^{[-2]})$, then (5.4.1) implies

$$\partial_u\partial_v\overset{(i)}{\Phi} = \frac{\Omega^2}{r^2}(\overset{\circ}{\Delta} - 4)\overset{(i)}{\Phi} + \frac{6M\Omega^2}{r^3}\overset{(i)}{\Phi}. \quad (5.4.2)$$

Equation (5.4.2), known as the Regge–Wheeler equation, is nothing but the tensorial linear wave equation with a positive potential and, as such, can be treated using well-established methods. In particular, (5.4.2) admits a conserved energy: If $T_{\mu\nu}$ denotes the canonical energy momentum tensor associated to (5.4.2), then $\nabla^\mu(T_{\mu\nu}(\partial_t)^\nu) = 0$.

To give context for the relevance of (5.4.2) within **(LGS)**, let's summarise the strategy of the proof of stability of **(LGS)** in [DHR19b]. First, the authors prove stability estimates for $\Phi^{(1)}$ using wave equation methods for (5.4.2). Then, they achieve from these estimates control over the extremal curvature components $\alpha^{[+2]}$ and $\alpha^{[-2]}$. Finally, they use the control over $\alpha^{[+2]}$ and $\alpha^{[-2]}$, together with delicate considerations of gauge, to derive estimates for the entire remainder of **(LGS)**.

5.4.2 Scattering theory for the Regge–Wheeler equation

The construction of a scattering theory for **(LGS)** follows a pattern similar to the proof of stability for **(LGS)**: First, given a seed scattering data set (Def. 5.3.1), we derive expressions for the induced for the corresponding $\Phi^{(1)} = (\Omega^{-2}r^2\partial_u)^2(r\Omega^4\alpha^{[+2]})$ along $\mathcal{C} \cup \mathcal{I}^-$.

Then, we temporarily ignore the rest of the system and only focus on the construction of the unique solution to (5.4.2) restricting to these data. This is done in the following way: We consider a sequence of finite characteristic initial value problems (as depicted in Fig. 5.4.1), where the data are posed on an outgoing null cone at finite retarded time $u = -n$ such that these data approach the scattering data along $\mathcal{I}^-$ in the limit $n \rightarrow \infty$. The unique existence of solutions to these finite characteristic initial value problems is ensured by standard theorems. We finally exploit the fact that (5.4.2) admits a conserved energy to show uniform-in- n estimates for these solutions, from which we deduce that the sequence of finite solutions constructed this way converges to a unique limiting solution—the scattering solution. In fact, the conserved energy associated to (5.4.2) defines *unitary* Hilbert space isomorphisms between spaces of scattering data along $\mathcal{C} \cup \mathcal{I}^-$ and spaces of restrictions of the scattering solution to any Cauchy hypersurface in $D^+(\mathcal{C} \cup \mathcal{I}^-)$. For details, see [Mas22a] or §6.6 of the next chapter.

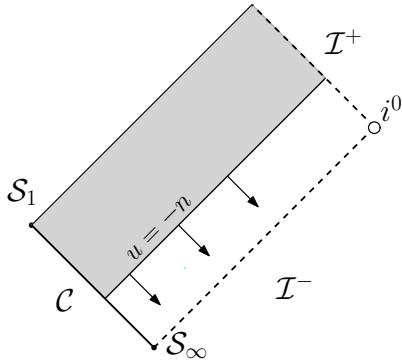


Figure 5.4.1 The limiting procedure that produces the scattering solution.

5.4.3 Scattering theory for the remainder of (LGS) :

At this point, we have a scattering solution $\overset{(i)}{\Phi} = (\Omega^{-2} r^2 \partial_u)^2 (r \Omega^4 \alpha^{[+2]})$ to (5.4.2) that restricts correctly to the data along $\mathcal{C} \cup \mathcal{I}^-$ induced by the seed scattering data. We can now construct the remaining quantities of (LGS) by systemically defining other quantities in terms of the seed scattering data and the solution $\overset{(i)}{\Phi}$. For instance, by integrating $\overset{(i)}{\Phi}$ from $\mathcal{I}^-$, and by computing the value of $\partial_u(r \Omega^4 \alpha^{[+2]})$ along $\mathcal{I}^-$ from the seed scattering data, we can define $\partial_u(r \Omega^4 \alpha^{[+2]})$ in all of $D^+(\mathcal{C} \cup \mathcal{I}^-)$. Similarly, we define $\alpha^{[+2]}$.

Owing to the high degree of redundancy in the system (LGS) (there are more equations than unknowns), this procedure is somewhat subtle. Let us illustrate the difficulty: The system (LGS) includes equations for the ∂_v- , the ∂_u- and angular derivatives for $\overset{(i)}{\hat{\chi}}$, one of them being written down in (5.3.5). We could thus choose to define $\overset{(i)}{\hat{\chi}}$ as the solution to the third of (5.3.5) (assuming we have already defined $\alpha^{[+2]}$), but we might then have difficulties proving that the other equations for $\overset{(i)}{\hat{\chi}}$ are also satisfied. Overcoming this problem requires being very careful about the order in which to define the various quantities and a precise understanding of the structure of the various equations. A detailed discussion of this is found in §6.7 of chapter 6.

We finish this section by remarking that the clean split that can be done for the linear problem, namely to first do a limiting argument to construct a solution $\overset{(i)}{\Phi}$ to (5.4.2) and to then construct all the remaining quantities from this limiting solution $\overset{(i)}{\Phi}$, will likely be no longer possible in the full, nonlinear theory. Instead, one will have to construct a sequence of finite solutions to the entire system, and then show that this entire sequence converges.

5.5 Solving the scattering problem II: Asymptotic analysis of solutions $\alpha^{[s]}$ to the Teukolsky equations

So far, we have written down scattering data along $\mathcal{C} \cup \mathcal{I}^-$ that capture the radiation emitted by an N -body system moving along approximately hyperbolic orbits in the infinite past (§5.3.5), and we have established that one can associate a unique scattering solution to these data (§5.4). With this solution obtained, we shall now sketch how to obtain asymptotic properties of the solution, focussing first on the extremal Weyl components $\alpha^{[+2]}$ and $\alpha^{[-2]}$.

5.5.1 The induced scattering data for $\alpha^{[+2]}$ and $\alpha^{[-2]}$

Since both $\alpha^{[+2]}$ and $\alpha^{[-2]}$ satisfy decoupled equations, we can also decouple the presentation of their analysis from the rest of the system. For this, we merely need to write down the scattering data for $\alpha^{[+2]}$ and $\alpha^{[-2]}$ that are induced by seed scattering data as in Def. 5.3.2. We will henceforth use the notation

$$r_0(u) := r(u, v = 1) = |u| - 2M \log |u| + \dots \quad (5.5.1)$$

Lemma 5.5.1. *Given seed data satisfying Def. 5.3.2, the induced data for $\alpha^{[+2]}$ are given by:*

$$\alpha_\ell^{[+2]}|_{\mathcal{C}} = \mathcal{A}_\ell r_0^{-3} - \frac{1}{6} \frac{(\ell+2)!}{(\ell-2)!} \overline{\mathcal{B}}_\ell r_0^{-4} \log r_0 + \mathcal{C}_\ell r_0^{-4} + \mathcal{O}(r_0^{-4-\epsilon}), \quad \partial_v^n(r\alpha^{[+2]}|_{\mathcal{I}^-} = 0, \quad (5.5.2)$$

for any $n \in \mathbb{N}_{\geq 1}$.

Similarly, the induced data for $\alpha^{[-2]}$ are given by

$$\alpha^{[-2]}|_{\mathcal{C}} = \mathcal{B}_\ell r_0^{-4} + \mathcal{O}(r_0^{-4-\epsilon}), \quad \partial_v^n(\Omega^{-2}r^2\partial_v)^2(r\alpha^{[-2]}|_{\mathcal{I}^-} = 0 \quad \forall n \in \mathbb{N}_{\geq 1}. \quad (5.5.3)$$

as well as the values on the spheres $\mathcal{S}_1, \mathcal{S}_\infty$ of the first two transversal derivatives:

$$(\Omega^{-2}r^2\partial_v)(r\alpha_\ell^{[-2]}|_{\mathcal{S}_1} = \mathcal{C}'_\ell, \quad (\Omega^{-2}r^2\partial_v)^2(r\alpha_\ell^{[-2]}|_{\mathcal{S}_\infty} = 2\overline{\mathcal{A}}_\ell. \quad (5.5.4)$$

Both $\mathcal{C}$ and $\mathcal{C}'$ are independent of u and their precise values won't matter for our analysis.

The information from this lemma will suffice to derive the asymptotics for $\alpha^{[+2]}$ and $\alpha^{[-2]}$. Notice that, even though $\alpha^{[+2]}$ and $\alpha^{[-2]}$ satisfy decoupled evolution equations, they are coupled at the level of the data. This is a manifestation of the Teukolsky–Starobinsky identities relating $\alpha^{[+2]}$ and $\alpha^{[-2]}$ [TP74].

5.5.2 A sketch of the problem for the lowest angular mode $\alpha_{\ell=2}^{[+2]}$

Recall that the elliptic operator $\mathring{\Delta} + s$ in (5.4.1) has eigenvalues $-\ell(\ell+1) + s(s+1) = -(\ell-s)(\ell+s+1)$, with $\ell \in \mathbb{N}_{\geq 2}$. In particular, for $\ell = s = 2$, $(\mathring{\Delta} + s)\alpha_\ell^{[s]} = 0$, and (5.4.1) simplifies to:

$$\partial_u(r^{-4}\Omega^4\partial_v(r^5\alpha_{\ell=2}^{[2]})) = -\frac{30M\Omega^6}{r^7} \cdot r^5\alpha_{\ell=2}^{[2]}. \quad (5.5.5)$$

The simple yet crucial observation now is that the RHS of (5.5.5) has an extra decay of r^{-3} compared to the LHS, even though the LHS only contains two derivatives. Thus, if we have any preliminary global decay estimate of $\alpha^{[s]}$, then we can insert this estimate into the RHS of (5.5.5) and integrate the equation in u and v to improve this estimate.

To illustrate this, let us assume that $|\alpha_{\ell=2}^{[2]}| \lesssim r^{-p}$ for some $p > 0$. We then have that $|\partial_u(r^{-4}\Omega^4\partial_v(r^5\alpha_{\ell=2}^{[2]}))| \lesssim Mr^{-2-p}$. We now integrate this from $\mathcal{I}^-$ in u , where $r^{-4}\partial_v(r^5\alpha_{\ell=2}^{[2]})$ vanishes by (5.5.2). The result is that $|\partial_v(r^5\alpha_{\ell=2}^{[2]})| \lesssim Mr^{3-p}$. Finally, we integrate this from $\mathcal{C}$ in v to obtain that $|r^5\alpha_{\ell=2}^{[2]} - r^5\alpha_{\ell=2}^{[2]}|_{\mathcal{C}}| \leq Mr^{4-p}$. Since $r^5\alpha_{\ell=2}^{[2]}|_{\mathcal{C}}$ is bounded by u^2 according to (5.5.2), we can divide by r^5 to improve the original estimate to $|\alpha_{\ell=2}^{[+2]}| \lesssim r^{-5}u^2 + r^{-p-1}$ and iterate to improve further.

In practice, the conservation of the energy associated to (5.4.2) gives us the initial estimate $|\alpha_{\ell=2}^{[2]}| \lesssim r^{-1}$, and iteratively inserting this estimate into (5.5.5) gives us the estimate

$$|r^5\alpha_{\ell=2}^{[s]} - r^5\alpha_{\ell=2}^{[s]}|_{\mathcal{C}}| \leq M \cdot r, \quad \implies \quad r^5\alpha_{\ell=2}^{[s]} = r^5\alpha_{\ell=2}^{[s]}|_{\mathcal{C}} + \mathcal{O}(M \cdot r) = \mathcal{A}_{\ell=2} \cdot u^2 + \dots \quad (5.5.6)$$

We now insert the last estimate into (5.5.5) one last time:

$$\partial_v(r^5\alpha_{\ell=2}^{[2]}) = -30M \cdot r^4 \int_{-\infty}^u \frac{\mathcal{A}_{\ell=2}u'^2}{r^7} + \dots du' = -\frac{M\mathcal{A}_{\ell=2}}{2} + M\mathcal{O}(|u|/r), \quad (5.5.7)$$

where the final equality follows from integrating by parts ($u^2/r^7 = \frac{1}{6}\partial_u(u^2/r^6) + \frac{1}{3}|u|/r^6 + \dots$ etc.). By integrating again in v , we finally deduce that

$$r^5\alpha_{\ell=2}^{[s]} = r^5\alpha_{\ell=2}^{[s]}|_C - \frac{1}{2}M \cdot \mathcal{A}_{\ell=2}r + \dots = \mathcal{A}_{\ell=2}u^2 - \frac{1}{2}M \cdot \mathcal{A}_{\ell=2}r + \dots \quad (5.5.8)$$

Using this procedure, we can find higher and higher order terms. For instance, the next-to-leading order term in $r^5\alpha_{\ell=2}^{[2]}|_C$, namely $-4\mathcal{B}_{\ell=2}r_0 \log r_0$, adds the term $2M\mathcal{B}_{\ell=2} \log^2 r$ to the expansion above. This proves (5.3.6) for $\ell = 2$.

5.5.3 A sketch for higher angular modes $\alpha_\ell^{[+2]}$

For $\ell = 2$, we crucially exploited the vanishing of $\mathring{\Delta} + s$ in (5.4.1). This made the RHS of (5.4.1) decay three powers faster in r than the LHS, allowing us to iteratively improve any estimate that we insert into the RHS.

For $\ell > 2$, it looks like this cannot be done, since the RHS now only decays 2 powers faster, which no longer yields an improvement after integrating in u and in v . This hurdle is, similarly to how we descended from the Teukolsky equations to the Regge–Wheeler equations, overcome by commuting: Denote $A^{[s],N} := (\Omega^{-2}r^2\partial_v)^N(r^{|s|+s+1}\alpha^{[s]})$, and, abusing notation, $\mathring{\Delta}A^{[s],N} := (\Omega^{-2}r^2\partial_v)^N(r^{|s|+s+1}\mathring{\Delta}\alpha^{[s]})$. A tedious computation, starting from (5.4.1), gives

$$\begin{aligned} \partial_u((\frac{\Omega^2}{r^2})^{N+s}\partial_v A^{[s],N}) &= (\frac{\Omega^2}{r^2})^{N+s+1}(N(N+1+2s) + (\mathring{\Delta} + s))A^{[s],N} \\ &\quad + 2M(\frac{\Omega^2}{r^2})^{N+s+1} \cdot (-c_N^{[s]}r^{-1}A^{[s],N} + N(N+s)(N+2s)A^{[s],N-1}), \end{aligned} \quad (5.5.9)$$

where $c_N^{[s]} = (1+s)(1+2s) + 3N(N+1+2s)$. But if $\alpha^{[s]}$ is supported on angular frequency ℓ , then the first line on the RHS of (5.5.9) vanishes, so we have

$$\partial_u((\frac{\Omega^2}{r^2})^\ell\partial_v A_\ell^{[s],\ell-s}) = 2M(\frac{\Omega^2}{r^2})^{\ell+1}(-c_{\ell-s}^{[s]}r^{-1}A_\ell^{[s],\ell-s} + (\ell-s)\ell(\ell+s)A_\ell^{[s],\ell-s-1}). \quad (5.5.10)$$

The first term on the RHS now features an r^{-3} weight multiplying $A_\ell^{[s],\ell-s}$, whereas the second term features an r^{-2} -weight that multiplies $A_\ell^{[s],\ell-s-1}$ instead of $A_\ell^{[s],\ell-s}$. But since $A^{[s],N-1} = A^{[s],N-1}|_C + \int \frac{\Omega^2}{r^2} A^{[s],N} dv$, $A^{[s],N-1}$ decays one power faster compared to $A^{[s],N}$, so we can view this term to also feature three extra powers in decay compared to the LHS of (5.5.10).

At this point, for $s = 2$, we can more or less mimic the argument presented in §5.5.2 for $\ell = 2$ to obtain an estimate for $A_\ell^{[s],\ell-s}$:

Firstly, we compute the transversal derivatives $A_\ell^{[s],N}|_{\mathcal{C}}$, $N \leq \ell - s$, along $\mathcal{C}$ by inductively integrating (5.5.9) from $\mathcal{I}^-$, where for $s = 2$ all boundary terms vanish. The result of this is that $A_\ell^{[s],\ell-s}|_{\mathcal{C}} = C_{\ell-s} \mathcal{A}_\ell r_0^\ell + \dots$ (and $A_\ell^{[s],N}|_{\mathcal{C}} \sim r_0^{N+s}$ for $N \leq \ell - s$), with $C_{\ell-s} = \frac{(-1)^\ell 2(\ell-s)!(2\ell)!}{\ell!(\ell+s)!}$.

Secondly, we perform an iterative argument as for $\ell = 2$ to produce a global estimate of the form

$$\partial_v A_\ell^{[s],\ell-s} = -M \mathcal{A}_\ell \frac{(-1)^\ell 2(\ell-s)!(\ell+1)!}{(\ell+s)!} r^{\ell-2} + M \mathcal{O} \left(\frac{|u|}{r} + \overline{\mathcal{B}}_\ell r^{\ell-3} \log r \right). \quad (5.5.11)$$

Thirdly and finally, we integrate this estimate $\ell - s + 1$ times from $\mathcal{C}$:

$$\begin{aligned} r^{|s|+s+1} \alpha_\ell^{[s]} &= \sum_{i=0}^{\ell-s} \frac{A_\ell^{[s],i}|_{\mathcal{C}}}{i!} \left(\frac{1}{r_0} - \frac{1}{r} \right)^i \\ &\quad + \underbrace{\int_{v|_{\mathcal{C}}}^v \frac{\Omega^2}{r^2} \cdots \int_{v|_{\mathcal{C}}}^{v(\ell-s)} \frac{\Omega^2}{r^2}}_{\ell-s+1 \text{ integrals}} \left(\frac{r^2}{\Omega^2} \partial_v A_\ell^{[s],\ell-s} \right) dv_{(\ell-s+1)} \cdots dv_1. \end{aligned} \quad (5.5.12)$$

The computations involved are now getting more and more involved, but it is relatively straight-forward to see that the nested integral in the second line produces a leading-order term $(-1)^{\ell+1} \frac{2M \mathcal{A}_\ell r}{(\ell-1)(\ell+2)}$, whereas the sum in the first line clearly remains bounded as $v \rightarrow \infty$ along $u = \text{const}$ —this sum exactly corresponds to the sum in the first line in (5.3.6). This proves (5.3.6).

5.5.4 A sketch for higher angular modes $\alpha_\ell^{[-2]}$

Compared to $s = +2$, the main difference in the case $s = -2$ is that, with the decay rate $\alpha^{[-2]}|_{\mathcal{C}} \sim r_0^{-4}$, the RHS of (5.4.1) decays like r^{-1} , so we cannot integrate (5.4.1) from $\mathcal{I}^-$ along $\mathcal{C}$ —instead, we have to integrate it from $\mathcal{S}_1$ in order to compute the transversal derivative $\partial_v(r\alpha^{[-s]})$ along $\mathcal{C}$. This is why we had to specify $\partial_v \alpha^{[-2]}$ along $\mathcal{S}_1$ as part of the seed data.

The commuted equation, (5.5.9) with $N = 1$, can then be integrated from $\mathcal{I}^-$, but the boundary term picked up from $\mathcal{I}^-$, $A_\ell^{[s=-2],2}|_{\mathcal{I}^-} = 2\overline{\mathcal{A}}_\ell \neq 0$ is a constant by (5.5.4).

At this point, all higher order transversal derivatives along $\mathcal{C}$ can be computed as for $s = 2$ —all further boundary terms vanish at $\mathcal{I}^-$ by (5.5.3). In particular, we find that $A_\ell^{[s=-2],\ell-s}|_{\mathcal{C}} = \frac{(\ell-2)!}{(\ell+2)!} C_{\ell-s} \overline{\mathcal{A}}_\ell r_0^{\ell-s} + \dots$, so the leading order decay of higher-order transversal derivatives along $\mathcal{C}$ is not dictated by the leading order decay of $\alpha^{[-2]}|_{\mathcal{C}}$ itself (i.e. it does not depend on $\overline{\mathcal{B}}!$).

We can now, in exact analogy to (5.5.11), show that

$$\partial_v A_\ell^{[-2],\ell-s} = (-1)^{\ell+1} M \overline{\mathcal{A}}_\ell \cdot (\ell+1)! r^{\ell-2} + \dots \quad (5.5.13)$$

from which we derive an expression for $\alpha^{[-2]}$ by again applying the formula (5.5.12):

$$r\alpha_\ell^{[-2]} = \sum_{i=0}^{\ell+2} \frac{1}{i!} \left(\frac{1}{r_0} - \frac{1}{r} \right)^i A_\ell^{[s],i}|_C + 2M\overline{\mathcal{A}}_\ell \frac{(-1)^{\ell+1} \cdot \ell(\ell+1)}{6r^3} \log\left(\frac{r}{r_0}\right) + \dots \quad (5.5.14)$$

The leading-order logarithmic behaviour is entirely governed by $\overline{\mathcal{A}}$, instead of the coefficient $\mathcal{B}$ that governs the leading order decay of $\alpha^{[-2]}$ towards $\mathcal{I}^-$, the reason being that this behaviour is determined by higher-order transversal derivatives along $\mathcal{C}$.¹

However, if we compute the expression for the radiation field of $\alpha_\ell^{[-2]}$, namely $\lim_{v \rightarrow \infty} r\alpha_\ell^{[-2]}$, the coefficient $\overline{\mathcal{A}}$ only enters at Schwarzschild order, that is to say: It does not contribute if $M = 0$. A very lengthy computation produces the second statement of Theorem 5.3.1, (5.3.7). See chapter 6 for details.

5.5.5 The summing of the ℓ -modes

The method we used to find the asymptotics for higher angular modes comes at a high price: Commuting with $(\Omega^{-2}r^2\partial_v)$ up to $\ell - s$ times generates a large amount of error terms that, although they all decay faster towards infinity (i.e. in $1/r$), individually grow extremely fast in ℓ . In particular, if we write $\alpha^{[s]} = \sum_{\ell=2}^{\infty} \alpha_\ell^{[s]}$, the bounds for the error terms in the estimates above are not summable even if the initial data are smooth.

We expect that this is only an artefact of the method that we use, and that there is a way to sum the estimates above. The investigation of this expectation is ongoing work [KK23]. Very roughly speaking, we hope to tackle the issue in four steps:

- (1) Consider the initial data for $\alpha^{[2]}$: $\alpha^{[2]}|_C = \mathcal{A}r_0^{-3} + \mathcal{B}r_0^{-4} \log r_0 + \mathcal{C}r_0^{-4} + \mathcal{O}(r_0^{-4-\epsilon})$. We write the solution $\alpha^{[2]}$ as a sum of the solution $\alpha_{\text{phg}}^{[2]}$ arising from the initial data $\alpha_{\text{phg}}^{[2]}|_C = \mathcal{A}r_0^{-3} + \mathcal{B}r_0^{-4} \log r_0 + \mathcal{C}r_0^{-4}$ (without the $\mathcal{O}$ -term) and the difference $\alpha_\Delta^{[2]}$.
- (2) A robust energy estimate on the difference $\alpha_\Delta^{[2]}$ then proves that $\alpha_\Delta^{[2]} = \mathcal{O}(r^{-4-\epsilon})$ in $D^+(\mathcal{C} \cup \mathcal{I}^-)$.
- (3) Next, we consider $\alpha_{\text{phg}}^{[2]}$. For this solution, we establish a persistence of polyhomogeneity result. Let's explain this: Loosely speaking, a function is said to be polyhomogeneous near a boundary $x = \infty$ if it has a series expansion in $x^{-p} \log^q x$, with a countable set of numbers $p \in \mathbb{R}$, $q \in \mathbb{N}$, in a robust, square-integrable sense. What we will show is that, since the initial data for $\alpha_{\text{phg}}^{[2]}$ are polyhomogeneous near $\mathcal{I}^-$, $\alpha_{\text{phg}}^{[s]}$ will also be polyhomogeneous near i^0 and near $\mathcal{I}^+$. This complements the result of [HV20].
- (4) We can finally determine the coefficients of the polyhomogeneous expansion of $\alpha^{[+2]}$ by decomposing $\alpha_{\text{phg}}^{[2]}$ into angular frequencies and using the methods of Section 5.5—since

¹This important detail was overlooked in the brief outview given in [GK22], which therefore made an incorrect prediction on $s = -2$.

we have already established that $\alpha_{\text{phg}}^{[2]}$ is polyhomogeneous, we no longer need to worry about issues of summability at this stage.

5.6 Solving the scattering problem III: Asymptotic analysis of the remainder of *(LGS)*

Equipped with asymptotic expressions for $\alpha^{[+2]}$ and $\alpha^{[-2]}$, we are now in a position to derive the asymptotic behaviour of all other quantities contained in *(LGS)*. For instance, asymptotic expressions for $\overset{(i)}{\hat{\chi}}$, $\overset{(i)}{\hat{\underline{\chi}}}$ and $\overset{(i)}{\hat{\not{g}}}$ follow from integrating the equations (5.3.5). To obtain the asymptotic behaviour for the remaining Weyl curvature quantities $\overset{(i)}{\Psi}_i$, $i = 1, 2, 3$, one needs to integrate the Bianchi identities; the result of this is that $\alpha^{[+2]} \sim r^{-4}$ near $\mathcal{I}^+$ (recall that $\alpha^{[+2]}$ is equivalent to $\overset{(i)}{\Psi}_0$) implies that $\overset{(i)}{\Psi}_1$, denoted $\overset{(i)}{\beta}$ in [DHR19b], goes like $r^{-4} \log r$ near $\mathcal{I}^+$ (as predicted by [Chr02]), whereas the peeling rates (0.1.3) hold true for $\overset{(i)}{\Psi}_2$, $\overset{(i)}{\Psi}_3$ and $\overset{(i)}{\Psi}_4$.

In order to obtain asymptotic expressions for all the other quantities contained in *(LGS)*, we would of course have to introduce *(LGS)* in much more detail. For the scope of the current chapter, suffice it to say that the constructed solution can be proved to be *extendable to $\mathcal{I}^+$ for any $s < 1$* in the sense of Def. 3.4 of [Hol16] (which also features a compact introduction of *(LGS)*.) We now move on to highlighting a few other points of particular interest.

5.6.1 Bondi normalising $\mathcal{I}^+$ in addition to $\mathcal{I}^-$

Provided we have Bondi normalised our seed scattering data at $\mathcal{I}^-$, i.e. made the metric perturbations $\overset{(i)}{\Omega}$ and $\overset{(i)}{\not{g}}$, as well as $r^2 \overset{(i)}{K}$ and $r(\Omega \text{tr} \chi)$ vanish at $\mathcal{I}^-$, it is natural to ask whether we can Bondi normalise the resulting solution along $\mathcal{I}^+$ as well. The answer is affirmative: Without going into too much detail, starting from Bondi normalised seed scattering data and our results for $\alpha^{[+2]}$ and $\alpha^{[-2]}$, we derive that the limits $r^2 \overset{(i)}{K}|_{\mathcal{I}^+}$ and $\overset{(i)}{\hat{\not{g}}}|_{\mathcal{I}^+}$ vanish automatically. On the other hand, we find that the limits $\overset{(i)}{\Omega}|_{\mathcal{I}^+}$ and $r(\Omega \text{tr} \underline{\chi})|_{\mathcal{I}^+}$ exist and decay in u . It is now possible to add a pure gauge solution that simultaneously kills off both of these terms. Since this gauge solution decays in u , it does not affect the Bondi normalisation of $\mathcal{I}^-$.

5.6.2 Corrections to the quadrupole formula: dE/du along $\mathcal{I}^+$

In the non-linear theory, the rate of change of energy along $\mathcal{I}^+$ is given by the Bondi mass loss formula [Bon60]: $\frac{dE}{du} = -\frac{1}{4} \langle |\mathbf{r} \hat{\underline{\chi}}|_{\mathcal{I}^+}^2 \rangle$, $\langle \cdot \rangle$ denoting the average over $\mathbb{S}^2$ and $\mathbf{r} \hat{\underline{\chi}}|_{\mathcal{I}^+}$ denoting the News function at $\mathcal{I}^+$. Of course, the linearisation of this formula is zero, but we can still define the linearised energy loss to be

$$\frac{dE}{dt} := -\frac{1}{16\pi} \int_{\mathbb{S}^2} |r \hat{\underline{\chi}}|_{\mathcal{I}^+}^2 \sin \theta \, d\theta \, d\varphi. \quad (5.6.1)$$

An expression for $r_{\hat{\chi}}^{(i)}$ can be inferred via integration of the second of (5.3.5) and our asymptotic expression (5.3.7) for $\alpha^{[-2]}$:

$$\lim_{u=\text{const}, v \rightarrow \infty} r_{\hat{\chi}}^{(i)} = (-1)^{\ell+1} |u|^{-2} \left(\frac{\mathcal{B}_\ell}{\ell(\ell+1)} + \frac{2M\overline{\mathcal{A}}_\ell(\ell-2)!}{(\ell+2)!} \right) + \mathcal{O}(|u|^{-2-\epsilon}). \quad (5.6.2)$$

For $\ell = 2$ and $M = 0$, this can be seen to exactly coincide with Einstein’s original quadrupole formula, though perhaps in a slightly less familiar form: Equation (5.6.2) relates the limiting behaviour near $\mathcal{I}^-$ (governed by $\mathcal{A}$ and $\mathcal{B}$ according to Def. 5.3.2) to the limiting behaviour along $\mathcal{I}^+$ as $u \rightarrow -\infty$. The full relation to the quadrupole formula is then established by relating the coefficients $\mathcal{A}_{\ell=2}$ and $\mathcal{B}_{\ell=2}$ to the Newtonian quadrupole moments (5.2.3) as described in §5.2.

More generally, (5.6.2) represents the linear corrections to (the multipole generalisation of) this formula when the linearisation is done around Schwarzschild instead of Minkowski.² Since the present work is meant as a precursor to the treatment of the full Einstein vacuum equations, we hope to eventually make the propagation part of this formula, i.e. going from $\mathcal{I}^-$ to $\mathcal{I}^+$, precise. Establishing a full, non-linear relation between the coefficients $\mathcal{A}$ and $\mathcal{B}$ and the physical multipole moments of the described matter distribution would be an exciting project on its own, cf. Open Problem 2 in chapter 0.

5.6.3 The antipodal matching condition

Since the analysis produces an arbitrarily precise understanding of the asymptotic behaviour of solutions near i^0 , it is also relevant for several questions concerning relations between the past limit point of $\mathcal{I}^+$ and the future limit point of $\mathcal{I}^-$. To give an example, we can prove that

$$\text{G.I.P.} \left[\lim_{v \rightarrow \infty} \lim_{u \rightarrow -\infty} r^2 \hat{\chi}_\ell^{(i)} \right] = \text{G.I.P.} \left[(-1)^\ell \lim_{u \rightarrow -\infty} \lim_{v \rightarrow \infty} r^2 \hat{\chi}_\ell^{(i)} \right], \quad (5.6.3)$$

where G.I.P. denotes the gauge independent part of the future and past News functions $\hat{\chi}_\ell^{(i)}$ and $\hat{\chi}_\ell^{(i)}$. This is closely related to Strominger’s antipodal matching condition. See also [Mas22b].

5.7 The question of late-time asymptotics

The entirety of the previous sections was concerned with the asymptotic behaviour near infinity restricted to $u \leq -1$. However, as we have seen in chapter 4 at the level of the scalar wave equation, this, in a certain sense, entirely fixes the possible asymptotics at late times, i.e. as $u \rightarrow \infty$, due to the existence of certain conserved, modified Newman–Penrose charges

²In parts of the literature, such corrections arising from backscattering off of the curvature of the spacetime are referred to as “tail effects”, see [BS93, BT23] and references therein.

along $\mathcal{I}^+$. We also gave a heuristic principle in chapter 4 to relate solutions to the Teukolsky equation to scalar waves.³

Thus, if we extend, for instance, the data along $\mathcal{C}$ all the way to the future event horizon, we end up with the following picture for global asymptotics (Fig. 5.7.1):

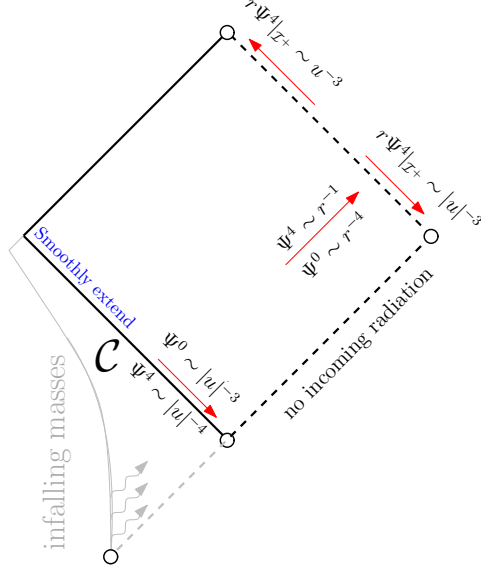


Figure 5.7.1 We can extend the scattering data along $\mathcal{C}$ all the way to the future event horizon in order to obtain the asymptotics near future timelike infinity. These turn out to be completely fixed by the asymptotics at early times, i.e., they are independent of how we extend the data along $\mathcal{C}$ to the future. We note here that, if the spacetime were stationary in the past, then the late-time asymptotics would be three powers faster [GK22, MZ22b].

5.8 Conclusion

Clearly, the results (5.3.6) and (5.5.8), which show that $\alpha^{[+2]} \sim r^{-4}$ near $\mathcal{I}^+$ (in complete agreement with [Dam86]), violate the peeling property (0.1.3) for $i = 0$ (recall that $\alpha^{[+2]}$ is equivalent to Ψ_0). As opposed to the violation of peeling near $\mathcal{I}^-$, which is derived using Post-Newtonian approximations around Minkowski (§5.2.1), the violation of peeling near $\mathcal{I}^+$ is caused by the mass term in the Schwarzschild metric (in the Post-Minkowskian picture, this effect would only be seen at order ϵ^2 in $g = \eta + \epsilon g^1 + \epsilon^2 g^2 + \mathcal{O}(\epsilon^3)$).

While our results thus rule out that the constructed spacetimes admit a smooth $\mathcal{I}^+$ (or $\mathcal{I}^-$) or admit Bondi coordinates near $\mathcal{I}^+$, both of these concepts can still be made sense of provided they are sufficiently weakened: In the case of Bondi coordinates, it suffices to drop condition (iii) from [Sac62b]. To be precise, the b -term in (4.1) of [Sac62b], which captures the $1/r^2$ fall-off of γ and δ in (2.9) of [Sac62b], does *not* vanish, it carries physical information. See [Kro99] for the computation of these terms if $\Psi_0 \sim r^{-4}$. The non-vanishing of this term then generates a $r^{-3} \log r$ in the expansion of U , a $r^{-1} \log r$ term in the expansion of $V = -r + 2M + \dots$, etc.

³Unfortunately, in the published [GK22] corresponding to chapter 4, a small mistake (that isn't present in chapter 4) crept into the latter heuristics: In the equation above (6.1) of chapter 4, it was assumed that the global asymptotic behaviour of Ψ_4 (a.k.a. $\alpha^{[-2]}$ —we now drop the ω) would be governed by its decay towards $\mathcal{I}^-$, so p_4 was taken to be 3. But as we stressed in §5.5.4, it is the decay of two transversal derivatives of Ψ_4 near $\mathcal{I}^-$ that determines the logarithmic behaviour near $\mathcal{I}^+$, and the value of p_4 should instead be 2.

In the case of Penrose’s asymptotic simplicity, the assumption on the regularity of the compactification needs to be weakened significantly. A nontrivial problem of interest to many would be to understand the optimal regularity with which a conformal boundary can be still be attached if $\Psi^0 = \mathcal{O}(r^{-4})$.⁴

Another notion of asymptotic flatness mentioned in the introduction (§0.1) is that of [CK93], which demands the spacetime to feature certain decay towards spatial infinity (along $t = 0$). The demanded decay in particular implies that $\alpha^{[+2]}$ would decay like $o(r^{-\frac{7}{2}})$ towards i^0 . Now, inspection of (5.3.6) or (5.5.5) immediately shows that $\alpha^{[+2]} \sim r^{-3}$ along $t = 0$. In other words, the class of spacetimes constructed in [CK93] decays too fast to capture the kinds of physics we are describing in this chapter. In particular, Christodoulou’s argument [Chr02] does not capture our construction. In fact, we can see that the argument, at least in the way it is paraphrased in §1.1.2 of chapter 1, is incorrect: For, if we set the coefficient $\mathcal{A}$ to vanish in Def. 5.3.2, this will neither affect the no incoming radiation condition on $\mathcal{I}^-$, nor the decay rate of the energy loss along $\mathcal{I}^+$, see eq. (5.6.2). Moreover, the induced data along $t = 0$ will decay fast enough for the framework of [CK93] to apply. But at the same time, $\alpha^{[+2]}$ will now decay like $r^{-5} \log^2 r$ towards $\mathcal{I}^+$ by (5.3.6), from which one easily derives that $\overset{(i)}{\beta} \sim r^{-4}$ near $\mathcal{I}^+$, which is in contradiction to the argument.

Finally, a notion of asymptotic flatness that does admit our constructed spacetimes is that of [DHR19b]. Indeed, as we said in §5.6, the constructed spacetime is “extendable to $\mathcal{I}^+$ ” in the sense of Def. 3.4 of [Hol16] for any $s < 1$.

The fact that our constructions decay slower near i^0 than demanded by [CK93] but still decay sufficiently fast near $\mathcal{I}^+$ is somewhat surprising as the spacetimes constructed in [CK93] are generically only “extendable to $\mathcal{I}^+$ ” for $s \leq 1/2$. This remarkable improvement in decay towards $\mathcal{I}^+$ compared to decay near i^0 is related to our constructions not having incoming radiation from $\mathcal{I}^-$ and will be further discussed elsewhere.

⁴Not even the question of the minimal conformal regularity that ensures peeling (0.1.3) is resolved, cf. [Fri02] and §2 of [Fri18].

Chapter 6

Early-time asymptotics for linearised gravity—The details

Abstract

In this final chapter, starting from the predictions of Post-Newtonian theory for a system of N infalling masses from the infinite past i^- , we formulate and solve a scattering problem for the Einstein vacuum equations linearised around the Schwarzschild solution, with the scattering data posed on a null hypersurface $\mathcal{C}$ emanating from a section of past null infinity $\mathcal{I}^-$, and on the part of $\mathcal{I}^-$ that lies to the future of this section: Along $\mathcal{C}$, we implement the Post-Newtonian theory-inspired hypothesis that the gauge-invariant Teukolsky quantities $\overset{(i)}{\alpha}$ and $\overset{(i)}{\underline{\alpha}}$ decay with certain inverse power rates $\overset{(i)}{\alpha} \sim r^{-3}$, $\overset{(i)}{\underline{\alpha}} \sim r^{-4}$, and we exclude incoming radiation from $\mathcal{I}^-$ by demanding the News function to vanish along $\mathcal{I}^-$.

After constructing the unique solution to this scattering problem, we then analyse the asymptotic behaviour of fixed angular frequencies near $\mathcal{I}^-$, i^0 and $\mathcal{I}^+$ by using a set of approximate conservation laws enjoyed by the Teukolsky quantities α and $\underline{\alpha}$. Confirming earlier heuristics due to Damour and Christodoulou, we find that the peeling property is violated both near $\mathcal{I}^-$ and near $\mathcal{I}^+$, with e.g. α near $\mathcal{I}^+$ only decaying like r^{-4} instead of r^{-5} . Surprisingly, we also find that the resulting solution decays slower towards i^0 than the solutions analysed in the monumental work of Christodoulou and Klainerman on the stability of the Minkowski space.

While we provide a sketch of how to resum the estimates obtained for the fixed angular frequencies, we leave the detailed study of this somewhat intricate issue to future work.

This chapter should be viewed as filling in the details to the overview given in the previous chapter 5.

6.1	Introduction	318
6.1.1	Structure of the chapter	318
6.1.2	At outlook on future work	320

6.2	The Schwarzschild family of spacetimes	321
6.2.1	The geometry of the Schwarzschild solution	321
6.2.2	The double null gauge	322
6.2.3	$\mathcal{S}_{u,v}$ -tangent tensors	322
6.2.4	Norms and the $\mathcal{O}$ -notation	324
6.2.5	Differential operators on the sphere	325
6.2.6	Definition of tensorial spherical harmonics	326
6.2.7	The Kerr family	328
6.3	The linearised Einstein vacuum equations in a double null gauge	329
6.3.1	The linearisation procedure	329
6.3.2	The system of equations	330
6.3.3	The Teukolsky equations and the Regge–Wheeler equations	332
6.4	Pure gauge and linearised Kerr solutions	337
6.4.1	The linearised Kerr solutions	337
6.4.2	The pure gauge solutions	337
6.4.3	Solutions supported on $\ell \leq 1$ are non-dynamical	339
6.4.4	Solutions supported on $\ell \geq 2$ with $\overset{(i)}{\alpha} = 0 = \overset{(i)}{\underline{\alpha}}$ are pure gauge	339
Part I: Setting up and solving the scattering problem		340
6.5	The general scattering data setup for (6.3.9)–(6.3.32)	340
6.5.1	Seed scattering data and the main theorem (Thm. 6.5.1)	341
6.5.2	Overview over the proof of Thm. 6.5.1	342
6.5.3	Defining the full scattering data along $\underline{\mathcal{C}}$	343
6.5.4	Bondi normalisation of seed scattering data	347
6.6	Scattering for the Regge–Wheeler equation and for the Teukolsky equation	350
6.6.1	Energy conservation for the Regge–Wheeler equation	350
6.6.2	Scattering for the Regge–Wheeler equation	351
6.6.3	Two uniqueness results for the Teukolsky equation (Teuk)	357
6.7	Construction of the (unique) scattering solution to (6.3.9)–(6.3.32)	361
6.7.1	Uniqueness of solutions to the scattering problem	361
6.7.2	Construction of the $\ell \geq 2$ -part of the solution to the scattering problem	362
6.7.3	Construction of the $\ell < 2$ -part of the solution to the scattering problem	380
6.8	The physical scattering data	383
6.8.1	The condition of no incoming radiation	383
6.8.2	Seed data describing the far-field region of the N -body problem	384
6.8.3	A preliminary description of solutions arising from seed data describing the far-field region of the N -body problem	386

Part II: Asymptotic Analysis of the Teukolsky equations and the Regge–Wheeler equations	388
6.9 The approximate conservation laws	389
6.9.1 The Teukolsky equations for general spin s	389
6.9.2 The commuted Teukolsky equations	390
6.9.3 The approximate conservation laws	393
6.9.4 The modified Newman–Penrose charges	395
6.9.5 The Regge–Wheeler equations and their conservation laws	395
6.10 Asymptotic analysis of $\alpha^{[s]}$ arising from general scattering data	398
6.10.1 The main theorem (Thm. 6.10.1)	399
6.10.2 Overview of the proof	400
6.10.3 Computing transversal derivatives along $\mathcal{C}$	401
6.10.4 A robust preliminary decay estimate based on energy conservation	401
6.10.5 Deriving asymptotic expressions using the approximate conservation laws	403
6.10.6 Asymptotic expansion for $\alpha_\ell^{[s]}$ and the proofs of Thm. 6.10.1 and Cor. 6.10.1	412
6.11 Asymptotics for $\alpha^{[2]} = r^5 \Omega^{-2} \overset{(i)}{\alpha}$ arising from physical data	414
6.11.1 The main theorem (Thm. 6.11.1)	414
6.11.2 Proof of Theorem 6.11.1	414
6.12 Asymptotics for $\Psi^{[\pm 2]}$ arising from physical data	418
6.13 Asymptotics for $\alpha^{[-2]} = r \Omega^2 \overset{(i)}{\underline{\alpha}}$ arising from physical data	419
6.13.1 The main theorem (Thm. 6.13.1)	419
6.13.2 Proof of Theorem 6.13.1	420
6.14 Post-Minkowskian expansions of the Teukolsky equation	424
6.14.1 The expansion	424
6.14.2 Explicitly solving the Minkowskian Teukolsky equation	425
6.14.3 Explicitly solving the inhomogeneous Minkowskian Teukolsky equation	428
6.15 Remarks on summing up the angular modes	432
Part III: Asymptotic analysis of the remainder of the system	434
6.16 Asymptotics for $\overset{(i)}{\hat{\chi}}$ and $\overset{(i)}{\hat{\underline{\chi}}}$	434
6.16.1 Asymptotics for $\overset{(i)}{\hat{\chi}}$	434
6.16.2 Asymptotics for $\overset{(i)}{\hat{\underline{\chi}}}$	435
6.17 Deriving asymptotics for $\overset{(i)}{\rho}$, $\overset{(i)}{\sigma}$, $\overset{(i)}{\beta}$ and $\overset{(i)}{\underline{\beta}}$	436
6.18 Deriving asymptotics for the remaining connection coefficients	437
6.19 Deriving asymptotics for the remaining metric coefficients and for $\overset{(i)}{K}$	437
Appendix 6.A Spin-weighted functions, the Newman–Penrose formalism, and the Christodoulou–Klainerman formalism	439

6.A.1	Miscellaneous calculations on $\mathbb{S}^2$	439
6.A.2	Spin-weighted functions, the operators $\bar{\partial}$, $\bar{\partial}'$, and the spin-weighted spherical harmonics ${}_sY_{\ell,m}$	440
6.A.3	A dictionary between the Christodoulou–Klainerman formalism and the Newman–Penrose formalism	443
Appendix 6.B Useful integral identities		446
6.B.1	Integrals in the u -direction	446
6.B.2	Integrals in the v -direction	449

6.1 Introduction

In this final chapter of the thesis, we give the full details corresponding to the previous overview chapter §5. We give a fully self-contained discussion of the semi-global scattering problem for the linearised Einstein vacuum equations around Schwarzschild, we make precise how we implement the physical assumptions discussed in the previous chapter as assumptions on the scattering data, and we then analyse in full detail the asymptotic properties of the resulting solution.

Even though this chapter can in principle be read on its own, we strongly recommend reading the previous chapter §5 first, as it provides a very detailed overview over the contents of the present chapter, as well as rough sketches of the arguments and results. For the same reason, we will keep the introduction to the present chapter to a minimum: We will restrict ourselves to providing an overview of its structure, and providing some commentary for context where appropriate.

We recall that this chapter, specifically its first part (§6.2–§6.8), is based on an ongoing collaboration with Hamed Masaoood [KM23].

6.1.1 Structure of the chapter

This chapter’s structure closely corresponds to the structure of chapter §5. It begins with some preliminaries:

First, in §6.2, we introduce the Schwarzschild family of spacetimes and give the geometric foundations for the remainder of the chapter. This section is largely informed by [Chr09, DHR19b, Czi17].

In §6.3, we then introduce the system of linearised gravity around Schwarzschild, following [DHR19b, Mas22b]. As this system is written down using the Christodoulou–Klainerman formalism, we provide a detailed dictionary between the Christodoulou–Klainerman formalism and the Newman–Penrose formalism in the appendix 6.A.

In §6.4, we recall the class of pure gauge and linearised Kerr solutions identified in [DHR19b] (cf. §5.3.2 of the previous chapter).

Beyond these preliminaries, the chapter is divided into three parts:

Part I: In Part I, we set up and solve the scattering problem for the system of linearised gravity around Schwarzschild, and we discuss the implementation of physically motivated assumptions for scattering data. This corresponds to §5.2–§5.4 of the previous chapter:

In §6.5, which corresponds to §5.3.3–§5.3.4 of the previous chapter, we give a definition of what constitutes a seed scattering data set, and we write down the main theorem of this part of the chapter (Theorem 6.5.1), namely that of the unique existence of a scattering solution realising a given seed scattering data set, together with an overview of its proof. We also give an account of how to remove the unphysical degrees of freedom from a seed scattering data set.

In §6.6, we then develop a *semi-global* scattering theory for the Regge–Wheeler equation (cf. §5.4 from the previous chapter). The main result of the section is a statement adapted from [Mas22a, Mas22b] (which developed a *global* scattering theory for the Regge–Wheeler equation), but we provide an alternative proof and also derive a few extra results that will prove useful for our schemes.

In §6.7, we then infer from this scattering theory for the Regge–Wheeler equation the existence of a unique scattering solution to the entire system of linearised gravity around Schwarzschild (cf. §5.4.3 from the previous chapter). We point out that this section exhibits some overlap with the second author’s recent [Mas22b] concerning the local existence part of the argument: During the collaboration on which this chapter is based, we noticed a gap in the local existence argument of [DHR19b]. We subsequently fixed this gap as part of the collaboration, and it was then similarly closed in [Mas22b].

In general, it is worth mentioning that the approach to the scattering problem taken here is somewhat different from that of [Mas22b]: In this chapter, we attempt to write down scattering data that model a certain class of spacetimes motivated by arguments from the physics literature (cf. §5.2 of the previous chapter), so the *scattering data* are the central object—these are where we implement our physically motivated assumptions. On the other hand, [Mas22b] studies the construction and properties of the *scattering map* in the entire exterior of Schwarzschild on more general grounds. But in order to study the scattering map, it suffices to only analyse scattering data in some convenient dense subset, namely that of smooth, compactly supported scattering data, and to then extend the results by density arguments, hence the asymptotics of the data considered in [Mas22b] require a less refined control compared to the data here. Since the semi-global scattering problem is simpler (and somewhat different) from the global scattering problem, we use the opportunity to give an entirely self-contained presentation of the former (even though it would be possible to derive our main result Theorem 6.5.1 from the results of [Mas22a, Mas22b]).

Finally, in §6.8, we give an explicit definition of the no incoming radiation condition (which is gauge independent and somewhat surprising), and we define what it means for a seed scattering data set to describe the exterior of N infalling masses following approximately hyperbolic orbits in the infinite past. We then provide a preliminary analysis of solutions arising from such data. This section corresponds to §5.3.5 and §5.5.1 from the previous chapter.

Part II: The second part of the chapter, Part II, is dedicated to the asymptotic analysis of only the Regge–Wheeler and the Teukolsky equations. This part corresponds to §5.5 of the previous chapter. This part of the chapter is entirely self-contained and, in fact, discusses the Teukolsky equations for any integer spin. It is similar in spirit to the analysis performed at the level of the linear wave equation in §3.10 of chapter 3, but goes in much more detail as we also need to understand higher-order asymptotics. We refer the reader to overview at the beginning of Part II.

Part III: The last part of the chapter, Part III, is what corresponds to §5.6 of the previous chapter: Taking the results of the previous part on the asymptotic behaviour of the Teukolsky and Regge–Wheeler quantities, we can now derive the semi-global asymptotics of the entire system of linearised gravity around Schwarzschild. While we don’t give a complete treatment of this problem (for that, the reader is referred to [KM23]), we give a sketch of its contents that suffices to explain the statements made in §5.6 of the previous chapter.

6.1.2 At outlook on future work

The only remaining ingredient *at the linear level* that is still missing in order to upgrade the results of this chapter to the full Einstein vacuum equations

$$\text{Ric}[g] = 0, \tag{6.1.1}$$

is the question of how to sum the fixed-angular-frequency estimates obtained in this chapter. We hope to address this in upcoming work [KK23], see already the brief summary in §5.5.5 of the previous chapter.

Once this issue is resolved at the linear level, the natural next step is to extend the work to the nonlinear theory, thus finally closing the circle to the very beginning of this thesis (Cf. Open Problem 1 and Conjectures 1.1.1, 1.1.2 of chapter 1.) As explained already in §5.8, this requires developing a scattering theory for the Einstein vacuum equations that admits slower decay near spatial infinity than assumed in most stability works, while at the same time having to prove stronger asymptotics than in most stability works. This exciting outlook marks the end of this introduction.

6.2 The Schwarzschild family of spacetimes

6.2.1 The geometry of the Schwarzschild solution

We define, for $M \in \mathbb{R}_{\geq 0}$, the Schwarzschild family of spacetimes as the family of smooth Lorentzian manifolds $(\mathcal{M}_M, g_M)$, where $\mathcal{M}_M = \{(t, r, \theta, \varphi) \in \mathbb{R} \times (2M, \infty) \times \mathbb{S}^2\}$, and where the metric g_M in coordinates (t, r, θ, φ) is given by

$$g_M = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + \mathring{g}, \quad (6.2.1)$$

where $\mathring{g}$ is simply the rescaled metric on the unit sphere:

$$\mathring{g} = r^2 \mathring{\mathring{g}} = r^2 (d\theta^2 + \sin^2 \theta d\varphi^2) \quad (6.2.2)$$

Next, we introduce the Eddington–Finkelstein double null coordinates

$$u = \frac{1}{2}(t - r_*), \quad v = \frac{1}{2}(t + r_*), \quad (6.2.3)$$

where r_* is the familiar radial function defined by

$$r_* = r + 2M \log \left(\frac{r}{2M} - 1 \right). \quad (6.2.4)$$

In this double null coordinate system (u, v, θ, φ) , the metric takes the form

$$g_M := -4\Omega^2 du dv + r^2 \mathring{\mathring{g}}, \quad \Omega = \sqrt{1 - \frac{2M}{r}}. \quad (6.2.5)$$

We denote by $\mathcal{C}_\lambda$ the outgoing null hypersurface $\{u = \lambda\}$, while $\underline{\mathcal{C}}_\nu$ will refer to the ingoing null hypersurface $\{v = \nu\}$. The sphere formed by the intersection of $\mathcal{C}_\lambda \cap \underline{\mathcal{C}}_\nu = \{(\lambda, \nu) \times \mathbb{S}^2\}$ will be denoted by $\mathcal{S}_{\lambda, \nu}$.

The Schwarzschild manifold can be extended to a manifold with boundary in various directions. Using the coordinate system (v, r, θ, φ) , we can attach the boundary $\mathcal{H}^+ := \{(v, r, \theta, \varphi) : r = 2M\}$, called the future event horizon, thus extending the manifold to $\mathbb{R} \times [2M, \infty) \times \mathbb{S}^2$. We can similarly attach $\mathcal{H}^-$, the past event horizon, by working in coordinates (u, r, θ, φ) .

In a similar fashion, we can attach boundaries at infinity (without viewing them as conformal boundaries). Working in the coordinate system (x, u, θ, φ) , where $x(u, v) := \frac{1}{r(u, v)}$, we attach the hypersurface $\mathcal{I}^+ := \{x_v, u, \theta, \varphi : x_v = 0\}$, called future null infinity. We can similarly attach $\mathcal{I}^-$, called past null infinity, by working in coordinates $(x_u, v, \theta, \varphi)$. See [Mas22a, DRS18] for an extended discussion, in particular Def. 4.2.1 in the latter.

A note on conventions: Even though we have now introduced several different coordinate systems; whenever we write ∂_u or ∂_v , it will always be with respect to the double null coordinate system (u, v, θ, φ) . Furthermore, we will allow ourselves to use the coordinates u, v to also refer to objects living in the extended manifold $\overline{\mathcal{M}}_M := \mathcal{M}_M \cup \mathcal{H}^- \cup \mathcal{H}^+ \cup \mathcal{I}^- \cup \mathcal{I}^+$ by permitting u, v to attain the values $\pm\infty$. Finally, when we talk about limits “as $u \rightarrow -\infty$ ” or “as $v \rightarrow \infty$ ” etc., it will always be understood that the respective other double null coordinates are kept fixed.

6.2.2 The double null gauge

The metric in the form (6.2.5) is an example of spacetime metrics g that are cast in a *double null gauge*, i.e. a coordinate system (u, v, θ^A) where u, v are null coordinates with respect to g and (θ^A) refers to an atlas on the topological spheres $\mathcal{S}_{u,v}$ that are formed by the intersections of the loci of u, v . In such a coordinate system, the metric g takes the form

$$g = -4\Omega^2 du dv + g_{AB} (d\theta^A - b^A dv)(d\theta^B - b^B dv). \quad (6.2.6)$$

In the above, for any (u, v) , $\Omega(u, v, \theta^A)$ is a scalar field on $\mathcal{S}_{u,v}$, $b(u, v, \theta^A)$ is a vector field tangent to $\mathcal{S}_{u,v}$ and $g(u, v, \theta^A)$ is a positive definite bilinear form on $\mathcal{S}_{u,v}$ which has vanishing projection in both null directions. Note that g is the metric induced by g on $\mathcal{S}_{u,v}$.

The metric (6.2.6) gives rise to an orthonormal frame (e_3, e_4, e_A) via

$$e_3 = du^\sharp = \Omega^{-1} \partial_u, \quad e_4 = dv^\sharp = \Omega^{-1} \partial_v - b^A e_A, \quad (6.2.7)$$

and e_A defined such that $(e_A|_{\mathcal{S}_{u,v}})$ is an orthonormal frame on $\mathcal{S}_{u,v}$. For example, in the double null gauge defined by the Eddington–Finkelstein coordinates on the Schwarzschild exterior, the orthonormal frame derived from the metric (6.2.5) is given by

$$\left(e_1 = \frac{1}{r} \partial_\theta, e_2 = \frac{1}{r \sin \theta} \partial_\varphi, e_3 = \frac{1}{\Omega} \partial_u, e_4 = \frac{1}{\Omega} \partial_v \right). \quad (6.2.8)$$

An important convention in this chapter is that uppercase Latin indices $A, B, C \dots$ will always refer to the indices on the spheres.

6.2.3 $\mathcal{S}_{u,v}$ -tangent tensors

When working in a double null gauge, it is convenient to decompose the metric, connection and curvature components into tensor fields that are everywhere tangent to the spheres $\mathcal{S}_{u,v}$. A beautiful and thorough presentation of this in generality is given in Chapter 1.2 of [Chr09].

Here, we restrict our attention to the Schwarzschild exterior in Eddington–Finkelstein coordinates, where the spheres $\mathcal{S}_{u,v}$ of the Eddington–Finkelstein foliation are round spheres with area-radius function given by $r = r(u, v)$ and metric given by $g = r^2 \mathring{g}$, $\mathring{g}$ being the standard

metric of the unit sphere. A one-form X is then said to be $\mathcal{S}_{u,v}$ -tangent if $X(e_3) = 0 = X(e_4)$ at each point, and a vector field V is said to be $\mathcal{S}_{u,v}$ -tangent if it is tangent to $\mathcal{S}_{u,v}$ for all u, v . These definitions can easily be generalised to include covariant, contravariant and mixed-type $\mathcal{S}_{u,v}$ -tensor fields, see [Chr09]. Moreover, it is easy to show that the musical isomorphism (i.e. the raising and lowering of indices via the metric g and its inverse g^{-1}) preserves the property of a tensor field to be $\mathcal{S}_{u,v}$ -tangent. We denote the bundle of $\mathcal{S}_{u,v}$ -tangent (p, q) -tensors on $\mathcal{M}_M$ by $\mathbb{H}T^{(p,q)}\mathcal{M}_M$, and similarly for any spherically symmetric subset of $\mathcal{M}_M$. In what follows, for any tensor bundle $\mathcal{T}$ over $\mathcal{M}_M$, we denote by $\Gamma^\infty(\mathcal{T})$ the space of smooth sections over $\mathcal{T}$.

A special role will be played by the tensor bundle $\mathbb{H}T_{stf}^{(0,2)}\mathcal{M}_M$, which consists of all elements of $\mathbb{H}T^{(0,2)}\mathcal{M}_M$ that are symmetric and trace-free with respect to $\mathring{g}$. We will also just call these tensors stf 2-tensors and, given $\alpha \in \Gamma^\infty(\mathbb{H}T^{(0,2)}\mathcal{M}_M)$, we will denote

$$(\alpha_{stf})_{AB} = \frac{1}{2}(\alpha_{AB} + \alpha_{BA}) - \mathring{g}_{AB}\mathring{g}^{CD}\alpha_{CD} = \frac{1}{2}(\alpha_{AB} + \alpha_{BA}) - \mathring{g}_{AB}\text{tr}\alpha. \quad (6.2.9)$$

Let ∇ be the Levi-Civita connection of the Schwarzschild spacetime $(\mathcal{M}_M, g)$. We denote by $\mathring{\nabla}$ the covariant derivative with respect to the metric $\mathring{g}$ on the spheres $\mathcal{S}_{u,v}$. Note that, at each point (u, v) , this is related to the Levi-Civita connection $\mathring{\nabla}$ on $\mathbb{S}^2$ simply via $\mathring{\nabla} = r(u, v) \cdot \mathring{\nabla}$.

For any tensor field $F \in \Gamma^\infty(\mathbb{H}T^{(p,q)}\mathcal{M}_M)$, we denote by $\mathring{\nabla}_3 F$ the component of $\nabla_3 F$ that is everywhere tangent to $\mathcal{S}_{u,v}$. An analogous definition gives $\mathring{\nabla}_4$. We will very often work with the weighted covariant derivatives

$$\mathring{\nabla}_u := \Omega \mathring{\nabla}_3 = \mathring{\nabla}_{\partial_u}, \quad \mathring{\nabla}_v := \Omega \mathring{\nabla}_4 = \mathring{\nabla}_{\partial_v}. \quad (6.2.10)$$

Note the relations

$$[\mathring{\nabla}_u, \mathring{\nabla}_v] = 0, \quad [\mathring{\nabla}_u, r\mathring{\nabla}_A] = 0 = [\mathring{\nabla}_v, r\mathring{\nabla}_A]. \quad (6.2.11)$$

In the computations of this chapter, the following simple fact will be used a lot: For any $\mathcal{S}_{u,v}$ -tangent $(0, q)$ -tensor Ξ , we have

$$\begin{aligned} (\mathring{\nabla}_u \Xi)(e_{A_1}, \dots, e_{A_q}) &= \partial_u(\Xi(e_{A_1}, \dots, e_{A_q})) \\ (\mathring{\nabla}_v \Xi)(e_{A_1}, \dots, e_{A_q}) &= \partial_v(\Xi(e_{A_1}, \dots, e_{A_q})) \end{aligned} \quad (6.2.12)$$

In words, evaluating the transversal covariant derivative $\mathring{\nabla}_u$ or $\mathring{\nabla}_v$ of an $\mathcal{S}_{u,v}$ -tensor in the frame (e_A, e_B) is the same as computing the partial derivative of the components of the tensor with respect to that frame. In the following, whenever we write down expressions such as $\int_{v_1}^{v_2} \mathring{\nabla}_v \Xi dv$ or similar, we will mean this component-wise.

6.2.4 Norms and the $\mathcal{O}$ -notation

Given $\mathcal{S}_{u,v}$ -tangent (p, q) -tensor fields $F, \tilde{F}$ on $\mathcal{M}_M$, we define the inner product

$$F_1 \cdot F_2 := (\not{g}^{-1})^{A_1 B_1} \dots (\not{g}^{-1})^{A_p B_p} \not{g}_{C_1 D_1} \dots \not{g}_{C_q D_q} F_{A_1, \dots, A_p}^{C_1, \dots, C_p} \tilde{F}_{B_1, \dots, B_p}^{D_1, \dots, D_p}, \quad (6.2.13)$$

and we define the pointwise norm via

$$|F|^2 := F \cdot F. \quad (6.2.14)$$

For $f \in \Gamma^\infty(\mathcal{M}_M)$, we then write

$$F = \mathcal{O}(f) \quad (6.2.15)$$

if there exists a positive uniform constant C such that $|F| \leq C \cdot f$. Speaking of constants, C will generally be a constant that is allowed to change from line to line, and instead of saying that $f \leq C \cdot g$, we will often just write $f \lesssim g$.

We will also use the $\mathcal{O}$ -notation restricted to subsets of $\mathcal{M}_M$. We will then also employ the $\mathcal{O}_\infty$ -notation: If F is an $\mathcal{S}_{u,v}$ -tangent tensor field on $\underline{\mathcal{C}}_v \cap \{u \leq -1\}$, then

$$F = \mathcal{O}_\infty(f) \quad (6.2.16)$$

will mean that for all $m, n \in \mathbb{N}$ there exists uniform positive constants $C_{m,n}$ s.t.

$$|\nabla_u^m \overset{\circ}{\nabla}^n F| \leq C_{m,n} |\partial_u^m f|. \quad (6.2.17)$$

(In particular, with this convention, $F = \mathcal{O}_\infty(1)$ will mean that all $\overset{\circ}{\nabla}_u$ -derivatives of F vanish identically.)

We define the following L^2 -norm (note that this has no r -weight!):

$$\|F(u, v, \cdot)\|_{L^2(\mathcal{S}_{u,v})}^2 := \int_{\mathcal{S}_{u,v}} |F|^2 \sin \theta \, d\theta \, d\varphi. \quad (6.2.18)$$

We similarly define

$$\|F(u, \cdot)\|_{L^2(\mathcal{C}_u \cap \{v_1 \leq v \leq v_2\})}^2 := \int_{v_1}^{v_2} \int_{\mathbb{S}^2} |F|^2(u, v, \theta, \varphi) \sin \theta \, d\theta \, d\varphi \, dv, \quad (6.2.19)$$

$$\|F(\cdot, v, \cdot)\|_{L^2(\underline{\mathcal{C}}_v \cap \{u_1 \leq u \leq u_2\})}^2 := \int_{u_1}^{u_2} \int_{\mathbb{S}^2} |F|^2(u, v, \theta, \varphi) \sin \theta \, d\theta \, d\varphi \, du. \quad (6.2.20)$$

We will often just write $\mathbb{S}^2$ rather than $\mathcal{S}_{u,v}$, and $[v_1, v_2] \times \mathbb{S}^2$ instead of $\mathcal{C}_u \cap \{v_1 \leq v \leq v_2\}$. We finally define the Sobolev norm for $H^1(\mathcal{C}_u \cap \{v_1 \leq v \leq v_2\})$ via

$$\|F\|_{H^1(\mathcal{C}_u \cap \{v_1 \leq v \leq v_2\})}^2 = \|F\|_{L^2(\mathcal{C}_u \cap \{v_1 \leq v \leq v_2\})}^2 + \|\nabla_v F\|_{L^2(\mathcal{C}_u \cap \{v_1 \leq v \leq v_2\})}^2 + \|\overset{\circ}{\nabla} F\|_{L^2(\mathcal{C}_u \cap \{v_1 \leq v \leq v_2\})}^2$$

and similarly for $\underline{\mathcal{C}}$ and higher order Sobolev norms. The corresponding spaces are defined as completions of the space of smooth tensor fields with respect to these norms.

6.2.5 Differential operators on the sphere

We now construct a variety of differential operators from ∇ . Since $\nabla = r^{-1} \cdot \mathring{\nabla}$, we will simply provide the definitions for the corresponding differential operators on the unit-sphere $\mathbb{S}^2$; lifting these definitions to differential operators acting on $\mathcal{S}_{u,v}$ -tangent tensors over $\mathcal{M}_M$ is straightforward. (Given a covariant tensor field on $\mathbb{S}^2$, we canonically identify it with a tensor field on $\mathcal{S}_{u,v}$ for each (u, v) , and we extend it to an $\mathcal{S}_{u,v}$ -tangent tensor field on $\mathcal{M}_M$ by demanding that it vanishes when contracted with e_3 or e_4 .)

We denote the volume form associated to the unit-sphere metric $\mathring{g}$ by $\mathring{\epsilon}$:

$$\mathring{\epsilon} = \sin \theta \, d\theta \wedge d\varphi, \quad (6.2.21)$$

we use $*$ to denote the Hodge dual on $(\mathbb{S}^2, \mathring{g})$, and we denote the unit-sphere Laplacian by $\mathring{\Delta}$ (so we have that $\mathring{\Delta} = r^2 \Delta$).

Definition 6.2.1. *We define for any covariant tensor field Ξ the divergence via*

$$\mathring{\text{div}} \Xi_{A_2 \dots A_p} = \mathring{g}^{A_0 A_1} \mathring{\nabla}_{A_0} \Xi_{A_1 A_2 \dots A_p}, \quad (6.2.22)$$

and the curl via

$$\mathring{\text{curl}} \Xi_{A_2 \dots A_p} = \mathring{\epsilon}^{A_0 A_1} \mathring{\nabla}_{A_0} \Xi_{A_1 A_2 \dots A_p}. \quad (6.2.23)$$

We then define

$$\mathring{\mathcal{P}}_1 : \Gamma^\infty(T^{(0,1)}\mathbb{S}^2) \rightarrow \Gamma^\infty(\mathbb{S}^2) \times \Gamma^\infty(\mathbb{S}^2), \quad \beta \mapsto (\mathring{\text{div}}\beta, \mathring{\text{curl}}\beta) \quad (6.2.24)$$

$$\mathring{\mathcal{P}}_2 : \Gamma^\infty(T_{\text{stf}}^{(0,2)}(\mathbb{S}^2)) \rightarrow \Gamma^\infty(T^{(0,1)}\mathbb{S}^2), \quad \alpha \mapsto \mathring{\text{div}}\beta \quad (6.2.25)$$

We further define the L^2 -adjoints of these operators:

$$\mathring{\mathcal{P}}_1^* : \Gamma^\infty(\mathbb{S}^2) \times \Gamma^\infty(\mathbb{S}^2) \rightarrow \Gamma^\infty(T^{(0,1)}\mathbb{S}^2), \quad (f, g) \mapsto -\mathring{\nabla} f + * \mathring{\nabla} g \quad (6.2.26)$$

$$\mathring{\mathcal{P}}_2^* : \Gamma^\infty(T^{(0,1)}\mathbb{S}^2) \rightarrow \Gamma^\infty(T_{\text{stf}}^{(0,2)}(\mathbb{S}^2)), \quad \beta \mapsto -(\mathring{\nabla} \beta)_{\text{stf}} \quad (6.2.27)$$

We finally define the magnetic adjoint of $\mathring{\mathcal{P}}_1$ to be

$$\overline{\mathring{\mathcal{P}}_1} : \Gamma^\infty(T^{(0,1)}\mathbb{S}^2) \rightarrow \Gamma^\infty(\mathbb{S}^2) \times \Gamma^\infty(\mathbb{S}^2), \quad \beta \mapsto (\mathring{\text{div}}\beta, -\mathring{\text{curl}}\beta). \quad (6.2.28)$$

For the relations between these operators and the “ $\mathring{\partial}$ ”-operators that scholars of the Newman–Penrose formalism may be more familiar with, see the Appendix 6.A.2.

The following simple lemma will be used very frequently throughout the chapter:

Lemma 6.2.1. *We have the relationships*

$$\begin{aligned} -\mathring{\mathcal{P}}_1^* \mathring{\mathcal{P}}_1 &= \mathring{\Delta} - 1, & -\mathring{\mathcal{P}}_1 \mathring{\mathcal{P}}_1^* &= \mathring{\Delta}, \\ -2\mathring{\mathcal{P}}_2^* \mathring{\mathcal{P}}_2 &= \mathring{\Delta} - 2, & -2\mathring{\mathcal{P}}_2 \mathring{\mathcal{P}}_2^* &= \mathring{\Delta} + 1. \end{aligned} \quad (6.2.29)$$

Proof. The results follow from relating the commutator of two covariant derivatives to the Riemann tensor, using that the Riemann tensor in two dimensions is determined by the Gauss curvature, and using that the Gauss curvature of $\mathbb{S}^2$ is 1. $\square$

6.2.6 Definition of tensorial spherical harmonics

Let $Y_{\ell,m} \in \Gamma^\infty(\mathbb{S}^2)$ be the usual real-valued spherical harmonics with $\ell \in \mathbb{N}$, $m = -\ell, -\ell+1, \dots, \ell$. We now define the corresponding 1-form spherical harmonics and symmetric traceless two-tensor spherical harmonics following [Czi17].

For this, we first recall the general fact that any smooth 1-form β on $\mathbb{S}^2$ can be written as

$$\beta = \mathring{\mathcal{P}}_1^*(f, g) \quad (6.2.30)$$

for a unique pair of smooth functions (f, g) that is supported on $\ell \geq 1$. Moreover, the kernel of the operator $\mathring{\mathcal{P}}_1^*$ is spanned by functions supported on $\ell = 0$.

Similarly, any smooth symmetric traceless two-tensor α on $\mathbb{S}^2$ can be uniquely represented as

$$\alpha = \mathring{\mathcal{P}}_2^* \mathring{\mathcal{P}}_1^*(f, g), \quad (6.2.31)$$

with (f, g) a pair of smooth functions supported on $\ell \geq 2$. Moreover, the kernel of the operator $\mathring{\mathcal{P}}_2^* \mathring{\mathcal{P}}_1^*$ is spanned by functions supported on $\ell = 0, 1$.

We can now already define what it means for a one-form or a symmetric, tracefree two-tensor to be supported on angular frequency ℓ .

Definition 6.2.2. *We say that a 1-form (or a traceless, symmetric two-tensor) is supported on angular frequency ℓ if the functions (f, g) in the unique representation (6.2.30) (or (6.2.31)) are supported on angular frequency ℓ . By definition, one-forms have no support on $\ell = 0$, and stf two-tensors have no support on $\ell = 0, 1$.*

It will, however, be convenient to also make a choice for an explicit orthonormal decomposition of the space of square-integrable one-forms and stf two-tensors on $\mathbb{S}^2$.

Definition 6.2.3. *Let $Y_{\ell,m}$ be the usual real-valued spherical harmonics. Then we define*

$$\begin{aligned} Y_{\ell,m}^{E,1} &= (\ell(\ell+1))^{-\frac{1}{2}} \mathring{\mathcal{P}}_1^*(Y_{\ell,m}, 0), & Y_{\ell,m}^{H,1} &= (\ell(\ell+1))^{-\frac{1}{2}} \mathring{\mathcal{P}}_1^*(0, Y_{\ell,m}), \\ Y_{\ell,m}^{E,2} &= (\tfrac{1}{2}(\ell-1)(\ell+2))^{-\frac{1}{2}} \mathring{\mathcal{P}}_2^* Y_{\ell,m}^{E,1}, & Y_{\ell,m}^{H,2} &= (\tfrac{1}{2}(\ell-1)(\ell+2))^{-\frac{1}{2}} \mathring{\mathcal{P}}_2^* Y_{\ell,m}^{H,1}. \end{aligned} \quad (6.2.32)$$

The superscripts E and H, standing for electric and magnetic part, respectively, have been chosen in view of the parity properties of these harmonics.

Definition 6.2.4. For any $\alpha = \mathring{\mathcal{P}}_2^* \mathring{\mathcal{P}}_1^*(f, g) \in \Gamma^\infty(T_{\text{stf}}^{(0,2)}(\mathbb{S}^2))$, we define its electric part α^E , its magnetic part α^H as well as its magnetic conjugate $\bar{\alpha}$ via

$$\alpha^E := \mathring{\mathcal{P}}_2^* \mathring{\mathcal{P}}_1^*(f, 0), \quad \alpha^H := \mathring{\mathcal{P}}_2^* \mathring{\mathcal{P}}_1^*(0, g), \quad \bar{\alpha} := \mathring{\mathcal{P}}_2^* \mathring{\mathcal{P}}_1^*(f, -g). \quad (6.2.33)$$

Remark 6.2.1. Up to r -scaling and factors of $\sqrt{2}$, these spherical harmonics are the same as Thorne's pure spin vector and tensor harmonics [Tho80]. We also provide a detailed discussion relating them to the Newman–Penrose spin-weighted spherical harmonics in the appendix 6.A.2.

It follows from either $[\mathring{\Delta}, \mathring{\mathcal{P}}_1^*] = \mathring{\mathcal{P}}_1^*$ or $[\mathring{\Delta}, \mathring{\mathcal{P}}_2^* \mathring{\mathcal{P}}_1^*] = 4\mathring{\mathcal{P}}_2^* \mathring{\mathcal{P}}_1^*$ that, for $s = 1, 2$:

$$\mathring{\Delta} Y_{\ell,m}^{E,s} = (-\ell(\ell+1) + s^2) Y_{\ell,m}^{E,s}, \quad \mathring{\Delta} Y_{\ell,m}^{H,s} = (-\ell(\ell+1) + s^2) Y_{\ell,m}^{H,s}. \quad (6.2.34)$$

We note that, in view of Def. 6.2.2, the same holds true more generally for 1-forms/traceless symmetric two-tensors supported on angular frequency ℓ .

We also have the following basic observations, which follow directly from Def. 6.2.3 and Lemma 6.2.1:

Lemma 6.2.2. *We have*

$$\begin{aligned} \mathring{\mathcal{P}}_1^*(Y_{\ell,m}, Y_{\ell,m}) &= \sqrt{\ell(\ell+1)} (Y_{\ell,m}^{E,1} + Y_{\ell,m}^{H,1}), \\ \mathring{\mathcal{P}}_1(Y_{\ell,m}^{E,1}, Y_{\ell,m}^{H,1}) &= \sqrt{\ell(\ell+1)} (Y_{\ell,m}, Y_{\ell,m}), \end{aligned} \quad (6.2.35)$$

as well as

$$\begin{aligned} \mathring{\mathcal{P}}_2^* Y_{\ell,m}^{E,1} &= \sqrt{\frac{1}{2}(\ell-1)(\ell+2)} Y_{\ell,m}^{E,2}, & \mathring{\mathcal{P}}_2^* Y_{\ell,m}^{H,1} &= \sqrt{\frac{1}{2}(\ell-1)(\ell+2)} Y_{\ell,m}^{H,2}, \\ \mathring{\mathcal{P}}_2 Y_{\ell,m}^{E,2} &= \sqrt{\frac{1}{2}(\ell-1)(\ell+2)} Y_{\ell,m}^{E,1}, & \mathring{\mathcal{P}}_2 Y_{\ell,m}^{H,2} &= \sqrt{\frac{1}{2}(\ell-1)(\ell+2)} Y_{\ell,m}^{H,1}. \end{aligned} \quad (6.2.36)$$

Next, we define the projection onto these spherical harmonics in the following way:

Definition 6.2.5. Let β be a smooth 1-form. Then we define

$$\beta_{\ell,m}^E := \int_{\mathbb{S}^2} \beta \cdot Y_{\ell,m}^{E,1} \sin \theta \, d\theta \, d\varphi, \quad \beta_{\ell,m}^H := \int_{\mathbb{S}^2} \beta \cdot Y_{\ell,m}^{H,1} \sin \theta \, d\theta \, d\varphi \quad (6.2.37)$$

as well as

$$\beta_\ell := \sum_{m=-\ell}^{\ell} (\beta_{\ell,m}^E Y_{\ell,m}^{E,1} + \beta_{\ell,m}^H Y_{\ell,m}^{H,1}). \quad (6.2.38)$$

Similarly, let α be a smooth symmetric traceless two-tensor. Then we define

$$\alpha_{\ell,m}^E := \int_{\mathbb{S}^2} \alpha \cdot Y_{\ell,m}^{E,2}, \quad \alpha_{\ell,m}^H := \int_{\mathbb{S}^2} \alpha \cdot Y_{\ell,m}^{H,2} \quad (6.2.39)$$

as well as

$$\alpha_\ell := \sum_{m=-\ell}^{\ell} (\alpha_{\ell,m}^E Y_{\ell,m}^{E,2} + \alpha_{\ell,m}^H Y_{\ell,m}^{H,2}). \quad (6.2.40)$$

In particular, while the $\alpha_{\ell,m}^E$ are scalars, the α_ℓ are traceless symmetric two-tensors.

The following standard proposition is proved in [Czi17]:

Proposition 6.2.1. *The family of electric and magnetic 1-form spherical harmonics together forms a complete, orthonormal basis of $L^2(T^{(0,1)}\mathbb{S}^2)$, and, for any $\beta \in L^2(T^{(0,1)}\mathbb{S}^2)$:*

$$\|\beta\|_{L^2(\mathbb{S}^2)}^2 = \sum_{\ell=1}^{\infty} \|\beta_\ell\|_{L^2(\mathbb{S}^2)}^2 = \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} (\beta_{\ell,m}^E)^2 + (\beta_{\ell,m}^H)^2. \quad (6.2.41)$$

Similarly, the family of electric and magnetic stf two-tensor spherical harmonics forms a complete, orthonormal basis of $L^2(T_{\text{stf}}^{(0,2)}(\mathbb{S}^2))$, and, for any $\alpha \in L^2(T_{\text{stf}}^{(0,2)}(\mathbb{S}^2))$,

$$\|\alpha\|_{L^2(\mathbb{S}^2)}^2 = \sum_{\ell=2}^{\infty} \|\alpha_\ell\|_{L^2(\mathbb{S}^2)}^2 = \sum_{\ell=2}^{\infty} \sum_{m=-\ell}^{\ell} (\alpha_{\ell,m}^E)^2 + (\alpha_{\ell,m}^H)^2. \quad (6.2.42)$$

As before, the definitions of these tensor spherical harmonics on $\mathbb{S}^2$ can easily be extended to tensor spherical harmonics on $\mathcal{M}_M$.

6.2.7 The Kerr family

The Schwarzschild family smoothly embeds into the two-parameter family of Kerr spacetimes $(\mathcal{M}_{M,a}, g_{M,a})$ (see, for instance, [DRSR16]). This means that when studying perturbations of the Schwarzschild family, one will, generically, see nearby members of the Kerr family. We will return to this point in §6.4, where we will write down the linearised version of the Kerr solution around Schwarzschild.

6.3 The linearised Einstein vacuum equations around Schwarzschild in a double null gauge

6.3.1 The linearisation procedure

We here give a rough outline of the linearisation procedure, the purpose being that this chapter can be read in a self-contained manner. For details, see [DHR19b].

Consider a one-parameter family of metrics $g(\epsilon)$ in double null gauge (6.2.6) solving the Einstein vacuum equations on $\mathcal{M}_M$ such that $g(\epsilon = 0)$ equals the Schwarzschild metric:

$$g(\epsilon) = -4\Omega^2(\epsilon) du dv + g_{AB}(\epsilon)(d\theta^A - b^A(\epsilon) dv)(d\theta^B - b^B dv). \quad (6.3.1)$$

The condition that $\epsilon = 0$ give the Schwarzschild metric means that $\Omega^2(0) = \Omega^2$, $b(0) = 0$ and $g(0) = g$. Next, we associate to this family of metrics a family of frames (e_1, e_2, e_3, e_4) according to (6.2.7), and we introduce the following decomposition of the connection coefficients:

$$\begin{aligned} \chi_{AB} &:= g(\nabla_A e_4, e_B), & \underline{\chi}_{AB} &:= g(\nabla_A e_3, e_B), \\ \eta_A &:= -\frac{1}{2}g(\nabla_3 e_A, e_4), & \underline{\eta}_A &:= -\frac{1}{2}g(\nabla_4 e_A, e_3), \\ \omega &:= \frac{1}{2}g(\nabla_4 e_3, e_4), & \underline{\omega} &:= \frac{1}{2}g(\nabla_3 e_4, e_3). \end{aligned} \quad (6.3.2)$$

Since the metric g is in double null gauge, all other connection coefficients (except for those on the sphere) either vanish or are easily related to the ones above. For instance, $\xi_A := \frac{1}{2}g(\nabla_4 e_4, e_A) = 0$, and $\zeta := \frac{1}{2}g(\nabla_A e_4, e_3) = \eta_A - \nabla_A \Omega$.

Similarly, we decompose the Riemann curvature tensor R into the coefficients ($\not\epsilon$ denoting the induced volume form on $\mathcal{S}_{u,v}$):

$$\begin{aligned} \alpha_{AB} &:= R_{A4B4}, & \underline{\alpha}_{AB} &:= R_{A3B3}, \\ \beta_A &:= R_{A434}, & \underline{\beta}_A &:= R_{A343}, \\ \rho &:= \frac{1}{4}R_{3434}, & \sigma \not\epsilon_{AB} &:= \frac{1}{2}R_{AB34}. \end{aligned} \quad (6.3.3)$$

The symmetries of the Riemann tensor allow to recast the last two definitions into the form

$$-\rho g_{AB} + \sigma \not\epsilon_{AB} = R_{A3B4}. \quad (6.3.4)$$

Finally, we denote the Gauss curvature on the spheres by K .

Remark 6.3.1. These decompositions are the basis of the Christodoulou–Klainerman formalism, which is essentially the same as the original Newman–Penrose formalism. In particular, writing

$\sqrt{2}m = e_1 + ie_2$, $\sqrt{2}\bar{m} = e_1 - ie_2$ ¹, we can define the famous Newman–Penrose scalars Ψ_i via:

$$\Psi_0 := \alpha(m, m), \quad \Psi_4 := \underline{\alpha}(\bar{m}, \bar{m}) \quad (6.3.5)$$

$$\Psi_1 := \beta(m) \quad \Psi_3 := \underline{\beta}(\bar{m}) \quad (6.3.6)$$

$$\Psi_2 := -\rho + i\sigma \quad (6.3.7)$$

For a detailed dictionary between the two, see the appendix 6.A.3.

At this point, we write down the Einstein vacuum equations as well as the Bianchi equations with respect to this decomposition, and linearise each of the resulting equations with respect to the parameter ε (by writing $\Omega = \Omega + \varepsilon \cdot \overset{(i)}{\Omega} + \mathcal{O}(\varepsilon^2)$ etc. and then discarding all terms of order ε^2). In the following, we will always use the superscript $\overset{(i)}$ to denote linearised quantities and bold font to denote nonlinear quantities.

6.3.2 The system of equations

Following the linearisation procedure outlined above gives the system of linearised gravity around Schwarzschild. In fact, this system can be formulated without reference to any linearisation by viewing it as a geometric system of partial differential equations on Schwarzschild for a set of unknown quantities:

Definition 6.3.1. *Let $\mathcal{D}$ be any spherically symmetric open subset of $\mathcal{M}_M$, and let*

$$\mathfrak{S} = \left(\overset{(i)}{\hat{\mathfrak{g}}}, \overset{(i)}{\text{tr}\hat{\mathfrak{g}}}, \overset{(i)}{\Omega}, \overset{(i)}{b}, \overset{(i)}{(\Omega\text{tr}\chi)}, \overset{(i)}{(\Omega\text{tr}\underline{\chi})}, \overset{(i)}{\hat{\chi}}, \overset{(i)}{\underline{\hat{\chi}}}, \overset{(i)}{\eta}, \overset{(i)}{\underline{\eta}}, \overset{(i)}{\omega}, \overset{(i)}{\underline{\omega}}, \overset{(i)}{\alpha}, \overset{(i)}{\beta}, \overset{(i)}{\rho}, \overset{(i)}{\sigma}, \overset{(i)}{\underline{\beta}}, \overset{(i)}{\underline{\alpha}}, \overset{(i)}{K} \right), \quad (6.3.8)$$

where

- $\overset{(i)}{\Omega}, \overset{(i)}{\text{tr}\hat{\mathfrak{g}}}, \overset{(i)}{(\Omega\text{tr}\chi)}, \overset{(i)}{(\Omega\text{tr}\underline{\chi})}, \overset{(i)}{\omega}, \overset{(i)}{\underline{\omega}}, \overset{(i)}{\rho}, \overset{(i)}{\sigma}$ and $\overset{(i)}{K} \in C^\infty(\mathcal{D})$,
- $\overset{(i)}{b}, \overset{(i)}{\eta}, \overset{(i)}{\underline{\eta}}$, as well as $\overset{(i)}{\beta}$ and $\overset{(i)}{\underline{\beta}} \in \Gamma^\infty(\mathbb{M}T^{(0,1)}\mathcal{D})$, and
- $\overset{(i)}{\hat{\mathfrak{g}}}, \overset{(i)}{\hat{\chi}}, \overset{(i)}{\underline{\hat{\chi}}}$ as well as $\overset{(i)}{\alpha}$ and $\overset{(i)}{\underline{\alpha}} \in \Gamma^\infty(\mathbb{M}T_{stf}^{(0,2)}\mathcal{D})$.

We say $\mathfrak{S}$ is a solution to the linearised Einstein vacuum equations around Schwarzschild if the components of $\mathfrak{S}$ satisfy the equations (6.3.9)–(6.3.32) everywhere on $\mathcal{D}$.

We now write down the equations of linearised gravity around Schwarzschild (6.3.9)–(6.3.32).

¹Note that $\bar{m}$ does *not* denote the magnetic conjugate (cf. Def. 6.2.4) here, but the complex conjugate. No further confusion should arise in the main body of the chapter, as we won't further mention these complex frame vector fields.

6.3.2.1 The equations governing the metric coefficients

First, we have the equations governing the metric coefficients:

$$\partial_u \text{tr} \overset{(1)}{\underline{\hat{g}}} = 2 \left(\overset{(1)}{\Omega} \text{tr} \underline{\chi} \right) \quad (6.3.9a) \quad \partial_v \text{tr} \overset{(1)}{\hat{g}} = 2 \left(\overset{(1)}{\Omega} \text{tr} \chi \right) - 2 \overset{\circ}{d} \overset{(1)}{v} r^{-1} \overset{(1)}{b} \quad (6.3.9b)$$

$$\nabla_u \overset{(1)}{\hat{g}} = 2 \overset{(1)}{\Omega} \underline{\chi} \quad (6.3.10a) \quad \nabla_v \overset{(1)}{\hat{g}} = 2 \overset{(1)}{\Omega} \chi + 2 \overset{\circ}{\mathcal{P}}_2^* r^{-1} \overset{(1)}{b} \quad (6.3.10b)$$

$$\partial_u \left(\frac{\overset{(1)}{\Omega}}{\underline{\Omega}} \right) = \underline{\underline{\omega}} \quad (6.3.11a) \quad \partial_v \left(\frac{\overset{(1)}{\Omega}}{\Omega} \right) = \overset{(1)}{\omega} \quad (6.3.11b)$$

$$2 \overset{\circ}{\nabla} \left(\frac{\overset{(1)}{\Omega}}{\underline{\Omega}} \right) = r \left(\overset{(1)}{\eta} + \underline{\eta} \right) \quad (6.3.12)$$

$$\nabla_u (r^{-1} \overset{(1)}{b}) = \frac{2 \overset{(1)}{\Omega}^2}{r} (\overset{(1)}{\eta} - \underline{\eta}) \quad (6.3.13)$$

6.3.2.2 The equations governing the connection coefficients

Next, we have the equations governing the connection coefficients:

$$\nabla_u \left(r \left(\overset{(1)}{\Omega} \text{tr} \chi \right) \right) = 2 \overset{(1)}{\Omega}^2 \left(\overset{\circ}{d} \overset{(1)}{v} \overset{(1)}{\eta} + r \overset{(1)}{\rho} - \frac{4M}{r^2} \left(\frac{\overset{(1)}{\Omega}}{\underline{\Omega}} \right) \right) - \overset{(1)}{\Omega}^2 \left(\overset{(1)}{\Omega} \text{tr} \underline{\chi} \right) \quad (6.3.14)$$

$$\nabla_v \left(r \left(\overset{(1)}{\Omega} \text{tr} \underline{\chi} \right) \right) = 2 \overset{(1)}{\Omega}^2 \left(\overset{\circ}{d} \overset{(1)}{v} \underline{\eta} + r \overset{(1)}{\rho} - \frac{4M}{r^2} \left(\frac{\overset{(1)}{\Omega}}{\Omega} \right) \right) + \overset{(1)}{\Omega}^2 \left(\overset{(1)}{\Omega} \text{tr} \chi \right) \quad (6.3.15)$$

$$\nabla_u \left(\frac{r^2}{\underline{\Omega}^2} \left(\overset{(1)}{\Omega} \text{tr} \underline{\chi} \right) \right) = -4r \underline{\underline{\omega}} \quad (6.3.16a) \quad \nabla_v \left(\frac{r^2}{\Omega^2} \left(\overset{(1)}{\Omega} \text{tr} \chi \right) \right) = 4r \overset{(1)}{\omega} \quad (6.3.16b)$$

$$\nabla_u \left(r \Omega \overset{(1)}{\hat{\chi}} \right) = -2 \overset{(1)}{\Omega}^2 \overset{\circ}{\mathcal{P}}_2^* \overset{(1)}{\eta} - \overset{(1)}{\Omega}^2 \Omega \overset{(1)}{\hat{\chi}} \quad (6.3.17a) \quad \nabla_v \left(\frac{r^2 \overset{(1)}{\hat{\chi}}}{\Omega} \right) = -r^2 \overset{(1)}{\alpha} \quad (6.3.17b)$$

$$\nabla_u \left(\frac{r^2 \overset{(1)}{\hat{\chi}}}{\underline{\Omega}} \right) = -r^2 \overset{(1)}{\underline{\alpha}} \quad (6.3.18a) \quad \nabla_v \left(r \Omega \overset{(1)}{\hat{\chi}} \right) = -2 \overset{(1)}{\Omega}^2 \overset{\circ}{\mathcal{P}}_2^* \underline{\eta} + \overset{(1)}{\Omega}^2 \Omega \overset{(1)}{\hat{\chi}} \quad (6.3.18b)$$

$$\nabla_u (r^2 \overset{(1)}{\eta}) = 2r \overset{\circ}{\nabla} \underline{\underline{\omega}} - r^2 \Omega \overset{(1)}{\underline{\beta}} \quad (6.3.19a) \quad \nabla_v (r \overset{(1)}{\eta}) = -r \Omega \overset{(1)}{\beta} + \overset{(1)}{\Omega}^2 \underline{\eta} \quad (6.3.19b)$$

$$\nabla_u (r \underline{\eta}) = r \Omega \overset{(1)}{\underline{\beta}} - \overset{(1)}{\Omega}^2 \overset{(1)}{\eta} \quad (6.3.20a) \quad \nabla_v (r^2 \underline{\eta}) = 2r \overset{\circ}{\nabla} \omega + r^2 \Omega \overset{(1)}{\beta} \quad (6.3.20b)$$

$$\partial_u \overset{(1)}{\omega} = -\overset{(1)}{\Omega}^2 \left(\overset{(1)}{\rho} - \frac{4M}{r^3} \left(\frac{\overset{(1)}{\Omega}}{\Omega} \right) \right) = \partial_v \underline{\underline{\omega}} \quad (6.3.21)$$

Furthermore, we have the following elliptic equations:

$$\mathring{d}\mathring{v}\hat{\chi}^{(1)} = -\Omega\hat{\eta}^{(1)} - r\hat{\beta}^{(1)} + \frac{1}{2\Omega}\mathring{\nabla}(\Omega\text{tr}\chi) \quad (6.3.22)$$

$$\mathring{d}\mathring{v}\hat{\chi}^{(1)} = \Omega\hat{\eta}^{(1)} + r\hat{\beta}^{(1)} + \frac{1}{2\Omega}\mathring{\nabla}(\Omega\text{tr}\chi) \quad (6.3.23)$$

$$\mathring{c}\mathring{u}\mathring{r}\mathring{l}\hat{\eta}^{(1)} = r\hat{\sigma}^{(1)} = -\mathring{c}\mathring{u}\mathring{r}\mathring{l}\hat{\eta}^{(1)} \quad (6.3.24)$$

We also have the following linearised equation for the Gaussian curvature:

$$\hat{K}^{(1)} = -\hat{\rho}^{(1)} + \frac{1}{2r}((\Omega\text{tr}\chi) - (\Omega\text{tr}\chi)) - \frac{2\Omega^2}{r^2} \left(\frac{\hat{\Omega}^{(1)}}{\hat{\Omega}} \right), \quad (6.3.25)$$

where $\hat{K}^{(1)}$ is defined via

$$\hat{K}^{(1)} := -\frac{1}{4r^2}(\hat{\Delta} + 2)\text{tr}\hat{\mathcal{G}}^{(1)} + \frac{1}{2}\mathring{d}\mathring{v}\mathring{d}\mathring{v}\hat{\mathcal{G}}^{(1)}. \quad (6.3.26)$$

6.3.2.3 The equations governing the curvature coefficients

The system of linearised gravity is completed by the linearised Bianchi equations:

$$\mathring{\nabla}_v(r\Omega^2\hat{\alpha}^{(1)}) = 2\hat{\mathcal{P}}_2^*\Omega^2\hat{\beta}^{(1)} + \frac{6M\Omega^2}{r^2}\Omega\hat{\chi}^{(1)} \quad (6.3.27)$$

$$\mathring{\nabla}_u\frac{r^4\hat{\beta}^{(1)}}{\Omega} = -\mathring{d}\mathring{v}r^3\hat{\alpha}^{(1)} \quad (6.3.28a) \quad \mathring{\nabla}_v(r^2\Omega\hat{\beta}^{(1)}) = \hat{\mathcal{P}}_1^*(r\Omega^2\hat{\rho}^{(1)}, r\Omega^2\hat{\sigma}^{(1)}) + \frac{6M\Omega^2}{r}\hat{\eta}^{(1)} \quad (6.3.28b)$$

$$\partial_u(r^3\hat{\rho}^{(1)}) = -\mathring{d}\mathring{v}r^2\Omega\hat{\beta}^{(1)} + 3M(\Omega\text{tr}\chi) \quad (6.3.29a) \quad \partial_v(r^3\hat{\rho}^{(1)}) = \mathring{d}\mathring{v}r^2\Omega\hat{\beta}^{(1)} + 3M(\Omega\text{tr}\chi) \quad (6.3.29b)$$

$$\partial_u(r^3\hat{\sigma}^{(1)}) = -\mathring{c}\mathring{u}\mathring{r}\mathring{l}r^2\Omega\hat{\beta}^{(1)} \quad (6.3.30a) \quad \partial_v(r^3\hat{\sigma}^{(1)}) = -\mathring{c}\mathring{u}\mathring{r}\mathring{l}r^2\Omega\hat{\beta}^{(1)} \quad (6.3.30b)$$

$$\mathring{\nabla}_u(r^2\Omega\hat{\beta}^{(1)}) = \hat{\mathcal{P}}_1^*(-r\Omega^2\hat{\rho}^{(1)}, r\Omega^2\hat{\sigma}^{(1)}) - \frac{6M\Omega^2}{r}\hat{\eta}^{(1)} \quad (6.3.31a) \quad \mathring{\nabla}_v\frac{r^4\hat{\beta}^{(1)}}{\Omega} = \mathring{d}\mathring{v}r^3\hat{\alpha}^{(1)} \quad (6.3.31b)$$

$$\mathring{\nabla}_u(r\Omega^2\hat{\alpha}^{(1)}) = -2\hat{\mathcal{P}}_2^*\Omega^2\hat{\beta}^{(1)} + \frac{6M\Omega^2}{r^2}\Omega\hat{\chi}^{(1)} \quad (6.3.32)$$

6.3.3 The Teukolsky equations and the Regge–Wheeler equations

Having written down the full system of linearised gravity around Schwarzschild, we now derive from (6.3.9)–(6.3.32) a set of equations that will play a central role in this chapter, among them the decoupled Teukolsky and the Regge–Wheeler equations.

6.3.3.1 The Teukolsky equations satisfied by $\overset{(i)}{\alpha}$ and $\underline{\alpha}$

Multiply (6.3.32) by $\frac{r^4}{\Omega^4}$ and apply ∇_v to it using (6.3.31b) and (6.3.17b) to obtain a decoupled wave equation for $\overset{(i)}{\alpha}$, known as the Teukolsky equation (recall $-2\mathring{\mathcal{P}}_2^*\mathring{\mathcal{P}}_2^* = \mathring{\Delta} - 2$ from (6.2.29)):

$$\nabla_v \left(\frac{r^4}{\Omega^4} \nabla_u (r\Omega^2 \overset{(i)}{\alpha}) \right) = r^3 (\mathring{\Delta} - 2) \overset{(i)}{\alpha} - 6Mr^2 \overset{(i)}{\alpha}. \quad (\text{Teuk})$$

Similarly, starting from (6.3.27), multiplying it by $\frac{r^4}{\Omega^4}$ and applying ∇_u to it using (6.3.28a) and (6.3.18a), we obtain

$$\nabla_u \left(\frac{r^4}{\Omega^4} \nabla_v (r\Omega^2 \underline{\alpha}) \right) = r^3 (\mathring{\Delta} - 2) \underline{\alpha} - 6Mr^2 \underline{\alpha} \quad (\underline{\text{Teuk}})$$

These equations will be discussed in much more detail in §6.9.

6.3.3.2 The Regge–Wheeler equations satisfied by $\overset{(i)}{\Psi}$ and $\underline{\Psi}$

In applications, the first order terms in (Teuk), ($\underline{\text{Teuk}}$) (which appear if one writes the equations as $\nabla_v \nabla_u \overset{(i)}{\alpha} = \dots$) give rise to several difficulties. Fortunately, these first-order terms can be removed by commutations with suitable vector fields. First, we note that,

$$\left(\frac{r^2 \nabla_u}{\Omega^2} \right) (r\Omega^2 \overset{(i)}{\alpha}) = -2\mathring{\mathcal{P}}_2^* r^2 \Omega \overset{(i)}{\beta} + 6M\Omega \overset{(i)}{\chi}, \quad (6.3.33)$$

$$\left(\frac{r^2 \nabla_u}{\Omega^2} \right)^2 (r\Omega^2 \overset{(i)}{\alpha}) = 2\mathring{\mathcal{P}}_2^* \mathring{\mathcal{P}}_1^* (r^3 \overset{(i)}{\rho}, -r^3 \overset{(i)}{\sigma}) + 6M(r\Omega \overset{(i)}{\chi} - r\Omega \underline{\overset{(i)}{\chi}}). \quad (6.3.34)$$

Eq. (6.3.33) is just (6.3.32) multiplied by $\frac{r^2}{\Omega^2}$, and (6.3.34) then follows from (6.3.33) by using (6.3.31a) and (6.3.17a).

Similarly, using (6.3.27) and (6.3.28b) with (6.3.18b), we get

$$\left(\frac{r^2 \nabla_v}{\Omega^2} \right) (r\Omega^2 \underline{\alpha}) = 2\mathring{\mathcal{P}}_2^* r^2 \Omega \underline{\overset{(i)}{\beta}} + 6M\Omega \underline{\overset{(i)}{\chi}}, \quad (6.3.35)$$

$$\left(\frac{r^2 \nabla_v}{\Omega^2} \right)^2 (r\Omega^2 \underline{\alpha}) = 2\mathring{\mathcal{P}}_2^* \mathring{\mathcal{P}}_1^* (r^3 \overset{(i)}{\rho}, r^3 \overset{(i)}{\sigma}) + 6M(r\Omega \overset{(i)}{\chi} - r\Omega \underline{\overset{(i)}{\chi}}) \quad (6.3.36)$$

We now define:

$$\begin{aligned} \overset{(i)}{\psi} &:= \left(\frac{r^2 \nabla_u}{\Omega^2} \right) (r\Omega^2 \overset{(i)}{\alpha}), & \underline{\overset{(i)}{\psi}} &:= \left(\frac{r^2 \nabla_v}{\Omega^2} \right) (r\Omega^2 \underline{\alpha}). \\ \overset{(i)}{\Psi} &:= \left(\frac{r^2 \nabla_u}{\Omega^2} \right)^2 (r\Omega^2 \overset{(i)}{\alpha}), & \underline{\overset{(i)}{\Psi}} &:= \left(\frac{r^2 \nabla_v}{\Omega^2} \right)^2 (r\Omega^2 \underline{\alpha}). \end{aligned} \quad (6.3.37)$$

An observation of purely algebraic nature (cf. Lemma 6.3.1) is that if ${}^{(i)}\alpha$ satisfies (Teuk) (or if $\underline{\alpha}$ satisfies (Teuk)), then the remarkably simple Regge–Wheeler equation,

$$\nabla_u \nabla_v \Psi - \frac{\Omega^2}{r^2} (\mathring{\Delta} - 4) \Psi - \frac{6M\Omega^2}{r^3} \Psi = 0, \quad (\text{RW})$$

is satisfied for $\Psi = \overset{(i)}{\Psi}, \underline{\Psi}$. (See §6.9 for much more details on commutations with $\frac{r^2}{\Omega^2} \nabla_u, \frac{r^2}{\Omega^2} \nabla_v$.)

As a consequence of

$$\overset{(i)}{\Psi} - \underline{\Psi} = -4\mathring{\mathcal{P}}_2^* \mathring{\mathcal{P}}_1^*(0, r^3 \overset{(i)}{\sigma}), \quad (6.3.38)$$

eq. (RW) is therefore also satisfied by $\mathring{\mathcal{P}}_2^* \mathring{\mathcal{P}}_1^*(0, r^3 \overset{(i)}{\sigma})$. In fact, in that case, (RW) is just the tensorialised version of the scalar Regge–Wheeler equation:

$$\partial_u \partial_v (r^3 \overset{(i)}{\sigma}) - \frac{\Omega^2}{r^2} \mathring{\Delta} (r^3 \overset{(i)}{\sigma}) - \frac{6M\Omega^2}{r^3} (r^3 \overset{(i)}{\sigma}) = 0, \quad (\text{RW-scalar})$$

which can also be verified directly by acting with ∂_v on (6.3.30a) and using (6.3.28b).

6.3.3.3 The Teukolsky–Starobinsky identities and further relations

We further note the relations (which follow directly from (Teuk))

$$\left(\frac{r^2 \nabla_v}{\Omega^2} \right) \overset{(i)}{\Psi} = \left((\mathring{\Delta} - 2) - 2 \left(1 - \frac{3M}{r} \right) \right) \frac{r^4}{\Omega^4} \nabla_u (r \Omega^2 \overset{(i)}{\alpha}) - \frac{6Mr^2}{\Omega^2} r \Omega^2 \overset{(i)}{\alpha} \quad (6.3.39)$$

$$\frac{\Omega^4}{r^4} \left(\frac{r^2 \nabla_v}{\Omega^2} \right)^2 \overset{(i)}{\Psi} = (\mathring{\Delta} - 2)(\mathring{\Delta} - 4) r \Omega^2 \overset{(i)}{\alpha} - 6M(\nabla_u + \nabla_v)(r \Omega^2 \overset{(i)}{\alpha}) \quad (6.3.40)$$

Similarly, we have (as a consequence of (Teuk))

$$\left(\frac{r^2 \nabla_u}{\Omega^2} \right) \underline{\Psi} = \left((\mathring{\Delta} - 2) - 2 \left(1 - \frac{3M}{r} \right) \right) \frac{r^4}{\Omega^4} \nabla_v (r \Omega^2 \underline{\alpha}) + \frac{6Mr^2}{\Omega^2} r \Omega^2 \underline{\alpha} \quad (6.3.41)$$

$$\frac{\Omega^4}{r^4} \left(\frac{r^2 \nabla_u}{\Omega^2} \right)^2 \underline{\Psi} = (\mathring{\Delta} - 2)(\mathring{\Delta} - 4) r \Omega^2 \underline{\alpha} + 6M(\nabla_u + \nabla_v)(r \Omega^2 \underline{\alpha}) \quad (6.3.42)$$

From the identity (6.3.38), we can now derive the celebrated Teukolsky–Starobinsky identities [TP74] by applying either $(\Omega^{-2} r^2 \nabla_v)^2$ to (6.3.42) or $(\Omega^{-2} r^2 \nabla_u)^2$ to (6.3.40):

$$\begin{aligned} & \frac{\Omega^4}{r^4} \left(\frac{r^2 \nabla_u}{\Omega^2} \right)^4 (r \Omega^2 \overset{(i)}{\alpha}) \\ &= (\mathring{\Delta} - 2)(\mathring{\Delta} - 4) r \Omega^2 \underline{\alpha} - 4\mathring{\mathcal{P}}_2^* \mathring{\mathcal{P}}_1^*(0, \text{curl} \nabla v r \Omega^2 \overset{(i)}{\alpha}) + 6M(\nabla_u + \nabla_v)(r \Omega^2 \underline{\alpha}), \end{aligned} \quad (6.3.43)$$

$$\begin{aligned} & \frac{\Omega^4}{r^4} \left(\frac{r^2 \nabla_v}{\Omega^2} \right)^4 (r \Omega^2 \underline{\alpha}) \\ &= (\dot{\Delta} - 2)(\dot{\Delta} - 4) r \Omega^2 \underline{\alpha} - 4 \dot{\mathcal{P}}_2^* \dot{\mathcal{P}}_1^* (0, \text{curl} \dot{\mathcal{P}}_1^* r \Omega^2 \underline{\alpha}) - 6M(\nabla_u + \nabla_v)(r \Omega^2 \underline{\alpha}). \end{aligned} \quad (6.3.44)$$

Remark 6.3.2. For any $F \in \Gamma^\infty(\mathbb{M} T^{(p,q)} \mathcal{M}_M)$, we have $r(\Omega^{-2} \nabla_u)^4 (r^4 F) = r^{-4} (\Omega^{-2} r^2 \nabla_u)^4 (r F)$.

Remark 6.3.3. The angular operator on the RHS of (6.3.43) or (6.3.44) can be rewritten: Writing $\alpha = \dot{\mathcal{P}}_2^* \dot{\mathcal{P}}_1^* (f, g)$, it follows from Lemma 6.2.1 that

$$-4 \dot{\mathcal{P}}_2^* \dot{\mathcal{P}}_1^* (0, \text{curl} \dot{\mathcal{P}}_2^* \dot{\mathcal{P}}_1^* (f, g)) = -2(\dot{\Delta} - 2)(\dot{\Delta} - 4) \dot{\mathcal{P}}_2^* \dot{\mathcal{P}}_1^* (0, g). \quad (6.3.45)$$

Conversely, one can show that for any $\alpha \in \Gamma^\infty(T_{\text{stf}}^{(0,2)}(\mathbb{S}^2))$, $2 \dot{\mathcal{P}}_2^* \dot{\mathcal{P}}_1^* \dot{\mathcal{P}}_1 \dot{\mathcal{P}}_2 \alpha = (\dot{\Delta} - 2)(\dot{\Delta} - 4) \alpha$. Thus, we can equivalently write the angular operator on the RHS of (6.3.43) or (6.3.44) as the operator that sends $\dot{\mathcal{P}}_2^* \dot{\mathcal{P}}_1^* (f, g) \mapsto (\dot{\Delta} - 2)(\dot{\Delta} - 4) \dot{\mathcal{P}}_2^* \dot{\mathcal{P}}_1^* (f, -g)$, or as $2 \dot{\mathcal{P}}_2^* \dot{\mathcal{P}}_1^* \overline{\dot{\mathcal{P}}_1} \dot{\mathcal{P}}_2$, i.e. we have

$$\frac{\Omega^4}{r^4} \left(\frac{r^2 \nabla_v}{\Omega^2} \right)^4 (r \Omega^2 \underline{\alpha}) = 2 \dot{\mathcal{P}}_2^* \dot{\mathcal{P}}_1^* \overline{\dot{\mathcal{P}}_1} \dot{\mathcal{P}}_2 r \Omega^2 \underline{\alpha} - 6M(\nabla_u + \nabla_v)(r \Omega^2 \underline{\alpha}). \quad (6.3.46)$$

In terms of the spin-weighted “eth”-operators $\eth, \eth'$ of the Newman–Penrose formalism, the operator $\dot{\mathcal{P}}_2^* \dot{\mathcal{P}}_1^* \overline{\dot{\mathcal{P}}_1} \dot{\mathcal{P}}_2$ corresponds to $\eth'^4$ (or $\eth^4$ when acting on $\underline{\alpha}$), see the appendix for details (Remark 6.A.3.)

A useful lemma

For future reference, let us here already collect the following statements concerning the relation between the Regge–Wheeler and Teukolsky operators:

Definition 6.3.2. Given $F \in \Gamma^\infty(\mathbb{M} T_{\text{stf}}^{(0,2)} \mathcal{M}_M)$, define

$$\text{Teuk}^{+2}[F] := \frac{\Omega^2}{r^2} \nabla_v \frac{r^4}{\Omega^4} \nabla_u F - \left(\dot{\Delta} - 2 - \frac{6M}{r} \right) F, \quad (6.3.47)$$

$$\text{Teuk}^{-2}[F] := \frac{\Omega^2}{r^2} \nabla_u \frac{r^4}{\Omega^4} \nabla_v F - \left(\dot{\Delta} - 2 - \frac{6M}{r} \right) F, \quad (6.3.48)$$

$$\text{RW}[F] := \frac{r^2}{\Omega^2} \left(\nabla_u \nabla_v - \frac{\Omega^2}{r^2} \left(\dot{\Delta} - 4 - \frac{6M}{r} \right) \right) F. \quad (6.3.49)$$

Lemma 6.3.1. For any $F \in \Gamma^\infty(\mathbb{M} T_{\text{stf}}^{(0,2)} \mathcal{M}_M)$, we have

$$\text{RW} \left[\left(\frac{r^2}{\Omega^2} \nabla_u \right)^2 (F) \right] = \left(\frac{r^2}{\Omega^2} \nabla_u \right)^2 \text{Teuk}^{+2}[F], \quad (6.3.50)$$

$$\text{RW} \left[\left(\frac{r^2}{\Omega^2} \nabla_v \right)^2 (F) \right] = \left(\frac{r^2}{\Omega^2} \nabla_v \right)^2 \text{Teuk}^{-2}[F], \quad (6.3.51)$$

as well as

$$\text{Teuk}^{+2} \left[\frac{\Omega^2}{r^2} \nabla_v \frac{r^2}{\Omega^2} \nabla_v F \right] = \frac{\Omega^2}{r^2} \nabla_v \frac{r^2}{\Omega^2} \nabla_v \{\text{RW}[F]\}. \quad (6.3.52)$$

Proof. The first two identities are proved in section 3 of [Mas22a], so we only prove the last. We first compute

$$\left[\frac{\Omega^2}{r^2} \nabla_v \frac{r^2}{\Omega^2} \nabla_v \right] \nabla_u = \nabla_u \left[\frac{\Omega^2}{r^2} \nabla_v \frac{r^2}{\Omega^2} \nabla_v \right] - \frac{2\Omega^2}{r^2} (3\Omega^2 - 2) \nabla_v. \quad (6.3.53)$$

From this, we similarly infer that

$$\begin{aligned} \left[\frac{\Omega^2}{r^2} \nabla_v \frac{r^2}{\Omega^2} \nabla_v \right] \left(\frac{r^2}{\Omega^2} \nabla_u \nabla_v \right) &= \frac{\Omega^2}{r^2} \nabla_v \frac{r^4}{\Omega^4} \left[\nabla_u \left(\frac{\Omega^2}{r^2} \nabla_v \frac{r^2}{\Omega^2} \nabla_v \right) - \frac{2\Omega^2}{r^2} (3\Omega^2 - 2) \nabla_v \right] \\ &= \left[\frac{\Omega^2}{r^2} \nabla_v \frac{r^4}{\Omega^4} \nabla_u \right] \left[\frac{\Omega^2}{r^2} \nabla_v \frac{r^2}{\Omega^2} \nabla_v \right] - 12M \frac{\Omega^2}{r^2} \nabla_v \\ &\quad - 2(3\Omega^2 - 2) \left[\frac{\Omega^2}{r^2} \nabla_v \frac{r^2}{\Omega^2} \nabla_v \right]. \end{aligned} \quad (6.3.54)$$

Finally, note that

$$\left[\frac{\Omega^2}{r^2} \nabla_v \frac{r^2}{\Omega^2} \nabla_v \right] \frac{6M}{r} = -12M \frac{\Omega^2}{r^2} \nabla_v + \frac{6M}{r} \left[\frac{\Omega^2}{r^2} \nabla_v \frac{r^2}{\Omega^2} \nabla_v \right]. \quad (6.3.55)$$

Applying the above relations to the expression

$$\frac{\Omega^2}{r^2} \nabla_v \frac{r^2}{\Omega^2} \nabla_v \left[\frac{r^2}{\Omega^2} \nabla_u \nabla_v - \dot{\Delta} + \left(4 - \frac{6M}{r} \right) \right] \quad (6.3.56)$$

proves the claim. $\square$

6.4 Pure gauge and linearised Kerr solutions

Central to the understanding of the system of linearised gravity around Schwarzschild is the existence of two classes of explicit solutions to it. The first of them arises by linearising either a Schwarzschild solution with nearby mass, or a Kerr solution with mass M near Schwarzschild (cf. §6.2.7)—written in double null gauge—around Schwarzschild. This class is the class of linearised Kerr solutions.

The other class, the class of pure gauge solutions, arises by considering coordinate transformations (at the nonlinear level) of the variables $\mathbf{u}, \mathbf{v}, \theta^A$ such that the double null form of the nonlinear metric (6.2.6) is preserved, and then linearising the metric in these new coordinates around the Schwarzschild metric. These solutions to (6.3.9)–(6.3.32), which represent the gauge ambiguity inherited from (6.1.1), have been derived and classified in Section 6 of [DHR19b]; we here write them down for completeness.

6.4.1 The linearised Kerr solutions

Proposition 6.4.1. *Let $\mathbf{m} \in \mathbb{R}$. Then the following solves (6.3.9)–(6.3.32):*

$$\begin{aligned} \text{tr} \overset{(i)}{\not{g}} &= -2\mathbf{m}, & \left(\frac{\overset{(i)}{\Omega}}{\Omega} \right) &= -\frac{1}{2}\mathbf{m}, \\ \overset{(i)}{\rho} &= -\frac{2M}{r^3}\mathbf{m}, & \overset{(i)}{K} &= \frac{\mathbf{m}}{r^2}, \end{aligned} \tag{6.4.1}$$

with all remaining components vanishing. Note that this solution is entirely supported on $\ell = 0$. We will refer to it as the linearised nearby Schwarzschild solution with parameter $\mathbf{m}$, or by $\mathfrak{S}_{\mathbf{m}}$.

Proposition 6.4.2. *Let $\mathbf{a} = \{\mathbf{a}_{-1}, \mathbf{a}_0, \mathbf{a}_{+1}\} \in \mathbb{R}^3$. Then the following is a solution to (6.3.9)–(6.3.32):*

$$\begin{aligned} \overset{(i)}{b} &= \frac{4M}{r^2} \sqrt{2} \sum_{m=-1}^1 \mathbf{a}_m Y_{\ell 1}^{\text{H},1}, & \overset{(i)}{\eta} &= \frac{3}{4r} \overset{(i)}{b} = -\underline{\eta}, \\ \overset{(i)}{\beta} &= \frac{\Omega}{r} \overset{(i)}{\eta} = -\underline{\beta}, & \overset{(i)}{\sigma} &= \frac{6M}{r^4} \sum_{m=-1}^1 \mathbf{a}_m Y_{\ell 1}, \end{aligned} \tag{6.4.2}$$

with all remaining components vanishing. Note that this solution is entirely supported on $\ell = 1$. We will refer to it as the linearised Kerr solution with parameter $\mathbf{a}$, or by $\mathfrak{S}_{\mathbf{a}}$.

6.4.2 The pure gauge solutions

6.4.2.1 The outgoing gauge solutions

The solution below arises from an infinitesimal coordinate transformation of $\mathbf{v}$:

Proposition 6.4.3. *Let $f(v, \theta^A)$ be a smooth scalar function on $\mathbb{R} \times \mathbb{S}^2$. Then the following solves (6.3.9)–(6.3.32):*

$$\begin{aligned}
\frac{\overset{(\iota)}{\Omega}}{\Omega} &= \frac{1}{2\Omega^2} \partial_v (f\Omega^2), & \text{tr} \overset{(\iota)}{\mathcal{G}} &= \frac{4}{r} \left(\overset{(\iota)}{\Delta} + \Omega^2 \right) f, & \overset{(\iota)}{\hat{\mathcal{G}}} &= \frac{4}{r} \overset{(\iota)}{\mathcal{P}}_2^* \overset{(\iota)}{\mathcal{P}}_1^* (f, 0), \\
\overset{(\iota)}{b} &= 2 \overset{(\iota)}{\mathcal{P}}_1^* \left(r \partial_v \left(\frac{f}{r} \right), 0 \right), & \overset{(\iota)}{\eta} &= -\frac{\Omega^2}{r^2} \overset{(\iota)}{\mathcal{P}}_1^* (f, 0), & \overset{(\iota)}{\underline{\eta}} &= -\overset{(\iota)}{\mathcal{P}}_1^* \left(\frac{1}{\Omega^2} \partial_v \left(\frac{\Omega^2}{r} f \right), 0 \right), \\
\overset{(\iota)}{\underline{\hat{\chi}}} &= 2 \frac{\Omega}{r^2} \overset{(\iota)}{\mathcal{P}}_2^* \overset{(\iota)}{\mathcal{P}}_1^* (f, 0), & (\Omega \text{tr} \chi) &= 2 \partial_v \left(\frac{\Omega^2}{r} f \right), & \left(\overset{(\iota)}{\Omega \text{tr} \underline{\chi}} \right) &= 2 \frac{\Omega^2}{r^2} \left(\overset{(\iota)}{\Delta} + 2\Omega^2 - 1 \right) f, \\
\overset{(\iota)}{\rho} &= \frac{6M\Omega^2}{r^4} f, & \overset{(\iota)}{\underline{\beta}} &= -\frac{6M\Omega}{r^4} \overset{(\iota)}{\mathcal{P}}_1^* (f, 0), & \overset{(\iota)}{K} &= -\frac{\Omega^2}{r^3} \left(\overset{(\iota)}{\Delta} + 2 \right) f,
\end{aligned} \tag{6.4.3}$$

with $\overset{(\iota)}{\omega}, \overset{(\iota)}{\underline{\omega}}$ related to $\overset{(\iota)}{\Omega}$ via (6.3.11), and with the remaining components $\overset{(\iota)}{\hat{\chi}}, \overset{(\iota)}{\alpha}, \overset{(\iota)}{\underline{\alpha}}, \overset{(\iota)}{\beta}, \overset{(\iota)}{\sigma}$ vanishing. We will refer to this solution as the outgoing gauge solution generated by f , or by $\mathfrak{S}_f$.

In practice, we will require that, as $v \rightarrow \infty$, $f(v, \theta^A) = f_0(\theta^A)v + \mathcal{O}_\infty(v^{1-\epsilon})$ for some $\epsilon > 0$ and some smooth, potentially identically vanishing $f_0(\theta^A)$.

6.4.2.2 The ingoing gauge solutions

The solution below arises from an infinitesimal coordinate transformation of $\mathbf{u}$:

Proposition 6.4.4. *Let $\underline{f}(u, \theta^A)$ be a smooth scalar function on $\mathbb{R} \times \mathbb{S}^2$. The following is a solution to (6.3.9)–(6.3.32):*

$$\begin{aligned}
\frac{\overset{(\iota)}{\Omega}}{\Omega} &= \frac{1}{2\Omega^2} \partial_u (\underline{f}\Omega^2), & \text{tr} \overset{(\iota)}{\mathcal{G}} &= -\frac{4\Omega^2}{r} \underline{f}, & \overset{(\iota)}{\hat{\mathcal{G}}} &= 0, \\
\overset{(\iota)}{b} &= -2 \frac{\Omega^2}{r} \overset{(\iota)}{\mathcal{P}}_1^* (\underline{f}, 0), & \overset{(\iota)}{\eta} &= -\overset{(\iota)}{\mathcal{P}}_1^* \left(\frac{1}{\Omega^2} \partial_u \left(\frac{\Omega^2}{r} \underline{f} \right), 0 \right), & \overset{(\iota)}{\underline{\eta}} &= +\frac{\Omega^2}{r^2} \overset{(\iota)}{\mathcal{P}}_1^* (\underline{f}, 0), \\
\overset{(\iota)}{\hat{\chi}} &= 2 \frac{\Omega}{r^2} \overset{(\iota)}{\mathcal{P}}_2^* \overset{(\iota)}{\mathcal{P}}_1^* (\underline{f}, 0), & (\Omega \text{tr} \chi) &= 2 \frac{\Omega^2}{r^2} \left(\overset{(\iota)}{\Delta} + 2\Omega^2 - 1 \right) \underline{f}, & \left(\overset{(\iota)}{\Omega \text{tr} \underline{\chi}} \right) &= -2 \partial_u \left(\frac{\Omega^2}{r} \underline{f} \right), \\
\overset{(\iota)}{\rho} &= -\frac{6M\Omega^2}{r^4} \underline{f}, & \overset{(\iota)}{\beta} &= +\frac{6M\Omega}{r^4} \overset{(\iota)}{\mathcal{P}}_1^* (\underline{f}, 0), & \overset{(\iota)}{K} &= \frac{\Omega^2}{r^3} \left(\overset{(\iota)}{\Delta} + 2 \right) \underline{f},
\end{aligned} \tag{6.4.4}$$

with $\overset{(\iota)}{\omega}, \overset{(\iota)}{\underline{\omega}}$ related to $\overset{(\iota)}{\Omega}$ via (6.3.11), and with the remaining components $\overset{(\iota)}{\hat{\chi}}, \overset{(\iota)}{\alpha}, \overset{(\iota)}{\underline{\alpha}}, \overset{(\iota)}{\beta}, \overset{(\iota)}{\sigma}$ vanishing. We will refer to this solution as the ingoing gauge solution generated by $\underline{f}$, or by $\mathfrak{S}_{\underline{f}}$.

In practice, we will require that, as $u \rightarrow -\infty$, $\underline{f}(u, \theta^A) = \underline{f}_0(\theta^A)u + \mathcal{O}_\infty(u^{1-\epsilon})$ for some $\epsilon > 0$ and some smooth, potentially identically vanishing $\underline{f}_0(\theta^A)$.

Remark 6.4.1. Notice that the outgoing gauge solution generated by a constant C is identical to the ingoing gauge solution generated by $-C$.

6.4.2.3 The sphere gauge solutions

The solution below arises from an infinitesimal coordinate transformation of $\{\theta^A\}$:

Proposition 6.4.5. *Let $q_1(v, \theta^A)$, $q_2(v, \theta^A)$ be two smooth scalar functions on $\mathbb{R} \times \mathbb{S}^2$. Then the following solves (6.3.9)–(6.3.32):*

$$\hat{g}^{(i)} = 2\mathring{\mathcal{P}}_2^* \mathring{\mathcal{P}}_1^*(q_1, q_2), \quad \text{tr} \hat{g}^{(i)} = 2\mathring{\Delta}(q_1), \quad r^{-1} \hat{b}^{(i)} = \mathring{\mathcal{P}}_1^*(\partial_v q_1, \partial_v q_2), \quad (6.4.5)$$

with all other components vanishing. We refer to these solutions as the sphere gauge solutions generated by (q_1, q_2) , or by $\mathfrak{S}_{(q_1, q_2)}$.

In practice, q_1 , q_2 and all its derivatives will be required to be bounded.

Definition 6.4.1. *A solution is said to be pure gauge if it is a linear combination of the solutions from Propositions 6.4.3–6.4.5.*

6.4.3 Solutions supported on $\ell \leq 1$ are non-dynamical

The following statement is proved (in slightly modified form) in Theorem 9.2 of [DHR19b]:

Proposition 6.4.6. *If a solution $\mathfrak{S}$ solves (6.3.9)–(6.3.32) on an open subset of $\overline{\mathcal{M}}_M$ and is supported on $\ell = 0, 1$, then, up to suitable addition of a pure gauge solution, $\mathfrak{S}$ can be written as a linearised nearby Schwarzschild solution (Prop. 6.4.1) plus a linearised Kerr solution (Prop. 6.4.2).*

Remark 6.4.2. The proposition cited above embodies the fact that the $\ell = 0, 1$ components of a solution to (6.3.9)–(6.3.32) are non-dynamical, and the dynamics of the system (6.3.9)–(6.3.32) are contained in the $\ell \geq 2$. **In light of this fact, any physical discussion of (6.3.9)–(6.3.32) is mostly a discussion of the $\ell \geq 2$ -modes.**

6.4.4 Solutions supported on $\ell \geq 2$ with $\overset{(i)}{\alpha} = 0 = \underline{\alpha}^{(i)}$ are pure gauge

The following is proved in Theorem B.1 of [DHR19b]:

Proposition 6.4.7. *If a solution $\mathfrak{S}$ solves (6.3.9)–(6.3.32) on an open subset of $\overline{\mathcal{M}}_M$, is supported on $\ell \geq 2$ and satisfies $\overset{(i)}{\alpha} = \underline{\alpha}^{(i)} = 0$, then $\mathfrak{S}$ is a pure gauge solution.*

Part I:

Setting up and solving the scattering problem

6.5 The general scattering data setup for (6.3.9)–(6.3.32)

Having given a thorough recap of the system of linearised gravity around Schwarzschild, we now have all ingredients to formulate and solve the scattering problem for this system. As opposed to the paper [Mas22b], which discusses the global scattering problem with data on $\mathcal{H}^-$ and $\mathcal{I}^-$, the present chapter discusses the semi-global scattering problem where scattering data are posed on $\mathcal{I}^-$ and an ingoing null hypersurface $\underline{\mathcal{C}}$ at finite $v = v_1$, restricted to negative values of u (see already Fig. 6.5.1). Here, we opt to give an entirely self-contained presentation of it with a somewhat different approach that, for instance, does not use a gauge fixing procedure and that proves a few additional statements.

We now give an overview of the next few sections:

The key to obtaining the scattering theory for (6.3.9)–(6.3.32) is the scattering theory for the Regge–Wheeler equation (RW), which we develop in §6.6.

In §6.5.1, we then define a notion of *seed scattering data* for (6.3.9)–(6.3.32), define what it means for a solution to (6.3.9)–(6.3.32) to be a scattering solution realising these seed data, and we write down Theorem 6.5.1, which expresses the solvability of the scattering problem under fairly general assumptions on the seed data. We then give an overview of the proof of this theorem in §6.5.2, we construct out of the seed data the remaining scattering data along $\underline{\mathcal{C}}$ in §6.5.3, and we discuss certain gauge considerations such as Bondi normalisation in §6.5.4.

In §6.7, we then present the full details of the proof of Theorem 6.5.1.

With this result having been proved under fairly general assumptions and clarifying the role of seed data in the scattering problem, we will then, in §6.8, write down a set of explicit assumptions on a seed scattering data set such that it can be said to describe the exterior of a system of N infalling masses with no incoming radiation from $\mathcal{I}^-$. These physical seed data will be at the heart of the entire chapter.

The setup: We begin by fixing our notation: We denote by $\underline{\mathcal{C}}$ the incoming null hypersurface given by

$$\underline{\mathcal{C}} := \{(u, v, \theta, \varphi) \in \overline{\mathcal{M}}_M \mid v = v_1, u \leq u_0 < 0\}.$$

We further denote $\mathcal{S}_1 := \{(u, v, \theta, \varphi) \in \mathcal{M}_M \mid v = v_1, u = u_0\}$, and we similarly denote the limiting sphere $\mathcal{S}_\infty := \{(u, v, \theta, \varphi) \in \overline{\mathcal{M}}_M \mid v = v_1, u = -\infty\}$. Finally, we denote the part of $\mathcal{I}^-$ that lies to the future of $\mathcal{S}_\infty$ by $\mathcal{I}_{v \geq v_1}^-$. See Fig. 6.5.1.

We will denote the future domain of dependence $D^+(\underline{\mathcal{C}} \cup \mathcal{I}_{v \geq v_1}^-)$ of $\underline{\mathcal{C}} \cup \mathcal{I}_{v \geq v_1}^-$ by $\mathcal{D}$.

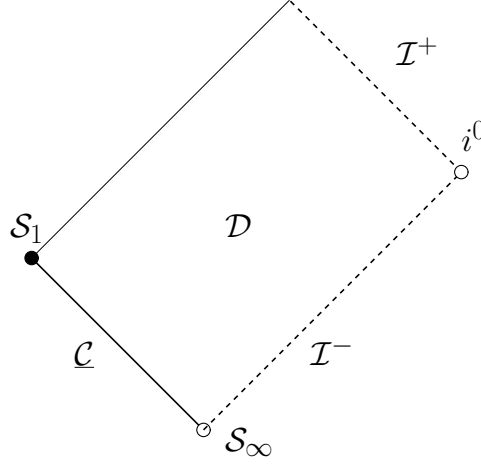


Figure 6.5.1 We will pose our scattering data on $\mathcal{C}$ and the part of $\mathcal{I}^-$ that lies to the future of $\mathcal{C}$ (i.e. $\mathcal{I}_{v \geq v_1}^-$).

6.5.1 Seed scattering data and the main theorem (Thm. 6.5.1)

Definition 6.5.1. A smooth seed scattering data set $\mathfrak{D}$ is a nonuple

$$\left(\Omega_{\mathcal{I}^-}^{(i)}, \hat{\chi}_{\mathcal{I}^-}^{(i)}, \mathbf{b}_{\mathcal{I}^-}^{(i)}, \hat{\mathfrak{g}}_{\mathcal{C}}^{(i)}, \underline{\omega}_{\mathcal{C}}^{(i)}, (\Omega \text{tr} \chi)_{\mathcal{S}_1}^{(i)}, (\Omega \text{tr} \underline{\chi})_{\mathcal{S}_1}^{(i)}, \underline{\beta}_{\mathcal{S}_1}^{(i)}, \text{tr} \hat{\mathfrak{g}}_{\mathcal{S}_1}^{(i)} \right), \quad (6.5.1)$$

where $\Omega_{\mathcal{I}^-}^{(i)}$, $\mathbf{b}_{\mathcal{I}^-}^{(i)}$ and $\hat{\chi}_{\mathcal{I}^-}^{(i)}$ are specified on $\mathcal{I}_{v \geq v_1}^-$:

$$\Omega_{\mathcal{I}^-}^{(i)} \in C^\infty(\mathcal{I}_{v \geq v_1}^-), \quad \mathbf{b}_{\mathcal{I}^-}^{(i)} \in \Gamma^\infty(\mathbb{M} T^{(0,1)} \mathcal{I}_{v \geq v_1}^-), \quad \hat{\chi}_{\mathcal{I}^-}^{(i)} \in \Gamma^\infty(\mathbb{M} T_{\text{stf}}^{(0,2)} \mathcal{I}_{v \geq v_1}^-),$$

where $\underline{\omega}_{\mathcal{C}}^{(i)}$, $\hat{\mathfrak{g}}_{\mathcal{C}}^{(i)}$ are specified along $\mathcal{C}$:

$$\underline{\omega}_{\mathcal{C}}^{(i)} \in C^\infty(\mathcal{C}), \quad \hat{\mathfrak{g}}_{\mathcal{C}}^{(i)} \in \Gamma^\infty(\mathbb{M} T_{\text{stf}}^{(0,2)} \mathcal{C}),$$

and, lastly, where $(\Omega \text{tr} \chi)_{\mathcal{S}_1}^{(i)}$, $(\Omega \text{tr} \underline{\chi})_{\mathcal{S}_1}^{(i)}$, $\text{tr} \hat{\mathfrak{g}}_{\mathcal{S}_1}^{(i)}$ as well as $\underline{\beta}_{\mathcal{S}_1}^{(i)}$ are specified on $\mathcal{S}_1$:

$$(\Omega \text{tr} \chi)_{\mathcal{S}_1}^{(i)}, (\Omega \text{tr} \underline{\chi})_{\mathcal{S}_1}^{(i)}, \text{tr} \hat{\mathfrak{g}}_{\mathcal{S}_1}^{(i)} \in C^\infty(\mathbb{S}^2), \quad \underline{\beta}_{\mathcal{S}_1}^{(i)} \in \Gamma^\infty(T^{(0,1)} \mathbb{S}^2).$$

Definition 6.5.2. Given a smooth seed scattering data set $\mathfrak{D}$, we call a solution $\mathfrak{S}$ to (6.3.9)–(6.3.32) on $\mathcal{D} = D^+(\mathcal{C} \cup \mathcal{I}^-)$ the scattering solution realising $\mathfrak{D}$ if

- Ω , $r^{-1}b$ and $r\hat{\chi}$ realise $\Omega_{\mathcal{I}^-}^{(i)}$, $\mathbf{b}_{\mathcal{I}^-}^{(i)}$ and $\hat{\chi}_{\mathcal{I}^-}^{(i)}$ as their pointwise limits at $\mathcal{I}^-$.
- $\underline{\omega}$, $\hat{\mathfrak{g}}$ restrict to $\underline{\omega}_{\mathcal{C}}^{(i)}$, $\hat{\mathfrak{g}}_{\mathcal{C}}^{(i)}$ on $\mathcal{C}$,

- $(\Omega \text{tr} \chi), \left(\Omega \text{tr} \underline{\chi} \right), \text{tr} \hat{g}^{(i)}$ and $\beta^{(i)}$ restrict to $(\Omega \text{tr} \chi)_{\mathcal{S}_1}, \left(\Omega \text{tr} \underline{\chi} \right)_{\mathcal{S}_1}, \text{tr} \hat{g}_{\mathcal{S}_1}^{(i)}$ and $\beta_{\mathcal{S}_1}^{(i)}$ on $\mathcal{S}_1$.

Over the course of the next few pages, we will prove the following theorem:

Theorem 6.5.1. *Let $\mathfrak{D}$ be a smooth scattering data set on $\underline{\mathcal{C}} \cup \mathcal{I}^-$ as in Def. 6.5.1. Suppose there exist positive numbers $\epsilon, \delta \in \mathbb{R}_{>0}$ as well as a smooth stf two-tensor $\hat{g}_{\mathcal{S}_\infty}^{(i)}$ such that, as $u \rightarrow -\infty$, the components of $\mathfrak{D}$ along $\underline{\mathcal{C}}$ satisfy*

$$\lim_{u \rightarrow -\infty} \hat{g}_{\underline{\mathcal{C}}}^{(i)} = \hat{g}_{\mathcal{S}_\infty}^{(i)}, \quad \hat{g}_{\underline{\mathcal{C}}}^{(i)} - \hat{g}_{\mathcal{S}_\infty}^{(i)} = \mathcal{O}_\infty \left(r^{-\frac{1}{2}-\delta} \right), \quad \underline{\omega}_{\underline{\mathcal{C}}} = \mathcal{O}_\infty \left(r^{-1-\epsilon} \right). \quad (6.5.2)$$

Then there exists a unique scattering solution $\mathfrak{S}$ realising $\mathfrak{D}$ in the sense of Def. 6.5.2, the uniqueness being understood w.r.t. the class of solutions with finite Regge–Wheeler energy (cf. Prop. 6.7.1).

Remark 6.5.1. We already point out that a large part of the seed scattering data does not carry physical information. For instance, by addition of pure gauge solutions (cf. §6.4), the quantities $\text{tr} \hat{g}_{\mathcal{S}_1}^{(i)}, \left(\Omega \text{tr} \underline{\chi} \right)_{\mathcal{S}_1}, \hat{g}_{\mathcal{S}_\infty}^{(i)}, \underline{\omega}_{\underline{\mathcal{C}}}, \Omega_{\mathcal{I}^-}$ as well as $\mathfrak{b}_{\mathcal{I}^-}$ can all simultaneously be set to 0. This will be discussed in detail in §6.5.4.

Remark 6.5.2. The assumed decay rate on $\hat{g}_{\underline{\mathcal{C}}}^{(i)}$ ensures that the seed scattering data set induces finite Regge–Wheeler energy along $\underline{\mathcal{C}}$ and cannot be improved without using completely different methods. This decay rate is also what corresponds to the decay assumed in [BZ09].

The assumed decay rate on $\underline{\omega}_{\underline{\mathcal{C}}}$, on the other hand, can be weakened due to the linearity of the system: Since this is not relevant for applications, we will stick with assuming $\underline{\omega}_{\underline{\mathcal{C}}}^{(i)}$ to be integrable.

6.5.2 Overview over the proof of Thm. 6.5.1

Our proof of Theorem 6.5.1 will be divided into the following steps.

- (i) Let $\mathfrak{D}$ be as in Thm. 6.5.1. From the components of $\mathfrak{D}$ on $\underline{\mathcal{C}}, \mathcal{S}_1$ and $\mathcal{I}^-$, we uniquely construct data for all the remaining components of the system (6.3.9)–(6.3.32) ($\hat{\chi}_{\underline{\mathcal{C}}}^{(i)}, \underline{\alpha}_{\underline{\mathcal{C}}}^{(i)}, \underline{\beta}_{\underline{\mathcal{C}}}^{(i)}, \underline{\Psi}_{\underline{\mathcal{C}}}^{(i)}$ etc.) such that any scattering solution realising $\mathfrak{D}$ must restrict to these constructed components along $\underline{\mathcal{C}}$. This is done in §6.5.3 (Prop. 6.5.1).
- (ii) We use a uniqueness clause for the Teukolsky equation (Cor. 6.6.2) to deduce the uniqueness of a solution realising $\mathfrak{D}$ in §6.7.1 (Prop. 6.7.1). This uniqueness clause for the Teukolsky equation, in turn, is derived in detail from uniqueness for the Regge–Wheeler equation in §6.6.
- (iii) Not having to worry about uniqueness anymore, we initiate the construction of a solution in §6.7.2. First, we define data at $\mathcal{I}^-$ for certain components of a solution to (6.3.9)–(6.3.32). In particular, we define data for $\nabla_v^{(i)} \Psi$ at $\mathcal{I}^-$. Combining this with (i), we now

have scattering data for the Regge–Wheeler equation (RW). We can thus construct a unique solution $\Psi^{(i)}$ to (RW) realising these scattering data using the results of [Mas22a] (see Thm. 6.6.1), of which we give a fully self-contained presentation in §6.6.

- (iv) We construct out of this solution $\Psi^{(i)}$ and the already defined data along $\underline{\mathcal{C}}$ and $\mathcal{I}^-$ first $\alpha^{(i)}$ by integrating (6.3.37), and then the entire $\ell \geq 2$ -part of a solution $\mathfrak{S}$ to (6.3.9)–(6.3.32) by suitably integrating the remaining equations of (6.3.9)–(6.3.32) to obtain the remaining components of a solution $\mathfrak{S}$. We finish by explicitly writing down the $\ell = 0, 1$ -part of $\mathfrak{S}$ (which is independent of $\Psi^{(i)}$).

Remark 6.5.3. The step to construct the entire $\ell \geq 2$ -part of $\mathfrak{S}$ starting from $\Psi^{(i)}$ is a combination of two ingredients: (1) Deriving local existence for (6.3.9)–(6.3.32) starting from local existence for $\underline{\Psi}^{(i)}$. (2) A certain level of a priori quantitative decay analysis as the data are posed at infinity. For instance, when defining $\alpha^{(i)}$ by integrating $\Psi^{(i)}$ twice from $\mathcal{I}^-$ using (6.3.37), we need to first prove that $\Psi^{(i)}$ decays sufficiently fast so that (6.3.37) is integrable.

We note that local existence for (6.3.9)–(6.3.32) starting from a solution $\Psi^{(i)}$ has hitherto only been written down in [Mas22b] after the first author pointed out that the sketch provided in [DHR19b] does not discuss several nontrivial problems that need to be overcome when proving it. A detailed overview is given in §6.7.2.

6.5.3 Defining the full scattering data along $\underline{\mathcal{C}}$

Given a seed scattering data set according to Definition 6.5.1, we may uniquely prescribe data for the full system on $\underline{\mathcal{C}}$ by suitably solving the constraint equations on each of $\underline{\mathcal{C}}$:

Proposition 6.5.1. *Let $\mathfrak{D}$ be a seed data set satisfying the assumption of Thm. 6.5.1. Then $\mathfrak{D}$ defines a unique tuple of functions, $\mathcal{S}_{u,v}$ -tangent one-forms and stf two-tensor fields along $\underline{\mathcal{C}}$*

$$\mathfrak{S}_{\underline{\mathcal{C}}} = \left(\hat{\mathfrak{g}}_{\underline{\mathcal{C}}}^{(i)}, \text{tr} \hat{\mathfrak{g}}_{\underline{\mathcal{C}}}^{(i)}, \hat{\Omega}_{\underline{\mathcal{C}}}^{(i)}, \hat{\mathfrak{b}}_{\underline{\mathcal{C}}}^{(i)}, (\hat{\Omega} \text{tr} \chi)_{\underline{\mathcal{C}}}^{(i)}, (\hat{\Omega} \text{tr} \chi)_{\underline{\mathcal{C}}}^{(i)}, \hat{\chi}_{\underline{\mathcal{C}}}^{(i)}, \hat{\chi}_{\underline{\mathcal{C}}}^{(i)}, \hat{\eta}_{\underline{\mathcal{C}}}^{(i)}, \hat{\eta}_{\underline{\mathcal{C}}}^{(i)}, \hat{\omega}_{\underline{\mathcal{C}}}^{(i)}, \hat{\beta}_{\underline{\mathcal{C}}}^{(i)}, \hat{\rho}_{\underline{\mathcal{C}}}^{(i)}, \hat{\sigma}_{\underline{\mathcal{C}}}^{(i)}, \hat{\beta}_{\underline{\mathcal{C}}}^{(i)}, \hat{\alpha}_{\underline{\mathcal{C}}}^{(i)}, \hat{\kappa}_{\underline{\mathcal{C}}}^{(i)} \right) \quad (6.5.3)$$

such that if $\mathfrak{S}$ is scattering solution realising $\mathfrak{D}$ as its seed scattering data set according to Definition 6.5.2, the restriction of $\mathfrak{S}$ to $\underline{\mathcal{C}}$ gives $\mathfrak{S}_{\underline{\mathcal{C}}}$. In particular, all ∇_u -equations of (6.3.9)–(6.3.32) except for (6.3.21) and (6.3.32), as well as the elliptic equations (6.3.12), (6.3.22)–(6.3.25) are satisfied along $\underline{\mathcal{C}}$ by $\mathfrak{S}_{\underline{\mathcal{C}}}$.

If $\mathfrak{S}$ is such that $r\alpha^{(i)}$ and $\hat{\omega}$ converge to $-\nabla_v \hat{\chi}_{\mathcal{I}^-}^{(i)}$, $\partial_v \hat{\Omega}_{\mathcal{I}^-}^{(i)}$ as $u \rightarrow -\infty$, respectively, then there additionally exist unique $\hat{\alpha}_{\underline{\mathcal{C}}}^{(i)}$ and $\hat{\omega}_{\underline{\mathcal{C}}}^{(i)}$ such that the restriction of $\mathfrak{S}$ to $\underline{\mathcal{C}}$ also gives $\hat{\alpha}_{\underline{\mathcal{C}}}^{(i)}$ and $\hat{\omega}_{\underline{\mathcal{C}}}^{(i)}$. In particular, the equations (6.3.21) and (6.3.32) are satisfied by $\mathfrak{S}_{\underline{\mathcal{C}}} \cup \left(\hat{\omega}_{\underline{\mathcal{C}}}^{(i)}, \hat{\alpha}_{\underline{\mathcal{C}}}^{(i)} \right)$ along $\underline{\mathcal{C}}$.

Proof. The proof proceeds by systematically defining the elements of $\mathfrak{S}_{\underline{\mathcal{C}}}$ from $\mathfrak{D}$ as solutions to a subset of the relevant equations of (6.3.9)–(6.3.32) along $\underline{\mathcal{C}}$, and by a posteriori proving that

the remaining of the ∇_u - and elliptic equations of (6.3.9)–(6.3.32) are automatically satisfied as well. The uniqueness clause is addressed at the end.

- 1) We define $\underline{\Omega}_{\mathcal{C}}$ along $\mathcal{C}$ via $\underline{\omega}_{\mathcal{C}}$ by integrating (6.3.11a) from $\mathcal{S}_{\infty}$ with $\underline{\Omega}_{\mathcal{I}^-}|_{\mathcal{S}_{\infty}}$ as data. (Condition (6.5.2) ensures that this is integrable.)
- 2) We define $\left(\underline{\Omega\mathrm{tr}\chi}\right)_{\mathcal{C}}$ along $\mathcal{C}$ via $\underline{\omega}_{\mathcal{C}}$ by integrating (6.3.16a) from $\mathcal{S}_1$ with $\left(\underline{\Omega\mathrm{tr}\chi}\right)_{\mathcal{S}_1}$ as data.
- 3) We then define $\mathrm{tr}\underline{\mathring{\mathcal{G}}}_{\mathcal{C}}$ along $\mathcal{C}$ by integrating (6.3.9a) with data $\mathrm{tr}\underline{\mathring{\mathcal{G}}}_{\mathcal{S}_1}$ on $\mathcal{S}_1$.
- 4) Next, we define $\underline{\hat{\chi}}_{\mathcal{C}}$ along $\mathcal{C}$ from $\underline{\hat{\mathcal{G}}}_{\mathcal{C}}$ via (6.3.10a), and we similarly define $\underline{\hat{\alpha}}_{\mathcal{C}}$ from $\underline{\hat{\chi}}_{\mathcal{C}}$ via (6.3.18a).
- 5) Given $\underline{\hat{\alpha}}_{\mathcal{C}}$, we define $\underline{\beta}_{\mathcal{C}}$ along $\mathcal{C}$ by integrating (6.3.28a) with data $\underline{\beta}_{\mathcal{S}_1}$ at $\mathcal{S}_1$.
- 6) Given $\underline{\hat{\chi}}_{\mathcal{C}}$, $\underline{\beta}_{\mathcal{C}}$ as well as $\left(\underline{\Omega\mathrm{tr}\chi}\right)_{\mathcal{C}}$, we now define $\underline{\eta}_{\mathcal{C}}$ along $\mathcal{C}$ as solution to the Codazzi equation (6.3.23). By multiplying (6.3.23) with $\frac{r^2}{\Omega}$ and acting with ∇_u , we directly verify that (6.3.19a) holds along $\mathcal{C}$. Indeed, using all our previous definitions:

$$\nabla_u(r^2\underline{\eta}_{\mathcal{C}}) = \mathrm{div}\nabla_u\left(\frac{r^2}{\Omega}\underline{\hat{\chi}}_{\mathcal{C}}\right) - \frac{1}{r}\nabla_u\left(\frac{r^4}{\Omega}\underline{\beta}_{\mathcal{C}}\right) - r^2\Omega\underline{\beta}_{\mathcal{C}} - \nabla\nabla_u\left(\frac{r^2}{2\Omega^2}\left(\underline{\Omega\mathrm{tr}\chi}\right)_{\mathcal{C}}\right) = -r^2\Omega\underline{\beta}_{\mathcal{C}} + 2r\nabla\underline{\omega}_{\mathcal{C}}. \quad (6.5.4)$$

- 7) We now define $\underline{\eta}_{\mathcal{C}}$ along $\mathcal{C}$ via (6.3.12) and our definitions of $\underline{\Omega}_{\mathcal{C}}$ and $\underline{\eta}_{\mathcal{C}}$. This directly implies that $\mathrm{curl}(\underline{\eta}_{\mathcal{C}} + \underline{\eta}_{\mathcal{C}}) = 0$. Moreover, inserting this definition into (6.3.19a) further shows that (6.3.20a) holds along $\mathcal{C}$.
- 8) Next, we define $\underline{\sigma}_{\mathcal{C}}$ along $\mathcal{C}$ as solution to (6.3.24): $r\underline{\sigma}_{\mathcal{C}} := \mathrm{curl}\underline{\eta}_{\mathcal{C}} (= -\mathrm{curl}\underline{\eta}_{\mathcal{C}})$, where the second equality follows by construction. Acting on this equation with $\nabla_u(r^2\cdot)$ and using (6.3.19b) proves that (6.3.30a) holds along $\mathcal{C}$ (note that $\mathrm{curl}\nabla f = 0$ for any smooth function f).
- 9) We define the Gaussian curvature $\underline{\mathcal{K}}_{\mathcal{C}}$ via (6.3.26).
- 10) We now need to define $\underline{\rho}_{\mathcal{C}}$: First, define $\underline{\rho}_{\mathcal{S}_1}$ on $\mathcal{S}_1$ as solution to the Gauss equation (6.3.25). Then, define $\underline{\rho}_{\mathcal{C}}$ along $\mathcal{C}$ by integrating (6.3.29a) from $\mathcal{S}_1$ with data $\underline{\rho}_{\mathcal{S}_1}$.
- 11) By construction, (6.3.25) holds on $\mathcal{S}_1$. We now define $\left(\underline{\Omega\mathrm{tr}\chi}\right)_{\mathcal{C}}$ along $\mathcal{C}$ as solution to (6.3.25) along $\mathcal{C}$. Multiplying (6.3.25) by r^2 and acting with ∇_u on it then shows, utilizing all the previous definitions, that (6.3.14) holds along $\mathcal{C}$.

At this point, it is left to define $\underline{\mathbf{b}}_{\mathcal{C}}$, $\underline{\hat{\chi}}_{\mathcal{C}}$, $\underline{\beta}_{\mathcal{C}}$ (as well as $\underline{\hat{\alpha}}_{\mathcal{C}}$ and $\underline{\omega}_{\mathcal{C}}$). In order to define these quantities, we need to integrate their relevant equations from $\mathcal{I}^-$ —we thus require that

these equations are integrable. Let us therefore list a few of the decay rates as $u \rightarrow -\infty$ of the quantities obtained so far:

◇ As $\underline{\omega}_{\mathcal{C}} = \mathcal{O}_{\infty}(r^{-1-\epsilon})$ according to (6.5.2), we find from (6.3.16a) that $r^2 \left(\Omega \text{tr} \underline{\chi} \right)_{\mathcal{C}}^{(1)} = \mathcal{O}_{\infty}(r^{-1-\epsilon} + 1)$. In particular, this decay rate together with (6.3.9a) implies that $\text{tr} \underline{\mathfrak{g}}_{\mathcal{C}}^{(1)}$ converges as $u \rightarrow -\infty$, and thus, by definition and by (6.5.2), $r^2 \underline{\mathfrak{K}}_{\mathcal{C}}^{(1)}$ also converges. We write:

$$\text{tr} \underline{\mathfrak{g}}_{\mathcal{S}_{\infty}}^{(1)} := \lim_{u \rightarrow -\infty} \text{tr} \underline{\mathfrak{g}}_{\mathcal{C}}^{(1)} = \text{tr} \underline{\mathfrak{g}}_{\mathcal{S}_1}^{(1)} - \frac{2r}{\Omega^2} \Big|_{\mathcal{S}_1} \left(\Omega \text{tr} \underline{\chi} \right)_{\mathcal{S}_1}^{(1)} + \lim_{u \rightarrow -\infty} \left(\frac{8}{r} \int_u^{u_0} r \underline{\omega}_{\mathcal{C}}^{(1)} du' \right) - 8 \int_{-\infty}^{u_0} \underline{\omega}_{\mathcal{C}}^{(1)} du'. \quad (6.5.5)$$

$$\underline{\mathfrak{K}}_{\mathcal{S}_{\infty}}^{(1)} := \lim_{u \rightarrow -\infty} r^2 \underline{\mathfrak{K}}^{(1)}(u, v_1, \theta^A) = \frac{1}{4} \left(\overset{\circ}{\Delta} + 2 \right) \text{tr} \underline{\mathfrak{g}}_{\mathcal{S}_{\infty}}^{(1)} + \frac{1}{2} \overset{\circ}{\text{div}} \overset{\circ}{\text{div}} \underline{\mathfrak{g}}_{\mathcal{S}_{\infty}}^{(1)}. \quad (6.5.6)$$

◇ The assumption (6.5.2) gives $\underline{\hat{\chi}}_{\mathcal{C}}^{(1)} = \mathcal{O}_{\infty}(r^{-\frac{3}{2}-\delta})$ via (6.3.10a) and thus $\underline{\alpha}_{\mathcal{C}}^{(1)} = \mathcal{O}_{\infty}(r^{-\frac{5}{2}-\delta})$ via (6.3.18a).

◇ We estimate $\underline{\beta}_{\mathcal{C}}^{(1)}$ and $\underline{\sigma}_{\mathcal{C}}^{(1)}$ via (6.3.28a) and (6.3.30a) to get $r^4 \underline{\beta}_{\mathcal{C}}^{(1)} = \mathcal{O}_{\infty}(r^{\frac{3}{2}-\delta} + 1)$, $\underline{\sigma}_{\mathcal{C}}^{(1)} = \mathcal{O}_{\infty}(r^{\frac{1}{2}-\delta} + 1)$.

◇ Therefore, the linearised Codazzi equation (6.3.23) implies $r^2 \underline{\eta}_{\mathcal{C}}^{(1)} = \mathcal{O}_{\infty}(r^{1-\min(\delta+\frac{1}{2}, \epsilon)} + 1)$.

◇ Equation (6.3.12) now implies $\underline{\eta}_{\mathcal{C}}^{(1)} = \frac{(2\overset{\circ}{\nabla} \underline{\Omega}_{\mathcal{I}^-}|_{\mathcal{S}_{\infty}})^{(1)}}{r} + \mathcal{O}_{\infty}(r^{-1-\min(\delta+\frac{1}{2}, \epsilon)} + r^{-2})$. We write

$$\underline{\eta}_{\mathcal{S}_{\infty}}^{(1)} := \lim_{u \rightarrow -\infty} r \underline{\eta}_{\mathcal{C}}^{(1)} = 2\overset{\circ}{\nabla} \underline{\Omega}_{\mathcal{I}^-}|_{\mathcal{S}_{\infty}}^{(1)} \quad (6.5.7)$$

◇ Equation (6.3.29a) implies $r^3 \underline{\rho}_{\mathcal{C}}^{(1)} = \mathcal{O}(r^{\frac{1}{2}-\min(\delta, \frac{1}{2}+\epsilon)} + 1)$. It follows via (6.3.25) that $r(\Omega \text{tr} \underline{\chi})_{\mathcal{C}}^{(1)}$ attains a limit as $u \rightarrow -\infty$:

$$(\Omega \text{tr} \underline{\chi})_{\mathcal{S}_{\infty}}^{(1)} := \lim_{u \rightarrow -\infty} r(\Omega \text{tr} \underline{\chi})_{\mathcal{C}}^{(1)} = 2\underline{\mathfrak{K}}_{\mathcal{S}_{\infty}}^{(1)} + 4\underline{\Omega}_{\mathcal{I}^-}|_{\mathcal{S}_{\infty}}^{(1)}. \quad (6.5.8)$$

We now define the remaining quantities along $\mathcal{C}$:

12) By the decay rates above, the RHS of (6.3.17a) is integrable. We thus define $\underline{\hat{\chi}}_{\mathcal{C}}^{(1)}$ along $\mathcal{C}$ via integration of (6.3.17a) with $\underline{\hat{\chi}}_{\mathcal{I}^-}|_{\mathcal{S}_{\infty}}^{(1)}$ as data for $\Omega r \cdot \underline{\hat{\chi}}_{\mathcal{C}}^{(1)}$ at $\mathcal{S}_{\infty}$. By construction, angular derivatives of $r \underline{\hat{\chi}}_{\mathcal{C}}^{(1)}$ will then converge to angular derivatives of $\underline{\hat{\chi}}_{\mathcal{I}^-}|_{\mathcal{S}_{\infty}}^{(1)}$.

13) Similarly, we define $\underline{\mathfrak{b}}_{\mathcal{C}}^{(1)}$ along $\mathcal{C}$ via integration of (6.3.13) with data $\underline{\mathfrak{b}}_{\mathcal{I}^-}|_{\mathcal{S}_{\infty}}^{(1)}$ for $r^{-1} \underline{\mathfrak{b}}_{\mathcal{C}}^{(1)}$.

14) Next, we define $\underline{\beta}_{\mathcal{C}}^{(1)}$ along $\mathcal{C}$ as the solution to the Codazzi equation (6.3.22). It follows that $r^2 \underline{\beta}_{\mathcal{C}}^{(1)}$ attains a limit as $u \rightarrow -\infty$:

$$\underline{\beta}_{\mathcal{S}_{\infty}}^{(1)} := \lim_{u \rightarrow -\infty} r^2 \underline{\beta}_{\mathcal{C}}^{(1)} = -\overset{\circ}{\text{div}} \underline{\hat{\chi}}_{\mathcal{I}^-}|_{\mathcal{S}_{\infty}}^{(1)} - \underline{\eta}_{\mathcal{S}_{\infty}}^{(1)} + \frac{1}{2} \overset{\circ}{\nabla} (\Omega \text{tr} \underline{\chi})_{\mathcal{S}_{\infty}}^{(1)} = -\overset{\circ}{\text{div}} \underline{\hat{\chi}}_{\mathcal{I}^-}|_{\mathcal{S}_{\infty}}^{(1)} + \overset{\circ}{\nabla} \underline{\mathfrak{K}}_{\mathcal{S}_{\infty}}^{(1)}. \quad (6.5.9)$$

Multiplying (6.3.22) with $r\Omega$ and acting with ∇_u , we then deduce that (6.3.31a) holds along all of $\underline{\mathcal{C}}$.

- 15) We define $\omega_{\underline{\mathcal{C}}}^{(i)}$ along $\underline{\mathcal{C}}$ via integration of (6.3.21) with $\partial_v \Omega_{\mathcal{I}^-}|_{\mathcal{S}_\infty}$ as data for $\omega_{\underline{\mathcal{C}}}^{(i)}$ at $\mathcal{S}_\infty$.
- 16) Finally, we define $\alpha_{\underline{\mathcal{C}}}^{(i)}$ along $\underline{\mathcal{C}}$ by integrating (6.3.32) with data $-\nabla_v \hat{\chi}_{\mathcal{I}^-}|_{\mathcal{S}_\infty}$ for $r\Omega^2 \alpha_{\underline{\mathcal{C}}}^{(i)}$ at $\mathcal{S}_\infty$.

This concludes the construction. The uniqueness part (for all quantities except $\alpha_{\underline{\mathcal{C}}}^{(i)}$ and $\omega_{\underline{\mathcal{C}}}^{(i)}$) goes as follows: If there are two different scattering solutions realising $\mathfrak{D}$, then their difference restricts to another solution with trivial seed scattering data. By demanding that this solution satisfies the equations (6.3.9)–(6.3.32), we can then repeat the procedure above but for trivial seed scattering data to deduce that all other quantities along $\underline{\mathcal{C}}$ must necessarily vanish identically.

The uniqueness of $\alpha_{\underline{\mathcal{C}}}^{(i)}$ and $\omega_{\underline{\mathcal{C}}}^{(i)}$ follows similarly under the additional convergence assumption of the proposition. $\square$

Remark 6.5.4. We see from the construction of the other quantities that, instead of specifying $r(\Omega \text{tr} \chi)^{(i)}$ and $\text{tr} \not\chi^{(i)}$ on $\mathcal{S}_1$, we can equivalently specify these quantities on $\mathcal{S}_\infty$.

Corollary 6.5.1. *The components of $\mathfrak{S}_{\underline{\mathcal{C}}}$ constructed in Proposition 6.5.1 satisfy, in particular,*

$$\begin{aligned} \hat{\chi}_{\underline{\mathcal{C}}}^{(i)} &= \mathcal{O}_\infty(r^{-\frac{3}{2}-\delta}), & \alpha_{\underline{\mathcal{C}}}^{(i)} &= \mathcal{O}_\infty(r^{-\frac{5}{2}-\delta}), \\ r^4 \underline{\beta}_{\underline{\mathcal{C}}}^{(i)} &= \mathcal{O}_\infty(r^{\frac{3}{2}-\delta} + 1), & r^3 \underline{\sigma}_{\underline{\mathcal{C}}}^{(i)} &= \mathcal{O}_\infty(r^{\frac{1}{2}-\delta} + 1), \\ r^3 \underline{\rho}_{\underline{\mathcal{C}}}^{(i)} &= \mathcal{O}_\infty(r^{\frac{1}{2}-\min(\delta, \epsilon+\frac{1}{2})} + 1). \end{aligned} \quad (6.5.10)$$

Proposition 6.5.1 allows us to also define transversal derivatives along $\underline{\mathcal{C}}$ to any order. In particular, we have:

Corollary 6.5.2. *Let $\mathfrak{D}$ be a seed data set satisfying the assumption of Thm. 6.5.1, and let $\mathfrak{S}_{\underline{\mathcal{C}}}$ be as in Prop. 6.5.1. Define the following tuple of $\mathcal{S}_{u,v}$ -tangent stf-tensors along $\underline{\mathcal{C}}$:*

$$\begin{aligned} \underline{\psi}_{\underline{\mathcal{C}}}^{(i)} &:= 2\mathring{\mathcal{P}}_2^* r^2 \Omega \underline{\beta}_{\underline{\mathcal{C}}}^{(i)} + 6M \Omega \hat{\chi}_{\underline{\mathcal{C}}}^{(i)}, \\ \underline{\psi}_{\underline{\mathcal{C}}}^{(i)} &:= -2\mathring{\mathcal{P}}_2^* r^2 \Omega \underline{\beta}_{\underline{\mathcal{C}}}^{(i)} + 6M \Omega \hat{\chi}_{\underline{\mathcal{C}}}^{(i)} \\ \underline{\Psi}_{\underline{\mathcal{C}}}^{(i)} &:= 2\mathring{\mathcal{P}}_2^* \mathring{\mathcal{P}}_1^* (r^3 \underline{\rho}_{\underline{\mathcal{C}}}^{(i)}, r^3 \underline{\sigma}_{\underline{\mathcal{C}}}^{(i)}) + 6M (r \Omega \hat{\chi}_{\underline{\mathcal{C}}}^{(i)} - r \Omega \hat{\chi}_{\underline{\mathcal{C}}}^{(i)}), \\ \underline{\Psi}_{\underline{\mathcal{C}}}^{(i)} &:= 2\mathring{\mathcal{P}}_2^* \mathring{\mathcal{P}}_1^* (r^3 \underline{\rho}_{\underline{\mathcal{C}}}^{(i)}, -r^3 \underline{\sigma}_{\underline{\mathcal{C}}}^{(i)}) + 6M (r \Omega \hat{\chi}_{\underline{\mathcal{C}}}^{(i)} - r \Omega \hat{\chi}_{\underline{\mathcal{C}}}^{(i)}) \end{aligned} \quad (6.5.11)$$

Then any smooth scattering solution realising $\mathfrak{D}$ will have $\frac{r^2}{\Omega^2} \nabla_v r \Omega^2 \alpha^{(i)}$, $(\frac{r^2}{\Omega^2} \nabla_v)^2 r \Omega^2 \underline{\alpha}^{(i)}$, $\frac{r^2}{\Omega^2} \nabla_u (r \Omega^2 \underline{\alpha}^{(i)})$ and $(\frac{r^2}{\Omega^2} \nabla_u)^2 (r \Omega^2 \underline{\alpha}^{(i)})$ restricting to $\underline{\psi}_{\underline{\mathcal{C}}}^{(i)}$, $\underline{\Psi}_{\underline{\mathcal{C}}}^{(i)}$, $\underline{\psi}_{\underline{\mathcal{C}}}^{(i)}$ and $\underline{\Psi}_{\underline{\mathcal{C}}}^{(i)}$, respectively.

Moreover, we have that

$$\underline{\Psi}_{\underline{\mathcal{C}}}^{(i)} = \mathcal{O}_\infty(r^{\frac{1}{2}-\delta} + 1) = \underline{\Psi}_{\underline{\mathcal{C}}}^{(i)}, \quad r^2 \underline{\psi}_{\underline{\mathcal{C}}}^{(i)} = \mathcal{O}_\infty(r^{\frac{3}{2}-\delta} + 1). \quad (6.5.12)$$

Proof. The first part of the statement follows directly from (6.3.33)–(6.3.36) and Proposition 6.5.1.

For (6.5.12), however, the rates derived in Proposition 6.5.1 would naively only give

$$\Psi_{\underline{\mathcal{C}}}^{(i)} = \mathcal{O}\left(r^{\frac{1}{2}-\min(\delta, \frac{1}{2}+\epsilon)-n} + 1\right) = \underline{\Psi}_{\underline{\mathcal{C}}}^{(i)}. \quad (6.5.13)$$

To show that these rates are in fact independent of ϵ , one first confirms that (6.3.41) is satisfied along $\underline{\mathcal{C}}$, and then integrates this equation from $\mathcal{S}_1$ together with the knowledge that $\underline{\alpha}_{\underline{\mathcal{C}}}^{(i)} = \mathcal{O}_{\infty}(r^{-\frac{5}{2}-\delta})$. This shows (6.5.12) for $\underline{\Psi}_{\underline{\mathcal{C}}}^{(i)}$. The result for $\Psi_{\underline{\mathcal{C}}}^{(i)}$ then follows from (6.3.38) and the rate (6.5.10) for $\underline{\sigma}_{\underline{\mathcal{C}}}^{(i)}$. $\square$

Finally, we deduce a corollary pertaining to lower angular modes. This result will be used for the construction of the $\ell = 0, 1$ -part of the solution in §6.7.3.

Corollary 6.5.3. *Let $\mathfrak{D}_{\ell=0,1}$ be a smooth seed scattering data set satisfying the assumptions of Thm. 6.5.1 that is supported only on $\ell = 0, 1$. Then, in addition to the limit $\mathbb{K}_{\mathcal{S}_{\infty}}^{(i)}$ defined in (6.5.6), the following limits exist as well:*

$$\rho_{\mathcal{S}_{\infty}}^{(i)} = \lim_{u \rightarrow -\infty} r^3 \rho_{\underline{\mathcal{C}}}^{(i)}, \quad \underline{\beta}_{\mathcal{S}_{\infty}}^{(i)} = \lim_{u \rightarrow -\infty} r^4 \underline{\beta}_{\underline{\mathcal{C}}}^{(i)}, \quad \lim_{u \rightarrow -\infty} r^4 \underline{\sigma}_{\underline{\mathcal{C}}}^{(i)} = -\mathring{\text{curl}} \underline{\beta}_{\mathcal{S}_{\infty}}^{(i)}. \quad (6.5.14)$$

Moreover, specifying $\mathfrak{D}_{\ell=0,1} = \{\Omega_{\mathcal{I}^-}^{(i)}, \mathbf{b}_{\mathcal{I}^-}^{(i)}, \underline{\omega}_{\mathcal{I}^-}^{(i)}, (\Omega \text{tr} \chi)_{\mathcal{S}_1}^{(i)}, (\Omega \text{tr} \underline{\chi})_{\mathcal{S}_1}^{(i)}, \underline{\beta}_{\mathcal{S}_1}^{(i)}, \text{tr} \underline{\mathbf{g}}_{\mathcal{S}_1}^{(i)}\}$ is equivalent to specifying the corresponding $\mathfrak{D}'_{\ell=0,1} = \{\Omega_{\mathcal{I}^-}^{(i)}, \mathbf{b}_{\mathcal{I}^-}^{(i)}, \underline{\omega}_{\mathcal{I}^-}^{(i)}, \rho_{\mathcal{S}_{\infty}}^{(i)}, (\Omega \text{tr} \underline{\chi})_{\mathcal{S}_1}^{(i)}, \underline{\beta}_{\mathcal{S}_{\infty}}^{(i)}, \text{tr} \underline{\mathbf{g}}_{\mathcal{S}_{\infty}}^{(i)}\}$. (Recall that $\hat{\mathbf{g}}$ and $\hat{\chi}$ are supported on $\ell \geq 2$ since they are stf two-tensors.)

Proof. Since $\mathfrak{D}_{\ell=0,1}$ is supported on $\ell = 0, 1$, we have that $\nabla_u \frac{r^4 \underline{\beta}_{\underline{\mathcal{C}}}^{(i)}}{\Omega} = 0$, and so $r^4 \underline{\beta}_{\underline{\mathcal{C}}}^{(i)}$ takes a limit. The claim for $\rho_{\underline{\mathcal{C}}}^{(i)}$ is similar. Finally, in order to compute the limit of $r^4 \underline{\sigma}_{\underline{\mathcal{C}}}^{(i)}$, we write

$$\lim_{u \rightarrow -\infty} r^4 \underline{\sigma}_{\underline{\mathcal{C}}}^{(i)} = \lim_{u \rightarrow -\infty} r^3 \mathring{\text{curl}} \underline{\eta}_{\underline{\mathcal{C}}}^{(i)} = - \lim_{u \rightarrow -\infty} r^4 \mathring{\text{curl}} \underline{\beta}_{\underline{\mathcal{C}}}^{(i)}, \quad (6.5.15)$$

where we used (6.3.24) and (6.3.23), together with the fact that $\mathring{\text{curl}} \nabla \left(\Omega \text{tr} \underline{\chi} \right)_{\underline{\mathcal{C}}}^{(i)} = 0 = (\hat{\chi}_{\underline{\mathcal{C}}})_{\ell < 2}^{(i)}$.

In order to see the equivalence of $\mathfrak{D}$ and $\mathfrak{D}'$, we note that by $\nabla_u \frac{r^4 \underline{\beta}_{\underline{\mathcal{C}}}^{(i)}}{\Omega} = 0$, specifying $\underline{\beta}_{\mathcal{S}_{\infty}}^{(i)}$ is equivalent to specifying $\underline{\beta}_{\mathcal{S}_1}^{(i)}$. Similarly, (6.3.9a) implies that specifying $\text{tr} \underline{\mathbf{g}}_{\mathcal{S}_1}^{(i)}$ is equivalent to specifying $\text{tr} \underline{\mathbf{g}}_{\mathcal{S}_{\infty}}^{(i)}$. Finally, via (6.3.29a), specifying $\rho_{\mathcal{S}_{\infty}}^{(i)}$ is equivalent to specifying $\rho_{\underline{\mathcal{C}}}^{(i)}|_{\mathcal{S}_1}$, which, in turn, is equivalent to specifying $(\Omega \text{tr} \chi)_{\mathcal{S}_1}^{(i)}$ by the Gauss equation (6.3.25). $\square$

6.5.4 Bondi normalisation of seed scattering data

We now describe how to *Bondi normalise* a seed scattering data set. This serves the purpose of identifying the physical degrees of freedom contained in a seed scattering data set.

Definition 6.5.3. A smooth seed scattering data set $\mathfrak{D}$ is said to be Bondi-normalised if the following conditions are satisfied:

- The lapse and the shift vector vanish at $\mathcal{I}^-$: $\Omega_{\mathcal{I}^-}^{(i)} = 0 = \mathfrak{b}_{\mathcal{I}^-}^{(i)}$.
- The limit $\hat{\mathfrak{g}}_{\mathcal{S}_\infty}^{(i)}$ of $\hat{\mathfrak{g}}_{\mathcal{C}}^{(i)}$, as well as the induced limits $\text{tr} \hat{\mathfrak{g}}_{\mathcal{S}_\infty}^{(i)}$, $\mathbb{K}_{\mathcal{S}_\infty}^{(i)}$ and $(\Omega \text{tr} \chi)_{\mathcal{S}_\infty}^{(i)}$ (defined in (6.5.5), (6.5.6), (6.5.8), respectively) vanish at $\mathcal{S}_\infty$.

We have the following proposition:

Proposition 6.5.2. For any smooth seed scattering data set $\mathfrak{D}$, there exist smooth functions $f(v, \theta^A)$, $\underline{f}(u, \theta^A)$ and $(q_1(v, \theta^A), q_2(v, \theta^A))$ such that the $\ell \geq 2$ -part of $\mathfrak{D} - \mathfrak{D}_f - \mathfrak{D}_{\underline{f}} - \mathfrak{D}_{(q_1, q_2)}$ is Bondi-normalised, where $\mathfrak{D}_f$, $\mathfrak{D}_{\underline{f}}$ and $\mathfrak{D}_{(q_1, q_2)}$ denote the seed scattering data belonging to the pure gauge solutions $\mathfrak{S}_f$, $\mathfrak{S}_{\underline{f}}$ and $\mathfrak{S}_{(q_1, q_2)}$.

An analogous statement is true for $\ell < 2$. In fact, given seed scattering data supported on $\ell < 2$, the explicit scattering solution to these data are written down in §6.7.3.

Proof. We first find $\underline{f}$ supported on $\ell \geq 2$ by demanding that

$$\lim_{u \rightarrow -\infty} r^{-3} (\mathring{\Delta} + 2) \underline{f} = (\mathbb{K}_{\mathcal{S}_\infty})_{\ell \geq 2}. \quad (6.5.16)$$

(Recall that $\mathring{\Delta} + 2$ is invertible on $\ell \geq 2$). This still leaves us with considerable freedom for $\underline{f}$.

Next, we find f such that

$$\partial_v f = (2\Omega_{\mathcal{I}^-})_{\ell \geq 2} + (\mathring{\Delta} + 2)^{-1} (\mathbb{K}_{\mathcal{S}_\infty})_{\ell \geq 2}. \quad (6.5.17)$$

Notice that $\mathfrak{D}_{\ell \geq 2} - \mathfrak{D}_f - \mathfrak{D}_{\underline{f}}$ now has vanishing Gaussian curvature $\overset{(i)}{K}$ at $\mathcal{S}_\infty$ and vanishing lapse $\overset{(i)}{\Omega}$ at $\mathcal{I}^-$. It follows that the corresponding $(\Omega \text{tr} \chi)_{\mathcal{S}_\infty}^{(i)}$ vanishes as well (by (6.5.8)).

Finally, we fix (q_1, q_2) such that

$$\hat{\mathfrak{g}}_{\mathcal{S}_\infty}^{(i)} = 2\mathring{\mathcal{P}}_2^* \mathring{\mathcal{P}}_1^*(q_1, q_2) \Big|_{v=v_1}, \quad \mathring{\mathcal{P}}_2^{*(i)} \mathfrak{b}_{\mathcal{I}^-} = \mathring{\mathcal{P}}_2^* \mathring{\mathcal{P}}_1^*(\partial_v q_1, \partial_v q_2). \quad (6.5.18)$$

It follows that $\mathfrak{D}_{\ell \geq 2} - \mathfrak{D}_f - \mathfrak{D}_{\underline{f}} - \mathfrak{D}_{(q_1, q_2)}$ has vanishing limits for the corresponding $\hat{\mathfrak{g}}_{\mathcal{S}_\infty}^{(i)}$ and $\mathfrak{b}_{\mathcal{I}^-}^{(i)}$. The vanishing of $\text{tr} \hat{\mathfrak{g}}_{\mathcal{S}_\infty}^{(i)}$ follows by (6.5.6). $\square$

Remark 6.5.5. The proof above shows that even after Bondi-normalising the seed scattering data, we still have a considerable amount of gauge freedom left: Changing f by a function independent of v or changing $\underline{f}$ by a function that grows slower than u does not affect the Bondi-normalisation of a seed scattering data set.²

²Contained in this freedom is the ($\ell \geq 2$ -part of the) BMS group at $\mathcal{I}^-$, see [Mas22b] for a more detailed discussion.

In particular, we can use this remaining gauge freedom to infer from the proof of Prop. 6.5.2 the following

Corollary 6.5.4. *For any smooth seed scattering data set $\mathfrak{D}$, there exist smooth functions $f(v, \theta^A)$, $\underline{f}(u, \theta^A)$ and $(q_1(v, \theta^A), q_2(v, \theta^A))$ such that the $\ell \geq 2$ -part of $\mathfrak{D} - \mathfrak{D}_f - \mathfrak{D}_{\underline{f}} - \mathfrak{D}_{(q_1, q_2)}$ is Bondi-normalised, and, in addition $(\underline{\omega}_{\underline{C}}^{(i)})_{\ell \geq 2} = 0$ and $\left(\left(\overset{(i)}{\Omega \text{tr} \chi} \right)_{\mathcal{S}_1} \right)_{\ell \geq 2} = 0$.*

Again, the analogous statement for $\ell < 2$ is shown in §6.7.3.

6.6 Scattering for the Regge–Wheeler equation and for the Teukolsky equation

The main ingredient for our construction of a scattering solution to (6.3.9)–(6.3.32) is the scattering theory for the Regge–Wheeler and Teukolsky equations developed in [Mas22a]. This, in turn, relies heavily on the fact that the Regge–Wheeler equation admits a conserved energy.

6.6.1 Energy conservation for the Regge–Wheeler equation

Definition 6.6.1. *For any u, U_1, U_2, v, V_1, V_2 with $U_1 \leq U_2$, $V_1 \leq V_2$, define*

$$E_u[\Psi](V_1, V_2) := \int_{\mathcal{C}_u \cap \{v \in [V_1, V_2]\}} d\bar{v} \sin \theta d\theta d\varphi \left[|\nabla_v \Psi|^2 + \frac{\Omega^2}{r^2} (|\dot{\nabla} \Psi|^2 + (3\Omega^2 + 1)|\Psi|^2) \right],$$

$$\underline{E}_v[\Psi](U_1, U_2) := \int_{\underline{\mathcal{C}}_v \cap \{u \in [U_1, U_2]\}} d\bar{u} \sin \theta d\theta d\varphi \left[|\nabla_u \Psi|^2 + \frac{\Omega^2}{r^2} (|\dot{\nabla} \Psi|^2 + (3\Omega^2 + 1)|\Psi|^2) \right].$$

Define furthermore

$$E_u[\dot{\nabla} \Psi](V_1, V_2) := \int_{\mathcal{C}_u \cap \{v \in [V_1, V_2]\}} d\bar{v} \sin \theta d\theta d\varphi \left[|\nabla_v \dot{\nabla} \Psi|^2 + \frac{\Omega^2}{r^2} (|\dot{\Delta} \Psi|^2 + (3\Omega^2 + 1)|\dot{\nabla} \Psi|^2) \right],$$

$$\underline{E}_v[\dot{\nabla} \Psi](U_1, U_2) := \int_{\underline{\mathcal{C}}_v \cap \{u \in [U_1, U_2]\}} d\bar{u} \sin \theta d\theta d\varphi \left[|\nabla_u \dot{\nabla} \Psi|^2 + \frac{\Omega^2}{r^2} (|\dot{\Delta} \Psi|^2 + (3\Omega^2 + 1)|\dot{\nabla} \Psi|^2) \right].$$

Finally, define, for $n \in \mathbb{N}$,

$$E_u^{(n)}[\Psi](V_1, V_2) := \begin{cases} E_u[\dot{\Delta}^{\frac{n}{2}} \Psi](V_1, V_2), & \text{if } \frac{n}{2} \in \mathbb{N}, \\ E_u[\dot{\nabla}(\dot{\Delta}^{\frac{n-1}{2}} \Psi)](V_1, V_2), & \text{if } \frac{n-1}{2} \in \mathbb{N}, \end{cases}$$

as well as

$$\underline{E}_v^{(n)}[\Psi](U_1, U_2) := \begin{cases} \underline{E}_v^{(n)}[\dot{\Delta}^{\frac{n}{2}} \Psi](U_1, U_2), & \text{if } \frac{n}{2} \in \mathbb{N}, \\ \underline{E}_v^{(n)}[\dot{\nabla}(\dot{\Delta}^{\frac{n-1}{2}} \Psi)](U_1, U_2), & \text{if } \frac{n-1}{2} \in \mathbb{N}. \end{cases}$$

Lemma 6.6.1. *Let Ψ be a smooth solution to the Regge–Wheeler equation (RW), and let $n \in \mathbb{N}$. Then Ψ satisfies*

$$E_{U_1}^{(n)}[\Psi](V_1, V_2) + \underline{E}_{V_1}^{(n)}[\Psi](U_1, U_2) = E_{U_2}^{(n)}[\Psi](V_1, V_2) + \underline{E}_{V_2}^{(n)}[\Psi](U_1, U_2). \quad (6.6.1)$$

6.6.2 Scattering for the Regge–Wheeler equation

We now give a compact, fully self-contained presentation of the scattering problem for the Regge–Wheeler equation adapted to our setting. Theorem 6.6.1 is a statement adapted from [Mas22a], but it's proved in an alternative way; all other statements in §6.6 are new.

Theorem 6.6.1 (Adapted from [Mas22a]). *For $P_{\mathcal{I}^-} \in L^2(\mathcal{I}^- \cap \{v \in [v_1, v_2]\})$ and $\Psi_{\mathcal{C}}$ with $E_{v_1}[\Psi_{\mathcal{C}}](-\infty, u_0) < \infty$, there exists a unique solution Ψ to the Regge–Wheeler equation (RW) such that for any $v \in [v_1, v_2]$ we have*

$$\lim_{u \rightarrow -\infty} \|\nabla_v \Psi(u, \cdot, \cdot) - P_{\mathcal{I}^-}\|_{L^2([v_1, v] \times \mathbb{S}^2)}^2 = 0, \quad \Psi|_{\mathcal{C}} = \Psi_{\mathcal{C}}. \quad (6.6.2)$$

This solution satisfies

$$\lim_{u \rightarrow -\infty} E_u[\Psi](v_1, v) = \|P_{\mathcal{I}^-}\|_{L^2([v_1, v] \times \mathbb{S}^2)}^2. \quad (6.6.3)$$

In addition, if $\Psi_{\mathcal{C}}$ and $P_{\mathcal{I}^-}$ are smooth, and if $E_{v_1}^{(n)}[\Psi_{\mathcal{C}}](-\infty, u_0) < \infty$ for any $n \in \mathbb{N}$, then Ψ is smooth as well and

$$\lim_{u \rightarrow -\infty} E_u^{(n)}[\Psi](v_1, v) = \|\mathring{\nabla}^s \mathring{\Delta}^{\frac{n-s}{2}} P_{\mathcal{I}^-}\|_{L^2([v_1, v] \times \mathbb{S}^2)}^2, \quad (6.6.4)$$

where $s = 1$ if n is odd and $s = 0$ if n is even.

We first specify and prove the uniqueness clause of Thm. 6.6.1:

Proposition 6.6.1. *Assume that Ψ is a solution to the Regge–Wheeler equation (RW) such that*

$$\lim_{u \rightarrow -\infty} \|\nabla_v \Psi(u, v, \cdot)\|_{L^2([v_1, v_2] \times \mathbb{S}^2)}^2 = 0, \quad \Psi|_{\mathcal{C}} = 0. \quad (6.6.5)$$

Assume furthermore that $\Psi|_{\mathcal{C}_{u_0} \cap \{v \in [v_1, v_2]\}} \in H_{loc}^1(\mathcal{C}_{u_0})$, $\Psi|_{\mathcal{C}_{v_2} \cap \{u \leq u_0\}} \in H_{loc}^1(\mathcal{C}_{v_2} \cap \{u \leq u_0\})$. Then $\Psi = 0$.

Proof. Let $\tilde{\Psi} \in H^2(\mathcal{S}_{u,v})$ such that $\mathring{\Delta} \tilde{\Psi} = \Psi$. Then $\tilde{\Psi}$ also satisfies the Regge–Wheeler equation with $\tilde{\Psi}|_{\mathcal{C}} = 0$. Applying Hardy's inequality on $\mathcal{C}_u \cap \{v \in [v_1, v_2]\}$ for any $v_2 \geq v_1$, $u \leq u_0$, together with Poincaré's inequality on $\mathbb{S}^2$ gives

$$\int_{v_1}^{v_2} \int_{\mathbb{S}^2} d\bar{v} \sin \theta d\theta d\varphi \frac{\Omega^2}{r^2} |\tilde{\Psi}|^2 \leq \int_{v_1}^{v_2} \int_{\mathbb{S}^2} d\bar{v} \sin \theta d\theta d\varphi |\nabla_v \tilde{\Psi}|^2 \leq \int_{v_1}^{v_2} \int_{\mathbb{S}^2} d\bar{v} \sin \theta d\theta d\varphi |\nabla_v \Psi|^2,$$

$$\begin{aligned} \int_{v_1}^{v_2} \int_{\mathbb{S}^2} d\bar{v} \sin \theta d\theta d\varphi \frac{\Omega^2}{r^2} |\mathring{\nabla} \tilde{\Psi}|^2 &\leq \int_{v_1}^{v_2} \int_{\mathbb{S}^2} d\bar{v} \sin \theta d\theta d\varphi |\nabla_v \mathring{\nabla} \tilde{\Psi}|^2 \\ &\leq \int_{v_1}^{v_2} \int_{\mathbb{S}^2} d\bar{v} \sin \theta d\theta d\varphi |\nabla_v \Psi|^2. \end{aligned}$$

Therefore, for any $v_2 \geq v_1$ we have

$$\lim_{u \rightarrow -\infty} \int_{v_1}^{v_2} \int_{\mathbb{S}^2} d\bar{v} \sin \theta d\theta d\varphi \left[|\nabla_v \tilde{\Psi}|^2 + \frac{\Omega^2}{r^2} |\tilde{\Psi}|^2 + \frac{\Omega^2}{r^2} |\dot{\nabla} \tilde{\Psi}|^2 \right] \Big|_{\mathcal{C}_u} = 0.$$

Energy conservation for Regge–Wheeler implies $\tilde{\Psi} = 0$, thus $\Psi = 0$. (Here, we used that $\dot{\Delta}$ is invertible on stf $\mathcal{S}_{u,v}$ -tensors, but the argument clearly also works for scalar functions by dealing with the spherically symmetric part in the same manner.) $\square$

Proof of the existence clause of Thm. 6.6.1. We provide an argument that avoids the use of the Arzelà–Ascoli theorem and instead directly constructs the solution.

We begin by assuming that $\mathbf{P}_{\mathcal{I}^-}$, $\Psi_{\mathcal{C}}$ are smooth and that $\Psi_{\mathcal{C}}$ is compactly supported. Let $\{u_n\}_{(n \in \mathbb{N})} \subset \mathbb{R}$ be a dyadic sequence with $u_n \rightarrow -\infty$ as $n \rightarrow \infty$, and with $\{u \leq u_1\}$ beyond the support of $\Psi_{\mathcal{C}}$. We define a sequence $\{\Psi_n\}_{(n \in \mathbb{N})}$ of smooth solutions to the Regge–Wheeler equation (RW), where Ψ_n is defined on $[u_n, u_0] \times [v_1, v_2] \times \mathbb{S}^2$, by taking Ψ_n to arise from characteristic data $\Psi_{\mathcal{C}}(u, \theta^A)$ at $\mathcal{C} \cap \{u \in [u_n, u_0]\}$ and $\int_{v_1}^v d\bar{v} \mathbf{P}_{\mathcal{I}^-}(\bar{v}, \theta^A)$ on $\mathcal{C}_{u_n} \cap \{v \in [v_1, v_2]\}$. (As part of the construction, we will also extend each Ψ_n to $u \leq u_n$ later on.) Note that the energy norm of Ψ_n on the initial outgoing cone $\mathcal{C}_{u_n} \cap \{v \in [v_1, v_2]\}$ is then bounded by:

$$E_{u_n}[\Psi_n](v_1, v_2) \leq \left(1 + \frac{(v_2 - v_1)^2}{r(u_n, v_1)^2}\right) \|\mathbf{P}_{\mathcal{I}^-}\|_{L^2([v_1, v_2] \times \mathbb{S}^2)}^2 + \frac{(v_2 - v_1)^2}{r(u_n, v_1)^2} \|\dot{\nabla} \mathbf{P}_{\mathcal{I}^-}\|_{L^2([v_1, v_2] \times \mathbb{S}^2)}^2. \quad (6.6.6)$$

The strategy now is to show that the restrictions of Ψ_n to the null cones $\mathcal{C}_{u_0} \cap \{v \in [v_1, v_2]\}$ and $\mathcal{C}_{v_2} \cap \{u \leq u_0\}$ converge as well, so that we can construct the limiting solution as the backwards solution arising from the limiting data $\Psi|_{\mathcal{C}_{u_0} \cap \{v \in [v_1, v_2]\}}$, $\Psi|_{\mathcal{C}_{v_2} \cap \{u \leq u_0\}}$.

First, we use the energy estimate to bound

$$E_{u_0}[\Psi_n - \Psi_{n+1}](v_1, v_2) + \underline{E}_{v_2}[\Psi_n - \Psi_{n+1}](u_n, u_0) \leq E_{u_n}[\Psi_n - \Psi_{n+1}](v_1, v_2). \quad (6.6.7)$$

In order to estimate the RHS, we first write

$$\begin{aligned} \Psi_n(u_n, v, \theta^A) - \Psi_{n+1}(u_n, v, \theta^A) &= \int_{v_1}^v \mathbf{P}_{\mathcal{I}^-}(\bar{v}, \theta^A) d\bar{v} - \Psi_{n+1}(u_n, v, \theta^A) \\ &= \Psi_{n+1}(u_{n+1}, v, \theta^A) - \Psi_{n+1}(u_n, v, \theta^A), \end{aligned} \quad (6.6.8)$$

and we then estimate, using the equation (RW) :

$$\begin{aligned}
& \int_{v_1}^{v_2} \int_{\mathbb{S}^2} d\bar{v} \sin \theta d\theta d\varphi |\nabla_v \Psi_{n+1}(u_n, \bar{v}, \theta^A) - \nabla_v \Psi_{n+1}(u_{n+1}, \bar{v}, \theta^A)|^2 \\
&= \int_{v_1}^{v_2} \int_{\mathbb{S}^2} d\bar{v} \sin \theta d\theta d\varphi \left| \int_{u_{n+1}}^{u_n} d\bar{u} \frac{\Omega^2}{r^2}(\bar{u}, \bar{v}) \left(\dot{\Delta} - (3\Omega^2 + 1) \right) \Psi_{n+1}(\bar{u}, \bar{v}, \theta^A) \right|^2 \\
&\lesssim \frac{1}{r(u_n, v_1)} \int_{v_1}^{v_2} d\bar{v} \int_{u_{n+1}}^{u_n} d\bar{u} \int_{\mathbb{S}^2} \sin \theta d\theta d\varphi \frac{\Omega^2}{r^2} |\dot{\Delta} \Psi_{n+1}|^2 \\
&\lesssim \frac{(v_2 - v_1)}{r(u_n, v_1)} \left[\|\dot{\nabla}^\gamma \mathbf{P}_{\mathcal{I}^-}\|_{L^2([v_1, v_2] \times \mathbb{S}^2)}^2 + \frac{(v_2 - v_1)^2}{r(u_n, v_1)^2} \|\dot{\Delta} \mathbf{P}_{\mathcal{I}^-}\|_{L^2([v_1, v_2] \times \mathbb{S}^2)}^2 \right],
\end{aligned} \tag{6.6.9}$$

where we used the energy estimate and (6.6.6) in the last line. This gives an estimate for the ∇_v -derivative part of the energy $E_{u_n}[\Psi_n - \Psi_{n+1}](v_1, v_2)$. The other terms can be estimated similarly (without having to invoke (RW)), and thus, by (6.6.9) and (6.6.8), we obtain (we hide $(v_2 - v_1)$ -weights inside $\lesssim$)

$$E_{u_n}[\Psi_n - \Psi_{n+1}](v_1, v_2) \lesssim_{v_2-v_1} \frac{1}{r(u_n, v_1)} \sum_{|\gamma| \leq 2} \|\dot{\nabla}^\gamma \mathbf{P}_{\mathcal{I}^-}\|_{L^2([v_1, v_2] \times \mathbb{S}^2)}^2. \tag{6.6.10}$$

Next, we bound

$$\underline{E}_{v_2}[\Psi_{n+1}](u_{n+1}, u_n) = E_{u_{n+1}}[\Psi_{n+1}](v_1, v_2) - E_{u_n}[\Psi_{n+1}](v_1, v_2) \tag{6.6.11}$$

by first estimating (notice that the estimate below is still valid for u_n replaced by any $u' \in [u_{n+1}, u_n]$)

$$\begin{aligned}
& \int_{v_1}^{v_2} \int_{\mathbb{S}^2} d\bar{v} \sin \theta d\theta d\varphi |\nabla_v \Psi_{n+1}(u_{n+1}, \bar{v}, \theta^A)|^2 - |\nabla_v \Psi_{n+1}(u_n, \bar{v}, \theta^A)|^2 \\
&= \int_{v_1}^{v_2} \int_{\mathbb{S}^2} d\bar{v} \sin \theta d\theta d\varphi \left[|\nabla_v \Psi_{n+1}(u_{n+1}, \bar{v}, \theta^A) - \nabla_v \Psi_{n+1}(u_n, \bar{v}, \theta^A)| \right. \\
&\quad \left. \cdot |\nabla_v \Psi_{n+1}(u_{n+1}, \bar{v}, \theta^A) + \nabla_v \Psi_{n+1}(u_n, \bar{v}, \theta^A)| \right] \\
&\leq \sqrt{\int_{v_1}^{v_2} \int_{\mathbb{S}^2} d\bar{v} \sin \theta d\theta d\varphi |\nabla_v \Psi_{n+1}(u_{n+1}, \bar{v}, \theta^A) - \nabla_v \Psi_{n+1}(u_n, \bar{v}, \theta^A)|^2} \\
&\quad \cdot \sqrt{\int_{v_1}^{v_2} \int_{\mathbb{S}^2} d\bar{v} \sin \theta d\theta d\varphi |\nabla_v \Psi_{n+1}(u_{n+1}, \bar{v}, \theta^A) + \nabla_v \Psi_{n+1}(u_n, \bar{v}, \theta^A)|^2} \\
&\lesssim_{v_2-v_1} \frac{1}{\sqrt{r(u_n, v_1)}} \left[\|\dot{\nabla}^\gamma \mathbf{P}_{\mathcal{I}^-}\|_{L^2([v_1, v_2] \times \mathbb{S}^2)}^2 + \frac{1}{r(u_n, v_1)^2} \|\dot{\Delta} \mathbf{P}_{\mathcal{I}^-}\|_{L^2([v_1, v_2] \times \mathbb{S}^2)}^2 \right],
\end{aligned} \tag{6.6.12}$$

where we used the estimates (6.6.6) and (6.6.9). We deal with the remaining terms in the difference $E_{u_{n+1}}[\Psi_{n+1}](v_1, v_2) - E_{u_n}[\Psi_{n+1}](v_1, v_2)$ similarly, hence we arrive at:

$$E_{v_2}[\Psi_{n+1}](u_{n+1}, u_n) \lesssim \frac{1}{\sqrt{r(u_n, v_1)}} \sum_{|\gamma| \leq 2} \|\mathring{\nabla}^\gamma \mathbf{P}_{\mathcal{I}^-}\|_{L^2([v_1, v_2] \times \mathbb{S}^2)}^2. \quad (6.6.13)$$

We now extend Ψ_n to the region $(-\infty, u_n] \times [v_1, v_2] \times \mathbb{S}^2$. We could do this by defining Ψ_n to be the backwards solution to (RW) with data $\Psi_n|_{\mathcal{C}_{u_n} \cap \{v \in [v_1, v_2]\}}$ on $\mathcal{C}_{u_n} \cap \{v \in [v_1, v_2]\}$ and constant data $\int_{v_1}^{v_2} \mathbf{P}_{\mathcal{I}^-} dv$ along $\mathcal{C}_{v_2} \cap \{u \leq u_n\}$, but then the extension would only be continuous.

Instead, we provide a *smooth* extension of $\Psi_n|_{\mathcal{C}_{v_2}}$ along $\mathcal{C}_{v_2}$: Let $h(u)$ be a smooth cutoff function which cuts off to 0 on $u \geq 1$ and is equal to 1 on $u \leq 0$, and let for any $n > 0$ $h_n := h\left(\frac{u - u_n}{u_{n-1} - u_n}\right)$, so that h_n cuts off on $u \geq u_{n-1}$ and is equal to 1 on $u \leq u_n$. We apply Seeley's extension theorem [See64] to extend $\Psi_n|_{\mathcal{C}_{v_2} \cap \{u \geq u_n\}} \cdot h_n$ to the region $u \leq u_n$. Denote this extension by $\mathbf{E}(\Psi_n|_{\mathcal{C}_{v_2} \cap \{u \geq u_n\}} \cdot h_n)$, in short just $\mathbf{E}$. Note that the extension satisfies [See64]

$$\begin{aligned} \|\mathbf{E}(\Psi_n|_{\mathcal{C}_{v_2} \cap \{u \geq u_n\}} \cdot h_n)\|_{H^k((-\infty, u_n] \times \mathbb{S}^2)} &\lesssim \|\Psi_n|_{\mathcal{C}_{v_2} \cap \{u \geq u_n\}} \cdot h_n\|_{H^k([u_n, \infty) \times \mathbb{S}^2)} \\ &\lesssim \|\Psi_n|_{\mathcal{C}_{v_2} \cap \{u \geq u_n\}}\|_{H^k([u_n, u_{n-1}] \times \mathbb{S}^2)} \end{aligned} \quad (6.6.14)$$

for any $k \geq 0$. Let $\tilde{h}_n$ be the cutoff function given by $\tilde{h}_n := h(u_n - u)$, which cuts off on $u \leq u_n - 1$ and is equal to 1 on $u \geq u_n$. We have by (6.6.14)

$$\begin{aligned} \|r^{-1} \tilde{h}_n \cdot \mathbf{E}\|_{L^2([u_n-1, u_n] \times \mathbb{S}^2)}^2 &\leq \frac{1}{r(u_n, v_2)^2} \|\tilde{h}_n \cdot \mathbf{E}\|_{L^2([u_n-1, u_n] \times \mathbb{S}^2)}^2 \\ &\lesssim \frac{1}{r(u_n, v_2)^2} \|\Psi_n|_{\mathcal{C}_{v_2} \cap \{u \geq u_n\}}\|_{L^2([u_n, u_{n-1}] \times \mathbb{S}^2)}^2 \\ &\lesssim \|r^{-1} \Psi_n|_{\mathcal{C}_{v_2} \cap \{u \geq u_n\}}\|_{L^2([u_n, u_{n-1}] \times \mathbb{S}^2)}^2. \end{aligned} \quad (6.6.15)$$

In the same way, we have

$$\begin{aligned} \|r^{-1} \tilde{h}_n \cdot \mathring{\nabla} \mathbf{E}\|_{L^2([u_n-1, u_n] \times \mathbb{S}^2)}^2 &\lesssim \|r^{-1} \mathring{\nabla} \Psi_n|_{\mathcal{C}_{v_2} \cap \{u \geq u_n\}}\|_{L^2([u_n, u_{n-1}] \times \mathbb{S}^2)}^2, \\ \|\mathring{\nabla}_u(\tilde{h}_n \cdot \mathbf{E})\|_{L^2([u_n-1, u_n] \times \mathbb{S}^2)}^2 &\lesssim \|\mathring{\nabla}_u \Psi_n|_{\mathcal{C}_{v_2} \cap \{u \geq u_n\}}\|_{L^2([u_n, u_{n-1}] \times \mathbb{S}^2)}^2 \end{aligned} \quad (6.6.16)$$

We can thus define data for Ψ_n on $\mathcal{C}_{v_2} \cap \{u \leq u_n\}$ via

$$\Psi_n|_{\mathcal{C}_{v_2} \cap \{u \leq u_n\}} := \tilde{h}_n \cdot \mathbf{E}(\Psi_n|_{\mathcal{C}_{v_2} \cap \{u \geq u_n\}} \cdot h_n). \quad (6.6.17)$$

Combining, (6.6.15), (6.6.16) and using (6.6.12) shows

$$E_{v_2}[\Psi_n|_{\mathcal{C}_{v_2}}](-\infty, u_n) \lesssim_{v_2-v_1} \frac{1}{\sqrt{r(u_n, v_2)}} \left(\|\mathring{\nabla} \mathbf{P}_{\mathcal{I}^-}\|_{L^2([v_1, v_2] \times \mathbb{S}^2)}^2 + \|\mathring{\Delta} \mathbf{P}_{\mathcal{I}^-}\|_{L^2([v_1, v_2] \times \mathbb{S}^2)}^2 \right). \quad (6.6.18)$$

The estimates (6.6.7), (6.6.10), (6.6.13), and (6.6.18) lead to

$$\underline{E}_{v_2}[\Psi_{n+1} - \Psi_n](-\infty, u_0) + E_{u_0}[\Psi_{n+1} - \Psi_n](v_1, v_2) \rightarrow 0 \quad (6.6.19)$$

as $n \rightarrow \infty$, and thus the restrictions $\Psi_n|_{\mathcal{C}_{u_0} \cap \{v \in [v_1, v_2]\}}$, $\Psi_n|_{\underline{\mathcal{C}}_{v_2} \cap \{u \leq u_0\}}$ converge with respect to the energy norms $E_{u_0}[\cdot](v_1, v_2)$, $E_{v_2}[\cdot](-\infty, u_0)$ to some limits $\Psi|_{\mathcal{C}_{u_0} \cap \{v \in [v_1, v_2]\}}$, $\Psi|_{\underline{\mathcal{C}}_{v_2} \cap \{u \leq u_0\}}$. (Notice that the above estimates carry over also to the commuted energies $\underline{E}_{v_2}^{(n)}[\Psi_{n+1} - \Psi_n](-\infty, u_0) + E_{u_0}^{(n)}[\Psi_{n+1} - \Psi_n](v_1, v_2)$.)

Let now Ψ be the backwards solution to Regge–Wheeler (RW) arising from the limiting data $\Psi|_{\mathcal{C}_{u_0} \cap \{v \in [v_1, v_2]\}}$, $\Psi|_{\underline{\mathcal{C}}_{v_2} \cap \{u \leq u_0\}}$, then Ψ satisfies

$$\underline{E}_{v_2}[\Psi](-\infty, u_0) + E_{u_0}[\Psi](v_1, v_2) < \infty, \quad (6.6.20)$$

and we have that $\Psi|_{\underline{\mathcal{C}}} = \Psi_{\underline{\mathcal{C}}}$. We also have

$$\lim_{u \rightarrow -\infty} \int_{v_1}^{v_2} \int_{\mathbb{S}^2} d\bar{v} \sin \theta d\theta d\varphi |\nabla_v \Psi(u, \bar{v}, \theta^A) - P_{\mathcal{I}^-}(\bar{v}, \theta^A)|^2 = 0 \quad (6.6.21)$$

by the following argument: for a given u , let n be such that $u \in [u_n, u_{n-1})$. Then we have

$$\begin{aligned} & \int_{v_1}^{v_2} \int_{\mathbb{S}^2} d\bar{v} \sin \theta d\theta d\varphi |\nabla_v \Psi(u, \bar{v}, \theta^A) - P_{\mathcal{I}^-}(\bar{v}, \theta^A)|^2 \\ & \leq \int_{v_1}^{v_2} \int_{\mathbb{S}^2} d\bar{v} \sin \theta d\theta d\varphi |\nabla_v \Psi_n(u, \bar{v}, \theta^A) - P_{\mathcal{I}^-}(\bar{v}, \theta^A)|^2 \\ & \quad + \int_{v_1}^{v_2} \int_{\mathbb{S}^2} d\bar{v} \sin \theta d\theta d\varphi |\nabla_v \Psi(u, \bar{v}, \theta^A) - \nabla_v \Psi_n(u, \bar{v}, \theta^A)|^2 \\ & \lesssim_{v_2-v_1} \frac{1}{\sqrt{r(u_n, v_1)}} \left[\|\mathring{\nabla} P_{\mathcal{I}^-}\|_{L^2([v_1, v_2] \times \mathbb{S}^2)}^2 + \frac{1}{r(u_n, v_1)^2} \|\mathring{\Delta} P_{\mathcal{I}^-}\|_{L^2([v_1, v_2] \times \mathbb{S}^2)}^2 \right] \\ & \quad + \int_{v_1}^{v_2} \int_{\mathbb{S}^2} d\bar{v} \sin \theta d\theta d\varphi |\nabla_v \Psi(u, \bar{v}, \theta^A) - \nabla_v \Psi_n(u, \bar{v}, \theta^A)|^2. \end{aligned} \quad (6.6.22)$$

In the last line, we used (6.6.12). The convergence (6.6.21) is now implied by the convergence of Ψ_n to Ψ with respect to the energy norms. The statement (6.6.3) is proved similarly.

Now, assume that $P_{\mathcal{I}^-} \in L^2([v_1, v_2] \times \mathbb{S}^2)$, $\Psi_{\underline{\mathcal{C}}}$ such that $\underline{E}_{v_1}[\Psi_{\underline{\mathcal{C}}}](-\infty, u_0) < \infty$. There exists a sequence $\{P_{\mathcal{I}^-, n}\}_{(n \in \mathbb{N})}$ of smooth data on $[v_1, v_2] \times \mathbb{S}^2$ that approximates $P_{\mathcal{I}^-}$ in $L^2([v_1, v_2] \times \mathbb{S}^2)$ and a sequence $\{\Psi_{\underline{\mathcal{C}}, n}\}_{(n \in \mathbb{N})}$ of smooth, compactly supported data on $\underline{\mathcal{C}}$ that approximates $\Psi_{\underline{\mathcal{C}}}$ in the norm given by $E_{v_1}[\cdot](-\infty, u_0)$. Let Ψ_n be the sequence of smooth solutions to Regge–Wheeler (RW) arising from data $P_{\mathcal{I}^-, n}$ on $\mathcal{I}_{v \geq v_1}^-$, $\Psi_{\underline{\mathcal{C}}, n}$ on $\underline{\mathcal{C}}$. Then $\Psi_n|_{\underline{\mathcal{C}}_{v_2}}$ converges to $\Psi_{\underline{\mathcal{C}}_{v_2}}$ with $\underline{E}_{v_2}[\Psi_{\underline{\mathcal{C}}_{v_2}}](-\infty, u_0) < \infty$ and $\Psi_n|_{\mathcal{C}_{u_0}}$ converges to $\Psi_{\mathcal{C}_{u_0}}$ with $E_{u_0}[\Psi_{\mathcal{C}_{u_0}}](v_1, v_2) < \infty$ by energy conservation. Let Ψ be the backwards solution to (RW) arising from characteristic data $\Psi|_{\mathcal{C}_{u_0}}$, $\Psi_{\underline{\mathcal{C}}_{v_2}}$. Then we have for any u

$$\begin{aligned}
& \int_{v_1}^{v_2} \int_{\mathbb{S}^2} d\bar{v} \sin \theta d\theta d\varphi |\nabla_v \Psi(u, \bar{v}, \theta^A) - P_{\mathcal{I}^-}(\bar{v}, \theta^A)|^2 \\
& \leq \int_{v_1}^{v_2} \int_{\mathbb{S}^2} d\bar{v} \sin \theta d\theta d\varphi |\nabla_v \Psi(u, \bar{v}, \theta^A) - \nabla_v \Psi_n(u, \bar{v}, \theta^A)|^2 \\
& \quad + \int_{v_1}^{v_2} \int_{\mathbb{S}^2} d\bar{v} \sin \theta d\theta d\varphi |\nabla_v \Psi_n(u, \bar{v}, \theta^A) - P_{\mathcal{I}^-,n}(\bar{v}, \theta^A)|^2 \\
& \quad + \int_{v_1}^{v_2} \int_{\mathbb{S}^2} d\bar{v} \sin \theta d\theta d\varphi |P_{\mathcal{I}^-}(\bar{v}, \theta^A) - P_{\mathcal{I}^-,n}(\bar{v}, \theta^A)|^2. \quad (6.6.23)
\end{aligned}$$

Choosing n large enough ensures that the first and third terms above are small. Choosing u large enough and using (6.6.21) then implies

$$\lim_{u \rightarrow -\infty} \int_{v_1}^{v_2} \int_{\mathbb{S}^2} d\bar{v} \sin \theta d\theta d\varphi |\nabla_v \Psi(u, \bar{v}, \theta^A) - P_{\mathcal{I}^-}(\bar{v}, \theta^A)|^2 = 0, \quad (6.6.24)$$

and the remaining statement that (6.6.3) holds can be proved similarly.

The final claim that the solution is smooth if the data are smooth and if the energies $E_{v_1}^{(n)}[\Psi_{\mathcal{C}}](-\infty, u_0)$ are finite for any $n \in \mathbb{N}$ follows by commuting with angular derivatives and Sobolev embedding (which automatically also proves (6.6.4)). $\square$

Two corollaries

We can directly infer from Thm. 6.6.1 the following

Proposition 6.6.2. *Assume that we are in the second clause of Thm. 6.6.1 (with smooth data). Then we have that, for any $n \in \mathbb{N}$,*

$$|\mathring{\nabla}^n \Psi|(u, v) \lesssim r^{\frac{1}{2}} \cdot \sqrt{\sum_{i \leq n+2} E_{v_1}^{(i)}[\Psi_{\mathcal{C}}](-\infty, u) + \|\mathring{\nabla}^s \mathring{\Delta}^{\frac{i-s}{2}} P_{\mathcal{I}^-}\|_{L^2([v_1, v] \times \mathbb{S}^2)}^2}, \quad (6.6.25)$$

where $s = 1$ if i is odd and 0 if i is even. Furthermore, for any $m \in \mathbb{N}_{\geq 1}$,

$$\nabla_v^m \mathring{\nabla}^n \Psi \rightarrow \nabla_v^{m-1} \mathring{\nabla}^n P_{\mathcal{I}^-}^{(v)}, \quad \text{as } u \rightarrow -\infty. \quad (6.6.26)$$

uniformly on compact v -intervals.

Remark 6.6.1. The convergence (6.6.26) would follow in the same way as (6.6.4) if we also assumed the energy to remain finite for an arbitrary amount of $\nabla_T = \nabla_u + \nabla_v$ -commutations.

Proof. The first statement is proved by first estimating, via the fundamental theorem of calculus:

$$|\Psi(u, v, \theta^A) - \Psi_{\mathcal{C}}(u, \theta^A)| \leq \int_{v_1}^v |\nabla_v^{(v)} \Psi(u, \bar{v}, \theta^A)| d\bar{v}, \quad (6.6.27)$$

from which it follows via Cauchy–Schwarz that

$$\begin{aligned} \|\Psi(u, v, \cdot) - \Psi_{\underline{C}}(u, \cdot)\|_{L^2(\mathbb{S}^2)}^2 &\leq \int_{v_1}^v d\bar{v} \cdot \int_{v_1}^v \int_{\mathbb{S}^2} |\nabla_v \Psi|^2 \sin \theta d\theta d\varphi d\bar{v} \\ &\leq (v - v_1) \cdot E_u[\Psi](v_1, v) \leq (v - v_1) \cdot \left[E_{v_1}[\Psi](-\infty, u) + \lim_{u \rightarrow -\infty} E_u[\Psi](v_1, v) \right], \end{aligned} \quad (6.6.28)$$

where we also used the energy estimate of Lemma 6.6.1 in the last estimate. We similarly estimate the initial data term $\|\Psi_{\underline{C}}\|_{L^2(\mathbb{S}^2)}^2$ against its energy by considering the fundamental theorem of calculus for $r^{-1}\Psi$:

$$\|r^{-1}\Psi_{\underline{C}}\|_{L^2(\mathbb{S}^2)}^2 \lesssim r^{-1} \int_{-\infty}^u \int_{\mathbb{S}^2} |r\nabla_u(r^{-1}\Psi_{\underline{C}})|^2 \sin \theta d\theta d\varphi du' \lesssim E_{v_1}[\Psi_{\underline{C}}](-\infty, u). \quad (6.6.29)$$

(The boundary term at $u = -\infty$ can be seen to vanish in the same manner.) Estimate (6.6.25) now follows by commuting and applying the standard Sobolev inequality on the sphere.

Finally, in order to prove the uniform convergence of $\nabla_v^m \nabla^n \Psi$ to $\nabla_v^{m-1} \nabla^n P_{\mathcal{I}^-}$ along compact intervals in v , we invoke the equation (RW) itself: Inserting the pointwise estimate (6.6.25) for $n = 0$ and integrating (RW) in u , it follows that $\nabla_v \Psi$ converges uniformly (on compact v -intervals) to a limit as $u \rightarrow -\infty$. By (6.6.4), this limit has to agree with $P_{\mathcal{I}^-}$. The commuted result follows inductively. $\square$

We also have the following enhanced uniqueness statement for solutions to (RW) of finite energy:

Corollary 6.6.1. *Assume Ψ is a solution to (RW) on $[v_1, v_2] \times (-\infty, u_0) \times \mathbb{S}^2$ for $v_2 > v_1$ such that $E_{v_2}[\Psi](-\infty, u_0) < \infty$ for some $v_2 > v_1$ and $\|\Psi\|_{H^1(\mathcal{C}_{u_0} \cap \{v \in [v_1, v_2]\})} < \infty$. Assume further that $\Psi|_{\underline{C}} = 0$ and that $\nabla_v \Psi(u, v, \theta^A) \rightarrow 0$ as $u \rightarrow -\infty$ for $v \in [v_1, v_2]$. Then $\Psi = 0$.*

Proof. Since $E_{v_2}[\Psi](-\infty, u_0) < \infty$, we have that $\nabla_v \Psi$ converges in $L^2([v_1, v_2] \times \mathbb{S}^2)$ as $u \rightarrow -\infty$ by a slight modification of the argument of (6.6.22) in Theorem 6.6.1, and so the family $\{\nabla_v \Psi(u, v, \theta^A)\}_{u \in (-\infty, u_0)}$ is uniformly integrable in $L^2([v_1, v_2] \times \mathbb{S}^2)$. As $\lim_{u \rightarrow -\infty} \nabla_v \Psi(u, v, \theta^A) = 0$ by assumption, Vitali's convergence theorem says that

$$\lim_{u \rightarrow -\infty} \int_{v_1}^{v_2} \int_{\mathbb{S}^2} d\bar{v} \sin \theta d\theta d\varphi |\nabla_v \Psi|^2 = 0. \quad (6.6.30)$$

Theorem 6.6.1 now implies $\Psi = 0$. $\square$

6.6.3 Two uniqueness results for the Teukolsky equation (Teuk)

We now study uniqueness for (Teuk).

Proposition 6.6.3. *Let α be a solution to (Teuk) on $(u, v, \theta^A) \in (-\infty, u_0) \times [v_1, v_2] \times \mathbb{S}^2$ such that $E_{v_2}[\Psi](-\infty, u_0) < \infty$, where $\Psi = (r^2 \Omega^{-2} \nabla_u)^2 (r \Omega^2 \alpha)$. If α satisfies*

$$\lim_{u \rightarrow -\infty} r\alpha(u, v, \theta^A) = 0, \quad \nabla_u(r\Omega^2 \alpha)|_{\mathcal{C}} = 0, \quad (6.6.31)$$

then $\alpha = 0$.

Proof. Recall that Ψ defined out of α via $\Psi = (r^2\Omega^{-2}\nabla_u)^2(r\Omega^2\alpha)$ satisfies the Regge–Wheeler equation (RW) in view of Lemma 6.3.1. Let $\psi = \frac{r^2}{\Omega^2}\nabla_u(r\Omega^2\alpha)$. We import the following estimates for arbitrary $\epsilon > 0$ from Propositions 12.1.1 and 12.1.2 of [DHR19b] (adapted to the region near $\mathcal{I}^-$):

$$\begin{aligned} & \int_{v_1}^{v_2} \int_{\mathbb{S}^2} d\bar{v} \sin \theta d\theta d\varphi \left[|r\Omega^2\alpha(u, \bar{v}, \theta^A)|^2 + |\psi(u, \bar{v}, \theta^A)|^2 \right] \\ & + \epsilon \int_u^{u_0} \int_{v_1}^{v_2} \int_{\mathbb{S}^2} d\bar{u} d\bar{v} \sin \theta d\theta d\varphi \frac{\Omega^2}{r^{1+\epsilon}} \left[|r\Omega^2\alpha(u, \bar{v}, \theta^A)|^2 + |\psi(u, \bar{v}, \theta^A)|^2 \right] \\ & \lesssim \int_{v_1}^{v_2} \int_{\mathbb{S}^2} d\bar{v} \sin \theta d\theta d\varphi \left[|r\Omega^2\alpha(u_0, \bar{v}, \theta^A)|^2 + |\psi(u_0, \bar{v}, \theta^A)|^2 \right] \\ & + E_{u_0}[\Psi](v_1, v_2) + \underline{E}_{v_2}[\Psi](-\infty, u_0). \end{aligned} \quad (6.6.32)$$

The above estimate remains valid with $\overset{\circ}{\nabla} r\Omega^2\alpha$, $\overset{\circ}{\nabla}\psi$ on both sides of the estimates *without requiring a higher order energy for Ψ* . The plan now is to use (6.6.32) to show that the following quantity vanishes: Define $\tilde{\Psi}$ via

$$\frac{\Omega^2}{r^2} \nabla_v \frac{r^2}{\Omega^2} \nabla_v \tilde{\Psi} = r\Omega^2\alpha, \quad (6.6.33)$$

with $\tilde{\Psi}$, $\nabla_v \tilde{\Psi}$ vanishing at v_1 . It is clear that

$$\text{RW}[\tilde{\Psi}]|_{\underline{\mathcal{C}}} = 0. \quad (6.6.34)$$

Note that

$$\nabla_v \text{RW}[\tilde{\Psi}] = \psi + \frac{r^2}{\Omega^2} (5 - 3\Omega^2 - \overset{\circ}{\Delta}) \nabla_v \tilde{\Psi} + 6M\tilde{\Psi}.$$

Since $\nabla_u(r\Omega^2\alpha) = 0$ at $\underline{\mathcal{C}}$, we also have

$$\nabla_v \text{RW}[\tilde{\Psi}]|_{\underline{\mathcal{C}}} = 0.$$

Therefore, we have by (6.3.52) from Lemma 6.3.1 that $\tilde{\Psi}$ satisfies the Regge–Wheeler equation (RW).

Now, by definition, we have

$$\nabla_v \tilde{\Psi} = \frac{\Omega^2}{r^2}(u, v) \int_{v_1}^v r(u, \bar{v})^3 \alpha(u, \bar{v}, \theta^A). \quad (6.6.35)$$

We make use of the fact that $r\alpha$ decays pointwise towards $\mathcal{I}^-$ to estimate

$$\begin{aligned}
\int_{\mathbb{S}^2} \sin \theta \, d\theta \, d\varphi \left| \frac{\Omega^2}{r^2} \int_{v_1}^v r^3 \alpha \right|^2 &= \frac{\Omega^4}{r^4} \int_{\mathbb{S}^2} \sin \theta \, d\theta \, d\varphi \left[\int_{v_1}^v d\bar{v} \frac{r^2}{\Omega^2} \int_{-\infty}^{u_0} d\bar{u} \frac{\Omega^2}{r^2} \psi \right]^2 \\
&\lesssim \int_{\mathbb{S}^2} \sin \theta \, d\theta \, d\varphi \left[\int_{v_1}^v d\bar{v} \int_{-\infty}^{u_0} d\bar{u} \frac{\Omega^2}{r^2} |\psi| \right]^2 \\
&\lesssim \frac{(v_2 - v_1)}{r(u, v_1)} \int_{v_1}^v d\bar{v} \int_{-\infty}^{u_0} d\bar{u} \int_{\mathbb{S}^2} \sin \theta \, d\theta \, d\varphi \frac{\Omega^2}{r^2} |\psi|^2 \\
&\lesssim \frac{(v_2 - v_1)}{r(u, v_1)} \times \left(\int_{v_1}^{v_2} \int_{\mathbb{S}^2} d\bar{v} \sin \theta \, d\theta \, d\varphi \left[|r\Omega^2 \alpha|_{u_0}|^2 + |\psi|_{u_0}|^2 \right] \right. \\
&\quad \left. + E_{u_0}[\Psi](v_1, v_2) + \underline{E}_{v_2}[\Psi](-\infty, u_0) \right), \tag{6.6.36}
\end{aligned}$$

where we used (6.6.32) in the last step. Thus $\|\tilde{\nabla}_v \tilde{\Psi}(u, \cdot, \cdot)\|_{L^2([v_1, v_2] \times \mathbb{S}^2)} \rightarrow 0$ as $u \rightarrow -\infty$. Combining this with $\tilde{\Psi}|_{\underline{\mathcal{C}}} = 0$, Proposition 6.6.1 implies $\tilde{\Psi} = 0$. Thus $\alpha = 0$. $\square$

The argument above allows us to conclude the following result, which will later on allow us to conclude uniqueness for the full system (6.3.9)–(6.3.32).

Corollary 6.6.2. *Let α be a solution to (Teuk) on $(u, v, \theta^A) \in (-\infty, u_0] \times [v_1, v_2] \times \mathbb{S}^2$ such that $\underline{E}_{v_2}[\Psi](-\infty, u_0) < \infty$, where $\Psi = (r^2 \Omega^{-2} \tilde{\nabla}_u)^2 r \Omega^2 \alpha$. If α satisfies*

$$\lim_{u \rightarrow -\infty} \frac{1}{r(u, v_1)} \int_{v_1}^v d\bar{v} r(u, \bar{v})^2 \alpha(u, \bar{v}, \theta^A) = 0, \quad \tilde{\nabla}_u(r\Omega^2 \alpha)|_{\underline{\mathcal{C}}} = 0, \tag{6.6.37}$$

then $\alpha = 0$.

Proof. Let $\tilde{\Psi}$ be as in the proof of Proposition 6.6.3. We compute $E_u[\tilde{\Psi}](v_1, v_2)$ by directly estimating

$$\begin{aligned}
\|\tilde{\nabla}_v \tilde{\Psi}(u, \cdot, \cdot)\|_{L^2([v_1, v_2] \times \mathbb{S}^2)}^2 &\lesssim \int_{v_1}^{v_2} dv \int_{\mathbb{S}^2} \sin \theta \, d\theta \, d\varphi \left(\int_{v_1}^v dv' r^3 \alpha \right)^2 \\
&\lesssim (v_2 - v_1)^2 \int_{v_1}^{v_2} dv \int_{\mathbb{S}^2} \sin \theta \, d\theta \, d\varphi (r\alpha)^2 \lesssim (v_2 - v_1)^2 \|r\alpha(u, \cdot, \cdot)\|_{L^2([v_1, v_2] \times \mathbb{S}^2)}^2, \tag{6.6.38}
\end{aligned}$$

and by similarly estimating

$$\|r^{-1} \tilde{\Psi}(u, \cdot, \cdot)\|_{L^2([v_1, v_2] \times \mathbb{S}^2)} \lesssim \frac{(v_2 - v_1)^{\frac{3}{2}}}{r(u, v_1)} \|r\alpha(u, \cdot, \cdot)\|_{L^2([v_1, v_2] \times \mathbb{S}^2)}, \tag{6.6.39}$$

$$\|r^{-1} \dot{\tilde{\nabla}} \tilde{\Psi}(u, \cdot, \cdot)\|_{L^2([v_1, v_2] \times \mathbb{S}^2)} \lesssim \frac{(v_2 - v_1)^{\frac{3}{2}}}{r(u, v_1)} \|\dot{\tilde{\nabla}} r\alpha(u, \cdot, \cdot)\|_{L^2([v_1, v_2] \times \mathbb{S}^2)}, \tag{6.6.40}$$

Combining (6.6.32) with (6.6.39), we see that $\|r^{-1} \tilde{\Psi}(u, \cdot, \cdot)\|_{L^2([v_1, v_2] \times \mathbb{S}^2)}$ decays to 0 as $u \rightarrow -\infty$. Thus, $r^{-1} \Psi$ decays in measure as $u \rightarrow -\infty$ by Vitali's convergence theorem. Furthermore, the

assumption that Ψ is of finite energy at v_2 means the same applies to $\tilde{\Psi}$ by combining (6.6.32) with (6.6.38), (6.6.39), (6.6.40). A computation shows

$$\frac{1}{r(u, v_1)} \int_{v_1}^v d\bar{v} r(u, \bar{v})^2 \alpha(u, \bar{v}, \theta^A) = \left[\frac{1}{\Omega^2} \nabla_v \tilde{\Psi} + \frac{1}{r} \tilde{\Psi} \right] \Big|_{(u, v, \theta^A)}, \quad (6.6.41)$$

hence $\nabla_v \tilde{\Psi}$, too, decays in measure as $u \rightarrow -\infty$. By Corollary 6.6.1, we then deduce that $\tilde{\Psi} = 0$. It follows that $\alpha = 0$. $\square$

6.7 Construction of the (unique) scattering solution to (6.3.9)–(6.3.32)

In this section, we present the full proof of Theorem 6.5.1. First, we prove uniqueness of scattering solutions in §6.7.1. We then present the construction of the $\ell \geq 2$ -part of the scattering solution in §6.7.2 and of the $\ell < 2$ -part in §6.7.3. The latter will be essentially trivial, all the difficulty is contained in $\ell \geq 2$.

6.7.1 Uniqueness of solutions to the scattering problem

We now prove the uniqueness clause of Theorem 6.5.1:

Proposition 6.7.1. *Assume that $\mathfrak{S}$ is a solution to (6.3.9)–(6.3.32) realising a vanishing seed scattering data set $\mathfrak{D}$. Assume moreover that*

$$\underline{E}_v[\Psi]^{(i)}(-\infty, u_0) < \infty \quad (6.7.1)$$

for all $v \geq v_1$. Then $\mathfrak{S}$ vanishes identically.

Proof. Let $\mathfrak{S}$ be a scattering solution realising $\mathfrak{D}$. Since $\mathfrak{D}$ is trivial, we deduce from Prop. 6.5.1 that, in particular, the restriction to $\underline{\mathcal{C}}$ of $\text{tr} \hat{\mathcal{G}}^{(i)}, \hat{\chi}^{(i)}, \hat{\underline{\chi}}^{(i)}, \hat{\beta}^{(i)}, \hat{\sigma}^{(i)}, K, \hat{\rho}^{(i)}, \hat{\underline{\beta}}^{(i)}, \hat{\underline{\alpha}}^{(i)}$ belonging to $\mathfrak{S}$ vanishes as well. By (6.3.32), we infer that $\nabla_u(r\Omega^2\hat{\alpha})$ vanishes along $\underline{\mathcal{C}}$ as well. The condition (6.7.1), together with

$$\lim_{u \rightarrow -\infty} \frac{1}{r(u, v)} \int_{v_1}^v d\bar{v} r^2 \hat{\alpha}^{(i)}(u, \bar{v}, \theta^A) \quad (6.7.2)$$

$$= - \lim_{u \rightarrow -\infty} \left(\frac{1}{r(u, v)} \left(\frac{r^2 \hat{\chi}^{(i)}}{\Omega}(u, v, \theta^A) - \frac{r^2 \hat{\underline{\chi}}^{(i)}}{\Omega}(u, v_1, \theta^A) \right) \right) = 0 \quad (6.7.3)$$

implies by Corollary 6.6.2 that $\hat{\alpha}^{(i)} \equiv 0$. It follows that $\hat{\Psi}^{(i)} \equiv 0$.

Consecutively integrating (6.3.17b), (6.3.31b) as well as (6.3.30b) from $\underline{\mathcal{C}}$ then yields that $\hat{\beta}^{(i)} \equiv 0 \equiv \hat{\chi}^{(i)}$ as well as $\hat{\sigma}^{(i)} \equiv 0$. It follows from (6.3.38) that $\hat{\Psi}^{(i)} \equiv 0$, and thus $\hat{\underline{\alpha}}^{(i)} \equiv 0$ (cf. Cor. 6.5.2).

With $\hat{\alpha}^{(i)}$ and $\hat{\underline{\alpha}}^{(i)}$ both vanishing identically, we now apply Proposition 6.4.7 to deduce that the $\ell \geq 2$ component of $\mathfrak{S}$ is pure gauge, i.e. $\mathfrak{S}_{\ell \geq 2} = \mathfrak{S}_{f_{\ell \geq 2}} + \mathfrak{S}_{\underline{f}_{\ell \geq 2}} + \mathfrak{S}_{(q_1, q_2)_{\ell \geq 2}}$ for some functions $f_{\ell \geq 2}$, $\underline{f}_{\ell \geq 2}$ and $(q_1, q_2)_{\ell \geq 2}$.

We now show that these functions vanish: Firstly, the vanishing of $\hat{\chi}^{(i)}$ forces $\underline{f}_{\ell \geq 2} = 0$. (Recall that the kernel of $\hat{\mathcal{P}}_2^* \hat{\mathcal{P}}_1^*$ is spanned by the $\ell = 0, 1$ -modes.) Secondly, the vanishing of $\lim_{u \rightarrow -\infty} \hat{\Omega}^{(i)}$ implies that the outgoing gauge solution must have $\partial_v f_{\ell \geq 2} = 0$, and the vanishing of, say, $\hat{\rho}|_{\underline{\mathcal{C}}}$, implies that $f_{\ell \geq 2} = 0$. Thirdly, the vanishing of $\lim_{u \rightarrow -\infty} r^{-1} \hat{b}^{(i)}$ implies that $\partial_v q_{1\ell \geq 2}$, $\partial_v q_{2\ell \geq 2}$ vanish, and the fact that $\hat{\mathcal{G}}^{(i)}$ is trivial on $\underline{\mathcal{C}}$ leads to $q_{1\ell \geq 2} = q_{2\ell \geq 2} = 0$.

At this point, we know that $\mathfrak{S}$ is supported only on $\ell = 0, 1$, so it is a linear combination of a pure gauge solution, a linearised nearby Schwarzschild solution, and a linearised Kerr solution by Prop. 6.4.6.

From the fact that $\mathring{\sigma}|_{\mathcal{C}} = 0$, we can infer that the linearised Kerr solution must vanish.

In order to also show that the linearised nearby Schwarzschild solution vanishes, we assume that it doesn't: Then this solution generates nonvanishing $\mathring{\rho}_{\mathfrak{m}} = -\frac{2M_{\mathfrak{m}}}{r^3} = -\frac{2}{r}\mathring{K}_{\mathfrak{m}}$ along $\mathcal{C}$, supported on $\ell = 0$, by Prop. 6.4.1. But since $\mathfrak{S}$ has $\mathring{\rho}|_{\mathcal{C}} = 0 = \mathring{K}|_{\mathcal{C}} = 0$, it must now be possible to find pure gauge solutions supported on $\ell = 0$ to kill off $\mathring{\rho}_{\mathfrak{m}}$ and $\mathring{K}_{\mathfrak{m}}$ along $\mathcal{C}$. An inspection of the expressions given in Propositions 6.4.3–6.4.4 shows that such solutions do not exist: Since the outgoing gauge solution decays too fast near $\mathcal{I}^-$ ($\mathring{\rho}_f \sim r^{-4}$, $\mathring{K} \sim r^{-3}$), the only candidate to kill off the leading order behaviour of $\mathring{\rho}_{\mathfrak{m}}$ and $\mathring{K}_{\mathfrak{m}}$ is the ingoing gauge solution, but this has $\mathring{\rho}_f = -\frac{3}{r}\mathring{K}_f$. Thus the linearised nearby Schwarzschild solution also vanishes.

We now know that $\mathfrak{S}$ is a pure gauge solution supported on $\ell = 0, 1$, i.e. $\mathfrak{S} = \mathfrak{S}_{f_{\ell \leq 1}} + \mathfrak{S}_{\underline{f}_{\ell \leq 1}} + \mathfrak{S}_{(q_1, q_2)_{\ell \leq 1}}$. The condition that $\mathring{\rho}|_{\mathcal{C}} = 0$ implies that $\underline{f}_{\ell \leq 1}(u, \theta^A) = f_{\ell \leq 1}(v = v_1, \theta^A)$, so $\underline{f}_{\ell \leq 1}$ is independent of u . The condition that $\mathring{\Omega} \rightarrow 0$ as $u \rightarrow -\infty$ then implies that $\partial_v f_{\ell \leq 1} = 0$, so we have that $\underline{f}_{\ell \leq 1}(u, \theta^A) = \tilde{f}_{\ell \leq 1}(\theta^A) = f_{\ell \leq 1}(v, \theta^A)$ for some $\tilde{f}$. Comparing now the expressions for $\text{tr} \mathring{g}$ generated by each of these solutions along $\mathcal{C}$, we deduce that $\tilde{f}_{\ell=1} = 0$. (The expression for $\text{tr} \mathring{g}$ generated by (q_1, q_2) comes with a different r -weight.) We thus have that $\underline{f}_{\ell \leq 1}(u, \theta^A) = C = f_{\ell \leq 1}(v, \theta^A)$ for some constant C , and hence, in view of Remark 6.4.1, $\mathfrak{S}_{f_{\ell \leq 1}} + \mathfrak{S}_{\underline{f}_{\ell \leq 1}} = 0$.

Thus, we have that $\mathfrak{S} = \mathfrak{S}_{(q_1, q_2)_{\ell \leq 1}}$. Since $r^{-1}\mathring{b} \rightarrow 0$ as $u \rightarrow -\infty$, and since $\text{tr} \mathring{g}$ vanishes along $\mathcal{C}$, it finally follows that $\mathfrak{S} = 0$. $\square$

6.7.2 Construction of the $\ell \geq 2$ -part of the solution to the scattering problem

We now present an explicit construction of a scattering solution $\mathfrak{S}$ to (6.3.9)–(6.3.32) realising a given smooth seed scattering data set $\mathfrak{D}$ such that $\mathfrak{S}$ and $\mathfrak{D}$ are related via Definition 6.5.2.

Proposition 6.7.2. *Given a smooth seed scattering data set $\mathfrak{D}_{\ell \geq 2}$ supported on $\ell \geq 2$ and satisfying the assumptions of Theorem 6.5.1, there exists a scattering solution $\mathfrak{S}_{\ell \geq 2}$ realising $\mathfrak{D}_{\ell \geq 2}$. By the previous Prop. 6.7.1, this is the unique scattering solution realising $\mathfrak{D}_{\ell \geq 2}$.*

Since the proof of Prop. 6.7.2 is quite long, we first give a detailed overview.

6.7.2.1 Overview of the construction

The construction is organised into the following steps:

- (I) Out of $\mathring{\chi}_{\mathcal{I}^-}$ and data on $\mathcal{C}$ (from Prop. 6.5.1), we define various tensor fields such as $\mathring{\beta}_{\mathcal{I}^-}$, $\mathring{\alpha}_{\mathcal{I}^-}$ and $\mathring{\mathbf{P}}_{\mathcal{I}^-}$ at $\mathcal{I}^-$. (These are to be thought of as the pointwise limits of $r^2\mathring{\beta}$, $r\mathring{\alpha}$ and $\mathring{\nabla}_v\mathring{\Psi}$ of the eventual solution.)

- (II) We now construct the unique scattering solution $\overset{(i)}{\Psi}$ to (RW) such that $\overset{(i)}{\Psi}$ restricts to $\overset{(i)}{\Psi}_{\underline{\mathcal{C}}}$ and $\nabla_v \overset{(i)}{\Psi}$ tends to $\overset{(i)}{P}_{\mathcal{I}^-}$ at $\mathcal{I}^-$ via Theorem 6.6.1, where $\overset{(i)}{\Psi}_{\underline{\mathcal{C}}}$ has been constructed in Cor. 6.5.2. Using (6.6.25) then gives us the bound $\overset{(i)}{\Psi} \lesssim r^{1/2}$
- (III) We then define $\overset{(i)}{\alpha}$ by integrating $\overset{(i)}{\Psi} = (\frac{r^2}{\Omega^2} \nabla_u)^2 (r \Omega^2 \overset{(i)}{\alpha})$ (cf. (6.3.37)) twice from $\mathcal{I}^-$, defining the boundary terms at $\mathcal{I}^-$ by using the definitions of $\overset{(i)}{\beta}_{\mathcal{I}^-}$ and $\overset{(i)}{\alpha}_{\mathcal{I}^-}$ from step (II) and demanding (6.3.32) to hold at $\mathcal{I}^-$. We then show that $\overset{(i)}{\alpha}$ satisfies (Teuk) by virtue of Lemma 6.3.1 and $\overset{(i)}{\Psi}$ satisfying (RW).
- (IV) Having constructed $\overset{(i)}{\alpha}$ in the previous step, we now define $\overset{(i)}{\chi}$ and $\overset{(i)}{\beta}$ by integrating (6.3.17b) and (6.3.31b) from $\underline{\mathcal{C}}$, respectively, with the boundary terms $\overset{(i)}{\hat{\chi}}_{\underline{\mathcal{C}}}$ and $\overset{(i)}{\hat{\beta}}_{\underline{\mathcal{C}}}$ defined in Prop. 6.5.1. The fact that $\overset{(i)}{\alpha}$ satisfies (Teuk) implies that (6.3.32) is satisfied. We then show that $r \overset{(i)}{\hat{\chi}}$ converges to $\overset{(i)}{\hat{\chi}}_{\mathcal{I}^-}$ as $u \rightarrow -\infty$, and similarly for $\overset{(i)}{\beta}$.
- (V) We define $\overset{(i)}{\sigma}$ by integrating (6.3.30b) from $\underline{\mathcal{C}}$, with boundary term $\overset{(i)}{\sigma}_{\underline{\mathcal{C}}}$ as in Prop. 6.5.1. We then prove that $\overset{(i)}{\sigma}$ satisfies the scalar Regge–Wheeler equation (RW-scalar). This is done as follows:

- We show that the $\overset{\circ}{\text{curl}}$ of (6.3.22), replacing $\overset{\circ}{\text{curl}} \overset{(i)}{\eta}$ with $r \overset{(i)}{\sigma}$, is satisfied:

$$\overset{\circ}{\text{curl}} \overset{\circ}{\text{div}} r^2 \Omega^{-1} \overset{(i)}{\hat{\chi}} = r^3 \overset{(i)}{\sigma} - \overset{\circ}{\text{curl}} r^3 \Omega^{-1} \overset{(i)}{\beta}. \quad (6.7.4)$$

This is proved by showing that ∇_v (6.7.4) is satisfied using (6.3.17b), (6.3.31b) and (6.3.30b) and using that (6.7.4) holds along $\underline{\mathcal{C}}$ by Prop. 6.5.1.

- We then show that the $\overset{\circ}{\text{curl}}$ of (6.3.28b) is satisfied:

$$\nabla_u \overset{\circ}{\text{curl}} r^2 \Omega \overset{(i)}{\beta} = -\overset{\circ}{\Delta} r \Omega^2 \overset{(i)}{\sigma} - 6M \Omega^2 \overset{\circ}{\text{curl}} \overset{(i)}{\sigma}. \quad (6.7.5)$$

This again follows by virtue of (6.7.5) being satisfied along $\underline{\mathcal{C}}$ and by showing that ∇_v (6.7.5) is satisfied using (6.3.31b), (6.3.30b), (6.3.17b) as well as (6.7.4).

Equations (6.3.30b) and (6.7.5) together imply that $\overset{(i)}{\sigma}$ solves (RW-scalar).

- (VI) Next, we define $\overset{(i)}{\underline{\Psi}} = \overset{(i)}{\Psi} + 4 \overset{\circ}{\mathcal{D}}_2^* \overset{\circ}{\mathcal{D}}_1^* (0, r^3 \overset{(i)}{\sigma})$. Then $\overset{(i)}{\underline{\Psi}}$ satisfies the Regge–Wheeler equation (RW). From here on, we can to some extent mirror the previous approach, with a few extra difficulties that we will highlight below: We define $\overset{(i)}{\underline{\alpha}}$ by integrating $(\frac{r^2}{\Omega^2} \nabla_v)^2 (r \Omega^2 \overset{(i)}{\underline{\alpha}})$ twice from $\underline{\mathcal{C}}$ with data terms coming from Prop. 6.5.1. The fact that $\overset{(i)}{\underline{\Psi}}$ satisfies (RW) is then used to show that $\overset{(i)}{\underline{\alpha}}$ satisfies (Teuk) via Lemma 6.3.1.
- (VII) We would like to define $\overset{(i)}{\underline{\chi}}$ by integrating (6.3.18a) from $\mathcal{I}^-$. Now, in general, we will have decay no better than $\overset{(i)}{\underline{\alpha}} = \mathcal{O}(|u|^{-\frac{5}{2}-})$ (cf. (6.5.10)), (6.3.18a), so $\nabla_u (\Omega^{-1} r^2 \overset{(i)}{\underline{\chi}}) = -r^2 \overset{(i)}{\underline{\alpha}}$

will not be integrable. This problem is resolved by commuting (6.3.18a) with ∇_v , using that $\nabla_v \underline{\alpha}^{(i)}$ decays one power faster than $\underline{\alpha}^{(i)}$: We define $\hat{\chi}^{(i)}$ by integrating

$$\nabla_u \nabla_v \frac{r^2 \hat{\chi}^{(i)}}{\Omega} = - \left(2 - \frac{1}{\Omega^2} \right) r \Omega^2 \underline{\alpha}^{(i)} - \frac{r}{\Omega^2} \nabla_v (r \Omega^2 \underline{\alpha}^{(i)}) \quad (6.7.6)$$

first in u from $\mathcal{I}^-$ and then in v from $\underline{\mathcal{C}}$. We can then define $\underline{\beta}^{(i)}$ via (6.3.27) and deduce that (6.3.28a) holds by virtue of $\underline{\alpha}^{(i)}$ satisfying (Teuk).

(VIII) The difficulty at this point is that we have no way of directly verifying (6.3.30a) ($\partial_u(r^3 \sigma^{(i)}) = -\mathring{\text{curl}} r^2 \Omega \underline{\beta}^{(i)}$). The work-around to this problem is to define a different $\sigma'^{(i)}$ that satisfies (6.3.30a), and to then show that $\sigma'^{(i)} = \sigma^{(i)}$ as follows:

- Define $r^3 \sigma'^{(i)}$ as the solution to $\mathring{\text{curl}}(6.3.23)$ (cf. (6.7.4)):

$$r^3 \sigma'^{(i)} := \mathring{\text{curl}} \mathring{\text{div}} \frac{r^2 \hat{\chi}^{(i)}}{\Omega} - \mathring{\text{curl}} \frac{r^3 \underline{\beta}^{(i)}}{\Omega}. \quad (6.7.7)$$

We directly deduce from the definition that $\sigma'^{(i)}$ satisfies (6.3.30a).

- As in step (V), we can then prove that $\mathring{\text{curl}}(6.3.28b)$ holds with $\sigma^{(i)}$ replaced by $\sigma'^{(i)}$, from which we can infer that $r^3 \sigma'^{(i)}$ satisfies (RW-scalar). We then show that $\sigma^{(i)}$ and $\sigma'^{(i)}$ attain the same data on $\underline{\mathcal{C}} \cup \mathcal{I}_{v \geq v_1}^-$ and appeal to the uniqueness clause of Theorem 6.6.1 to deduce $\sigma^{(i)} = \sigma'^{(i)}$.

(IX) We define $\underline{\eta}^{(i)}, \underline{\eta}^{(i)}$ via (6.3.17a) and (6.3.18b), respectively. It is then a simple computation to confirm that (6.3.19b) and (6.3.20a) hold as well. Then, using, in particular, equations (6.3.30a) and (6.3.30b), we deduce (6.3.24). Since thus $\mathring{\text{curl}}(\underline{\eta}^{(i)} + \underline{\eta}^{(i)}) = 0$, we can define $\underline{\Omega}^{(i)}$ as solution to (6.3.12). From $\underline{\Omega}^{(i)}$, we define $\underline{\omega}^{(i)}$ and $\underline{\omega}^{(i)}$ via (6.3.11), and (6.3.20b), (6.3.19a) immediately follow from (6.3.20a), (6.3.19b).

(X) Similarly, since we have already proved that the $\mathring{\text{curl}}$ -parts of (6.3.23) and (6.3.22) vanish in (6.7.4), (6.7.7), we can define $(\Omega \text{tr} \chi)^{(i)}$ via (6.3.22) and $(\Omega \text{tr} \underline{\chi})^{(i)}$ via (6.3.23). Directly from this definition, we infer (6.3.16) by appropriately differentiating (6.3.22) in v and (6.3.23) in u .

(XI) As the penultimate step, we define $\rho^{(i)}$ and prove that all remaining equations of (6.3.9)–(6.3.32) featuring $\rho^{(i)}$ are satisfied (except the Gauss equation, since we have not yet defined the metric components $\hat{g}^{(i)}$ and $\text{tr} \hat{g}^{(i)}$). The easiest way to do this is as follows:

- Prove that $\mathring{\text{curl}} \mathring{\text{div}}(6.3.34)$ is satisfied. This allows to define $\mathring{\mathcal{P}}_2^* \mathring{\mathcal{P}}_1^*(\rho^{(i)}, 0)$, and thus $\rho^{(i)}$, via (6.3.34), i.e. as solution to

$$\Psi = 2 \mathring{\mathcal{P}}_2^* \mathring{\mathcal{P}}_1^*(r^3 \rho^{(i)}, -r^3 \sigma^{(i)}) + 6M(r \Omega \hat{\chi}^{(i)} - r \Omega \underline{\chi}^{(i)}). \quad (6.7.8)$$

- By acting with ∇_u or ∇_v on (6.7.8) and using all the previous equations, we can deduce that (6.3.29a), (6.3.29b) are satisfied. We can then deduce the remaining Bianchi equations (6.3.31a), (6.3.28b).
- We deduce (6.3.14) by multiplying (6.3.22) by $r\Omega$ and then acting with ∇_u . We similarly prove (6.3.15).
- We finally prove (6.3.21) by acting on the definition (6.3.12) of $\overset{(i)}{\Omega}$ with $\nabla_u \nabla_v$ and using the already established equations.

(XII) We define the remaining metric coefficients $\overset{(i)}{b}$, $\overset{(i)}{\hat{g}}$ and $\text{tr} \overset{(i)}{\hat{g}}$ via integration of (6.3.13) from $\mathcal{I}^-$ with $\overset{(i)}{b}_{\mathcal{I}^-}$ as data, and integration of (6.3.10b) and (6.3.9b) from $\mathcal{C}$ with $\overset{(i)}{\hat{g}}_{\mathcal{C}}$ and $\text{tr} \overset{(i)}{\hat{g}}_{\mathcal{C}}$ (defined in Prop. 6.5.1) as data. By further taking the ∇_v -derivative of (6.3.9a) and (6.3.10a), we can establish that (6.3.9b) and (6.3.10b) hold everywhere since they hold along $\mathcal{C}$. The final thing left to do is to prove the Gauss equation (6.3.25). For this, we prove that $\nabla_v(r^2(6.3.25))$ holds and that, by construction, (6.3.25) holds along $\mathcal{C}$. We have now constructed a solution to (6.3.9)–(6.3.32). By construction and Prop. 6.5.1, this solution realises the prescribed seed data. This completes the proof.

6.7.2.2 The full details of the construction

Proof of Proposition 6.7.2. We now present the full details of the construction. The reader already convinced by the overview should feel free to skip this section.

Step (I): Defining scattering data at $\mathcal{I}^-$

We first define a number of fields at $\mathcal{I}^-$ that will play the role of data at $\mathcal{I}^-$ in the construction carried out in the subsequent sections:

Definition 6.7.1. *Given a smooth seed scattering data set $\mathfrak{D}$, we define the following fields at $\mathcal{I}^-$:*

$$\overset{(i)}{\beta}_{\mathcal{I}^-} = \overset{\circ}{\text{div}} \overset{(i)}{\hat{\chi}}_{\mathcal{I}^-} - \overset{\circ}{\nabla} \overset{(i)}{K}_{\mathcal{S}_\infty}, \quad \overset{(i)}{\alpha}_{\mathcal{I}^-} = -\overset{\circ}{\nabla}_v \overset{(i)}{\hat{\chi}}_{\mathcal{I}^-}, \quad \overset{(i)}{\psi}_{\mathcal{I}^-} = -2\overset{\circ}{\mathcal{P}}_2^* \overset{(i)}{\beta}_{\mathcal{I}^-}, \quad (6.7.9)$$

where $\overset{(i)}{K}_{\mathcal{S}_\infty}$ has been defined in (6.5.6), as well as

$$\overset{(i)}{S}_{\mathcal{I}^-} = -4\overset{\circ}{\mathcal{P}}_2^* \overset{\circ}{\nabla} \overset{(i)}{\Omega}_{\mathcal{I}^-} + \overset{(i)}{\hat{\chi}}_{\mathcal{I}^-}, \quad \overset{(i)}{P}_{\mathcal{I}^-} = -2(\overset{\circ}{\Delta} - 4)\overset{\circ}{\mathcal{P}}_2^* \overset{(i)}{\beta}_{\mathcal{I}^-} - 6M \overset{(i)}{\alpha}_{\mathcal{I}^-}. \quad (6.7.10)$$

Remark 6.7.1. The above definitions will be used as data at $\mathcal{I}^-$ for $r^2 \overset{(i)}{\beta}$, $r \overset{(i)}{\alpha}$, $\overset{(i)}{\psi}$, $\nabla_v(r^2 \Omega^{-1} \overset{(i)}{\hat{\chi}})$, and $\overset{\circ}{\nabla}_v \overset{(i)}{\Psi}$, respectively.

The definition above will be used explicitly to define the scattering solution over the next few pages. On the other hand, we will now define a number of fields at $\mathcal{I}^-$ for which it will later follow that they are attained as limits of the constructed solution.

Definition 6.7.2. *Given a smooth seed scattering data set $\mathfrak{D}$, we define the following fields at $\mathcal{I}^-$:*

$$\begin{aligned} \hat{\mathfrak{g}}_{\mathcal{I}^-}^{(i)}(v) &= \hat{\mathfrak{g}}_{\mathcal{S}_\infty}^{(i)} + \int_{v_1}^v 2\mathring{\mathcal{D}}_2^* \mathfrak{b}_{\mathcal{I}^-}^{(i)}(\bar{v}) \, d\bar{v}, & \text{tr} \hat{\mathfrak{g}}_{\mathcal{I}^-}^{(i)}(v) &= \text{tr} \hat{\mathfrak{g}}_{\mathcal{S}_\infty}^{(i)} - \int_{v_1}^v \mathring{\text{div}} \mathfrak{b}_{\mathcal{I}^-}^{(i)}(\bar{v}) \, d\bar{v}, \\ \omega_{\mathcal{I}^-}^{(i)} &= \partial_v \Omega_{\mathcal{I}^-}^{(i)}, & \underline{\eta}_{\mathcal{I}^-}^{(i)} &= 2\mathring{\nabla} \Omega_{\mathcal{I}^-}^{(i)}, \\ \mathfrak{K}_{\mathcal{I}^-}^{(i)} &= \mathfrak{K}_{\mathcal{S}_\infty}^{(i)}, & (\Omega \text{tr} \chi)_{\mathcal{I}^-}^{(i)} &= 2\mathfrak{K}_{\mathcal{I}^-}^{(i)} + 4\Omega_{\mathcal{I}^-}^{(i)}. \end{aligned} \quad (6.7.11)$$

Remark 6.7.2. We will show in our construction that the fields above are attained as limits of the quantities $\hat{\mathfrak{g}}^{(i)}$, $\text{tr} \hat{\mathfrak{g}}^{(i)}$, $\omega^{(i)}$, $r\eta^{(i)}$, $r^2 K^{(i)}$ and $r(\Omega \text{tr} \chi)^{(i)}$. On the other hand, it will follow from our construction that the limits of the following quantities vanish as $u \rightarrow -\infty$: $\underline{\omega}^{(i)}$, $r(\Omega \text{tr} \chi)^{(i)}$, $r\eta^{(i)}$, $r\hat{\chi}^{(i)}$, $r^2 \rho^{(i)}$, $r^2 \sigma^{(i)}$, $r^2 \beta^{(i)}$, $r^2 \alpha^{(i)}$.

Step (II): Constructing $\Psi^{(i)}$ using scattering theory for (RW)

We now construct $\Psi^{(i)}$ via the scattering theory for the Regge–Wheeler equation (RW) given by Theorem 6.6.1.

Proposition 6.7.3. *For a smooth seed scattering data $\mathfrak{D}$, there exists a unique smooth finite energy solution $\Psi^{(i)}$ to (RW) such that, for any $v \geq v_1$:*

$$\lim_{u \rightarrow -\infty} \|\mathring{\nabla}_v \Psi^{(i)} - \mathring{\mathcal{P}}_{\mathcal{I}^-}^{(i)}\|_{L^2([v_1, v] \times \mathbb{S}^2)} = 0, \quad \Psi^{(i)}|_{\underline{\mathcal{C}}} = \underline{\Psi}^{(i)}_{\underline{\mathcal{C}}}, \quad (6.7.12)$$

where $\underline{\Psi}^{(i)}_{\underline{\mathcal{C}}}$ is defined in Cor. 6.5.2 and $\mathring{\mathcal{P}}_{\mathcal{I}^-}^{(i)}$ is defined in Def. 6.7.1. Moreover, we have the estimate

$$|\mathring{\nabla}^n \Psi^{(i)}| \leq C_n(v) r^{1/2}, \quad \forall n \in \mathbb{N} \quad (6.7.13)$$

for some $C_n(v)$ depending continuously on v , and for any $n \in \mathbb{N}$, $m \in \mathbb{N}_{\geq 1}$, we have

$$\lim_{u \rightarrow -\infty} \mathring{\nabla}_v^m \mathring{\nabla}^n \Psi^{(i)} = \mathring{\nabla}_v^{m-1} \mathring{\nabla}^n \mathring{\mathcal{P}}_{\mathcal{I}^-}^{(i)} \quad (6.7.14)$$

as $u \rightarrow -\infty$, uniformly on $[v_1, v]$ for any $v \geq v_1$.

Proof. By the decay rate in Cor. 6.5.2, we deduce that $\underline{\Psi}^{(i)}_{\underline{\mathcal{C}}}$ has finite energy along $\underline{\mathcal{C}}$. The result thus follows from the scattering theory for (RW) (Thm. 6.6.1) as well as (6.6.25) and (6.6.26). $\square$

Step (III): Constructing $\overset{(i)}{\alpha}$, $\overset{(i)}{\beta}$ and $\overset{(i)}{\chi}$

Next, we construct $\overset{(i)}{\alpha}$ by integrating (6.3.34) from $\mathcal{I}^-$:

Definition 6.7.3. For a smooth seed scattering data $\mathfrak{D}$ and $\overset{(i)}{\Psi}$ arising via Proposition 6.7.3, define $\overset{(i)}{\psi}$ by integrating

$$\frac{\Omega^2}{r^2} \overset{(i)}{\Psi} = \nabla_u \overset{(i)}{\psi} \quad (6.7.15)$$

in u from $\mathcal{I}^-$ with data $-2\overset{\circ}{\mathcal{P}}_2^* \overset{(i)}{\beta}_{\mathcal{I}^-} = \overset{(i)}{\Psi}_{\mathcal{I}^-}$. Similarly, we define $\overset{(i)}{\alpha}$ by integrating

$$\frac{\Omega^2}{r^2} \overset{(i)}{\psi} = \nabla_u r \Omega^2 \overset{(i)}{\alpha} \quad (6.7.16)$$

with data $\overset{(i)}{\alpha}_{\mathcal{I}^-}$ at $\mathcal{I}^-$ (defined in Def. 6.7.1). By construction, $\overset{(i)}{\alpha}|_{\underline{\mathcal{C}}} = \overset{(i)}{\alpha}_{\underline{\mathcal{C}}}$.

Note that these definitions are well-defined in view of estimate (6.7.13).

We now show that $\overset{\circ}{\nabla}_v, \overset{\circ}{\nabla}$ -derivatives of $r^3 \Omega \overset{(i)}{\psi}$ converge to $\overset{\circ}{\nabla}_v, \overset{\circ}{\nabla}$ -derivatives of $-2\overset{\circ}{\mathcal{P}}_2^* \overset{(i)}{\beta}_{\mathcal{I}^-}$ as $u \rightarrow -\infty$, and a similar statement for $\overset{(i)}{\alpha}$:

Lemma 6.7.1. For a smooth seed scattering data $\mathfrak{D}$ and $\underline{\overset{(i)}{\psi}}, \overset{(i)}{\alpha}$ arising via Definition 6.7.3, we have, for any $n, m \in \mathbb{N}$,

$$\overset{\circ}{\nabla}^n \overset{\circ}{\nabla}_v^m \overset{(i)}{\psi}(u, v, \theta^A) \rightarrow \overset{\circ}{\nabla}^n \overset{\circ}{\nabla}_v^m \left(-2\overset{\circ}{\mathcal{P}}_2^* \overset{(i)}{\beta}_{\mathcal{I}^-} \right)(v, \theta^A), \quad (6.7.17)$$

$$\overset{\circ}{\nabla}^n \overset{\circ}{\nabla}_v^m r \Omega^2 \overset{(i)}{\alpha}(u, v, \theta^A) \rightarrow -\overset{\circ}{\nabla}^n \overset{\circ}{\nabla}_v^{m+1} \overset{(i)}{\chi}_{\mathcal{I}^-}(v, \theta^A) \quad (6.7.18)$$

as $u \rightarrow -\infty$. For any $v \geq v_1$, this convergence is uniform on $[v_1, v]$.

Proof. We compute that

$$\frac{r^2}{\Omega^2} \overset{\circ}{\nabla}_u \overset{\circ}{\nabla}_v \overset{(i)}{\psi} = \frac{r^2}{\Omega^2} \overset{\circ}{\nabla}_v \overset{\circ}{\nabla}_u \overset{(i)}{\psi} = \overset{\circ}{\nabla}_v \overset{(i)}{\Psi} - \frac{3\Omega^2 - 1}{r} \overset{(i)}{\Psi}. \quad (6.7.19)$$

Integrating in u and using (6.7.13) as well as (6.7.14), we see that $\overset{\circ}{\nabla}_v \overset{(i)}{\psi}$ converges to a limit as $u \rightarrow -\infty$ and that this convergence is uniform on $[v_1, v]$ for any $v \geq v_1$. Since $\overset{(i)}{\psi} \rightarrow -\overset{\circ}{\mathcal{P}}_2^* \overset{(i)}{\beta}_{\mathcal{I}^-}$, it follows that $\overset{\circ}{\nabla}_v \overset{(i)}{\psi} \rightarrow -\overset{\circ}{\nabla}_v \overset{\circ}{\mathcal{P}}_2^* \overset{(i)}{\beta}_{\mathcal{I}^-}$. The result for $\overset{\circ}{\nabla}_v$ - and $\overset{\circ}{\nabla}$ -derivatives follows by straightforward commutation.

The result for $\overset{(i)}{\alpha}$ follows analogously. $\square$

Next, we want to show that $\overset{(i)}{\alpha}$ solves the Teukolsky equation (Teuk). For this, we first show:

Lemma 6.7.2. *For a smooth seed scattering data $\mathfrak{D}$ and $\psi^{(i)}, \alpha^{(i)}$ arising via Definition 6.7.3, we have that*

$$\lim_{u \rightarrow -\infty} \text{Teuk}^{+2}[r\Omega^2 \alpha^{(i)}] = \lim_{u \rightarrow -\infty} \frac{r^2}{\Omega^2} \nabla_u \text{Teuk}^{+2}[r\Omega^2 \alpha^{(i)}] = 0 \quad (6.7.20)$$

for any $v \geq v_1$, where the operator Teuk^{+2} was defined in Def. 6.3.2.

Proof. We can re-write the operators above as follows:

$$\text{Teuk}^{+2}[r\Omega^2 \alpha^{(i)}] = \frac{3\Omega^2 - 1}{r} \psi^{(i)} + \nabla_v \psi^{(i)} + \left(-\mathring{\Delta} + 2 + \frac{6M}{r} \right) r\Omega^2 \alpha^{(i)}, \quad (6.7.21)$$

$$\frac{r^2}{\Omega^2} \nabla_u \text{Teuk}^{+2}[r\Omega^2 \alpha^{(i)}] = \nabla_v \Psi^{(i)} - \left(\mathring{\Delta} - 3\Omega^2 - 1 \right) \psi^{(i)} + 6Mr\Omega^2 \alpha^{(i)}. \quad (6.7.22)$$

Therefore, using the previous Lemma 6.7.1:

$$\lim_{u \rightarrow -\infty} \text{Teuk}^{+2}[r\Omega^2 \alpha^{(i)}] = -2\mathring{\mathcal{P}}_2^* \nabla_v \beta_{\mathcal{I}^-}^{(i)} - \left(\mathring{\Delta} - 2 \right) \alpha_{\mathcal{I}^-}^{(i)}. \quad (6.7.23)$$

$$\lim_{u \rightarrow -\infty} \frac{r^2}{\Omega^2} \nabla_u \text{Teuk}^{+2}[r\Omega^2 \alpha^{(i)}] = \mathring{\mathcal{P}}_{\mathcal{I}^-}^{(i)} + 2 \left(\mathring{\Delta} - 4 \right) \mathring{\mathcal{P}}_2^* \beta_{\mathcal{I}^-}^{(i)} - 6M \nabla_v \hat{\chi}_{\mathcal{I}^-}^{(i)}. \quad (6.7.24)$$

The right hand sides above both vanish by construction (cf. Def. 6.7.1). $\square$

We may now infer:

Corollary 6.7.1. *For a smooth seed scattering data $\mathfrak{D}$ and $\alpha^{(i)}$ defined in Definition 6.7.3, we have that $\alpha^{(i)}$ satisfies the Teukolsky equation (Teuk).*

Proof. Using Lemma 6.3.1, we have that

$$\left(\frac{r^2}{\Omega^2} \nabla_u \right)^2 \text{Teuk}^{+2}[r\Omega^2 \alpha^{(i)}] = \text{RW}[\Psi^{(i)}] = 0 \quad (6.7.25)$$

since $\Psi^{(i)}$ satisfies the Regge–Wheeler equation (RW). We integrate this equation twice from $\mathcal{I}^-$, where the boundary terms vanish by Lemma 6.7.2. $\square$

Step (IV): Constructing of $\hat{\chi}^{(i)}$ and $\beta^{(i)}$

We now construct $\hat{\chi}^{(i)}, \beta^{(i)}$ and show that (6.3.32) is satisfied:

Definition 6.7.4. *With $\alpha^{(i)}$ defined in Def. 6.7.3, define $\hat{\chi}^{(i)}, \beta^{(i)}$ to be the unique solutions to (6.3.17b), (6.3.31b) with data $\frac{r^2}{\Omega} \hat{\chi}_{\mathcal{C}}^{(i)}, \frac{r^4}{\Omega} \beta_{\mathcal{C}}^{(i)}$ respectively.*

Lemma 6.7.3. *The quantities $\hat{\chi}^{(i)}, \beta^{(i)}$ and $\alpha^{(i)}$ satisfy eq. (6.3.32)*

Proof. Def. 6.7.4, together with the Teukolsky equation (Teuk) satisfied by $\alpha^{(i)}$, implies that

$$\nabla_v \left[\frac{r^4}{\Omega^4} \nabla_u r \Omega^2 \alpha^{(i)} + 2 \mathring{\mathcal{D}}_2^* \frac{r^4 \beta^{(i)}}{\Omega} - 6M \frac{r^2 \hat{\chi}^{(i)}}{\Omega} \right] = 0. \quad (6.7.26)$$

Integrating this in v from $\underline{\mathcal{C}}$, and using that $\alpha_{\underline{\mathcal{C}}}^{(i)}$ is related to $\beta_{\underline{\mathcal{C}}}^{(i)}$, $\hat{\chi}_{\underline{\mathcal{C}}}^{(i)}$ via (6.3.32) by Proposition 6.5.1 proves the lemma. $\square$

For later purposes, we will also need to verify that $r \hat{\chi}^{(i)}$ and $r^2 \beta^{(i)}$ realise $\hat{\chi}_{\mathcal{I}^-}^{(i)}$ and $\beta_{\mathcal{I}^-}^{(i)}$ (which we have used to define $\psi^{(i)}$ and $\alpha^{(i)}$) as their limits at $\mathcal{I}^-$:

Corollary 6.7.2. *For $\hat{\chi}^{(i)}$, $\beta^{(i)}$ constructed in Definition 6.7.4, we have that for any $m, n \in \mathbb{N}$,*

$$\lim_{u \rightarrow -\infty} \nabla_v^m \mathring{\nabla}_v^m r \hat{\chi}^{(i)} = \nabla_v^m \mathring{\nabla}_v^m \hat{\chi}_{\mathcal{I}^-}^{(i)}, \quad \lim_{u \rightarrow -\infty} \nabla_v^m \mathring{\nabla}_v^m r^2 \beta^{(i)} = \nabla_v^m \mathring{\nabla}_v^m \beta_{\mathcal{I}^-}^{(i)}, \quad (6.7.27)$$

the convergence being uniform in v in compact v -intervals.

Proof. We will prove the statement for $\hat{\chi}^{(i)}$; the proof for $\beta^{(i)}$ being similar: First of all, we have, by definition,

$$\frac{1}{r} \left(\frac{r^2}{\Omega} \hat{\chi}^{(i)}(u, v) - \frac{r^2}{\Omega} \hat{\chi}_{\underline{\mathcal{C}}}^{(i)}(u) \right) = -\frac{1}{r} \int_{v_1}^v r^2 \alpha^{(i)}(u, \bar{v}) d\bar{v}. \quad (6.7.28)$$

Since $r \alpha^{(i)}(u, v)$ converges uniformly to $-\hat{\chi}_{\mathcal{I}^-}^{(i)}$ on $[v_1, v]$ for any fixed $v \geq v_1$ by Lemma 6.7.1, and since the ratio $\frac{r(u, \bar{v})}{r(u, v)}$ tends to 1 uniformly in on $[v_1, v]$ for any fixed v , we have that

$$-\frac{1}{r(u, v)} \int_{v_1}^v r(u, \bar{v})^2 \alpha^{(i)}(u, \bar{v}) d\bar{v} \rightarrow -\int_{v_1}^v \hat{\alpha}_{\mathcal{I}^-}^{(i)}(\bar{v}) d\bar{v} = \hat{\chi}_{\mathcal{I}^-}^{(i)}(v) - \hat{\chi}_{\mathcal{I}^-}^{(i)}(v_1) \quad (6.7.29)$$

as $u \rightarrow -\infty$ for any $v \geq v_1$. Using that $r \hat{\chi}_{\underline{\mathcal{C}}}^{(i)} \rightarrow \hat{\chi}_{\mathcal{I}^-}^{(i)}(v = v_1)$ by construction (Prop. 6.5.1), it follows that $r \hat{\chi}^{(i)} \rightarrow \hat{\chi}_{\mathcal{I}^-}^{(i)}$ as claimed. The proof for higher derivatives follows from Lemma 6.7.1 and commuting. $\square$

Step (V): Constructing $\sigma^{(i)}$

Definition 6.7.5. *Define $\sigma^{(i)}$ to be the unique solution to (6.3.30b) with data $r^3 \sigma_{\underline{\mathcal{C}}}^{(i)}$ on $\underline{\mathcal{C}}$.*

Corollary 6.7.3. *We have for any $n \in \mathbb{N}$ and some $C_n(v)$ depending continuously on v that*

$$|\mathring{\nabla}^n(r^3 \sigma^{(i)})| \leq C_n(v) r^{\frac{1}{2}}. \quad (6.7.30)$$

In addition, the following convergence is uniform in v on any compact v -interval:

$$\lim_{u \rightarrow -\infty} \nabla_v^m \mathring{\nabla}_v^n r^3 \sigma^{(i)} = -\nabla_v^{m-1} \mathring{\nabla}_v^n \text{curl} \beta_{\mathcal{I}^-}^{(i)}, \quad \text{for all } m \in \mathbb{N}_{\geq 1}, n \in \mathbb{N}. \quad (6.7.31)$$

Proof. The convergence (6.7.31) follows from the definition of $\overset{(i)}{\sigma}$ and Cor. 6.7.2. The upper bound (6.7.30) follows from the bound (6.7.13), the definition of $\overset{(i)}{\sigma}$ and the rates (6.5.10) for $\overset{(i)}{\sigma}_{\mathcal{C}}$. $\square$

Lemma 6.7.4. *The quantity $\overset{(i)}{\sigma}$ defined in Definition 6.7.5 satisfies $\mathring{\text{curl}}(6.3.22)$ as well as $\mathring{\text{curl}}(6.3.31a)$:*

$$\mathring{\text{curl}} \mathring{d}\overset{(i)}{v} r \Omega \overset{(i)}{\chi} = r^2 \Omega^2 \overset{(i)}{\sigma} - \mathring{\text{curl}} r^2 \Omega \overset{(i)}{\beta}, \quad (6.7.32)$$

$$\partial_u \mathring{\text{curl}} r^2 \Omega \overset{(i)}{\beta} = -\mathring{\Delta} r \Omega^2 \overset{(i)}{\sigma} - 6M \Omega^2 \overset{(i)}{\sigma}. \quad (6.7.33)$$

Proof. Equations (6.3.31b), (6.3.30b) imply

$$\partial_v \left(r^3 \overset{(i)}{\sigma} - \mathring{\text{curl}} r^3 \Omega^{-1} \overset{(i)}{\beta} \right) = -\mathring{\text{curl}} r^2 \overset{(i)}{\alpha} = \partial_v \mathring{\text{curl}} \mathring{d}\overset{(i)}{v} \frac{r^2 \overset{(i)}{\chi}}{\Omega} \quad (6.7.34)$$

and, since $\overset{(i)}{\chi}$, $\overset{(i)}{\sigma}$, $\overset{(i)}{\beta}$ satisfy (6.7.32) at $\mathcal{C}$, equation (6.7.32) propagates for $v \geq v_1$.

The proof for (6.7.33) is similar, but involves more computations: We first multiply the LHS of (6.7.33) and differentiate in v :

$$\begin{aligned} \partial_v \left(\frac{r^2}{\Omega^2} \partial_u \mathring{\text{curl}} r^2 \Omega \overset{(i)}{\beta} \right) &= \partial_v \left(\frac{r^2}{\Omega^2} \right) \partial_u \mathring{\text{curl}} r^2 \Omega \overset{(i)}{\beta} + \partial_u \mathring{\text{curl}} \nabla_v (r^2 \Omega \overset{(i)}{\beta}) \\ &= \partial_u \mathring{\text{curl}} \mathring{d}\overset{(i)}{v} r^2 \overset{(i)}{\alpha} + \frac{3\Omega^2 - 1}{r} \mathring{\text{curl}} \mathring{d}\overset{(i)}{v} r^3 \overset{(i)}{\alpha} - 2(3\Omega^2 - 2) \mathring{\text{curl}} r^2 \Omega \overset{(i)}{\beta} \quad (6.7.35) \\ &= \mathring{\text{curl}} \mathring{d}\overset{(i)}{v} \left[-2\mathring{\mathcal{P}}_2^* r^2 \Omega \overset{(i)}{\beta} + 6M \Omega \overset{(i)}{\chi} \right] - 2(3\Omega^2 - 2) \mathring{\text{curl}} r^2 \Omega \overset{(i)}{\beta}, \end{aligned}$$

where we used (6.3.31b) in the second line and (6.3.32) in the third line. We re-write the angular operator (recall the notation $\mathring{d}\overset{(i)}{v} = \mathring{\mathcal{P}}_2$) using eq. (6.2.29):

$$\mathring{\mathcal{P}}_1 \left[-2\mathring{\mathcal{P}}_2 \mathring{\mathcal{P}}_2^* \right] = \mathring{\mathcal{P}}_1 \left[\mathring{\Delta} + 1 \right] = \mathring{\mathcal{P}}_1 \left[2 - \mathring{\mathcal{P}}_1^* \mathring{\mathcal{P}}_1 \right] = \left[2 - \mathring{\mathcal{P}}_1 \mathring{\mathcal{P}}_1^* \right] \mathring{\mathcal{P}}_1 = \left[\mathring{\Delta} + 2 \right] \mathring{\mathcal{P}}_1,$$

so we have, in particular, that

$$\mathring{\text{curl}} \mathring{d}\overset{(i)}{v} \left(-2\mathring{\mathcal{P}}_2^* r^2 \Omega \overset{(i)}{\beta} \right) = \left(\mathring{\Delta} + 2 \right) \mathring{\text{curl}} r^2 \Omega \overset{(i)}{\beta},$$

and we can write

$$\partial_v \frac{r^2}{\Omega^2} \partial_u \mathring{\text{curl}} r^2 \Omega \overset{(i)}{\beta} = \left(\mathring{\Delta} + \frac{12M}{r} \right) \mathring{\text{curl}} r^2 \Omega \overset{(i)}{\beta} + 6M \mathring{\text{curl}} \mathring{d}\overset{(i)}{v} \Omega \overset{(i)}{\chi}.$$

Finally, we apply (6.7.32) to this:

$$\partial_v \left(\frac{r^2}{\Omega^2} \partial_u \mathring{\text{curl}} r^2 \Omega \overset{(i)}{\beta} \right) = \left(\mathring{\Delta} + \frac{12M}{r} \right) \mathring{\text{curl}} r^2 \Omega \overset{(i)}{\beta} + 6M \left(\Omega^2 \overset{(i)}{\sigma} - \mathring{\text{curl}} r \Omega \overset{(i)}{\beta} \right). \quad (6.7.36)$$

Next, we apply the operator $\partial_v \frac{r^2}{\Omega^2}$ to the right hand side of (6.7.33):

$$\partial_v \left(\mathring{\Delta} r^3 \sigma + 6Mr^2 \sigma \right) = -\mathring{\Delta} \mathring{\mathcal{C}} \mathring{\mathcal{U}} \mathring{\mathcal{L}} r^2 \Omega^{(i)} \beta - 6Mr \Omega^2 \sigma - 6M \mathring{\mathcal{C}} \mathring{\mathcal{U}} \mathring{\mathcal{L}} r \Omega \beta. \quad (6.7.37)$$

Combining (6.7.36) and (6.7.37), we deduce that

$$\partial_v \left(\frac{r^2}{\Omega^2} \partial_u \mathring{\mathcal{C}} \mathring{\mathcal{U}} \mathring{\mathcal{L}} r^2 \Omega^{(i)} \beta + \mathring{\Delta} r^3 \sigma + 6Mr^2 \sigma \right) = 0. \quad (6.7.38)$$

But since (6.7.33) holds along $\underline{\mathcal{C}}$, we can now integrate the above from $\underline{\mathcal{C}}$ to deduce that it holds everywhere. $\square$

Corollary 6.7.4. *The quantity $\sigma^{(i)}$ constructed in Definition 6.7.5 satisfies the scalar Regge–Wheeler equation (RW-scalar).*

Proof. This follows from (6.3.30b) and (6.7.33). $\square$

Step (VI): Constructing $\underline{\Psi}^{(i)}$ and $\underline{\alpha}^{(i)}$.

We are now in position to define $\underline{\Psi}^{(i)}$ and thus $\underline{\alpha}^{(i)}$:

Definition 6.7.6. Define $\underline{\Psi}^{(i)}$ via

$$\underline{\Psi}^{(i)} := \Psi^{(i)} + 2\mathring{\mathcal{P}}_2^* \mathring{\mathcal{P}}_1^*(0, r^3 \sigma). \quad (6.7.39)$$

Define $\underline{\psi}^{(i)}$ to be the unique solution to

$$\frac{\Omega^2}{r^2} \underline{\Psi}^{(i)} = \mathring{\nabla}_v \underline{\psi}^{(i)}, \quad (6.7.40)$$

with data $\underline{\psi}_{\underline{\mathcal{C}}}^{(i)} = 2\mathring{\mathcal{P}}_2^* r^2 \Omega \underline{\beta}_{\underline{\mathcal{C}}}^{(i)} + 6M \Omega \hat{\chi}_{\underline{\mathcal{C}}}^{(i)}$ at $\underline{\mathcal{C}}$. Define $\underline{\alpha}^{(i)}$ to be the unique solution to

$$\frac{\Omega^2}{r^2} \underline{\psi}^{(i)} = \mathring{\nabla}_v r \Omega^2 \underline{\alpha}^{(i)}, \quad (6.7.41)$$

with data $r \Omega^2 \underline{\alpha}_{\underline{\mathcal{C}}}^{(i)}$ at $\underline{\mathcal{C}}$.

Corollary 6.7.5. *Since $\Psi^{(i)}$ and $\mathring{\mathcal{P}}_2^* \mathring{\mathcal{P}}_1^*(0, r^3 \sigma)$ satisfy the Regge–Wheeler equation (RW), $\underline{\Psi}^{(i)}$ also satisfies (RW). Moreover, we have for any $n \in \mathbb{N}$ and for some $C_n(v)$ depending continuously on v the estimates*

$$|\mathring{\nabla}^n \underline{\Psi}| \leq C_n(v) r^{\frac{1}{2}}, \quad |\mathring{\nabla}^n \underline{\psi}| \leq C_n(v) r^{-\frac{1}{2}}, \quad |\mathring{\nabla}^n \underline{\alpha}| \leq C_n(v) r^{-5/2}, \quad (6.7.42)$$

as well as the uniform convergence on compact v -intervals of

$$\lim_{u \rightarrow -\infty} \nabla_v^m \dot{\nabla}^n \underline{\Psi} = \nabla_v^{m-1} \dot{\nabla}^n (\underline{\Psi}_{\mathcal{I}^-} - \dot{\mathcal{D}}_2^* \dot{\mathcal{D}}_1^*(0, \text{curl} \underline{\beta}_{\mathcal{I}^-})), \quad \text{for all } m \in \mathbb{N}_{\geq 1}, n \in \mathbb{N}. \quad (6.7.43)$$

Proof. The first of (6.7.42) follows from the estimates (6.7.13), (6.7.30). Then using the definition of $\underline{\psi}^{(i)}$ and $\underline{\alpha}^{(i)}$ together with the rates (6.5.10) and (6.5.12), the other two rates follow as well.

The convergence (6.7.43) follows directly from (6.7.14) and (6.7.31). $\square$

As we did for $\underline{\alpha}^{(i)}$, we now show that $\underline{\alpha}^{(i)}$ satisfies (Teuk) by appealing to equation (6.3.51) from Lemma 6.3.1.

Corollary 6.7.6. *The quantity $\underline{\alpha}^{(i)}$ constructed in Def. 6.7.6 satisfies the Teukolsky equation (Teuk).*

Proof. Since $\underline{\Psi}^{(i)}$ satisfies the Regge–Wheeler equation (RW), we have by Lemma 6.3.1 that

$$\left(\frac{r^2}{\Omega^2} \nabla_v \right)^2 \text{Teuk}^{-2}[r\Omega^2 \underline{\alpha}^{(i)}] = 0. \quad (6.7.44)$$

A computation analogous to that in the proof of Lemma 6.7.2 shows that $\text{Teuk}^{-2}[r\Omega^2 \underline{\alpha}^{(i)}]$, $\nabla_v \text{Teuk}^{-2}[r\Omega^2 \underline{\alpha}^{(i)}]$ vanish at $\underline{\mathcal{C}}$. $\square$

Step (VII): Constructing $\underline{\hat{\chi}}^{(i)}$ and $\underline{\beta}^{(i)}$

We now construct $\underline{\hat{\chi}}^{(i)}$, $\underline{\beta}^{(i)}$. Note that since $|\underline{\alpha}^{(i)}| \lesssim C(v)r^{-\frac{5}{2}}$, we cannot integrate (6.3.18a) in u from $\mathcal{I}^-$. However, since $|\nabla_v \underline{\alpha}^{(i)}| \lesssim C(v)r^{-7/2}$ (by (6.7.42)), we can instead integrate the ∇_v -commuted equation:

Definition 6.7.7. Define $\underline{S}^{(i)}$ to be the unique solution to

$$\nabla_u \underline{S}^{(i)} = -r^{-1} \underline{\psi}^{(i)} - \left(2 - \frac{1}{\Omega^2} \right) r\Omega^2 \underline{\alpha}^{(i)}, \quad (6.7.45)$$

with data $\underline{S}_{\mathcal{I}^-}^{(i)}$ at $\mathcal{I}^-$ defined in Def. 6.7.1. Define $\underline{\hat{\chi}}^{(i)}$ to be the unique solution to

$$\nabla_v \frac{r^2 \underline{\hat{\chi}}^{(i)}}{\Omega} = \underline{S}^{(i)}, \quad (6.7.46)$$

with data $r^2 \Omega^{-1} \underline{\hat{\chi}}_{\underline{\mathcal{C}}}^{(i)}$ at $\underline{\mathcal{C}}$.

Lemma 6.7.5. *The quantity $\underline{\hat{\chi}}^{(i)}$ constructed in Definition 6.7.7 satisfies*

$$\nabla_u \frac{r^2 \underline{\hat{\chi}}^{(i)}}{\Omega} = -r^2 \underline{\alpha}^{(i)}. \quad (6.7.47)$$

Moreover, we have the following uniform convergence along compact v -intervals:

$$\lim_{u \rightarrow -\infty} \nabla_v^m \overset{\circ}{\nabla}_v^n r^2 \hat{\underline{\chi}} = \nabla_v^{m-1} \overset{\circ}{\nabla}_v^n \mathbf{S}_{\mathcal{I}^-}, \quad \text{for all } m \in \mathbb{N}_{\geq 1}, n \in \mathbb{N}. \quad (6.7.48)$$

Proof. The first statement follows by integrating the identity

$$\nabla_v r^2 \overset{(i)}{\underline{\alpha}} = r^2 \Omega \overset{(i)}{\psi} + \left(2 - \frac{1}{\Omega^2}\right) r \Omega^2 \overset{(i)}{\underline{\alpha}} = -\nabla_u \overset{(i)}{S} = -\nabla_v \nabla_u \frac{r^2 \overset{(i)}{\hat{\chi}}}{\Omega} \quad (6.7.49)$$

in v from $\mathcal{C}$, using that (6.7.47) is satisfied along $\mathcal{C}$ by $\overset{(i)}{\underline{\alpha}}_{\mathcal{C}}$ and $\overset{(i)}{\hat{\chi}}_{\mathcal{C}}$ as a consequence of Prop. 6.5.1.

The proof of (6.7.48) proceeds exactly as the proof of Lemma 6.7.1, using the rates (6.7.42). $\square$

Definition 6.7.8. Define $\overset{(i)}{\underline{\beta}}$ to be the unique solution to (6.3.27), with $\overset{(i)}{\hat{\chi}}$ defined as in Definition 6.7.7 and $\overset{(i)}{\underline{\alpha}}$ defined as in Definition 6.7.6. By construction and Prop. 6.5.1, $\overset{(i)}{\underline{\beta}}|_{\mathcal{C}} = \overset{(i)}{\underline{\beta}}_{\mathcal{C}}$.

Corollary 6.7.7. The quantity $\overset{(i)}{\underline{\beta}}$ constructed in Definition 6.7.8 satisfies (6.3.28a).

Proof. This follows by computing $\nabla_u(\frac{r^4}{\Omega^4} (6.3.27))$, using (6.3.18a) for $\overset{(i)}{\hat{\chi}}$ as well as the Teukolsky equation (Teuk) for $\overset{(i)}{\underline{\alpha}}$. (Recall that the operator $\overset{\circ}{\mathcal{D}}_2^*$ is invertible on $\ell \geq 2$.) $\square$

Step (VIII): Constructing $\overset{(i)'}{\sigma}$ and proving that $\overset{(i)'}{\sigma} = \overset{(i)}{\sigma}$

Recall the difficulties addressed in the overview of Step (VIII) in §6.7.2.1: Since we have no direct way of inferring (6.3.30b) at this point, we instead define a different $\overset{(i)'}{\sigma}$:

Definition 6.7.9. Define $\overset{(i)'}{\sigma}$ via the $\overset{\circ}{\text{curl}}$ of (6.3.23):

$$r^3 \overset{(i)'}{\sigma} := \overset{\circ}{\text{curl}} \overset{\circ}{\nabla}_v r^2 \Omega^{-1} \overset{(i)}{\hat{\chi}} - \overset{\circ}{\text{curl}} r^3 \Omega^{-1} \overset{(i)}{\underline{\beta}}. \quad (6.7.50)$$

By construction and Prop. 6.5.1, $\overset{(i)'}{\sigma}$ restricts to $\overset{(i)}{\sigma}_{\mathcal{C}}$ on $\mathcal{C}$.

Lemma 6.7.6. The quantity $\overset{(i)'}{\sigma}$ satisfies

$$\partial_u(r^3 \overset{(i)'}{\sigma}) = \overset{\circ}{\text{curl}} r^2 \Omega \overset{(i)}{\underline{\beta}}. \quad (6.7.51)$$

Proof. This follows by taking the ∂_u -derivative of the LHS of (6.7.50) using the already established equations (6.3.18a), (6.3.28a). $\square$

In order to infer that $\overset{(i)'}{\sigma} = \overset{(i)}{\sigma}$, we first show that their radiation fields at $\mathcal{I}^-$ are identical:

Lemma 6.7.7. For $\overset{(i)'}{\sigma}$, we have

$$\lim_{u \rightarrow -\infty} \partial_v(r^3 \overset{(i)'}{\sigma}) = -\overset{\circ}{\text{curl}} \overset{(i)}{\beta}_{\mathcal{I}^-} = \lim_{u \rightarrow -\infty} \partial_v(r^3 \overset{(i)}{\sigma}). \quad (6.7.52)$$

This convergence is uniform in v on compact v -intervals.

Proof. Note that we have already shown this for $\sigma^{(i)}$. For $\sigma'^{(i)}$, we first take the ∂_v -derivative of (6.7.50) and use (6.7.48) to find

$$\lim_{u \rightarrow -\infty} \partial_v(r^3 \sigma'^{(i)}) = \mathring{\text{curl}} \mathring{\text{div}} \mathring{\mathbf{S}}_{\mathcal{I}^-}^{(i)} - \lim_{u \rightarrow -\infty} \mathring{\text{curl}} \nabla_v(r^3 \underline{\beta}^{(i)}). \quad (6.7.53)$$

In order to evaluate the limit on the RHS, we compute, using (6.3.27):

$$\lim_{u \rightarrow -\infty} \mathring{\mathcal{P}}_2^* \nabla_v(r^3 \Omega^{-1} \underline{\beta}^{(i)}) = \lim_{u \rightarrow -\infty} (\nabla_v(r \underline{\psi}^{(i)}) - 6M \nabla_v(r \Omega \underline{\hat{\chi}}^{(i)})). \quad (6.7.54)$$

But

$$\nabla_v(r \underline{\psi}^{(i)}) = r^{-1} \Omega^2 \underline{\Psi}^{(i)} + \Omega^2 \underline{\psi}^{(i)}$$

tends to 0 as $u \rightarrow -\infty$ by (6.7.42) and (6.7.43), and so does $\nabla_v(r \Omega \underline{\hat{\chi}}^{(i)})$ by (6.7.48). It thus follows that $\lim_{u \rightarrow -\infty} \mathring{\mathcal{P}}_2^* \nabla_v(r^3 \Omega^{-1} \underline{\beta}^{(i)}) = 0$. In order to deduce that $\lim_{u \rightarrow -\infty} \mathring{\text{curl}} r^3 \Omega^{-1} \underline{\beta}^{(i)} = 0$ as well, we simply apply $\mathring{\text{curl}} \mathring{\mathcal{P}}_2$: To be precise, we use that for any 1-form β :

$$2 \mathring{\text{curl}} \mathring{\mathcal{P}}_2 \mathring{\mathcal{P}}_2^* \beta = -(\mathring{\Delta} + 2) \mathring{\text{curl}} \beta. \quad (6.7.55)$$

It follows that

$$\lim_{u \rightarrow -\infty} -(\mathring{\Delta} + 2) \partial_v(r^3 \sigma'^{(i)}) = -(\mathring{\Delta} + 2) \mathring{\text{curl}} \mathring{\text{div}} \mathring{\mathbf{S}}_{\mathcal{I}^-}^{(i)} - \lim_{u \rightarrow -\infty} 2 \mathring{\text{curl}} \mathring{\mathcal{P}}_2 \mathring{\mathcal{P}}_2^* \nabla_v(r^3 \underline{\beta}^{(i)}), \quad (6.7.56)$$

The term on the RHS now vanishes by the same argument as above, and, by the invertibility of $\mathring{\Delta} + 2$ on $\ell \geq 2$, a Poincaré inequality and Sobolev embedding:

$$\lim_{u \rightarrow -\infty} \partial_v(r^3 \sigma'^{(i)}) = \mathring{\text{curl}} \mathring{\text{div}} \mathring{\mathbf{S}}_{\mathcal{I}^-}^{(i)} = \mathring{\text{curl}} \mathring{\text{div}} \mathring{\hat{\chi}}_{\mathcal{I}^-}^{(i)} = \mathring{\text{curl}} \mathring{\beta}_{\mathcal{I}^-}^{(i)}, \quad (6.7.57)$$

where the last two equalities follow from definitions of $\mathring{\mathbf{S}}_{\mathcal{I}^-}^{(i)}$ and $\mathring{\beta}_{\mathcal{I}^-}^{(i)}$ in Def. 6.7.1 (and the fact that $\mathring{\text{curl}} \mathring{\text{div}} \mathring{\mathcal{P}}_2^* \nabla f = 0 = \mathring{\text{curl}} \nabla f$ for any $f \in \Gamma^\infty(\mathbb{S}^2)$). $\square$

We can now show that $\sigma'^{(i)}$ also satisfies (RW-scalar):

Lemma 6.7.8. *The quantities $\underline{\beta}^{(i)}$, $\sigma'^{(i)}$ satisfy the $\mathring{\text{curl}}$ of (6.3.28a), i.e.:*

$$\partial_v \mathring{\text{curl}} r^2 \Omega \underline{\beta}^{(i)} = -\mathring{\Delta} r \Omega^2 \sigma'^{(i)} - 6M \Omega^2 \sigma'^{(i)}. \quad (6.7.58)$$

In particular, in view of Lemma 6.7.6, $\sigma'^{(i)}$ satisfies (RW-scalar).

Proof. This is similar to the proof of (6.7.33). In the same way in which we proved (6.7.38), we first show that

$$\partial_u \left[\frac{r^2}{\Omega^2} \partial_v \mathring{\text{curl}} r^2 \Omega \underline{\beta}^{(i)} - \mathring{\Delta} r^3 \sigma^{(i)'} - 6M r^2 \sigma^{(i)'} \right] = 0. \quad (6.7.59)$$

Notice that we cannot integrate this from $\mathcal{I}^-$, because the object on which ∂_u is acting does not converge near $\mathcal{I}^-$. We again resolve this problem by considering the ∂_v -derivative:

$$\partial_u \partial_v \left[\frac{r^2}{\Omega^2} \partial_v \mathring{\text{curl}} r^2 \Omega \underline{\beta}^{(i)} - \mathring{\Delta} r^3 \sigma^{(i)'} - 6M r^2 \sigma^{(i)'} \right] = 0. \quad (6.7.60)$$

We integrate this in u , using that the boundary term at $\mathcal{I}^-$ vanishes. Indeed:

Subclaim 2.

$$\lim_{u \rightarrow -\infty} \partial_v \frac{r^2}{\Omega^2} \partial_v \mathring{\text{curl}} r^2 \Omega \underline{\beta}^{(i)} = \mathring{\Delta} \mathring{\text{curl}} \beta_{\mathcal{I}^-}^{(i)} = \lim_{u \rightarrow -\infty} \mathring{\Delta} r^3 \sigma^{(i)'}. \quad (6.7.61)$$

Proof of the sublemma. As in the proof of Lemma 6.7.7, we use (6.3.27) to compute

$$2\mathring{\mathcal{P}}_2^* \nabla_v \frac{r^2}{\Omega^2} \nabla_v r^2 \Omega \underline{\beta}^{(i)} = \nabla_v \underline{\Psi}^{(i)} - 6M \nabla_v^2 \frac{r^2 \hat{\chi}^{(i)}}{\Omega} + 6M \frac{(3\Omega^2 - 1)}{r} \nabla_v \frac{r^2 \hat{\chi}^{(i)}}{\Omega} - 12M(3\Omega^2 - 2) \Omega \hat{\chi}^{(i)}; \quad (6.7.62)$$

so we have

$$\lim_{u \rightarrow -\infty} 2\mathring{\mathcal{P}}_2^* \nabla_v \frac{r^2}{\Omega^2} \nabla_v r^2 \Omega \underline{\beta}^{(i)} = \lim_{u \rightarrow -\infty} \nabla_v \underline{\Psi}^{(i)} - 6M \nabla_v \mathring{\mathcal{S}}_{\mathcal{I}^-}^{(i)}. \quad (6.7.63)$$

In view of (6.7.43), we have

$$\begin{aligned} \lim_{u \rightarrow -\infty} \nabla_v \underline{\Psi}^{(i)} &= \mathring{\mathcal{P}}_{\mathcal{I}^-}^{(i)} - 4\mathring{\mathcal{P}}_2^* \mathring{\mathcal{P}}_1^*(0, \mathring{\text{curl}} \beta_{\mathcal{I}^-}^{(i)}) = -2(\mathring{\Delta} - 4) \mathring{\mathcal{P}}_2^* \beta_{\mathcal{I}^-}^{(i)} - 4\mathring{\mathcal{P}}_2^* \mathring{\mathcal{P}}_1^*(0, \mathring{\text{curl}} \beta_{\mathcal{I}^-}^{(i)}) - 6M \alpha_{\mathcal{I}^-}^{(i)} \\ &= 2\mathring{\mathcal{P}}_2^* \mathring{\mathcal{P}}_1^* \left(\mathring{\text{div}} \beta_{\mathcal{I}^-}^{(i)}, -\mathring{\text{curl}} \beta_{\mathcal{I}^-}^{(i)} \right) + 6M \nabla_v \hat{\chi}_{\mathcal{I}^-}^{(i)}, \end{aligned} \quad (6.7.64)$$

where we also used (6.2.29). Acting now on this with $\mathring{\text{curl}} \mathring{\mathcal{P}}_2$, we obtain

$$\lim_{u \rightarrow -\infty} 2\mathring{\text{curl}} \mathring{\mathcal{P}}_2 \mathring{\mathcal{P}}_2^* \nabla_v \frac{r^2}{\Omega^2} \nabla_v r^2 \Omega \underline{\beta}^{(i)} = -(\mathring{\Delta} + 2)(\mathring{\Delta}) \mathring{\text{curl}} \beta_{\mathcal{I}^-}^{(i)}. \quad (6.7.65)$$

The result then follows using (6.7.55) (and that we are supported on $\ell \geq 2$). $\square$

Using this sublemma, we can now integrate (6.7.60) from $\mathcal{I}^-$ to deduce that

$$\partial_v \left[\frac{r^2}{\Omega^2} \partial_v \mathring{\text{curl}} r^2 \Omega \underline{\beta}^{(i)} - \mathring{\Delta} r^3 \sigma^{(i)'} - 6M r^2 \sigma^{(i)'} \right] = 0. \quad (6.7.66)$$

In turn, integrating this from $\underline{\mathcal{C}}$, and verifying that $\partial_v \mathring{\text{curl}} r^2 \Omega^{(\iota)} \underline{\beta} - \mathring{\Delta} r \Omega^2 \sigma^{(\iota)} - 6M \Omega^2 \sigma^{(\iota)} = 0$ at $\underline{\mathcal{C}}$ by computing the ∇_v derivative of $\underline{\beta}$ in spacetime via (6.3.27), restricting to $\underline{\mathcal{C}}$ and using Proposition 6.5.1, we conclude the proof of (6.7.58).

Acting now with (6.7.58) on (6.7.51), we infer that $\sigma^{(\iota)}$ satisfies (RW-scalar). $\square$

Corollary 6.7.8. *With $\sigma^{(\iota)}$ defined in Def. 6.7.9 and σ defined in Def. 6.7.5, we have $\sigma^{(\iota)} = \sigma$.*

Proof. Since $\sigma^{(\iota)} = \sigma$ on $\underline{\mathcal{C}}$ and $\lim_{u \rightarrow -\infty} \partial_v r^3 \sigma^{(\iota)} = \lim_{u \rightarrow -\infty} \partial_v r^3 \sigma$ uniformly in v , the uniqueness clause of Theorem 6.6.1 implies $\sigma^{(\iota)} = \sigma$ by virtue of both σ and $\sigma^{(\iota)}$ satisfying (RW-scalar). $\square$

Step (IX): Constructing $\overset{(\iota)}{\eta}$, $\overset{(\iota)}{\underline{\eta}}$ as well as $\overset{(\iota)}{\Omega}$, $\overset{(\iota)}{\omega}$ and $\overset{(\iota)}{\underline{\omega}}$

Definition 6.7.10. *Recall that the kernel of $\mathring{\mathcal{P}}_2^*$ is spanned by $\ell = 0, 1$. Define $\overset{(\iota)}{\eta}$, $\overset{(\iota)}{\underline{\eta}}$ to be the unique solutions to (6.3.17a) and (6.3.18b).*

$$-2\mathring{\mathcal{P}}_2^* \Omega^2 \overset{(\iota)}{\eta} = \nabla_u r \Omega \overset{(\iota)}{\hat{\chi}} + \Omega^3 \overset{(\iota)}{\hat{\chi}}, \quad -2\mathring{\mathcal{P}}_2^* \Omega^2 \overset{(\iota)}{\underline{\eta}} = \nabla_v r \Omega \overset{(\iota)}{\hat{\chi}} - \Omega^3 \overset{(\iota)}{\hat{\chi}}. \quad (6.7.67)$$

Corollary 6.7.9. *The one-forms $\overset{(\iota)}{\eta}$ and $\overset{(\iota)}{\underline{\eta}}$ satisfy (6.3.20a), (6.3.19b) and (6.3.24). Moreover, we have $\overset{(\iota)}{\eta}|_{\underline{\mathcal{C}}} = \overset{(\iota)}{\eta}|_{\underline{\mathcal{C}}}$, $\overset{(\iota)}{\underline{\eta}}|_{\underline{\mathcal{C}}} = \overset{(\iota)}{\underline{\eta}}|_{\underline{\mathcal{C}}}$ as well as*

$$\lim_{u \rightarrow -\infty} r \overset{(\iota)}{\eta} = 0, \quad \lim_{u \rightarrow -\infty} r \overset{(\iota)}{\underline{\eta}} = \overset{(\iota)}{\underline{\eta}}_{\mathcal{I}^-}, \quad (6.7.68)$$

where $\overset{(\iota)}{\underline{\eta}}_{\mathcal{I}^-}$ is defined in Def. 6.7.2. This convergence is uniform on compact v -intervals and commutes with $\mathring{\nabla}$ and ∇_v -derivatives.

Proof. We prove (6.3.19b) by multiplying the first of (6.7.67) (i.e. (6.3.17a)) with $\Omega^{-2}r$ and then acting with ∇_v , using (6.3.17b), (6.3.32) as well as the second of (6.7.67) (i.e. (6.3.18b)).

The proof of (6.3.20a) is analogous.

In order to prove (6.3.24), we take the $\mathring{\text{curl}} \mathring{\text{div}}$ of (6.3.17a). From there, using (6.7.32), (6.7.50) and Corollary 6.7.8, we get

$$\partial_u \left(r^2 \Omega^2 \sigma^{(\iota)} - \mathring{\text{curl}} r^2 \Omega \overset{(\iota)}{\beta} \right) = \left(\mathring{\Delta} + 2 \right) \mathring{\text{curl}} \Omega^2 \overset{(\iota)}{\eta} - \frac{\Omega^2}{r} \left(r^2 \Omega \sigma^{(\iota)} + \mathring{\text{curl}} r^2 \Omega \overset{(\iota)}{\beta} \right). \quad (6.7.69)$$

We then utilise (6.3.30a) and (6.7.33) to obtain

$$\left(\mathring{\Delta} + 2 \right) \mathring{\text{curl}} r \overset{(\iota)}{\eta} = \left(\mathring{\Delta} + 2 \right) r^2 \sigma^{(\iota)}, \quad (6.7.70)$$

which implies $\mathring{\text{curl}} r \overset{(\iota)}{\eta} = r \sigma^{(\iota)}$ since $\mathring{\Delta} + 2$ is invertible on $\ell \geq 2$.

The claim for $\mathring{\text{curl}} \overset{(\iota)}{\underline{\eta}}$ follows in the same way, using equations (6.3.18b), (6.7.50), (6.3.30b), Lemma 6.7.8 and Corollary 6.7.8.

The claim that $\overset{(i)}{\eta}$ restricts to $\overset{(i)}{\eta}_{\underline{\mathcal{C}}}$ follows from the definition, the analogous statements for $\overset{(i)}{\chi}$ and $\overset{(i)}{\underline{\chi}}$ as well the fact that the first of (6.7.67) is satisfied along $\underline{\mathcal{C}}$ by Prop. 6.5.1. Furthermore, it is easy to see that $r\overset{(i)}{\eta}$ tends to 0 as $u \rightarrow -\infty$.

The uniform convergence of $r\overset{(i)}{\underline{\eta}}$ follows from its definition and the uniform convergence of $\overset{(i)}{\underline{S}}$ and $\overset{(i)}{\chi}$ to $\overset{(i)}{S}_{\mathcal{I}^-}$ and $\overset{(i)}{\chi}_{\mathcal{I}^-}$, respectively. By integrating (6.3.20a), we then infer that $\overset{(i)}{\eta}|_{\underline{\mathcal{C}}} = \overset{(i)}{\eta}_{\underline{\mathcal{C}}}$. $\square$

In particular, since we have now shown that $\overset{\circ}{\text{curl}}(\overset{(i)}{\eta} + \overset{(i)}{\underline{\eta}}) = 0$, the following definition is well-defined:

Definition 6.7.11. Define $\overset{(i)}{\Omega}$ via $2\overset{\circ}{\nabla}\left(\frac{\overset{(i)}{\Omega}}{\overset{(i)}{\Omega}}\right) = r(\overset{(i)}{\eta} + \overset{(i)}{\underline{\eta}})$, and define $\overset{(i)}{\omega} = \partial_v\left(\frac{\overset{(i)}{\Omega}}{\overset{(i)}{\Omega}}\right)$, $\overset{(i)}{\underline{\omega}} = \partial_u\left(\frac{\overset{(i)}{\Omega}}{\overset{(i)}{\Omega}}\right)$.

Corollary 6.7.10. Equations (6.3.19a) and (6.3.20b) are satisfied. Moreover, $\lim_{u \rightarrow -\infty} \overset{(i)}{\Omega} = \overset{(i)}{\Omega}_{\mathcal{I}^-}$ (this convergence being uniform on compact v -intervals and commuting with $\overset{\circ}{\nabla}^n$ and $\overset{\circ}{\nabla}_v^m$), and $\overset{(i)}{\underline{\omega}}|_{\underline{\mathcal{C}}} = \overset{(i)}{\underline{\omega}}_{\underline{\mathcal{C}}}$.

Proof. Equation (6.3.19a) follows directly from (6.3.20a), and the definition of $\overset{(i)}{\Omega}$ and $\overset{(i)}{\underline{\omega}}$. Equation (6.3.20b) follows similarly. The convergence of $\overset{\circ}{\nabla}\overset{(i)}{\Omega}$ to $\overset{\circ}{\nabla}\overset{(i)}{\Omega}_{\mathcal{I}^-}$ follows from the limiting behaviour of $\overset{(i)}{\eta}$ and $\overset{(i)}{\underline{\eta}}$, and the convergence of $\overset{(i)}{\Omega}$ to $\overset{(i)}{\Omega}_{\mathcal{I}^-}$ then follows straight-forwardly (using that we are supported on $\ell \geq 2$). The restriction of $\overset{(i)}{\underline{\omega}}$ to $\underline{\mathcal{C}}$ follows from $\overset{(i)}{\Omega}$ restricting to $\overset{(i)}{\Omega}_{\underline{\mathcal{C}}}$ by definition and Prop. 6.5.1. $\square$

Step (X): Constructing $(\overset{(i)}{\Omega\text{tr}\chi})$ and $(\overset{(i)}{\Omega\text{tr}\underline{\chi}})$.

Definition 6.7.12. We define $(\overset{(i)}{\Omega\text{tr}\chi})$ and $(\overset{(i)}{\Omega\text{tr}\underline{\chi}})$ according to (6.3.22) and (6.3.23), i.e.

$$\frac{1}{2\Omega}\overset{\circ}{\nabla}(\overset{(i)}{\Omega\text{tr}\chi}) = \overset{\circ}{\text{div}}\overset{(i)}{\chi} + \Omega\overset{(i)}{\underline{\eta}} + r\overset{(i)}{\beta}, \quad \frac{1}{2\Omega}\overset{\circ}{\nabla}(\overset{(i)}{\Omega\text{tr}\underline{\chi}}) = \overset{\circ}{\text{div}}\overset{(i)}{\underline{\chi}} - \Omega\overset{(i)}{\eta} + r\overset{(i)}{\beta}. \quad (6.7.71)$$

By construction and Prop. 6.5.1, $(\overset{(i)}{\Omega\text{tr}\chi})$ and $(\overset{(i)}{\Omega\text{tr}\underline{\chi}})$ restrict to $(\overset{(i)}{\Omega\text{tr}\chi})_{\underline{\mathcal{C}}}$ and $(\overset{(i)}{\Omega\text{tr}\underline{\chi}})_{\underline{\mathcal{C}}}$ along $\underline{\mathcal{C}}$, respectively.

Remark 6.7.3. These definitions are well-defined since we already know that the $\overset{\circ}{\text{curl}}$ of the respective RHS's vanish by (6.7.32), (6.7.50) and $\overset{\circ}{\text{curl}}\overset{(i)}{\eta} = r\overset{(i)}{\sigma} = r\overset{(i)}{\sigma}' = -\overset{\circ}{\text{curl}}\overset{(i)}{\underline{\eta}}$.

Corollary 6.7.11. The quantities $(\overset{(i)}{\Omega\text{tr}\chi})$ and $(\overset{(i)}{\Omega\text{tr}\underline{\chi}})$ defined above satisfy equations (6.3.16), and $\lim_{u \rightarrow -\infty} r(\overset{(i)}{\Omega\text{tr}\underline{\chi}}) = 0$, $\lim_{u \rightarrow -\infty} r(\overset{(i)}{\Omega\text{tr}\chi}) = (\overset{(i)}{\Omega\text{tr}\chi})_{\mathcal{I}^-}$, these convergences being uniform on compact v -intervals and commuting with $\overset{\circ}{\nabla}_v$ - and $\overset{\circ}{\nabla}$ -derivatives.

Proof. Equation (6.3.16a) follows by acting on the first of (6.7.71) with $\overset{\circ}{\nabla}_u(\frac{r^2}{\Omega}\cdot)$ and using (6.3.17a), (6.3.20a) as well as (6.3.31a). Equation (6.3.16b) follows analogously.

The limiting behaviour follows from the definitions and the previous estimates. $\square$

Step (XI): Constructing $\overset{(i)}{\rho}$ and proving all equations featuring $\overset{(i)}{\rho}$

Before we can define $\overset{(i)}{\rho}$, we need to prove the following

Lemma 6.7.9. *We have*

$$\mathring{\text{cyl}}\mathring{\text{d}}\mathring{\text{v}}\overset{(i)}{\Psi} = 2\mathring{\text{cyl}}\mathring{\text{d}}\mathring{\text{v}}\mathring{\mathcal{P}}_2^*\mathring{\mathcal{P}}_1^*(0, -r^3\overset{(i)}{\sigma}) + 6M\mathring{\text{cyl}}\mathring{\text{d}}\mathring{\text{v}}(r\Omega\overset{(i)}{\hat{\chi}} - r\Omega\overset{(i)}{\underline{\hat{\chi}}}). \quad (6.7.72)$$

Proof. First, note that this equation holds along $\mathcal{C}$ by construction (cf. Proposition 6.5.1 and Corollary 6.5.2). Next, we confirm that $\nabla_v(6.7.72)$ holds by computing the ∇_v -derivative of the LHS (6.7.72) using the definition of $\overset{(i)}{\Psi}$ and the Teukolsky equation (Teuk) (the result is (6.3.39)), and using (6.3.30b), (6.3.17b) and (6.3.18b) for the RHS. $\square$

The above lemma enables the following definition:

Definition 6.7.13. *Define $\overset{(i)}{\rho}$ to be the unique solution to*

$$\overset{(i)}{\Psi} = 2\mathring{\mathcal{P}}_2^*\mathring{\mathcal{P}}_1^*(r^3\overset{(i)}{\rho}, -r^3\overset{(i)}{\sigma}) + 6M(r\Omega\overset{(i)}{\hat{\chi}} - r\Omega\overset{(i)}{\underline{\hat{\chi}}}). \quad (6.7.73)$$

By construction and by Prop. 6.5.1, $\overset{(i)}{\rho}|_{\mathcal{C}} = \overset{(i)}{\rho}_{\mathcal{C}}$.

Remark 6.7.4. Definition 6.7.13 implies, via (6.3.38):

$$\underline{\overset{(i)}{\Psi}} = 2\mathring{\mathcal{P}}_2^*\mathring{\mathcal{P}}_1^*(r^3\overset{(i)}{\rho}, r^3\overset{(i)}{\sigma}) + 6M(r\Omega\overset{(i)}{\hat{\chi}} - r\Omega\overset{(i)}{\underline{\hat{\chi}}}). \quad (6.7.74)$$

Lemma 6.7.10. *The quantity $\overset{(i)}{\rho}$ defined above satisfies equations (6.3.31a), (6.3.28b) as well as (6.3.29).*

Proof. We first prove (6.3.31a): We take the definition (6.7.73), and rewrite its LHS as $\overset{(i)}{\Psi} = (\frac{r^2}{\Omega^2}\nabla_u)^2(r\Omega^2\overset{(i)}{\alpha})$. We now rewrite this as the $\frac{r^2}{\Omega^2}\nabla_u$ -derivative of the RHS of (6.3.33). Equation $\mathring{\mathcal{P}}_2^*(6.3.31a)$ then follows by taking into account the already established equation (6.3.17a), and the result follows from the invertibility of $\mathring{\mathcal{P}}_2^*$.

Equation (6.3.28b) follows similarly, starting from (6.7.74).

Next, we prove (6.3.29b). Using (6.3.31b) and (6.3.32), we compute:

$$\nabla_v \frac{r^2}{\Omega^2} \nabla_u r^2 \Omega \overset{(i)}{\beta} = -2\mathring{\mathcal{P}}_2^*\mathring{\mathcal{P}}_2^* r^2 \Omega \overset{(i)}{\beta} + 6M\mathring{\text{d}}\mathring{\text{v}}\Omega\overset{(i)}{\hat{\chi}} - 2(3\Omega^2 - 2)r^2 \Omega \overset{(i)}{\beta}. \quad (6.7.75)$$

Therefore, by (6.3.31a) and (6.3.19b) we have

$$\mathring{\mathcal{P}}_1^*(-\partial_v r^3 \overset{(i)}{\rho}, \partial_v r^3 \overset{(i)}{\sigma}) = \left(\mathring{\Delta} + 1\right) r^2 \Omega \overset{(i)}{\beta} + 6M\mathring{\text{d}}\mathring{\text{v}}\Omega\overset{(i)}{\hat{\chi}} - 2(3\Omega^2 - 2)r^2 \Omega \overset{(i)}{\beta} + 6M(r\Omega\overset{(i)}{\hat{\beta}} - \Omega^2\overset{(i)}{\underline{\eta}}). \quad (6.7.76)$$

Taking the divergence of both sides and using $\mathring{\mathcal{D}}\text{iv} \left(\mathring{\Delta} + 1 \right) \beta = \left(\mathring{\Delta} + 2 \right) \mathring{\mathcal{D}}\text{iv} \beta$ and equation (6.3.22), we get

$$\mathring{\Delta} \partial_v r^3 \rho^{(i)} = \mathring{\Delta} \mathring{\mathcal{D}}\text{iv} r^2 \Omega \beta^{(i)} + 3M \mathring{\Delta} (\Omega \text{tr} \chi)^{(i)}. \quad (6.7.77)$$

This shows that (6.3.29b) is satisfied by the invertibility of $\mathring{\Delta}$ on $\ell \geq 2$.

The argument to prove (6.3.29a) is analogous. $\square$

Lemma 6.7.11. *The equations (6.3.14) and (6.3.15) are satisfied by $(\Omega \text{tr} \chi)^{(i)}$ and $(\Omega \text{tr} \underline{\chi})^{(i)}$.*

Proof. Let's first prove (6.3.14): We differentiate the definition of $(\Omega \text{tr} \chi)^{(i)}$, (6.3.22):

$$\begin{aligned} \frac{1}{2} \mathring{\nabla} \mathring{\nabla}_u (r (\Omega \text{tr} \chi)^{(i)}) &= \mathring{\nabla}_u (\mathring{\mathcal{D}}\text{iv} r \Omega \underline{\chi}^{(i)}) + \mathring{\nabla}_u (r \Omega^2 \underline{\eta}^{(i)}) + \mathring{\nabla}_u (r^2 \Omega \beta^{(i)}) \\ &= -2\Omega^2 \mathring{\mathcal{D}}_2 \mathring{\mathcal{D}}_2^* \eta^{(i)} - \Omega^3 \mathring{\mathcal{D}}\text{iv} \underline{\chi}^{(i)} + \mathring{\nabla}_u (\Omega^2) r \underline{\eta}^{(i)} + r \Omega^3 \underline{\beta}^{(i)} - \Omega^4 \eta^{(i)} + \mathring{\mathcal{D}}_1^* (-r \Omega^2 \rho^{(i)}, r \Omega^2 \sigma^{(i)}) - \frac{6M\Omega^2}{r} \eta^{(i)}, \end{aligned} \quad (6.7.78)$$

where we used (6.3.17a), (6.3.20a) and (6.3.31a). We now use the fact that

$$-2\Omega^2 \mathring{\mathcal{D}}_2 \mathring{\mathcal{D}}_2^* \eta^{(i)} = -\Omega^2 \mathring{\mathcal{D}}_1^* \mathring{\mathcal{D}}_1 \eta^{(i)} + 2\Omega^2 \eta^{(i)} = \Omega^2 (\mathring{\nabla} \mathring{\mathcal{D}}\text{iv} \eta^{(i)} - * \mathring{\nabla} \text{curl} \eta^{(i)} + 2\eta^{(i)}) = \Omega^2 (\mathring{\nabla} \mathring{\mathcal{D}}\text{iv} \eta^{(i)} - * \mathring{\nabla} r \sigma^{(i)} + 2\eta^{(i)}),$$

as well as equations (6.3.12) and (6.3.23) to conclude that $\mathring{\nabla}$ (6.3.14) holds. Eq. (6.3.15) follows similarly. $\square$

Lemma 6.7.12. *The equations (6.3.21) are satisfied by $\omega^{(i)}$ and $\underline{\omega}^{(i)}$, respectively.*

Proof. We compute $\mathring{\nabla}_u \mathring{\nabla} \omega^{(i)} = \mathring{\nabla}_u \mathring{\nabla}_v \mathring{\nabla} \left(\frac{\Omega}{\Omega} \right)^{(i)} = \mathring{\nabla}_v \mathring{\nabla} \omega^{(i)}$ by using (6.3.12), (6.3.20a), (6.3.19b) to compute $\mathring{\nabla} \partial_v \left(\frac{\Omega}{\Omega} \right)^{(i)}$, and then using (6.3.28b), (6.3.31a), (6.3.20a) and (6.3.19b) to compute $\mathring{\nabla}_u \mathring{\nabla}_v \mathring{\nabla} \left(\frac{\Omega}{\Omega} \right)^{(i)}$. $\square$

Step (XII): Constructing the remaining metric components $\hat{\mathcal{G}}^{(i)}$, $\text{tr} \hat{\mathcal{G}}^{(i)}$ and $\hat{b}^{(i)}$

Definition 6.7.14. *Define $\hat{b}^{(i)}$ to be the solution to $\mathring{\nabla}_u (r^{-1} \hat{b}^{(i)}) = 2r^{-1} \Omega^2 (\eta^{(i)} - \underline{\eta}^{(i)})$ (i.e. (6.3.13)) with data $\mathring{\mathbf{b}}_{\mathcal{I}^-}^{(i)}$ for $r^{-1} \hat{b}^{(i)}$ at $\mathcal{I}^-$.*

Definition 6.7.15. *Define $\text{tr} \hat{\mathcal{G}}^{(i)}$ and $\hat{\mathcal{G}}^{(i)}$ as solutions to (6.3.9b) and (6.3.10b) with $\text{tr} \hat{\mathcal{G}}_{\mathcal{C}}^{(i)}$ and $\hat{\mathcal{G}}_{\mathcal{C}}^{(i)}$ as data along $\mathcal{C}$.*

Lemma 6.7.13. *The equations (6.3.9a) and (6.3.10a) are satisfied by $\text{tr} \hat{\mathcal{G}}^{(i)}$ and $\hat{\mathcal{G}}^{(i)}$.*

Proof. We present the proof for (6.3.9a); the proof for (6.3.10a) is similar. Since (6.3.9a) is satisfied along $\mathcal{C}$, it suffices to prove that ∂_v (6.3.9a) holds. We prove the latter by computing

$\partial_v \partial_u \text{tr} \overset{(1)}{g} = \partial_u \partial_v \text{tr} \overset{(1)}{g}$ via (6.3.14) and (6.3.13), and by computing $\partial_v(2(\Omega \text{tr} \overset{(1)}{\chi}))$ via (6.3.15). The result then follows. $\square$

Corollary 6.7.12. *The following convergence is uniform in v on compact v -intervals:*

$$\lim_{u \rightarrow -\infty} \text{tr} \overset{(1)}{g} = \text{tr} \overset{(1)}{g}_{\mathcal{I}^-}, \quad \lim_{u \rightarrow -\infty} \overset{(1)}{g} = \overset{(1)}{g}_{\mathcal{I}^-}, \quad \lim_{u \rightarrow -\infty} r^2 \overset{(1)}{K} = \overset{(1)}{K}_{\mathcal{I}^-}. \quad (6.7.79)$$

These limits commute with $\overset{\circ}{\nabla}_v$ - and $\overset{\circ}{\nabla}$ -derivatives.

From the definition of b and the decay rates of $\overset{(1)}{\eta}$ and $\overset{(1)}{\underline{\eta}}$, it is easy to see that $r^{-1}b$ converges uniformly in v to $\overset{(1)}{b}$ as $u \rightarrow -\infty$, and it thus follows that $\overset{(1)}{g}$, $\text{tr} \overset{(1)}{g}$ as well as $r^2 \overset{(1)}{K}$ tend to $\overset{(1)}{g}_{\mathcal{I}^-}$, $\text{tr} \overset{(1)}{g}_{\mathcal{I}^-}$ and $\overset{(1)}{K}_{\mathcal{I}^-}$, respectively.

The final equation of (6.3.9)–(6.3.32) that we have to prove is the Gauss equation:

Lemma 6.7.14. *The Gauss equation (6.3.25) is satisfied, with $\overset{(1)}{K}$ defined in (6.3.26).*

Proof. We compute the ∂_v -derivative of the Gauss equation using (6.3.9b), (6.3.10b), (6.3.29b), (6.3.16b), (6.3.15) and (6.3.11b). This shows that $\partial_v(6.3.25)$ is satisfied. The result now follows, as (6.3.25) is satisfied along $\mathcal{C}$ by Prop 6.5.1. $\square$

We have now constructed a solution to (6.3.9)–(6.3.32), and we have shown that it satisfies Definition 6.5.2. This concludes the proof of Prop. 6.7.2. $\square$

6.7.3 Construction of the $\ell < 2$ -part of the solution to the scattering problem

We now present the construction of the $\ell < 2$ -part of the solution: This construction will be entirely explicit:

Proposition 6.7.4. *Given a smooth seed scattering data set $\mathfrak{D}_{\ell=0}$ supported on $\ell = 0$, let $\overset{(1)}{\rho}_{\mathcal{C}}$, $\overset{(1)}{K}_{\mathcal{S}_{\infty}}$ be as in Prop. 6.5.1 (see (6.5.6)), and let $\overset{(1)}{\rho}_{\mathcal{S}_{\infty}}$ be as in Cor. 6.5.3. Then $\mathfrak{S}_{\mathfrak{m}} + \mathfrak{S}_f + \mathfrak{S}_{\underline{f}}$ is a scattering solution realising $\mathfrak{D}_{\ell=0}$, with*

$$\begin{aligned} \mathfrak{m} &= \frac{\overset{(1)}{\rho}_{\mathcal{S}_{\infty}}}{M} + 3\overset{(1)}{K}_{\mathcal{S}_{\infty}} \\ \underline{f} &= \frac{r}{6M\Omega^2} (-r^3 \overset{(1)}{\rho}_{\mathcal{C}} - 2\overset{(1)}{\rho}_{\mathcal{S}_{\infty}} + 6M\overset{(1)}{K}_{\mathcal{S}_{\infty}}) \\ f &= \int_{v_1}^v 2(\overset{(1)}{\Omega}_{\mathcal{I}^-} + \overset{(1)}{K}_{\mathcal{S}_{\infty}} + \frac{\overset{(1)}{\rho}_{\mathcal{S}_{\infty}}}{4M}) dv'. \end{aligned} \quad (6.7.80)$$

The decomposition $\mathfrak{S}_{\mathfrak{m}} + \mathfrak{S}_f + \mathfrak{S}_{\underline{f}}$ is unique up to the ambiguity addressed in Remark 6.4.1.

Proof. It is easy to directly verify that this is a scattering solution realising $\mathfrak{D}_{\ell=0}$. Still, we find it insightful to provide the construction: We begin by realising that only the ingoing gauge

solution $\mathfrak{S}_{\underline{f}}$ and the nearby Schwarzschild solution $\mathfrak{S}_{\mathfrak{m}}$ have non-trivial limits for $r^3 \rho^{(1)}|_{\underline{\mathcal{C}}}$ and $r^2 K^{(1)}|_{\underline{\mathcal{C}}}$. If we write $\lim_{u \rightarrow -\infty} r^{-1} \underline{f} = \underline{f}_0$, we then find the system of equations

$$\rho_{\mathcal{S}_\infty}^{(1)} = -6M \underline{f}_0 - 2M \mathfrak{m}, \quad K_{\mathcal{S}_\infty}^{(1)} = 2 \underline{f}_0 + \mathfrak{m}, \quad (6.7.81)$$

which is solved by $\mathfrak{m} = \frac{\rho_{\mathcal{S}_\infty}^{(1)}}{M} + 3K_{\mathcal{S}_\infty}^{(1)}, \underline{f}_0 = -K_{\mathcal{S}_\infty}^{(1)} - \frac{\rho_{\mathcal{S}_\infty}^{(1)}}{2M}$. We have thus already determined $\mathfrak{S}_{\mathfrak{m}}$.

Next, we fully fix $\underline{f}$ by demanding that

$$\rho_{\underline{\mathcal{C}}}^{(1)} = -\frac{6M\Omega^2}{r^4} \underline{f} - \frac{2M\mathfrak{m}}{r^3} = -\frac{6M\Omega^2}{r^4} \underline{f} - \frac{2\rho_{\mathcal{S}_\infty}^{(1)}}{r^3} - \frac{6MK_{\mathcal{S}_\infty}^{(1)}}{r^3}. \quad (6.7.82)$$

Finally, we fix f by noticing that both $\mathfrak{m}$ and $\underline{f}$ generate a limit of Ω at $\mathcal{I}^-$, by demanding that $f|_{\underline{\mathcal{C}}} = 0$ and by

$$\Omega_{\mathcal{I}^-}^{(1)} = \frac{1}{2} \partial_v f + \frac{1}{2} \left(K_{\mathcal{S}_\infty}^{(1)} + \frac{\rho_{\mathcal{S}_\infty}^{(1)}}{2M} \right) - \frac{\rho_{\mathcal{S}_\infty}^{(1)}}{2M} - \frac{3K_{\mathcal{S}_\infty}^{(1)}}{2}. \quad (6.7.83)$$

It is left to show that this is indeed a scattering solution realising $\mathfrak{D}_{\ell=0}$: By Corollary 6.5.3, it suffices to show that the constructed solution realises $\rho_{\mathcal{S}_\infty}^{(1)}, \text{tr} \mathfrak{g}_{\mathcal{S}_\infty}^{(1)}, \Omega_{\mathcal{I}^-}^{(1)}, \underline{\omega}_{\underline{\mathcal{C}}}^{(1)}$ and $(\Omega \text{tr} \chi)_{\mathcal{S}_1}^{(1)}$: Now, by construction, the solution $\mathfrak{S}_{\mathfrak{m}} + \mathfrak{S}_f + \mathfrak{S}_{\underline{f}}$ realises $\rho_{\mathcal{S}_\infty}^{(1)}, K_{\mathcal{S}_\infty}^{(1)} = -\frac{1}{2} \text{tr} \mathfrak{g}_{\mathcal{S}_\infty}^{(1)}$ as well as $\Omega_{\mathcal{I}^-}^{(1)}$. Moreover, since it realises $\rho_{\underline{\mathcal{C}}}^{(1)}$, we can deduce from (6.3.29a) that it also realises $(\Omega \text{tr} \chi)_{\underline{\mathcal{C}}}^{(1)}$ (and thus $(\Omega \text{tr} \chi)_{\mathcal{S}_1}^{(1)}$) by Prop. 6.5.1. Finally, we deduce from (6.3.16a) that $\mathfrak{S}_{\mathfrak{m}} + \mathfrak{S}_f + \mathfrak{S}_{\underline{f}}$ also realises $\underline{\omega}_{\underline{\mathcal{C}}}^{(1)}$. $\square$

Proposition 6.7.5. *Given a smooth seed scattering data set $\mathfrak{D}_{\ell=1}$ supported on $\ell = 1$, let $\rho_{\underline{\mathcal{C}}}^{(1)}, K_{\mathcal{S}_\infty}^{(1)}$ be as in Prop. 6.5.1, and let $\rho_{\mathcal{S}_\infty}^{(1)}, \underline{\beta}_{\mathcal{S}_\infty}^{(1)}$ be as in Cor. 6.5.3. Then $\mathfrak{S}_{\mathfrak{a}} + \mathfrak{S}_f + \mathfrak{S}_{\underline{f}} + \mathfrak{S}_{(q_1, q_2)}$ is a scattering solution realising $\mathfrak{D}_{\ell=1}$ for*

$$\begin{aligned} \mathfrak{a}_m &= -3M\sqrt{2}(\underline{\beta}_{\mathcal{S}_\infty}^{(1)})_{\ell=1, m}^{\text{H},1}, \\ (q_1, q_2)_{\ell=1, m} &= \left(\frac{1}{4} \left(\frac{3}{2M} \rho_{\mathcal{S}_\infty}^{(1)} - \text{tr} \mathfrak{g}_{\mathcal{S}_\infty}^{(1)} \right), 0 \right)_{\ell=1, m} + \int_{v_1}^v \sqrt{2} ((\mathfrak{b}_{\mathcal{I}^-})_{1, m}^{\text{E},1}, (\mathfrak{b}_{\mathcal{I}^-})_{1, m}^{\text{H},1}) dv', \\ f_{\ell=1, m} &= -\frac{1}{6M\sqrt{2}} (\underline{\beta}_{\mathcal{S}_\infty}^{(1)})_{\ell=1, m}^{\text{E},1} + \int_{v_1}^v 2 \left(\Omega_{\mathcal{I}^-}^{(1)} - \frac{\rho_{\mathcal{S}_\infty}^{(1)}}{6M} \right)_{\ell=1, m} dv', \\ \underline{f}_{\ell=1, m} &= -\frac{r^4}{6M\Omega^2} (\rho_{\mathcal{S}_\infty}^{(1)})_{\ell=1, m} + \frac{1}{\sqrt{2} \cdot 6M} (\underline{\beta}_{\mathcal{S}_\infty}^{(1)})_{\ell=1, m}^{\text{E},1}. \end{aligned} \quad (6.7.84)$$

Proof. We uniquely determine $\mathfrak{a}$ directly from Prop. 6.4.2 and $\underline{\beta}_{\mathcal{S}_\infty}^{(1)}$. (Note that none of the pure gauge solutions affect the magnetic part of $\underline{\beta}$.)

Now, we choose a preliminary definition of $\underline{f}_p$ by demanding that $\mathfrak{S}_{\underline{f}_p}$ restricts to $\rho_{\underline{c}}^{(i)}$:

$$\underline{f}_p = -\frac{r^4}{6M\Omega^2}\rho_{\underline{c}}^{(i)} + \tilde{f}(\theta^A). \quad (6.7.85)$$

We will determine $\tilde{f}(\theta^A)$ later. This choice for the ingoing gauge function induces a limit for $\text{tr}\mathring{g}^{(i)}$ at $\mathcal{I}^-$. This limit is independent of $\tilde{f}(\theta^A)$. Since the outgoing gauge function always induces a vanishing limit for $\text{tr}\mathring{g}^{(i)}$ at $\mathcal{I}^-$, we can thus fix $q_1(v = v_1, \theta^A)$ by demanding that $\mathfrak{S}_{\underline{f}_p} + \mathfrak{S}_{(q_1, q_2)}$ restrict to $\text{tr}\mathring{g}_{\mathcal{S}_\infty}^{(i)}$:

$$\text{tr}\mathring{g}_{\mathcal{S}_\infty}^{(i)} = \frac{3}{2M}\rho_{\mathcal{S}_\infty}^{(i)} - 4q_1(v = v_1, \theta^A). \quad (6.7.86)$$

We now fully fix (q_1, q_2) by demanding that $q_2(v = v_1, \theta_A) = 0$ and that $\mathring{\mathcal{P}}_1^*(\partial_v q_1, \partial_v q_2) = \mathfrak{b}_{\mathcal{I}^-}^{(i)}$.

Next, we fix $\partial_v f$ by demanding that the limit $\Omega_{\mathcal{I}^-}^{(i)}$ is met:

$$\frac{\partial_v f}{2} = \Omega_{\mathcal{I}^-}^{(i)} - \frac{\rho_{\mathcal{S}_\infty}^{(i)}}{12M}, \quad (6.7.87)$$

where the second term comes from $\underline{f}_p$. We fully fix f by demanding that the limit $\underline{\beta}_{\mathcal{S}_\infty}^{(i)}$ is attained, i.e.:

$$f_{\ell=1,m}(v = v_1) = -\frac{1}{6M\sqrt{2}}(\underline{\beta}_{\mathcal{S}_\infty}^{(i)})_{\ell=1,m}^{\text{E},1}. \quad (6.7.88)$$

Finally, we fix $\tilde{f}$ by demanding that $\rho_{\underline{c}}^{(i)}$ is attained.

It is left to show that the constructed solution realises $\mathfrak{D}_{\ell=1}$. By construction, $\mathfrak{S}_a + \mathfrak{S}_f + \mathfrak{S}_{\underline{f}} + \mathfrak{S}_{(q_1, q_2)}$ realises $\Omega_{\mathcal{I}^-}^{(i)}$ and $\mathfrak{b}_{\mathcal{I}^-}^{(i)}$ as limits at $\mathcal{I}^-$, and it realises $\underline{\beta}_{\mathcal{S}_\infty}^{(i)}$, $\text{tr}\mathring{g}_{\mathcal{S}_\infty}^{(i)}$ as well as $\rho_{\underline{c}}^{(i)}$. Moreover, in view of $\nabla_u \frac{r^4 \underline{\beta}_{\ell=1}^{(i)}}{\Omega} = 0$, it follows that $\underline{\omega}_{\underline{c}}^{(i)}$ and $(\Omega \text{tr}\underline{\chi})_{\mathcal{S}_1}^{(i)}$ are also attained by (6.3.29a) and (6.3.16a). The result now follows from Corollary 6.5.3 (as in the case $\ell = 0$). $\square$

6.8 The physical scattering data

The previous sections featured a general discussion of how to set up and solve the scattering problem for a general class of seed scattering data.

In the present section, we specify this discussion to physically motivated seed data. Let us briefly recall from chapter 5 that these physically motivated seed data ought to describe the correct scattering setup describing the far-field region of a system of N infalling masses which follow approximately hyperbolic Keplerian orbits in the infinite past and whose near field is well described by Post-Newtonian theory. We will first define the condition of no incoming radiation, then give and physically justify a definition of seed scattering data describing the far-region of a system of N infalling masses, and finally collect a few basic consequences arising from this definition that will be useful for the analysis in the later parts of this chapter.

6.8.1 The condition of no incoming radiation

We first define the condition of no incoming radiation from $\mathcal{I}^-$ (this definition is stated without any prior gauge fixing!):

Definition 6.8.1. *A set of smooth seed scattering data $\mathfrak{D}$ is said to satisfy the no incoming radiation condition if the following condition is satisfied:*

$$\mathring{\text{div}} \hat{\chi}_{\mathcal{I}^-}^{(i)} = \mathring{\nabla} \mathring{\text{K}}_{\mathcal{S}_\infty}^{(i)}, \quad (6.8.1)$$

where $\mathring{\text{K}}_{\mathcal{S}_\infty}^{(i)}$ is defined in (6.5.6). This is equivalent to saying

$$\hat{\chi}_{\mathcal{I}^-}^{(i)} = \mathring{\mathcal{D}}_2^* \mathring{\mathcal{D}}_1^*(f_x, 0), \quad (\mathring{\Delta} + 2)f_x = 2\mathring{\text{K}}_{\mathcal{S}_\infty}^{(i)}. \quad (6.8.2)$$

for some $f_x \in \Gamma^\infty(\mathcal{I}_{v \geq v_1}^-)$ with $\partial_v f_x = 0$.

The property of a seed scattering data set to have no incoming radiation cannot be changed by addition of a pure gauge solution (Prop. 6.4.3–Prop. 6.4.5). We are therefore justified in calling it a **gauge invariant condition**.

Conversely, we recall from §6.5.4 that the other quantities of $\mathfrak{D}$ at $\mathcal{I}^-$, $\mathring{\text{b}}_{\mathcal{I}^-}^{(i)}$ and $\mathring{\Omega}_{\mathcal{I}^-}^{(i)}$, can be set to 0 by addition of pure gauge solutions. Since $\mathring{\text{K}}_{\mathcal{S}_\infty}^{(i)}$ can also be set to zero, the no incoming radiation condition therefore **completely eliminates the radiative, physical degrees of freedom at $\mathcal{I}^-$** . In particular, if $\mathfrak{D}$ is as in Thm. 6.5.1 and satisfies the no incoming radiation condition, then all the radiation fields defined in Def. 6.7.1 vanish. For later reference, we highlight that this in particular implies that $r\mathring{\alpha}^{(i)}$, $\mathring{\nabla}_v \mathring{\Psi}^{(i)}$ and $\mathring{\nabla}_v \mathring{\underline{\Psi}}^{(i)} = \mathring{\nabla}_v (r^2 \mathring{\nabla}_v)(r\Omega^2 \mathring{\underline{\alpha}}^{(i)})$ all attain vanishing limits at $\mathcal{I}^-$.

Remark 6.8.1. The definition of no incoming radiation here is written down in such a way as to be gauge invariant. Most readers will be more familiar with a version of the no incoming

radiation condition in the nonlinear theory where a gauge has already been fixed: Provided one works in a gauge where the spheres at $\mathcal{I}^-$ are round (i.e. $r^2 K \rightarrow 1$ as $u \rightarrow -\infty$), the no incoming radiation condition is captured by the requirement that the Bondi mass along $\mathcal{I}^-$ be constant. In view of the mass growth formula (this is simply the time reversal of the Bondi mass loss formula [BVdBM62, Sac62b, CK93]),

$$\frac{dM_{\text{Bondi}}}{dv} = \lim_{u \rightarrow -\infty} \frac{1}{16\pi} \int_{\mathbb{S}^2} |r\hat{\chi}|^2 d\Omega, \quad (6.8.3)$$

this clearly requires $r\hat{\chi} = 0$ at $\mathcal{I}^-$. This is exactly what our condition restricts to in the case of Bondi-normalised data (which have $\overset{(i)}{K}_{S_\infty} = 0$).

Clearly, it would not be enough to replace the condition of no incoming radiation (6.8.1) with a weaker condition such as demanding that $\overset{(i)}{\alpha}_{\mathcal{I}^-} = 0$ along $\mathcal{I}^-$, as this would be consistent with a constant $\overset{(i)}{P}_{\mathcal{I}^-}$ at past null infinity, which would correspond to a constant growth in energy. However, it is interesting to note that condition (6.8.1) is, in fact, equivalent to demanding that $\overset{(i)}{\alpha}_{\mathcal{I}^-} = 0$ and $\lim_{u \rightarrow -\infty} \frac{r^2}{\Omega^2} \nabla_u (r\Omega^2 \overset{(i)}{\alpha}_{\mathcal{C}}) = 0$, as can be seen from (6.3.32) and Definition 6.7.1. In other words, the no incoming radiation condition forces $\overset{(i)}{\alpha}_{\mathcal{C}} = o(r^{-2})$ along $\mathcal{C}$.

6.8.2 Seed data describing the far-field region of the N -body problem

The link to the physical scenario of N infalling masses (with negligible internal structure, i.e. point masses) moving along approximately hyperbolic orbits in the infinite past is made by introducing Post-Newtonian approximations to the Post-Minkowskian multipolar expansion framework; this has been described in detail in §5.2 of the previous chapter. In short, the argument in the Post-Minkowskian setting consists of writing down a general expression of an outgoing vacuum solution to the Einstein equations, which takes the following form for the curvature components of $\overset{(i)}{\alpha}$ and $\overset{(i)}{\underline{\alpha}}$:³

$$\overset{(i)}{\alpha}_\ell = \frac{(\ell+2)!}{(\ell-2)!} r^{\ell-2} \sum_{m=-\ell}^{\ell} \nabla_u^{\ell-2} \left(\frac{I_{\ell,m}(u) Y_{\ell,m}^{\text{E},2} - S_{\ell,m}(u) Y_{\ell,m}^{\text{H},2}}{r^{3+\ell}} \right), \quad (6.8.5)$$

$$\overset{(i)}{\underline{\alpha}}_\ell = r^{\ell-2} \sum_{m=-\ell}^{\ell} \nabla_u^{\ell+2} \left(\frac{I_{\ell,m}(u) Y_{\ell,m}^{\text{E},2} + S_{\ell,m}(u) Y_{\ell,m}^{\text{H},2}}{r^{\ell-1}} \right). \quad (6.8.6)$$

for some real-valued functions $I_{\ell,m}, S_{\ell,m}$ depending only on u . Notice that (6.8.5) and (6.8.6) are exactly related via the Teukolsky–Starobinsky identity (6.3.43) with $M = 0$.

³Note that this is exactly analogous to the statement that the general fixed-angular frequency solution ϕ_ℓ to the Minkowskian wave equation $\square_\eta \phi = 0$ takes the form

$$\phi_\ell = r^\ell \sum_{m=-\ell}^{\ell} \partial_u^\ell \left(\frac{f_{\ell,m}(u) Y_{\ell,m}}{r^{\ell+1}} \right) + \partial_v^\ell \left(\frac{\bar{f}_{\ell,m}(v) Y_{\ell,m}}{r^{\ell+1}} \right), \quad (6.8.4)$$

with the ∂_v -part of the solution vanishing because it is demanded that the solution be of purely outgoing type.

These expressions, which are valid in the vacuum region of spacetime, are then matched to expressions for the matter region of spacetime (i.e. the region of the N bodies), which, in turn, are derived under a number of approximations and matching to the Newtonian theory. The result is that the $I_{\ell,m}$ acquire the interpretation of the ℓ -th Newtonian mass multipole moment, and the $S_{\ell,m}$ acquire the interpretation of the ℓ -th Newtonian current multipole moment. Finally, these moments are computed using the Newtonian theory (or perturbations thereof). Since, schematically, the ℓ -th mass multipole moment goes like $m \cdot r^\ell$, where m and r denote mass and size of the system, and since, for a system of masses following hyperbolic orbits, the size grows linearly in time, we thus find that $I_{\ell,m}(u) \sim |u|^\ell$ as $u \rightarrow -\infty$. The same rate is found for $S_{\ell,m}$, and it can thus be seen that $\alpha_\ell^{(i)}$, within this framework, is predicted to decay like $|u|^2/r^5 \sim r^{-3}$ near past null infinity. Similarly, one finds that $\underline{\alpha}_\ell^{(i)}$ decays like r^{-4} near past null infinity, and that $\underline{\chi}^{(i)}$ decays like r^{-2} near past null infinity. We refer the reader to chapter 5 for details.

We will now assume that this general information obtained from a perturbative framework around Minkowski is valid up until some finite advanced time, and we implement it in the context of linearised gravity around Schwarzschild in the following way:

Definition 6.8.2. *A seed scattering data set $\mathfrak{D}$ satisfying the no incoming radiation condition is said to describe the far-field region of a system of N infalling masses following approximately hyperbolic Keplerian orbits if the following conditions are satisfied:*

- (I) *There exists $\delta > 0$ and some $\underline{\mathcal{B}} \in \Gamma^\infty(\mathbb{A}T_{stf}^{(0,2)}(\underline{\mathcal{C}}))$ with $\nabla_u \underline{\mathcal{B}} = 0$ (and supported on all angular modes) such that*

$$\underline{\alpha}_\underline{\mathcal{C}}^{(i)} = -6\underline{\mathcal{B}}r^{-4} + \mathcal{O}_\infty(r^{-4-\delta}). \quad (6.8.7)$$

- (II) *The limit $\lim_{u \rightarrow -\infty} r^2 \underline{\hat{\chi}}_\underline{\mathcal{C}}^{(i)} = \underline{\hat{\chi}}_{\mathcal{S}_\infty}^{(i)}$ exists and is non-vanishing (and is supported on all angular modes).*

- (III) *The limit $\lim_{u \rightarrow -\infty} r^3 \underline{\alpha}_\underline{\mathcal{C}}^{(i)} = \mathcal{A}$ exists and is non-vanishing (and is supported on all angular modes).*

Here, the quantities $\underline{\alpha}_\underline{\mathcal{C}}^{(i)}$, $\underline{\alpha}_\underline{\mathcal{C}}^{(i)}$ and $\underline{\hat{\chi}}_\underline{\mathcal{C}}^{(i)}$ are as defined in Prop. 6.5.1.

Remark 6.8.2. The factor -6 is introduced for later notational convenience. Condition (III) already partially follows from the previous conditions: The no incoming radiation condition implies that $\lim_{u \rightarrow -\infty} r \underline{\alpha}_\underline{\mathcal{C}}^{(i)} = 0 = \lim_{u \rightarrow -\infty} r^2 \underline{\alpha}_\underline{\mathcal{C}}^{(i)}$, and condition (I) implies that the limits $\lim_{u \rightarrow -\infty} r^3 (\underline{\rho}_\underline{\mathcal{C}}^{(i)}, \underline{\sigma}_\underline{\mathcal{C}}^{(i)}) = (\underline{\rho}_{\mathcal{S}_\infty}^{(i)}, \underline{\sigma}_{\mathcal{S}_\infty}^{(i)})$ exist. By condition (II), and equations (6.3.23) and (6.3.24), we moreover have that $\underline{\sigma}_{\mathcal{S}_\infty}^{(i)} = \text{curl div} \underline{\hat{\chi}}_{\mathcal{S}_\infty}^{(i)} \neq 0$. It then follows by construction ((6.3.32) and (6.3.31a)) that $\lim_{u \rightarrow -\infty} r^3 \underline{\alpha}_\underline{\mathcal{C}}^{(i)} = \mathring{\mathcal{D}}_2^* \mathring{\mathcal{D}}_1^* (\underline{\rho}_{\mathcal{S}_\infty}^{(i)}, -\underline{\sigma}_{\mathcal{S}_\infty}^{(i)})$. Thus, condition (III) only adds the additional requirement that $\underline{\rho}_{\mathcal{S}_\infty}^{(i)} \neq 0$.

6.8.3 A preliminary description of solutions arising from seed data describing the far-field region of the N -body problem

We now offer a preliminary description of solutions arising from seed data as in Def. 6.8.2.

Proposition 6.8.1. *Let $\mathfrak{D}$ be a smooth seed scattering data set as in Def. 6.8.2. By Prop. 6.5.2, we can without loss of generality assume it to be Bondi-normalised in the sense of Def. 6.5.3. Moreover, by Cor. 6.5.4, we can additionally assume that $\underline{\omega}_{\mathcal{C}}^{(i)} = 0$ and that $\left(\Omega \text{tr} \chi\right)_{\mathcal{S}_1}^{(i)} = 0$.*

By Theorem 6.5.1, there exists a unique scattering solution $\mathfrak{S}$ realising $\mathfrak{D}$. The data along $\mathcal{C}$ induced by $\mathfrak{S}$ satisfy:

$$\underline{\alpha}_{\mathcal{C}}^{(i)} = -6\underline{\mathcal{B}}r^{-4} + \mathcal{O}_{\infty}(r^{-4-\delta}), \quad (6.8.8)$$

$$\underline{\beta}_{\mathcal{C}}^{(i)} = -6\mathring{\text{d}}\text{iv}\underline{\mathcal{B}}r^{-4} \log r + \mathcal{C}_1 r^{-4} + \mathcal{O}_{\infty}(r^{-4-\delta}), \quad (6.8.9)$$

$$\underline{\sigma}_{\mathcal{C}}^{(i)} = \underline{\sigma}_{\mathcal{S}_{\infty}}^{(i)} r^{-3} + 6\mathring{\text{curl}}\mathring{\text{d}}\text{iv}\underline{\mathcal{B}}r^{-4} \log r + \mathcal{C}_2 r^{-4} + \mathcal{O}_{\infty}(r^{-4-\delta}), \quad (6.8.10)$$

$$\underline{\rho}_{\mathcal{C}}^{(i)} = \underline{\rho}_{\mathcal{S}_{\infty}}^{(i)} r^{-3} + 6\mathring{\text{d}}\text{iv}\mathring{\text{d}}\text{iv}\underline{\mathcal{B}}r^{-4} \log r + \mathcal{C}_3 r^{-4} + \mathcal{O}_{\infty}(r^{-4-\delta}), \quad (6.8.11)$$

$$\underline{\beta}_{\mathcal{C}}^{(i)} = \mathring{\mathcal{P}}_1^*(-\underline{\rho}_{\mathcal{S}_{\infty}}^{(i)}, \underline{\sigma}_{\mathcal{S}_{\infty}}^{(i)})r^{-3} - 3\mathring{\mathcal{P}}_1^*\mathring{\mathcal{P}}_1\mathring{\mathcal{P}}_2\underline{\mathcal{B}}r^{-4} \log r + \mathcal{C}_4 r^{-4} + \mathcal{O}_{\infty}(r^{-4-\delta}), \quad (6.8.12)$$

$$\underline{\alpha}_{\mathcal{C}}^{(i)} = \mathcal{A}r^{-3} + 2\mathring{\mathcal{P}}_2^*\mathring{\mathcal{P}}_1^*\mathring{\mathcal{P}}_1\mathring{\mathcal{P}}_2\underline{\mathcal{B}}r^{-4} \log r + \mathcal{C}_5 r^{-4} + \mathcal{O}_{\infty}(r^{-4-\delta}), \quad (6.8.13)$$

$$\underline{\Psi}_{\mathcal{C}}^{(i)} = 2\mathcal{A} + 12\mathring{\mathcal{P}}_2^*\mathring{\mathcal{P}}_1^*\mathring{\mathcal{P}}_1\mathring{\mathcal{P}}_2\underline{\mathcal{B}}r^{-1} \log r + \mathcal{C}_6 r^{-1} + \mathcal{O}_{\infty}(r^{-1-\delta}), \quad (6.8.14)$$

$$\underline{\Psi}_{\mathcal{C}}^{(i)} = 2\overline{\mathcal{A}} + 12\mathring{\mathcal{P}}_2^*\mathring{\mathcal{P}}_1^*\mathring{\mathcal{P}}_1\mathring{\mathcal{P}}_2\underline{\mathcal{B}}r^{-1} \log r + \mathcal{C}_7 r^{-1} + \mathcal{O}_{\infty}(r^{-1-\delta}), \quad (6.8.15)$$

$$\underline{\Psi}_{\mathcal{C}}^{(i)} = -12\mathring{\mathcal{P}}_2^*\mathring{\mathcal{P}}_2\underline{\mathcal{B}} + \mathcal{C}_8 r^{-2} + \mathcal{O}_{\infty}(r^{-2-\delta}), \quad (6.8.16)$$

where we recall the notation $\overline{\mathcal{A}}$ to denote the magnetic conjugate (cf. Def. 6.2.4), where $\nabla_u \mathcal{C}_n = 0$ for $n = 1, \dots, 11$ —the precise value of these constants will not play a role in this chapter—and where

$$\mathcal{A} = \mathring{\mathcal{P}}_2^*\mathring{\mathcal{P}}_1^*(\underline{\rho}_{\mathcal{S}_{\infty}}^{(i)}, -\underline{\sigma}_{\mathcal{S}_{\infty}}^{(i)}) = \mathring{\mathcal{P}}_2^*\mathring{\mathcal{P}}_1^*(\underline{\rho}_{\mathcal{S}_{\infty}}^{(i)}, -\mathring{\text{curl}}\mathring{\text{d}}\text{iv}\underline{\chi}_{\mathcal{S}_{\infty}}^{(i)}). \quad (6.8.17)$$

Similarly, we have for the connection coefficients:

$$\left(\Omega \text{tr} \chi\right)_{\mathcal{C}}^{(i)} = 0 = \text{tr} \mathring{\mathcal{G}}_{\mathcal{C}}^{(i)} = \underline{\omega}_{\mathcal{C}}^{(i)} = \underline{\Omega}_{\mathcal{C}}^{(i)}, \quad (6.8.18)$$

$$\underline{\hat{\chi}}_{\mathcal{C}}^{(i)} = \underline{\hat{\chi}}_{\mathcal{S}_{\infty}}^{(i)} r^{-2} + 6\underline{\mathcal{B}}r^{-3} + \mathcal{O}_{\infty}(r^{-3-\delta}), \quad (6.8.19)$$

$$\underline{\eta}_{\mathcal{C}}^{(i)} = -\underline{\eta}_{\mathcal{C}}^{(i)} = \mathring{\text{d}}\text{iv}\underline{\hat{\chi}}_{\mathcal{S}_{\infty}}^{(i)} r^{-2} + 6\mathring{\text{d}}\text{iv}\underline{\mathcal{B}}r^{-3} \log r + \mathcal{C}_8 r^{-3} + \mathcal{O}_{\infty}(r^{-3-\delta}), \quad (6.8.20)$$

$$\underline{\hat{\chi}}_{\mathcal{C}}^{(i)} = (-2\mathring{\mathcal{P}}_2^*\mathring{\mathcal{P}}_2 - 1)\underline{\hat{\chi}}_{\mathcal{S}_{\infty}}^{(i)} r^{-2} - 6\mathring{\mathcal{P}}_2^*\mathring{\mathcal{P}}_2\underline{\mathcal{B}}r^{-3} \log r + \mathcal{C}_9 r^{-3} + \mathcal{O}_{\infty}(r^{-3-\delta}), \quad (6.8.21)$$

$$(\Omega \text{tr} \chi)_{\mathcal{C}}^{(i)} = (2\mathring{\text{d}}\text{iv}\mathring{\text{d}}\text{iv}\underline{\hat{\chi}}_{\mathcal{S}_{\infty}}^{(i)} + 2\underline{\rho}_{\mathcal{S}_{\infty}}^{(i)})r^{-2} + 12\mathring{\text{d}}\text{iv}\mathring{\text{d}}\text{iv}\underline{\mathcal{B}}r^{-3} \log r + \mathcal{C}_{10} r^{-3} + \mathcal{O}_{\infty}(r^{-3-\delta}), \quad (6.8.22)$$

$$\underline{\omega}_{\mathcal{C}}^{(i)} = -\underline{\rho}_{\mathcal{S}_{\infty}}^{(i)} r^{-2} - 3\mathring{\text{d}}\text{iv}\mathring{\text{d}}\text{iv}\mathcal{A}r^{-3} \log r + \mathcal{C}_{11} r^{-3} + \mathcal{O}_{\infty}(r^{-3-\delta}). \quad (6.8.23)$$

On the other hand, all radiation fields defined in Def. 6.7.1, 6.7.2 vanish, and we additionally have the following limits at $\mathcal{I}^-$:

$$\lim_{u \rightarrow -\infty} r^3 \overset{(i)}{\sigma} = \overset{(i)}{\sigma}_{\mathcal{I}^-} = \overset{(i)}{\sigma}_{\mathcal{S}_\infty}, \quad \lim_{u \rightarrow -\infty} r^3 \overset{(i)}{\rho} = \overset{(i)}{\rho}_{\mathcal{I}^-} = \overset{(i)}{\rho}_{\mathcal{S}_\infty}, \quad \lim_{u \rightarrow -\infty} r^2 \overset{(i)}{\underline{\chi}} = \overset{(i)}{\underline{\chi}}_{\mathcal{I}^-} = \overset{(i)}{\underline{\chi}}_{\mathcal{S}_\infty}. \quad (6.8.24)$$

Proof. Equations (6.8.8)–(6.8.13) follow from Prop. 6.5.1 and by consecutively integrating (6.3.28a)–(6.3.32). Equations (6.8.14)–(6.8.16) follow from Cor. 6.5.2.

The last equality in (6.8.17) follows from (6.3.24) and (6.3.23).

Equations (6.8.18)–(6.8.23) also follow from Prop. 6.5.1 and by consecutive integration: Equation (6.8.18) follows from (6.3.16a) and (6.3.9a), equation (6.8.20) follows from (6.3.22), equation (6.8.21) follows from (6.3.17a), and so on.

Finally, (6.8.24) follows from inspection of the proof of Theorem 6.5.1. (Consider the ∇_v -derivatives of the $\overset{(i)}{\rho}$, $\overset{(i)}{\sigma}$ and $\overset{(i)}{\underline{\chi}}$ via (6.3.29b), (6.3.30b) and (6.3.18b). It follows from the proof of the theorem that these converge uniformly to some limit at $\mathcal{I}^-$, which, by virtue of all limits in Definitions 6.7.1 and 6.7.2 vanishing, vanish as well.) $\square$

Part II:

Asymptotic analysis of the Teukolsky equations and the Regge–Wheeler equations

In the previous part of this chapter, we carefully set up the scattering problem for linearised gravity around Schwarzschild, with scattering data posed on an ingoing null cone $\underline{\mathcal{C}}$ and $\mathcal{I}^-$, see Fig. 6.5.1, and we showed in §§6.5–6.7 how to construct the corresponding scattering solution in the domain of dependence $\mathcal{D} := D^+(\underline{\mathcal{C}} \cup \mathcal{I}_{v \geq v_1}^-)$. Finally, in §6.8, we introduced the notion of scattering data describing the exterior of a system of N infalling masses following approximately hyperbolic orbits—which we will henceforth refer to as “the physical data”—and gave a preliminary characterisation of the corresponding scattering solution.

We will now set the foundations for finding the asymptotic properties in all of $\mathcal{D}$ for general scattering solutions, and we will apply this in particular to the scattering solutions arising from the physical data of §6.8. The key to this is an asymptotic analysis of the Teukolsky equations and the Regge–Wheeler equation (introduced in §6.3.3)—this is the content of the present part of the chapter. Since both the Teukolsky and the Regge–Wheeler equations decouple from the rest of the system of equations (6.3.9)–(6.3.32), this part of the chapter will have to make very little reference to the previous part of the chapter, and it can essentially be viewed as independent work on the Teukolsky equations on Schwarzschild. In fact, we will present the results of this part of the chapter so as to be valid for the Teukolsky equations of any spin.

Overview of the structure of this part of the chapter: In §6.9, we take the Teukolsky equations (Teuk), (Teuk), re-write them in a unified way as equations for Teukolsky quantities $\alpha^{[s]}$ of general spin s ($\overset{(i)}{\alpha}$ corresponding to $s = +2$, $\underline{\alpha}$ corresponding to $s = -2$), and derive from them an infinite set of approximate/asymptotic conservation laws satisfied by angular modes of $\alpha^{[s]}$. We deduce similar conservation laws for the Regge–Wheeler equation and provide definitions of the generalised Newman–Penrose charges $\mathbf{I}_\ell^{p,q}[\alpha^{[s]}]$ (recall the discussion in chapter 4).

In the remaining sections, we then consider a scattering setup for the Teukolsky equation for $\alpha^{[s]}$ with scattering data on $\underline{\mathcal{C}} \cup \mathcal{I}_{v \geq v_1}^-$, and we use the aforementioned conservation laws to derive asymptotic expressions for angular modes of the resulting scattering solutions:

First, in §6.10, we prescribe $\alpha^{[s]}$ to decay with some general polynomial decay along $\underline{\mathcal{C}}$, and we prescribe a condition along $\mathcal{I}^-$ that, in the case $s \geq 0$ is the no incoming radiation condition, but, in the case $s < 0$, is slightly stronger than the no incoming radiation condition of Def. 6.8.1. This will make it easier for us to later highlight the somewhat different behaviour of $\alpha^{[s]}$ for negative s . Under this setup, we prove a result for the general asymptotic behaviour throughout $\mathcal{D}$ of fixed angular modes of $\alpha^{[s]}$.

In §6.11, we then apply these results to $\overset{(i)}{\alpha}$ ($s = +2$) with the data that we physically motivated in §6.8, i.e. with $\overset{(i)}{\alpha}$ along $\underline{\mathcal{C}}$ decaying as in (6.8.13). Due to the presence of the

logarithmic term in the decay (6.8.13), this requires a very minor modification of the result of §6.10.

We can similarly apply the results of §6.10 to deduce the asymptotic behaviour of solutions to the Regge–Wheeler equation arising from physical initial data ((6.8.14) and (6.8.15)). This is done in §6.12.

In §6.13, we then apply the results of §6.10 in the case $s = -2$ to find the asymptotics of $\underline{\alpha}^{(i)}$ for the physical data of §6.8 (eq. (6.8.8)). Again, a minor modification is necessary to account for the fact that our assumption in §6.10 is slightly stronger than the no incoming radiation condition. Furthermore, we will find that, for the special data under consideration, a cancellation occurs, and the leading-order asymptotic results of §6.10 are no longer sufficient: To be precise, the leading-order results of §6.10 only imply that $\lim_{v \rightarrow \infty} r \underline{\alpha}^{(i)} = C \cdot |u|^{-2} + \dots$, but this constant C is found to be vanishing.

We find the next-to leading-order decay of $\lim_{v \rightarrow \infty} r \underline{\alpha}^{(i)}$ by introducing a decomposition of the Teukolsky equation that is similar to a Post–Minkowskian expansion and which allows us to find the next-to leading-order asymptotics of $\underline{\alpha}^{(i)}$ by solving a few recurrence relations. This is the content of §6.14.

Finally, with all the results of the previous sections having been obtained for angular modes of $\alpha^{[s]}$, we comment (but do not yet resolve) in §6.15 on the issue of summing these fixed angular frequency estimates obtained in the previous sections. The resolution of this problem will be the subject of future work.

6.9 The approximate conservation laws

In this section, we will derive an infinite set of *approximate* conservation laws (which imply exact *asymptotic* conservation laws, cf. the discussion of chapter 4) for $\alpha^{(i)}$ and $\underline{\alpha}^{(i)}$ starting from the Teukolsky equations (Teuk), (Teuk). Let us first fix some notation: Since Ω only appears with even integer powers in this part of the chapter, we denote

$$D := \Omega^2 = 1 - \frac{2M}{r}. \quad (6.9.1)$$

Recall that $\partial_v r = D = -\partial_u r$.

6.9.1 The Teukolsky equations for general spin s

We begin by massaging the equation (Teuk) into a form that is more convenient for our purposes: First, we rewrite

$$\begin{aligned} & \nabla_v (r^4 D^{-2} \nabla_u (r D \alpha^{(i)})) \\ &= \nabla_v (\nabla_u (r^5 \alpha^{(i)} D^{-1})) - \nabla_v (D^2 r^{-4} \partial_u (r^4 D^{-2}) (r^5 \alpha^{(i)} D^{-1})) \end{aligned}$$

$$\begin{aligned}
&= \nabla_u(\nabla_v(r^5 \overset{(i)}{\alpha} D^{-1})) - \partial_v \partial_u \log(r^4 D^{-2})(r^5 \overset{(i)}{\alpha} D^{-1}) - \partial_u \log(D^{-2} r^4) \nabla_v(r^5 \overset{(i)}{\alpha} D^{-1}) \\
&= \frac{r^4}{D^2} \nabla_u\left(\frac{D^2}{r^4} \nabla_v\left(\frac{r^5 \overset{(i)}{\alpha}}{D}\right)\right) - \partial_v \partial_u \log(r^4 D^{-2}) \frac{r^5 \overset{(i)}{\alpha}}{D}
\end{aligned}$$

We further have

$$\partial_v \partial_u \log(D^{-2} r^4) = \frac{4D}{r^2} - \frac{24MD}{r^3}.$$

Using the two equations above, we rewrite the Teukolsky equation (Teuk) as follows:

$$\nabla_u(r^{-4} D^2 \nabla_v \frac{r^5 D \overset{(i)}{\alpha}}{D^2}) = \frac{D}{r^{2+4}} (\mathring{\Delta} - 2 + 4)(r^5 D \overset{(i)}{\alpha}) - \frac{30MD}{r^{3+4}} r^5 D \overset{(i)}{\alpha} \quad (6.9.2)$$

Next, we suggestively write (Teuk) as

$$\nabla_u(r^4 D^{-2} \nabla_v(D^2 \frac{r \overset{(i)}{\alpha}}{D})) = \frac{D}{r^{2-4}} (\mathring{\Delta} - 2) \frac{r \overset{(i)}{\alpha}}{D} - \frac{6MD}{r^{3-4}} \frac{r \overset{(i)}{\alpha}}{D}, \quad (6.9.3)$$

and introduce the notation

$$\mathring{\Delta}_{[s]} := \mathring{\Delta} + s, \quad \alpha^{[s]} := \begin{cases} r^5 D^{-1} \overset{(i)}{\alpha}, & s = 2, \\ r D \overset{(i)}{\alpha}, & s = -2 \end{cases} \quad (6.9.4)$$

The two equations (6.9.2) and (6.9.3) can then be formulated in a unified way as

$$\nabla_u(r^{-2s} D^s \nabla_v \alpha^{[s]}) = \frac{D^{s+1}}{r^{2+2s}} \mathring{\Delta}_{[s]} \alpha^{[s]} - \frac{2MD^{s+1}}{r^{3+2s}} (1+s)(1+2s) \alpha^{[s]}, \quad s = \pm 2. \quad (\text{Teuk:s})$$

We note that the operator $\mathring{\Delta}_{[s]}$ has eigenvalues

$$\Lambda_\ell^{[s]} := -\ell(\ell+1) + s(s+1) = -(\ell-s)(\ell+s+1), \quad \ell \geq 2. \quad (6.9.5)$$

Remark 6.9.1. Note that if a scalar function ϕ solves $\square_g \phi = 0$, then $r\phi$ satisfies (Teuk:s) for $s = 0$. Similarly, the Teukolsky equation for $s = \pm 1$ has the interpretation of describing the dynamics of electromagnetic perturbations on Schwarzschild, $\alpha^{[\pm 1]}$ then being a 1-form. We can also make sense of (Teuk:s) for $|s| \geq 2$ by regarding $\alpha^{[s]}$ as an stf $\mathcal{S}_{u,v}$ -tangent $|s|$ -tensor, though the physical interpretation of the equation for $s > 2$ is not clear.

All computations presented in the remainder of this and the next section are valid for arbitrary integer values of s , though the main interest of course lies in $s = \pm 2$.

6.9.2 The commuted Teukolsky equations

We now prove a simple commutation formula, from which we will later derive an approximate conservation law.

Proposition 6.9.1. *Let $N \in \mathbb{N}$, and let $\alpha^{[s]}$ be a smooth solution to (Teuk:s). Then*

$$\begin{aligned} & \nabla_u \left(\left(\frac{D}{r^2} \right)^{N+s} \nabla_v \left(\frac{r^2}{D} \nabla_v \right)^N \alpha^{[s]} \right) \\ &= \left(\frac{D}{r^2} \right)^{N+s+1} \sum_{j=0}^1 \left(a_{N,j}^{[s]} + b_{N,j}^{[s]} \overset{\circ}{\Delta}_{[s]} - c_{N,j}^{[s]} \frac{2M}{r} \right) (2M)^j \left(\frac{r^2}{D} \nabla_v \right)^{N-j} \alpha^{[s]}. \end{aligned} \quad (6.9.6)$$

Similarly, we have

$$\begin{aligned} & \nabla_v \left(\left(\frac{D}{r^2} \right)^{N-s} \nabla_u \left(\frac{r^2}{D} \nabla_u \right)^N (r^{-2s} D^s \alpha^{[s]}) \right) \\ &= \left(\frac{D}{r^2} \right)^{N-s+1} \sum_{j=0}^1 (-1)^j \left(a_{N,j}^{[-s]} + b_{N,j}^{[-s]} \overset{\circ}{\Delta}_{[-s]} - c_{N,j}^{[-s]} \frac{2M}{r} \right) (2M)^j \left(\frac{r^2}{D} \nabla_u \right)^{N-j} (r^{-2s} D^s \alpha^{[s]}). \end{aligned} \quad (6.9.7)$$

The constants $a_{N,j}^{[s]}, b_{N,j}^{[s]}, c_{N,j}^{[s]}$ are given explicitly by

$$a_{N,0}^{[s]} = N(N+1+2s), \quad b_{N,0}^{[s]} = 1, \quad c_{N,0}^{[s]} = (1+s)(1+2s) + 3N(N+1+2s) \quad (6.9.8)$$

and

$$a_{N,1}^{[s]} = N(N+s)(N+2s), \quad b_{N,1}^{[s]} = 0 = c_{N,1}^{[s]}. \quad (6.9.9)$$

Proof of Proposition 6.9.1. We first prove (6.9.6). Note that for $N = 0$, it reduces to (Teuk:s). We now establish an inductive relation to prove it for all $N \in \mathbb{N}$. Observe:

$$\begin{aligned} & \nabla_u \left(\left(\frac{D}{r^2} \right)^{N+s+1} \nabla_v \left(\frac{r^2}{D} \nabla_v \right)^{N+1} \alpha^{[s]} \right) \\ &= \nabla_u \nabla_v \left(\left(\left(\frac{D}{r^2} \right)^{N+s+1} \left(\frac{r^2}{D} \nabla_v \right)^{N+1} \alpha^{[s]} \right) \right) - \nabla_u \left(\partial_v \left(\left(\frac{D}{r^2} \right)^{N+1+s} \right) \left(\frac{r^2}{D} \nabla_v \right)^{N+1} \alpha^{[s]} \right) \\ &= \left(\frac{D}{r^2} \right)^{N+s+1} \nabla_v \left(\left(\frac{r^2}{D} \right)^{N+s+1} \nabla_u \left(\left(\frac{D}{r^2} \right)^{N+s+1} \left(\frac{r^2}{D} \nabla_v \right)^{N+1} \alpha^{[s]} \right) \right) \\ &\quad - \partial_v \log \left(\left(\frac{r^2}{D} \right)^{N+s+1} \right) \nabla_u \left(\left(\frac{D}{r^2} \right)^{N+s+1} \left(\frac{r^2}{D} \nabla_v \right)^{N+1} \alpha^{[s]} \right) \\ &\quad - \nabla_u \left(\partial_v \log \left(\left(\frac{r^2}{D} \right)^{-N-s-1} \right) \left(\frac{D}{r^2} \right)^{N+s+1} \left(\frac{r^2}{D} \nabla_v \right)^{N+1} \alpha^{[s]} \right) \end{aligned}$$

$$\begin{aligned}
&= \left(\frac{D}{r^2}\right)^{N+s+1} \nabla_v \left(\left(\frac{r^2}{D}\right)^{N+s+1} \nabla_u \left(\left(\frac{D}{r^2}\right)^{N+s+1} \left(\frac{r^2}{D} \nabla_v\right)^{N+1} \alpha^{[s]} \right) \right) \\
&\quad + \partial_u \partial_v \log \left(\left(\frac{r^2}{D}\right)^{N+s+1} \right) \cdot \left(\frac{D}{r^2}\right)^{N+s+1} \left(\frac{r^2}{D} \nabla_v\right)^{N+1} \alpha^{[s]}.
\end{aligned}$$

Since $\partial_u \partial_v \log r = \frac{D}{r^2} - \frac{4MD}{r^3}$, and since $\partial_u \partial_v \log D = \frac{4MD}{r^3}$, we have

$$\partial_u \partial_v \log \left(\left(\frac{r^2}{D}\right)^{N+s+1} \right) = (N+s+1) \left(\frac{2D}{r^2} - \frac{12MD}{r^3} \right).$$

In conclusion, we thus obtain

$$\begin{aligned}
&\nabla_u \left(\left(\frac{D}{r^2}\right)^{N+s+2} \left(\frac{r^2}{D} \nabla_v\right)^{N+2} \alpha^{[s]} \right) \\
&= \left(\frac{D}{r^2}\right)^{N+s+1} \nabla_v \left(\left(\frac{r^2}{D}\right)^{N+s+1} \nabla_u \left(\left(\frac{D}{r^2}\right)^{N+s+1} \left(\frac{r^2}{D} \nabla_v\right)^{N+1} \alpha^{[s]} \right) \right) \\
&\quad + \left(\frac{D}{r^2}\right)^{N+s+2} \left(\frac{r^2}{D} \nabla_v\right)^{N+1} \alpha^{[s]} \left(2(N+1+s) - 6 \cdot \frac{2M}{r} (N+1+s) \right). \quad (6.9.10)
\end{aligned}$$

We now assume (6.9.6) to hold for some fixed $N \in \mathbb{N}$. Using the inductive relation (6.9.10), we then arrive at (6.9.6) with N replaced by $N+1$, where

$$a_{N+1,0}^{[s]} = a_{N,0}^{[s]} + 2(s+N+1), \quad b_{N+1,0}^{[s]} = b_{N,0}^{[s]}, \quad c_{N+1,0}^{[s]} = c_{N,0}^{[s]} + 6(N+1+s)$$

and

$$a_{N+1,1}^{[s]} = a_{N,1}^{[s]} + c_{N,0}^{[s]}.$$

It is left to solve these recurrence relations, whose initial values are provided by (Teuk:s):

$a_{0,0}^{[s]} = 0 = a_{0,1}^{[s]}$, $b_{0,0}^{[s]} = 1$, $c_{0,0}^{[s]} = (1+s)(1+2s)$, the result being

$$a_{N,0}^{[s]} = N(N+1+2s), \quad b_{N,0}^{[s]} = 1, \quad c_{N,0}^{[s]} = (1+s)(1+2s) + 3N(N+1+2s) \quad (6.9.11)$$

and

$$\begin{aligned}
a_{N,1}^{[s]} &= \sum_{i=0}^{N-1} c_{i,0}^{[s]} = \sum_{i=0}^{N-1} (1+2s)(1+s) + 6si + 3i(i+1) \\
&= (N-1)N(N+1) + 3s(N-1)N + (1+s)(1+2s)N \\
&= N(N^2 - 1 + 3sN - 3s + 1 + 3s + 2s^2) = N(N^2 + 3sN + 2s^2) = N(N+s)(N+2s).
\end{aligned} \tag{6.9.12}$$

This concludes the proof of (6.9.6).

Let's now move on to prove (6.9.7). We recall the following computation:

$$\left(\frac{r^2}{D}\right)^s \nabla_u \left(\left(\frac{D}{r^2}\right)^s \nabla_v \alpha^{[s]} \right) = \nabla_v \left(\left(\frac{r^2}{D}\right)^s \nabla_u (r^{-2s} D^s \alpha^{[s]}) \right) + s \left(\partial_v \partial_u \log \frac{r^2}{D} \right) \alpha^{[s]}.$$

As a consequence, we have

$$\begin{aligned}
&\nabla_v \left(\left(\frac{r^2}{D}\right)^s \nabla_u (r^{-2s} D^s \alpha^{[s]}) \right) \\
&= \frac{r^{2s}}{D^s} \left(\frac{D}{r^2} \mathring{\Delta}_{[s]} (r^{-2s} D^s \alpha^{[s]}) - \frac{2MD}{r^3} ((1+s)(1+2s)) (r^{-2s} D^s \alpha^{[s]}) \right) - s \left(\frac{2D}{r^2} - \frac{12MD}{r^3} \right) \alpha^{[s]} \\
&= \frac{r^{2s}}{D^s} \left(\frac{D}{r^2} \mathring{\Delta}_{[-s]} (r^{-2s} D^s \alpha^{[s]}) - \frac{2MD}{r^3} ((1-s)(1-2s)) (r^{-2s} D^s \alpha^{[s]}) \right),
\end{aligned} \tag{6.9.13}$$

where we used (Teuk:s) in the second line as well as the relations $\mathring{\Delta}_{[s]} - 2s = \mathring{\Delta}_{[-s]}$ and $(1+s)(1+2s) - 6s = (1-s)(1-2s)$ in the third. From this, we obtain the inductive relationship:

$$\begin{aligned}
&\nabla_v \left(\left(\frac{D}{r^2}\right)^{N-s+2} \left(\frac{r^2}{D} \nabla_u\right)^{N+2} (r^{-2s} D^s \alpha^{[s]}) \right) \\
&= \left(\frac{D}{r^2}\right)^{N-s+1} \nabla_u \left(\left(\frac{r^2}{D}\right)^{N-s+1} \nabla_v \left(r^{2s} \left(\frac{D}{r^2}\right)^{N+1} \left(\frac{r^2}{D} \nabla_u\right)^{N+1} (r^{-2s} D^s \alpha^{[s]}) \right) \right) \\
&\quad + \left(\frac{D}{r^2}\right)^{N-s+2} \left(\frac{r^2}{D} \nabla_u\right)^{N+1} (r^{-2s} D^s \alpha^{[s]}) (N-s+1) \left(2 - 6 \cdot \frac{2M}{r} \right).
\end{aligned} \tag{6.9.14}$$

Equation (6.9.7) then follows in the same way as equation (6.9.6) did. $\square$

6.9.3 The approximate conservation laws

Consider a solution $\alpha^{[s]}$ to (Teuk:s) supported only on angular frequency ℓ . For such solutions, we have $(a_{\ell-s,0}^{[s]} + b_{\ell-s,0}^{[s]} \mathring{\Delta}_{[s]}) \alpha_\ell^{[s]} = 0$, so there is a cancellation in equation (6.9.6) for $N = \ell - s$. The remaining term with a bad r -weight, the $a_{\ell-s,1}^{[s]}$ -term, can then be removed by iteratively

subtracting suitable multiples of (6.9.6) with $N = \ell - s - i$, $i \in \{1, \dots, \ell - s\}$. This leads to the following class of approximate conservation laws:

Corollary 6.9.1. *Let $\alpha_\ell^{[s]}$ be a smooth solution to (Teuk:s) that is supported on some fixed $\ell \in \mathbb{N}_{\geq |s|}$. Define*

$$\tilde{a}_{N,0,\ell}^{[s]} := a_{N,0}^{[s]} - b_{N,0}^{[s]} \Lambda_\ell^{[s]} = (N + s - \ell)(N + s + \ell + 1). \quad (6.9.15)$$

If $x_{i,\ell}^{[s]}$ is any collection of constants with $x_{i,\ell}^{[s]} = 0$ if $i > \ell - s$, then, as a direct corollary of Proposition 6.9.1,

$$\begin{aligned} & \nabla_u \left(\left(\frac{D}{r^2} \right)^\ell \nabla_v \left(\sum_{i=0}^{\ell-s} x_{i,\ell}^{[s]} (2M)^i \left(\frac{r^2}{D} \nabla_v \right)^{\ell-s-i} \alpha_\ell^{[s]} \right) \right) \\ &= \sum_{i=0}^{\ell-s} \left(2(i+1) x_{i+1,\ell}^{[s]} \frac{2M}{r} \left(1 - \frac{3M}{r} \right) + x_{i,\ell}^{[s]} \cdot \tilde{a}_{\ell-s-i,0,\ell}^{[s]} + x_{i-1,\ell}^{[s]} a_{\ell-s-i+1,1}^{[s]} - x_{i,\ell}^{[s]} \frac{2M}{r} \cdot c_{\ell-s-i,0}^{[s]} \right) \\ & \quad \cdot \left(\frac{D}{r^2} \right)^{\ell+1} (2M)^i \left(\frac{r^2}{D} \nabla_v \right)^{\ell-s-i} \alpha_\ell^{[s]}. \end{aligned} \quad (6.9.16)$$

We now fix the constants $x_{i,\ell}^{[s]}$ via

$$x_{i,\ell}^{[s]} := (-1)^i \prod_{j=0}^{i-1} \frac{a_{\ell-s-j,1}^{[s]}}{\tilde{a}_{\ell-s-j-1,0,\ell}^{[s]}} = \frac{1}{i!} \frac{(2\ell-i)!}{(2\ell)!} \frac{(\ell-s)! \ell! (\ell+s)!}{(\ell-s-i)! (\ell-i)! (\ell+s-i)!} \quad (6.9.17)$$

such that the a -terms in (6.9.16) vanish. (Note that $x_{i,\ell}^{[s]} = x_{i,\ell}^{[-s]}$, so, in particular $x_{i,\ell}^{[s]} = 0$ for $i \geq \ell - |s|$.) Finally, define

$$A_\ell^{[s]} := \sum_{i=0}^{\ell-|s|} x_{i,\ell}^{[s]} (2M)^i \left(\frac{r^2}{D} \nabla_v \right)^{\ell-s-i} \alpha_\ell^{[s]}. \quad (6.9.18)$$

This quantity then satisfies

$$\begin{aligned} & \nabla_u \left(\left(\frac{D}{r^2} \right)^\ell \nabla_v A_\ell^{[s]} \right) \\ &= \sum_{k=0}^{\ell-|s|} \left(2(k+1) x_{k+1,\ell}^{[s]} \frac{2M}{r} \left(1 - \frac{3M}{r} \right) - x_{k,\ell}^{[s]} \frac{2M}{r} \cdot c_{\ell-s-k,0}^{[s]} \right) \\ & \quad \cdot \left(\frac{D}{r^2} \right)^{\ell+1} (2M)^k \left(\frac{r^2}{D} \nabla_v \right)^{\ell-s-k} \alpha_\ell^{[s]}. \end{aligned} \quad (6.9.19)$$

Remark 6.9.2. An analogous result holds with u and v interchanged.

6.9.4 The modified Newman–Penrose charges

The approximate conservation laws (6.9.19) of the previous subsection imply the conservation of weighted limits of $A_\ell^{[s]}$. The following corollary mainly serves the purpose of providing us with some nomenclature for these limits. We therefore state some fairly strong assumptions that guarantee the existence and the conservation of these limits.

Corollary 6.9.2. *Let $\alpha^{[s]}$ be a smooth solution to (Teuk:s) arising from scattering data on $\underline{\mathcal{C}} \cup \mathcal{I}^-$, and let $p \in \mathbb{R}$, $q \in \mathbb{N}_{\geq 0}$. Suppose that the following bound holds in the domain of dependence $\mathcal{D}$ of $\underline{\mathcal{C}} \cup \mathcal{I}^-$:*

$$\left| \nabla_v \left(\frac{r^2}{D} \nabla_v \right)^{\ell-s} \alpha_\ell^{[s]} \right| \leq C(u) \frac{\log^q r}{r^p} \quad (6.9.20)$$

for some constant that is allowed to depend on u . If there exists a $u' \leq u_0$ such that the limit $\lim_{v \rightarrow \infty} \frac{r^p}{\log^q r} \nabla_v \left(\frac{r^2}{D} \nabla_v \right)^{\ell-s} \alpha_\ell^{[s]}(u', v)$ exists, then the limit

$$\mathbf{I}_\ell^{p,q}[\alpha^{[s]}](u) := \lim_{v \rightarrow \infty} \frac{r^p}{\log^q r} \nabla_v A_\ell^{[s]}(u, v) \quad (6.9.21)$$

exists for all $u \leq u_0$ and is independent of u , $\nabla_u \mathbf{I}_\ell^{p,q}[\alpha^{[s]}] = 0$. We call this limit the (p, q) -modified Newman–Penrose charge of $\alpha^{[s]}$.

Furthermore, if we only assume (6.9.20) for $s = +2$, and assume that $\left| \nabla_v^2 \left(\frac{r^2}{D} \nabla_v \right)^{\ell-2} \alpha^{[+2]} \right| = o\left(\frac{\log^q r}{r^p}\right)$ and that $\alpha^{[+2]}$ and $\alpha^{[-2]}$ are related via (6.3.44), then

$$\mathbf{I}_\ell^{p,q}[\alpha^{[-2]}] = 2\mathring{\mathcal{P}}_2^* \mathring{\mathcal{P}}_1^* \overline{\mathring{\mathcal{P}}_1} \mathring{\mathcal{P}}_2 \mathbf{I}_\ell^{p,q}[\alpha^{[+2]}]. \quad (6.9.22)$$

6.9.5 The Regge–Wheeler equations and their conservation laws

We observe that if $s < 0$, then the commuted equation (6.9.6) for $N = |s| = -s$ gives the Regge–Wheeler equation (cf. (RW)), and if $s \geq 0$, then the same holds for (6.9.7) with $N = |s| = s$. To be precise, we define

$$\Psi^{[s]} := \begin{cases} \left(\frac{r^2}{D} \nabla_v \right)^{|s|} \alpha^{[s]}, & \text{if } s \leq 0, \\ \left(\frac{r^2}{D} \nabla_u \right)^s (r^{-2s} D^s \alpha^{[s]}), & \text{if } s > 0, \end{cases} \quad (6.9.23)$$

and this quantity satisfies

$$\nabla_u \nabla_v \Psi^{[s]} = \frac{D}{r^2} \left((\mathring{\Delta}_{[s]} - \Lambda_0^{[s]}) - c_{0,0}^{[s],\text{RW}} \frac{2M}{r} \right) \Psi^{[s]}, \quad (6.9.24)$$

where $\Lambda_0^{[s]} = -s(s+1)$ is the lowest eigenvalue of $\mathring{\Delta}_{[s]}$ (eq. (6.9.5)), and where $c_{0,0}^{[s],\text{RW}} := c_{[s],0}^{[-|s|]} = 1 - s^2$.

Starting from (6.9.24), we can then, just as before, derive its commuted versions along with the resulting conservation laws:

Proposition 6.9.2. *Let $N \in \mathbb{N}$ and let $\Psi^{[s]}$ be a smooth solution to the Regge–Wheeler equation (6.9.24). Then*

$$\begin{aligned} \nabla_u \left(\left(\frac{D}{r^2} \right)^N \nabla_v \left(\frac{r^2}{D} \nabla_v \right)^N \Psi^{[s]} \right) \\ = \left(\frac{D}{r^2} \right)^{N+1} \sum_{j=0}^1 \left(a_{N,j}^{[s],\text{RW}} + b_{N,j}^{[s],\text{RW}} (\mathring{\Delta}_{[s]} - \Lambda_0^{[s]}) - c_{N,j}^{[s],\text{RW}} \frac{2M}{r} \right) (2M)^j \left(\frac{r^2}{D} \nabla_v \right)^{N-j} \Psi^{[s]}, \end{aligned} \quad (6.9.25)$$

with $a_{N,0}^{[s],\text{RW}} = a_{N,0}^{[0]} = N(N+1)$, $b_{N,0}^{[s],\text{RW}} = b_{N,0}^{[0]} = 1$, $c_{N,0}^{[s],\text{RW}} = 3N(N+1) + (1-s^2)$ and $a_{N,1}^{[s],\text{RW}} = (N-s)N(N+s)$, and $b_{N,1}^{[s],\text{RW}} = 0 = c_{N,1}^{[s],\text{RW}}$.

Furthermore, for $x_{i,\ell}^{[s],\text{RW}} := x_{i,\ell}^{[s]}$, the quantity

$$\Psi_\ell^{[s]} := \sum_{i=0}^{\ell-|s|} x_{i,\ell}^{[s],\text{RW}} (2M)^i \left(\frac{r^2}{D} \nabla_v \right)^{\ell-i} \Psi_\ell \quad (6.9.26)$$

satisfies (note that $c_{N,0}^{[s],\text{RW}} = c_{N-s,0}^{[s]}$ for $N \geq s$)

$$\begin{aligned} \nabla_u \left(\left(\frac{D}{r^2} \right)^\ell \nabla_v \Psi_\ell^{[s]} \right) \\ = \sum_{k=0}^{\ell-|s|} \left(2(k+1) x_{k+1,\ell}^{[s],\text{RW}} \frac{2M}{r} \left(1 - \frac{3M}{r} \right) - x_{k,\ell}^{[s],\text{RW}} \frac{2M}{r} \cdot c_{\ell-k,0}^{[s],\text{RW}} \right) \\ \cdot \left(\frac{D}{r^2} \right)^{\ell+1} (2M)^k \left(\frac{r^2}{D} \nabla_v \right)^{\ell-k} \Psi_\ell^{[s]}. \end{aligned} \quad (6.9.27)$$

Finally, provided the limit exists, we denote

$$\mathbf{I}_\ell^{p,q}[\Psi](u) := \lim_{v \rightarrow \infty} \frac{r^p}{\log^q r} \nabla_v \Psi_\ell^{[s]}(u, v) \quad (6.9.28)$$

and call it the modified Newman–Penrose charge for $\Psi^{[s]}$.

Remark 6.9.3. If $s \leq 0$, we see immediately that $\Psi_\ell^{[s]} = \Lambda_\ell^{[s]}$. For $s > 0$, a similar statement is valid only to leading order, up to application of some angular operator of order $2s$. For instance, for $s = 2$, we have (cf. eq. (6.3.40))

$$\Psi_\ell^{[s]} = (\mathring{\Delta} - 2)(\mathring{\Delta} - 4)\Lambda_\ell^{[s]} - 6M(\nabla_u + \nabla_v)\Lambda_\ell^{[s]}.$$

In particular, $\mathbf{I}_\ell^{p,q}[\Psi^{[2]}](u) = (\ell-1)\ell(\ell+1)(\ell+2)\mathbf{I}_\ell^{p,q}[\alpha^{[2]}](u)$ since the limit of $\frac{r^p}{\log^q r}\nabla_v(-6M)(\nabla_u + \nabla_v)A_\ell^{[s]}$ will vanish (for the same reason that $\mathbf{I}_\ell^{p,q}[\alpha^{[2]}](u)$ is conserved).

6.10 Asymptotic analysis of $\alpha^{[s]}$ arising from general scattering data

Having derived a class of approximate conservation laws (eq. (6.9.19)) satisfied by angular modes $\alpha_\ell^{[s]}$ of solutions $\alpha^{[s]}$ to the Teukolsky equation (Teuk:s) with arbitrary integer spin s , we now use these to derive precise asymptotic expressions for fixed angular modes $\alpha_\ell^{[s]}$ of solutions $\alpha^{[s]}$ arising from scattering data prescribed on $\underline{\mathcal{C}} \cup \mathcal{I}_{v \geq v_1}^-$. We introduce the notation

$$r_0(u, v) := r(u, v = v_1). \quad (6.10.1)$$

Throughout the entire part of this chapter, we shall only study solutions to (Teuk:s) that satisfy the no incoming radiation condition (cf. Def. 6.8.1), namely:

$$\begin{aligned} \lim_{u \rightarrow -\infty} \nabla_v \Psi^{[s]} &= 0, \\ \lim_{u \rightarrow -\infty} \left(\frac{r^2}{D} \nabla_u \right)^{s-1} (r^{-2s} \alpha^{[s]}) &= 0, \quad \text{if } s \geq 0, \end{aligned} \quad (6.10.2)$$

where $\Psi^{[s]}$ is related to $\alpha^{[s]}$ via (6.9.23). We note that the second condition of (6.10.2) can be shown to be implied by the first one, but we do not show this here.

We complement these data at $\mathcal{I}_{v \geq v_1}^-$ by specifying along $\underline{\mathcal{C}}$ that

$$\alpha_{\underline{\mathcal{C}}}^{[s]} = \mathcal{A} r_0^{-p+s} + \mathcal{O}_{\max(s+1,0)}(r_0^{-p-\epsilon}) \quad (6.10.3)$$

for some $\epsilon > 0$, $p > \max(-s-1, -1/2)$, and for some $\mathcal{A} \in \Gamma^\infty(\mathbb{M} T_{stf}^{(0,|s|)} \underline{\mathcal{C}})$ with $\nabla_u \mathcal{A} = 0$.

For $s < 0$, conditions (6.10.2) and (6.10.3) together are insufficient to solve the scattering problem, it is additionally necessary to specify all transversal derivatives up to order $-s$ along $\mathcal{S}_1$ or $\mathcal{S}_\infty$. For the present section, we will simplify our lives by assuming that

$$r^{-2s} \nabla_v^i \alpha_{\mathcal{S}_\infty}^{[s]} = 0, \quad \text{for } i = 1, \dots, -s+1. \quad (6.10.4)$$

This is a simplifying assumption that is independent of the condition of no incoming radiation. We will remove it in §6.13, where the case $s < 0$ is discussed in the context of the physical data of §6.8. (Notice that the “physical” decay rate of $\alpha^{[-2]}$ along $\underline{\mathcal{C}}$ is given by $r_0^{-3} = r_0^{-1-2}$ (cf. (6.8.8)), which violates the condition (6.10.3) that $p > -s-1 = 1$.)

A note on conventions In the asymptotic computations to follow, we will often derive asymptotic decay/growth rates for expressions without computing the precise coefficients. For these situations, we will always just denote the coefficients by $\mathcal{E}$. On the other hand, to express *upper bounds* where the precise constant doesn’t matter, we will make use of the letter C . This

will always denote a global constant that is allowed to change from line to line. Occasionally, we will also write $C(a_1, a_2, \dots)$ to express that C depends only on $a_1, a_2, \dots$.

6.10.1 The main theorem (Thm. 6.10.1)

The content of §6.10 is the proof of the following

Theorem 6.10.1. *Let $s \in \mathbb{Z}$. Then there exists a unique smooth scattering solution $\alpha^{[s]}$ to (Teuk:s) that satisfies the no incoming radiation condition (6.10.2), restricts along $\underline{\mathcal{C}}$ to $\alpha_{\underline{\mathcal{C}}}^{[s]}$ as specified in (6.10.3), and, for $s < 0$, satisfies (6.10.4). Moreover, for $\ell \geq |s|$, each angular mode $\alpha_\ell^{[s]}$ of this solution has the following asymptotic expression throughout $\mathcal{D} = D^+(\underline{\mathcal{C}} \cup \mathcal{I}_{v \geq v_1}^-)$, provided that $\ell - p \geq 0$:*

$$\begin{aligned} \alpha_\ell^{[s]} = & \mathcal{A}_\ell r_0^{s-p} \sum_{n=0}^{\max(\ell-s, [p-s])} \cdot \left(\frac{r_0}{r}\right)^n (S_{\ell,p,\ell-s,n,s} + \mathcal{O}(r_0^{-\epsilon} + M r_0^{-1})) \\ & - (-1)^{\ell-s} M \mathcal{A}_\ell \frac{(\ell-s)!(s+p)!}{(\ell+s)!} (\ell+p+1) \cdot r^{s-1-p} \\ & \cdot \begin{cases} (s-p-2)! \cdot \{(\ell-p) \\ + \mathcal{O}((\frac{r_0}{r})^{s-p-1} + \frac{r_0}{r}(1 + \delta_{s-2-p,0} \log r/r_0) + r^{-\epsilon}(1 + \delta_{s-p-1,\epsilon} \log r/r_0))\}, & 1+p-s \notin \mathbb{N}_{\geq 0}, \\ \frac{(-1)^{p+1-s}}{(p+1-s)!} ((\ell-p) \log r/r_0 + \mathcal{O}(\frac{r_0}{r} \log r/r_0 + r^{-\epsilon}(1 + \delta_{\epsilon,1} \log r/r_0))), & 1+p-s \in \mathbb{N}_{\geq 0}, \end{cases} \end{aligned} \quad (6.10.5)$$

where the constants $S_{\ell,p,j,n,s}$ are defined in (6.10.55) and vanish for $n > \ell - s$.

Moreover, if $\ell - p \geq -1$, then the solution has finite, non-zero and conserved ℓ -th Newman–Penrose charge $\mathbf{I}_\ell^{p',q}[\alpha^{[s]}]$ for $p' = 2 - \ell + p$, $q = \delta_{\ell-p,-1}$; its value can be read off from Proposition 6.10.4.

Finally, the form of the expansion (6.10.5) (without specifying the coefficients in the third line) also holds for $p > \ell + 1$.

Remark 6.10.1. The uniqueness asserted in Theorem 6.10.1 is with respect to the class of finite Regge–Wheeler energy solutions.

Remark 6.10.2. Regarding the last sentence of the theorem, we in fact also expect the explicit form of the coefficients in the third line (6.10.5) to be valid for $p > \ell + 1$ as well.

Remark 6.10.3. The ℓ -dependent constants hiding in the $\mathcal{O}$ terms in (6.10.5) grow too fast to directly extract an asymptotic expression for the entire solution $\alpha^{[s]} = \sum_\ell \alpha_\ell^{[s]}$. We return to this point in §6.15.

The theorem allows to also directly infer statements about solutions to the Regge–Wheeler equation (6.9.24):

Corollary 6.10.1. *Prescribe scattering data for $\Psi^{[s]}$ satisfying $\Psi_{\underline{\mathcal{C}}}^{[s]} = \mathcal{A}r_0^{-p} + \mathcal{O}_1(r_0^{-p-\epsilon})$ together with the no incoming radiation condition (6.10.2). Then the resulting scattering solution satisfies the estimate (6.10.5) with $\alpha^{[s]}$ replaced by $\Psi^{[s]}$ and with s on the RHS of (6.10.5) replaced by 0.*

We will prove this corollary at the end of this section.

6.10.2 Overview of the proof

The proof consists of the following steps:

- I First, in §6.10.3, we compute sufficiently many transversal derivatives $(\frac{r^2}{D}\nabla_v)^n\alpha^{[s]}$ along $\underline{\mathcal{C}}$ (in an a priori fashion) by inductively integrating equation (6.9.6) from $\mathcal{S}_\infty$.
- II We then establish the existence of a scattering solution and prove a preliminary pointwise decay estimate (this decay is sharp near $\mathcal{I}^-$ but fails to be sharp near $\mathcal{I}^+$) for $\alpha^{[s]}$ and its derivatives using the Regge–Wheeler energy estimate (Lemma 6.6.1) in §6.10.4. Note that control over the Regge–Wheeler energy along $\underline{\mathcal{C}}$ is obtained from tangential ∇_u -derivatives along $\underline{\mathcal{C}}$ in the case $s \geq 0$, whereas, in the case of $s < 0$, it is obtained from *transversal* ∇_v -derivatives. Cf. Section 6.9.5.
- III Starting from this preliminary decay estimate, we next use the approximate conservation law (6.9.19) to derive an asymptotic expression for the quantity $\nabla_v A_\ell^{[s]}$ in §6.10.5.
- IV The asymptotic estimate for $\nabla_v A_\ell^{[s]}$ allows us to obtain an asymptotic estimate for $\nabla_v \left(\frac{r^2}{D}\nabla_v\right)^{\ell-s}\alpha^{[s]}$. We then integrate this latter estimate up to $\ell - s$ terms from $\underline{\mathcal{C}}$, using the estimates obtained in step a.), to obtain asymptotic estimates for all lower order derivatives, including $\alpha^{[s]}$ itself. This final step is carried out in §6.10.6.
- V In principle, we could then use all the asymptotic estimates for the lower-order derivatives to obtain a more precise asymptotic estimate for $\nabla_v A_\ell^{[s]}$ and iterate, but we will not do this here.

Remark 6.10.4. In the following, all the estimates arising from integrating along characteristics (e.g. $\int(\nabla_v\alpha^{[s]})dv$) will be comparing the components of $\alpha^{[s]}$ in the parallelly propagated frame (e_1, e_2) ; we will be using that, in view of (6.2.12),

$$\nabla_u(D^\ell r^{-2\ell}\nabla_v(D^{-1}r^2\nabla_v)^{\ell-s}\alpha^{[s]})_{A\dots B} = \partial_u(D^\ell r^{-2\ell}\partial_v(D^{-1}r^2\partial_v)^{\ell-s}\alpha_{A\dots B}^{[s]}). \quad (6.10.6)$$

For fixed angular modes $\alpha_\ell^{[s]}$, another way of viewing this is to write $\alpha_\ell^{[s]} = \sum_{m=-\ell}^\ell (\alpha_{\ell,m}^{[s],E} Y_{\ell,m}^{E,[s]} + \alpha_{\ell,m}^{[s],H} Y_{\ell,m}^{H,[s]})$, then we are just comparing the scalar functions $\alpha_{\ell,m}^{[s],E}(u, v)$, $\alpha_{\ell,m}^{[s],H}(u, v)$ at different points in spacetime.

6.10.3 Computing transversal derivatives along $\underline{\mathcal{C}}$

The following is to be understood as an a priori estimate:

Proposition 6.10.1. *Assume that a scattering solution $\alpha^{[s]}$ attaining the data of Theorem 6.10.1 exists, and assume that $\lim_{u \rightarrow -\infty} r^{-2s} \nabla_v^N \alpha^{[s]} = 0$ for any $N \geq 0$. Then we have the following expressions for transversal derivatives along $\underline{\mathcal{C}}$ if $N \leq \ell - s$:*

$$(D^{-1}r^2\nabla_v)^N \alpha_\ell^{[s]} \Big|_{\underline{\mathcal{C}}} = \underbrace{\mathcal{A}_\ell r_0^{N-p+s} \frac{(s+p)!}{(N+s+p)!} \prod_{i=0}^{N-1} \tilde{a}_{i,0,\ell}^{[s]}}_{:=C_{(N,p)}^\nearrow} + \mathcal{O}(r^{N-p+s-\epsilon} + Mr^{N-p+s-1}). \quad (6.10.7)$$

The product on the RHS can be computed as

$$\prod_{i=0}^{N-1} \tilde{a}_{i,0,\ell}^{[s]} = (-1)^N \frac{(\ell-s)!(N+s+\ell)!}{(\ell-s-N)!(\ell+s)!}. \quad (6.10.8)$$

In particular,

$$C_{(\ell-s,p)}^\nearrow = (-1)^{\ell-s} \frac{(s+p)!(\ell-s)!(2\ell)!}{(\ell+p)!(\ell+s)!}. \quad (6.10.9)$$

On the other hand, if $N > \ell - s$, then we have that

$$\left(\frac{r^2}{D} \nabla_v \right)^N \alpha_\ell^{[s]} \Big|_{\underline{\mathcal{C}}} = M \cdot \mathcal{E} r_0^{N-p+s-1} + M \mathcal{O}(r^{N-p+s-1-\epsilon} + Mr^{N-p+s-2}) \quad (6.10.10)$$

for some constants $\mathcal{E}$ whose precise values won't matter.

Proof. The proof proceeds by inductively integrating (6.9.6) from $\mathcal{S}_\infty$. The estimate (6.10.7) holds true for $N = 0$. Assume now that $N < \ell - s$ is fixed, and assume (6.10.7) to hold for all $n \leq N$. Then we can integrate (6.9.6) from $\mathcal{S}_\infty$, where we use the assumption that $\lim_{u \rightarrow -\infty} r^{-2s} \nabla_v^i \alpha_{\mathcal{I}^-}^{[s]} = 0$ for any $i \geq 0$, to obtain:

$$\begin{aligned} & \left(\frac{D}{r^2} \right)^{N+s} \nabla_v (D^{-1}r^2\nabla_v)^N \alpha^{[s]} \\ &= \int \left(\frac{D}{r^2} \right)^{N+s+1} \tilde{a}_{N,0,\ell}^{[s]} \cdot \mathcal{A}_\ell r_0^{N+s-p} \frac{(s+p)!}{(N+s+p)!} \prod_{i=0}^{N-1} \tilde{a}_{i,0,\ell}^{[s]} du + \mathcal{O}(r^{N+1+s-p-\epsilon} + Mr^{N+s-p}). \end{aligned}$$

This proves (6.10.7). Equation (6.10.8) is a direct computation. The last claim follows in a similar fashion, using now that $\tilde{a}_{\ell-s,0,\ell}^{[s]} = 0$. $\square$

6.10.4 A robust preliminary decay estimate based on energy conservation

Proposition 6.10.2. *There exists a unique, smooth scattering solution in $\mathcal{D}$ attaining the data described in Theorem 6.10.1. This solution satisfies the assumption of Prop. 6.10.1 and*

moreover satisfies the following pointwise bound throughout $\mathcal{D}$:

$$|\alpha_\ell^{[s]}| \leq \ell^{1+\max(-s,0)} C(\mathcal{A}_\ell, p, s, M) r_0^{-p-\frac{1}{2}} \max(r^{s+\frac{1}{2}}, r_0^{s+\frac{1}{2}}). \quad (6.10.11)$$

If $p > 0$, then the factor $\sqrt{r/r_0}$ can be dropped, and we have for any $N \geq 0$:

$$\left| \left(\frac{r^2}{D} \nabla_v \right)^N \alpha_\ell^{[s]} \right| \leq \ell^{1+2N+\max(-s,0)} \max\left(\left(\frac{r}{r_0}\right)^s, 1\right) r_0^{s-p} \max\left(r_0^N, r^N \cdot \left(\frac{r_0}{r}\right)^{\lfloor p-s \rfloor}\right). \quad (6.10.12)$$

Proof. Recall that, given a smooth solution $\alpha^{[s]}$ to the Teukolsky equation (Teuk:s), we know that the derived quantity $\Psi^{[s]}$, defined in (6.9.23), solves the Regge–Wheeler equation (6.9.24). Either by assumption (6.10.3) in the case $s \geq 0$, or by Prop. 6.10.1, we then know that $\Psi^{[s]}|_{\underline{\mathcal{C}}} = \mathcal{O}_1(r^{1/2-\epsilon'})$ for some $\epsilon' > 0$; in particular, it has finite Regge–Wheeler energy $E_{v_1}[\Psi^{[s]}|_{\underline{\mathcal{C}}}](-\infty, u_0)$ as defined in Def. 6.6.1. This means that our scattering theory for the Regge–Wheeler equation applies and that it provides us with a unique scattering solution $\Psi^{[s]}$ (cf. Thm 6.6.1). It is then trivial to reconstruct out of this the unique solution $\alpha^{[s]}$ by appropriately integrating the definition (6.9.24) either from $\underline{\mathcal{C}}$ or from $\mathcal{I}^-$, and it is moreover standard to show that all derivatives $r^{-2s} \nabla_v^N \alpha^{[s]} \rightarrow 0$ as $u \rightarrow -\infty$ for any $N \geq 0$.

We now move on to prove the estimate (6.10.11): By the Regge–Wheeler T -energy identity (cf. Lemma 6.6.1), we have that for any $v \geq v_1$:

$$\underline{E}_v[\Psi^{[s]}](-\infty, u) \leq \underline{E}_{v_1}[\Psi^{[s]}](-\infty, u) + \lim_{u' \rightarrow -\infty} E_{u'}[\Psi^{[s]}](v_1, v). \quad (6.10.13)$$

The limit on the RHS vanishes by assumption (6.10.2).

We now focus on fixed angular modes: Either directly by the initial data assumption in the case $s \geq 0$, or by Prop. 6.10.1 if $s < 0$, it follows that, along $\underline{\mathcal{C}}$,

$$\begin{aligned} \underline{E}_{v_1}[\Psi_\ell^{[s]}](-\infty, u) &= \int_{v=v_1, u' \in (-\infty, u)} |\nabla_u \Psi_\ell^{[s]}|^2 + \frac{D\ell(\ell+1)|\Psi_\ell^{[s]}|^2}{r^2} + \frac{6MD|\Psi_\ell^{[s]}|^2}{r^3} du' \\ &\leq \ell^{2+\max(-2s,0)} C(\mathcal{A}_\ell, p, s, M) r_0^{-2p-1} \end{aligned} \quad (6.10.14)$$

for some constant $C(\mathcal{A}_\ell, p, s, M)$ depending only on $\mathcal{A}_\ell, p, s, M$ and which is allowed to change from line to line. We can exploit the positive potential term in the estimate above to further estimate for any $v \geq v_1$:

$$\int_{v=\text{const}, u' \in (-\infty, u)} r^2 |\nabla_u (r^{-1} \Psi_\ell^{[s]})|^2 du' \lesssim \underline{E}_v[\Psi^{[s]}](-\infty, u). \quad (6.10.15)$$

Now, applying a simple fundamental theorem of calculus argument together with Cauchy–Schwarz and the previous three estimates, we obtain

$$|r^{-1}\Psi_\ell^{[s]}| \lesssim r^{-1/2} \sqrt{\underline{E}_v[\Psi_\ell^{[s]}](-\infty, u)} \leq \ell^{1+\max(-s,0)} C(\mathcal{A}_\ell, p, s, M) r_0^{-p-1/2}. \quad (6.10.16)$$

Integrating this $|s|$ times either from $\underline{\mathcal{C}}$ or from $\mathcal{I}^-$, we then obtain the estimate

$$|\alpha_\ell^{[s]}| \leq \ell^{1+\max(-s,0)} \max(r^{s+1/2}, r_0^{s+1/2}) r_0^{-1/2-p}. \quad (6.10.17)$$

This proves (6.10.11).

Let us now assume that $p > 0$. In order to then remove the factor $\sqrt{r/r_0}$, we need to consider a u -weighted version of the energy estimate Lemma 6.6.1: Indeed, it is easy to show that

$$\begin{aligned} & \underline{E}_v^{|u|^{1+\delta}}[\Psi_\ell^{[s]}](-\infty, u) \\ &:= \int_{v=\text{const}, u' \in (-\infty, u)} |u'|^{1+\delta} \left(|\nabla_u \Psi_\ell^{[s]}|^2 + \frac{D\ell(\ell+1)|\Psi_\ell^{[s]}|^2}{r^2} + \frac{6MD|\Psi_\ell^{[s]}|^2}{r^3} \right) du' \end{aligned} \quad (6.10.18)$$

decays in v for any $\delta > -1$. Since for any $p > 0$, there exists a $\delta > 0$ such that the quantity $\underline{E}_{v_1}^{|u|^{1+\delta}}[\Psi_\ell^{[s]}](-\infty, u)$ is finite at $\underline{\mathcal{C}}$, we can then repeat the previous argument, but starting with

$$|\Psi_\ell^{[s]}| \leq |u|^{-\delta/2} \sqrt{\underline{E}_{v_1}^{|u|^{1+\delta}}[\Psi_\ell^{[s]}](-\infty, u)} \quad (6.10.19)$$

instead of (6.10.16).

The statement about higher-order derivatives is then proved similarly to how we proved the estimate (6.10.7). $\square$

6.10.5 Deriving asymptotic expressions using the approximate conservation laws

Equipped with the pointwise estimate (6.10.11) derived from the Regge–Wheeler energy estimate, we shall now use the approximate conservation law (6.9.19) to derive asymptotic expressions for $\alpha_\ell^{[s]}$. Appealing to (6.9.19), while providing us with a simple and intuitive understanding for these asymptotic expressions, means that we cannot directly make statements uniform in ℓ , we will therefore stop keeping track of ℓ -dependencies of constants in estimates.

6.10.5.1 A preliminary estimate on $(\frac{r^2}{D}\nabla_v)^{\ell-s+1}\alpha^{[s]}$

We start by proving the following estimate:

Proposition 6.10.3. *The solution from Theorem 6.10.1 satisfies throughout $\mathcal{D}$:*

$$\left| \left(\frac{r^2}{D} \nabla_v \right)^{\ell-s+1} \alpha_\ell^{[s]} \right| \leq M \cdot C \max(r^{\ell-p}, r_0^{\ell-p}). \quad (6.10.20)$$

Proof. As a first step, notice that estimate (6.10.11) can be cast into the form

$$\left| \left(\frac{r^2}{D} \nabla_v \right)^N \alpha_\ell^{[s]} \right| \leq_N C(v) \max(r^{N+s-p}, r_0^{N+s-p}), \quad \forall N \in \mathbb{N}, \quad (6.10.21)$$

for $C_N(v)$ continuously depending on v . We will now show that there exists a sufficiently large global constant C_{bs} (depending also on M), to be determined later, such that

$$\left| \left(\frac{r^2}{D} \nabla_v \right)^{\ell+1-s} \alpha_\ell^{[s]} \right| \leq C_{\text{bs}} \max(r^{\ell+1-p}, r_0^{\ell+1-p}). \quad (6.10.22)$$

Consider the set $X := \{v \in [v_1, \infty) \text{ s.t. (6.10.22) holds } \forall v' \leq v, u \leq u_0\}$. If C_{bs} is sufficiently large, then either $\sup_X = \infty$, or $\sup_X = v_2 > v_1$ is finite. If the first case holds, we are done, so let us assume the second case.

In view of (6.10.7), integrating (6.10.22) immediately implies

$$\left| \left(\frac{r^2}{D} \nabla_v \right)^{\ell-s+1-i} \alpha_\ell^{[s]} \right| \leq C \cdot C_{\text{bs}} \max(r^{\ell+1-i-p}, r_0^{\ell+1-i-p}) \quad (6.10.23)$$

for all $i \in \{1, \dots, \ell - s + 1\}$. Inserting these estimates for $i = 1, 2$ into the approximate conservation law (6.9.19), we then obtain that

$$\left| r^2 D^\ell \nabla_v A_\ell^{[s]} \right| \lesssim r^{2\ell+2} \int \frac{1}{r^{3+2\ell}} C_{\text{bs}} \max(r^{\ell-p}, r_0^{\ell-p}) \lesssim C_{\text{bs}} \max(r^{\ell-p}, r_0^{\ell-p}), \quad (6.10.24)$$

where we used that the boundary term at $\mathcal{I}^-$ vanishes. Hence, recalling the definition of $A_\ell^{[s]}$, (6.9.18) and using the estimates (6.10.22), we deduce that

$$\left| \left(\frac{r^2}{D} \nabla_v \right)^{\ell-s+1} \alpha_\ell^{[s]} \right| \lesssim C_{\text{bs}} \max(r^{\ell-p}, r_0^{\ell-p}). \quad (6.10.25)$$

Estimate (6.10.25) improves (6.10.22) within X , provided that u_0 is sufficiently large. We can now integrate (6.10.21) with $N = \ell + 2 - s$ from $v = v_2$ to $v = v_2 + \delta$ for a sufficiently small distance $\delta > 0$ to show that $(v_2 + \delta) \in X$, a contradiction. Thus, $\sup_X = \infty$, and (6.10.25) holds throughout $\mathcal{D}$. $\square$

6.10.5.2 An asymptotic estimate for $\nabla_v A_\ell^{[s]}$

Next, we upgrade (6.10.20) to an asymptotic expression for $\nabla_v A_\ell^{[s]}$:

Proposition 6.10.4. *The solution from Theorem 6.10.1 satisfies the following estimates throughout $\mathcal{D}$: If $\ell - p > -1$, then*

$$\begin{aligned} \frac{r^2}{D} \nabla_v A_\ell^{[s]} &= 2M \mathcal{A}_\ell C_{(\ell-s,p)}^\nearrow (2x_{1,\ell}^{[s]} - c_{\ell-s,0}^{[s]}) \cdot \frac{(\ell-p)!(\ell+p+1)!}{(2\ell+2)!} \cdot r^{\ell-p} \\ &+ M \cdot \mathcal{O} \left(\frac{r_0 + M(1 + \delta_{\ell-p,0} \log r/r_0)}{r^{1-(\ell-p)}} + \frac{r_0^{\ell-p+1}}{r} + \frac{\delta_{\ell-p-\epsilon,-1} \log(r/r_0) + 1}{r^{\epsilon-(\ell-p)}} \right). \end{aligned} \quad (6.10.26)$$

If $\ell - p \in \mathbb{N}_{\geq 0}$, then we have more precisely:

$$\begin{aligned} \frac{r^2}{D} \nabla_v A_\ell^{[s]} &= 2M \mathcal{A}_\ell C_{(\ell-s,p)}^\nearrow (2x_{1,\ell}^{[s]} - c_{\ell-s,0}^{[s]}) \cdot \frac{(\ell-p)!(\ell+p+1)!}{(2\ell+2)!} \cdot r^{\ell-p} \\ &\cdot \left(1 + \sum_{i=1}^{\ell-p} \frac{1}{i!} \left(\frac{r_0}{r} \right)^i \prod_{j=1}^i (\ell+1+p+j) \right) + M \cdot \mathcal{O} \left(\frac{M(1 + \delta_{\ell-p,0} \log r/r_0)}{r^{1-(\ell-p)}} + \frac{\delta_{\ell-p-\epsilon,-1} \log(r/r_0) + 1}{r^{\epsilon-(\ell-p)}} \right). \end{aligned} \quad (6.10.27)$$

Note that $\frac{(2x_{1,\ell}^{[s]} - c_{\ell-s,0}^{[s]})}{(2\ell+2)!} = -\frac{1}{2(2\ell)!}$.

If $\ell - p = -1$, then

$$\frac{r^2}{D} \nabla_v A_\ell^{[s]} = 2M \mathcal{A}_\ell C_{(\ell-s,p)}^\nearrow (2x_{1,\ell}^{[s]} - c_{\ell-s,0}^{[s]}) \cdot \frac{\log(r/r_0)}{r} + M \cdot \mathcal{O}(r^{-1}). \quad (6.10.28)$$

If $\ell - p < -1$, then

$$\left| \frac{r^2}{D} \nabla_v A_\ell^{[s]} \right| \leq M C \frac{r_0^{\ell-p+1}}{r}. \quad (6.10.29)$$

Finally, we have that

$$\begin{aligned} \frac{r^2}{D} \nabla_v A_\ell^{[s]} &= \left(\frac{r^2}{D} \nabla_v \right)^{\ell-s+1} \alpha_\ell^{[s]} + 2M x_{1,\ell}^{[s]} \mathcal{A}_\ell C_{(\ell-s,p)}^\nearrow \cdot r_0^{\ell-p} \\ &+ M \mathcal{O}(r_0^{\ell-p-\epsilon} + \max(r^{\ell-p-1}, r_0^{\ell-p-1})(1 + \delta_{\ell-p,1} \log r/r_0)) \end{aligned} \quad (6.10.30)$$

Proof. Integrating the estimate (6.10.20) from $\underline{\mathcal{C}}$, we obtain that

$$\left| \left(\frac{r^2}{D} \nabla_v \right)^{\ell-s} \alpha_\ell^{[s]} - \left(\frac{r^2}{D} \nabla_v \right)^{\ell-s} \alpha_\ell^{[s]} \Big|_{\underline{\mathcal{C}}} \right| \lesssim M \cdot C \max(r^{\ell-p-1}, r_0^{\ell-p-1})(1 + \delta_{\ell-p,1} \log r/r_0), \quad (6.10.31)$$

and integrating (6.10.31) from $\underline{\mathcal{C}}$, using also the estimates (6.10.7) to estimate terms on $\underline{\mathcal{C}}$, gives

$$\left| \left(\frac{r^2}{D} \nabla_v \right)^{\ell-s-i} \alpha_\ell^{[s]} \right| \leq C r_0^{\ell-p-i} + M \cdot C \max(r^{\ell-p-i-1}, r_0^{\ell-p-i-1})(1 + \delta_{\ell-p-i,1} \log r/r_0). \quad (6.10.32)$$

(Here, we used Lemma 6.B.7 and the fact that $\frac{1}{r} \log \frac{r}{r_0} \lesssim \frac{1}{r_0}$.) We can now prove the proposition by re-inserting the improved estimates (6.10.31), (6.10.32) into (6.9.19):

$$\begin{aligned} \nabla_u(r^{-2\ell} D^\ell \nabla_v A_\ell^{[s]}) &= \frac{2M}{r^{2\ell+3}} (2x_{1,\ell}^{[s]} - c_{\ell-s,0}^{[s]}) \left(\frac{r^2}{D} \nabla_v \right)^{\ell-s} \alpha_\ell^{[s]}|_{\mathcal{C}} \\ &\quad + \frac{M^2}{r^{2\ell+3}} \mathcal{O}(\max(r^{\ell-p-1}, r_0^{\ell-p-1})(1 + \delta_{\ell-p,1} \log r/r_0)) \end{aligned}$$

Integrating in u , we thus obtain that

$$\begin{aligned} \left| r^{-2\ell} \nabla_v A_\ell^{[s]} - \mathcal{A}_\ell C_{(\ell-s,p)}^\nearrow \int_{-\infty}^u \frac{2Mr_0^{\ell-p}}{r^{2\ell+3}} (2x_{1,\ell}^{[s]} - c_{\ell-s,0}^{[s]}) du' \right| \\ \leq \frac{M^2 C(1 + \delta_{\ell-p,0} \log r/r_0)}{r^{\ell+3+p}} + \frac{MC(1 + \delta_{\ell-p-\epsilon,-1} \log \frac{r}{r_0})}{r^{\ell+2+p+\epsilon}}, \end{aligned} \quad (6.10.33)$$

where we used Lemmata 6.B.1–6.B.4 to estimate the RHS (note, in particular, that $\int r^{-N} \log r/r_0 du \leq r^{-N+1}$ for any $N \geq 1$ by Lemma 6.B.4).

We now need to evaluate the integral on the LHS of (6.10.33). If $\ell - p < -1$, then (6.10.29) follows immediately. If $\ell - p = -1$, then (6.10.28) follows from Lemma 6.B.2. If $\ell - p > -1$, then (6.10.26) follows from Lemma 6.B.1, (6.B.1).

The refined statement (6.10.27) for $\ell - p \in \mathbb{N}$ follows from (6.B.2), which implies:

$$\int \frac{r_0^{\ell-p}}{r^{2\ell+3}} du = \frac{(\ell-p)!(\ell+1+p)!}{(2\ell+2)!} \frac{1}{r^{\ell+2+p}} \left(1 + \sum_{i=1}^{\ell-p} \frac{1}{i!} \left(\frac{r_0}{r} \right)^i \frac{(\ell+1+p+i)!}{(\ell+1+p)!} + \mathcal{O}(M/r) \right).$$

□

6.10.5.3 Asymptotics for $\nabla_v(r^2/D \nabla_v)^{\ell-s} \alpha_\ell^{[s]}$ and higher derivatives

We now compute the asymptotics for $\nabla_v(r^2/D \nabla_v)^{\ell-s} \alpha_\ell^{[s]}$. Even though we could just deduce them from the asymptotics of $A_\ell^{[s]}$, we find it easier to compute them directly.

Proposition 6.10.5. *The solution from Theorem 6.10.1 satisfies the following estimates throughout $\mathcal{D}$: If $\ell - p > 0$, then*

$$\begin{aligned} \left(\frac{r^2}{D} \nabla_v \right)^{\ell-s+1} \alpha_\ell^{[s]} &= -M \mathcal{A}_\ell C_{(\ell-s,p)}^\nearrow \frac{(\ell-p)!(\ell+p+1)!}{(2\ell)!} r^{\ell-p} + M \mathcal{O}(r_0^{\ell-p} + (r_0 + M)r^{\ell-p-1}) \\ &\quad + M^2 \mathcal{O}(\max(r^{\ell-p-1}, r_0^{\ell-p-1})) + M \mathcal{O}(\max(r^{\ell-p-\epsilon}, r_0^{\ell-p-\epsilon})(1 + \delta_{\ell-p,\epsilon} \log r/r_0)). \end{aligned} \quad (6.10.34)$$

If $\ell - p \in \mathbb{N}_{\geq 0}$, then we have more precisely:

$$\left(\frac{r^2}{D} \nabla_v \right)^{\ell-s+1} \alpha_\ell^{[s]} = -M \mathcal{A}_\ell C_{(\ell-s,p)}^\nearrow \frac{c_{\ell-s,0}^{[s]} + a_{\ell-s,1}^{[s]}}{\ell+1} r_0^{\ell-p} - M \mathcal{A}_\ell C_{(\ell-s,p)}^\nearrow \frac{(\ell-p)!(\ell+p+1)!}{(2\ell)!} r^{\ell-p}$$

$$\cdot \left(1 - \delta_{\ell-p,0} + \sum_{i=1}^{\ell-p} \frac{1}{i!} \left(\frac{r_0}{r} \right)^i \prod_{j=1}^i (\ell + 1 + p + j) \right) + M \cdot \mathcal{O} \left(\frac{M}{r^{1-(\ell-p)}} + \frac{\delta_{\ell-p,\epsilon} \log(r/r_0) + 1}{r^{\epsilon-(\ell-p)}} \right). \quad (6.10.35)$$

If $\ell - p < 0$, then, for all $1 \leq i < [p - \ell] + 1$, the limit of $(\frac{r^2}{D} \nabla_v)^{\ell-s+i} \alpha_\ell^{[s]}$ as $v \rightarrow \infty$ exists, and

$$\lim_{v \rightarrow \infty} \left(\frac{r^2}{D} \nabla_v \right)^{\ell-s+i} \alpha_\ell^{[s]} = \mathcal{E} \cdot M r_0^{\ell-p+i-1} + M \mathcal{O}(r_0^{\ell-p+i-2} + r_0^{\ell-p+i-1-\epsilon}), \quad (6.10.36)$$

for some constants $\mathcal{E}$ whose values we don't determine. For $1 \leq i \leq [p - \ell]$, we have

$$\left(\frac{r^2}{D} \nabla_v \right)^{\ell-s+i} \alpha_\ell^{[s]} = \lim_{v \rightarrow \infty} \left(\frac{r^2}{D} \nabla_v \right)^{\ell-s+i} \alpha_\ell^{[s]} + M \mathcal{O}(r^{-1} r_0^{\ell-p+i}), \quad (6.10.37)$$

and, for $i = [p - \ell] + 1$, we have

$$\begin{aligned} \left(\frac{r^2}{D} \nabla_v \right)^{\ell-s+i} \alpha_\ell^{[s]} &= \lim_{v \rightarrow \infty} \left(\frac{r^2}{D} \nabla_v \right)^{\ell-s+i} \alpha_\ell^{[s]} \\ &\quad + M \cdot \mathcal{E} r^{\ell-p-1+[p-\ell]} (1 + \delta_{p-\ell, [p-\ell]} \log r/r_0) + M \mathcal{O}(r^{-1} r_0^{\ell-p+1}). \end{aligned} \quad (6.10.38)$$

Proof. The first part of the proof is very similar to the proof of Proposition 6.10.4, with the difference that now we want to integrate (6.9.6) rather than (6.9.19). Recall the estimate (6.10.31): Integrating it, we get

$$\begin{aligned} \left(\frac{r^2}{D} \nabla_v \right)^{\ell-s-1} \alpha_\ell^{[s]} &= \left(\frac{r^2}{D} \nabla_v \right)^{\ell-s-1} \alpha_\ell^{[s]} \Big|_{\underline{\mathcal{C}}} + \left(\frac{1}{r_0} - \frac{1}{r} \right) \left(\frac{r^2}{D} \nabla_v \right)^{\ell-s} \alpha_\ell^{[s]} \Big|_{\underline{\mathcal{C}}} \\ &\quad + M \mathcal{O}(\max(r^{\ell-p-2}, r_0^{\ell-p-2})(1 + \delta_{\ell-p,2} \log r/r_0)). \end{aligned} \quad (6.10.39)$$

We insert these estimates into (6.9.6), with $N = \ell - s$, to obtain

$$\begin{aligned} &\nabla_u \left(\frac{D^\ell}{r^{2\ell}} \nabla_v \left(\frac{r^2}{D} \nabla_v \right)^{\ell-s} \alpha_\ell^{[s]} \right) \\ &= - \frac{2M}{r} \left(\frac{D}{r^2} \right)^{\ell+1} (c_{\ell-s,0}^{[s]} + a_{\ell-s,1}^{[s]}) \left(\frac{r^2}{D} \nabla_v \right)^{\ell-s} \alpha_\ell^{[s]} \Big|_{\underline{\mathcal{C}}} \\ &\quad + 2M a_{\ell-s,1}^{[s]} \left(\frac{D}{r^2} \right)^{\ell+1} \left(\left(\frac{r^2}{D} \nabla_v \right)^{\ell-s-1} \alpha_\ell^{[s]} \Big|_{\underline{\mathcal{C}}} + \frac{1}{r_0} \left(\frac{r^2}{D} \nabla_v \right)^{\ell-s} \alpha_\ell^{[s]} \Big|_{\underline{\mathcal{C}}} \right) \\ &\quad + M^2 r^{-2\ell-2} \mathcal{O}(\max(r^{\ell-p-2}, r_0^{\ell-p-2})(1 + \delta_{\ell-p,2} \log r/r_0)) \\ &= - 2M (c_{\ell-s,0}^{[s]} + a_{\ell-s,1}^{[s]}) C_{(\ell-s,p)}^{\nearrow} \frac{r_0^{\ell-p}}{r^{2\ell+3}} + a_{\ell-s,1}^{[s]} (C_{(\ell-s-1,p)}^{\nearrow} + C_{(\ell-s,p)}^{\nearrow}) \frac{r_0^{\ell-p-1}}{r^{2\ell+2}} \\ &\quad + M^2 r^{-2\ell-2} \mathcal{O}(\max(r^{\ell-p-2}, r_0^{\ell-p-2})(1 + \delta_{\ell-p,2} \log r/r_0) + M \mathcal{O}(\frac{r_0^{\ell-p-1-\epsilon}}{r^{2\ell+2}})). \end{aligned} \quad (6.10.40)$$

Now, since

$$\frac{r_0^{\ell-p}}{r^{2\ell+3}} = \partial_u \left(\frac{1}{2\ell+2} \frac{r_0^{\ell-p}}{r^{2\ell+2}} \right) + \frac{\ell-p}{2\ell+2} \frac{r_0^{\ell-p-1}}{r^{2\ell+2}} + M\mathcal{O}\left(\frac{r_0^{\ell-p-2}}{r^{2\ell+2}}\right),$$

and since $C_{(\ell-s-1,p)}^\nearrow = -\frac{\ell+p}{2\ell} C_{(\ell-s,p)}^\nearrow$, integrating (6.10.40) gives

$$\begin{aligned} \left(\frac{r^2}{D}\nabla_v\right)^{\ell-s+1}\alpha_\ell^{[s]} &= -2M \frac{c_{\ell-s,0}^{[s]} + a_{\ell-s,1}^{[s]}}{2(\ell+1)} C_{(\ell-s,p)}^\nearrow r_0^{\ell-p} \\ &\quad - 2MC_{(\ell-s,p)}^\nearrow \left(\frac{\ell-p}{2(\ell+1)} (c_{\ell-s,0}^{[s]} + a_{\ell-s,1}^{[s]}) - a_{\ell-s,1}^{[s]} \left(1 - \frac{\ell+p}{2\ell}\right) \right) r^{2\ell+2} \int_{-\infty}^u \frac{r_0^{\ell-p-1}}{r^{2\ell+2}} du' \\ &\quad + M^2\mathcal{O}(\max(r^{\ell-p-1}, r_0^{\ell-p-1})) + M\mathcal{O}(\max(r^{\ell-p-\epsilon}, r_0^{\ell-p-\epsilon})(1 + \delta_{\ell-p,\epsilon} \log r/r_0)). \end{aligned} \quad (6.10.41)$$

Note that the second line in the estimate above vanishes if $\ell - p = 0$. (The integral itself would produce a logarithmic leading order term in that case!) Indeed, we have the following simple expression:

$$\begin{aligned} \left(\frac{\ell-p}{2(\ell+1)} (c_{\ell-s,0}^{[s]} + a_{\ell-s,1}^{[s]}) - a_{\ell-s,1}^{[s]} \left(1 - \frac{\ell+p}{2\ell}\right) \right) &= \frac{\ell-p}{2(\ell+1)} (c_{\ell-s,0}^{[s]} + a_{\ell-s,1}^{[s]}) - \frac{\ell+1}{\ell} a_{\ell-s,1}^{[s]} \\ &= \frac{\ell-p}{2(\ell+1)} ((1+s)(1+2s) + 3(\ell-s)(\ell+1+s) - (\ell-s)(\ell+s)) = \frac{(\ell-p)(2\ell+1)}{2}. \end{aligned} \quad (6.10.42)$$

The statements (6.10.34) and (6.10.35) then follow from Lemma 6.B.1.

In order to prove the remaining statements, we proceed inductively. Assume that $\ell - p < 0$. Our induction assumption is the following: For all $1 \leq i < [p - \ell]$, the limit $\lim_{v \rightarrow \infty} (\frac{r^2}{D}\nabla_v)^{\ell-s+i}\alpha_\ell^{[s]}(u, v)$ exists, and we have the estimates

$$\lim_{v \rightarrow \infty} \left(\frac{r^2}{D}\nabla_v\right)^{\ell-s+i}\alpha_\ell^{[s]} = \mathcal{E} \cdot Mr_0^{\ell-p+i-1} + M\mathcal{O}(Mr_0^{\ell-p+i-2} + r_0^{\ell-p+i-1-\epsilon}), \quad (6.10.43)$$

$$\left(\frac{r^2}{D}\nabla_v\right)^{\ell-s+i}\alpha_\ell^{[s]} = \lim_{v \rightarrow \infty} \left(\frac{r^2}{D}\nabla_v\right)^{\ell-s+i}\alpha_\ell^{[s]} + M\mathcal{O}(r^{-1}r_0^{\ell-p+i}). \quad (6.10.44)$$

We first establish the case $i = 1$: If $\ell - p < 0$, then (6.10.41) implies that $(\frac{r^2}{D}\nabla_v)^{\ell-s+1}\alpha_\ell^{[s]} = M\mathcal{O}(r_0^{\ell-p})$. We moreover have that $(\frac{r^2}{D}\nabla_v)^{\ell-s}\alpha_\ell^{[s]} = \mathcal{O}(r^{\ell-p})$ from (6.10.31). Inserting both of these estimates into (6.9.6) with $N = \ell - s + 1$ and integrating in u , we then obtain that

$$\begin{aligned} \nabla_v \left(\frac{r^2}{D}\nabla_v\right)^{\ell-s+1}\alpha_\ell^{[s]} &= M\mathcal{O} \left(r^{2\ell+2} \int \frac{r_0^{\ell-p}}{r^{2\ell+4}} du' \right) \\ &= M\mathcal{O} \left(\max \left(\frac{r_0^{\ell-p+1}}{r^2}, r^{\ell-p-1} (1 + \delta_{\ell-p,-1} \log r/r_0) \right) \right). \end{aligned}$$

Since $\ell - p < 0$, the RHS is integrable in v , and hence $(\frac{r^2}{D}\nabla_v)^{\ell-s+1}\alpha_\ell^{[s]}$ attains a limit at $\mathcal{I}^+$. If $1 < [p - \ell]$, then (6.10.43) and (6.10.44) follow directly, also taking into account (6.10.10). On the other hand, if $0 > \ell - p \geq -1$, then we obtain that⁴

$$\lim_{v \rightarrow \infty} (\frac{r^2}{D}\nabla_v)^{\ell-s+1}\alpha_\ell^{[s]} = \mathcal{E} \cdot Mr_0^{\ell-p} + M\mathcal{O}(r_0^{\ell-p+1-2} + r_0^{\ell-p+1-1-\epsilon}), \quad (6.10.45)$$

$$(\frac{r^2}{D}\nabla_v)^{\ell-s+1}\alpha_\ell^{[s]} = \lim_{v \rightarrow \infty} (\frac{r^2}{D}\nabla_v)^{\ell-s+1}\alpha_\ell^{[s]} + M\mathcal{O}(r^{\ell-p}(1 + \delta_{\ell-p,-1} \log r/r_0)). \quad (6.10.46)$$

Inserting the expressions (6.10.45) and (6.10.46) back into (6.9.6) with $N = \ell - s + 1$ then gives that

$$\begin{aligned} (\frac{r^2}{D}\nabla_v)^{\ell-s+1}\alpha_\ell^{[s]} &= \lim_{v \rightarrow \infty} (\frac{r^2}{D}\nabla_v)^{\ell-s+1}\alpha_\ell^{[s]} \\ &\quad + \mathcal{E} \cdot Mr^{\ell-p}(1 + \delta_{\ell-p,-1} \log r/r_0) + M\mathcal{O}(r^{-1}r_0^{\ell-p+1}). \end{aligned} \quad (6.10.47)$$

We have at this point proved (6.10.36)–(6.10.38) for $i = 1$.

Let now $\ell - p < -1$, and let us inductively assume that there exists $n \in \mathbb{N}$, $1 \leq n < [p - \ell] - 1$ such that (6.10.43) and (6.10.44) hold for all $1 \leq i \leq n$. We will show that (6.10.43) and (6.10.44) then also hold for $i = n + 1$. For this, we insert the estimates (6.10.43), (6.10.44) into (6.9.6) with $N = \ell - s + n$ and integrate in u ; this gives:

$$\frac{r^2}{D}\nabla_v(\frac{r^2}{D}\nabla_v)^{\ell-s+n}\alpha_\ell^{[s]} = r^{2\ell+2n+2}M \cdot \mathcal{O}\left(\int \frac{r_0^{\ell-p+n-1}}{r^{2\ell+2n+2}} du'\right) = M \cdot \mathcal{O}(r_0^{\ell-p+n}). \quad (6.10.48)$$

We now insert this estimate into (6.9.6) with $N = \ell - s + n + 1$, which similarly gives that

$$\frac{r^2}{D}\nabla_v(\frac{r^2}{D}\nabla_v)^{\ell-s+n+1}\alpha_\ell^{[s]} = r^{2\ell+2n+4}M \cdot \mathcal{O}\left(\int \frac{r_0^{\ell-p+n}}{r^{2\ell+2n+4}} du'\right) = M \cdot \mathcal{O}(r_0^{\ell-p+n+1}). \quad (6.10.49)$$

Integrating (6.10.49) from $\mathcal{C}$ (where we use estimate (6.10.10)) then establishes that $(\frac{r^2}{D}\nabla_v)^{\ell-s+n+1}\alpha_\ell^{[s]}$ attains a limit at $\mathcal{I}^+$, which satisfies (6.10.43). Integrating (6.10.49) from $\mathcal{I}^+$ then shows that (6.10.44) holds as well, thus completing the inductive argument.

In order to show the final missing part of the proposition, namely to show that (6.10.36) holds for $i = [p - \ell] + 1$ and that (6.10.38) holds, we proceed in the same way in which we proved (6.10.45)–(6.10.47). □

⁴Here, we use that, for any $q > 0$,

$$-(q-1) \int_{r_0}^r \frac{\log(r'/r_0)}{r'^q} dr' = \frac{\log(r/r_0)}{r^{q-1}} + \frac{1}{q-1} \left(\frac{1}{r^{q-1}} - \frac{1}{r_0^{q-1}} \right)$$

6.10.5.4 Asymptotic estimates for $(\frac{r^2}{D}\nabla_v)^j \alpha_\ell^{[s]}$ for $j \leq \ell - s$

We now compute the asymptotic expressions for all the lower order derivatives. In particular, this requires explicitly solving the Minkowskian problem, i.e. it requires explicitly solving the Teukolsky equation for $M = 0$. We begin with the following simple observation:

Proposition 6.10.6. *Let $j \in \{0, \dots, \ell - s\}$. The following identity holds for any smooth $\alpha_\ell^{[s]}$:*

$$\begin{aligned} \left(\frac{r^2}{D}\nabla_v\right)^{\ell-s-j}\alpha_\ell^{[s]}(u, v) &= \sum_{i=0}^j \frac{1}{i!} \left(\frac{1}{r_0} - \frac{1}{r}\right)^i \left(\frac{r^2}{D}\nabla_v\right)^{\ell-s-j+i}\alpha_\ell^{[s]} \Big|_{\underline{\mathcal{C}}} \\ &\quad + \underbrace{\int_{v_1}^v \frac{D}{r^2} \cdots \int_{v_1}^{v(j)} \frac{D}{r^2}}_{j+1 \text{ integrals}} \left(\frac{r^2}{D}\nabla_v\right)^{\ell-s+1}\alpha_\ell^{[s]} dv_{(j+1)} \cdots dv_1. \end{aligned} \quad (6.10.50)$$

Proof. Fundamental theorem of calculus and Lemma 6.B.5. □

We now make use of Proposition 6.10.6 to obtain an asymptotic estimate for all lower-order derivatives, and, in particular, for $\alpha_\ell^{[s]}$ itself. We first turn our attention to the second line of (6.10.50).

Proposition 6.10.7. *Let $j \in \{0, \dots, \ell - s\}$. The solution of Theorem 6.10.1 satisfies the following estimates: If $\ell - p > 0$ and if $j < \ell - p - 1$, then*

$$\begin{aligned} \left(\frac{r^2}{D}\nabla_v\right)^{\ell-s-j}\alpha_\ell^{[s]} &= \sum_{i=0}^j \frac{1}{i!} \left(\frac{1}{r_0} - \frac{1}{r}\right)^i \left(\frac{r^2}{D}\nabla_v\right)^{\ell-s-j+i}\alpha_\ell^{[s]} \Big|_{\underline{\mathcal{C}}} \\ &\quad - M\mathcal{A}_\ell C_{(\ell-s,p)}^\nearrow \frac{(\ell-p)(\ell+p+1)!}{(2\ell)!} (\ell-2-p-j)! r^{\ell-1-p-j} \\ &\quad \cdot (1 + \mathcal{O}((r_0/r)^{\ell-1-p-j} + r_0/r(1 + \delta_{\ell-2-p,j} \log r/r_0) + r^{-\epsilon}(1 + \delta_{\ell-p-1-\epsilon,j} \log r/r_0))). \end{aligned} \quad (6.10.51)$$

If $\ell - p > 0$ and $j = \ell - p - 1$, then

$$\begin{aligned} \left(\frac{r^2}{D}\nabla_v\right)^{\ell-s-j}\alpha_\ell^{[s]} &= \sum_{i=0}^j \frac{1}{i!} \left(\frac{1}{r_0} - \frac{1}{r}\right)^i \left(\frac{r^2}{D}\nabla_v\right)^{\ell-s-j+i}\alpha_\ell^{[s]} \Big|_{\underline{\mathcal{C}}} \\ &\quad - M\mathcal{A}_\ell C_{(\ell-s,p)}^\nearrow \frac{(\ell-p)(\ell+p+1)!}{(2\ell)!} (\log(r/r_0) + \mathcal{O}(r_0/r \log r/r_0 + r^{-\epsilon}(1 + \delta_{\ell,1} \log r/r_0))). \end{aligned} \quad (6.10.52)$$

If $j > \ell - p - 1$, then

$$\left(\frac{r^2}{D}\nabla_v\right)^{\ell-s-j}\alpha_\ell^{[s]} = \sum_{i=0}^j \frac{1}{i!} \left(\frac{1}{r_0} - \frac{1}{r}\right)^i \left(\frac{r^2}{D}\nabla_v\right)^{\ell-s-j+i}\alpha_\ell^{[s]} \Big|_{\underline{\mathcal{C}}}$$

$$+ M\mathcal{O}(r_0^{(\ell-1-p)-j} + (\ell-p)r^{\ell-1-p-j}(1 + \delta_{\ell-p \in \mathbb{Z}_{\neq 0}} \log r/r_0)). \quad (6.10.53)$$

Proof. The proof follows from taking the expression (6.10.50) and computing the integrals in the second line using the estimates from Proposition 6.10.5 and Lemmata 6.B.5–6.B.7. $\square$

Remark 6.10.5. Starting from the more precise estimate (6.10.35) in the case of $\ell - p \in \mathbb{N}_0$, we can also show in exactly the same way in which we proved the estimates of Proposition 6.10.7 that (setting $\epsilon = \infty$ for simplicity)

$$\begin{aligned} \alpha_\ell^{[s]} = & \sum_{i=0}^{\ell-s} \frac{1}{i!} \left(\frac{1}{r_0} - \frac{1}{r} \right)^i \left(\frac{r^2}{D} \nabla_v \right)^{\ell-s-j+i} \alpha_\ell^{[s]} \Big|_{\underline{\mathcal{C}}} \\ & + Mr_0^{s-p-1} \sum_{n=0}^{\ell-s+1} (Q_n + S'_n \log r_0/r + \mathcal{O}(Mr_0^{-1})) \left(\frac{r_0}{r} \right)^n \end{aligned}$$

for some (potentially vanishing) constants Q_n, S'_n that we do not yet determine. We will return to this point later, when we need to compute higher-order asymptotics (cf. §6.13).

We now turn our attention to the first line of (6.10.50):

Proposition 6.10.8. *Let $j \in \{0, \dots, \ell - s\}$. The solution of Theorem 6.10.1 satisfies the following estimates:*

$$\sum_{i=0}^j \frac{1}{i!} \left(\frac{1}{r_0} - \frac{1}{r} \right)^i \left(\frac{r^2}{D} \nabla_v \right)^{\ell-s-j+i} \alpha_\ell^{[s]} \Big|_{\underline{\mathcal{C}}} = \mathcal{A}_\ell r_0^{\ell-j-p} \sum_{n=0}^j \left(\frac{r_0}{r} \right)^n \cdot S_{\ell,p,j,n,s} (1 + \mathcal{O}(r_0^{-\epsilon} + Mr_0^{-1})), \quad (6.10.54)$$

where the constants $S_{\ell,p,j,n,s}$ are defined for $0 \leq n \leq j \leq \ell - s$ via

$$S_{\ell,p,j,n,s} := \frac{(-1)^{\ell-s+n} (s+p)! (\ell-s)! (2\ell-j+n)!}{n! (\ell+s)! (\ell+p)!} \binom{\ell-p}{\ell-p-j+n}. \quad (6.10.55)$$

Note that

$$S_{\ell,p,j,n,s} = 0 \iff (\ell-p \in \mathbb{N}_{\geq 0} \text{ and } n < j + p - \ell). \quad (6.10.56)$$

Proof. For simpler notation, we consider the case where $M = 0$ and $\epsilon = \infty$. Inserting the expressions from Proposition 6.10.1, we have

$$\begin{aligned}
& \sum_{i=0}^j \frac{1}{i!} \left(\frac{1}{r_0} - \frac{1}{r} \right)^i \left(\frac{r^2}{D} \nabla_v \right)^{\ell-s-j+i} \alpha_\ell^{[s]} \Big|_{\mathcal{C}} \\
&= \mathcal{A}(-1)^{\ell-s-j} r_0^{\ell-j-p} \frac{(s+p)!(\ell-s)!}{(\ell+s)!} \sum_{i=0}^j \frac{(-1)^i}{i!} \left(1 - \frac{r_0}{r} \right)^i \frac{(2\ell-j+i)!}{(\ell-j+i+p)!} \\
&= \mathcal{A}(-1)^{\ell-s-j} r_0^{\ell-j-p} \frac{(s+p)!(\ell-s)!}{(\ell+s)!} \sum_{n=0}^j \left(\frac{r_0}{r} \right)^n \sum_{i=0}^j \binom{i}{n} \frac{(-1)^{i+n}}{i!} \frac{(2\ell-j+i)!}{(\ell-j+i+p)!} \\
&= \mathcal{A} r_0^{\ell-j-p} \sum_{n=0}^j \left(\frac{r_0}{r} \right)^n \cdot S_{\ell,p,j,n,s},
\end{aligned} \tag{6.10.57}$$

where $S_{\ell,p,j,n,s}$ are constants that we now determine by observing that the expression above, for $j = \ell - s$, solves (Teuk:s) with $M = 0$, i.e. it solves (6.14.3). We can thus appeal to Proposition 6.14.1:⁵ Since $(r^2 \nabla_v)^{\ell-s} \alpha_\ell^{[s]} = (r^2 \nabla_v)^{\ell-s} \alpha_\ell^{[s]}|_{\mathcal{C}} = \mathcal{A}_\ell C_{(\ell-s,p)}^\nearrow$, we infer the value of $S_{\ell-s} (= S_{\ell,p,\ell-s,n=\ell-s,s})$ via (6.14.9): $S_{\ell-s} = \frac{(s+p)!}{(\ell+p)!} \frac{(2\ell)!}{(\ell+s)!}$, and (6.10.55) thus follows for $j = \ell - s$.

In order to compute the constants $S_{\ell,p,j,n,s}$ for $j < \ell - s$, we simply observe that $S_{\ell,p,j,n,s} = (-1)^{\ell-s-j} \frac{(n+\ell-s-j)!}{n!} S_{\ell,p,\ell-s,n,s}$, as can be seen by differentiating the expression (6.10.54).

Considering now the case $M \neq 0 \neq \epsilon$ gives the error terms $\mathcal{O}(Mr_0^{-1} + r_0^{-\epsilon})$. $\square$

6.10.6 Asymptotic expansion for $\alpha_\ell^{[s]}$ and the proofs of Thm. 6.10.1 and Cor. 6.10.1

We combine the results of the previous sections into the following proposition:

Proposition 6.10.9. *The scattering solution $\alpha_\ell^{[s]}$ described in Theorem 6.10.1 satisfies the following expansion throughout $\mathcal{D}$. Suppose that $\ell - p \geq -1$.*

$$\begin{aligned}
\alpha_\ell^{[s]} &= \mathcal{A}_\ell r_0^{s-p} \sum_{n=0}^{\max(\ell-s, \lceil p-s \rceil)} S_{\ell,p,\ell-s,n,s} \cdot \left(\frac{r_0}{r} \right)^n (1 + \mathcal{O}(r_0^{-\epsilon} + Mr_0^{-1})) \\
&\quad - M \mathcal{A}_\ell C_{(\ell-s,p)}^\nearrow \frac{(\ell+p+1)!}{(2\ell)!} \cdot r^{s-1-p} \\
&\cdot \begin{cases} (s-p-2)! \{(\ell-p) \\ + \mathcal{O}((\frac{r_0}{r})^{s-p-1} + \frac{r_0}{r} (1 + \delta_{s-2-p,0} \log r/r_0) + r^{-\epsilon} (1 + \delta_{s-p-1,\epsilon} \log r/r_0))\}, & 1+p-s \notin \mathbb{N}_{\geq 0} \\ \frac{(-1)^{p+1-s}}{(p+1-s)!} ((\ell-p) \log r/r_0 + \mathcal{O}(\frac{r_0}{r} \log r/r_0 + r^{-\epsilon} (1 + \delta_{\epsilon,1} \log r/r_0))), & 1+p-s \in \mathbb{N}_{\geq 0} \end{cases}
\end{aligned} \tag{6.10.58}$$

⁵For the reader who cannot be bothered to skip to section 6.14, all we are doing here is inserting the above expression into the Teukolsky equation with $M = 0$ and solving the resulting recurrence relations for $S_{\ell,p,j=\ell-s,n,s}$.

If, on the other hand, $\ell - p < 0$, then we define $S_{\ell,p,j,n,s}$ to be zero whenever $n > j$, and the above formula still applies, though we make no claims on the coefficients in the third line.

Proof. If $\ell - s \leq \ell - p - 1$, i.e., if $s \geq p + 1$, then the result directly follows from setting $j = \ell - s$ in (6.10.51) or (6.10.52).

If $\ell - p > 0$ and $\ell - s > \ell - p - 1$, then integrate either (6.10.51) or (6.10.52), with $j = \ell - [p+1]$, $[p+1] - s$ times from $\mathcal{I}^+$ (instead of from $\mathcal{C}$), using (6.10.53) to estimate the boundary terms on $\mathcal{I}^+$. If $\ell - p \leq 0$, then integrate instead the expressions from Proposition 6.10.5 an appropriate amount of times from $\mathcal{I}^+$. $\square$

This concludes the proof of Thm. 6.10.1. We finish the section by proving its Corollary 6.10.1:

Proof of Cor. 6.10.1. In order to prove the corollary, we repeat the proof of Theorem 6.10.1, but instead of using the approximate conservation law (6.9.19) for $\alpha^{[s]}$, we appeal to the approximate conservation law (6.9.27) for $\Psi^{[s]}$. Of course, if $s = 0$, then $\Psi^{[s]} = \alpha^{[s]}$ and the result trivially follows. But if we now consider the Regge–Wheeler equation for general spin $s \neq 0$, we can simply exploit that the coefficients $a_{N,0}^{[s],\text{RW}}$ and $b_{N,0}^{[s],\text{RW}}$ as well as the eigenvalues of $\mathring{\Delta}_{[s]} - \Lambda_0^{[s]}$ in (6.9.27) are independent of s , so the contribution in the first line of (6.10.58), which is entirely Minkowskian, is as for $s = 0$.

The contribution in the last two lines of (6.10.58), on the other hand, is generated by integration of (6.9.19) or (6.9.27) to obtain the leading-order behaviour of $\nabla_v A_\ell^{[s]}$ or $\nabla_v \Psi_\ell^{[s]}$, respectively. This, in turn, is now governed by the coefficient $2x_{1,\ell}^{[s]} - c_{\ell-s,0}^{[s]}$ in the case of $\nabla_v A_\ell^{[s]}$ (cf. Prop. 6.10.4), or, in the case of $\nabla_v \Psi_\ell^{[s]}$, by the coefficient $2x_{1,\ell}^{[s],\text{RW}} - c_{\ell,0}^{[s],\text{RW}}$. But since $2x_{1,\ell}^{[s],\text{RW}} - c_{\ell,0}^{[s],\text{RW}} = 2x_{1,\ell}^{[s]} - c_{\ell-s,0}^{[s]} = -(\ell+1)(2\ell+1)$ is independent of s , the result is then identical to that of $s = 0$. $\square$

6.11 Asymptotics for $\alpha^{[2]} = r^5 \Omega^{-2} \overset{(i)}{\alpha}$ arising from physical data

We now apply the results of the previous section to find the asymptotics of $\overset{(i)}{\alpha}$ arising from the physical data described in §6.8 (cf. Prop. 6.8.1). Recall from (6.8.13) that these data satisfy the following expansion along $\underline{\mathcal{C}}$:

$$r^{-s} \alpha_{\underline{\mathcal{C}}}^{[s=2]} = \frac{r^3}{\Omega^2} \overset{(i)}{\alpha}_{\underline{\mathcal{C}}} = \mathcal{A} + \mathcal{B} \frac{\log r_0}{r_0} + \frac{\mathcal{C}}{r_0} + \mathcal{O}(r_0^{-1-\delta}), \quad (6.11.1)$$

with

$$\mathcal{B} = 2\overset{\circ}{\mathcal{P}}_2^* \overset{\circ}{\mathcal{P}}_1^* \overline{\overset{\circ}{\mathcal{P}}_1} \overset{\circ}{\mathcal{P}}_2 \mathcal{B}. \quad (6.11.2)$$

The presence of the logarithmic term in (6.11.1) means that we will have to modify the proof of Theorem 6.10.1 in a minor way. The description of this modification and the result of it is the content of this section.

6.11.1 The main theorem (Thm. 6.11.1)

Theorem 6.11.1. *Let $\overset{(i)}{\alpha}$ be the unique, smooth scattering solution to (Teuk) arising from the physical data of §6.8, i.e. arising from (6.10.2) with $s = 2$ and (6.11.1). Then each angular mode $\overset{(i)}{\alpha}_\ell$ of $\overset{(i)}{\alpha}$ satisfies the following asymptotic expansion throughout $\mathcal{D}$:*

$$\begin{aligned} \frac{r^5 \overset{(i)}{\alpha}_\ell}{\Omega^2} &= \sum_{n=0}^{\ell-2} \left(\frac{r_0}{r} \right)^n \{ \mathcal{A}_\ell r_0^2 S_{\ell,0,\ell-2,n,2} + \mathcal{B}_\ell r_0 \log r_0 S_{\ell,1,\ell-2,n,2} \\ &\quad + \mathcal{O}(M \mathcal{A}_\ell r_0) + \mathcal{O}(\mathcal{B}_\ell r_0) + \mathcal{O}(\mathcal{C}_\ell r_0) + \mathcal{O}(r_0^{1-\delta}) \} \\ &\quad - \frac{(-1)^\ell 2M \mathcal{A}_\ell r}{(\ell-1)(\ell+2)} - \frac{(-1)^\ell 3M \mathcal{B}_\ell \log^2 r}{\ell(\ell+1)} + M \log r (O(\mathcal{A}_\ell) + O(\mathcal{B}_\ell) + O(\mathcal{C}_\ell)), \end{aligned} \quad (6.11.3)$$

where the constants $S_{\ell,p,\ell-s,n,s}$ are defined in (6.10.55).

Moreover, the following limits are conserved along $\mathcal{I}^+$:

$$\mathbf{I}_\ell^{2-\ell,0}[\alpha^{[2]}] = \lim_{v \rightarrow \infty} r^{2-\ell} \nabla_v \mathbf{A}_\ell^{[2]} = (-1)^{\ell+1} 2M \mathcal{A}_\ell \frac{(\ell-2)!(\ell+1)}{(\ell+2)!}, \quad (6.11.4)$$

$$\lim_{v \rightarrow \infty} r^4 \overset{(i)}{\alpha}_\ell = (-1)^{\ell+1} \frac{2M \mathcal{A}_\ell}{(\ell-1)(\ell+2)}. \quad (6.11.5)$$

6.11.2 Proof of Theorem 6.11.1

Proof. This is essentially an application of Theorem 6.10.1 with $s = 2$ and a superposition of $(p = 0, \epsilon = 1 + \delta)$ and $(p = 1, \epsilon = \infty)$. The only modification comes from the logarithmic term in (6.11.1), $\mathcal{B} r_0^{-1} \log r_0$. Throughout the rest of this proof, let us therefore assume initial data

satisfying

$$r^{-s} \alpha_{\underline{\mathcal{C}}}^{[s]} = \mathcal{B} r_0^{-p} \log r_0 + \mathcal{C} r_0^{-p}, \quad p > \max(-1/2, -s + 1). \quad (6.11.6)$$

For convenience, we keep the exponent p general for now and only restrict to $p = 1$ in the end. In order to avoid case distinctions, we will in the propositions below also make the additional assumption that $\ell > p$.

For logarithmically decaying data as in (6.11.6), we first need an analogue of Proposition 6.10.1:

Proposition 6.11.1. *For $N \leq \ell - s$, transversal derivatives along $\underline{\mathcal{C}}$ satisfy:*

$$(D^{-1} r^2 \nabla_v)^N \alpha_{\underline{\mathcal{C}}}^{[s]} \Big|_{\underline{\mathcal{C}}} = C_{(N,p)}^{\nearrow} r_0^{N-p+s} \log r_0 \cdot \left(\mathcal{B}_\ell + \frac{\mathcal{B}_\ell}{\log r_0} \sum_{k=1}^N \frac{1}{k+s+p} + \frac{\mathcal{C}_\ell}{\log r_0} \right) + \mathcal{O}(M r_0^{N-p+s-1} \log r_0). \quad (6.11.7)$$

Proof. The proof goes by induction as in Proposition 6.10.1, using that

$$\int_{-\infty}^u r_0^{-k-s-p-1} \log r_0 \, du' = \frac{r_0^{-k-s-p}}{k+s+p} \left(\log r_0 + \frac{1}{k+s+p} \right) + M \mathcal{O}(r_0^{-k-s-p-1} \log r_0).$$

□

Analogously to how we proved Proposition 6.10.3, we then show that

$$\left| \left(\frac{r^2}{D} \nabla_v \right)^{\ell-s+1} \alpha_{\underline{\mathcal{C}}}^{[s]} \right| \leq M C \max(r^{\ell-p} \log r, r_0^{\ell-p} \log r_0), \quad (6.11.8)$$

and, integrating this from $\underline{\mathcal{C}}$, we obtain a logarithmically adorned version of (6.10.31).

We can now prove analogues of Propositions 6.10.4 and 6.10.5:

Proposition 6.11.2. *Suppose that $\ell - p > 0$. Then*

$$\lim_{v \rightarrow \infty} \frac{r^{p-\ell}}{\log r} \frac{r^2}{D} \nabla_v A_{\underline{\mathcal{C}}}^{[s]} = -M \mathcal{B}_\ell C_{(\ell-s,p)}^{\nearrow} \frac{(\ell-p)!(\ell+p+1)!}{(2\ell)!}, \quad (6.11.9)$$

and

$$\begin{aligned} \left(\frac{r^2}{D} \nabla_v \right)^{\ell-s+1} \alpha_{\underline{\mathcal{C}}}^{[s]} &= -M C_{(\ell-s,p)}^{\nearrow} \frac{(\ell-p)!(\ell+p+1)!}{(2\ell)!} r^{\ell-p} \log r \\ &\quad \cdot \left(\mathcal{B}_\ell \left(1 + \frac{1}{\log r} \left(d_{2\ell+3,\ell-p} + \sum_{k=1}^{\ell-s} \frac{1}{k+s+p} \right) \right) + \mathcal{C}_\ell + \mathcal{O}(r_0/r) \right), \end{aligned} \quad (6.11.10)$$

where $d_{2\ell+3,\ell-p}$ is defined in Lemma 6.B.1. If $p \in \mathbb{N}$, then $d_{2\ell+3,\ell-p} = -\sum_{k=\ell-p+1}^{\ell+p+1} \frac{1}{k}$.

Proof. The proof proceeds in the same manner as the proof of Proposition 6.10.4, with the difference that we now need to compute the integral

$$\int \left(\mathcal{B}_\ell \frac{r_0^{\ell-p} \log r_0}{r^{2\ell+3}} + \left(\mathcal{B}_\ell \sum_{k=1}^{\ell-s} \frac{1}{k+s+p} + \mathcal{C}_\ell \right) \frac{r_0^{\ell-p}}{r^{2\ell+3}} \right) du,$$

which we do using (6.B.1) and (6.B.3) for the second and first term, respectively. $\square$

We are left with having to integrate (6.11.10) $\ell + 1$ times from $\underline{\mathcal{C}}$. For the Minkowskian part of the expression, we have

Proposition 6.11.3. *Suppose that $\ell - p \geq 0$ and that $p \in \mathbb{N}$. Then we have:*

$$\begin{aligned} & \sum_{i=0}^{\ell-s} \frac{1}{i!} \left(\frac{1}{r_0} - \frac{1}{r} \right)^i \left(\frac{r^2}{D} \nabla_v \right)^{\ell-s-j+i} \alpha_\ell^{[s]} \Big|_{\underline{\mathcal{C}}} \\ &= (\mathcal{B}_\ell \log r_0 + \mathcal{C}_\ell) r_0^{s-p} \sum_{n=0}^{\ell-s} \left(\frac{r_0}{r} \right)^n \cdot S_{\ell,p,\ell-s,n,s} (1 + \mathcal{O}(Mr_0^{-1})) \\ &+ \mathcal{B}_\ell r_0^{s-p} \sum_{n=0}^{p-s-1} \left(\frac{r_0}{r} \right)^n R_{\ell,p,\ell-s,n,s} \\ &+ \mathcal{B}_\ell r_0^{s-p} \sum_{n=p-s}^{\ell-s} \left(\frac{r_0}{r} \right)^n S_{\ell,p,\ell-s,n,s} \left(\sum_{k=0}^{\ell-s-1-n} \frac{1}{\ell-p-k} + \sum_{k=1}^{\ell-s} \frac{1}{k+s+p} \right), \end{aligned} \quad (6.11.11)$$

where

$$R_{\ell,p,\ell-s,n,s} := (-1)^{\ell-p+1} \frac{(\ell-p)!(\ell-s)!(p+s)!}{(\ell+p)!(\ell+s)!} \frac{(p-n-s-1)!(\ell+n+s)!}{n!(\ell-n-s)!}. \quad (6.11.12)$$

Proof. As in the proof of Prop. 6.10.8, we show the result for $M = 0$: Let $j \in \{0, \dots, \ell - s\}$. Then we have

$$\begin{aligned} & \sum_{i=0}^j \frac{1}{i!} \left(\frac{1}{r_0} - \frac{1}{r} \right)^i \left(\frac{r^2}{D} \nabla_v \right)^{\ell-s-j+i} \alpha_\ell^{[s]} \Big|_{\underline{\mathcal{C}}} \\ &= (\mathcal{B}_\ell \log r_0 + \mathcal{C}_\ell) r_0^{\ell-j-p} \sum_{n=0}^j \left(\frac{r_0}{r} \right)^n \cdot S_{\ell,p,j,n,s} \\ &+ (-1)^{\ell-s-j} r_0^{\ell-j-p} \frac{(s+p)!(\ell-s)!}{(\ell+s)!} \\ &\cdot \sum_{n=0}^j \left(\frac{r_0}{r} \right)^n \sum_{i=0}^j \binom{i}{n} \frac{(-1)^{i+n}}{i!} \frac{(2\ell-j+i)!}{(\ell-j+i+p)!} \left(\sum_{k=1}^{\ell-s-j+i} \frac{1}{k+p+s} \right). \end{aligned} \quad (6.11.13)$$

For $j = \ell - s$, we now apply Proposition 6.14.2 to get a simpler expression for the last line: By comparing (6.11.7), with $N = \ell - s$, and (6.14.14), we can read off the constants $S_{\ell-s}$ and $R_{\ell-s}$; in particular, we have $R_{\ell-s}/S_{\ell-s} = \sum_{k=1}^{\ell-s} \frac{1}{k+s+p}$. The result then follows. $\square$

Lastly, we prove the analogue of Proposition 6.10.7:

Proposition 6.11.4. *Let $\ell - p \in \mathbb{N}_{>0}$. Then*

$$\begin{aligned} \frac{r^5 \overset{(i)}{\alpha}_\ell}{\Omega^2} &= \sum_{i=0}^{\ell-s} \frac{1}{i!} \left(\frac{1}{r_0} - \frac{1}{r} \right)^i \left(\frac{r^2}{D} \nabla_v \right)^i \alpha_\ell^{[s]}|_{\mathcal{C}} \\ &\quad - MC_{(\ell-s,p)}^{\nearrow} \frac{(\ell-p)(\ell+p+1)!}{(2\ell)!} \frac{(-1)^{p+1-s}}{2(p+1-s)!} r^{s-1-p} \\ &\quad \cdot \left(\mathcal{B}_\ell \log^2 r + \log r \left(\mathcal{C}_\ell + \mathcal{B}_\ell \left(1 - \sum_{k=1}^{p+1-s} \frac{1}{k} - \sum_{k=\ell-p+1}^{\ell+p+1} \frac{1}{k} + \sum_{k=1+s+p}^{\ell+p} \frac{1}{k} \right) \right) + \mathcal{O}(\log^2 r_0) \right). \end{aligned} \quad (6.11.14)$$

Proof. The proof is similar to the one of Proposition 6.10.7, with the difference being that we now have to appeal to Lemma 6.B.8 as well (which is responsible for the $\mathcal{B}_\ell(\log^2 r + \log r(1 - \sum_{k=1}^{p+1-s} \frac{1}{k}))$ -term). $\square$

Superposing Theorem 6.10.1 with $p = 0$ and $\epsilon = 1 + \delta$ with the combined results of Propositions 6.10.6, 6.11.3 and 6.11.4 with $p = 1$ proves (6.11.3). The other two equations (6.11.4) and (6.11.5) follow directly from Thm. 6.10.1 (as they are independent of $\mathcal{B}$). $\square$

6.12 Asymptotics for $\Psi^{[\pm 2]}$ arising from physical data

We now apply the results of §6.10 to the solutions to the Regge–Wheeler equations $\Psi^{[\pm 2]}$ arising from the physical data described in §6.8 (cf. Prop. 6.8.1). As we have seen in (6.8.14) and (6.8.15), respectively, both Regge–Wheeler quantities, $\overset{(i)}{\Psi} = \Psi^{[2]}$ and $\underline{\overset{(i)}{\Psi}} = \Psi^{[-2]}$, will have the following expansion along $\underline{\mathcal{C}}$:

$$\Psi_{\underline{\mathcal{C}}}^{[s]} = \mathcal{P}^{1,[s]} + \mathcal{P}^{2,[s]} r_0^{-1} \log r_0 + \mathcal{P}^{3,[s]} r_0^{-1} + \mathcal{O}(r_0^{-1-\delta}), \quad (6.12.1)$$

where the precise values of $\mathcal{P}^{3,[s]}$ do not matter and where

$$\mathcal{P}^{1,[2]} = 2\mathcal{A} = \overline{\mathcal{P}^{1,[-2]}}, \quad \mathcal{P}^{2,[2]} = 12\mathring{\mathcal{P}}_2^* \mathring{\mathcal{P}}_1^* \overline{\mathring{\mathcal{P}}_1} \mathring{\mathcal{P}}_2 \mathcal{B} = \overline{\mathcal{P}^{2,[2]}}. \quad (6.12.2)$$

In just the same way in which Cor. 6.10.1 followed from Thm. 6.10.1, we can directly infer the following result from Thm. 6.11.1 by setting $s = 0$:

Theorem 6.12.1. *For $\ell \geq |s|$, the unique, smooth scattering solution $\Psi^{[s]}$ restricting to (6.12.1) along $\underline{\mathcal{C}}$ and satisfying the no incoming radiation condition (6.10.2) at $\mathcal{I}^-$ has the following asymptotic expansion for each angular mode $\Psi_\ell^{[s]}$ throughout $\mathcal{D}$:*

$$\begin{aligned} \Psi_\ell^{[s]} = & \sum_{n=0}^{\ell} \left(\frac{r_0}{r} \right)^n \left(\mathcal{P}_\ell^{1,[s]} S_{\ell,0,\ell,n,0} + \mathcal{P}_\ell^{2,[s]} S_{\ell,1,\ell,n,0} r_0^{-1} \log r_0 + \mathcal{O}(r_0^{-1}) \right) \\ & + M(-1)^\ell \left(\ell(\ell+1) \mathcal{P}_\ell^{1,[s]} r^{-1} \log r / r_0 - \frac{1}{2}(\ell-1)(\ell+2) \mathcal{P}_\ell^{2,[s]} r^{-2} \log^2 r + \mathcal{O}(r^{-2} \log r) \right), \end{aligned} \quad (6.12.3)$$

where the constants $S_{\ell,p,\ell-s,n,s}$ are defined in (6.10.55).

Remark 6.12.1. Notice, in particular, that

$$\lim_{v \rightarrow \infty} \Psi_\ell^{[s]} = \mathcal{P}_\ell^{1,[s]} S_{\ell,0,\ell,0,0} + \mathcal{O}(r_0^{-1} \log r_0) = (-1)^\ell \mathcal{P}_\ell^{1,[s]} + \mathcal{O}(r_0^{-1} \log r_0), \quad (6.12.4)$$

and hence

$$\lim_{u \rightarrow -\infty} \lim_{v \rightarrow \infty} \Psi_\ell^{[s]} = (-1)^\ell \lim_{v \rightarrow \infty} \lim_{u \rightarrow -\infty} \Psi_\ell^{[s]}. \quad (6.12.5)$$

This is, in essence, the antipodal matching condition at the level of linearised gravity. See [Mas22b] for an extended discussion.

Proof. The statement follows by setting $s = 0$ in the proof of Theorem 6.11.1, cf. the proof of Cor. 6.10.1 at the end of §6.10. $\square$

6.13 Asymptotics for $\alpha^{[-2]} = r\Omega^2 \underline{\alpha}$ arising from physical data

We now move to the asymptotic analysis of $\underline{\alpha}$ in the case of the physical data of §6.8. Recall from (6.8.8) that these satisfy the following expansion along $\underline{\mathcal{C}}$:

$$r^{-s} \alpha_{\underline{\mathcal{C}}}^{[s=-2]} = r^3 \Omega^2 \underline{\alpha}_{\underline{\mathcal{C}}} = -6 \underline{\mathcal{B}} r^{-p} + \mathcal{O}(r^{-p-\delta}) \quad (6.13.1)$$

with $p = 1$. We thus already see that, since the condition $p > -s - 1$ is violated (we have $p = 1$ and $s = -2$), Theorem 6.10.1 does not directly apply (cf. (6.10.3)).

We recall that the condition of no incoming radiation (6.10.2) in the case $s < 0$ reads

$$\lim_{u \rightarrow -\infty} \nabla_v \left(\frac{r^2}{D} \nabla_v \right)^{|s|} \alpha^{[s]} \equiv 0. \quad (6.13.2)$$

It is thus clear that the specification of (6.13.1) and (6.13.2) does not suffice to determine a solution $\alpha^{[s]}$ to the scattering problem if $s < 0$ —one needs to additionally specify all lower-order transversal derivatives $(\frac{r^2}{D} \nabla_v)^i \alpha^{[s]}$, $i \in \{1, \dots, |s|\}$ as seed data at a sphere along $\underline{\mathcal{C}}$.

In Theorem 6.10.1, we simplified our lives by imposing $\nabla_v^i (r^5 \underline{\alpha})|_{\mathcal{I}^-} = 0$ for $i = 1, \dots, s$, but this condition had no physical motivation.

In the present section, we instead discuss the physically motivated setup for the data of §6.8: Since, for general p violating $p > -s - 1$ in (6.13.1), the integral along $\underline{\mathcal{C}}$ of (Teuk:s) for $s = -2$ diverges, the seed data for the transversal derivatives cannot be specified at $\mathcal{S}_\infty$. Instead, they have to be specified at the finite sphere $\mathcal{S}_1$ (or at any other finite sphere). In the context of our physical data, these data for transversal derivatives are encoded in (6.8.15) and (6.8.16) (recall the definitions (6.3.37)). We have:

$$\frac{r^2}{D} \nabla_v \alpha^{[-2]}|_{\underline{\mathcal{C}}} = \underline{\Psi}_{\underline{\mathcal{C}}} = 6(-2\mathring{\mathcal{P}}_2^* \mathring{\mathcal{P}}_2) \underline{\mathcal{B}} \frac{\log r_0}{r_0^2} + \mathcal{C}_8 r_0^{-2} + \mathcal{O}(r^{-2-\delta}), \quad (6.13.3)$$

(note that $(-2\mathring{\mathcal{P}}_2^* \mathring{\mathcal{P}}_2 \underline{\mathcal{B}})_\ell = \tilde{a}_{0,0,\ell}^{[-2]} \underline{\mathcal{B}}_\ell = -(\ell-1)(\ell+2) \underline{\mathcal{B}}_\ell$) and

$$\left(\frac{r^2}{D} \nabla_v \right)^2 \alpha^{[-2]}|_{\underline{\mathcal{C}}} = \underline{\Psi}_{\underline{\mathcal{C}}} = 2\mathring{\mathcal{P}}_2^* \mathring{\mathcal{P}}_1^* \mathring{\mathcal{P}}_1 \mathring{\mathcal{P}}_2 (2\underline{\mathcal{A}} + 6\underline{\mathcal{B}} r_0^{-1} \log r_0) + \mathcal{C}_7 r_0^{-1} + \mathcal{O}(r_0^{-1-\delta}), \quad (6.13.4)$$

where $\underline{\mathcal{A}}$ is such that $2\mathring{\mathcal{P}}_2^* \mathring{\mathcal{P}}_1^* \mathring{\mathcal{P}}_1 \mathring{\mathcal{P}}_2 \underline{\mathcal{A}} = \overline{\underline{\mathcal{A}}}$. Notice that

$$(2\mathring{\mathcal{P}}_2^* \mathring{\mathcal{P}}_1^* \mathring{\mathcal{P}}_1 \mathring{\mathcal{P}}_2 \underline{\mathcal{A}})_\ell = \tilde{a}_{0,0,\ell}^{[-2]} \tilde{a}_{1,0,\ell}^{[-2]} \underline{\mathcal{A}}_\ell = \frac{(\ell+2)!}{(\ell-2)!} \underline{\mathcal{A}}_\ell = \overline{\underline{\mathcal{A}}}_\ell. \quad (6.13.5)$$

6.13.1 The main theorem (Thm. 6.13.1)

With the preliminaries discussed in the preceding subsection, let us now state the main result of this section:

Theorem 6.13.1. *Let $\underline{\alpha}^{(i)}$ be the unique smooth scattering solution to (Teuk) arising from the physical data of §6.8, i.e. from (6.10.2) with $s = -2$ and (6.13.1), (6.13.3)–(6.13.4). Then each angular mode $\underline{\alpha}_\ell^{(i)}$ of $\underline{\alpha}$ satisfies the following asymptotic expansion throughout $\mathcal{D}$.*

$$\begin{aligned} \Omega^2 r \underline{\alpha}_\ell^{(i)} = \alpha_\ell^{[-2]} &= \underline{\mathcal{A}}_\ell r_0^{-2} \sum_{n=2}^{\ell+2} \underline{S}_{\ell,0,\ell+2,n,-2} \left(\frac{r_0}{r}\right)^n + (\underline{\mathcal{B}} \log r_0 + \underline{\mathcal{C}}) r_0^{-3} \sum_{n=3}^{\ell+2} \underline{S}_{\ell,1,\ell+2,n,-2} \left(\frac{r_0}{r}\right)^n \\ &+ r_0^{-3} \sum_{n=0}^{\ell+3} (\underline{\mathcal{B}}_\ell R'_{\ell,n} + M \underline{\mathcal{A}}_\ell S'_{\ell,n} + \mathcal{O}(r_0^{-\delta})) \left(\frac{r_0}{r}\right)^n \\ &+ (-1)^\ell M \underline{\mathcal{A}}_\ell \frac{\ell(\ell+1)(\ell+2)!}{3(\ell-2)!} r^{-3} (\log r - \log r_0) \\ &- (-1)^\ell M \underline{\mathcal{B}}_\ell \frac{(\ell-1)(\ell+2)(\ell+2)!}{8(\ell-2)!} r^{-4} \log^2 r + M \mathcal{O}(r^{-4}(\log r + r_0^{1-\delta})), \end{aligned} \quad (6.13.6)$$

where the constants $\underline{S}_{\ell,p,\ell-s,n,s}$ are defined in (6.13.14).

Moreover, the following limits are conserved along $\mathcal{I}^+$ (cf. (6.11.4), (6.11.5)):

$$\mathbf{I}_\ell^{2-\ell,0}[\alpha^{[-2]}] = \lim_{v \rightarrow \infty} r^{2-\ell} \nabla_v \mathbf{A}_\ell^{[-2]} = (-1)^{\ell+1} 2M \underline{\mathcal{A}}_\ell \frac{(\ell+2)!(\ell+1)}{(\ell-2)!} = \frac{(\ell+2)!}{(\ell-2)!} \overline{\mathbf{I}_\ell^{2-\ell,0}[\alpha^{[2]}]}, \quad (6.13.7)$$

$$\lim_{v \rightarrow \infty} \frac{1}{r} \left(\frac{r^2}{D} \nabla_v\right)^4 (\Omega^2 r \underline{\alpha}^{(i)}) = (-1)^{\ell+1} 2M \underline{\mathcal{A}}_\ell \frac{\ell(\ell+1)(\ell+2)!}{(\ell-2)!} = \frac{(\ell+2)!}{(\ell-2)!} \lim_{v \rightarrow \infty} r^4 \underline{\alpha}_\ell^{(i)}. \quad (6.13.8)$$

Finally, the limit $\lim_{\mathcal{I}^+} r \underline{\alpha}_\ell^{(i)}$ satisfies:

$$\lim_{v \rightarrow \infty} r \underline{\alpha}_\ell^{(i)} = (-1)^\ell \left(\frac{12 \underline{\mathcal{B}}_\ell}{\ell(\ell+1)} - 4M \underline{\mathcal{A}}_\ell \right) r_0^{-3} + \mathcal{O}(r_0^{-3-\delta}). \quad (6.13.9)$$

6.13.2 Proof of Theorem 6.13.1

Compared to the proof of Theorem 6.11.1, there are two differences: Firstly, the computation of transversal derivatives along $\underline{\mathcal{C}}$ is very slightly different, as has already been mentioned in the beginning of this section.

The other, more notable, difference is that we here need to understand more refined asymptotic estimates since we also want to compute the coefficient in front of the r_0^{-3} -decay of the limit $\lim_{v \rightarrow \infty} r \underline{\alpha}^{(i)}$ (cf. (6.13.9)). Indeed, as can be seen from (6.13.6), the leading order contributions from the first line all vanish at $\mathcal{I}^+$, as $\underline{S}_{\ell,0,\ell+2,0,-2} = \underline{S}_{\ell,1,\ell+2,0,-2} = 0$. We will come back to this at the end of the proof.

Proof. First, we compute the transversal derivatives. In the same way in which we proved Propositions 6.10.1 and 6.11.1, we have:

Proposition 6.13.1. *Let $N \geq 2$, $s = -2$. Define the constants*

$$\underline{C}_{(N,p)}^{\nearrow} = \frac{(|s| + p)!}{(N + s + p)!} \prod_{j=0}^{N-1} \tilde{a}_{j,\ell}^{[s]} = \frac{(|s| + p)!}{(N + s + p)!} (-1)^N \frac{(\ell - s)!(N + s + \ell)!}{(\ell - s - N)!(\ell + s)!}. \quad (6.13.10)$$

Then, for some constants $\underline{E}'_n$ that we shall not yet compute,

$$\begin{aligned} \left(\frac{r^2}{D} \nabla_v \right)^N \alpha^{[s=-2]} \Big|_{\underline{C}} &= \underline{\mathcal{A}} r_0^{s-p+N} \underline{C}_{(N,p)}^{\nearrow} \Big|_{p=0} + \left(\underline{\mathcal{B}} r_0^{s-p+N} \log r_0 + \underline{\mathcal{C}} r_0^{-s-p+N} \right) \underline{C}_{(N,p)}^{\nearrow} \Big|_{p=1} \\ &+ \underline{\mathcal{B}} r_0^{s-p+N} (-\delta_{N,0} \cdot (|s| + p)! + \underline{C}_{(N,p)}^{\nearrow} \left(+ \sum_{i=p+s+2}^{N+s+p} \frac{1}{i} \right) \Big|_{p=1} + M \underline{E}'_n \underline{\mathcal{A}} r_0^{-3+N} + \mathcal{O}(r_0^{-3+N-\delta}). \end{aligned} \quad (6.13.11)$$

In fact, the formula is valid also for $N = 0, 1$.

Proof. For $N \geq 2$, the proof proceeds inductively in the same way as the proof of Proposition 6.11.1, starting from (6.13.4). The cases $N = 0, 1$ follow from direct comparison with (6.13.3) and (6.13.4). $\square$

Next, we again compute the Minkowskian part of the solution.

Proposition 6.13.2. *Let $s = -2$:*

$$\begin{aligned} \sum_{i=0}^{\ell-s} \frac{1}{i!} \left(\frac{1}{r_0} - \frac{1}{r} \right)^i \left(\frac{r^2}{D} \nabla_v \right)^{\ell-s-j+i} \alpha^{[s]} \Big|_{\underline{C}} &= \underline{\mathcal{A}} r_0^{s-p} \sum_{n=0}^{\ell-s} \left(\frac{r_0}{r} \right)^n \cdot \left(\underline{S}_{\ell,p,\ell-s,n,s} + \mathcal{O}(M r_0^{-1}) \right) \Big|_{p=0} \\ &+ \left(\underline{\mathcal{B}} \log r_0 + \underline{\mathcal{C}} \right) r_0^{s-p} \sum_{n=0}^{\ell-s} \left(\frac{r_0}{r} \right)^n \cdot \underline{S}_{\ell,p,\ell-s,n,s} \Big|_{p=1} \\ &+ \underline{\mathcal{B}} r_0^{s-p} \sum_{n=0}^{p-s-1} \left(\frac{r_0}{r} \right)^n \underline{R}_{\ell,p,\ell-s,n,s} \Big|_{p=1} \\ &+ \underline{\mathcal{B}} r_0^{s-p} \sum_{n=p-s}^{\ell-s} \left(\frac{r_0}{r} \right)^n \underline{S}_{\ell,p,\ell-s,n,s} \left(\sum_{k=0}^{\ell-s-1-n} \frac{1}{\ell-p-k} + \sum_{k=2}^{\ell-s} \frac{1}{k+s+p} \right) \Big|_{p=1}, \end{aligned} \quad (6.13.12)$$

where

$$\underline{R}_{\ell,p,\ell-s,n,s} := (-1)^{\ell-p+1} \frac{(\ell-p)!(\ell-s)!(p+|s|)!}{(\ell+p)!(\ell+s)!} \frac{(p-n-s-1)!(\ell+n+s)!}{n!(\ell-n-s)!}, \quad (6.13.13)$$

and

$$\underline{S}_{\ell,p,j,n,s} := \frac{(-1)^{\ell-s+n} (|s| + p)! (\ell - s)! (2\ell - j + n)! (\ell - p)!}{n! (\ell + s)! (\ell + p)! (\ell - p - j + n)! (j - n)!}. \quad (6.13.14)$$

Notice, in particular, that

$$\lim_{v \rightarrow \infty} \sum_{i=0}^{\ell-s} \frac{1}{i!} \left(\frac{1}{r_0} - \frac{1}{r} \right)^i \left(\frac{r^2}{D} \nabla_v \right)^{\ell-s-j+i} \alpha^{[s]} \Big|_{\underline{\mathcal{C}}} = (-1)^\ell \frac{12 \underline{\mathcal{B}}_\ell r_0^{-3}}{\ell(\ell+1)} + M \mathcal{O}(r_0^{-3}). \quad (6.13.15)$$

Proof. We again set $M = 0$ and argue by linearity. We have

$$\begin{aligned} (r^2 \nabla_v)^{\ell-s} \alpha^{[s=-2]} \Big|_{\underline{\mathcal{C}}} &= \underline{\mathcal{A}} r_0^{\ell-p} \underline{\mathcal{C}}_{(\ell-s,p)}^\nearrow \Big|_{p=0} \\ &+ r_0^{\ell-p} (\underline{\mathcal{B}} \log r_0 + \underline{\mathcal{C}}) \underline{\mathcal{C}}_{(\ell-s,p)}^\nearrow \Big|_{p=1} + \underline{\mathcal{B}} r_0^{\ell-p} \underline{\mathcal{C}}_{(\ell-s,p)}^\nearrow \sum_{i=p+s+2}^{N+s+p} \frac{1}{i} \Big|_{p=1} \end{aligned} \quad (6.13.16)$$

The result now follows by applying Prop. 6.14.1 to the $\underline{\mathcal{A}}$ and the $\underline{\mathcal{C}}$ -terms, and by applying Prop. 6.14.2 to the $\underline{\mathcal{B}}$ -terms. (Cf. the proofs of Propositions 6.10.8 and 6.11.3.) $\square$

Finally, we have

Proposition 6.13.3. *Let $s = -2$. Then*

$$\begin{aligned} \alpha_\ell^{[s]} &= \Omega^2 r \underline{\alpha}_\ell = \sum_{i=0}^{\ell-s} \frac{1}{i!} \left(\frac{1}{r_0} - \frac{1}{r} \right)^i \left(\frac{r^2}{D} \nabla_v \right)^i \alpha^{[s]} \Big|_{\underline{\mathcal{C}}} \\ &+ M r_0^{s-p-1} \sum_{n=0}^{p-s} \left(\underline{\mathcal{C}}_n \frac{r_0}{r} \right)^n - M \underline{\mathcal{A}}_\ell \underline{\mathcal{C}}_{(\ell-s,p)}^\nearrow \frac{(\ell-p)(\ell+p+1)!}{(2\ell)!} \frac{(-1)^{p+1-s}}{(p+1-s)!} r^{s-p-1} (\log r - \log r_0) \Big|_{p=0} \\ &- M \underline{\mathcal{B}}_\ell \underline{\mathcal{C}}_{(\ell-s,p)}^\nearrow \frac{(\ell-p)(\ell+p+1)!}{(2\ell)!} \frac{(-1)^{p+1-s}}{2(p+1-s)!} r^{s-p-1} \log^2 r \Big|_{p=1} + M \mathcal{O}(r^{-4}(r_0^{1-\delta} + \log r)). \end{aligned} \quad (6.13.17)$$

Moreover, we have the following estimate for the limit $r \underline{\alpha}|_{\mathcal{I}^+}$:

$$\lim_{v \rightarrow \infty} r \underline{\alpha}_\ell = (-1)^\ell r_0^{-3} \left(\frac{12 \underline{\mathcal{B}}_\ell}{\ell(\ell+1)} - 4M \underline{\mathcal{A}}_\ell \right). \quad (6.13.18)$$

Proof. The first statement is proved exactly as in the previous sections, the only difference being that $\underline{\mathcal{C}}_{(\ell-s,p)}^\nearrow$ is now replaced by $\underline{\mathcal{C}}_{(\ell-s,p)}^\nearrow$: First, we prove that

$$\begin{aligned} \left(\frac{r^2}{D} \nabla_v \right)^{\ell-s+1} \alpha^{[s]} &= -M \underline{\mathcal{A}}_\ell \underline{\mathcal{C}}_{(\ell-s,p)}^\nearrow \frac{(\ell-p)(\ell+p+1)!}{(2\ell)!} r^{\ell-p} \Big|_{p=0} \\ &- M \underline{\mathcal{B}}_\ell \underline{\mathcal{C}}_{(\ell-s,p)}^\nearrow \frac{(\ell-p)(\ell+p+1)!}{(2\ell)!} r^{\ell-p} \log r \Big|_{p=1} + M \mathcal{O}(r_0^\ell) \end{aligned}$$

as in Propositions 6.10.5 and 6.11.2 for the $\underline{\mathcal{A}}$ and the $\underline{\mathcal{B}}$ terms, respectively. We subsequently compute the nested integrals from Proposition 6.10.6 as in Propositions 6.10.7 and 6.11.4, and the result follows.

For the second statement (6.13.18), the $\underline{\mathcal{B}}$ -term can be read off directly from Proposition 6.13.2. For the $M\underline{\mathcal{A}}$ -term, we need to work a bit harder:

Since we have already captured the contribution coming from the $\underline{\mathcal{B}}$ - and the $\underline{\mathcal{C}}$ -terms (the latter one being vanishing), it now suffices to consider the problem where $\underline{\mathcal{B}}, \underline{\mathcal{C}} = 0$ and $\delta = \infty$. We then do the following Post-Minkowskian type argument: We split the Teukolsky equation up into equations (6.14.3), (6.14.6) and (6.14.5), where we only pose non-trivial data for (the homogeneous) equation (6.14.3), and pose trivial data for the other two (inhomogeneous) equations.

To be precise, the data we pose for (6.14.3) is that $\alpha_{M=0}^{[s]}|_{\underline{\mathcal{C}}} = 0$, that $\nabla_v \alpha_{M=0}^{[s]}|_{\underline{\mathcal{C}}} = 0$ and that $\lim_{u \rightarrow -\infty} (\frac{r^2}{D} \nabla_v)^2 \alpha_{M=0}^{[s]}|_{\underline{\mathcal{C}}} = 4\dot{\mathcal{P}}_2^* \dot{\mathcal{P}}_1^* \dot{\mathcal{P}}_1 \dot{\mathcal{P}}_2 \underline{\mathcal{A}}$ (cf. (6.13.1), (6.13.3), (6.13.4)), together with the no incoming radiation condition (6.13.2). The solution is then given by $\alpha_{M=0}^{[s]} = \sum_{i=0}^{\ell-s} \frac{1}{i!} \left(\frac{1}{r_0} - \frac{1}{r} \right)^i (\frac{r^2}{D} \nabla_v)^i \alpha_{M=0}^{[s]}|_{\underline{\mathcal{C}}}$, and is determined by Proposition 6.14.1, the result being the first line of (6.13.12); in particular, the coefficients S_n in (6.14.8) are given by (6.13.14). We read off that $\lim_{\mathcal{I}^+} \alpha_{M=0}^{[s]} = 0$.

We now insert $\alpha_{M=0}^{[s]}$ as an *inhomogeneity* into (6.14.4). We can easily see from Proposition 6.10.5, eq. (6.10.27) that the solution $\alpha_{M^1}^{[s]}$ must thus be of the form assumed in Proposition 6.14.4, see also Remark 6.10.5. Applying the result of Proposition 6.14.4 then gives us a handy expression for the corresponding solution $\alpha_{M^1}^{[s]}$, and we read off that

$$\lim_{v \rightarrow \infty} \alpha_{M^1}^{[s]} = 2M \underline{\mathcal{A}}_\ell \cdot r_0^{s-p-1} Q_0|_{p=0}, \quad (6.13.19)$$

with Q_0 given by (6.14.24) (where $S_{\ell-s} = \underline{S}_{\ell,p,\ell-s,\ell-s,s}|_{p=0}$), i.e. $Q_0 = -2 \cdot (-1)^\ell$.

Finally, it is easy to see that the remaining difference $\alpha_{\mathcal{O}(M^2)}^{[s]} = \alpha^{[s]} - \alpha_{M=0}^{[s]} - \alpha_{M^1}^{[s]}$, as a solution to (6.14.5) with the previous $\alpha_{M=0}^{[s]}$ and $\alpha_{M^1}^{[s]}$ as inhomogeneities and with trivial data, has $\lim_{v \rightarrow \infty} \alpha_{\mathcal{O}(M^2)}^{[s]} = \mathcal{O}(r_0^{-4})$. This concludes the proof of the proposition ... $\square$

...and thus the proof of Thm. 6.13.1. $\square$

6.14 Post-Minkowskian expansions of the Teukolsky equation

This section presents a Post-Minkowskian expansion of (Teuk:s). The purpose of this is for us to reduce certain somewhat tedious computations to simply solving Minkowskian Teukolsky equations with inhomogeneities. The results of this section have already been used and referenced in the previous sections for the proofs of Prop. 6.10.8, Prop. 6.11.3 and Prop. 6.13.2, i.e. to compute the Minkowskian parts of the solution. We have also used them to prove (6.13.18).

6.14.1 The expansion

Recall (Teuk:s). We re-write it in coordinates $(r_0, r, \theta, \varphi)$, denoting $\partial_u r_0 = -(1 - \frac{2M}{r_0}) = -D_0$. Since $\partial_u = -D_0 \partial_{r_0} - D \partial_r$ and $\partial_v = D \partial_r$, we have

$$-(D_0 \nabla_{r_0} + D \nabla_r)(r^{-2s} D^{s+1} \nabla_r \alpha^{[s]}) = \frac{D^{s+1}}{r^{2+2s}} \mathring{\Delta}_{[s]} \alpha^{[s]} - \frac{2MD^{s+1}}{r^{3+2s}}(1+s)(1+2s)\alpha^{[s]}. \quad (6.14.1)$$

Note that

$$-D \nabla_r(r^{-2s} D^{s+1} \nabla_r \alpha^{[s]}) = (2sr^{-2s-1} D^{s+2} - (s+1)2Mr^{-2s-2} D^{s+1}) \nabla_r \alpha^{[s]} - D^{s+2} r^{-2s} \nabla_r^2 \alpha^{[s]}.$$

Dividing by $D^{s+1} r^{-2s}$ thus casts (6.14.1) into the form

$$\begin{aligned} -(\nabla_{r_0} + \nabla_r) \alpha^{[s]} + \frac{D_0 - D}{D_0} \nabla_r^2 \alpha^{[s]} + 2s \frac{D}{D_0} \nabla_r \alpha^{[s]} - (s+1) \frac{2M}{r^2} \nabla_r \alpha^{[s]} \\ = \frac{\mathring{\Delta}_{[s]} \alpha^{[s]}}{D_0 r^2} - \frac{2M}{D_0 r^3} (1+s)(1+2s) \alpha^{[s]}. \end{aligned} \quad (6.14.2)$$

We now decompose the equation above into the following three equations: Firstly:

$$-(\nabla_{r_0} + \nabla_r)(r^{-2s} \nabla_r \alpha_{M=0}^{[s]}) = \frac{\mathring{\Delta}_{[s]} \alpha_{M=0}^{[s]}}{r^{2+2s}}, \quad (6.14.3)$$

the Minkowskian Teukolsky equation. Secondly:

$$\begin{aligned} &-(\nabla_{r_0} + \nabla_r)(r^{-2s} \nabla_r \alpha_{M^1}^{[s]}) \\ &\quad + r^{-2s} \left(\frac{2M}{r_0} + (s+1) \frac{2M}{r} \right) \nabla_{r_0} \nabla_r \alpha_{M=0}^{[s]} \\ &\quad + r^{-2s} \left((s+2) \frac{2M}{r} \right) \nabla_r^2 \alpha_{M=0}^{[s]} - \left(2sr^{-2s-1} (s+2) \frac{2M}{r} + (s+1) 2Mr^{-2s-2} \right) \nabla_r \alpha_{M=0}^{[s]} \\ &\quad = \frac{\mathring{\Delta}_{[s]} \alpha_{M^1}^{[s]}}{r^{2+2s}} \\ &\quad - \frac{2M(s+1) \mathring{\Delta}_{[s]} \alpha_{M=0}^{[s]}}{r^{3+2s}} - \frac{2M}{r^{3+2s}} (1+s)(1+2s) \alpha_{M=0}^{[s]}, \end{aligned} \quad (6.14.4)$$

the Minkowskian Teukolsky equation with inhomogeneity, and thirdly:

$$\begin{aligned}
& - (D_0 \nabla_{r_0} + D \nabla_r) (r^{-2s} D^{s+1} \nabla_r \alpha_{\mathcal{O}(M^2)}^{[s]}) \\
& \quad - r^{-2s} (D_0 D^{s+1} - 1) \nabla_{r_0} \nabla_r \alpha_{M^1}^{[s]} - r^{-2s} (D^{s+2} - 1) \nabla_r^2 \alpha_{M^1}^{[s]} \\
& \quad + 2s r^{-2s-1} (D^{s+2} - 1) \nabla_r \alpha_{M^1}^{[s]} - (s+1) 2M r^{-2s-2} \nabla_r \alpha_{M^1}^{[s]} \\
& - r^{-2s} (D_0 D^{s+1} - (1 - \frac{2M}{r_0} - (s+1) \frac{2M}{r})) \nabla_{r_0} \nabla_r \alpha_{M=0}^{[s]} - r^{-2s} (D^{s+2} - (1 - (s+2) \frac{2M}{r})) \nabla_r^2 \alpha_{M=0}^{[s]} \\
& \quad + 2s r^{-2s-1} (D^{s+2} - (1 - (s+2) \frac{2M}{r})) \nabla_r \alpha_{M=0}^{[s]} - (s+1) 2M r^{-2s-2} (D^{s+1} - 1) \nabla_r \alpha_{M=0}^{[s]} \\
& = \frac{D^{s+1}}{r^{2+2s}} \mathring{\Delta}_{[s]} \alpha_{\mathcal{O}(M^2)}^{[s]} - \frac{2MD^{s+1}}{r^{3+2s}} (1+s)(1+2s) \alpha_{\mathcal{O}(M^2)}^{[s]} \\
& \quad + \frac{(D^{s+1} - 1) \mathring{\Delta}_{[s]} \alpha_{M^1}^{[s]}}{r^{2+2s}} - \frac{2MD^{s+1}}{r^{3+2s}} (1+s)(1+2s) \alpha_{M^1}^{[s]} \\
& \quad + \frac{(D^{s+1} - (1 - (s+1) \frac{2M}{r})) \mathring{\Delta}_{[s]} \alpha_{M=0}^{[s]}}{r^{3+2s}} - \frac{2M(D^{s+1} - 1)}{r^{3+2s}} (1+s)(1+2s) \alpha_{M=0}^{[s]}, \quad (6.14.5)
\end{aligned}$$

the Schwarzschild Teukolsky equation with inhomogeneity. Note that adding (6.14.3)–(6.14.5) gives back (6.14.1) with $\alpha^{[s]} = \alpha_{M=0}^{[s]} + \alpha_{M^1}^{[s]} + \alpha_{\mathcal{O}(M^2)}^{[s]}$. The decomposition $\alpha^{[s]} = \alpha_{M=0}^{[s]} + \alpha_{M^1}^{[s]} + \alpha_{\mathcal{O}(M^2)}^{[s]}$ corresponds to a Post-Minkowskian decomposition in (r_0, r) -coordinates, with an exact keeping track of the error term $\alpha_{\mathcal{O}(M^2)}^{[s]}$. In applications, equation (6.14.3)—the Teukolsky equation in Minkowski—is solved first, equation (6.14.4) then becomes a Minkowskian Teukolsky equation with inhomogeneity depending on the already solved for $\alpha_{M=0}^{[s]}$, and (6.14.5) is a Schwarzschild Teukolsky equation with inhomogeneity depending on $\alpha_{M=0}^{[s]}$ and $\alpha_{M^1}^{[s]}$.

We observe that by eq. (6.14.3), the equation (6.14.4) can be rewritten as

$$\begin{aligned}
& - (\nabla_{r_0} + \nabla_r) (r^{-2s} \nabla_r \alpha_{M^1}^{[s]}) = \frac{\mathring{\Delta}_{[s]} \alpha_{M^1}^{[s]}}{r^2} \\
& \quad + \frac{2M \mathring{\Delta}_{[s]} \alpha_{M=0}^{[s]}}{r^2 r_0} - \frac{2M(1+s)(1+2s) \alpha_{M=0}^{[s]}}{r^3} \\
& \quad + (s+1) \frac{2M}{r^2} \nabla_r \alpha_{M=0}^{[s]} + \frac{2s}{r} (D_0 - D) \nabla_r \alpha_{M=0}^{[s]} - (D_0 - D) \nabla_r^2 \alpha_{M=0}^{[s]}. \quad (6.14.6)
\end{aligned}$$

6.14.2 Explicitly solving the Minkowskian Teukolsky equation

We now present explicit solutions, supported on fixed angular frequency, of (6.14.3). The following proposition was used in the proof of Propositions 6.10.8 and 6.13.2:

Proposition 6.14.1. *Suppose that $\mathcal{A}_\ell$ is supported on angular frequency $\ell \geq |s|$, that $\ell \geq p \in \mathbb{N}_{\geq 0}$, and that*

$$\alpha_{M=0}^{[s]} = \mathcal{A}_\ell r_0^{s-p} \sum_{n=0}^{\ell-s} S_n \left(\frac{r_0}{r} \right)^n \quad (6.14.7)$$

solves (6.14.3). Then

$$S_n = S_{\ell-s} \frac{(-1)^{\ell-s}(\ell-s)!(\ell-p)!}{(2\ell)!} \cdot (-1)^n \frac{(\ell+s+n)!}{n!(s-p+n)!(\ell-s-n)!}, \quad (6.14.8)$$

and (6.14.7) indeed solves (6.14.3). **The result holds true for any** $p \in \mathbb{R}$ if we replace the factor $\frac{(\ell-p)!}{(s-p+n)!(\ell-s-n)!}$ by $\binom{\ell-p}{s-p+n}$ and extend the binomial coefficient to non-integer arguments in the usual fashion. Moreover,

$$(r^2 \nabla_r)^{\ell-s} \alpha^{[s]} = (-1)^{\ell-s} \mathcal{A}_\ell r_0^{\ell-p} (\ell-s)! S_{\ell-s}. \quad (6.14.9)$$

Proof. The proof is strictly simpler than the proof of the next proposition. $\square$

The following proposition was used in the proof of Propositions 6.11.3 and 6.13.2.

Proposition 6.14.2. Suppose that $\mathcal{B}_\ell$ is supported on angular frequency $\ell \geq |s|$, that $\ell \geq p \in \mathbb{N}_{\geq 0}$, and that

$$\alpha_{M=0}^{[s]} = \mathcal{B}_\ell r_0^{s-p} \left(\sum_{n=0}^{\ell-s} S_n \left(\frac{r_0}{r} \right)^n \log r_0 + \sum_{n=0}^{\ell-s} R_n \left(\frac{r_0}{r} \right)^n \right) \quad (6.14.10)$$

solves (6.14.3). Then

$$S_n = S_{\ell-s} \frac{(-1)^{\ell-s}(\ell-s)!(\ell-p)!}{(2\ell)!} \cdot (-1)^n \frac{(\ell+s+n)!}{n!(s-p+n)!(\ell-s-n)!}, \quad (6.14.11)$$

and, for $n \leq p-s-1$,

$$R_n = S_{\ell-s} (-1)^{\ell-p+1} \frac{(\ell-s)!(\ell-p)!}{(2\ell)!} \frac{(p-n-1-s)!(n+\ell+s)!}{n!(\ell-s-n)!}, \quad (6.14.12)$$

whereas for $p-s \leq n \leq \ell-s$:

$$R_n = S_n \frac{R_{\ell-s}}{S_{\ell-s}} + S_n \sum_{i=0}^{\ell-s-1-n} \frac{1}{\ell-p-i}. \quad (6.14.13)$$

Moreover, (6.14.10) indeed solves (6.14.3), and

$$(r^2 \nabla_r)^{\ell-s} \alpha_{M=0}^{[s]} = (-1)^{\ell-s} \mathcal{B}_\ell r_0^{\ell-p} ((\ell-s)! S_{\ell-s} \log r_0 + (\ell-s)! R_{\ell-s}). \quad (6.14.14)$$

Proof. We compute

$$\begin{aligned} \nabla_r \alpha_{M=0}^{[s]} &= \mathcal{B}_\ell r_0^{s-p} \sum_{n=0}^{\ell-s} \frac{-n}{r} \left(S_n \left(\frac{r_0}{r} \right)^n \log r_0 + R_n \left(\frac{r_0}{r} \right)^n \right), \\ -\nabla_r^2 \alpha_{M=0}^{[s]} &= \mathcal{B}_\ell r_0^{s-p} \sum_{n=0}^{\ell-s} \frac{-n(n+1)}{r^2} \left(S_n \left(\frac{r_0}{r} \right)^n \log r_0 + R_n \left(\frac{r_0}{r} \right)^n \right), \end{aligned}$$

and

$$\begin{aligned} -\nabla_{r_0} \nabla_r \alpha_{M=0}^{[s]} &= r_0^{s-p} \sum_{n=0}^{\ell-s} \left(\frac{r_0}{r} \right)^n \left((n+s+p)n \frac{1}{r_0 r} (S_n \log r_0 + R_n) + \frac{n}{r_0 r} S_n \right) \\ &= r_0^{s-p} \sum_{n=0}^{\ell-s-1} \left(\frac{r_0}{r} \right)^n \left((n+1+s+p)n \frac{1}{r^2} (S_{n+1} \log r_0 + R_{n+1}) + \frac{n+1}{r^2} S_{n+1} \right). \end{aligned}$$

Inserting the above expressions into (6.14.3) and equating coefficients, we obtain the system of equations (using $n(n+2s+1) - (\ell-s)(\ell+s+1) = (n-(\ell-s))(n+\ell+s+1)$):

$$(n-(\ell-s))(n+\ell+s+1)S_n = S_{n+1}(n+1)(n+1+s-p), \quad (6.14.15)$$

$$(n-(\ell-s))(n+\ell+s+1)R_n = R_{n+1}(n+1)(n+1+s-p) + (n+1)S_{n+1}. \quad (6.14.16)$$

The first of these inductively implies (6.14.11).

For the second equation (6.14.16), we first set $n = p-s-1$:

$$R_{p-s-1} = \frac{p-s}{(p+\ell)(p-\ell-1)} S_{p-s}. \quad (6.14.17)$$

For $n < p-s-1$, we then simply have $(\ell-s-n)(n+\ell+s+1)R_n = R_{n+1}(n+1)(p-s-1-n)$, which is solved by

$$\begin{aligned} R_n &= R_{p-s-1} \frac{(p-s-1)!(\ell-p+1)!}{(\ell+p-1)!} \frac{(p-n-1-s)!(n+\ell+s)!}{n!(\ell-s-n)!} \\ &= -S_{p-s} \frac{(p-s)!(\ell-p)!}{(\ell+p)!} \frac{(p-n-1-s)!(n+\ell+s)!}{n!(\ell-s-n)!} \\ &= S_{\ell-s} (-1)^{\ell-p+1} \frac{(\ell-s)!(\ell-p)!}{(2\ell)!} \frac{(p-n-1-s)!(n+\ell+s)!}{n!(\ell-s-n)!}. \end{aligned} \quad (6.14.18)$$

To compute the remaining coefficients, we observe that (6.14.16) implies that $R_n = 0$ for $n > \ell-s$. Finally, for $p-s \leq n \leq \ell-s$, we write $R_n = S_n \cdot P_n$ in (6.14.12), which gives:

$$S_n P_n = S_n P_{n+1} + \frac{1}{n+1+s-p} S_n,$$

which is solved by $P_{n-j} = P_{n+1} + \sum_{i=0}^j \frac{1}{s-p+n+1-i}$ or, equivalently,

$$P_n = P_{\ell-s} + \sum_{i=0}^{\ell-s-1-n} \frac{1}{\ell-p-i}. \quad (6.14.19)$$

□

6.14.3 Explicitly solving the inhomogeneous Minkowskian Teukolsky equation

We now prove two statements about eq. (6.14.6). The relevance of the first statement will be to compute the inhomogeneity of (6.14.6), provided that $\alpha_{M=0}^{[s]}$ is as in Proposition 6.14.1. The purpose of the second proposition will be to provide a simple expression for solutions of (6.14.6), provided that the inhomogeneity is as computed in the first proposition. The result of the second proposition has already been used in the proof of (6.13.18).

Proposition 6.14.3. *Define the following functional, denoting the inhomogeneity of (6.14.6):*

$$g[\alpha_{M=0}^{[s]}] := \frac{2M\mathring{\Delta}_{[s]}\alpha_{M=0}^{[s]}}{r^2r_0} - \frac{2M(1+s)(1+2s)\alpha_{M=0}^{[s]}}{r^3} + (s+1)\frac{2M}{r^2}\nabla_r\alpha_{M=0}^{[s]} + \frac{2s}{r}(D_0-D)\nabla_r\alpha_{M=0}^{[s]} - (D_0-D)\nabla_r^2\alpha_{M=0}^{[s]}. \quad (6.14.20)$$

Then, for $\alpha_{M=0}^{[s]}$ as in Proposition 6.14.1, we have that

$$g[\alpha_{M=0}^{[s]}] \cdot \frac{r^2r_0^{1+p-s}}{2M} = \sum_{n=0}^{\ell-s+1} \left(\frac{r_0}{r}\right)^n \underbrace{((n-\ell+s)(n+\ell+s+1)S_n - (n+s)(n+2s)S_{n-1})}_{:=T_n}. \quad (6.14.21)$$

Proof. By definition of g , we have that

$$g[\alpha_{M=0}^{[s]}] \cdot \frac{r^2r_0^{1+p-s}}{2M} = -(\ell-s)(\ell+s+1)\alpha_{M=0}^{[s]} - \frac{r_0}{r}(1+s)(1+2s)\alpha_{M=0}^{[s]} + (3s+1)\nabla_r\alpha_{M=0}^{[s]} \cdot r_0 - 2s\nabla_r\alpha_{M=0}^{[s]} \cdot r + (r^2 - rr_0)\nabla_r^2\alpha_{M=0}^{[s]}.$$

We now insert the specific form of $\alpha_{M=0}^{[s]}$ and repeat the calculations done in the proof of the previous proposition to get

$$\begin{aligned} g[\alpha_{M=0}^{[s]}] \cdot \frac{r^2r_0^{1+p-s}}{2M} &= \sum_{n=0}^{\ell-s} S_n [-(\ell-s)(\ell+s+1)] \left(\frac{r_0}{r}\right)^n - (1+s)(1+2s)S_n \left(\frac{r_0}{r}\right)^{n+1} \\ &\quad - \frac{n}{r}[(3s+1)r_0 - 2sr] \left(\frac{r_0}{r}\right)^n + n(n+1) \left(1 - \frac{r_0}{r}\right) S_n \left(\frac{r_0}{r}\right)^n \\ &= \sum_{n=0}^{\ell-s} S_n \left(\frac{r_0}{r}\right)^n [-(\ell-s)(\ell+s+1) + 2sn + n(n+1)] \\ &\quad + \sum_{n=1}^{\ell-s+1} S_{n-1} \left(\frac{r_0}{r}\right)^n [-(1+s)(1+2s) - (n-1)(3s+1) - (n-1)n]. \end{aligned}$$

The result follows from $-(\ell-s)(\ell+s+1) + 2sn + n(n+1) = (n-\ell+s)(n+\ell+s+1)$ and $-(1+s)(1+2s) - (n-1)(3s+1) - (n-1)n = -(n+s)(n+2s)$. $\square$

Proposition 6.14.4. *Let $\alpha_{M=0}^{[s]}$ be as in Proposition 6.14.1, and let $\ell \geq p \in \mathbb{N}_{\geq 0}$. Then, if $\alpha_{M^1}^{[s]}$ is of the form*

$$\alpha_{M^1}^{[s]} = 2Mr_0^{s-p-1} \left(\sum_{n=0}^{\ell-s+1} S'_n \left(\frac{r_0}{r} \right)^n \log r_0/r + Q_n \left(\frac{r_0}{r} \right)^n \right) \quad (6.14.22)$$

and solves (6.14.6), we have

$$S'_n = S_{\ell-s} \frac{(-1)^{\ell-s}(\ell-s)!(\ell-p)!}{2(2\ell)!} \cdot \frac{(-1)^n(\ell+s+n)!}{n!(s-p+n-1)!(\ell-s-n)!} = \frac{s-p+n}{2} S_n \quad (6.14.23)$$

and, if $n \leq p-s$:

$$Q_n = S_{\ell-s} (-1)^{\ell-p+1} \frac{(\ell-s)!(\ell-p)!}{2(2\ell)!} \frac{(p-n-s)!(n+\ell+s)!}{n!(\ell-s-n)!}, \quad (6.14.24)$$

so $Q_n = -\frac{s-p+n}{2} R_n$ if $n \leq p-s-1$. For $\ell-s \geq n > p-s$, the coefficients Q_n can all be explicitly expressed in terms of a free constant, say, $Q_{\ell-s}$. Finally, $Q_{\ell-s+1} = \frac{\ell+s+1}{2} S_{\ell-s}$. With these expressions, (6.14.22) indeed solves (6.14.6).

Proof. Inserting the ansatz into equation (6.14.6), we obtain

$$\begin{aligned} & \sum \left(\frac{r_0}{r} \right)^n \log r_0/r [(-n(n+1) - 2sn)S'_n + (n+1)(n+s-p)S'_{n-1}] \\ & + \sum \left(\frac{r_0}{r} \right)^n [(-2n-1-2s)S'_n + ((n+1) + (n+s-p))S'_{n-1}] \\ & + \sum \left(\frac{r_0}{r} \right)^n [(-n(n+1) - 2sn)Q_n + (n+1)(n+s-p)Q_{n-1}] \\ & = -(\ell-s)(\ell+s+1) \left(\sum S'_n \left(\frac{r_0}{r} \right)^n \log r_0/r + Q_n \left(\frac{r_0}{r} \right)^n \right) + g[\alpha_{M=0}^{[s]}] \cdot \frac{r^2 r_0^{1+p-s}}{2M}, \end{aligned}$$

Let us, for now, just write

$$g[\alpha_{M=0}^{[s]}] \cdot \frac{r^2 r_0^{1+p-s}}{2M} = \sum_{n=0}^{\ell-s+1} T_n \left(\frac{r_0}{r} \right)^n.$$

(Recall that $T_n = 0$ if $n < p-s$ from (6.14.21).) Then we get the following system of equations for the coefficients S'_n and Q_n :

$$(n - (\ell-s))(n + \ell + s + 1)S'_n = S'_{n+1}(n+1)(n+s-p), \quad (6.14.25)$$

$$\begin{aligned} (n - (\ell-s))(n + \ell + s + 1)Q_n &= Q_{n+1}(n+1)(n+s-p) \\ &\quad - (2n+2s+1)S'_n + (2n+1+s-p)S'_{n+1} - T_n \end{aligned} \quad (6.14.26)$$

We can immediately solve the first equation to obtain (6.14.23). For the second equation (6.14.26), we observe that, setting $n = \ell - s$,

$$Q_{\ell-s+1}(\ell - s + 1)(\ell - p) = (2\ell + 1)S'_{\ell-s} + T_{\ell-s}, \quad (6.14.27)$$

and, setting $n = \ell - s + 1$,

$$(2\ell + 2)Q_{\ell-s+1} = Q_{\ell-s+2}(\ell - s + 2)(\ell - p + 1) - T_{\ell-s+1}. \quad (6.14.28)$$

In order to ensure that $Q_{\ell-s+2} = 0$ (and thus $Q_{\ell-s+i} = 0$ for all $i \geq 2$), we demand that

$$Q_{\ell-s+1} = \frac{(2\ell + 1)S'_{\ell-s}}{(\ell - s + 1)(\ell - p)} + \frac{T_{\ell-s}}{(\ell - s + 1)(\ell - p)} = \frac{-T_{\ell-s+1}}{2\ell + 2}, \quad (6.14.29)$$

which is a condition on $S'_{\ell-s}$:

$$\begin{aligned} S'_{\ell-s} &= -\frac{(\ell - s + 1)(\ell - p)}{(2\ell + 1)(2\ell + 2)}T_{\ell-s+1} - \frac{T_{\ell-s}}{2\ell + 1} \\ &= \left(\frac{(\ell - s + 1)(\ell - p)(\ell + 1)(\ell + s + 1)}{(2\ell + 1)(2\ell + 2)} - \frac{(\ell - s)(\ell + s)(\ell - p)\ell}{2\ell(2\ell + 1)} \right) S_{\ell-s} \\ &= \frac{(\ell - p)S_{\ell-s}}{2(2\ell + 1)}((\ell - s + 1)(\ell + s + 1) - (\ell - s)(\ell + s)) = \frac{\ell - p}{2}S_{\ell-s} \end{aligned} \quad (6.14.30)$$

Here, we used that $T_{\ell-s} = -\ell(\ell + s)S_{\ell-s-1} = (\ell - s)\ell(\ell + s)(\ell - p)S_{\ell-s}/(2\ell)$ and $T_{\ell-s+1} = -(\ell + 1)(\ell + s + 1)S_{\ell-s}$.

We can now freely specify a value for $Q_{\ell-s}$ and then solve (6.14.26) to obtain all Q_n for $p - s < n \leq \ell - s$. Since we won't need their specific values in this chapter, we won't compute them.

Next, assuming that $p - s \geq 0$ (otherwise we'd be done), we insert $n = p - s$ into (6.14.26):

$$(p - \ell)(p + \ell + 1)Q_{p-s} = (p - s + 1)S'_{p-s+1} - T_{p-s}, \quad (6.14.31)$$

so Q_{p-s} can be directly determined: Since $S'_{p-s+1} = S_{p-s+1}/2$, and since

$$T_{p-s} = (n - \ell - s)(n + \ell + s + 1)S_n|_{n=p-s} = S_{n+1}(n + 1)(n + 1 + s - p)|_{n=p-s} = S_{p-s+1}(p - s + 1),$$

we get

$$Q_{p-s} = \frac{p - s + 1}{2(\ell - p)(\ell + p + 1)}S_{p-s+1}. \quad (6.14.32)$$

Pleasantly, we also have for any $n < p - s$ that

$$(n - (\ell - s))(n + \ell + s + 1)Q_n = Q_{n+1}(n + 1)(n + s - p), \quad (6.14.33)$$

so we can inductively determine all Q_n for $n \leq p - s$:

$$\begin{aligned}
 Q_n &= Q_{p-s} \frac{(p-s)!(\ell-p)!}{(\ell+p)!} \frac{(p-n-s)!(n+\ell+s)!}{n!(\ell-s-n)!} \\
 &= S_{p-s+1} \frac{(p-s+1)(\ell-p-1)!}{2(\ell+p+1)!} \frac{(p-n-s)!(n+\ell+s)!}{n!(\ell-s-n)!} \\
 &= S_{\ell-s} (-1)^{\ell-p+1} \frac{(\ell-s)(\ell-p)!}{2(2\ell)!} \frac{(p-n-s)!(n+\ell+s)!}{n!(\ell-s-n)!}.
 \end{aligned} \tag{6.14.34}$$

This concludes the proof. □

6.15 Remarks on summing up the angular modes

As we have already said in Remark 6.10.3, the estimates obtained in this part of the chapter cannot directly be summed in ℓ . For instance, if we write, say, $\overset{(i)}{\alpha} = \sum_{\ell=2}^{\infty} \overset{(i)}{\alpha}_{\ell}$, and insert the estimates for $\overset{(i)}{\alpha}_{\ell}$ that we have obtained in Theorem 6.11.1, then the ℓ -dependent constants hiding in the $\mathcal{O}(\dots)$ -terms of Theorem 6.11.1 will not be summable in ℓ . To give a concrete example: We cannot directly infer from (6.11.5) of Thm. 6.11.1 that

$$\lim_{v \rightarrow \infty} r^4 \overset{(i)}{\alpha} = \sum_{\ell=2}^{\infty} (-1)^{\ell+1} \frac{2M \mathcal{A}_{\ell}}{(\ell-1)(\ell+2)}. \quad (6.15.1)$$

The full resolution of this problem will be left to future work [KK23], but we here already give an idea how to approach the problem, slightly expanding on the strategy that we have already briefly mentioned in §5.5.5 of the previous chapter.

Taking $\overset{(i)}{\alpha}$ to be as in §6.11, we first write the initial data along $\underline{\mathcal{C}}$ (6.11.1) as

$$\overset{(i)}{\alpha}_{\underline{\mathcal{C}}} = \overset{(i)}{\alpha}_{\underline{\mathcal{C}}, \text{phg}} + \overset{(i)}{\alpha}_{\underline{\mathcal{C}}, \Delta}, \quad (6.15.2)$$

where

$$\overset{(i)}{\alpha}_{\underline{\mathcal{C}}, \text{phg}} = \mathcal{A} + \mathcal{B} \frac{\log r_0}{r_0} + \frac{\mathcal{C}}{r_0}. \quad (6.15.3)$$

We now define $\overset{(i)}{\alpha}_{M=0, \text{phg}}$ as the scattering solution to the *Minkowskian* Teukolsky equation (6.14.3) with scattering data $\overset{(i)}{\alpha}_{\underline{\mathcal{C}}, \text{phg}}$ and no incoming radiation.

From $\overset{(i)}{\alpha}_{M=0, \text{phg}}$, we then define $\overset{(i)}{\alpha}_{M^1, \text{phg}}$ as the scattering solution to the *inhomogeneous Minkowskian* Teukolsky equation (6.14.4) with trivial scattering data and with the inhomogeneity given by $\overset{(i)}{\alpha}_{M=0, \text{phg}}$. (To be precise, the inhomogeneity is given by $g[r^5 \Omega^{-2} \overset{(i)}{\alpha}_{M=0, \text{phg}}]$, with g defined in Prop. 6.14.3).

Lastly, we define $\overset{(i)}{\alpha}_{\Delta}$ as the scattering solution to the *inhomogeneous Schwarzschildian* Teukolsky equation (6.14.5), with scattering data $\overset{(i)}{\alpha}_{\underline{\mathcal{C}}, \Delta}$ and with the inhomogeneity sourced by $\overset{(i)}{\alpha}_{M=0, \text{phg}}$ and $\overset{(i)}{\alpha}_{M^1, \text{phg}}$. We note that $\overset{(i)}{\alpha} = \overset{(i)}{\alpha}_{M=0, \text{phg}} + \overset{(i)}{\alpha}_{M^1, \text{phg}} + \overset{(i)}{\alpha}_{\Delta}$.

Of course, in order for these definitions to be well-defined, one briefly needs to strengthen out scattering theory to allow for suitably decaying inhomogeneous terms.

The point is now that one can use the estimate (6.10.11), the only estimate of this part of the chapter where we have sufficiently strong control in ℓ in order to admit summation in ℓ , to prove the following statements:

- First, we can show: $|(\overset{(i)}{\alpha}_{M=0, \text{phg}})_{\ell}| \leq C_{\ell} r^{-3}$.
- Inserting this estimate into (6.14.4) and proceeding as in the proof of (6.10.11), we can then show: $|(\overset{(i)}{\alpha}_{M^1, \text{phg}})_{\ell}| \leq C_{\ell} r^{-3} r_0^{-1}$.

- Inserting both of the above estimates, we can similarly show that $|(\overset{(i)}{\alpha}_\Delta)_\ell| \leq C_\ell r^{-3} r_0^{-1-\delta}$.
It is relatively straight-forward to then improve this estimate to $|(\overset{(i)}{\alpha}_\Delta)_\ell| \leq C_\ell r^{-4-\delta}$.

In each of the above estimates, the constant C_ℓ is some constant changing from line to line which, importantly, can be bounded against ℓ^N for an ℓ -independent integer N .

The upshot is that the issue of proving, say, (6.15.3) now entirely reduces to proving robust estimates for the quantities $(\overset{(i)}{\alpha}_{M=0,\text{phg}})_\ell$ and $(\overset{(i)}{\alpha}_{M^1,\text{phg}})_\ell$, which can, for instance, be done by proving that these quantities satisfy certain persistence of polyhomogeneity results, cf. the brief discussion in §5.5.5 of the previous chapter.

In order to prove more refined asymptotic statements, e.g. that

$$\lim_{v \rightarrow \infty} r^5 \log^{-2} r \left(\overset{(i)}{\alpha} - r^{-4} \lim_{v \rightarrow \infty} r^4 \overset{(i)}{\alpha} \right) = \sum_{\ell=2}^{\infty} (-1)^{\ell+1} \frac{3M \mathcal{A}_\ell \mathcal{B}_\ell}{\ell(\ell+1)}, \quad (6.15.4)$$

one can follow a similar pattern: However, one now needs make a stronger assumption on initial data, namely that they admit an expansion up to an error term $\overset{(i)}{\alpha}_{\mathcal{C},\Delta} = \mathcal{O}(r^{-5})$, and one needs to consider higher-order expansions of the Teukolsky equation (up to order $\mathcal{O}(M^3)$) as well.

Part III:

Asymptotic analysis of the remainder of the system

In this third part of the chapter, we roughly outline⁶ how to obtain the asymptotics for the remaining quantities of the system (6.3.9)–(6.3.32) starting from the physical data of §6.8 in the gauge of Prop. 6.8.1 and our asymptotic estimates for $\overset{(1)}{\alpha}$, $\overset{(1)}{\underline{\alpha}}$, $\overset{(1)}{\Psi}$ and $\overset{(1)}{\underline{\Psi}}$ from sections 6.11, 6.12 and 6.13. This section ties up with the discussion in §5.6 of the previous chapter.

We begin with the asymptotics of the null shears:

6.16 Asymptotics for $\overset{(1)}{\hat{\chi}}$ and $\overset{(1)}{\underline{\hat{\chi}}}$

Asymptotic expressions for the in- and outgoing null shears follow by straight-forwardly integrating the transport equations (6.3.17b) and (6.3.18b) from $\underline{\mathcal{C}}$ and $\mathcal{I}^-$, respectively.

6.16.1 Asymptotics for $\overset{(1)}{\hat{\chi}}$

By (6.3.17b), we have

$$\frac{r^2 \overset{(1)}{\hat{\chi}}}{\Omega} = \frac{r^2 \overset{(1)}{\hat{\chi}_{\mathcal{C}}}}{\Omega} - \int_{v_1}^v r^2 \overset{(1)}{\alpha} dv'. \quad (6.16.1)$$

Using the asymptotic estimate (6.11.3) for $\overset{(1)}{\alpha}$, we first compute the limit of $r^2 \overset{(1)}{\hat{\chi}}_{\ell}$ at $\mathcal{I}^+$:

$$\begin{aligned} \lim_{v \rightarrow \infty} r^2 \overset{(1)}{\hat{\chi}}_{\ell} &= \frac{r^2 (\overset{(1)}{\hat{\chi}_{\mathcal{C}}})_{\ell}}{\Omega} - \int_{v_1}^{\infty} \sum_{n=0}^{\ell-2} \frac{r_0^n}{r^{n+3}} \mathcal{A}_{\ell} r_0^2 S_{\ell,0,\ell-2,n,2} dv + \mathcal{O}(r_0^{-1} \log r_0) \\ &= \frac{r^2 (\overset{(1)}{\hat{\chi}_{\mathcal{C}}})_{\ell}}{\Omega} - (-1)^{\ell} \frac{2(\ell-2)!}{(\ell+2)!} \mathcal{A}_{\ell} \sum_{n=0}^{\ell-2} \frac{(-1)^n}{n+2} \frac{(\ell+2+n)!}{n!(n+2)!(\ell-2-n)!} + \mathcal{O}(r_0^{-1} \log r_0). \end{aligned} \quad (6.16.2)$$

The proof of the following identity is left as an exercise to the reader:

$$\sum_{n=0}^{\ell-2} \frac{(-1)^n}{n+2} \frac{(\ell+2+n)!}{n!(n+2)!(\ell-2-n)!} = -1 + (-1)^{\ell} + \ell(\ell+1). \quad (6.16.3)$$

Now, on the one hand, we have by (6.8.21) that

$$\begin{aligned} \frac{r^2 (\overset{(1)}{\hat{\chi}_{\mathcal{C}}})_{\ell}}{\Omega} &= ((-2\overset{\circ}{\mathcal{D}}_2^* \overset{\circ}{\mathcal{D}}_2 - 1) \overset{(1)}{\hat{\chi}}_{\mathcal{S}_{\infty}})_{\ell} + \mathcal{O}(r_0^{-1} \log r_0) \\ &= ((\overset{\circ}{\Delta} - 3) \overset{(1)}{\hat{\chi}}_{\mathcal{S}_{\infty}})_{\ell} + \mathcal{O}(r_0^{-1} \log r_0) = (1 - \ell(\ell+1)) (\overset{(1)}{\hat{\chi}}_{\mathcal{S}_{\infty}})_{\ell}. \end{aligned} \quad (6.16.4)$$

⁶More details will be given in the upcoming [KM23].

On the other hand, we have by (6.8.17)

$$\mathcal{A} = \mathring{\mathcal{D}}_2^* \mathring{\mathcal{D}}_1^* (\mathring{\rho}_{\mathcal{S}_\infty}, -\mathring{\sigma}_{\mathcal{S}_\infty}) = \mathring{\mathcal{D}}_2^* \mathring{\mathcal{D}}_1^* (\mathring{\rho}_{\mathcal{S}_\infty}, -\mathring{\text{curl}} \mathring{\text{div}} \hat{\chi}_{\mathcal{S}_\infty}^{(1)}). \quad (6.16.5)$$

Let now $\mathcal{Y}^E$ be the unique stf $\mathcal{S}_{u,v}$ -tangent two-tensor field that satisfies

$$\mathring{\text{curl}} \mathring{\text{div}} \mathcal{Y}^E = 0, \quad \mathring{\text{div}} \mathring{\text{div}} \mathcal{Y}^E = \mathring{\rho}_{\mathcal{S}_\infty}. \quad (6.16.6)$$

Then we have

$$\mathcal{A}_\ell = \frac{(\ell+2)!}{2(\ell-2)!} (\mathcal{Y}_\ell^E - (\hat{\chi}_{\mathcal{S}_\infty}^{(1)})_\ell^H). \quad (6.16.7)$$

Putting all the above equalities together, we conclude that, for the magnetic part of $\hat{\chi}^{(1)}$:

$$\lim_{v \rightarrow \infty} r^2 \hat{\chi}_\ell^H = (-1)^\ell (\hat{\chi}_{\mathcal{S}_\infty}^{(1)})_\ell^H + \mathcal{O}(r_0^{-1} \log r_0), \quad (6.16.8)$$

or, to phrase it slightly differently:

$$\lim_{v \rightarrow \infty} \lim_{u \rightarrow -\infty} r^2 \hat{\chi}_\ell^H = (-1)^\ell \lim_{u \rightarrow -\infty} \lim_{v \rightarrow \infty} r^2 \hat{\chi}_\ell^H. \quad (6.16.9)$$

This is Strominger's antipodal matching condition, see also [Mas22b].

Similarly, for the electric part of $\hat{\chi}^{(1)}$, we obtain

$$\lim_{v \rightarrow \infty} r^2 \hat{\chi}_\ell^E = (1 - \ell(\ell+1)) (\hat{\chi}_{\mathcal{S}_\infty}^{(1)})_\ell^E - (-1 + (-1)^\ell + \ell(\ell+1)) \mathcal{Y}_\ell^E \quad (6.16.10)$$

We note that it's possible to add a pure gauge solution such that $\mathcal{Y}^E = -(\hat{\chi}_{\mathcal{S}_\infty}^{(1)})^E$.

Having obtained an estimate for the limit of $r^2 \hat{\chi}$ towards $\mathcal{I}^+$, we can now integrate (6.3.17b) from $\mathcal{I}^+$ to compute the next terms in the expansion of $\hat{\chi}^{(1)}$ towards $\mathcal{I}^+$, this gives

$$r^2 \hat{\chi}_\ell^{(1)} = \lim_{v \rightarrow \infty} r^2 \hat{\chi}_\ell^{(1)} - \frac{(-1)^\ell 2M \mathcal{A}_\ell}{(\ell-1)(\ell+2)} \frac{1}{r} - \frac{(-1)^\ell 3M \mathcal{B}_\ell}{\ell(\ell+1)} \frac{\log^2 r}{r^2} + \mathcal{O}\left(M \frac{\log r}{r^2} + \frac{r_0^2}{r^2}\right). \quad (6.16.11)$$

6.16.2 Asymptotics for $\hat{\chi}^{(1)}$

We similarly find the asymptotics for $\hat{\chi}^{(1)}$ by using (6.3.18a) and writing

$$\frac{r^2 \hat{\chi}_\ell^{(1)}}{\Omega} = (\hat{\chi}_{\mathcal{S}_\infty}^{(1)})_\ell - \int_{-\infty}^u r^2 \hat{\alpha}_\ell^{(1)} du'. \quad (6.16.12)$$

In particular, we obtain the following expression for the limit towards $\mathcal{I}^+$:

$$\lim_{v \rightarrow \infty} r \hat{\chi}_\ell^{(1)} = - \int_{-\infty}^u \lim_{v \rightarrow \infty} r \hat{\alpha}_\ell^{(1)} du', \quad (6.16.13)$$

which we can compute using (6.13.9).

Furthermore, by commuting (6.3.18a) with $\frac{r^2}{\Omega^2} \nabla_v$, we can compute that the following limits exist as $v \rightarrow \infty$:

$$\frac{r^2}{\Omega^2} \nabla_v (r \Omega^{(i)} \underline{\hat{\chi}}_\ell) \rightarrow \lim_{v \rightarrow \infty} \frac{r^2}{\Omega^2} \nabla_v (r \Omega^{(i)} \underline{\hat{\chi}}_\ell) \quad \text{as } v \rightarrow \infty, \quad (6.16.14)$$

$$\frac{1}{\log r} \left(\frac{r^2}{\Omega^2} \nabla_v \right)^2 (r \Omega^{(i)} \underline{\hat{\chi}}_\ell) \rightarrow \lim_{v \rightarrow \infty} \frac{1}{\log r} \left(\frac{r^2}{\Omega^2} \nabla_v \right)^2 (r \Omega^{(i)} \underline{\hat{\chi}}_\ell) \quad \text{as } v \rightarrow \infty. \quad (6.16.15)$$

In fact, obtaining the second limit requires some amount of computations. Indeed, if we write

$$\begin{aligned} \nabla_v \left(\frac{r^2}{\Omega^2} \nabla_v (r \Omega^{(i)} \underline{\hat{\chi}}_\ell) \right) &= r \nabla_v^2 \left(\frac{r^2}{\Omega} \underline{\hat{\chi}} \right) + \frac{2M}{r} \nabla_v \left(\frac{r^2}{\Omega} \underline{\hat{\chi}} \right) - \frac{4M\Omega^2}{r^2} \frac{r^2}{\Omega} \underline{\hat{\chi}} \\ &= -r \left(\int \nabla_v^2 (r^2 \underline{\hat{\alpha}}) du \right) - \frac{2M}{r} \left(\int \nabla_v (r^2 \underline{\hat{\alpha}}) du \right) + \frac{4M}{r^2} \left(\int r^2 \underline{\hat{\alpha}} du - \underline{\hat{\chi}}_{\mathcal{S}_\infty} \right) \\ &= -r \left(\int \frac{\Omega^2}{r^3} \left(\frac{r^2}{D} \nabla_v \right)^2 (r \Omega^{(i)} \underline{\hat{\chi}}_\ell) - \frac{2M}{r^3} \left(\frac{r^2}{D} \nabla_v \right) (r \Omega^{(i)} \underline{\hat{\chi}}_\ell) + \frac{2M}{r^2 \Omega^2} r \Omega^{(i)} \underline{\hat{\chi}}_\ell du \right) \\ &\quad - \frac{2M}{r} \left(\int \frac{1}{r} \left(\frac{r^2}{D} \nabla_v \right) (r \Omega^{(i)} \underline{\hat{\chi}}_\ell) + \left(1 - \frac{2M}{\Omega^2 r} \right) r \Omega^{(i)} \underline{\hat{\chi}}_\ell du \right) \\ &\quad + \frac{4M}{r^2} \left(\int \frac{r}{\Omega^2} r \Omega^{(i)} \underline{\hat{\chi}}_\ell du - \underline{\hat{\chi}}_{\mathcal{S}_\infty} \right), \end{aligned}$$

then it at first glance looks like the RHS decays like r^{-1} as $v \rightarrow \infty$. For instance, the term $-r \int \frac{\Omega^2}{r^3} \left(\frac{r^2}{D} \nabla_v \right)^2 (r \Omega^{(i)} \underline{\hat{\chi}}_\ell) du$ would naively seem to give a contribution at order $-r \int r^{-3} du \sim r^{-1}$. However, inserting the precise asymptotics of (6.13.6), one will encounter a cancellation that will mean this term behaves like $r^{-2} \log r$ towards $\mathcal{I}^-$. Similarly, all the terms featuring an integral of $\underline{\hat{\alpha}}$ (with no ∇_v -derivatives acting on $\underline{\hat{\alpha}}$) give a $1/r$ -contribution as well, but these contributions cancel each other as can be seen by integration by parts. Together, the two observations above ensure the existence of the limit (6.16.15). We leave the details to the upcoming [KM23].

6.17 Deriving asymptotics for $\rho^{(i)}$, $\sigma^{(i)}$, $\beta^{(i)}$ and $\underline{\beta}^{(i)}$

From now on, we will only provide a rough sketch of the problem of obtaining asymptotics for the remaining quantities.

- As we already have an asymptotic estimate for $\Psi_\ell^{(i)}$ and $\underline{\Psi}_\ell^{(i)}$ by virtue of Thm. 6.12.1, we immediately also get an estimate for $\dot{\mathcal{P}}_2^* \dot{\mathcal{P}}_1^*(0, r^3 \sigma_\ell^{(i)})$ by virtue of (6.3.38). In particular, for Ψ_ℓ denoting either of $\Psi_\ell^{(i)}$, $\underline{\Psi}_\ell^{(i)}$ and $r^3 \sigma_\ell^{(i)}$, we have by (6.12.5) that

$$\lim_{u \rightarrow -\infty} \lim_{v \rightarrow \infty} \Psi_\ell = (-1)^\ell \lim_{v \rightarrow \infty} \lim_{u \rightarrow -\infty} \Psi_\ell = (-1)^\ell \lim_{u \rightarrow -\infty} \Psi_\ell|_{\mathcal{C}}. \quad (6.17.1)$$

- Next, we can use (6.3.34) (or, equivalently, (6.3.36)) and our asymptotic estimates for $\overset{(1)}{\Psi}_\ell$, $\overset{(1)}{\underline{\Psi}}_\ell$, $\overset{(1)}{\sigma}_\ell$, $\overset{(1)}{\hat{\chi}}_\ell$ and $\overset{(1)}{\underline{\hat{\chi}}}_\ell$ to immediately obtain an asymptotic expression for $\overset{(1)}{\rho}_\ell$, with $r^3 \overset{(1)}{\rho}_\ell$ attaining a limit at $\mathcal{I}^+$.
- We compute, using (6.3.31b)

$$\frac{r^4 \overset{(1)}{\beta}_\ell}{\Omega} = \frac{r^4 (\overset{(1)}{\beta}_{\mathcal{C}})_\ell}{\Omega} + \int \mathring{\mathrm{d}}v r^3 \overset{(1)}{\alpha}_\ell \, \mathrm{d}v. \quad (6.17.2)$$

In particular, we see that $\frac{r^4}{\log r} \overset{(1)}{\beta}_\ell$ takes a limit at $\mathcal{I}^+$.

- Finally, we obtain an asymptotic expression for $\overset{(1)}{\underline{\beta}}$ directly from (6.3.27), from which we see that $r^2 \overset{(1)}{\underline{\beta}}_\ell$ attains a limit at $\mathcal{I}^+$.

6.18 Deriving asymptotics for the remaining connection coefficients

We now provide instructions to compute the remaining connection coefficients:

- From our asymptotic expressions for $\nabla_v(r\Omega^2 \overset{(1)}{\hat{\chi}}_\ell)$, $\overset{(1)}{\hat{\chi}}_\ell$ and $\overset{(1)}{\hat{\chi}}_\ell$, we can obtain an asymptotic estimate for $\overset{\circ}{\mathcal{P}}_{2\eta_\ell}^{*(1)}$ via (6.3.18b). In particular, we can see that $r^2 \overset{(1)}{\eta}_\ell$ attains a limit at $\mathcal{I}^+$.
- We can now obtain an expression for $\overset{(1)}{\omega}_\ell$ by considering (6.3.20b) and differentiating (6.3.18b) in v . In particular, as a consequence of (6.16.15), we can show that $\frac{r^3}{\log r} \overset{(1)}{\omega}_\ell$ attains a limit at $\mathcal{I}^+$.
- We can now integrate the asymptotic estimate for $\overset{(1)}{\omega}_\ell$ in v from $\mathcal{C}$ and use (6.8.23) to compute $\lim_{v \rightarrow \infty} \overset{(1)}{\Omega}_\ell$.
- We can then read off $\overset{(1)}{\eta}_\ell$ from (6.3.12) and compute the limit $\lim_{v \rightarrow \infty} r \overset{(1)}{\eta}_\ell$.
- From the Codazzi equations (6.3.22) and (6.3.23), we can now read off that the limits $\lim_{v \rightarrow \infty} r^2 (\Omega \mathrm{tr} \overset{(1)}{\chi})_\ell$ and $\lim_{v \rightarrow \infty} r (\Omega \mathrm{tr} \overset{(1)}{\underline{\chi}})_\ell$ exist. In fact, it is relatively straight-forward to see that $r (\Omega \mathrm{tr} \overset{(1)}{\underline{\chi}})_\ell \lesssim r^{-1} r_0^{-1}$.
- Finally, an estimate for $\overset{(1)}{\underline{\omega}}$ can be obtained by integrating (6.3.21).

6.19 Deriving asymptotics for the remaining metric coefficients and for $\overset{(1)}{K}$

Finally, we sketch how to compute the remaining metric coefficients and, thus, the Gaussian curvature:

- The quantity $\hat{\mathcal{G}}_\ell^{(1)}$ can be directly obtained from (6.3.10a). As $\lim_{u \rightarrow -\infty} \hat{\mathcal{G}}_\ell^{(1)} = 0$, it immediately follows that $\lim_{v \rightarrow \infty} \hat{\mathcal{G}}_\ell^{(1)}$ vanishes as well.
- Similarly, by commuting (6.3.10a) with ∇_v , we can obtain an asymptotic expression for $\nabla_v \hat{\mathcal{G}}_\ell^{(1)}$, and thus, via (6.3.10b) an asymptotic estimate for $b_\ell^{(1)}$. One can thus show that $r b_\ell^{(1)}$ attains a limit at $\mathcal{I}^+$.
- Finally, we compute $\text{tr} \hat{\mathcal{G}}_\ell^{(1)}$ via integration of (6.3.9a) from $\mathcal{I}^-$. It follows immediately that $\lim_{v \rightarrow \infty} \text{tr} \hat{\mathcal{G}}_{\ell=0}^{(1)}$, and thus, by (6.3.26), that $\lim_{v \rightarrow \infty} r^2 \hat{K}_\ell^{(1)} = 0$.
- With a bit more work, we can then, in fact, show that $r^3 \hat{K}^{(1)}$ attains a limit as $v \rightarrow \infty$.

We conclude from this sketch that the constructed solution corresponding to the physical scattering data of section 6.8 is *extendable to null infinity for any $s < 1$* in the sense of Def. 3.4 of [Hol16]. In fact, the limits of the quantities $r^{3+s} \alpha_\ell^{(1)}$, $r^{3+s} \beta_\ell^{(1)}$ and $r^{2+s} \omega_\ell^{(1)}$ towards $\mathcal{I}^+$ all vanish for any $s < 1$, and, in their place, the limits of the quantities $r^4 \alpha_\ell^{(1)}$, $\frac{r^4}{\log r} \beta_\ell^{(1)}$ and $\frac{r^3}{\log r} \omega_\ell^{(1)}$ are finite and nonvanishing.

Appendix 6.A Spin-weighted functions, the Newman–Penrose formalism, and the Christodoulou–Klainerman formalism

In this section, we relate the formalism of the present chapter to the Newman–Penrose formalism by defining spin-weighted quantities, the spin-weighted spherical harmonics and by explicitly writing down a dictionary between the notation used by Newman and Penrose and that of the present chapter (which goes back to Christodoulou and Klainerman). We begin with a few calculations.

6.A.1 Miscellaneous calculations on $\mathbb{S}^2$

We work with the metric $\dot{g} = d\theta^2 + \sin^2 \theta d\varphi^2$ on $\mathbb{S}^2$. We recall from (6.2.31) that any symmetric tracefree two-tensor α can be written as $\dot{\mathcal{P}}_2^* \dot{\mathcal{P}}_1^*(f, g)$ for two smooth functions f and g on $\mathbb{S}^2$ that are uniquely specified up to $\ell \leq 1$ -modes. Let us spell out what this operator looks like in components: The only non-vanishing Christoffel symbols are

$$\Gamma_{\varphi\varphi}^\theta = -\cos \theta \sin \theta, \quad \Gamma_{\varphi\theta}^\varphi = \cot \theta = \Gamma_{\theta\varphi}^\varphi.$$

Then, working in the orthonormal frame $e_1 = \partial_\theta$, $e_2 = \frac{1}{\sin \theta} \partial_\varphi$, the Christoffel symbols ($\dot{\nabla}_{e_i} e_j = \Gamma_{ij}^k e_k$) become

$$\Gamma_{21}^2 = \cot \theta, \quad \Gamma_{22}^1 = -\cot \theta, \quad \Gamma_{12}^2 = 0 = \text{all others.} \quad (6.A.1)$$

We can thus compute (using that $\dot{\nabla}_1 \dot{\nabla}_2 f = \dot{\nabla}_2 \dot{\nabla}_1 f$)

$$\begin{aligned} \dot{\mathcal{P}}_2^* \dot{\mathcal{P}}_1^*(f, g)_{11} &= \dot{\nabla}_1 \dot{\nabla}_1 f - \frac{1}{2} \dot{\Delta} f - \dot{\nabla}_1 \dot{\nabla}_2 g = -\dot{\mathcal{P}}_2^* \dot{\mathcal{P}}_1^*(f, g)_{22}, \\ \dot{\mathcal{P}}_2^* \dot{\mathcal{P}}_1^*(f, g)_{12} &= \dot{\nabla}_1 \dot{\nabla}_2 f - \frac{1}{2} (\dot{\nabla}_2 \dot{\nabla}_2 g - \dot{\nabla}_1 \dot{\nabla}_1 g), \end{aligned}$$

and, using (6.A.1),

$$\begin{aligned} \dot{\nabla}_1 \dot{\nabla}_1 f &= \partial_\theta^2 f, \\ \dot{\nabla}_1 \dot{\nabla}_2 f &= \frac{1}{\sin \theta} (\partial_\theta \partial_\varphi f - \cot \theta \partial_\varphi f) = \dot{\nabla}_2 \dot{\nabla}_1 f, \\ \dot{\nabla}_2 \dot{\nabla}_2 f &= \frac{1}{\sin^2 \theta} (\partial_\varphi^2 f + \cos \theta \sin \theta \partial_\theta f). \end{aligned}$$

In summary, we have

$$\dot{\mathcal{P}}_2^* \dot{\mathcal{P}}_1^*(f, g)_{11} = \partial_\theta^2 f - \frac{1}{2} \dot{\Delta} f - \frac{1}{\sin \theta} (\partial_\theta \partial_\varphi g - \cot \theta \partial_\varphi g) = -(\dot{\mathcal{P}}_2^* \dot{\mathcal{P}}_1^*(f, g))_{22},$$

$$\mathring{\mathcal{P}}_2^* \mathring{\mathcal{P}}_1^*(f, g)_{12} = \frac{\partial_\theta \partial_\varphi f}{\sin \theta} - \frac{\cot \theta}{\sin \theta} \partial_\varphi f - \frac{1}{2 \sin^2 \theta} (\partial_\varphi^2 g + \cos \theta \sin \theta \partial_\theta g) + \frac{1}{2} \partial_\theta^2 g.$$

We can similarly compute that the components of the Laplacian $\mathring{\Delta}$ acting on symmetric traceless two-tensors are given by

$$\begin{aligned} (\mathring{\Delta} \alpha)_{11} &= \mathring{\Delta}(\alpha_{11}) - 4 \cot^2 \theta \alpha_{11} - 4 \frac{\cot \theta}{\sin \theta} \alpha_{12}, \\ (\mathring{\Delta} \alpha)_{12} &= \mathring{\Delta}(\alpha_{12}) - 4 \cot^2 \theta \alpha_{12} + 4 \frac{\cot \theta}{\sin \theta} \alpha_{11}, \end{aligned} \tag{6.A.2}$$

where we used

$$\begin{aligned} (\mathring{\nabla}_1 \mathring{\nabla}_1 \alpha)_{11} &= \partial_\theta^2 \alpha_{11}, \\ (\mathring{\nabla}_2 \mathring{\nabla}_2 \alpha)_{11} &= \frac{1}{\sin^2 \theta} \partial_\varphi^2 \alpha_{11} - 4 \frac{\cot \theta}{\sin \theta} \partial_\varphi \alpha_{11} + \cot \theta \partial_\theta \alpha_{11} - 4 \cot^2 \theta \alpha_{11}, \\ (\mathring{\nabla}_1 \mathring{\nabla}_1 \alpha)_{12} &= \partial_\theta^2 \alpha_{12}, \\ (\mathring{\nabla}_2 \mathring{\nabla}_2 \alpha)_{12} &= \frac{1}{\sin^2 \theta} \partial_\varphi^2 \alpha_{12} + 4 \frac{\cot \theta}{\sin \theta} \partial_\varphi \alpha_{11} + \cot \theta \partial_\theta \alpha_{12} - 4 \cot^2 \theta \alpha_{12}. \end{aligned}$$

6.A.2 Spin-weighted functions, the operators $\mathring{\partial}$, $\mathring{\partial}'$, and the spin-weighted spherical harmonics ${}_s Y_{\ell, m}$

Rather than working in the simple geometric framework of 1-forms and stf two-tensor fields, large parts of the literature work within the closely related framework of spin s -weighted functions. We here define the spaces of spin s -weighted functions following [Sbi22]. Introduce the following complex frame vector fields on $\mathbb{S}^2$:

$$m = \frac{1}{\sqrt{2}}(e_1 + ie_2), \quad \bar{m} = \frac{1}{\sqrt{2}}(e_1 - ie_2). \tag{6.A.3}$$

We again stress that the notation $\bar{m}$ does not refer to magnetic conjugation (cf. 6.2.4), $\bar{m}$ is simply the complex conjugate of m .

Definition 6.A.1. *We define*

- the space of smooth spin 2-weighted functions on $\mathbb{S}^2$ as the image of $\Gamma^\infty(T_{\text{stf}}^{(0,2)}(\mathbb{S}^2))$ under the map sending $\alpha \in \Gamma^\infty(T_{\text{stf}}^{(0,2)}(\mathbb{S}^2))$ to $\alpha(m, m) = \alpha_{11} + i\alpha_{12}$.
- the space of spin (-2)-weighted functions on $\mathbb{S}^2$ as the image of $\Gamma^\infty(T_{\text{stf}}^{(0,2)}(\mathbb{S}^2))$ under the map sending α to $\alpha(\bar{m}, \bar{m}) = \alpha_{11} - i\alpha_{12}$.
- the space of smooth spin 1-weighted functions on $\mathbb{S}^2$ as the image of $\Gamma^\infty(T^{(0,1)}\mathbb{S}^2)$ under the map sending $\beta \in \Gamma^\infty(T^{(0,1)}\mathbb{S}^2)$ to $\sqrt{2}\beta(m) = \beta_1 + i\beta_2$,
- the space of spin (-1)-weighted functions as the image of $\Gamma^\infty(T^{(0,1)}\mathbb{S}^2)$ under the map sending β to $\sqrt{2}\beta(\bar{m})$.

- the space of smooth spin 0-weighted functions on $\mathbb{S}^2$ as the image of $\Gamma^\infty(\mathbb{S}^2)^2$ under the map $\mathfrak{c}$ sending $(f, g) \mapsto \mathfrak{c}(f, g) := (f + ig)$, or, equivalently, under the map $\bar{\mathfrak{c}}$ sending $(f, g) \mapsto \bar{\mathfrak{c}}(f, g) := (f - ig)$.

For intrinsic definitions and properties of these spaces, see also the original work [NP66], or [DHR19a] or [Sbi22]. In large parts of the physics literature on spin s -weighted functions, readers will likely also encounter the eth operators (eth referring to the Old English letter $\eth$):

We here provide an extrinsic, geometric definition of these operators in terms of the formalism of the present chapter:

Definition 6.A.2. Let $f \in \Gamma^\infty(\mathbb{S}^2)$, let $\beta \in \Gamma^\infty(T^{(0,1)}\mathbb{S}^2)$, and let $\alpha \in \Gamma^\infty(T_{\text{stf}}^{(0,2)}(\mathbb{S}^2))$. We define the operators $\eth_{s \rightarrow s+1}$ (sending spin s -weighted functions to spin $s+1$ -weighted functions) and $\eth'_{s \rightarrow s-1}$ (sending spin s -weighted functions to spin $s-1$ -weighted functions, respectively, via:

$$\begin{aligned} \sqrt{2}(\eth_1^*(f, 0))(m) &=: \eth_{0 \rightarrow 1}f, & \sqrt{2}(\eth_1^*(f, 0))(\bar{m}) &=: \eth'_{0 \rightarrow -1}f, \\ \sqrt{2}(\eth_2^*(\beta))(m, m) &=: \eth_{1 \rightarrow 2}(\beta(m)), & \sqrt{2}(\eth_2^*(\beta))(\bar{m}, \bar{m}) &=: \eth'_{-1 \rightarrow -2}(\beta(\bar{m})), \\ \sqrt{2}(\eth_2\alpha)(m) &=: -\eth'_{2 \rightarrow 1}(\alpha(m, m)), & \sqrt{2}(\eth_2\alpha)(\bar{m}) &=: -\eth_{-2 \rightarrow -1}(\alpha(\bar{m}, \bar{m})), \end{aligned} \quad (6.A.4)$$

and, finally,

$$\mathfrak{c}(\eth_1\beta) = \mathfrak{d}\mathfrak{v}\beta + i\mathfrak{c}\mathfrak{v}\mathfrak{r}\mathfrak{l}\beta =: -\sqrt{2}\eth'_{1 \rightarrow 0}(\beta(m)), \quad \bar{\mathfrak{c}}(\eth_1\beta) = \mathfrak{d}\mathfrak{v}\beta - i\mathfrak{c}\mathfrak{v}\mathfrak{r}\mathfrak{l}\beta := -\sqrt{2}\eth_{-1 \rightarrow 0}(\beta(\bar{m})). \quad (6.A.5)$$

Remark 6.A.1. Using that $*e_1 = -e_2$, $*e_2 = -e_1$ and thus $*m = i \cdot m$, $*\bar{m} = -i\bar{m}$, (6.A.4) implies that $(\eth_1^*(0, g))(m) = i\eth g$, $(\eth_1^*(0, g))(\bar{m}) = -i\eth' g$, so

$$\sqrt{2}(\eth_1^*(f, g))(m) = \eth_{0 \rightarrow -1}(\mathfrak{c}(f, g)), \quad \sqrt{2}(\eth_1^*(f, g))(\bar{m}) = \eth'_{0 \rightarrow 1}(\bar{\mathfrak{c}}(f, g)). \quad (6.A.6)$$

Remark 6.A.2. It is easy to check that, in (θ, φ) coordinates, Def. 6.A.2 implies

$$\eth_{s \rightarrow s+1} = -\partial_\theta - \frac{i}{\sin \theta} \partial_\varphi + s \cot \theta, \quad \eth'_{s \rightarrow s-1} = -\partial_\theta + \frac{i}{\sin \theta} \partial_\varphi - s \cot \theta. \quad (6.A.7)$$

This is how most readers will have encountered the operator before (with the $s \rightarrow s+1$ subscript suppressed). For instance, we compute

$$\begin{aligned} \frac{1}{\sqrt{2}}\eth_{1 \rightarrow 2}\beta(m) &:= (\eth_2^*\beta)(m, m) = (\eth_2^*\beta)_{11} + i(\eth_2^*\beta)_{12} \\ &= -\frac{1}{2} \left(\partial_\theta \beta_1 - \frac{1}{\sin \theta} \partial_\varphi \beta_2 \right) - \frac{i}{2} \left(\partial_\theta + \frac{1}{\sin \theta} \partial_\varphi \beta_1 - \cot \theta \beta_2 \right) \\ &= -\frac{1}{2} \left(\partial_\theta (\beta_1 + i\beta_2) + \frac{i}{\sin \theta} \partial_\varphi (\beta_1 + i\beta_2) - \cot \theta (\beta_1 + i\beta_2) \right). \end{aligned}$$

Remark 6.A.3. Recall the angular operator $\mathring{\mathcal{P}}_2^* \mathring{\mathcal{P}}_1^* \overline{\mathring{\mathcal{P}}_1} \mathring{\mathcal{P}}_2$ appearing on the RHS of (6.3.44). We can rewrite this operator in terms of spin-weighted functions as follows:

$$\begin{aligned} 2(\mathring{\mathcal{P}}_2^* \mathring{\mathcal{P}}_1^* \overline{\mathring{\mathcal{P}}_1} \mathring{\mathcal{P}}_2 \alpha)(\overline{m}, m) &= \sqrt{2} \mathring{\partial}'_{-1 \rightarrow -2} ((\mathring{\mathcal{P}}_1^* \overline{\mathring{\mathcal{P}}_1} \mathring{\mathcal{P}}_2 \alpha)(\overline{m})) \\ &= \mathring{\partial}'_{-1 \rightarrow -2} \mathring{\partial}'_{0 \rightarrow -1} (\overline{\mathfrak{c}(\mathring{\mathcal{P}}_1 \mathring{\mathcal{P}}_2 \alpha)}) \\ &= \mathring{\partial}'_{-1 \rightarrow -2} \mathring{\partial}'_{0 \rightarrow -1} (\mathfrak{c}(\mathring{\mathcal{P}}_1 \mathring{\mathcal{P}}_2 \alpha)) \\ &= -\sqrt{2} \mathring{\partial}'_{-1 \rightarrow -2} \mathring{\partial}'_{0 \rightarrow -1} \mathring{\partial}'_{1 \rightarrow 0} ((\mathring{\mathcal{P}}_2 \alpha)(m)) \\ &= \mathring{\partial}'_{-1 \rightarrow -2} \mathring{\partial}'_{0 \rightarrow -1} \mathring{\partial}'_{1 \rightarrow 0} \mathring{\partial}'_{2 \rightarrow 1} (\alpha(m, m)). \end{aligned}$$

We now use the $\mathring{\partial}$ -operators to define the spin s -weighted spherical harmonics:

Definition 6.A.3. Defining⁷ ${}_0 Y_{\ell, m} = Y_{\ell, m}$, the spin s -weighted spherical harmonics ${}_s Y_{\ell, m}$ are inductively defined for any $s \in \mathbb{Z}$ and $\ell \geq |s|$ via:

$$\begin{aligned} {}_{s+1} Y_{\ell, m} &:= \frac{1}{\sqrt{(\ell - s)(\ell + s + 1)}} \mathring{\partial}_{s \rightarrow s+1} {}_s Y_{\ell, m}, \\ {}_{s-1} Y_{\ell, m} &:= -\frac{1}{\sqrt{(\ell + s)(\ell - s + 1)}} \mathring{\partial}'_{s \rightarrow s-1} {}_s Y_{\ell, m}. \end{aligned} \quad (6.A.8)$$

To give an example, we have

$${}_{\pm 2} Y_{\ell, m} = \frac{1}{\sqrt{(\ell - 1)\ell(\ell + 1)(\ell + 2)}} \left(\mp \partial_\theta - \frac{i}{\sin \theta} \partial_\varphi \pm \cot \theta \right) \left(\mp \partial_\theta - \frac{i}{\sin \theta} \partial_\varphi \right) Y_{\ell, m} = \overline{{}_{\mp 2} Y_{\ell, m}}.$$

Remark 6.A.4. With these definitions at hand, we can now relate the spin s -weighted spherical harmonics to the tensorial spherical harmonics of Def. 6.2.3:

$${}_1 Y_{\ell, m} = \sqrt{2} Y_{\ell, m}^{\text{E}, 1}(m) = -i \sqrt{2} Y_{\ell, m}^{\text{H}, 1}(m) = -\overline{{}_{-1} Y_{\ell, m}}, \quad (6.A.9)$$

$${}_2 Y_{\ell, m} = \sqrt{2} Y_{\ell, m}^{\text{E}, 2}(m, m) = -i \sqrt{2} Y_{\ell, m}^{\text{H}, 2}(m, m) = \overline{{}_{-2} Y_{\ell, m}}. \quad (6.A.10)$$

The spin s -weighted spherical harmonics are eigenfunctions of the spin s -weighted spherical Laplacian:

Definition 6.A.4. For $|s| = 2$, the spin s -weighted Laplacian is defined via

$$\mathring{\Delta}^{[+2]}(\alpha(m, m)) := ((\mathring{\Delta} + 2)\alpha)(m, m), \quad \mathring{\Delta}^{[-2]}(\alpha(\overline{m}, \overline{m})) := ((\mathring{\Delta} - 2)\alpha)(\overline{m}, \overline{m}), \quad (6.A.11)$$

and analogously for $|s| = 1, 0$. In particular, $\mathring{\Delta}^{[0]}$ is simply the Laplacian acting on scalar functions.

⁷In most parts of the literature, one typically starts with $\mathfrak{c}(Y_{\ell, m}) = Y_{\ell, m} + i Y_{\ell, m}$ as ${}_0 Y_{\ell, m}$.

The eigenvalues of $\overset{\circ}{\Delta}^{[s]}$ are $-\ell(\ell+1) + s(s+1)$. In coordinates (θ, φ) , the spin s -weighted Laplacian for $s = 2$ reads (cf. to (6.A.2))

$$\overset{\circ}{\Delta}^{[s]} = \overset{\circ}{\Delta}^{[0]} - \frac{2si \cot \theta}{\sin \theta} \partial_\varphi - 4 \cot^2 \theta + s. \quad (6.A.12)$$

Finally, we remark that it can also be written as

$$\overset{\circ}{\Delta}^{[s]} = \eth'_{s+1 \rightarrow s} \eth_{s \rightarrow s+1} = \eth_{s-1 \rightarrow s} \eth'_{s \rightarrow s-1} + 2s. \quad (6.A.13)$$

6.A.3 A dictionary between the Christodoulou–Klainerman formalism and the Newman–Penrose formalism

In this part of the appendix, we provide an explicit translation of the Christodoulou–Klainerman framework and notation, employed in the present chapter, to the very closely related Newman–Penrose formalism.

Since many people are familiar with only one of them, even though both formalisms are essentially the same, we hope that this will make it easier for people to read papers written in the respective other formalism.

The C–K formalism In the Christodoulou–Klainerman formalism [CK93], given a spacetime $(\mathcal{M}, g)$, a local orthonormal null frame (e_1, e_2, e_3, e_4) is picked, with e_1, e_2 spacelike and e_3, e_4 null. That is to say, letting upper-case Latin letters denote indices $\in \{1, 2\}$, $g(e_A, e_B) = \delta_{AB}$, $g(e_A, e_3) = 0 = g(e_A, e_4)$, and $g(e_3, e_4) = -2$. We further introduce the notation that $\not{g}$ and $\not{g}$ denote the metric and volume form on the orthogonal complement $\langle e_3, e_4 \rangle^\perp$ induced by the spacetime metric and spacetime volume form, respectively.

Then, the Ricci/connection coefficients are decomposed as follows:

$$\begin{aligned} \chi_{AB} &= g(\nabla_A e_4, e_B), & \underline{\chi}_{AB} &= g(\nabla_A e_3, e_B), \\ \zeta_A &= \frac{1}{2} g(\nabla_A e_4, e_3), & \underline{\zeta}_A &= \frac{1}{2} g(\nabla_A e_3, e_4) = -\zeta_A, \\ \omega &= \frac{1}{2} g(\nabla_4 e_3, e_4), & \underline{\omega} &= \frac{1}{2} g(\nabla_3 e_4, e_3), \\ \eta &= -\frac{1}{2} g(\nabla_3 e_A, e_4), & \underline{\eta} &= -\frac{1}{2} g(\nabla_4 e_A, e_3), \\ \xi_A &= \frac{1}{2} g(\overset{\circ}{\nabla}_4 e_4, e_A), & \underline{\xi}_A &= \frac{1}{2} g(\nabla_3 e_3, e_A), \end{aligned} \quad (6.A.14)$$

and, finally⁸,

$$\mathring{\nabla}_A = g(\nabla_A e_1, e_2), \quad \mathring{\nabla}_3 = g(\nabla_3 e_1, e_2), \quad \mathring{\nabla}_4 = g(\nabla_4 e_1, e_2). \quad (6.A.15)$$

⁸We are not aware of a standard convention for the notation for the Christoffel symbols on the sphere within the C–K framework, so we here introduce the notation $\mathring{\nabla}$.

All other connection coefficients either vanish or can be derived from the ones above by an application of the Leibniz rule, using the orthonormality of the frame. Notice that, in general, neither χ nor $\underline{\chi}$ are symmetric.

Similarly, the Weyl curvature tensor C (which equals the Riemann curvature tensor if $(\mathcal{M}, g)$ solves the Einstein vacuum equations) is decomposed as follows:

$$\begin{aligned}\alpha_{AB} &= C(e_A, e_4, e_B, e_4), & \underline{\alpha}_{AB} &= C(e_A, e_3, e_B, e_3) \\ \beta_A &= C(e_A, e_4, e_3, e_4), & \underline{\beta}_A &= C(e_A, e_3, e_4, e_3) \\ -\rho\delta_{AB} + \sigma\epsilon_{AB} &= C(e_A, e_3, e_B, e_4).\end{aligned}\tag{6.A.16}$$

Notice that α and $\underline{\alpha}$ are symmetric, and also tracefree by definition of the Weyl tensor.

The N–P formalism On the other hand, in the Newman–Penrose formalism [NP62], one works with a null tetrad $(l, n, m, \bar{m})$ which is related to a frame (e_1, e_2, e_3, e_4) as in the C–K formalism via $e_4 = l$, $e_3 = n$, $\frac{1}{\sqrt{2}}(e_1 + ie_2) = m$, $\frac{1}{\sqrt{2}}(e_1 - ie_2) = \bar{m}$. In order to closely resemble the standard works employing this formalism, we will now use ; subscripts to denote covariant differentiation. Notice that the C–K and the N–P formalisms use an overlapping set of symbols for different quantities, it should however always be clear from context which ones we are talking about. The Ricci coefficients are then decomposed into the following complex scalars:

$$\begin{aligned}\rho &= l_{\mu;\nu} m^\mu \bar{m}^\nu, & \sigma &= l_{\mu;\nu} m^\mu m^\nu, \\ \mu &= -n_{\mu;\nu} \bar{m}^\mu m^\nu, & \lambda &= -n_{\mu;\nu} \bar{m}^\mu \bar{m}^\nu, \\ \alpha &= \frac{1}{2}(l_{\mu;\nu} n^\mu \bar{m}^\nu - m_{\mu;\nu} \bar{m}^\mu \bar{m}^\nu), & \beta &= \frac{1}{2}(l_{\mu;\nu} n^\mu m^\nu - m_{\mu;\nu} \bar{m}^\mu m^\nu), \\ \epsilon &= \frac{1}{2}(l_{\mu;\nu} n^\mu l^\nu - m_{\mu;\nu} \bar{m}^\mu l^\nu), & \gamma &= \frac{1}{2}(l_{\mu;\nu} n^\mu n^\nu - m_{\mu;\nu} \bar{m}^\mu n^\nu), \\ \tau &= l_{\mu;\nu} m^\mu n^\nu, & \pi &= -n_{\mu;\nu} \bar{m}^\mu l^\nu, \\ \kappa &= l_{\mu;\nu} m^\mu l^\nu, & \nu &= -n_{\mu;\nu} \bar{m}^\mu n^\nu.\end{aligned}\tag{6.A.17}$$

Finally, Newman and Penrose decompose the Weyl curvature coefficients according to

$$\begin{aligned}\Psi_0 &= C(l, m, l, m), & \Psi_4 &= C(\bar{m}, n, \bar{m}, n), \\ \Psi_1 &= C(m, l, n, l), & \Psi_3 &= C(\bar{m}, n, l, n), \\ \Psi_2 &= -C(m, n, \bar{m}, l).\end{aligned}\tag{6.A.18}$$

The dictionary We explicitly relate the definitions of Newman and Penrose to the C–K quantities as follows:

$$\begin{aligned}
 \rho &= \chi(\bar{m}, m) = \frac{1}{2}(\not{\partial} \cdot \chi + i \not{\partial} \cdot \chi), & \sigma &= \chi(m, m) = \hat{\chi}(m, m), \\
 \mu &= -\underline{\chi}(m, \bar{m}) = -\frac{1}{2}(\not{\partial} \cdot \underline{\chi} - i \not{\partial} \cdot \underline{\chi}), & \lambda &= -\underline{\chi}(\bar{m}, \bar{m}) = -\hat{\chi}(\bar{m}, \bar{m}), \\
 \bar{\alpha} + \beta &= 2\zeta(m) = -2\underline{\zeta}(m), & -\bar{\alpha} + \beta &= i \cdot \not{\nabla}(m) = \not{\nabla}^*(m), \\
 \epsilon + \bar{\epsilon} &= -2\omega, & \gamma + \bar{\gamma} &= 2\underline{\omega}, \\
 \epsilon - \bar{\epsilon} &= i\not{\nabla}_4, & \gamma - \bar{\gamma} &= i\not{\nabla}_3, \\
 \tau &= 2\eta(m), & \pi &= -2\underline{\eta}(\bar{m}), \\
 \kappa &= 2\xi(m), & \nu &= -2\underline{\xi}(\bar{m}).
 \end{aligned} \tag{6.A.19}$$

Let us give one detailed example for the required computations:

$$\begin{aligned}
 -\bar{\alpha} + \beta &= \frac{1}{2}\bar{m}_{\mu;\nu}m^\mu m^\nu - \frac{1}{2}m_{\mu;\nu}\bar{m}^\mu m^\nu = \frac{1}{2}g(\nabla_m \bar{m}, m) - \frac{1}{2}g(\nabla_m m, \bar{m}) = -g(\nabla_m m, \bar{m}) \\
 &= -\frac{1}{2\sqrt{2}}g(\nabla_{e_1+ie_2}e_1+ie_2, e_1-ie_2) = -\frac{1}{2\sqrt{2}}(-ig(\nabla_1 e_1, e_2)+ig(\nabla_1 e_2, e_1)+g(\nabla_2 e_1, e_2)-g(\nabla_2 e_2, e_1)) \\
 &= -\frac{1}{\sqrt{2}}(g(\nabla_2 e_1, e_2) - ig(\nabla_1 e_1, e_2)) = \frac{i}{\sqrt{2}}(g(\nabla_1 e_1, e_2) + ig(\nabla_2 e_1, e_2)) = i\not{\nabla}_A m^A = \not{\nabla}^*(m)
 \end{aligned}$$

Finally, we evidently have

$$\begin{aligned}
 \Psi_0 &= \alpha(m, m), & \Psi_4 &= \underline{\alpha}(\bar{m}, \bar{m}), \\
 \Psi_1 &= \beta(m), & \Psi_3 &= \beta(\bar{m}), \\
 \Psi_2 &= \rho + i\sigma.
 \end{aligned} \tag{6.A.20}$$

Appendix 6.B Useful integral identities

Recall the notation $r_0(u) = r(u, v = v_1)$. Note that $r_0/r \leq 1$. Recall further the notation $D = 1 - 2M/r = \partial_v r = -\partial_u r$, and $D_0 = 1 - 2M/r_0$. Finally, we write, for any $q \in \mathbb{R}$ such that $-q \notin \mathbb{N}_{>0}$, $q!$ to mean $q! := \Gamma(q + 1)$.

6.B.1 Integrals in the u -direction

The following lemmata are used for integrating the approximate conservation law (6.9.16) or similar equations from $\mathcal{I}^-$:

Lemma 6.B.1. *Let $q \in \mathbb{R}$, $N \in \mathbb{N}$, with $-1 < q < N - 1$. Then*

$$\int_{-\infty}^u \frac{r(u', 1)^q}{r(u', v)^N} du' = \frac{q!(N - q - 2)!}{(N - 1)!} \frac{1}{r(u, v)^{N-q-1}} + \mathcal{O}\left(\frac{r_0}{r^{N-q}} + \frac{r_0^{q+1}}{r^N} + \frac{M}{r^{N-q}}\right). \quad (6.B.1)$$

In the special case where $q \in \mathbb{N}$, we have more precisely that

$$\int_{-\infty}^u \frac{r_0^q}{r^N} du' = \frac{q!(N - q - 2)!}{(N - 1)!} \frac{1}{r^{N-q-1}} \cdot \left(1 + \sum_{i=1}^q \frac{r_0^i}{r^i} \prod_{j=1}^i (N - q - 2 + j)\right) + \mathcal{O}\left(\frac{M}{r^{N-q}}\right). \quad (6.B.2)$$

Finally, we have

$$\int_{-\infty}^u \frac{r^q \log r}{r^N} du' = \frac{q!(N - q - 2)!}{(N - 1)!} \frac{\log r}{r^{N-q-1}} + \frac{d_{N,q}}{r^{N-q-1}} + \mathcal{O}\left(\frac{r_0 \log r_0}{r^{N-q}} + \frac{r_0^{q+1} \log r_0}{r^N} + \frac{M \log r}{r^{N-q}}\right), \quad (6.B.3)$$

where the constant $d_{N,q}$ is given by $\int_0^\infty \frac{x^q \log x}{(1+x)^N} dx = \frac{q!(N-q-2)!}{(N-1)!} (H_q - H_{N-q-2})$.

Proof. The second statement, (6.B.2), follows from

$$\begin{aligned} \int \frac{r_0^q}{r^N} du &= \int \frac{d}{du} \left(\frac{1}{N-1} \frac{r_0^q}{r^{N-1}} \right) + \frac{qD_0}{N-1} \frac{r_0^{q-1}}{r^{N-1}} - \frac{r_0^q}{r^N} (1-D) \\ &= \frac{1}{N-1} \frac{r_0^q}{r^{N-1}} + \frac{q}{N-1} \int \frac{r_0^{q-1}}{r^{N-1}} du + M \cdot \mathcal{O}\left(\int \frac{r_0^q}{r^{N+1}} + \frac{r_0^{q-2}}{r^{N-1}} du\right) \end{aligned}$$

and complete induction.

Moving to the first statement, (6.B.1), we first prove it in the case $M = 0$ where $r - r_0$ is independent of u :

$$\begin{aligned} \int_{-\infty}^u \frac{r_0^q}{r^N} du' &= \int_{-\infty}^u \frac{r_0^q}{((r - r_0) + r_0)^N} du' = \frac{1}{(r - r_0)^{N-q-1}} \int_{r_0/(r-r_0)}^\infty \frac{x^q}{(1+x)^N} dx \\ &= \frac{1}{(r - r_0)^{N-q}} \int_0^\infty \frac{x^q}{(1+x)^N} dx - \frac{1}{(r - r_0)^{N-q-1}} \int_0^{r_0/(r-r_0)} \frac{x^q}{(1+x)^N} dx. \end{aligned}$$

The first integral on the RHS can be looked up or be computed by *mathematica* and evaluates to $\int_0^\infty \frac{x^q}{(1+x)^N} dx = q!(N-q-2)!/(N-1)!$. On the other hand, the second integral can be bounded as

$$\int_0^{r_0/(r-r_0)} \frac{x^q}{(1+x)^N} dx = \frac{(r_0/(r-r_0))^{q+1}}{q+1} + \mathcal{O}\left((r_0/(r-r_0))^{q+2}\right).$$

This proves the claim if $M = 0$.

In order to prove the claim for $M > 0$, we copy the above proof and use that $r - r_0 = v + M\mathcal{O}(\log r)$. This immediately gives the claim, but with a logarithmic error bound of order $M\mathcal{O}(r^{q-N} \log r)$. We remove this logarithmic error term by slightly deforming the path of integration: Instead of integrating along constant v , we integrate along constant $r - r_0$: We work in coordinates $z = r - r_0$, $y = r_0$. Then $\partial_u = -D_0\partial_y + (D_0 - D)\partial_z$, $\partial_v = D\partial_z$. Clearly, the integral $\int_{-\infty}^u r_0^q/r^N du'$ then satisfies

$$-D_0\partial_y \left(\int_{-\infty}^u r_0^q/r^N du' \right) = r_0^q/r^N + (D - D_0)\partial_z \left(\int_{-\infty}^u r_0^q/r^N du' \right).$$

We can now divide by D_0 and integrate in y (along constant z). By using the estimate with the logarithmic error term for the second term on the RHS, we can show that that term is subleading:

$$\int_{-\infty}^u \frac{r_0^q}{r^N} du' = \int_{r_0}^\infty \frac{y^q}{(z+y)^N} dy + M \cdot \mathcal{O}\left((z+y)^{-N} y^{q-1}\right) dy.$$

The first integral on the RHS is now the same one we had for $M = 0$:

$$\int_{r_0}^\infty \frac{y^q}{(z+y)^N} dy = \frac{q!(N-q-2)!}{(N-1)!(r-r_0)^{N-q-1}} + \mathcal{O}\left(\frac{r_0^{q+1}}{(r-r_0)^N}\right).$$

The proof concludes by expanding $1/(r-r_0)^{N-q-1} = 1/(r^{N-q-1} \cdot (1-r_0/r)^{N-q-1})$.

Finally, we prove (6.B.3). We only consider the case $M = 0$: Then

$$\begin{aligned} \int_{-\infty}^u \frac{r_0^q \log r_0}{r^N} du' &= \int_{r_0}^\infty \frac{y^q \log y}{(z+y)^N} dy \\ &= \frac{1}{(r-r_0)^{N-q-1}} \int_{r_0/(r-r_0)}^\infty \frac{x^q}{(1+x)^N} dx + \frac{\log(r-r_0)}{(r-r_0)^{N-q-1}} \int_{r_0/(r-r_0)}^\infty \frac{x^q}{(1+x)^N} dx. \end{aligned}$$

The proof concludes by again writing both integrals as $\int_{r_0/(r-r_0)}^\infty = \int_0^\infty - \int_0^{r_0/(r-r_0)}$, and by using that $\log(r-r_0) = \log r + \mathcal{O}(r_0/r)$. $\square$

Lemma 6.B.2. *Let $N \in \mathbb{N}_{>0}$. Then*

$$\int_{-\infty}^u \frac{r_0^{-1}}{r^N} du = r^{-N} \log(r/r_0) + \mathcal{O}(r^{-N}). \quad (6.B.4)$$

Proof. We only give the proof for $M = 0$. The extension to $M > 0$ is immediate using $r - r_0 = v + \mathcal{O}(\log r)$.

The integral is computed using partial fractions. Denote again $r_0 = y$, $r - r_0 = z$. Then

$$\frac{1}{r_0 r^N} = \frac{1}{y(z+y)^N} = \frac{1}{z^N} \left(\frac{1}{y} + \frac{1}{y+z} \right) - \sum_{i=1}^{N-1} \frac{1}{z^{N+1-i}} \frac{1}{(z+y)^{i+1}}.$$

We now compute

$$z^{-N} \int_{-\infty}^u \left(\frac{1}{y} + \frac{1}{y+z} \right) du' = z^{-N} \int_{r_0}^{\infty} \left(\frac{1}{y} + \frac{1}{y+z} \right) dy = \frac{1}{(r-r_0)^N} \log r/r_0.$$

The statement then follows from $(r-r_0)^{-N} \log(r/r_0) = r^{-N} \log(r/r_0)(1+r_0/r)^N$ and the fact that $-x \log x$ is bounded by e^{-1} on $x \in [0, 1]$. $\square$

The previous lemmata can be extended to the entire range of $q \in \mathbb{R}$ using integration by parts:

Lemma 6.B.3. *Suppose that $N \in \mathbb{N}$, $q \in \mathbb{R}$, with $-q < -1$. Then*

$$\begin{aligned} \int \frac{x^{-q}}{(x+y)^N} dx' &= \frac{1}{(-q+1)} \sum_{i=0}^{\lceil q-2 \rceil} \frac{x^{-q+1+i}}{(x+y)^{N+i}} \prod_{j=0}^{i-1} \frac{(N+j)}{(-q+j+2)} \\ &\quad + \prod_{j=0}^{\lceil q-2 \rceil} \frac{(N+j)}{(-q+j+1)} \int \frac{x^{-q+\lceil q-1 \rceil}}{(x+y)^{N+\lceil q-1 \rceil}} dx. \end{aligned} \quad (6.B.5)$$

Proof. Induction. $\square$

Lemma 6.B.4. *We have*

$$\int_{-\infty}^u \frac{\log r - \log r_0}{r} du' = 1 + \mathcal{O}(r_0/r). \quad (6.B.6)$$

Proof. We again first prove this for $M = 0$. Using the dilogarithmic identity $\sum_{k=1}^{\infty} \frac{x^k}{k!} = -\int_0^x \frac{\log(1-x')}{x'} dx' (= \text{Li}_2(x))$ for $|x| \leq 1$ (page 1004 in [AS74]), we have that

$$\int_{-\infty}^u \frac{\log(v-u') - \log|u'|}{(v-u')} du' = -\int_0^{v/(v-u)} \frac{\log(1-x')}{x'} dx' = \frac{v}{v-u} + \sum_{k=2}^{\infty} (v/(v-u))^{k-1}/k^2,$$

so the result follows. In order to elevate the result to $M \neq 0$, we follow the same procedure as in the proof of Lemma 6.B.1. $\square$

6.B.2 Integrals in the v -direction

The next lemmata are relevant for integrating in v from $\underline{\mathcal{C}}$.

Lemma 6.B.5. Define $x = 1/r$ and $x_0 = 1/r_0$. Let $N \in \mathbb{N}$, and let $q \in \mathbb{R}$. Then

$$\underbrace{\int_{v_{\underline{\mathcal{C}}}}^v \frac{D}{r^2}(u, v_{(1)}) \cdots \int_{v_{\underline{\mathcal{C}}}}^{v_{(n-1)}} \frac{D}{r^2}(u, v_{(n)}) \int_{v_{\underline{\mathcal{C}}}}^{v_{(n)}} \frac{D}{r^2} r^q(u, v_{(n+1)}) \, dv_{(n+1)} \, dv_{(n)} \cdots dv_{(1)}}_{N \text{ integrals}} = (-1)^N \int_{x_0(u)}^x \cdots \int_{x_0}^{x_{(n-1)}} \int_{x_0}^{x_{(n)}} x_{(n+1)}^{-q} \, dx_{(n+1)} \cdots dx_{(1)}. \quad (6.B.7)$$

Proof. Direct substitution of the variable $x = 1/r$. □

Lemma 6.B.6. Let $N \in \mathbb{N}$. Then

$$(-1)^N \underbrace{\int_{x_0}^x \cdots \int_{x_0}^{x_{(n-1)}}}_{N \text{ times}} dx_{(n)} \cdots dx_{(1)} = \frac{(x_0 - x)^N}{N!}. \quad (6.B.8)$$

Proof. Induction. □

Lemma 6.B.7. Let $N \in \mathbb{N}$, and let $q \in \mathbb{R}$. If $N \leq q - 1$, or if $q \notin \mathbb{N}_{>0}$, then

$$\begin{aligned} (-1)^N \underbrace{\int_{x_0}^x \cdots \int_{x_0}^{x_{(n-1)}}}_{N \text{ times}} x_{(n)}^{-q} \, dx_{(n)} \cdots dx_{(1)} \\ = \frac{(q - N - 1)!}{(q - 1)!} x^{N-q} - x_0^{N-q} \sum_{i=0}^{N-1} \frac{(q + i - N - 1)!}{(q - 1)! i!} \left(1 - \frac{x}{x_0}\right)^i. \end{aligned} \quad (6.B.9)$$

Otherwise, if $N > q - 1$ and $q \in \mathbb{N}_{>0}$, then

$$\begin{aligned} (-1)^N \underbrace{\int_{x_0}^x \cdots \int_{x_0}^{x_{(n-1)}}}_{N \text{ times}} x_{(n)}^{-q} \, dx_{(n)} \cdots dx_{(1)} \\ = \frac{(-1)^{N+1-q}}{(q - 1)!(N - q)!} x^{N-q} \log(x/x_0) - x_0^{N-q} \sum_{i=0}^{N-1} d_{N,q,i} \left(1 - \frac{x}{x_0}\right)^i, \end{aligned} \quad (6.B.10)$$

with some coefficients $d_{N,q,i} \in \mathbb{Q}$.

Proof. The first statement (6.B.9) is easy to prove (via induction).

In order to show (6.B.10), one first shows that

$$\underbrace{\int_{x_0}^x \cdots \int_{x_0}^{x_{(n-1)}}}_{N \text{ times}} x_{(n)}^{-1} \, dx_{(n)} \cdots dx_{(1)} = \frac{x^{N-1} \log(x/x_0)}{(N - 1)!} - x_0^{N-1} \sum_{i=1}^{N-1} \frac{(\frac{x}{x_0} - 1)^i}{i!(N - 1 - i)!} \sum_{j=0}^{i-1} \frac{1}{(N - 1 - j)}.$$

Combining this with the first statement of the lemma, and appropriately splitting up the integrals, we then obtain

$$\begin{aligned}
(-1)^N \underbrace{\int_{x_0}^x \cdots \int_{x_0}^{x_{(n-1)}}}_{N \text{ times}} x_{(n)}^{-q} dx_{(n)} \cdots dx_{(1)} &= \frac{(-1)^{N+1-q}}{(q-1)!(N-q)!} x^{N-q} \log(x/x_0) \\
&\quad - \frac{(-1)^{N+1-q}}{(q-1)!(N-q)!} x_0^{N-q} \sum_{i=1}^{N-q} \binom{N-q}{i} \left(\frac{x}{x_0} - 1\right)^i \sum_{j=0}^{i-1} \frac{1}{N-q-j} \\
&\quad - \frac{1}{(q-1)!} \sum_{i=0}^{q-2} \frac{i!}{(N+1-q+i)!} x_0^{-1-i} (x_0 - x)^{N+1-q+i}.
\end{aligned}$$

We note that it's much easier to prove only the schematic form of (6.B.10), without keeping track of the precise expressions of the coefficients. $\square$

We finally record a version of the above Lemma when logarithms are present:

Lemma 6.B.8. *Let $N \in \mathbb{N}$, and let $q \in \mathbb{R}$. If $N \leq q-1$, or if $q \notin \mathbb{N}_{>0}$, then*

$$\begin{aligned}
(-1)^N \underbrace{\int_{x_0}^x \cdots \int_{x_0}^{x_{(n-1)}}}_{N \text{ times}} x_{(n)}^{-q} \log(x_{(n)}) dx_{(n)} \cdots dx_{(1)} \\
= \frac{(q-N-1)!}{(q-1)!} x^{N-q} \log x - x_0^{N-q} \sum_{i=0}^{N-1} \frac{(q+i-N-1)!}{(q-1)!i!} \left(1 - \frac{x}{x_0}\right)^i \log x_0 + d'_{N,q,i} \left(\frac{x}{x_0}\right)^i
\end{aligned} \tag{6.B.11}$$

Otherwise, if $N > q-1$ and $q \in \mathbb{N}_{>0}$, then

$$\begin{aligned}
(-1)^N \underbrace{\int_{x_0}^x \cdots \int_{x_0}^{x_{(n-1)}}}_{N \text{ times}} x_{(n)}^{-q} \log x_{(n)} dx_{(n)} \cdots dx_{(1)} \\
= \frac{(-1)^{N+1-q}}{2(q-1)!(N-q)!} x^{N-q} \left(\log^2 x + \log x \left(1 - \sum_{i=1}^{N-q} \frac{1}{i}\right) \right) \\
- x_0^{N-q} \sum_{i=0}^{N-1} (d''_{N,q,i} + d'''_{N,q,i} \log x_0 + d_{N,q,i} \log^2 x_0) \left(1 - \frac{x}{x_0}\right)^i, \tag{6.B.12}
\end{aligned}$$

with some coefficients $d_{N,q,i} \in \mathbb{Q}$.

Proof. The proof is similar to that of Lemma 6.B.7, using also that $\int x^{-1} \log x dx = \frac{1}{2} \log^2 x$ and

$$\int x^n \log^2 x dx = \frac{x^{n+1}}{n+1} \left(\log^2 x - \frac{2}{n+1} \log x + \frac{2}{(n+1)^2} \right), \quad \forall n \neq -1.$$

$\square$

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