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QUANTUM DEFORMATION OF BOSE PARASTATISTICS

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Abstract: A q-deformation of the transformation of the Chevalley basis to an odd basis of generators of the universal enveloping of the Lie superalgebra $B(0,n)$ is presented. It is shown that one thus obtains a reasonable quantum deformation of the algebra of n para-Bose oscillators.

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1 Introduction

It has been realized by E.Wigner [1], back in 1950, that the canonical commutation relation, written in terms of creation and annihilation operators as

$$[a^-, a^+] = 1 \tag{1.1}$$

is by no means the only solution of the Heisenberg equations of motion

$$[H, a^\pm] = \pm a^\pm \ , \tag{1.2}$$

where

$$H = \frac{1}{2}(a^- a^+ + a^+ a^-) \equiv \frac{1}{2}\{a^-, a^+\} \tag{1.3}$$

is the (Bose-) harmonic oscillator Hamiltonian. A more general, "paraquantization" scheme was proposed in 1953 by H.S.Green [2]. From the algebraic point of view the main difference between the canonical quantization approach and the paraquantization scheme comes from the replacement of the standard bilinear relation (1.1) by a pair of relations, a threelinear one and its conjugate, which has additional solutions.

This idea has been extended to the case of several degrees of freedom (as well as to parafermionic systems which will not be discussed here). It has been shown that physical (lowest weight unitary) representations of the parastatistics algebra are labeled by a positive integer p called the order. It turned out that the most appropriate way of defining Bose-parastatistics algebra for $n > 1$ degrees of freedom was to attract some generalized symmetry argumentation and it has been found [3] that the resulting parabosonic algebra is just the (universal enveloping algebra of the) Lie superalgebra $osp(1|2n)$, appearing also as $B(0, n)$ in the Kac classification list [4] (see [5] both for an introduction or an advanced guidance in parastatistics). Therefore, it would be natural to expect that a deformation of Bose-parastatistics algebra should be related to a deformation of $B(0, n)$.

Quantum deformations of universal enveloping algebras of Lie superalgebras have been defined in [6], [7] (see also [8]). They naturally appear as a graded generalization of the notion of a quantum universal enveloping algebra (QUEA) $U_q(\mathcal{G})$ of a Lie algebra $\mathcal{G}$ defined by Jimbo [9], which contains perhaps the most popular examples of nontrivial i.e., both noncommutative and noncocommutative, quasitriangular Hopf algebras ("quantum groups", [10], [11]). Quantum generalizations of the Jordan-Schwinger-Bargmann construction have been also proposed — first for the simplest case of $U_q\big(su(2)\big)$ realization ([12], [13], [14]) — in terms of the so called quantum "Biedenharn-Macfarlane" oscillators, the QUEA generators being expressed, like in the undeformed case, as bilinear combinations of these. This construction has been further generalized for the quantum deformations of a broad class of Lie algebras and superalgebras [15], [6], [7].

Our interest in this subject evolved from an attempt to interpret quantum groups as generalized internal symmetries in two dimensional conformal field theory [16], [17]. An approach to quantum oscillators describing specific "quantum internal degrees of freedom" has been developed in [18] and extended further in [19], [20]. The basic assumption about the oscillators in this setting concerns their covariance properties with respect to the corresponding QUEA (an idea which, for the $U_q\big(su(2)\big)$ case, has been first realized in [21]); one derives then the corresponding relations requiring their compatibility with the transformation laws of the creation and annihilation operators, and some other natural properties. We shall refer to the oscillators constructed this way as to "covariant oscillators". Their relation to the Biedenharn-Macfarlane ones is displayed in [18].

The situation is somewhat different in the case of the q-deformation of $U\big(B(0,n)\big)$. In the undeformed case the odd generators to which creation and annihilation operators correspond in representations enter the Cartan-Weyl-Kac basis of $B(0,n)$ so that their (anti-)commutation relations turn out to be fixed from the outset. The q-deformed version being formulated in terms of the deformed analogs of the Chevalley-Kac basis, one has to define appropriately the q-counterparts for the odd generators, relations among which would give a sensible deformed version of the Bose-parastatistics algebra. Here the prescriptions of [22] (see also references therein) for the appropriate definition of the quantum counterpart of the Cartan-Weyl-Kac generators turn out to be helpful.

Previous attempts to define a $q-$deformed Bose-parastatistics deal with the comparatively easy $n=1$ case [23], [24]. There are also results for the $n=2$ case [25]; the author has been informed about partial results for the general case as well [26].

The paper is organized as follows. After reviewing the undeformed case we display in details the construction of the odd generators of $U_q\big(B(0,n)\big)$ which correspond to the "classical" ($q=1$) para-Bose creation and annihilation operators (we shall refer to them as to "totally odd basis"), together with certain threelinear relations among them (with coefficients in the Cartan subalgebra). We prove that this construction can serve as an alternative definition of $U_q\big(B(0,n)\big)$; in particular, the standard set of relations for the Chevalley generators is in one-to-one correspondence with the set of relations we derive for the totally odd basis. We also give an interpretation of the notion of parastatistics order of a representation which seems to be useful, leading to compact and suggestive formulae (the latter being perhaps new even for the $q=1$ case).

Apart from the aspect of q-parastatistics, constructing representations of $U_q\big(B(0,n)\big)$ could have various other applications. Since the sub-Hopf-algebraic structure in the deformed case differs from that in the undeformed one, it seems to be useful to have at hand alternative constructions of quantum group representations which coinside in the $q\to 1$ limit. One should mention among the latter — for $n=2$ — a construction of the state space of the "q-deformed top" [27] which describes the zero modes of the WZNW model (see also [28] where a q- deformed chiral version of the classical finite-dimensional model is considered), the regular representations of $U_q(sl(2))$ for q a root of unity [29], a natural q-deformation of the supersingletons [30] and of the important from the physical point of view massless representations of the four dimensional Poincaré (and conformal) algebras [31] — see [20], etc.

2 Scope of the Undeformed Case

It would be useful for what follows to introduce the notation

$$a_{|\mu|}^{sgn(\mu)} = \pi(e_\mu)\,, \quad \mu \in \{\pm1, \pm2, \dots, \pm n\} \tag{2.1}$$

where a_i^+, resp. a_i^-, $1 \leq i \leq n$ are n pairs of para-Bose creation, resp. annihilation operators, assuming that e_μ generate the abstract parabosonic algebra and π stands for a vacuum representation (there is no need to fix it for the moment), the vacuum being just a lowest weight vector annihilated by all a_i^-. The parabosonic relations for the case of n oscillators can be now written in the form

$$[\{e_\mu, e_\nu\}, e_\rho] = 2(\delta'_{\rho\mu} e_\nu + \delta'_{\rho\nu} e_\mu)\,; \quad \mu, \nu, \rho \in \{\pm1, \pm2, \dots, \pm n\} \tag{2.2}$$

where

$$\delta'_{\mu\nu} := sgn(\mu)\delta_{\mu\, -\nu} \tag{2.3}$$

is a sort of a finite-dimensional analog of the δ' function.

One can easily prove [3] that if one defines

$$E_i = \frac{1}{2}\{e_i, e_{-i-1}\}\,, \quad E_{-i} = \frac{1}{2}\{e_{-i}, e_{i+1}\}\,, \quad 1 \leq i \leq n-1\,, \quad E_{\pm n} = e_{\pm n}\,, \tag{2.4}$$

then $\{E_\mu\}_{\mu=\pm1,\pm2,\dots,\pm n}$ form a Chevalley-Kac basis for the Lie superalgebra $osp(1|2n) = B(0,n)$. Here $\{e_\mu\}_{\mu=\pm1,\pm2,\dots,\pm n}$ and $E_{\pm n}$ are assumed to be odd, and all other $E_{\pm i}$ — even generators. Since we shall need the notion of a root system of a simple Lie (super-)algebra and, moreover, since it is in fact the same in the deformed as in the undeformed case, we shall briefly spell out its most important features for our case (see [4] for more details).

The rank of $B(0,n)$ being equal to n, there are n simple roots, $\{\alpha_i\}_{i=1,2,\dots,n}$; the set of odd simple roots τ contains only one element, $\tau = \{\alpha_n\}$. The Cartan matrix $c_{ij} \equiv 2(\alpha_i, \alpha_j)/(\alpha_i, \alpha_i)$ is given by

$$(c_{ij}) = \begin{pmatrix} 2 & -1 & 0 & \dots & \dots & \dots \\ -1 & 2 & -1 & \dots & \dots & \dots \\ 0 & -1 & 2 & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & 2 & -1 & 0 \\ \dots & \dots & \dots & -1 & 2 & -1 \\ \dots & \dots & \dots & 0 & -2 & 2 \end{pmatrix}. \tag{2.5}$$

The correspondence between the roots α_i and the Chevalley generators $E_{\pm i}$ in Eq.(2.4) is $E_{\pm i} \leftrightarrow \pm\alpha_i$.

One has now all the data needed to write down the commutation and Serre relations for the universal enveloping algebra of the Lie superalgebra $U\big(B(0,n)\big)$ in terms of $\{E_\mu\}$ assuming that the n Cartan generators are *defined* — up to normalization — by the (anti-)commutators of E_i and E_{-i}, $1 \leq i \leq n$. However, it is instructive to have the Cartan-Weyl basis of $B(0,n)$as well. There are $2n^2$ even roots (all of them real) and 2n odd ones; in standard notations [4] the set of all roots Δ is decomposed as $\Delta = \Delta_0 \cup \Delta_1$, where

$$\Delta_0 = \{\pm 2\beta_i \,; \pm(\beta_j + \beta_k) \,; \pm(\beta_j - \beta_k) | 1 \le i \le n \,; 1 \le k < j \le n\} \,, \tag{2.6a}$$

$$\Delta_1 = \{\pm\beta_i | 1 \le i \le n\} \,. \tag{2.6b}$$

Adding the n Cartan generators, one obtains

$$dim B(0,n) = n(2n+3)\ (= 5, 14, 27, \ldots \text{for } n = 1, 2, 3 \ldots) \,. \tag{2.7}$$

Accordingly, the Cartan-Weyl basis is most conveniently expressed in terms of the generators corresponding to the odd roots, the correspondence being given by $\pm\beta_i \leftrightarrow\ e_{\pm i}$. It is not surprising at all — taking into account Eq.(2.4) — that the relation between the simple roots and the odd ones is

$$\alpha_i = \beta_i - \beta_{i+1} \,, 1 \le i \le n-1 \,; \quad \alpha_n = \beta_n \,. \tag{2.8}$$

We shall call $\{e_\mu\}_{\mu=\pm1,\pm2,\ldots,\pm n}$ "the totally odd basis". In some respects it is even more convenient than the Chevalley basis. The one-to-one correspondence between both is becoming transparent if we write down the relations inverse to Eq.(2.4):

$$e_{\pm n} = E_{\pm n} \,; \tag{2.9a}$$

$$e_i = [E_i, e_{i+1}] \,, \quad e_{-i} = [e_{-(i+1)}, E_{-i}] \,, \quad i = 1, 2, \ldots, n-1 \,. \tag{2.9b}$$

The recursive relations (2.9b) follow directly from Eqs. (2.4), (2.2).

Note that, as a consequence of (2.5) and (2.8), the odd roots corresponding to the generators $\{e_i\}_{i=1,2,\ldots,n}$ form an orthonormal coordinate system. Indeed, let us symmetrize the Cartan matrix (2.5) by defining [1]

$$a_{ij} := d_i c_{ij} \,, \quad d_i = \frac{1}{2}(\alpha_i, \alpha_i) := 2 - \delta_{in} \,, \quad 1 \le i, j \le n \,. \tag{2.10}$$

Then the symmetric Cartan matrix is given by $a_{ij} = (\alpha_i, \alpha_j)$,

$$(a_{ij}) = \begin{pmatrix} 4 & -2 & 0 & \ldots & \ldots & \ldots \\ -2 & 4 & -2 & \ldots & \ldots & \ldots \\ 0 & -2 & 4 & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & 4 & -2 & 0 \\ \ldots & \ldots & \ldots & -2 & 4 & -2 \\ \ldots & \ldots & \ldots & 0 & -2 & 2 \end{pmatrix} \tag{2.11}$$

[1] One only has the freedom of choosing an overall normalization factor; our $\{d_i\}$ are twice bigger than those in [22] and [7].

and it is amusing to check that, since

$$\beta_i = \sum_{k=i}^{n} \alpha_k \ , \quad 1 \le i \le n \ , \tag{2.12}$$

the scalar products (β_i, β_j) obey

$$(\beta_i, \beta_j) = \sum_{k=i}^{n} \sum_{l=j}^{n} a_{kl} = 2\delta_{ij} \ . \tag{2.13}$$

Hence, in the undeformed case one can alternatively define $U\big(B(0,n)\big)$ in terms of its Chevalley or, respectively, totally odd bases.

Note that, according to (2.6a), *all* even Cartan-Weyl generators are given (or, in fact, can be *defined*) in terms of the anticommutators of the odd ones, the Cartan subalgebra being spanned by $\{\{e_i, e_{-i}\} | 1 \le i \le n\}$ (this last property does not hold in the q-deformed case). On the other hand, (2.6a) implies also that relations (2.2) may be reformulated as defining the commutator (in the proper sence) of *any pair* of an even and an odd member of the Cartan-Weyl basis (surely, in terms of a linear combination of odd ones). Finally, *any* of the even-even commutation relations can be deduced from the even-odd ones by expressing one of the even generators as an anticommutator of odds and then using the generalized Jacobi identity

$$[A, \{B, C\}] = \{B, [A, C]\} + \{C, [A, B]\} \ . \tag{2.14}$$

Indeed, assuming that A is even and B,C odd, only known quantities appear in the right-hand side of (2.14). Defining $U\big(B(0,n)\big)$ as a free associative algebra with generators $\{e_\mu\}_{\mu=\pm1,\pm2,\ldots,\pm n}$, one has to impose appropriate relations among them which could be identified with any generating subset of the parabosonic ones (2.1) (not all of these are independent because of the generalized Jacobi identities); choosing its standard definition in terms of the Chevalley basis, one considers the commutation and Serre relations for $\{E_\mu\}_{\mu=\pm1,\pm2,\ldots,\pm n}$ instead.

3 $U_q\big(B(0,n)\big)$ in Two Different Bases

We have collected in this section all the relevant formulae concerning the two bases of $U_q\big(B(0,n)\big)$, the Chevalley and the totally odd one (their equivalence is proved in the next section).

The commutation relations for the Chevalley generators are

$$[2][E_i, E_{-j}] = \delta_{ij}[H_i] \, ; \quad 1 \le i,j \le n \, , \quad (i,j) \neq (n,n) \tag{3.1a}$$

where $[x] := \frac{q^x - q^{-x}}{q-q^{-1}}$,

$$\{E_n, E_{-n}\} = [H_n] \tag{3.1b}$$

$$q^{H_i} E_{\pm j} q^{-H_i} = q^{\pm a_{ij}} E_{\pm j} \, , \quad 1 \le i,j \le n \, . \tag{3.1c}$$

Trivial Serre relations:

$$[E_{\pm i}, E_{\pm j}] = 0 \, ; \quad |i-j| \ge 2 \, , \quad 1 \le i,j \le n \, . \tag{3.2}$$

Nontrivial Serre relations:

$$S(E_{\pm i}, E_{\pm(i+1)}) = 0 \, , \quad 1 \le i \le n-1 \, , \tag{3.3}$$

$$S(E_{\pm(i+1)}, E_{\pm i}) = 0 \, , \quad 1 \le i \le n-2 \tag{3.4}$$

$$T(E_{\pm n}, E_{\pm(n-1)}) = 0 \, , \tag{3.5}$$

where

$$S(x,y) = x^2 y + y x^2 - (q^2 + q^{-2}) xyx \, , \tag{3.6}$$

$$T(x,y) = x^3 y + y x^3 - (q^2 + q^{-2} - 1)(xyx^2 + x^2 yx) \, , \tag{3.7}$$

are the Serre polynomials for this case.

Denote, assuming

$$[A,B]_{q^\alpha} := AB - q^\alpha BA$$

(analogous convention will be used further also for anticommutators). Then the totally odd generators (and the corresponding Cartan generators) are defined by (cf. [22])

$$e_{\pm n} := E_{\pm n} \tag{3.8a}$$

$$e_i := [E_i, e_{i+1}]_{q^{-2}} \tag{3.8b}$$

$$e_{-i} := [e_{-(i+1)}, E_{-i}]_{q^2} \, , \tag{3.8c}$$

$$q^{\pm h_i} := \prod_{k=i}^{n} q^{\pm H_k} \, . \tag{3.8d}$$

The following relations are counterparts of (3.1) - (3.5) for the totally odd basis:

$$\{e_i, e_{-i}\} = [h_i]\,, \quad 1 \le i \le n \tag{3.9a}$$

$$q^{h_i} e_{\pm j} q^{-h_i} = q^{\pm 2\delta_{ij}} e_{\pm j}\;, \quad 1 \le i, j \le n\;, \tag{3.9b}$$

and, introducing the convenient notation

$$\mathcal{E}_i := \frac{1}{[2]} q^{-h_{i+1}} \{e_i, e_{-(i+1)}\}\;, \quad \mathcal{E}_{-i} := \frac{1}{[2]} \{e_{-i}, e_{i+1}\} q^{h_{i+1}}\;, \quad 1 \le i \le n-1\;;$$

$$\mathcal{E}_{\pm n} := e_{\pm n}\;, \tag{3.10a}$$

$$q^{\pm \mathcal{H}_i} := q^{\pm h_i} q^{\pm h_{i+1}}\;, \quad 1 \le i \le n-1\;; \quad q^{\pm \mathcal{H}_n} := q^{\pm h_n}\;, \tag{3.10b}$$

we have also

$$[\mathcal{E}_i, e_{-j}] = -\delta_{ij} q^{\mathcal{H}_i} e_{-(i+1)}\,, \quad [\mathcal{E}_{-i}, e_j] = \delta_{ij} e_{i+1} q^{-\mathcal{H}_i}\,, \quad 1 \le i \le n-1\,, \quad 1 \le j \le n \tag{3.11}$$

$$[\mathcal{E}_{\pm i}, e_{\pm j}] = 0\,, \quad 3 \le i+2 \le j \le n \tag{3.12}$$

$$[\mathcal{E}_i, e_{i+1}]_{q^{-2}} = e_i\;, \quad [e_{-(i+1)}, \mathcal{E}_{-i}]_{q^2} = e_{-i} \tag{3.13}$$

$$[\mathcal{E}_{\pm i}, e_{\pm i}]_{q^2} = 0\,, \quad 1 \le i \le n-1 \tag{3.14}$$

$$[\mathcal{E}_{\pm(i+1)}, e_{\pm i}] = 0\,, \quad 1 \le i \le n-2\,, \tag{3.15a}$$

$$[\{e_{\pm(n-1)}, e_{\pm n}\}, e_{\pm n}]_{q^2} = 0\,. \tag{3.15b}$$

The super-Hopf algebraic structure is given by
- the gradation

$$degE_{\pm i} = 0\,, \quad 1 \le i \le n-1\,; \quad degE_{\pm n} = 1\,; \tag{3.16a}$$

$$degq^{\pm H_i} = 0\,, \quad 1 \le i \le n \tag{3.16b}$$

$$dege_{\pm i} = 1\,; \quad degq^{\pm h_i} = 0\,, \quad 1 \le i \le n\,, \tag{3.16c}$$

- the comultiplication

$$\Delta(E_i) = E_i \otimes 1 + q^{H_i} \otimes E_i \tag{3.17a}$$

$$\Delta(E_{-i}) = 1 \otimes E_{-i} + E_{-i} \otimes q^{-H_i} \tag{3.17b}$$

$$\Delta(q^{\pm H_i}) = q^{\pm H_i} \otimes q^{\pm H_i}\;, \tag{3.17c}$$

note that in super-Hopf algebras, for any A, D and for homogeneous B, C

$$(A \otimes B)(C \otimes D) = (-1)^{degBdegC} AC \otimes BD \tag{3.18}$$

- the antipode

$$\gamma(E_i) = -q^{-H_i} E_i \tag{3.19a}$$

$$\gamma(E_{-i}) = -E_{-i} q^{H_i} \tag{3.19b}$$

$$\gamma(q^{\pm H_i}) = q^{\mp H_i} \tag{3.19c}$$

(in super-Hopf algebras γ should be a graded antihomorphism i.e., that

$$\gamma(AB) = (-1)^{degA degB} \gamma(B)\gamma(A) , \tag{3.20}$$

otherwise e.g. Eq.(3.1b) would be inconsistent), and

- the counit,

$$\varepsilon(E_{\pm i}) = 0 = \varepsilon(e_{\pm i}) , \tag{3.21a}$$

$$\varepsilon(q^{\pm H_i}) = 1 = \varepsilon(q^{\pm h_i}) . \tag{3.21b}$$

For e_μ , the formulae for the coproduct and the antipode are more involved — additional terms (vanishing for $q = 1$) appear, e.g., in

$$\Delta(e_n) = e_n \otimes 1 + q^{h_n} \otimes e_n , \tag{3.22a}$$

$$\Delta(e_{n-1}) = e_{n-1} \otimes 1 + q^{h_{n-1}} \otimes e_{n-1} + (q - q^{-1})\{e_{n-1}, e_{-n}\} \otimes e_n , \tag{3.22b}$$

$$\Delta(e_{n-2}) = e_{n-2} \otimes 1 + q^{h_{n-2}} \otimes e_{n-2} + (q - q^{-1})\big(\{e_{n-2}, e_{-(n-1)}\} \otimes e_{n-1} + \{e_{n-2}, e_{-n}\} \otimes e_n\big) , \tag{3.22c}$$

etc. The corresponding formulae for the antipode can be also easily derived from the definitions and (3.19) - (3.20).

Note also the existence of a *non-graded* algebraic antiinvolution $*$,

$$(AB)^* = B^* A^* , \tag{3.23}$$

(the "Cartan-Planck conjugation" of [22]) acting on the generators as

$$(E_{\pm i})^* = E_{\mp i} , \quad (q^{\pm H_i})^* = q^{\mp H_i} , \quad q^* = q^{-1} . \tag{3.24}$$

In other words, relations obtained by applying $*$ to true relations are also true; we shall refer to them as to the "conjugate relations", correspondingly. For q on the unit circle we define $*$ as a coalgebraic antihomomorphism ([32], [19]; see also [33]).

The action of the antiinvolution $*$ in the totally odd basis is quite similar to (3.24) — one has

$$(e_{\pm i})^* = e_{\mp i} , \quad (q^{\pm \mathcal{H}_i})^* = q^{\mp \mathcal{H}_i} . \tag{3.25}$$

4 Proof of the Equivalence

This is the main part of the paper. We shall prove here that the two systems of generators and relations of Section 3 are equivalent, e.g. that starting from the standard Chevalley basis with relations

$$\mathbf{I}: \quad (3.1) - (3.5)$$

one obtains (with the conventions (3.8) and (3.10)) the q-deformed totally odd basis satisfying

$$\mathbf{II}: \quad (3.9), (3.11) - (3.15)$$

and, vice versa, that — assuming that (3.8) and (3.10) hold — relations **II** imply the system **I** (with the identifications $E_{\pm i} \to \mathcal{E}_{\pm i}$, $q^{\pm H_i} \to q^{\pm \mathcal{H}_i}$).

The idea of the proof is to use the induction suggested by the natural "tower" of inclusions[2]

$$U_q\big(B(0,1)\big) \subset U_q\big(B(0,2)\big) \subset \ldots \subset U_q\big(B(0,k+1)\big) \subset \ldots \subset U_q\big(B(0,n)\big)$$

adding at any subsequent level a new quadruple of generators

$$\{E_{\pm(n-k)}, q^{\pm H_{n-k}}\} ,$$

or

$$\{e_{\pm(n-k)}, q^{\pm h_{n-k}}\} ,$$

respectively.[3]

$$Proof \quad of \quad \mathbf{I} \quad \Longrightarrow \quad \mathbf{II} .$$

Let us define $e_{\pm i}, q^{\pm h_i}$ through (3.8), and, further, $\mathcal{E}_i$ — through (3.10a). Then, for $i = n$ (3.9a) is trivial, and for $1 \le i \le n-1$ it follows by induction. Indeed, the following chain of relations for $i = k$ can be derived from that for $i \ge k+1$:

$$[E_i, e_{-j}] = 0 , \quad i < j , \tag{4.1a}$$

$$q^{h_{i+1}} E_i q^{-h_{i+1}} = q^{-2} E_i , \tag{4.1b}$$

$$E_i = \frac{1}{[2]} q^{-h_{i+1}} \{e_i, e_{-(i+1)}\} \equiv \mathcal{E}_i , \tag{4.1c}$$

$$q^{H_i} e_{i+1} q^{-H_i} = q^{-2} e_{i+1} , \tag{4.1d}$$

$$[E_{-i}, e_i] = e_{i+1} q^{-H_i} , \tag{4.1e}$$

[2] Having in mind quantum field theory applications, one can consider the infinite "tower" of inclusions as well (it would be better then to reverse the ordering of the generators).

[3] In the first case nontrivial commutations appear only among "nearest neighbor" (quadruples of) generators; for the totally odd basis the structure is more involved.

$$\{e_i, e_{-i}\} = [h_i]\,. \tag{4.1f}$$

(We shall prove below that (4.1a) is in fact valid for any $1 \le i < j \le n$, the equations from (4.1b) to (4.1e) — for $1 \le i \le n-1$, and (4.1f) — for $1 \le i \le n$).

Let first $k = n-1$. Then,

$$[E_{n-1}, e_{-n}] \equiv [E_{n-1}, E_{-n}] = 0 \tag{4.2a}$$

according to (3.1a). Further,

$$q^{h_n} E_{n-1} q^{-h_n} \equiv q^{H_n} E_{n-1} q^{-H_n} = q^{a_{n\,n-1}} E_{n-1} = q^{-2} E_{n-1} \tag{4.2b}$$

because of (3.1c) (cf. (2.11)). Now (3.8b), (4.2a) and (4.2b) imply

$$\{E_{n-1}, e_{-n}\} = \{E_{n-1} e_n - q^{-2} e_n E_{n-1}, e_{-n}\} =$$

$$= E_{n-1}\{e_n, e_{-n}\} - q^{-2}\{e_n, e_{-n}\} E_{n-1} = E_{n-1}[h_n] - q^{-2}[h_n] E_{n-1} =$$

$$= \big([h_n + 2] - q^{-2}[h_n]\big) E_{n-1} = [2] q^{h_n} E_{n-1}\,, \tag{4.2c}$$

i.e., (4.1c) for $i = n-1$ as claimed. We also have

$$q^{H_{n-1}} e_n q^{-H_{n-1}} \equiv q^{H_{n-1}} E_n q^{-H_{n-1}} = q^{a_{n-1\,n}} E_n = q^{-2} e_n \tag{4.2d}$$

and, as a corollary,

$$[E_{-(n-1)}, e_{n-1}] = [E_{-(n-1)}, E_{n-1} e_n - q^{-2} e_n E_{n-1}] =$$

$$= [E_{-(n-1)}, E_{n-1}] e_n - q^{-2} e_n [E_{-(n-1)}, E_{n-1}] = \frac{1}{[2]}\big(q^{-2} e_n [H_{n-1}] - [H_{n-1}] e_n\big) =$$

$$= \frac{1}{[2]} e_n \big(q^{-2}[H_{n-1}] - [H_{n-1} - 2]\big) = e_n q^{-H_{n-1}}\,. \tag{4.2e}$$

All this implies

$$\{e_{n-1}, e_{-(n-1)}\} = \{e_{n-1}, e_{-n} E_{-(n-1)} - q^2 E_{-(n-1)} e_{-n}\} =$$

$$= \{e_{n-1}, e_{-n}\} E_{-(n-1)} + e_{-n} [E_{-(n-1)}, e_{n-1}] -$$

$$- q^2 E_{-(n-1)} \{e_{n-1}, e_{-n}\} + q^2 [E_{-(n-1)}, e_{n-1}] e_{-n} \equiv$$

$$\equiv [\{e_{n-1}, e_{-n}\}, E_{-(n-1)}]_{q^2} + \{e_{-n}, [E_{-(n-1)}, e_{n-1}]\}_{q^2} =$$

$$= [2][q^{h_n} E_{n-1}, E_{-(n-1)}]_{q^2} + \{e_{-n}, e_n q^{-H_{n-1}}\}_{q^2} =$$

$$= [2] q^{h_n} [E_{n-1}, E_{-(n-1)}] + \{e_{-n}, e_n\} q^{-H_{n-1}} = q^{h_n} [H_{n-1}] + [h_n] q^{-H_{n-1}} =$$

$$= [H_{n-1} + h_n] \equiv [h_{n-1}]\,; \tag{4.2f}$$

it has been used that

$$\{A, BC\} = \{A, B\}C + B[C, A] = B\{A, C\} - [B, A]C \ . \tag{4.3}$$

Thus, the relations (4.1) for $i = n - 1$ follow indeed from those for $i = n$. This step (namely, the proof that (4.1) for $i \geq k+1$ imply those for $i \geq k$) can be made with almost no changes for any $1 \leq k \leq n-1$, the only thing to be taken into account — in the derivation of (4.1a), (4.1b), (4.1d) — on top of the arguments used for $k = n - 1$ being the trivial remark that $e_{\pm i}, q^{\pm h_i}$ for any $1 \leq i \leq n-1$ may be expressed entirely in terms of $\{E_{\pm l}, q^{\pm H_l}\}$ with $l \geq i$ (we shall call this "the triangular property"). The latter follows directly from the definitions (3.8); e.g., (3.8b), (3.8c) imply

$$e_i = [E_i, [E_{i+1}, \dots [E_{n-1}, E_n]_{q^{-2}} \dots]_{q^{-2}}]_{q^{-2}} \ , \tag{4.4a}$$

$$e_{-i} = [[\dots [E_{-n}, E_{-(n-1)}]_{q^2} \dots, E_{-(i+1)}]_{q^2}, E_{-i}]_{q^2} \ , \tag{4.4b}$$

respectively.

So we have proved (3.9a). Having in mind (4.4) (see also (2.13)), it is quite easy now to derive (3.9b):

$$q^{h_i} e_{\pm j} q^{-h_i} \equiv \prod_{k=i}^{n} q^{H_k} e_{\pm j} \prod_{m=i}^{n} q^{-H_m} =$$

$$= q^{\pm \sum_{k=i}^{n} \sum_{l=j}^{n} a_{kl}} e_{\pm j} = q^{\pm 2\delta_{ij}} e_{\pm j} \ . \tag{4.5}$$

Applying the antiinvolution $*$ to (4.1), we obtain the following conjugate relations (see (3.23), (3.24), (3.25)) :

$$[E_{-i}, e_j] = 0 \ , \quad 1 \leq i < j \leq n, \tag{4.6a}$$

$$q^{h_{i+1}} E_{-i} q^{-h_{i+1}} = q^2 E_{-i} \ , \tag{4.6b}$$

$$E_{-i} = \frac{1}{[2]} \{e_{-i}, e_{i+1}\} q^{h_{i+1}} \equiv \mathcal{E}_{-}i \ , \tag{4.6c}$$

$$q^{H_i} e_{-(i+1)} q^{-H_i} = q^2 e_{-(i+1)} \ , \tag{4.6d}$$

$$[e_{-i}, E_i] = q^{H_i} e_{-(j+1)} \ , \tag{4.6e}$$

the last four equalities being valid for $1 \leq i \leq n-1$. Relation (4.1f) is self-conjugate.

To prove the first part of the equalities (3.11), since we have already (4.1c) and (4.6e) (see also (3.10b)), we must only check that

$$[E_i, e_{-j}] = 0 \ , 1 \leq j < i \leq n-1 \ ;$$

the second part is obtained by conjugation. Let us first consider the case $2 \leq j+1 = i \leq n-1$ .Using (3.1a), (3.1c) and (3.2), we get

$$[E_{j+1}, e_{-j}] \equiv [E_{j+1}, [e_{-(j+1)}, E_{-j}]_{q^2}] = [[E_{j+1}, e_{-(j+1)}], E_{-j}]_{q^2} =$$

$$= [q^{H_{j+1}} e_{-(j+2)}, E_{-j}]_{q^2} = q^{H_{j+1}} [e_{-(j+2)}, E_{-j}] = 0\,. \tag{4.7}$$

With (3.2) and (4.7), one can proceed further to obtain — step by step — the remaining relations of the first part of (3.11):

$$[E_{j+2}, e_{-j}] = [E_{j+2}, [e_{-(j+1)}, E_{-j}]_{q^2}] = [[E_{j+2}, e_{-(j+1)}], E_{-j}]_{q^2} = 0 \tag{4.8}$$

etc.

The vanishing of the commutators $[\mathcal{E}_{\pm i}, e_{\pm j}]$ for $3 \le i+2 \le j \le n$ — Eqs. (3.12) — is a direct consequence of (4.1c) and (4.6c), the trivial Serre relations (3.2) and the "triangular property" (4.4).

Proceeding further, one sees that (3.13) follows immediately from the defining recursion relations (3.8b), (3.8c) (due to (4.1c), (4.6c)). To establish (3.14), one should first note that the two sets of equations are conjugate to each other. On the other hand, one can use again (4.1c) and then convert the commutator $[E_i, e_i]_{q^2}$ by expressing

$$e_i \equiv [E_i, e_{i+1}]_{q^{-2}} = [E_i, [E_{i+1}, e_{i+2}]_{q^{-2}}]_{q^{-2}} \tag{4.9}$$

to

$$[E_i, [E_i, [E_{i+1}, e_{i+2}]_{q^{-2}}]_{q^{-2}}]_{q^2} = [[E_i, [E_i, E_{i+1}]_{q^{-2}}]_{q^2}, e_{i+2}]_{q^{-2}} \tag{4.10a}$$

(E_i and e_{i+2} commute, due to (3.12)), for $1 \le i \le n-2$, or to

$$[E_{n-1}, [E_{n-1}, e_n]_{q^{-2}}]_{q^2} \equiv [E_{n-1}, [E_{n-1}, E_n]_{q^{-2}}]_{q^2}\ , \tag{4.10b}$$

for $i = n-1$. The proof that both (4.10a) and (4.10b) vanish can be done by representing the Serre polynomial as

$$S(x,y) \equiv x^2 y + y x^2 - (q^2 + q^{-2}) x y x =$$

$$= [x, [x,y]_{q^\alpha}]_{q^{-\alpha}}\ , \quad \alpha = \pm 2 \tag{4.11}$$

and then using (3.3).

The following chain of equalities,

$$(q^2 + q^{-2})[E_{i+1}, e_i] = (q^2 + q^{-2})[E_{i+1}, [E_i, e_{i+1}]_{q^{-2}}] =$$

$$= (q^2 + q^{-2}) E_{i+1} E_i e_{i+1} - E_{i+1} e_{i+1} E_i - q^{-4} E_{i+1} e_{i+1} E_i -$$

$$- q^2 E_i e_{i+1} E_{i+1} - q^{-2} E_i e_{i+1} E_{i+1} + q^{-2}(q^2 + q^{-2}) e_{i+1} E_i E_{i+1} =$$

$$= (q^2 + q^{-2}) E_{i+1} E_i e_{i+1} - E_{i+1} e_{i+1} E_i - q^{-2} e_{i+1} E_{i+1} E_i -$$

$$- E_i E_{i+1} e_{i+1} - q^{-2} E_i e_{i+1} E_{i+1} + q^{-2}(q^2 + q^{-2}) e_{i+1} E_i E_{i+1} =$$

$$= (q^2 + q^{-2}) E_{i+1} E_i (E_{i+1} e_{i+2} - q^{-2} e_{i+2} E_{i+1}) - E_{i+1}(E_{i+1} e_{i+2} - q^{-2} e_{i+2} E_{i+1}) E_i -$$

$$- q^{-2}(E_{i+1} e_{i+2} - q^{-2} e_{i+2} E_{i+1}) E_{i+1} E_i - E_i E_{i+1}(E_{i+1} e_{i+2} - q^{-2} e_{i+2} E_{i+1}) -$$

$$- q^{-2} E_i (E_{i+1} e_{i+2} - q^{-2} e_{i+2} E_{i+1}) E_{i+1} +$$

$$+q^{-2}(q^2+q^{-2})(E_{i+1}e_{i+2}-q^{-2}e_{i+2}E_{i+1})E_iE_{i+1}=$$
$$=\big((q^2+q^{-2})E_{i+1}E_iE_{i+1}-E_{i+1}^2E_i-E_iE_{i+1}^2\big)e_{i+2}+$$
$$+e_{i+2}\big(q^{-4}E_{i+1}^2E_i+q^{-4}E_iE_{i+1}^2-q^{-4}(q^2+q^{-2})E_{i+1}E_iE_{i+1}\big)=$$
$$=q^{-4}e_{i+2}S(E_{i+1},E_i)-S(E_{i+1},E_i)e_{i+2}=0\,, \qquad (4.12)$$

proves (in fact, for $q^4\neq -1$) relations (3.15a) (the conjugate equations should be also valid) — see (3.4), (3.6). Relations (3.15b) follow directly from (3.5), (3.7), e.g.

$$[\{e_{n-1},e_n\},e_n]_{q^2}\equiv[\{[E_{n-1},E_n]_{q^{-2}},E_n\},E_n]_{q^2}=$$
$$=([E_{n-1},E_n]_{q^{-2}}E_n+E_n[E_{n-1},E_n]_{q^{-2}})E_n-q^2E_n([E_{n-1},E_n]_{q^{-2}}E_n+E_n[E_{n-1},E_n]_{q^{-2}})=$$
$$=E_n^3E_{n-1}+E_{n-1}E_n^3-(q^2+q^{-2}-1)(E_n^2E_{n-1}E_n+E_nE_{n-1}E_n^2)\equiv T(E_n,E_{n-1})=0, \quad (4.13)$$

and one also has the conjugate relation. This completes the proof that relations **I** imply **II**.

Proof of **II** $\Longrightarrow$ **I** .

We are going to prove now that, within the free associative algebra with generators $\{e_{\pm i},q^{\pm h_i}\}$, $1\le i\le n$ and relations (3.9), one can recover, by (3.10), the Chevalley generators of $U_q\big(B(0,n)\big)$,i.e., that $\mathcal{E}_{\pm i}$, $q^{\pm\mathcal{H}_i}$, $1\le i\le n$ obey the defining commutation relations (3.1) and the Serre relations (3.2) - (3.7). Let us start with the proof of (3.1a) for $1\le i=j\le n-1$. We have

$$[2][\mathcal{E}_i,\mathcal{E}_{-i}]=\mathcal{E}_i(e_{-i}e_{i+1}+e_{i+1}e_{-i})q^{h_{i+1}}-(e_{-i}e_{i+1}+e_{i+1}e_{-i})q^{h_{i+1}}\mathcal{E}_i=$$
$$=(e_{-i}\mathcal{E}_i-q^{h_i}q^{-h_{i+1}}e_{-(i+1)})e_{i+1}q^{h_{i+1}}+(q^{-2}e_{i+1}\mathcal{E}_i+e_i)e_{-i}q^{h_{i+1}}-$$
$$-(e_{-i}e_{i+1}+e_{i+1}e_{-i})q^{h_{i+1}}\mathcal{E}_i=$$
$$=e_{-i}(q^{-2}e_{i+1}\mathcal{E}_i+e_i)q^{h_{i+1}}-q^{h_i}e_{-(i+1)}e_{i+1}+$$
$$+e_{i+1}(q^{-2}e_{-i}\mathcal{E}_iq^{h_{i+1}}-q^{h_i}e_{-(i+1)})+$$
$$+e_ie_{-i}q^{h_{i+1}}-(e_{-i}e_{i+1}+e_{i+1}e_{-i})q^{h_{i+1}}\mathcal{E}_i=$$
$$=\{e_i,e_{-i}\}q^h_{i+1}-\{e_{i+1},e_{-(i+1)}\}q^{h_i}=[h_i]q^{h_{i+1}}-[h_{i+1}]q^{h_i}=[\mathcal{H}_i]\,. \qquad (4.14)$$

Relation (3.1b) is just (3.9a) for $i=j=n$. Let now $1\le i\neq j\le n$ and, e.g., $i+1\le j$. Then, for $j=n$, we use directly (3.11), and for $j\le n-1$ we first rewrite

$$[2][\mathcal{E}_i,\mathcal{E}_{-j}]=[\mathcal{E}_i,\{e_{-j},e_{j+1}\}q^{h_{j+1}}] \qquad (4.15)$$

and then apply (3.9b), (3.11) and (3.12) (the latter — because $i+2\le j+1$) to prove that the commutator vanishes. If, on the contrary, $i\ge j+1$, we express $\mathcal{E}_i$ according to (3.10a) and apply the conjugate relations instead.

Relations (3.1c) are trivial corollaries of (3.9b); in fact, when $1 \leq i,j \leq n-1$, one just has to check that

$$q^{h_i} q^{-h_{i+1}} \{e_j, e_{-(j+1)}\} q^{h_{i+1}} q^{-h_i} = q^{a_{ij}} \{e_j, e_{-(j+1)}\} \tag{4.16}$$

and when i and/or j are equal to n, the derivation is even simpler.

The trivial Serre relations (3.2) follow from (3.9b), (3.11) and (3.12).

Let us consider, finally, the nontrivial Serre relations. We shall display the proof for those involving $\{\mathcal{E}_i\}$ with positive i; the ones for $\{\mathcal{E}_{-i}\}$ can be obtained by conjugation. One has (see (4.11))

$$S(\mathcal{E}_i, \mathcal{E}_{i+1}) = \frac{1}{[2]} [\mathcal{E}_i, [\mathcal{E}_i, q^{-h_{i+2}} \{e_{i+1}, e_{-(i+2)}\}]_{q^{-2}}]_{q^2} =$$

$$= \frac{1}{[2]} q^{-h_{i+2}} [\mathcal{E}_i, \{e_i, e_{-(i+2)}\}]_{q^2} = \frac{1}{[2]} q^{-h_{i+2}} \{[\mathcal{E}_i, e_i]_{q^2}, e_{-(i+2)}\} = 0 \, , 1 \leq i \leq n-2 \tag{4.17a}$$

(due to (3.9b), (3.11) and (3.14)), and

$$S(\mathcal{E}_{n-1}, \mathcal{E}_n) = [\mathcal{E}_{n-1}, [\mathcal{E}_{n-1}, e_n]_{q^{-2}}]_{q^2} = [\mathcal{E}_{n-1}, e_{n-1}]_{q^2} = 0 \, . \tag{4.17b}$$

This proves (3.3); relations (3.4) for $1 \leq i \leq n-2$ follow from

$$S(\mathcal{E}_{i+1}, \mathcal{E}_i) = \frac{1}{[2]} [\mathcal{E}_{i+1}, [\mathcal{E}_{i+1}, q^{-h_{i+1}} \{e_i, e_{-(i+1)}\}]_{q^2}]_{q^{-2}} =$$

$$= \frac{q^2}{[2]} [\mathcal{E}_{i+1}, q^{-h_{i+1}} \{e_i, [\mathcal{E}_{i+1}, e_{-(i+1)}]\}]_{q^{-2}} = \frac{q^4}{[2]} q^{-h_{i+1}} \{e_i, [\mathcal{E}_{i+1}, [\mathcal{E}_{i+1}, e_{-(i+1)}]]_{q^{-4}}\} =$$

$$= -\frac{q^4}{[2]} q^{-h_{i+1}} \{e_i, [\mathcal{E}_{i+1}, q^{h_{i+1}} q^{-h_{i+2}} e_{-(i+2)}]_{q^{-4}}\} = -\frac{1}{[2]} q^{-h_{i+2}} \{e_i, [\mathcal{E}_{i+1}, e_{-(i+2)}]\} = 0 \tag{4.18}$$

(we have used (4.11), (3.9b), (3.15a) and (3.11)). The last Serre relation (3.5) can be derived from the definitions and (3.15b):

$$T(\mathcal{E}_n, \mathcal{E}_{n-1}) = [\{[\mathcal{E}_{n-1}, \mathcal{E}_n]_{q^{-2}}, \mathcal{E}_n\}, \mathcal{E}_n]_{q^2} =$$
$$= [\{e_{n-1}, e_n\}, e_n]_{q^2} = 0 \, . \tag{4.19}$$

Hence, we have proved the equivalence of the both systems of generators and relations which produce the super-Hopf quantum algebra $U_q\big(B(0,n)\big)$. Note the differences between the relations for $\{e_\mu\}_{\mu=\pm1,\pm2,\ldots,\pm n}$ in the deformed and undeformed cases — first, in the deformed case we have in general coefficients in the Cartan subalgebra, and second — this is the reason for not having displayed (for $n \geq 2$) the full set of analogs of (2.2) — not all of the relations are of the typical for the $q=1$ para-Bose algebra type (i.e., threelinear in the odd generators in the left-hand side and linear in the right-hand side). Although it is obvious that one could, in principle, *compute* all the expressions of the type $[\{e_\mu, e_\nu\}, e_\rho]$, e.g., by expressing the odd generators as in (4.4), one has to expect that, in general, threelinear terms (with coefficients that vanish in the limit $q \to 1$) would appear in the right-hand side, too [4]. Indeed, one can show, as an example — for the first nontrivial case $n=3$ — that

$$[\{e_{-1}, e_3\}, e_2] = -(q^2 - q^{-2}) \{e_{-1}, e_2\} e_3 \, . \tag{4.20}$$

[4] The author owes this observation to T.Palev [26].

5 Q-deformed Para-Bose Oscillators

We consider in this section, for any $n-$tuple of real numbers $\underline{p} = \{p_1, p_2, \ldots, p_n\}$, vacuum representations $\pi^{(\underline{p})}$ of the super-Hopf algebra $U_q\big(B(0,n)\big)$

$$\pi^{(\underline{p})}(e_{\pm i}) \equiv a_i^{\pm}\,, \quad i = 1, 2, \ldots, n\,, \tag{5.1}$$

characterized by the existence of a lowest weight vector $|0>$, the vacuum, such that

$$a_i^- |0>= 0\,, \quad q^{\pm h_i}|0>= q^{\pm p_i}|0> \tag{5.2}$$

(we are not going to introduce special notations for the Cartan generators in the representation $\pi^{(\underline{p})}$). One has then (cf. (3.9) - (3.15))

$$\{a_i^-, a_i^+\} = [h_i]\,, \quad 1 \le i \le n \tag{5.3a}$$

$$q^{h_i} a_j^{\pm} q^{-h_i} = q^{\pm 2\delta_{ij}} a_j^{\pm}\,, \quad 1 \le i,j \le n \tag{5.3b}$$

$$[\{a_i^+, a_{i+1}^-\}, a_j^-]_{q^{-2\delta_{i+1\,j}}} = -[2]\delta_{ij} q^{h_i} a_{i+1}^-\,, \quad 1 \le i \le n-1, \quad 1 \le j \le n \tag{5.4}$$

$$[\{a_i^-, a_{i+1}^+\}, a_j^+]_{q^{-2\delta_{i+1\,j}}} = [2]\delta_{ij} a_{i+1}^+ q^{-h_i}\,, \quad 1 \le i \le n-1, \quad 1 \le j \le n \tag{5.4*}$$

$$[\{a_i^+, a_{i+1}^-\}, a_j^+] = 0\,, \quad 3 \le i+2 \le j \le n \tag{5.5}$$

$$[\{a_i^-, a_{i+1}^+\}, a_j^-] = 0\,, \quad 3 \le i+2 \le j \le n \tag{5.5*}$$

$$[\{a_i^+, a_{i+1}^-\}, a_{i+1}^+] = [2] q^{h_{i+1}} a_i^+\,, \quad 1 \le i \le n-1 \tag{5.6}$$

$$[\{a_i^-, a_{i+1}^+\}, a_{i+1}^-] = -[2] a_i^+ q^{-h_{i+1}}\,, \quad 1 \le i \le n-1 \tag{5.6*}$$

$$[\{a_i^+, a_{i+1}^-\}, a_i^+]_{q^2} = 0\,, \quad 1 \le i \le n-1 \tag{5.7}$$

$$[\{a_i^-, a_{i+1}^+\}, a_i^-]_{q^2} = 0\,, \quad 1 \le i \le n-1 \tag{5.7*}$$

$$[\{a_{i+1}^+, a_{i+2}^-\}, a_i^+] = 0\,, \quad 1 \le i \le n-2 \tag{5.8a}$$

$$[\{a_{i+1}^-, a_{i+2}^+\}, a_i^-] = 0\,, \quad 1 \le i \le n-2 \tag{5.8a*}$$

$$[\{a_{n-1}^+, a_n^+\}, a_n^+]_{q^2} = 0 \tag{5.8b}$$

$$[\{a_{n-1}^-, a_n^-\}, a_n^-]_{q^2} = 0\,. \tag{5.8b*}$$

Let us consider the representation of the subalgebra $U_q\big(B(0,1)\big)$ generated by some quadruple $\{e_{\pm i}, q^{\pm h_i}\}$; since they are all isomorphic, we shall skip the index i altogether. As a corollary of (5.2), (5.3) one has

$$\{a^-, a^+\}|0>= [p]|0> \tag{5.9}$$

which is an expected generalization of the known for $q = 1$ notion of the order of parastatistics [5]. It is obvious that the set of eigenvectors of $q^{\pm h}$

$$|n>:= (a^+)^n|0> \, , \quad q^{\pm h}|n>= q^{\pm(p+2n)}|n> \, , \quad n = 0,1,2,\ldots \tag{5.10}$$

is nondegenerate for generic q . Hence, the action of $a^\pm$ is given by

$$a^+|n> = |n+1> \, , \tag{5.11a}$$

$$a^-|n> = x_n|n-1> \, , \tag{5.11b}$$

where x_n obeys the finite difference equation following from

$$x_n|n-1>\equiv a^-|n>= a^-a^+|n-1>=$$

$$= \big(-a^+a^- + [h]\big)|n-1>= \big(-x_{n-1} + [p+2(n-1)]\big)|n-1> \tag{5.12a}$$

together with the initial condition

$$x_0 = 0 \, . \tag{5.12b}$$

The unique solution of (5.12a), (5.12b) is

$$x_n = \frac{1}{[2]}\big([2n+p-1] - (-1)^n[p-1]\big) \, . \tag{5.13}$$

Hence, if we introduce the $q-$analogs of the number operators $q^{\pm N_i}$,$i = 1,2,\ldots,n$ by defining

$$q^{N_i}a_j^\pm q^{-N_i} = q^{\pm\delta_{ij}}a_j^\pm \, , \quad [q^{\pm N_i}, q^{h_j}] = 0 \tag{5.14a}$$

$$q^{\pm N_i}|0>= |0> \, , \tag{5.14b}$$

we can identify

$$h_i = 2N_i + P_i \, , \quad i = 1,2,\ldots,n \tag{5.15}$$

where the operators $\{P_i\}$ are in the commutant of the representation $\pi^{(\underline{p})}$. Taking them to be all equal,[5] $P_i \equiv P$, leads to

$$a_i^-a_i^+ = \frac{1}{[2]}\big([2N_i+P+1] + (-1)^{N_i}[P-1]\big) \tag{5.16a}$$

$$a_i^+a_i^- = \frac{1}{[2]}\big([2N_i+P-1] - (-1)^{N_i}[P-1]\big) \tag{5.16b}$$

(all N_i have nonnegative-integer spectrum). These compact expressions are perhaps new for the undeformed case as well. It is quite clear that P plays the role of an order operator.

[5] In the undeformed case this is a theorem following from the requirement of the positivity of the metric and the existence of a lowest weight state [5]; it turns out that there is also a straightforward q-deformed version of it which fails, however, for q a root of unity.

Choosing P to be equal to the unit operator, i.e. $h_i = 2N_i + 1$, one obtains

$$a_i^- a_i^+ = \frac{1}{[2]}[2N_i + 2] = [N_i + 1]_{q^2} \tag{5.17a}$$

$$a_i^+ a_i^- = \frac{1}{[2]}[2N_i] = [N_i]_{q^2} \ , \tag{5.17b}$$

or

$$a_i^- a_i^+ - q^{\pm 2} a_i^+ a_i^- = q^{\pm 2N_i} \ . \tag{5.18a}$$

Let now $n = 2$;it is trivial to show that (5.18a) and

$$\begin{aligned} a_1^+ a_2^+ &= q^2 a_2^+ a_1^+ \\ a_1^+ a_2^- &= q^{-2} a_2^- a_1^+ \\ a_1^- a_2^+ &= q^{-2} a_2^+ a_1^- \\ a_1^- a_2^- &= q^2 a_2^- a_1^- \end{aligned} \tag{5.18b}$$

imply relations (5.3) - (5.8). The operators $a_i^\pm, i = 1, 2$ obeying (5.18) are the "covariant" (Pusz-Woronowicz type, see [21]) $U_q\big(sl(2)\big)$ -oscillators of [18]. They are related to the Biedenharn-Macfarlane ones [12]-[14] by

$$\begin{aligned} b_1 &= q^{N_1 + 2N_2} a_1 \\ b_1^* &= a_1^+ q^{-N_1 - 2N_2} \\ b_2^- &= q^{N_2} a_2^- \\ b_2^* &= a_2^+ q^{-N_2} \\ N_{b_1} &= N_1 \ , \quad N_{b_2} = N_2 \ . \end{aligned} \tag{5.18c}$$

6 Outlook

There are several problems which deserve future consideration, e.g.

- a more detailed investigation of the structure of the q-deformed parabosonic relations
- representations of the algebra (5.3) - (5.8) with $h_i = 2N_i + P$, $i = 1, 2, \ldots, n$ for higher values of the parastatistics order p
- representations for q - root of unity
- applications to specific two-dimensional models,

etc. We postpone the discussion of these questions to a future publication.

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References

1. E.P.Wigner, Phys.Rev. **77** (1950) 711.
2. H.S.Green, Phys.Rev. **90** (1953) 270.
3. A.Ganchev, T.Palev, J.Math.Phys. **21** (1980) 797.
4. V.G.Kac, Adv.Math. **26** (1977) 8.
5. Y.Ohnuki, S.Kamefuchi, "Quantum Field Theory and Parastatistics", Univ. of Tokyo Press, Springer-Verlag, Berlin 1982.
6. M.Chaichian, P.Kulish, Phys.Lett. **234B** (1990) 72.
7. R.Floreanini, V.Spiridonov, L.Vinet, Commun.Math.Phys. **137** (1991) 149.
8. R.Floreanini, D.Leites, L.Vinet, Lett.Math.Phys. **23** (1991) 127.
9. M.Jimbo, Lett.Math.Phys. **10** (1985) 63; Lett.Math.Phys. **11** (1986) 247; Commun.Math.Phys. **102** (1986) 537.
10. V.G.Drinfeld, Quantum Groups, *in:* Proc. ICM Berkeley 1986, vol.1 (AMS, Providence, R.I. 1987) pp. 798 – 820.
11. L.D.Faddeev, N.Yu.Reshetikhin, L.A.Takhtajan, Algebra and Analysis **1** (1989) 178 (Engl. translation: Leningrad Math.J. **1** (1990) 193).
12. L.C.Biedenharn, J.Phys.**A22** (1989) L873.
13. A.J.Macfarlane, J.Phys.**A22** (1989) 4581.
14. C.P.Sun, H.C.Fu, J.Phys.**A22** (1989) L983.
15. T.Hayashi, Commun.Math.Phys. **127**(1990) 129.
16. L.K.Hadjiivanov, R.R.Paunov, I.T.Todorov, Nucl.Phys. **B356** (1991) 387.
17. Ya.S.Stanev, I.T.Todorov, L.K.Hadjiivanov, Phys.Lett. **B276** (1992) 87.
18. L.K.Hadjiivanov, R.R.Paunov, I.T.Todorov, J.Math.Phys. **33** (1992) 1379.
19. P.Furlan, L.K.Hadjiivanov, I.T.Todorov, J.Math.Phys. **33** (1992) 4255.
20. M.Flato, L.K.Hadjiivanov, I.T.Todorov, Found.Phys. **23** (1993) 571.
21. W.Pusz, S.L.Woronowicz, Rep.Math.Phys. **27** (1989) 231.
22. S.M.Khoroshkin, V.N.Tolstoy, Commun.Math.Phys. **141** (1991) 599.
23. R.Floreanini, L.Vinet, J.Phys. **A23** (1990) L1019.

24.E.Celeghini, T.Palev, M.Tarlini, Mod.Phys.Lett. **B5** (1991) 187.

25.T.Palev, N.Stoilova, Lett.Math.Phys. (to appear).

26.T.Palev, private communication.

27.A.Yu.Alekseev, L.D.Faddeev, Commun.Math.Phys. **141** (1991) 413.

28.A.Yu.Alekseev, I.T.Todorov, "Quadratic brackets from symplectic forms", ESI Vienna preprint, in preparation.

29.A.Yu.Alekseev, D.V.Glushenkov, A.V.Lyakhovskaya, Paris preprint PAR-LPTHE 92-24 (June 1992), Algebra and Analysis (to appear).

30.M.Flato, C.Fronsdal, Singletons: Fundamental Gauge Theory, *in:* Topological and Geometrical Methods in Field Theory, Symposium in Espoo, Finland 1986, Eds. J.Hietarinta, J.Westerholm (World Scientific, Singapore 1986) pp.273 – 290.

31.G.Mack, I.T.Todorov, J.Math.Phys. **10** (1969) 2078.

32.G.Mack, V.Schomerus, Nucl.Phys. **B370** (1992) 185.

33.M.Scheunert, J.Math.Phys. **34** (1993) 320.