Negative tension branes as stable thin shell wormholes

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We investigate negative tension branes as stable thin shell wormholes in Reissner-Nordström-(anti) de Sitter spacetimes in d dimensional Einstein gravity. Imposing Z_2 symmetry, we construct and classify traversable static thin shell wormholes in spherical, planar (or cylindrical) and hyperbolic symmetries. In spherical geometry, we find the higher dimensional counterpart of Barceló and Visser's wormholes, which are stable against spherically symmetric perturbations. We also find the classes of thin shell wormholes in planar and hyperbolic symmetries with a negative cosmological constant, which are stable against perturbations preserving symmetries. In most cases, stable wormholes are found with the combination of an electric charge and a negative cosmological constant. However, as special cases, we find stable wormholes even with vanishing cosmological constant in spherical symmetry and with vanishing electric charge in hyperbolic symmetry.

Keywords: Wormhole; stability; thin-shell.

1. Construction

We obtain wormholes by operating three steps invoking junction conditions [1]. Firstly, consider a couple of d dimensional manifolds, $\mathcal{V}_\pm$. We assume $d \geq 3$. The d dimensional Einstein equations are given by

$$G_{\mu\nu\pm} + \frac{(d-1)(d-2)}{6}\Lambda_\pm g_{\mu\nu\pm} = 8\pi T_{\mu\nu\pm}, \tag{1}$$

where $G_{\mu\nu\pm}$, $T_{\mu\nu\pm}$ and $\Lambda_\pm$ are Einstein tensors, stress-energy tensors and cosmological constants in the manifolds $\mathcal{V}_\pm$, respectively. The metrics on $\mathcal{V}_\pm$ are given by $g^\pm_{\mu\nu}(x^\pm)$. The metrics for static and spherically, planar and hyperbolically symmetric spacetimes on $\mathcal{V}_\pm$ are written as

$$ds^2_\pm = -f_\pm(r_\pm)dt^2_\pm + f_\pm(r_\pm)^{-1}dr^2_\pm + r^2_\pm(d\Omega^k_{d-2})^2_\pm, \tag{2}$$

$$f_\pm(r_\pm) = k - \frac{\Lambda_\pm r^2_\pm}{3} - \frac{M_\pm}{r^{d-3}_\pm} + \frac{Q^2_\pm}{r^{2(d-3)}_\pm}, \tag{3}$$

respectively. $M_\pm$ and $Q_\pm$ correspond to the masses and charges in $\mathcal{V}_\pm$, respectively. k is a constant that determines the geometry of the $(d-2)$ dimensional space and takes ± 1 or 0. $k = +1$, 0 and -1 correspond to a sphere, plane (or cylinder) and a hyperboloid, respectively.

Secondly, we construct a manifold $\mathcal{V}$ by gluing $\mathcal{V}_\pm$ at their boundaries. We choose the boundary hypersurfaces $\partial\mathcal{V}_\pm \equiv \{r_\pm = a \mid f_\pm(a) > 0\}$, where a is called the thin shell radius. Then, by gluing the two regions with matching their boundaries, $\partial\mathcal{V}_+ = \partial\mathcal{V}_- \equiv \partial\mathcal{V}$, we can construct a new manifold $\mathcal{V}$ which has a wormhole throat at $\partial\mathcal{V}$. $\partial\mathcal{V}$ should be a timelike hypersurface, on which the line element is given by

$$ds^2_{\partial\mathcal{V}} = -d\tau^2 + a(\tau)^2(d\Omega^k_{d-2})^2. \tag{4}$$

τ stand for proper time on the junction surface $\partial\mathcal{V}$.

Thirdly, by using the junction conditions, we derive the equations for the manifold $\partial\mathcal{V}$.

1.1. Z_2 *wormholes made of pure tension*

From now on, we assume Z_2 symmetric wormholes for simplicity. Since our metrics Eq. (2) are diagonal, S^i_j, the stress-energy tensor on the shell, is also diagonalized and written as $S^i_j = \mathrm{diag}(-\sigma, p, p, \dots, p)$, where p is the surface pressure and σ is the surface energy density living on the thin shell. The explicit form of junction conditions yield

$$\sigma = -\frac{d-2}{4\pi a}A, \tag{5}$$

$$p = \frac{1}{4\pi}\left(\frac{B}{A} + \frac{d-3}{a}A\right), \tag{6}$$

where $A(a) \equiv \sqrt{f + \dot{a}^2}$, $B(a) \equiv \ddot{a} + \frac{1}{2}f'$. Thus, we deduce a critical property of wormholes that σ must be negative.

Then the τ-component of the Hamiltonian constraints, $S^{ij}{}_{|j} + [T^\alpha_\beta e^i_\alpha n^\beta]_\pm = 0$, reduces to

$$\sigma' = -\frac{d-2}{a}(\sigma + p). \tag{7}$$

From Eq. (7) we see that $\sigma = const.$ if we choose pure tension $p = -\sigma$ as a equation of state on the shell. In this setup of pure (negative) tension, we get the equation of motion for radially moving shell by squaring Eq. (5). The result is $\dot{a}^2 + V(a) = 0$, where

$$V(a) = f(a) - \left(\frac{4\pi a\sigma}{d-2}\right)^2. \tag{8}$$

1.2. *Static solutions, stability criterion and the horizon-avoidance condition*

Suppose a thin shell throat be static at $a = a_0$ and its throat radius satisfy the relation $f(a_0) > 0$. This condition is called the horizon-avoidance condition in [2].

We analyze stability against small perturbations preserving symmetries [3]. To determine whether the shell is stable or not against the perturbation, we use Taylor expansion of the potential $V(a)$ around the static radius a_0 as

$$V(a) = V(a_0) + V'(a_0)(a - a_0) + \frac{1}{2}V''(a_0)(a - a_0)^2 + \mathcal{O}((a - a_0)^3). \tag{9}$$

One find $\dot{a}_0 = 0, \ddot{a}_0 = 0 \Leftrightarrow V(a_0) = 0, V'(a_0) = 0$ at $a = a_0$ so the leading-order term for the potential given by Eq. (9) is proportional to $V''(a_0)(a-a_0)^2$. Therefore, the stability condition against radial perturbations for the thin shell is given by

$$V''(a_0) > 0. \tag{10}$$

The present system may have static solutions $a = a_0$. Then, from Eqs. (5) and (6) we get the equation which determines a_0 as

$$2ka_0^{2(d-3)} - (d-1)Ma_0^{d-3} + 2(d-2)Q^2 = 0. \tag{11}$$

For $k = \pm 1$, Eq. (11) has two solutions:

$$a_{0\pm}^{d-3} = \frac{d-1}{4k} M(1 \pm b), \tag{12}$$

where

$$b := \sqrt{1 - k\frac{q^2}{q_c^2}}, q := \frac{|Q|}{|M|}, q_c := \frac{(d-1)}{4\sqrt{d-2}}. \tag{13}$$

2. Classification of static wormholes

By studying both the existence of static solutions and stability conditions, we can search static and stable wormholes. The results are summarized in Table 1 – 4.

Table 1. The existence and stability of Z_2 symmetric static wormholes in three dimensions. $k = 1$, 0 and -1 correspond to spherical, planar (cylindrical) and hyperbolic symmetries, respectively.

	Static solution	Horizon avoidance	Stability
$k - M + Q^2 = 0$	$\forall a_0 > 0$	Satisfied	Marginally stable
$k - M + Q^2 \neq 0$	None	–	–

Table 2. The existence and stability of Z_2 symmetric static wormholes in spherical symmetry in four and higher dimensions. The expressions for the static solutions $a_{0\pm}$ $(0 < a_{0-} < a_{0+})$ are given by Eqs. (12) and (13) with $k = 1$. $H_\pm(d, q)$ and $R(d)$ are given by Eq. (A.3) and Eq. (A.2), respectively. Note that $H_+ > 0$ for $0 \leq q \leq q_c$, while $H_- > 0$ only for $1/2 < q \leq q_c$. Therefore, if $\Lambda = 0$, the horizon-avoidance condition holds for a_{0+} for $0 \leq q \leq q_c$, while it does for a_{0-} only for $1/2 < q \leq q_c$. For $M > 0$ and $q = q_c$, the double root solution $a = a_{0\pm}$ is linearly marginally stable but nonlinearly unstable.

		Static solution	Horizon avoidance	Stability
$M > 0$	$q = 0$	$[(d-1)M/2]^{1/(d-3)}$	$\lambda < H_+(d, 0)$	Unstable
	$0 < q < q_c$	$a_{0\pm}$	$\lambda < H_\pm(d, q)$ for $a_{0\pm}$	a_{0-}: Stable, a_{0+}: Unstable
	$q = q_c$	$[(d-1)M/4]^{1/(d-3)}$	$\lambda < R(d)$	Unstable
	$q_c < q$	None	–	–
$M < 0$		None	–	–
$M = 0$		None	–	–

Table 3. The existence and stability of Z_2 symmetric static wormholes in hyperbolic symmetry in four and higher dimensions. The definitions for q and λ are same as in Table 2. The expressions for the static solutions $a_{0\pm}$ are given by Eqs. (12) and (13) with $k = -1$. Since all of I, N and S are negative, the horizon-avoidance condition cannot be satisfied with $\Lambda = 0$ for any cases in hyperbolic symmetry.

		Static solution	Horizon avoidance	Stability
$M > 0$	$q = 0$	None	–	–
	$q > 0$	a_{0-}	$\lambda < I(d, q)$	Stable
$M < 0$	$q = 0$	$[(d-1)\|M\|/2]^{1/(d-3)}$	$\lambda < N(d, 0)$	Stable
	$q > 0$	a_{0+}	$\lambda < N(d, q)$	Stable
$M = 0$	$Q = 0$	None	–	–
	$\|Q\| > 0$	$[\sqrt{d-2}\|Q\|]^{1/(d-3)}$	$\Lambda/3 < S(d, q)$	Stable

Table 4. The existence and stability of Z_2 symmetric static wormholes in planar or cylindrical symmetry in four and higher dimensions. The definitions for q and λ are same as in Table 2. Note that we assume that the bulk spacetime is described by the Reissner-Nordstöm-(anti) de Sitter metric or its higher dimensional counterpart. $J(d, q)$ is given by Eq. (A.7). Since J is negative, the horizon-avoidance condition cannot be satisfied with $\Lambda = 0$ for $M > 0$ and $q > 0$ in planar or cylindrical symmetry.

		Static solution	Horizon avoidance	Stability
$M > 0$	$q = 0$	None	–	–
	$q > 0$	$[2(d-2)q^2 M/(d-1)]^{1/(d-3)}$	$\lambda < J(d, q)$	Stable
$M < 0$	$q = 0$	None	–	–
	$q > 0$	None	–	–
$M = 0$	$Q = 0$	$\forall a_0 > 0$	Satisfied	Marginally stable
	$\|Q\| > 0$	None	–	–

3. Summary and discussion

We developed the thin shell formalism for d dimensional spacetimes and investigated spherically, planar (cylindrically) and hyperbolically symmetric wormholes with a pure negative tension brane and found and classifies Z_2 symmetric static solutions which are stable against radial perturbations. We found that in most cases charge is needed to keep the static throat radius positive and that a negative cosmological constant tends to decrease the radius of the black hole horizon and then to achieve the horizon avoidance. So the combination of an electric charge and a negative cosmological constant makes it easier to construct stable wormholes. However, a negative cosmological constant is unnecessary in a certain situation of $k = +1$ and $M > 0$ and charge is unnecessary in a certain situation of $k = -1$ and $M < 0$. We summarize the results in Tables 1, 2, 3 and 4.

In three dimensions, there is only possibility to have a marginally stable wormhole. The ingredients of this wormhole are a couple of AdS space-times.

Then, we restrict the spacetime dimensions to be higher than or equal to four. For $k = +1$, spherically symmetric thin shell wormholes which are made with a negative tension brane are investigated. It turns out that the mass must be positive, i.e., $M > 0$. The obtained wormholes can be interpreted as the higher dimensional counterpart of Barceló-Visser wormholes [2]. As a special case, if $1/2 < q < q_c$, one can obtain a stable wormhole without a cosmological constant. This wormhole consists of a negative tension brane and a couple of over-charged Reissner-Nordström space-times.

For $k = -1$, though it is hard to imagine how such symmetry is physically realized, they are interesting from the viewpoint of stability analyses. It turns out that M can be positive, zero and negative for stable wormholes. In this geometry, there is no upper limit for $|Q|$ for stable wormholes. There is possibility for a stable wormhole without charge if $M < 0$ and $\lambda < N(d, 0)$ is satisfied.

For $k = 0$, the geometry is planar symmetric or cylindrically symmetric. In this case, since the generalized Birkhoff's theorem does not apply, we should regard the Reissner-Nordström-(anti) de Sitter spacetime as a special solution to the electrovacuum Einstein equations. This means that the present analysis only covers a part of possible static thin shell wormholes and the stability against only a part of possible radial perturbations. Under such a restriction, we find that we need $Q \neq 0$ and $\Lambda < 0$ to have stable wormholes. There is no upper limit for $|Q|$. In the zero mass case, the wormhole is marginally stable.

We would note that the existence and stability of negative tension branes as thin shell wormholes crucially depend on the curvature of the maximally symmetric $(d-2)$ dimensional manifolds. On the other hand, they do not qualitatively but only quantitatively depend on the number of space time dimensions.

Acknowledgments

T.K. was supported in part by Rikkyo University Special Fund for Research. T.H. was partially supported by the Grant-in-Aid No. 26400282 for Scientific Research Fund of the Ministry of Education, Culture, Sports, Science and Technology, Japan.

Appendix A. Definition of functions

Functions presented in tables are defined by

$$\lambda := \frac{\Lambda}{3} |M|^{\frac{2}{d-3}}, \tag{A.1}$$

$$R(d) \equiv \left(\frac{4}{d-1}\right)^{\frac{2}{d-3}} \frac{(d-3)^2}{(d-1)(d-2)}, \tag{A.2}$$

$$H_{\pm}(d, q) \equiv - \left\{\frac{4}{(d-1)(1 \pm b)}\right\}^{\frac{2}{d-3}} \frac{d-3}{(d-1)(d-2)(1 \pm b)} \left[2 - (d-1)(1 \pm b)\right], \tag{A.3}$$

$$I(d, q) := - \frac{d-3}{(d-1)(d-2)} \left\{\frac{4}{(d-1)(b-1)}\right\}^{-\frac{2}{d-3}} \left[\frac{2}{b-1} + d - 1\right], \tag{A.4}$$

$$N(d,q) \equiv \left\{ \frac{4}{(d-1)(1+b)} \right\}^{\frac{2}{d-3}} \frac{d-3}{(d-1)(d-2)(1+b)} \left[2-(d-1)(1+b)\right], \quad \text{(A.5)}$$

$$S(d,q) \equiv -\,|Q|^{-\frac{2}{d-3}}(d-3)(d-2)^{d-4}, \quad \text{(A.6)}$$

$$J(d,q) \equiv -\,\frac{1}{4}\left(\frac{1}{q}\right)^{2\frac{d-1}{d-3}} \left(\frac{d-1}{2(d-2)}\right)^{\frac{2}{d-3}} \frac{(d-1)(d-3)}{(d-2)^2}. \quad \text{(A.7)}$$

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