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Emergent geometry, torsion and anomalies in non-relativistic topological matter

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Abstract. I review and discuss aspects of the interplay of emergent geometry and anomalies in topological semimetals and insulators, focusing on effects of torsion. This correspondence identifies torsional topological responses in terms of anomalies and anomaly related hydrodynamic phenomena involving gauge fields and geometry. I discuss how torsional emergent geometry arises from elastic deformations in crystalline materials and how this background couples to thee low-energy continuum models inherited from lattice models, utilizing the semiclassical expansion. Via the coupling of momentum space topology and emergent vielbein geometry, non-relativistic topological matter can realise new geometrical responses of mixed gauge-gravitational character. The topological low-energy torsional responses depend momentum space geometry, lattice momenta and the regularization and UV completion, provided by the non-relativistic physics and symmetries of topological materials.

1. Introduction: Topological matter and anomalies

Topological matter has robust, protected quantum responses and associated zero modes that are insensitive to microscopic details, demarcating them from trivial ground states such as ordinary insulators, superfluids and normal metals. The integer quantized conductivity (in units of e^2/h) of the quantum Hall effect in two spatial dimensions (2+1d) is a prime example. See more from e.g. the reviews [1, 2, 3, 4] along with other more recent gapped and gapless symmetry protected topological states.

Quantum field theory (QFT) anomalies can be utilized to classify topological matter [5, 6, 7, 8], at least in terms of representative if yet idealized low-energy field theory models. The link to symmetry based classifications of topological matter [9, 10] follows from the topological responses needed for the anomalous symmetries of the quantum theory [11]. In general, anomalies can be defined as the sensitivity and interplay of classical gauge and global symmetries to quantization, such as the chiral anomaly in 1+1d and 3+1d [12] and the closely related parity anomaly in 2+1d [13]. When mixed gauge-gravitational anomalies are considered in addition [14], the tenfold classification of symmetry protected topological phases relates to relativistic anomalies and their descent relations [5]. In this way, early on since the discoveries of topological matter, the interelations and correspondence between anomalies and topology has been well-appreciated, much like for the QFT anomalies.

Nevertheless, at least two things in this correspondence are immediately non-evident: QFT anomalies are characterised as robust and in-escapable consequences of retaining or gauging certain symmetries in relativistic QFTs, whereas topological phases are (mostly) found in non-relativistic condensed matter systems which are finite and with well-defined UV completions, e.g. certain lattice models and, eventually, atomistic many-body physics. The latter, by definition, formally do not posses any anomalies. This discrepancy is of course explained by the fact that anomalies creep in via lattice regularization terms and fermion doubling theorems [15, 16, 17, 18]. The anomalies, chiral and/or doubler fermions can be placed on e.g. spatially separated boundaries, interfaces and different lattice momenta, realising the domain wall idea of anomaly inflow [11]. So, by the so-called t' Hooft anomaly matching and anomaly inflow, the low-energy theory simultaneously reproduces the macroscopic topological response and bulk-boundary correspondence in terms of the anomaly and boundary degrees of freedom saturating the conservation laws. Yet, for anomalies to classify all topological phases, as they now are understood, they must go beyond the standard anomalies and the tenfold classification. In particular, more recent examples of topological phases are often protected by crystalline symmetries, see e.g. [19], which are non-relativistic discrete spatial symmetries different to both gravity and internal gauge symmetries, and feature weaker topology than the tenfold classification. The topological responses for these are missing for the most part, despite the extensive classifications.

The purpose of this short article is twofold. I place topological semimetals and insulators with quasirelativistic low-energy models on a background constituting an emergent geometry with torsion, which can be taken to originate from a crystalline lattice and elastic deformations (or order parameter textures in topological superconductors and superfluids) with dislocations (vortices). First, I compare and contrast this low-energy field theory on curved space with that of strictly relativistic field theories. Second, from this background, I explicitly compute and identify new classes of mixed gauge-torsional anomalous responses that are allowed by non-relativistic crystalline symmetries, e.g. in Chern (quantum Hall) insulators [20, 21] and Weyl and Dirac semimetals [1, 4] (and other related topological phases). This is done directly by using the semi-classical quasirelativistic continuum models on emergent geometries with torsion. Given a lattice model whose low-energy theory matches to the quasirelativistic continuum theory, this gives an approach that is valid both at low and high energies which, in principle, allows to explicitly compute several proposed anomalous responses in topological semimetals related to geometry and torsion.

I expect that the correspondence between anomalies and topology is general, although not yet complete in terms of all known topological phases — this hypothesis somewhat depending on the working definitions of a topological phase ("weak"and "strong" symmetry protected topology [3, 19]) and anomalies as well. Relatedly, these anomalous conservation laws are what hydrodynamics encompasses and there is a growing set of new anomalous phenomena in relativistic hydrodynamics beyond QFT [23, 24, 25, 26, 27, 28]. Here anomalous hydrodynamics is extended for non-relativistic topological matter with torsion. In this respect, one motivation for the consideration of torsion is to enlarge hydrodynamic anomalies to encompass new representatives of non-relativistic topological matter, especially crystalline phases. In a strict sense, any QFT anomaly should be defined in terms of (at least) two "competing" symmetries that cannot be simultaneously realised (or gauged) in the quantum theory. The competition is summarized by associated anomaly polynomials and descent relations. For the purposes of this article, especially with regards to torsional anomalies, I will be slightly imprecise with this terminology and refer to an anomaly as the breaking of any classical conservation law. Nevertheless, only the consideration of perturbative 1+1d chiral anomaly, 2+1d parity anomaly, and the 3+1d chiral anomaly from QFT are sufficient for this article, see the Appendix for a quick review. These correspond to the 2+1-dimensional quantum Hall effect and 3+1-dimensional Weyl and Dirac semimetals, respectively. In particular, comparing the torsional Nieh-Yan anomaly and the anomalous quantum Hall effect in 3+1d, it can be seen that the non-relativistic torsional anomalies in the hydrodynamic responses of topological phases are closely related to chiral U(1) QFT anomalies, yet are distinct, in the same way as anomalous hydrodynamics

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encompasses new phenomena outside but related to QFT anomalies. See Refs. [29, 28] for discussions about the relativistic case with torsion.

2. Emergent geometry and torsion in topological semimetals and insulators

The topological states sufficient for our purposes are encapsulated by the following family of lattice model Hamiltonians, in momentum space $H = \sum_{\boldsymbol{k}} \Psi^\dagger \mathcal{H}(\boldsymbol{k})\Psi$,

$$\mathcal{H}(\boldsymbol{k}) = \sum_i d_\mu(\boldsymbol{k})\Gamma^\mu = \frac{1}{a}\sum_i^d v_i\Gamma^i \sin(\boldsymbol{k}_i a) + m\Gamma^0 + \frac{r}{a}\sum_i^d (1-\cos(\boldsymbol{k}_i a))\Gamma^0 \tag{1}$$

where a is the lattice constant (say, of a cubic lattice for simplicity) and $\{\Gamma^\mu, \Gamma^\nu\} = 2\delta^{\mu\nu}$ are anticommuting Dirac matrices. The Ψ is a non-relativistic fermion whose indices do not necessarily correspond to spin but instead e.g., say, to atomic orbitals in a material. The v_i are velocities from nearest neighbour hopping terms. Allowing for anisotropy, the most general coupling would be $v_{ij}\Gamma^i \sin(k_j a)$. The lattice introduces a UV cutoff $\sim 1/a$ and the $O(r)$ mass terms (isotropic here for simplicity) decouple the Dirac fermions around $\boldsymbol{k} = 0$ from those at some $k_i = \pi/a$ at the expense of (explicit) axial (chiral) symmetry breaking. Note that this model is nothing else than lattice Dirac-Wilson fermions and the Lorentz-breaking lattice and $O(r)$ mass terms vanish in the continuum limit $a \to 0$ [15, 18]. According to the fermion doubling theorem, it is impossible to retain a single (chiral) fermion while at the same time preserving all (chiral) symmetries [16].

Although the simple lattice model (1) is not realistic for general real materials, its low-energy behaviour is realized in several topological phases [9, 21]. Accordingly, at large and positive $m \gg |v_i|, |r|$, the system is a trivial insulator. The continuum Dirac theory emerges as $a \to 0$ when the mass parameter m/r is close to isolated critical values where some of the eigenvalues $\pm|d(\boldsymbol{k})|$ of $\mathcal{H}(\boldsymbol{k})$ cross zero. When all the eigenvalues are non-zero but some levels have crossed (so-called band inversions), the system is a topological insulator. At the corresponding critical points, it becomes a topological Dirac (or Weyl) semimetal. In 2+1d (or $2n$+1d), the Dirac model is a representative for the integer anomalous quantum Hall or Chern insulator, while for 3+1d it is a model for the time-reversal symmetric topological insulator [21, 22]. Extending the 2+1d insulator model to 3+1d with $v_z = 0, -1 < m/r < 1$, we get a Weyl semimetal with Weyl Hamiltonian [1, 4]

$$\sum_\pm H_{\pm\boldsymbol{k}_\star}(\boldsymbol{k}) \sim \sum_{\pm,i} \pm v_i^\pm \sigma^i (k \mp k_\star)_i + \cdots \tag{2}$$

close to two Weyl nodes at $\pm a\boldsymbol{k}_\star = (0, 0, \pm \arccos[(r+m)/r])$ with $\boldsymbol{v}^\pm = (v_x, v_y, \pm r \sin ak_{\star z})$, and massive doublers when any $k_{x,y} = \pi$. The linear continuum expansion is valid when $|k - k_\star| \ll k_\star$ and higher-order terms are negligible. The Dirac Fermion at $\boldsymbol{k} = 0$ has been split to two Weyl fermions at $\pm\boldsymbol{k}_\star$ via choosing $v_z = 0$, breaking time-reversal and preserving inversion symmetry. The isolated points $\pm\boldsymbol{k}_\star$ are a topological Fermi surface [1, 30]. This is therefore at time-reversal breaking Weyl semimetal with a minimum number of two nodes of opposite chirality. When parity (inversion) is broken instead, the minimum number of nodes is four [4]. This topological phase is well-defined, since the Weyl Hamiltonian is stable in 3+1d for small perturbations, since there are 3 parameters $\boldsymbol{k}$ and Pauli matrices involved in the condition $|d_i(\boldsymbol{k})| = 0$ for (2).

2.1. Coupling topological matter to emergent geometry and gauge fields

The low-energy dispersion of quasiparticles in a topological semimetal or insulator takes the form of a Weyl/Dirac dispersion by general principles from topology [1, 30]. Working directly in the continuum limit, let us therefore consider more general Weyl/Dirac action

$$i\partial_t - \mathcal{H}(\boldsymbol{k}) = \hat{m} + \Gamma^a e^\mu_a (k - k_\star)_\mu + \cdots \tag{3}$$

where e.g. $\hat{m} = m\mathbf{1}$, in general a matrix of momentum (in)dependent "masses", and $e^\mu_a = \frac{\partial d_a}{\partial k_\mu}$ are the linear momentum expansion coefficients (previously the v_{ij} in (1)). The low-energy dispersion $\omega^2 - k^2 - m^2 = g^{\mu\nu}k_\mu k_\nu - m^2 = 0$ with metric $g^{\mu\nu}$ is determined by the (inverse) vielbein e^μ_a and is in general anisotropic. A small external electromagnetic field is introduced as $k_\mu \to k_\mu - A_\mu$, in addition to the shift $k_{\star\mu}$, and the spin-connection can enter as well.

Before describing the elastic geometry in detail, let us discuss coupling non-trivial, weakly varying vielbein coefficients $e^a_\mu(x)$ to the lattice fermions, using the Dirac/Weyl Hamiltonian as an example [31, 32]. The geometry is $e^i_a(x) = e^{(0)i}_a + \delta e^i_a(x)$, where a denotes some lattice indices with undeformed basis $e^{(0)a}_i$, $a = 1,2,3$ w.r.t. some fixed laboratory coordinates $i = x, y, z$. Notably, we keep momenta k_a as a referring to the undeformed lattice, while deformed momenta are $k_i = e^a_i k_a$ in the local coordinates i. For elasticity, a is spatial; a non-trivial "convective" $e_0 = e^\mu_0 \partial_\mu \equiv D_t$ follows from e.g. a moving frame w.r.t. to the medium or periodical driving with weak coordinate dependence. Schematically, in a semi-classical expansion, the Hamiltonian changes as [32]

$$\Gamma^a(k - k_\star)_a \to \Gamma^a e^i_a(\hat{k} - k_\star)_i + \cdots = \Gamma^a(e^i_a \hat{k}_i - k_{\star a}) + \cdots \tag{4}$$

$$\approx \Gamma^a(e^i_a \hat{q}_i - \delta e^a_i k_{\star a}) + \cdots \tag{5}$$

with constant $k_{\star a} = e^i_a k_{\star i}$. The lower line (5) follows by expanding $\hat{k}_i = k_{\star a} e^{(0)a}_i + \hat{q}_i$, i.e. around the original node $k^{(0)}_{\star i}$, where now $\tilde{A}_i = \delta e^a_i k_{\star a}$ is small elastic gauge field and $\hat{q}_i = (k - k_\star)_i$ is the small momentum coupled to the vielbein e^a_i; to first order $\delta e^i_a \hat{q}_i$ vanishes. The "minimal momentum coupling" (4) was studied in the Ref. [31] and compared to explicit lattice construction in the presence of strain and found to agree with it and the continuum limit. On the other hand, the elastic gauge field of (5) is studied in [33, 1, 34, 35, 36, 37] derived utilizing the *same* lattice construction. We shall adopt the formalism (4) onwards here, stressing that they differ only via the approximation from (4) to (5) and/or non-universal constant from the lattice phonon coupling [32]. For both, the deformation comes from δe^a_i but either e^a_i or $\tilde{A}_i$ as the explicit source.

Indeed, it seems that the constant $k_{\star a}$ can be shifted away by the rotation to $\psi_{\boldsymbol{k}_\star} \sim e^{i\boldsymbol{k}_\star \cdot \boldsymbol{x}}\Psi$ and the difference of (4) to the approximation (5) is innocuous. This however is not the case in the presence of anomalies (nor the shift for general momentum space integrals)! The related couplings (4) and (5) have important differences: $e^i_a \hat{k}_i$ in (4) depends on momentum and has the original constant shift $k_{a\star}$, whereas the gauge field $\tilde{A}_i[\delta e^a_i]$ in (5) has just constant frame $e^{(0)a}_i$ (to lowest order) and no shift $k_{\star a}$ since only q_i enters. Moreover, one gets different results for geometric phenomena related to the chiral anomaly with the momentum dependence of (4) leading to non-universal regularization dependent terms, the hallmark of torsional anomalies. In constrast, anomalies follow from (5) with universal coupling $k_\star$, playing the role both universal emergent electric charge and the dimensionful UV scale of $\tilde{A}_i$, i.e. $\tilde{A}_i \sim eA_i$ is a pseudo U(1) gauge field [37]. Different anomaly expressions derived using the related (4), (5) expressions have created some controversy in the literature regarding the UV coefficients proportional to $k_\star$. A particularly simple way to derive torsional chiral anomalies is to compute the Landau level spectral flow from (4), see e.g. the Refs. [33, 38, 32]. Any discrepancy to (5) should be addressed since a common starting point [31, 35] is in terms of the strain induced modified hopping parameters, e.g. $v_i \to v_{ij}[u_{mn}]$ in Eq. (1), with u_{ij} in (8). For now, we exclusively use the "minimal momentum coupling" (4) and defer more comments on (4) vs. (5) until in

Sec. 6. Along with the results of [31, 32], we extend it to continuum models derived from incarnations of (1) and the semiclassical expansion for the effective action finding agreement when applicable. Since a lot about anomalies and topological phases have been learnt by studying (weakly) coordinate dependent masses $\hat{m}(x)$, a natural question is to ask what happens if the emergent frame fields e_a^μ become coordinate dependent $e_a^\mu(x)$. What is note-worthy is that the emergent e_a^μ are more fundamental that the emergent metric $g_{\mu\nu}$ in the dispersion, opening the possibility for torsional physics as the e_μ^a independent from the connection.

2.2. Torsional elastic geometry

For lattice systems, the continuum formulation allows to interpret the elastic distortions as sources for an effective vielbein, spin connection and metric. In general, this background is torsionful and curved in the presence of translational and rotational lattice defects, i.e. dislocations and disclinations, see e.g. [39, 40, 41, 42] and [43, 44, 38]. For the fermions, this correspondence amounts to the gauging of smooth translations and rotations at the level of the background and sources in the continuum theory. This remains valid topological responses of the effective theory, although of course on the fixed background the smooth lattice transformations have finite elastic energy breaking the symmetry. We now briefly review the geometric background from elasticity.

The elastic distortion is given as

$$x'^a = x^a + u^a(x) \tag{6}$$

where $a = 0, \ldots, d$ label lattice directions (again, for spatial lattices $u^0 = 0$). The x^a are the undeformed, reference lattice directions which we for simplicity take to be aligned with the reference (i.e. laboratory) spacetime coordinates x^μ, $\mu = t, x, y, z, \ldots$, i.e. $x^a = \delta_\mu^a x^\mu \equiv e_\mu^{(0)a} x^\mu$. With respect to this coordinate reference frame, the elastic deformation introduces the change of coordinates

$$e_\mu^a = \frac{\partial x'^a}{\partial x^\mu} = e_\mu^{(0)a} + \partial_\mu u^a \tag{7}$$

from which the symmetric and antisymmetric strains are

$$u_{ab} := \frac{1}{2}(\partial_a u_b + \partial_b u_a), \quad \tilde{u}_{ab} := \frac{1}{2}(\partial_a u_b - \partial_b u_a). \tag{8}$$

The $\tilde{u}_{ab}$ is a rotation and since uniform rotations do not contribute to elastic energy, can be dropped out from first-order elasticity. For this reason u_{ab}, representing acoustic phonons, is often sufficient. For generality, however, we retain the coupling with the emergent vielbein e_μ^a, never only the symmetric strain. Dislocation defects are encoded as multivalued $\delta e_\mu^a = \partial_\mu u^a$ such that $T^a = (de^a)_{\mu\nu} = \frac{1}{2}(\partial_\mu e_\nu^a - \partial_\nu e_\mu^a) = (d^2 u^a)_{\mu\nu} \neq 0$. The rotational disclination defects $R^a{}_b \sim d^2(\tilde{u}^a{}_b)_{\mu\nu}$, equal to dislocation dipoles, are confined [45] in the presence of crystalline order. These tensors correspond, respectively, to emergent torsion and and curvature from elastic deformations, see (9), (10) and (11) below.

We now add local microstructure corresponding to a local orientation degree of freedom and parametrize the local orientation with the Galilean spacetime transformation $x \to x'' = \Theta x + u$, where $\Theta^a{}_b = \mathbf{1} + \theta^a{}_b$ is a (spatial) rotation independent of u^a. Note that e_μ^a in (7) does not include the full change x'' of x, only the translation part u^a. The central quantity is the (unsymmetrized) strain tensor from the $x \to x''$ transformation,

$$w_\mu^a = x'' - x = \partial_\mu u^a + \theta_\mu^a = u_\mu^a + \tilde{u}_\mu^a + \theta_\mu^a \tag{9}$$

where, for small u^a and $\theta^a{}_b$, the total local rotation is $\tilde{\theta}^a_\mu \equiv \tilde{u}^a_\mu + \theta^a_\mu$ is the sum the antisymmetrized strain and rotation tensors. We added the antisymmetric field θ_{ab} to keep the total local rotation independent of u^a (i.e. $\tilde{u}_{ab}$). Rotations θ_{ab} are zeroth order in gradients and power counting. This translates to the vielbein e^a_μ and spin connection $\omega_{\mu ab}$ being independent. The non-zero θ_{ab} has various physical interpretations: test disclinations, the microrotation field of Cosserat elasticity [46], the gauge freedom related to the interrelations translations and rotations (i.e. torsion and curvature) [40], the ambient curvature of the space where the lattice is embedded, or finally, contribution from corners, vortices and other singular points [47, 48].

Gravitationally, the geometry from the transformation (9) seems harmless and "pure gauge" but the incorporation of dislocations and disclinations necessitates torsion and curvature, meaning multivalued/singular u^a and $\tilde{\theta}^a{}_b$ [40] and test disclinations can be introduced via non-trivial θ within first order elasticity. Depending on the detailed application and elastodynamics, the contributions from the local rotations $\tilde{\theta}$ needs to be analyzed case-by-case. In the simplest case, local hopping overlaps change as a function of the distance only, i.e. only u_{ij} [31], however for anisotropic orbitals, also the local rotation θ_{ab} affects the hopping elements.

To summarize the elastic quantities and the emergent geometry, we let $x'' = \Theta x + u$ around undeformed flat space lattice. Then

$$e^a_\mu = \delta^a_\mu + \partial_\mu u^a, \quad \Gamma^\lambda_{\mu\nu} = \partial_\mu\partial_\nu x''^\lambda = \partial_\mu\partial_\nu u^\lambda + \partial_\mu \theta^\lambda{}_\nu, \quad \omega^a_{\mu b} = e^a_\lambda \Gamma^\lambda_{\mu\nu} e^\nu_b + e^a_\lambda \partial_\mu e^\lambda_b = \partial_\mu \theta^a{}_b. \quad (10)$$

The equations must be supplemented by local continuity conditions, so that the local metric compatible geometry is consistent with (10) and the parallel transport, $\nabla_\mu = \partial_\mu + \hat{\Gamma}_\mu$,

$$[\nabla_\mu, \nabla_\nu]V^\lambda = R^\lambda_{\mu\nu\rho}V^\rho - T^\lambda_{\mu\nu}\nabla_\lambda V^\rho. \quad (11)$$

where torsion is $T^a \equiv \frac{1}{2}T^a_{\mu\nu}dx^\mu \wedge dx^\nu = \frac{1}{2}(\Gamma^a_{\mu\nu} - \Gamma^a_{\nu\mu})dx^\mu \wedge dx^\nu = de^a + \omega^a{}_b \wedge e^b$ and curvature $R^a{}_b \equiv \frac{1}{2}R^a{}_{b\mu\nu}dx^\mu \wedge dx^\nu = \frac{1}{2}[\partial_\mu\Gamma^a_{\nu b} + \Gamma^a_{\mu c}\Gamma^c_{\nu b} - (\mu \leftrightarrow \nu)]dx^\mu \wedge dx^\nu = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b$, in differential form notation. For example, we can require that $O(\partial^3 u, \partial^3\theta)$ terms are zero so that the metric, curvature and torsion are continuous and consistent with parallel transport with the tetrad and connection in (10) to that order in derivatives. See e.g. [40, 38] for more discussions about torsional geometry in our context.

3. Semiclassical expansion of the quasirelativistic low-energy continuum theory

3.1. Low-energy continuum theory

Assuming the elastic frame coupling (4) and armed with the background (10), I write the low-energy theory for a non-relativistic fermion Ψ with semi-classical Hamiltonian $\mathcal{H}(x;k)$ as [1, 30, 21]

$$S = \int d^3x dt \Psi^\dagger[iD_t - \mathcal{H}(\boldsymbol{x};\boldsymbol{k})]\Psi \simeq \sum_{\boldsymbol{k}_\star} \int e d^4x \overline{\psi}_{\boldsymbol{k}_\star}(x)[iD_{\boldsymbol{k}_\star}(x) + \hat{m}_{\boldsymbol{k}_\star}]\psi_{\boldsymbol{k}_\star}(x) + \cdots \quad (12)$$

where $\simeq$ represents the (semi-classical, weakly coordinate dependent) low-energy continuum limit and $\cdots$ are non-linear corrections and/or interactions, see e.g. Refs. [31, 34, 49, 32, 50, 51, 52, 53] for extensive discussions how this limit can be taken.

The original low-energy fermions are $\Psi(x) \sim \sum_{\boldsymbol{k}_\star} \frac{1}{\sqrt{e}}\psi_{\boldsymbol{k}_\star}(x)$ and $\psi_{\boldsymbol{k}_\star} = e^{i\boldsymbol{k}_\star\cdot\boldsymbol{x}}\phi(x)$, where ϕ is slowly varying, are close around some collection of inequivalent Brillouin zone (BZ) wave vectors $\boldsymbol{k}_\star$, such as the origin, corners or a Fermi node (or in general any Fermi surface). Note the shift of $\psi_{\boldsymbol{k}_\star}$ by $\boldsymbol{k}_\star$ in momentum space, so that The Dirac mass $\hat{m}_{\boldsymbol{k}_\star}$ is the energy scale of the low-energy quasiparticles, representing the energy gap for insulators and absent for semimetals. Correspondingly, the $D_{\boldsymbol{k}_\star}(x)$ is the emergent low-energy Dirac operator close to $\boldsymbol{k}_\star$

$$D_{\boldsymbol{k}_\star}(x) = \gamma^a e^\mu_a(\partial_\mu + \hat{\omega}_\mu + iqA_\mu + ik_{\star\mu}). \quad (13)$$

The low-energy Dirac form follows by general principles from topology [30] and the various quantities entering this operator are:

$$\begin{aligned} \{\gamma^a,\gamma^b\} &= 2\eta^{ab} && \text{flat space gamma matrices} \\ \eta^{ab}e^\mu_a e^\nu_b &= g^{\mu\nu} && \text{emergent vielbein} \\ A_\mu, k_{\star\mu} &\equiv e^a_\mu k_{\star a} && \text{U(1) gauge field and Fermi/BZ momentum} \\ \hat\omega_\mu &\equiv \tfrac{1}{2}\omega^{ab}_\mu \Sigma_{ab} && \text{emergent spin connection with generators } \Sigma_{ab} \end{aligned} \tag{14}$$

and $\hat\omega_\mu \equiv \omega^a_{\mu b} = e^a_\nu \partial_\mu e^\nu_b + e^a_\lambda \Gamma^\lambda_{\mu\nu} e^\nu_b$ a metric compatible spin connection, which however, does not necessarily coincide with the torsion-free Christoffel connection, $\Gamma^\lambda_{\mu\nu} \neq \mathring{\Gamma}^\lambda_{\mu\nu}$. In the same fashion, the emergent rotation generators Σ_{ab} do not necessarily match the Lorentz generators $\gamma_{ab} = \frac{i}{2}[\gamma_a, \gamma_b]$ [48]. I emphasize that the original system (12) is in flat space and the EM gauge field A_μ is fundamental, in contrast to the emergent geometric fields.

3.2. Semiclassical expansion
The gauge invariant (one-loop) effective action for anomalous currents is summarized as

$$S_{\text{eff}}[\Phi] = i \,\text{Tr} \ln[\not{D} - \hat M[\Phi]] = i \sum_n \frac{(-1)^n}{n} \text{Tr}\left[G_0(p)\not\Phi(x)\right]^n. \tag{15}$$

Here Tr includes phase space integrals over coordinate $d^{d+1}x$ and momentum space $d^{d+1}p/(2\pi)^{d+1}$, $G_0(p)$ is the (time-ordered, Feynmann) propagator of the unperturbed, translation invariant Hamiltonian and $\hat M[\Phi]$ is any classical coordinate dependent background field Φ perturbation, contracted in the trace with proper (gamma) matrix representations of (12). Once the denominators are expanded, we are left with expressions like $\frac{1}{p^2-m^2}\Phi = \Phi\frac{1}{p^2-m^2} + \frac{1}{(p^2-m^2)^2}[p^2,\Phi]$ and $[p_\mu, \Phi(x)] = -i\partial_\mu \Phi(x)$ from which the momentum and coordinate space traces can be disentangled by moving all momentum terms, say, to the left. $S_{\text{eff}}[\Phi]$ reduces to a gradient expansion in the perturbing fields $\Phi(x)$ times momentum space integrals [54]

$$S_{\text{eff}}[\Phi] = \int \frac{d^4p}{(2\pi)^4} \boldsymbol{F}(p_\mu) \cdot \int d^4x \boldsymbol{f}(\Phi, \partial_\mu\Phi, \dots). \tag{16}$$

Here $\boldsymbol{F}(p)$ is a polynomial momentum space tensor contracted with $\boldsymbol{f}(\Phi,\partial_\mu\Phi,\dots)$, a local field tensor expression, as allowed by the (anisotropic) non-relativistic symmetries. Of course, in relativistic theories, $\boldsymbol{F}(p) = F(p^2)$ and $\boldsymbol{f}(\Phi,\partial_\mu\Phi,\dots) = f(\Phi,\partial_\mu\Phi,\dots)$ by Lorentz invariance. In calculations, we further expand $\boldsymbol{f}(\Phi,\partial_\mu\Phi)$ in $\Phi = \Phi_0 + \delta\Phi$, where Φ_0 represent some reference background fields. It is important that the momentum space prefactor $\int_{\text{BZ}} d^4p\boldsymbol{F}(p_a)$ retains its from under (small) elastic deformations which enter the coordinate space expression in terms of the $\Phi(x) = e^a[u^a]$, where u^a are the (small) distortion fields in coordinate space. Finally, the expansion (15), (16) is equivalent to semi-classical Greens functions with Moyal products, or the computation of vacuum polarization diagrams in the limit of external momenta $q \to 0$. See the Appendix for a review of chiral anomalies and the expansion (16).

For the background (14) in (12), $\Phi(x) = A_\mu(x), e^\mu_a p_\mu, \omega^a_{\mu b}$ etc. and we can evaluate

$$J^\mu = \frac{1}{e}\frac{\delta S_{\text{eff}}}{\delta A_\mu}, \quad \mathcal{J}^\mu_a = \frac{1}{e}\frac{\delta S_{\text{eff}}}{\delta e^a_\mu}, \quad \mathcal{S}^\mu_{ab} = \frac{1}{e}\frac{\delta S_{\text{eff}}}{\delta \omega^{ab}_\mu} \tag{17}$$

corresponding to electric current, emergent stress-momentum and angular-momentum. Since e^a_μ couples to $i\partial_\mu \sim p_\mu$, J^a_μ is always a stress-momentum. However, the emergent S^μ_{ab} depends on the physical identification of the symmetry generators Σ_{ab} (the degree of freedom of the colums of Ψ). More generally, the emergent fields in (12) can be of various origin, I focus on crystalline systems. Another example is inhomogeneous tensor order parameters in unconventional (non s-wave) superconductors and superfluids and/or coupling to ambient geometry [1, 55, 56]. The vielbein and spin-connection could also simply be taken to represent abstract quantities set to flat space values at the end of the calculation or e.g. thermal gradients [57, 58, 27].

4. 2+1d quantum Hall effect and polarization with torsion

Now I calculate electromagnetic and geometric responses utilizing the semiclassical expansion for continuum models with simplest ingredients from the non-relativistic lattice. That is, we assume linearized spectrum but allow for anisotropies, finite node momenta, as well as a finite validity for the linear models. The linear approximation can be extended if all Fermi momenta are small and the boundary conditions for propagators of the all relevant bands tend to some constant values at higher momenta, allowing the unwinding of the BZ $T^3 \times \mathbb{R} \to \mathbb{R}^{3,1}$. This is often unrealistic in materials, in contrast to lattice models of relativistic theories. Any simple lattice model like (1) should be matched to detailed $k \cdot p$ expansions of realistic band structures, including many bands. In addition, the presence of extra massive fermions in the periodic BZ should be accounted for while keeping track of possible UV divergences or cutoffs for torsion in the low-energy theory.

4.1. Quantum Hall conductivity and torsional Hall viscosity in 2+1d Chern insulator

As a warm-up to using (16), let us take a 2+1d Chern insulator with the continuum model of (1) near $\boldsymbol{k} = 0$ with mass $\hat{m} = m\mathbf{1}$,

$$\mathcal{L}_{2+1\text{d}} = \overline{\psi}(\gamma^a e^\mu_a \partial_\mu - m)\psi. \tag{18}$$

The lattice model (1) this corresponds was detailed in [22, 21, 38, 31] and including doublers, has four massive Dirac fermions in total. The terms from slowly varying perturbation $\delta \not{e}^\mu p_\mu = -\delta \not{e}^a p_a$ are

$$S_{\text{eff}}[e^a, A] = \frac{i}{2}\,\text{Tr}\left[\frac{1}{\not{p}-m}\delta\not{e}^a p_a \frac{1}{\not{p}-m}\delta\not{e}^b p_b\right] + \frac{i}{2}\,\text{Tr}\left[\frac{1}{\not{p}-m}\not{A}\frac{1}{\not{p}-m}\not{A}\right]$$
$$+\frac{i}{2}\,\text{Tr}\left[\frac{1}{\not{p}-m}\delta\not{e}^a p_a \frac{1}{\not{p}-m}\not{A}\right] + \frac{i}{2}\,\text{Tr}\left[\frac{1}{\not{p}-m}\not{A}\frac{1}{\not{p}-m}\delta\not{e}^a p_a\right],$$

The cross terms vanish if we assume that $p_\star = 0$ by antisymmetry of $\int dp_a$, see below, leaving

$$\frac{i}{2}\,\text{Tr}[\not{A} i\not{\partial}\not{A}\frac{m}{(p^2-m^2)^2}] + \frac{i}{2}\,\text{Tr}[\not{e} i\not{\partial}\not{e}\frac{m p_a p_b}{(p^2-m^2)^2}]. \tag{19}$$

We use $\text{tr}(\gamma^\mu\gamma^\nu\gamma^\lambda) = -2i\epsilon^{\mu\nu\lambda}$ in 2+1d and performing the momentum space integrals with standard regularization [59], we get

$$\boxed{S_{\text{eff}}[A, e^a] = C_A(m)\int A \wedge dA + C_T(m,\Lambda)\eta_{ab}\int e^a \wedge de^b,} \tag{20}$$

where $A \wedge dA = \epsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda d^3x$, $e^a \equiv e^a_\mu dx^\mu$ etc. in differential form notation, and

$$C_A(m) = 4\pi\int_0^\infty \frac{dp}{(2\pi)^3}\frac{mp^2}{(p^2-m^2)^2} = -\frac{1}{8\pi}\text{sgn }(m) \tag{21}$$

$$C_T(m,\Lambda) = -\frac{4\pi}{3}\int_0^\Lambda \frac{dp}{(2\pi)^3}\frac{mp^4}{(p^2-m^2)^2} = \frac{\Lambda m}{6\pi^2} + \text{sgn}(m)\frac{m^2}{8\pi} + O(m/\Lambda). \tag{22}$$

This corresponds to the results in Ref. [38] and need to be further regularized. According to the parity anomaly [13], the C_A must be $1/4\pi$ quantized in the absence of strong interactions (fractionalization) and, even worse, the torsional term is diverges with UV cutoff $\propto \Lambda$. In Ref. [38], these coefficient were regulated with massive Pauli-Villars fermions, e.g. the $O(r)$ doublers of the lattice model (1), leaving just

$$\tilde{C}_A(m) = \frac{\text{sgn}(m) \pm 1}{8\pi}, \quad \tilde{C}_T(m) = \frac{\text{sgn}(m) \pm 1}{8\pi} m^2. \tag{23}$$

Since this is a (torsional) quantum Hall response, there are massless boundary 1+1d fermions with the chiral and torsional momentum anomaly related to the gauge non-invariances at the boundary. In contrast to the full coordinate transformations, the elastic dreibein can be transformed independently of the coordinates. Accordingly, the non-trivial e^a as elastic dreibein transform $e^a \to e^a + du^a$ under smooth elastic transformation in 2+1d bulk, to arrive to

$$\delta S_{\text{eff},1+1\text{d}}[A, e^a] = -\tilde{C}_A(m) \int \lambda F - \tilde{C}_T(m) \eta_{ab} \int u_a T^b, \tag{24}$$

This leads to the the anomalies

$$d \star J = \frac{1}{4\pi} F, \quad d \star \mathcal{J}_a = -\frac{m^2}{8\pi} T^a \ . \tag{25}$$

where $d \star J \equiv \partial_\mu (e J^\mu) d^4x$ and $d \star \mathcal{J}_a \equiv \partial_\mu (e \mathcal{J}^\mu_a) d^4x$ are the current and elastic stress-momentum covariant divergences, respectively. The former is the standard (covariant) anomaly from the 2+1d U(1) Chern-Simons term. The latter torsional boundary anomaly follows also via the anomaly inflow picture for general coordinate transformations [38], although the torsional CS term is both gauge and general coordinate covariant and therefore does not have any consistent anomaly in the 2+1d bulk. Interestingly, it seems that the emergent, elastic gauge transformation $\delta e^a = du^a$ captures the same information as gauging full coordinate transformations in the presence of the U(1) anomaly. Similar gauge transformations were discussed in terms of chiral elasticity in [60].

4.2. Topological polarization in a 2+1d TRB semimetal

There is a symmetry protected semimetal in 2+1d with a closely related response, which serves as an illustration of a gapless model with geometric response related to torsion. The Dirac model for this is (found e.g. low-energy graphene [43, 44])

$$\mathcal{L}'_{2\text{d}+1} = \sum_{p_\star} \overline{\psi} [\gamma^a e^\mu_a (i\partial_\mu - p_{\star\mu} - A_\mu) + \delta m_\star] \psi \tag{26}$$

there are symmetry protected Dirac nodes at $p_\star$, where the system is gapless. The mass $\delta m_\star \to \pm 0$ is a small symmetry breaking parameter (e.g. PT symmetry), whose role will become clear below. Now consider also the mixed terms, with fixed $p_\star$,

$$S_{\text{eff}} = i \, \text{Tr} \left[\frac{1}{\not{p} - \not{p}_\star + \delta m_\star} \delta \not{e}^a p_a \frac{1}{\not{p} - \not{p}_\star + \delta m_\star} \not{A} \right] = \text{Tr} [\frac{(\not{p} - \delta m_\star) p_a}{[(p-p_\star)^2 - \delta m_\star^2]^2} \not{e}^a \not{\partial} \not{A}] \tag{27}$$

The momentum space integral is for the mixed term and the torsional Hall viscosity term, similar to (23), are

$$\int \frac{d^3p}{(2\pi)^3} \frac{p_a \delta m_\star}{(p-p_\star)^2 - \delta m_\star^2} = \frac{\text{sgn}(\delta m_\star)}{8\pi} p_{\star a}, \quad \int \frac{d^3p}{(2\pi)^3} \frac{\delta m_\star p_a p_b}{(p-p_\star)^2 + \delta m_\star^2} = \frac{\text{sgn}(\delta m_\star)}{8\pi} p_{\star a} p_{\star b}, \tag{28}$$

by a shift the integration variable to $p \to p + p_\star$ and small $\delta m_\star$. We now want to consider difference between phases with opposite signs of small $\delta m_\star$, similar to the well-defined differences (23). All possible symmetry breaking terms give

$$\boxed{S_{\text{eff}} = \frac{\text{sgn}(\delta m_\star)}{8\pi} \sum_{p_\star} \int A \wedge dA + 2p_{\star a} e^a \wedge dA + p_{\star a} p_{\star b} e^a \wedge de^b} \tag{29}$$

see the recent papers [61, 62] for a lattice calculations of these terms. The new mixed term represent the jump of the topological polarization dipole-moment with value $P_a = \sum_{p_\star} \text{sgn}(\delta m_\star) \frac{p_{\star a}}{2\pi}$, summed over the nodes across the $\delta m_\star = 0$ [63, 64, 61]. Note that usually there are several nodes by fermion doubling. In the presence of the parameter $\delta m_\star \to \pm 0$ the link between the Chern number $\tilde{C}_A$ and polarization difference P_a is well-known [21], here the torsional terms are in addition regulated via the momentum scale $p_\star$. For non-zero torsion $T^a = de^a$, the action is not gauge invariant and requires jump of charge $j^0 = -P_a b^a$ at dislocations with Burgers vector b^a [65, 66, 67] due to zero modes. The last term is an analog of the torsional Hall viscosity term, the jump of which across $\delta m = 0$ is possible for the gapless system with non-zero $p_\star$ and implies a stress density on dislocations [38, 68].

5. Time-reversal breaking Weyl semimetal: anomalous quantum Hall effect and torsional anomaly

Let us derive the 3+1d anomalous quantum Hall and torsional Hall viscosity Weyl semimetal responses building on the 2+1d results of Sec. 4.

Consider the time-reversal breaking (TRB) Weyl semimetal (WSM) i.e. two Weyl nodes at $p = \pm p_\star$, with node separation $2p_\star$. The low-energy continuum model is, valid around some finite neighbourhood around the nodes,

$$S_{\text{TRB}} = \int d^4x e \, \overline{\psi} \gamma^a e_a^\mu (D_\mu - \gamma^5 p_{\star\mu}) \psi + \dots \tag{30}$$

where with $D_\mu = \partial_\mu - iA_\mu$ and $\psi = (\psi_+ \ \ \psi_-)$ is the sum of two antichiral Weyl fermions $\psi_\pm$ of (2) at $\pm \boldsymbol{p}_\star = \pm p_\star \hat{\boldsymbol{z}}$. The idea is to do the phase space integrals at constant $(p - p_\star)_z \neq 0$ where the 2+1d model is massive.

5.1. AQHE in TRB Weyl semimetal

For this subsection, the tetrad e_a^μ and $p_\star$ are constants and the electromagnetic A_μ is a slowly varying field in the gradient expansion. The relevant term for AQHE is

$$S_{\text{eff}}[A] = \frac{i}{2} \text{Tr} \left[\frac{1}{\not{p} - \not{p}_\star \gamma^5} \not{A} \frac{1}{\not{p} - \not{p}_\star \gamma^5} \not{A} \right] = \sum_{\gamma^5 = \pm 1} \frac{i}{2} \text{Tr} \left[\frac{(\not{p} - \not{p}_i \gamma^5)}{p_\perp^2 - p_i^2} \not{A}_\pm \frac{(\not{p} - \not{p}_i \gamma^5)}{p_\perp^2 - p_i^2} \not{A}_\pm \right]$$

Now we use $p_\mu A_\nu - i\partial_\mu A_\nu + A_\mu p_\nu$, pick the $\gamma^5 = +1$ part and $\text{tr}(\overline{\sigma}^\mu \sigma^\nu \overline{\sigma}^\lambda \sigma^\rho) = 2i\epsilon^{\mu\nu\lambda\rho}$, leading to

$$2i\epsilon^{\mu\nu\lambda z} \text{Tr} \left[\frac{(p - p_\star)_z}{(p_\perp^2 - p_\star^2)^2} A_\mu \partial_\nu A_\lambda \right].$$

As advertised, the momentum space integral splits and we are left with, $m_\pm = (p \mp p_\star)$,

$$\int \frac{dp^z}{2\pi} \int_{\text{BZ}} \frac{d^3 p_\perp}{(2\pi)^3} \frac{-m_+(p_z)}{(p_\perp^2 - m_+^2(p_z))^2} = -\int \frac{dp^z}{2\pi} \frac{i\text{sgn} \ m_+(p_z)}{8\pi}.$$

This integral is zero when integrated over the BZ by antisymmetry around $p_\star$. This is expected since original integral cancels by antisymmetry in p_z as well, after a shift of integration variable. In the limit where we extend to $p_z \in [-\infty, \infty]$, this becomes the ambigious cancellation of two shifted and linearly divergent integrals, much like the original chiral anomaly [12]. Nevertheless, we proceed and add the other chirality $\gamma^5 = -1$, giving

$$S_{\text{eff}}[A] = \frac{1}{8\pi^2} \oint dp_z [\text{sgn } m_-(p_z) - \text{sgn } m_+(p_z)] \int d^4x \ \epsilon^{z\mu\nu\lambda} A_\mu \partial_\nu A_\lambda$$
$$= \frac{1}{8\pi^2} \int d^4x \ \epsilon^{z\mu\nu\lambda} 2p^{(0)}_{\star z} A_\mu \partial_\nu A_\lambda. \tag{31}$$

Surprisingly, this integral is finite. More carefully, we regularize the $d^3p_\perp, dp_z$ integrals, e.g. by including by including the two-dimensional doublers with mass $O(r)$ from (1) and (2), and obtain the finite 2+1d Hall conductance (23). We could pick either sign, which differ by a sign and a translation by a reciprocal lattice vector, corresponding to additional 3+1d integer QH states, cf. (44) and [69]. Here the upper sign occurs, yielding an identical answer to (31). In general, this response can be written as

$$\boxed{S_{\text{eff}}[A] = \sum_{i \in \text{nodes}} \frac{\chi_i p_{ia}}{8\pi^2} \int e^a \wedge A \wedge dA} \tag{32}$$

where $p_{ia} e^a = p_{ia} e^a_\mu dx^\mu$ is along the node vector p_i. Moreover, noting that $p^{(0)}_{\star\mu}$ couples like $A_{5\mu}$, (31) gives the (covariant) axial anomaly (A.8). The coefficient is

$$\sum_i \frac{\chi_i p_{ia}}{8\pi^2} = \frac{N_{a3}(p)}{8\pi^2} = \int \frac{dp_a}{2\pi} \int \frac{d^3p_\perp}{24\pi^2} \text{tr}[(G_0 dG_0^{-1})^3]. \tag{33}$$

$N_{a3}(p)$ is the invariant that counts the right-handed minus left-handed nodes $G_0^{-1}(\omega = 0, p) = 0$ along p_a [1]. Physically this is the Chern number or Berry flux [4] between the Weyl nodes at $\pm p_\star$, where each section is a 2+1d integer quantum Hall state, cf. (23) and [70, 69]. The relation of (32) was linked to the chiral anomaly [17] in [71, 72] and implies the existence of chiral boundary modes connecting the projections of the nodes at surface BZ called Fermi arcs. It also implies that in the presence of screw dislocations, there must be dislocation bound Fermi arc-like modes in order to cancel the gauge non-invariance for $T^a \neq 0$ [73, 68, 74].

5.2. Nieh-Yan torsional anomaly in a TRB WSM

The AQHE response features the elastic tetrad field as a "spectator" and implies dislocation bound zero modes. A natural question is whether and intrisically torsional response related to the chiral Nieh-Yan anomaly [75] is possible [38, 76, 49]. To derive this, assume the model (30) with non-trivial $e^a = e^{(0)a} + \delta e^a$. The response is

$$S_{\text{eff}}[e^a] = \sum_{i \in \text{nodes}} \frac{i}{2} \text{Tr}[\frac{1}{\not{p} - \not{p}_i} \sigma^a \delta e^\mu_a p_\mu \frac{1}{\not{p} - \not{p}_i} \sigma^b \delta e^\nu_b p_\nu] \tag{34}$$

Again, we use $\sigma^a \delta e^\mu_a p_\mu = -\sigma^\mu \delta e^a_\mu p_a$ and arrive to

$$S_{\text{eff}}[e^a] = \sum_i \frac{i}{2} \text{Tr}[\frac{p_a p_b}{(p - p_i)^4} (\not{p} - \not{p}_i)(-i\not{\partial} \delta \not{e}^a \delta \not{e}^b)] \tag{35}$$

Performing the matrix traces, the momentum space integral of interest is of the form

$$-\operatorname{Tr}\left[\frac{p_{\perp a}p_{\perp b}+p^2_{\|a,b}}{[p_\perp^2+(p-p_i)^2_\|]^2}(p-p_1)_\|\delta_{\mu,\|}e^a_\nu\partial_\lambda e^b_\rho\right]i\epsilon^{\mu\nu\lambda\rho}\chi_i \tag{36}$$

where χ_i is the chirality of the node i. The first and second terms are proportional to

$$\begin{aligned}\int\frac{d^3p_\perp}{(2\pi)^3}\frac{p_{\perp a}p_{\perp b}(p-p_i)_\|}{[p_\perp^2+(p-p_i)^2_\|]^2} &= \frac{4\pi}{(2\pi)^3}\int_0^{\Lambda_\perp}dp_\perp\frac{p_\perp^4\eta_{ab}/3(p-p_i)_\|}{[p_\perp^2+(p-p_i)^2_\|]^2}\\ &= \left[\frac{\Lambda_\perp(p-p_i)_\|}{6\pi^2}-\operatorname{sgn}\frac{(p-p_i)^2_\|}{8\pi}+O(m/\Lambda_\perp)\right]\eta^\perp_{ab}\end{aligned} \tag{37}$$

$$\int\frac{d^3p_\perp}{(2\pi)^3}\frac{p^2_\|(p-p_i)_\|}{[p_\perp^2+(p-p_i)^2_\|]^2} = \frac{4\pi}{(2\pi)^3}\int_0^\infty dp_\perp\frac{p_\perp^2p_\|^2(p-p_i)_\|}{[p_\perp^2+(p-p_i)^2_\|]^2}=\operatorname{sgn}\frac{p_\|^2}{8\pi} \tag{38}$$

in the directions $p_\perp$ perpendicular to p_i, where $\Lambda_\perp \gg (p-p_i)_\|$ is a UV-cutoff and $\operatorname{sgn} \equiv \operatorname{sgn}(p-p_i)_\|$. These are diverging and need regularization. Integrating over the node direction, we are left with

$$\int_{-\Lambda_\|}^{\Lambda_\|}\frac{dp_\|}{2\pi}\frac{\Lambda_\perp(p-p_i)_\|}{6\pi^2}=-\frac{p_i\Lambda_\perp\Lambda_\|}{6\pi^3}\to 0, \tag{39}$$

$$\int_{-\Lambda_\|}^{\Lambda_\|}\frac{dp_\|}{2\pi}\frac{\operatorname{sgn}}{8\pi}(p-p_i)^2_\|=\frac{p_i\Lambda_\|^2}{8\pi^2} \tag{40}$$

$$\int_{-\Lambda_\|}^{\Lambda_\|}\frac{dp_\|}{2\pi}\frac{\operatorname{sgn}}{8\pi}p^2_\|=\frac{p_i\Lambda_\|^2}{8\pi^2}+\text{even in } p_i \text{ terms} \tag{41}$$

where $\Lambda_\|$ is a UV cutoff along the Weyl node vector p_i. Performing the integrals, we used the regularized 2+1d Hall viscosity result (23) for (37) and (38). In general the cutoff $\Lambda=\Lambda_\|$ is dictated by the validity of the linear model (30), even though momentum integrals are finite on the lattice with $\Lambda\sim 1/a$. Summing over the nodes, this leads to

$$\boxed{S_{\text{eff}}[e^a]=\frac{\Lambda^2\eta_{ab}}{8\pi^2}\sum_i\int ed^4x\;\epsilon^{\mu\nu\lambda\rho}\chi_ip^{(0)}_{i\mu}e^a_\nu\partial_\lambda e^b_\rho=\sum_i\frac{\Lambda^2}{4\pi^2}\int p_{ia}e^a\wedge N} \tag{42}$$

where in terms of the full Nieh-Yan form $N=e^a\wedge T_a$ [75].

Noting $p^{(0)}_{i\mu}$ plays also the role of the axial gauge field $A_{5\mu}$, this effective action implies a non-relativistic version of the covariant torsional Nieh-Yan anomaly (A.13) [75, 76, 49, 32]. In terms of torsional Hall viscosity [38, 31, 35, 49], it implies that $d\star\mathcal{J}_a=\frac{p_{\star a}\Lambda}{4\pi^2}dN$ for $\mathcal{J}_a=\frac{\delta S_{\text{eff}}}{\delta e^a}$, cf. Eq. (A.13). Interestingly, by antisymmetry of $e^a\wedge e^b$, the non-linear effective action is zero if e^a is non-zero only along p_{Fa}. The ensuing 3+1d anomaly, however, can be then derived from a $e^a\wedge dN$ term in 4+1d, mimicking the 4+1d $U(1)$ CS term for e^a. Finally, Note however that the expression (42) rests on Lorentz breaking symmetries, making the generalization to the RHS of (A.13) not in general well-defined. For more discussion see [49].

5.3. Related crystalline insulators in 3+1d

Finally, I discuss how to connect the results (32), (42) to results on other 3+1d topological states. The TRB WSM is an intermediate state between two time-reversal breaking insulators, the trivial and weak 3+1d topological Chern insulator. Namely the two Weyl nodes can be gapped with mass m and annihilated when they overlap in the BZ. The trivial insulator is obtained when the nodes meet at $p_\star = 0$, whereas the non-trivial state follows when the nodes meet at the zone boundary as $\pm p_\star \to \pm\pi/a$. Effectively this a chiral rotation equal to $e^{i\boldsymbol{G}_a\cdot\boldsymbol{x}/2}$ along $p_{\star a}$, where $G_a = 2\pi/a$ is a reciprocal lattice vector. Then, see the Appendix,

$$S_{\text{eff}}[e^a, A] = \sigma_{Ha}\int e^a \wedge A \wedge dA + \eta_{Ha}\int e^a \wedge N \tag{43}$$

where now, m being the insulating gap,

$$\sigma_{Ha} = \frac{G_a}{8\pi^2}, \quad \eta_{Ha} = G'_a\frac{\Lambda^2}{8\pi^2} + G''_a\frac{m^2\log(\frac{\Lambda}{m})}{8\pi^2}. \tag{44}$$

These are the 3+1d Hall conductivities [77, 69, 78] and viscosities [38, 31] with in general integer multiplicities of elementary reprocical lattice vectors G_a, G'_a, G''_a and protected by crystalline symmetries like $p_{\star a}$. Note that due to the mass term, there is additional logarithmic term which was studied in [68]. Our results imply that $G_a = G'_a = G''_a$, i.e. the coefficient are equal up to the unknown scales Λ^2 and $m^2\log(\frac{\Lambda}{m})$ for the simple TRB Weyl model. The continuity of both σ_H and η_H is similar to the continuity the torsional Hall viscosity (23) in terms of the gap/mass parameter m. In contrast, from 2+1d parity anomaly, there is a jump for the Hall conductivity in 2+1d. Here it is removed by the fact that in 3+1d, the anomaly is integrated along the the third direction $\boldsymbol{p}_\star \sim \boldsymbol{G}_a$, signalling weaker crystalline protected topology [3] (the preferred lattice direction).

6. Relation to other recent work

With the main results Eqs. (23) [38], (29) and (42) [49] involving torsion, their relation to (44), I now discuss some overlapping results from the literature. Torsion in topological matter has been discussed in many references, e.g. [43, 44, 79, 38, 80, 32, 50, 81]. These feature different results and models, from strictly relativistic models to non-relativistic systems similar to this review. A common approach to elastic deformations involves so-called pseudo-gauge fields [37, 82], related to torsion by (4) and (5). These are nothing else that the translational gauge field $\tilde{A}_i \equiv p_{\star a}\delta e^a_i$, due to the elastic deformation. This gauge field introduces the UV cutoff $p_\star$, which here is replaced by Λ with the assumption that $\Lambda \ll p_\star$ due to the validity of the linear expansion $|p - p_\star| \ll p_\star$. The $\tilde{A}$ becomes independent from the elastic geometry only if we allow deformations $\delta p_{\star a}(x)$ of the Fermi momentum independent from δe^a. While this is certainly feasible, this was not considered here. Instead, the Fermi node momentum $p_{\star a}$ was a constant UV parameter of the low-energy theory, and also subject to elastic deformations in terms of non-trivial e^a. In the Ref. [52], it was moreover shown that the correct expansion of the continuum theory is around the original, undeformed Fermi point $p_{\star a}$. Finally, the recent paper [53] discussed precisely this UV sensitivity in lattice model but without torsion. They also took the spin-connection to depend strictly on e^a, in contrast to Eq. (10). Their result was that due to the UV parameter $p_\star \sim 1/a$, the consistent truncation of the theory is in terms of the translational gauge field $\tilde{A}_i$, while all other terms are small in gradient expansion. This is precisely what we here discussed in terms of emergent torsion of (4), (5) and is absent in relativistic models, since a non-zero $p_{\star a}$ breaks Lorentz symmetry. Indeed, for the continuum limit derived from lattice, one usually needs to assume $p_\star \ll 1/a$ so that the linear expansion remains valid in the presence of deformation.

Here I discussed lattice systems (1) with massive doublers as regulators, while torsional emergent geometry emerges in directly in many-body continuum systems in topological superfluids (and superconductors) [49]. Namely, the torsional anomaly is realized in chiral

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Weyl superfluid ^{3}He-A and the anomaly coefficient has been experimentally verified [49, 83], see also [33, 1]. Finally, in relativistic systems, the diverging torsional anomaly term (A.12) can be removed by a counter term without breaking any additional symmetries. In non-relativistic topological matter, this is determined by the UV completion and UV cutoff parameters Λ are well-defined a priori. Relativistic models with UV torsional terms were discussed also in [84]. Notwithstanding, at finite temperatures and chemical potentials, relativistic and non-relativistic systems feature anomalous currents from torsion [85, 86, 87, 88, 50], similar to the chiral vortical effect induced in hydrodynamics by chiral QFT anomaly [23, 29, 27]. These were discussed in the Ref. [28] to which we guide the reader. The non-relativistic anomaly terms are similar to these in that they feature UV parameters in the hydrodynamic responses.

7. Conclusions and Discussion

I reviewed how geometry and torsion enters the low-energy field theories corresponding to non-relativistic topological matter. The torsional background geometry is provided by the continuum elasticity with dislocations (and disclinations) and is emergent, i.e. the ambient geometry is flat and the fields are provided by the surrounding material medium. In particular, the torsion couples to the finite node momenta $p_\star$ allowed in non-relativistic systems. The ensuing torsional responses include 2+1d Chern insulators and 3+1d Weyl/Dirac semimetals and their descriptions in terms of the parity and chiral anomalies, respectively. The main results are Eqs. (23), (29) and (42). These all feature non-universal UV parameters $p_\star$ and Λ, needed for torsional anomalies. Along the way, I compared and highlighted the differences of non-relativistic continuum models of topological matter to relativistic field theory and lattice models. Given a (realistic) lattice model, the anomalies should be possible to compute explicitly and compared with experiments, much like [49, 83] in chiral Weyl superfluid ^{3}He-A. Interestingly, the torsional anomalies are not sharply quantized and can be non-zero even in the trivial phases [31]. This translates to the non-universal dimensionful coefficients needed for torsion and the associated non-quantized stress-energy-momentum transport, as compared to gauge fields and charge transport. The mixed torsional responses with electromagnetic fields are interesting, since e.g. the AQHE requires Fermi arc states at all momenta $-p_\star \leq p \leq p_\star$ [69], potentially restricting also purely torsional transport coefficients as in Eq. (23) and the related expressions (32), (42) in 3+1d. An immediate consequence of the mixed torsional responses is that they modify the electro- and elastodynamics of these materials [31, 81, 64] and are inherently tunable due to the non-universal, "unquantized" coefficients, yet being protected by topology.

I should note that the relativistic torsional terms have been long controversial and relativistically, only recently elucidated in the hydrodynamic form at finite temperature and chemical potentials, see e.g. [87] and references therein. The results discussed here can, however, be linked to other more well-defined anomalous responses and, importantly, are well-defined in non-relativistic models with cutoffs to the low-energy effective theory in contrast to relativistic QFT. The responses were derived from the semiclassical expansion and correspond to hydrodynamic responses sensitive to QFT anomalies. They essentially follow from the chiral anomaly for U(1) fields with some important differences related to the dimensionless nature of e^a, its coupling to momentum p_a, and the presence of extra UV coefficients, present in any realistic model for non-relativistic topological matter. Here the effects concretely followed from the finite Fermi momenta in the BZ and anisotropic lattice symmetries. I focused on torsion, whereas related crystalline curvature terms were discussed in [48].

Put differently, the characteristic of torsional terms is their sensitivity to momentum space topology and geometry. The torsional terms (29), (42) couple to the chiral Weyl dipole and it is possible to describe similar terms for other types of momentum space multipole charges as well, providing different crystalline symmetries unique anomalous responses in terms of the emergent crystalline geometry. For so-called higher-order insulators and semimetals, see e.g. [19], even more terms are possible and will likely involve combinations of several different responses related to the emergent geometry. Notwithstanding, any true (hydrodynamic) anomaly term should come with its respective anomaly polynomial and descent relations that quantifies the impossibility of realizing or gauging all symmetries in the quantum theory [25, 26]. Here we just directly derived the effective actions in 3+1d, utilizing knowledge of the familiar 3+1d chiral anomaly and parity anomaly in 2+1d. The extension of the effective actions in terms of anisotropic 4+1d Chern-Simons like terms with vielbeins and torsion is desirable. In general, the elastic gauge symmetries $e^a \to e^a + du^a$ are related to higher-form symmetries that enumerate different lattice directions, planes and (hyper)surfaces [89]. The associated responses require torsional geometric gauge theories with translational and rotational fields in a non-relativistic and crystalline setting, see e.g. [78, 64, 90, 48, 63, 89, 62]. These responses, symmetries, anomalies and their anomaly polynomials in topological matter will be discussed elsewhere, see [84] for discussion for a relativistic model.

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Appendix A. The chiral anomaly

Here we review the chiral anomaly term to introduce the standard anomaly term and fix notations. The following sections detail the anomaly and the semi-classical expansion.

The Dirac action coupled to a axial mass term, U(1) gauge field and gravity is

$$S_{\mathrm{Dirac}}[\overline{\psi},\psi] = \int d^4x e \left[\frac{1}{2}\overline{\psi}\gamma^a e^\mu_a iD_\mu\psi + \text{h.c.} - \overline{\psi}M(\phi)\psi\right]$$

where e^μ_a is the inverse vielbein, the metric is $e^a_\mu e^b_\nu \eta_{ab} = g_{\mu\nu}$ and $D_\mu = \partial_\mu + iA_\mu + i\hat{\omega}$, cf. (13) and (14). The axial mass term is $M(\phi) = me^{i\phi\gamma^5} = m(\cos\phi + i\gamma^5\sin\phi)$ where $0 \leq \phi < 2\pi$ is a parameter.

In general, the axial current is not conserved but suffers from quantum anomalies,

$$\partial_\mu(ej^\mu_5) = 2iem\overline{\psi}\gamma^5\psi + \text{anomaly}(\phi) \ . \tag{A.1}$$

The mass term explicitly breaks the chiral symmetry, while the anomaly term is proportional to ϕ, since naively $M(\phi) \to m$ via axial gauge transformations $\delta A_5 = mi\partial_\mu\phi/2$. This breaks the conservation law by quantum fluctuations and classifies topological responses [5]. Alternatively, the anomaly term is given as the Jacobian of the path-integral measure, cf. [59],

$$\ln\mathcal{D}[\overline{\psi},\psi] \to \ln\mathcal{D}[\overline{\psi},\psi] + S_{\mathrm{anom}}[\phi], \tag{A.2}$$

where

$$S_{\mathrm{anom}}[\phi] = \phi\left[\frac{q^2}{8\pi^2}\int F\wedge F + \frac{1}{192\pi^2}\int \mathrm{tr}\, R\wedge R\right]. \tag{A.3}$$

The trivial insulator and topological insulator are found for $m > 0$ and $m < 0$, and cannot be continuously rotated while preserving the symmetries. We can start with the mass $M(\phi = 0) = m$ and consider the chiral rotation $\psi \to e^{i\phi/2\gamma_5}\psi$ with $m \to M(\phi)$. The axial mass $M(\phi)$ adiabatically connects e.g. the trivial TRI insulator with $m > 0$, i.e. $\phi = 0$ to the topological TRI insulator with $m < 0$, i.e. $\phi = \pi$ by breaking the TR symmetry for $\phi \neq 0, \pi$ [21].

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Appendix A.1. U(1) and gravitational anomaly terms from the semiclassical gradient expansion
We can derive the gauge and gravitational anomaly contributions in a mechanical fashion by using the gradient expansion (15) and $M(\phi) \approx m(1 + i\phi\gamma^5)$. The relevant terms give

$$S_{\text{eff}}[A] = -i\,\text{Tr}[\frac{1}{\not{p} - m} q\not{A} \frac{1}{\not{p} - m} q\not{A} \frac{1}{\not{p} - m} im\phi\gamma^5] \tag{A.4}$$

separating momentum and coordinate space traces gives

$$4i^5 q^2 m^2 \phi \epsilon^{\mu\nu\lambda\rho}\,\text{Tr}[\frac{1}{(p^2 - m^2)^3}\partial_\mu A_\nu \partial_\lambda A_\rho] = \frac{\phi q^2}{8\pi^2}\int d^4x \epsilon^{\mu\nu\lambda\rho}\partial_\mu A_\nu \partial_\lambda A_\rho, \tag{A.5}$$

in agreement with Eq. (A.3). For the gravitational term, we can evaluate $S_{\text{eff}}[\omega]$ in the special case where the spin-connection $\omega^{ab} = \omega^{ab}_\mu dx^\mu$ and $R^{ab} = d\omega^{ab} + \omega^a{}_c \wedge \omega^{cb}$ are non-trivial but the vielbein e^a_μ and metric are constant. To lowest order, the terms in $R \wedge R$ are

$$-\frac{i}{3}\,\text{Tr}[\frac{1}{\not{p} - m}\tfrac{1}{2}\not{\omega}^{ab}\sigma_{ab}\frac{1}{\not{p} - m}\tfrac{1}{2}\not{\omega}^{cd}\sigma_{cd}\frac{1}{\not{p} - m} im\phi\gamma^5]$$
$$= \frac{i^3 m^2 \phi}{3}\frac{4}{2^2 4^2} 2^3 \epsilon^{\mu\nu\lambda\rho}\,\text{Tr}[\frac{1}{(p^2 - m^2)^3}\partial_\mu \omega^{ab}_\nu \partial_\lambda \omega_{\rho ba}] = \frac{\epsilon^{\mu\nu\lambda\rho}}{192\pi^2}\int d^4x\,\text{tr}(\partial_\mu\omega_\nu\partial_\lambda\omega_\rho). \tag{A.6}$$

This term gives directly the 2nd order term of $\frac{1}{192\pi^2} R \wedge R$ in (A.3). Covariant and consistent mixed gravitational anomalies differ by an arbitrary multiplicative coefficient, depending whether the mixed anomaly is derived from anomaly polynomial with either unbroken gauge or diffeomorphism transformations or their mixture. In (A.6) we computed just the coefficient of the lowest order term.

Appendix A.2. U(1) axial anomaly
Finally, to gain more insight to the derivative expansion, let's evaluate the contributions to the effective action related to the axial anomaly. The axial U(1) coupling is $\not{A}_5 \equiv \gamma^\mu\gamma^5 A_{5\mu}$ and for the purpose quickly obtaining the anomaly, we simply replace $i\partial_\mu\phi/2 \sim A_5$ in (A.5),

$$S^{(m)}_{\text{eff}}[A, A_5] = \frac{q^2 \epsilon^{\mu\nu\lambda\rho}}{4\pi^2}\int d^4x A_{5\mu} A_\nu \partial_\lambda A_\rho, \tag{A.7}$$

$$d \star J_5 = \frac{q^2}{4\pi^2} F \wedge F. \tag{A.8}$$

where $d \star J_5 = \partial_\mu(e j^\mu_5) d^4x$. In detail, this effective action is derived from the five-dimensional analog of (A.4) with the replacement of $i\partial_\mu\phi/2 \sim A_5$, since the 4+1d Chern-Simons term is the only local effective action that can produce the anomaly in 3+1d [11, 25, 26]. This action is gauge invariant up to boundary terms and produces the consistent anomaly in 3+1d; when also the contribution from the boundary fermions is added, the covariant anomaly (A.8) follows [11, 26]. We recall that the anomaly is determined by the m-independent contribution of finite m terms, in accord with the fact that the anomalous Ward identity arises from the regularization dependent cancellation of linearly diverging diagrams or, equivalently, from the contribution of Pauli-Villars regulator fermion introduced to cancel the diverging terms. This will be different for torsion, see (A.13).

Appendix A.3. Semiclassical expansion with torsion

Let us now assume $\hat{\omega} = 0$ and non-trivial torsional tetrad $e^a_\mu = e^{a(0)}_\mu + \delta e^a_\mu(x)$, where $\mathrm{e}^{(0)a}_\mu$ are flat space vielbein. To first order, the distinction between coordinate and local orthonormal indices of $\gamma^\mu = e^\mu_a \gamma^a$ or $p_a = e^\mu_a p_\mu$ is not important. The anomaly term is now

$$-i\,\mathrm{Tr}[\frac{1}{\not{p}-m} i\phi\gamma^5 \frac{1}{\not{p}-m}\gamma^a \delta e^\mu_a p_\mu \frac{1}{\not{p}-m}\gamma^b \delta e^\nu_b p_\nu] \tag{A.9}$$

We use the relation $\gamma^a \delta e^\mu_a p_\mu = -\gamma^\mu \delta e^a_\mu p_a = \delta \not{e}^a p_a$. The relevant term is UV diverging,

$$S_{\text{eff}}[e^a, \phi] = -4i\; m^2 \phi \epsilon^{\mu\nu\lambda\rho}\,\mathrm{Tr}[\frac{p_a p_b}{(p^2-m^2)^3}\partial_\mu e^a_\nu \partial_\lambda e^b_\rho] = \frac{-\phi m^2 \log(\frac{\Lambda}{m})}{8\pi^2}\int d^4x \epsilon^{\mu\nu\lambda\rho}\frac{1}{4}T^a_{\mu\nu}T_{a\lambda\rho}. \tag{A.10}$$

where we used the fact that the extra momentum factors p_a, p_b need to contract pairwise to η_{ab} for the desired term. Now the anomaly induced axial mass term depends explicitly on m^2 and logarithmically on the UV cutoff Λ.

We derived the responses assuming vanishing spin-connection. By Lorentz invariance, the Nieh-Yan form $N = e^a \wedge T^a$ with $dN(e^a, \omega_{ab}) = T^a \wedge T_a - R_{ab} \wedge e^a \wedge e^b$ enters. The axial anomaly terms becomes the same with logatrithmic divergence

$$S^{(m)}_{\text{eff}}[A_5, e^a] = \frac{-m^2 \log(\frac{\Lambda}{m})}{4\pi^2}\int A_5 \wedge e^a \wedge T_a. \tag{A.11}$$

this term was studied in [68] and replaces the term (A.7) for gauge fields. Moreover, disctinct from gauge fields, the m-independent torsional anomaly term is non-zero and quadratically diverging. Similarly, the massive 4+1d Chern-Simons like effective action from five dimensions is linearly UV diverging. The divergence can be cancelled with appropriate regularization, e.g. Pauli-Villars fields. The resulting finite coefficient is proportional to the five-dimensional mass-squared m^2_{UV}, an UV scale for the four-dimensional fermions. In a condensed matter setting, this represent a higher-dimensional topological insulator with gap $m_{\text{UV}} \sim \Lambda$ in terms of the boundary fermions, as related by anomaly inflow. Accordingly, gathering all diverging terms, we arrive to the Nieh-Yan anomaly term [75]

$$S_{\text{eff}}[A_5, e^a] = \frac{\Lambda^2 - m^2 \log(\frac{\Lambda}{m})}{4\pi^2}\int A_5 \wedge e^a \wedge T_a \tag{A.12}$$

with the anomaly in covariant form

$$d \star J_5 = \frac{\Lambda^2 - m^2 \log(\frac{\Lambda}{m})}{4\pi^2}(T^a \wedge T_a - R_{ab} \wedge e^a \wedge e^b). \tag{A.13}$$

Notice the dimensionful and non-universal coefficient, needed due to the canonical dimensions of e^a. The above terms are anomalies from geometry (i.e. gravity), with UV sensitivity that break no other symmetries than the already anomalous A_5. Moreover, if Lorentz invariance holds to arbitrary scales for Dirac/Weyl fermions, $\Lambda \to \infty$ in vacuum. Therefore, they can and should be subtracted with counter terms [29], unlike the finite gauge or gravitational anomalies. However, at non-zero chemical potential and temperature, torsional contributions are possible even in relativistic systems [28]. In contradistinction, for condensed matter systems there is the possibility of anisotropic Lorentz breaking terms and UV completion of the (quasi)relativistic fermionic theory, where Λ is a well-defined and finite cutoff parameter [38, 49].

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