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THERMODYNAMICS OF STRONG INTERACTIONS AT HIGH ENERGY AND ITS CONSEQUENCES FOR ASTROPHYSICS *)

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ABSTRACT

Statistical thermodynamics in a particular form derived from high energy physics is used to describe the thermodynamical properties of what might have been our universe before its energy density became much lower than nuclear density. The main features are :

* even if it started with infinite energy density, it never had a temperature greater than $T_0 = 160$ MeV (1.86×10^{12} °K), which is the universal highest temperature in this theory;
* for very large energy density the pressure is not, as in usual theories, proportional to the energy density but only to its logarithm;
* inside each elementary volume V_0 ($\approx$nucleon volume) the energy fluctuates by an amount of the order of the total energy contained in V_0. For infinite energy density this fluctuation does not vanish as in ordinary theories, but tends to $\Delta E/E \sim 0.4$. The conjecture is proposed that smaller but still substantial fluctuations of the baryonic quantum number may go along with the energy fluctuations.

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1. INTRODUCTION

Methods of statistical thermodynamics have recently been applied with success to high-energy interactions of hadrons. It seems that the same methods can be applied to macroscopic situations as they may appear in astrophysics under extreme circumstances.

The results presented here will modify, in particular, the behaviour of "the early universe". The consequences for very dense matter in the ordinary sense (superdense stars) are not yet clear. The early universe, to which the present thermodynamics would apply, would be one with total baryon number zero (or at least very small compared to the total number of baryons + antibaryons). It would be a universe that can be described as a very general black-body radiation containing all sorts of particles including hadrons as the most essential ingredient.

We shall use natural units $\hbar = c = k$ (Boltzmann's constant) = 1. A table in the Appendix gives the conversion between these units and CGS units for some of the more interesting quantities. Here is just an illustration:

$$\text{density: } 1 \text{ g cm}^{-3} \leftrightarrow 4.311 \times 10^{-6} \text{ MeV}^4 \qquad (1)$$

so that the density of a nucleon, $m_p/V_0 \approx 6 \times 10^8$ MeV^4, corresponds to about 1.4×10^{14} g cm^{-3}. Here we made use of the "nucleon volume" $V_0 = (4\pi/3)\, m_\pi^{-3} = 1.59 \times 10^{-6}$ MeV^{-3} which will turn up frequently in the following.

In the new accelerator at Serpukhov (Russia), protons of 70 GeV hit others at rest; there, during $\approx 10^{-23}$ sec, the total energy density in the centre-of-mass frame of the collision reaches twelve times that of the proton at rest. In cosmic-ray collisions, densities a hundred times greater occur. We try to describe what happens in such situations.

In these collisions two things are mixed up with each other: a particular kind of thermodynamics and strong collective motions along the collision axis. While the latter are of no interest in astrophysical applications, we must mention them here because they have been used -- and are being used -- as an argument against employing statistical thermodynamics to such processes. If that were so, then we could not reach any conclusion about what kind of thermodynamics applies. What I wish to affirm is that these collective motions have only the effect of Lorentz-transforming isotropic thermodynamic momentum distributions into the observed very forward-backward peaked ones. There does not exist any theory that allows one to calculate the velocity distribution of this collective motion from first principles, but let us be content with the fact that such a velocity distribution can be described phenomenologically, and that therefore we have a means of disentangling thermodynamics and collective motions.

Incidentally, the simplest way of disentangling them is to consider only quantities that are independent of the collective motion, such as

* multiplicities of various kinds of particles;
* distribution of the transverse components of the momentum of the particles produced in the collision.

The very fact that particles -- at not very high energies, mainly pions -- are abundantly produced has led Koppe[1] in 1948 to apply statistical thermodynamics to the production process. Two years later, Fermi[2] published something similar, and for that reason one speaks not of the "Koppe statistical model" but of Fermi's[*]. These models and many later refinements were rather successful in those features that do not depend on collective motions. The most important of these refinements was that excited states of baryons[3] and mesons[4] had to be treated as particles -- in spite of their short lifetime. In our present thermodynamics we go even further in admitting all excited states of hadrons, including those not yet discovered. This thermodynamics, which is a consequent extrapolation of earlier ones (including phase-space models), explains very simply and in agreement with experiments the behaviour of the above-mentioned multiplicities and transverse momentum distributions. The latter are of primary interest because the transverse momentum distribution follows closely a Boltzmann distribution (which is already a strong hint towards thermodynamics) $\sim \exp(-p_{\perp}/T)$. The difficulty only is that one should expect T to grow with the fourth root of the total collision energy as in a decent black-body radiation; but instead, accumulated experience over the last 10 years shows that T tends rapidly to a limiting value T_0 of the order of 160 MeV, when the energy goes to the highest values hitherto encountered in cosmic-ray experiments. Already at 30 GeV we have $T \sim 120$ MeV.

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*) I am grateful to V. Telegdi for having made me aware of that historical injustice.

The multiplicities again follow a law $\sim \exp(-m/T)$, as one expects in a situation where thermodynamics governs; but here again: $T \to T_0$.

If thermodynamics can be applied at all -- and this is postulated on the basis of the experimental evidence -- then it has to explain the above-mentioned features. It turns out that the "limiting temperature" is the simple consequence of taking all resonance states into account as if they were particles. This has far-reaching consequences for the equation of state. It also leads to the conjecture that there are no elementary hadrons, but that each one consists of all others (at least for $m \to \infty$).

In this paper we cannot enter into details about foundations of the new thermodynamics. The whole theory has been worked out and applied to high-energy in physics in five papers[5]. These are

"Statistical Thermodynamics of strong interactions at high energies I, II*), and III" (henceforward referred to as I,II,III).

"On the hadronic mass spectrum".

"Hadronic matter near the boiling point"; this latter and the first chapter of II can serve as an easily readable introduction.

2. THE THERMODYNAMICAL MODEL OF STRONG INTERACTIONS

In astrophysical applications we can forget about the collective motions. For the moment we shall concern ourselves with hadronic black-body radiation, and only at the end add the contributions due to weak and electromagnetic interactions. In our hadronic black-body radiation, all kinds of hadrons are produced in just the same way as light quanta are produced in the usual electromagnetic black-body radiation. Of course we shall employ relativistic statistical mechanics so that the energy of a particle with mass m is $\sqrt{p^2 + m^2}$. We shall use the Gibbs canonical ensemble, and the main tool will be the partition function $Z(T,V)$, which is related to the free energy $F(T,V)$ by

$$F(T,V) = -T \ln Z(T,V) . \qquad (2)$$

The numbers N_γ of particles of the kind γ are not given numbers but are determined by the thermal equilibrium, therefore

*) Together with J. Ranft.

$$\left(\frac{\partial F}{\partial N_\gamma}\right)_{T,V} \equiv \mu_\gamma = 0 \tag{3}$$

That is, all chemical potentials for all kinds of particles are zero (our chemical potentials are relativistic; calling the ordinary ones μ_0, ours are $\mu_\gamma = \mu_0 + m_\gamma = 0$ which corresponds to $\mu_{0\gamma} = -m_\gamma$ in the ordinary case). Putting all $\mu_\gamma = 0$ leads immediately to baryon conservation, since $m_N = m_{\bar{N}}$. Similarly, strangeness and charge are conserved. Hence, total baryon number = total strangeness = total charge = 0. All these conservation laws are fulfilled only in the average, because "using" chemical potentials and putting them zero means treating particle numbers on the level of a canonical ensemble just as if they were on the same footing as energy. One can deal with particle numbers also on the micro-canonical level by requiring exact conservation laws, for instance if one wishes to calculate production rates; this has been done in (III).

In order to calculate the partition function, we need a model of the hadronic black-body radiation. It is the following: we treat the particles present in V as free, non-interacting particles. The whole of strong interaction dynamics then enters in just two conditions:

* Particles can be created or absorbed spontaneously and instantaneously according to the local conditions; thus heat need not travel via kinetic energy transport, as in a classical gas, but can propagate via virtual particles with the velocity of light.
* The kinds of particles -- uniquely identified by their mass -- that are allowed to participate are exactly those that are the massive eigenstates of strong interactions: pions, kaons, nucleons, hyperons, and all - all! - their resonant states.

By allowing spontaneous creation and annihilation of all possible hadrons, we have eliminated the strong interactions from our partition function and can now deal with undetermined numbers of all kinds of free hadrons. We have, of course, to know the allowed kinds of particles, namely the hadronic mass spectrum $\rho(m)\,dm$, which says how many different states there are between m and m + dm. Once this spectrum is known, the partition function can be calculated (I):

$$Z(V,T) = \sum_{\text{all states } \psi} \exp(-E(\psi)/T) = \prod_{\alpha\gamma} \left[\sum_{\nu_{\alpha\gamma}} x_{\alpha\gamma}^{\nu_{\alpha\gamma}} \right]$$

$$= \exp\left\{ \frac{VT}{2\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \int_0^{\infty} \rho(m;n)\, m^2 K_2\left(n \frac{m}{T}\right) dm \right\} \quad (4)$$

$$\rho(m;n) = \rho_{Bose}(m) - (-1)^n \rho_{Fermi}(m)$$

α labels the momenta and γ the kinds of particles; $x_{\alpha\gamma} \equiv \exp(-\sqrt{p_\alpha^2 + m_\gamma^2}/T)$
$\nu_{\alpha\gamma}$ = occupation numbers; $0 \ldots \infty$ for bosons, 0,1 for fermions.

From this partition function we get all interesting quantities along standard rules.

What, then, is the hadronic mass spectrum on which everything now depends?

For the low-lying states it is known experimentally; in recent years the resonance hunting has yielded an ever-growing population of states, but below 1000-1200 MeV this population is now constant. Above this region new states continue to be discovered, and the heaviest ones reach a mass of more than 3000 MeV in the baryon case and more than 2000 MeV in the meson family. No known principle of physics stands against the possibility that this goes on and on -- in fact we have every reason to believe that it goes on, since generally in nature everything that is not explicitly forbidden is realized. Let us now remember the object, which we wish to describe by statistical thermodynamics as hadronic black-body radiation: it is physically nothing other than a lump of highly excited hadronic matter staying together just long enough, in a small enough space region, to allow the strong interactions to establish an equilibrium between their possible modes of existence (asymptotic states with definite particle numbers).

Thus our very postulate that statistical thermodynamics can be applied implies that we believe in the existence of such highly excited lumps of hadronic matter, which then decay within $\sim 10^{-23}$ sec according to just the statistical laws derivable from the partition function (4). The lumps of matter at which we aim have masses from a few to a few hundred GeV and higher.

Now comes the main point: these lumps of hadronic matter -- we shall call them fireballs -- decay statistically into pions, kaons, nucleons, hyperons, and -- according to hypothesis -- into resonances of those. But so also do the high-mass resonances that we know experimentally; a glance at the tables published by the Particle Data Group[6)] shows this. We claim then that between the resonances and our fireballs there is no difference of principle, and that the latter are the natural continuation of the former. Now, calling all of them fireballs (including pions, nucleons, etc.), we formulate the basic postulate, valid in the limit of large mass:

A fireball is

→ a statistical equilibrium (hadronic black-body radiation) of undetermined numbers of all kinds of fireballs, each of which, in turn, is considered to be ↩ (P)

It turns out that this postulate, which interpolates between two established experimental facts, determines our thermodynamics completely.

This is seen by rewriting the partition function (4) in the simplified form (which contains all essentials)

$$Z(V,T) = \exp\left\{ \int_0^\infty \rho(m)\, F(m,T)\, dm \right\} \tag{5a}$$

with a known function F(m,T). On the other hand, every book on statistical mechanics tells us that one may also write

$$Z(V,T) = \int_0^\infty \sigma(m)\, e^{-\frac{m}{T}}\, dm \quad , \tag{5b}$$

where $\sigma(m)\, dm$ is the number of states of the "main" fireball between m and m + dm. Since $\rho(m)$ counts also the number of states between m and m + dm, these two functions must in some sense be the same, if our postulate (P) shall hold -- at least for $m \to \infty$. A detailed discussion [see (I)] shows that one can only require

$$\frac{\ln \rho(m)}{\ln \sigma(m)} \Rightarrow 1 \quad \text{for } m \to \infty \tag{6}$$

which says that the entropy of a fireball is a unique function of its mass, and that this function is the same for the main fireball as for its constituents; thus in the sense of thermodynamics all fireballs are alike, except for their mass. I like to call the postulate (P) "asymptotic bootstrap" because it says that there are no elementary hadrons; each one consists of all others.

The mathematical conclusions drawn from formulae (5a,b) and (6) are straightforward [cf. (I)]: from closer inspection we see that F(m,T) falls off asymptotically like $m^{3/2}$ exp (−m/T), so that

$$Z(V,T) \Rightarrow \exp\left[\int_0^\infty m^{3/2}\rho(m)e^{-\frac{m}{T}}dm\right] \Longleftrightarrow \int_0^\infty \sigma(m)e^{-\frac{m}{T}}dm \tag{7}$$

This is consistent with formula (6) if and only if both ρ and σ grow exponentially; in particular

$$\rho(m) \underset{m\to\infty}{\Longrightarrow} \frac{\text{const}}{m^{5/2}}\, e^{\frac{m}{T_0}} \tag{8}$$

Now we have everything we want: for low lying resonances the mass spectrum is experimentally known; for very large m we know its asymptotic behaviour. It is easy to guess a formula that has the right asymptotic form and even fits the actual (smoothed) mass spectrum down to m = 0; one such formula is

$$\rho(m) = \frac{a}{(m_0^2+m^2)^{5/4}}\, e^{\frac{m}{T_0}}$$

$$a = 2.63 \cdot 10^4\ \text{MeV}^{3/2};\qquad T_0 = 160\ \text{MeV};\qquad m_0 = 500\ \text{MeV} \tag{9}$$

Figure 1 shows how nicely this fits. One can read off the constants a and T_0 rather accurately (m_0 has no physical significance). The figure shows: the part of the spectrum ($\lesssim$ 1200 MeV) that is well known, indeed grows exponentially, and the higher part of the spectrum tries to improve its behaviour over the years.

If we now use this information, then we can calculate ln Z explicitly, and although it is apparently a partition function of free particles, the main properties of strong interactions are present in it because of the mass spectrum and its peculiar asymptotic behaviour.

The most striking consequence of formula (8) is, of course, that T_0 is the highest temperature allowed in this thermodynamics, because ln Z exists only for $T < T_0$ and diverges logarithmically when $T \to T_0$.

As strong interactions are virtually present everywhere, this T_0 is a universal highest temperature for equilibrium states (an electron-positron gas initially at $T > T_0$ will, after some time, create pions, etc., and then cool down to $T < T_0$; that is: it was not in an equilibrium state initially).

Our results are then:

a) The three propositions:

* there are no elementary hadrons; each of them consists of all others [postulate (P)];
* the hadronic mass spectrum grows exponentially;
* there exists a universal highest temperature ;

these three propositions are nothing other than three different wordings of one -- perhaps the -- basic property of strong interactions.

b) The partition function and all thermodynamic distributions derivable from it are now well-defined calculable functions of T and V.

3. CHECKING AGAINST EXPERIMENTS

In papers (II) and (III) we have worked out the detailed consequences of this model and compared them to the experiments. We list here a few of these checks, in a summary way, in order to give the reader some confidence in the model.

First of all: the exponential growth of the mass spectrum is a somewhat shocking result*), and it is reassuring that the experimental spectrum indeed grows exponentially in the region below 1200 MeV (that is: where we are rather sure to know it well) (see Fig. 1). It could have behaved very differently! (Think of electromagnetic and weak interactions.)

Secondly: our mass spectrum implies the existence of a highest temperature T_0 -- and that is just what has been observed

*) Which very recently has been derived independently and differently also in other theories of strong interactions[7].

in the transverse momentum distribution of secondaries produced in high-energy collisions. But if our model is any good, then our limiting temperature T_0, as read off from the mass spectrum, must coincide numerically with the one observed in the transverse momentum distribution when the collision energy becomes very large -- indeed, both turn out to be about 160 MeV. In (II) we have derived a formula for the average transverse momentum as a function of the mass of the emitted particle and of the temperature T (which is itself a known function of the energy density). It reads

$$\langle p_\perp (m,T) \rangle = \sqrt{\tfrac{1}{2}\pi m T}\ \frac{K_{5/2}\left(\frac{m}{T}\right)}{K_2\left(\frac{m}{T}\right)} \tag{10}$$

and the comparison with experiments is shown in Fig. 2, where indeed pions from the highest analysed cosmic-ray energies lie on the curve for T = 160 MeV. At 10-30 GeV collisions, T should be about 120 MeV (this follows from the calculable energy-temperature relation, see Figs. 9 to 11) and π, K, $\bar{p}$ lie on the corresponding curve. A nice detail: protons, in pp collisions, need not be created; therefore they can emerge from the "cold" peripheral regions of the collision; antiprotons, on the other hand, are newly created and since the creation rate goes like exp (-2m/T) they can come only from the "hottest" interior region. Therefore $\bar{p}$ should have larger transverse momenta (heat motion) than protons, as indeed they do in Fig. 2.

Thirdly: the total production rates of all kinds of particles can be calculated from the partition function. They are proportional to exp (-m/T) for single production (like pions) and to exp (-2m/T) for pair production (like proton-antiproton pairs). Details have been worked out in (II) and (III), and there the conservation laws for baryon number and strangeness have been treated on the microcanonical level; the factor 2 in the exponent of pair production rates comes from that. Figure 3 shows the total charged multiplicities as calculated from the theory, compared to bubble chamber and emulsion results. Figure 4 shows the same for heavy pair production rates. Taking the two figures together, the production rates from pions to antideuterons drop by a factor $\sim 10^{10}$, and over this whole range the calculated rates agree with the experimental ones within some 30%.

Fourthly: if pairs of particles -- such as $K + \bar{K}$ -- are produced, one can calculate their invariant mass distribution $f(M^2)\ dM^2$, where $M^2 = (P_K + P_{\bar{K}})^2$ (P = four momentum). Figure 5 shows the comparison with the experimental distribution in 1.2 GeV/c $p\bar{p}$ annihilation. The two curves correspond to the assumptions that the two particles are created at the same place a), or far apart from each other b). "Local" strangeness conservation is clearly favoured. This may have the consequence for astrophysics that in spite of large energy-fluctuations (see below), we must not expect large fluctuations of the baryon number within volumes much larger than 1 fermi in radius.

Fifthly: by supplementing the thermodynamical model with a phenomenological description of collective motions, detailed momentum distributions (magnitude and angle) can be calculated. The velocity distribution of these collective motions is a function whose form one can guess by intuition and then parametrize. In fact one needs two such functions of quite different shape, one for the newly created particles and one for the throughgoing ones (protons in pp collisions). It was possible to describe with two such functions (the same two for all kinds of secondaries and of all energies from 12 to 70 GeV) the detailed angular and momentum distribution of all measured spectra, with satisfactory to excellent agreement. Figures 6 and 7 show comparison with experiments in two cases.

Whilst so far all things that one can reasonably calculate from such a model agree with the corresponding experiments, it is not true that this proves the model to be correct. The point is this: the model affirms that the mass spectrum grows exponentially *ad infinitum*, and this can clearly never be verified by any experiment. Indeed, if we assume that $\rho(m)$ grows exponentially up to some mass M and then is cut off, we can always choose M large enough to fit all experiments. In that case, T_0 is no longer the absolute highest temperature, but still at $T = T_0$ a marked change of behaviour will occur. To illustrate this I have assumed such an exponential spectrum growing like $\exp(m/T_0)$ but cut off at M, and calculated a slightly simplified partition function and from it the energy density ε, which then becomes a function of T and M. Figure 8 shows the resulting set of curves $\varepsilon(T,M)$ in a logarithmic scale. $M = \infty$ is our model. One sees that for finite M the curves change slope near $T = T_0$. One can then ask: how large must M be in order that T cannot much exceed the value of 160 MeV for the highest analysed cosmic-ray energies? Now in such events (primary energy $\sim 10^5$ GeV) the energy density is of the order of 10^{11} MeV^4, and the measured transverse momenta limit T to $\lesssim$ 170 MeV. Thus M $\gtrsim$ 10,000 MeV. In other words: all presently known experiments could be fitted with a theory of our kind but with a mass spectrum growing exponentially up to M $\gtrsim$ 10 GeV. It is important to note that the actually known mass spectrum grows exponentially only up to 1200 MeV, and that the corresponding curve of Fig. 8 does not show the slightest indication of anything particular happening around $T \approx T_0$. The conclusion is then that the mass spectrum must grow exponentially much further beyond 1200 MeV, in fact up to at least 10,000 MeV.

Corresponding curves can easily be drawn for the pressure, etc. -- they will all look quite similar. Thus we have to expect that for all relevant quantities the deviations of our present theory from conventional ones will start at about T $\approx$ 100 MeV or so, and become very marked above 150 MeV -- even if M should not be infinite but only $\gtrsim$ 10 GeV.

It should be stressed that although M $\gtrsim$ 10 GeV can fit all present experiments, will furthermore yield practically the same quantitative results as our theory ($M = \infty$), and will even have remarkable effects on functions interesting to astrophysics (such as pressure), it is an unsatisfactory and inconsistent assumption. This has been discussed in the paper "Hadronic Matter Near the Boiling Point" [5] (p. 1035), and will not be repeated here. The main argument is that assuming any finite value for M would divide the world of fireballs into two classes with fundamentally different properties, one with $m < M$ and the other with $m > M$. As long as no physical principle is known, which would justify such a distinction of two classes, the only self-consistent assumption is $M = \infty$; this is equivalent to our postulate (P). And that means:

> As long as our theory (with $M = \infty$) is not ruled out by contradictory experimental evidence, fireballs of arbitrarily large mass are allowed. In particular the early universe may be considered to be just such a fireball, and its properties can be described by the partition function as long as its density is not much lower than the nucleon energy density; that is, as long as strong interactions are prevalent.

If we say that there is no limitation to the mass of a fireball, then this is understood as a statement about strong interactions. Clearly the total mass of the universe is an upper limit, and general relativity may even provide a distinction of two classes of fireballs. We do not consider these problems here.

4. THERMODYNAMICAL PROPERTIES OF A FIREBALL

The programme of this section is to calculate first the partition function and from it the energy and the pressure. We shall express the pressure as a function of T and also as a function of the energy density. For all these quantities we give asymptotic formulae for $T \to T_0$. As the "universal fireball" (as we shall call it) has time enough to develop an over-all equilibrium between strong, electromagnetic, and weak forces, the latter two contributions will be made explicit.

4.1 The partition function of hadrons

The partition function as given by formula (4) has been calculated as follows: below $m \cong 1000$ MeV we know the detailed mass spectrum $\rho(m;n)$ from the compiled particle data[6]. There we write the mass spectrum as a finite set of δ functions at the right mass values and with the right multiplicities; this yields a finite sum. Above m = 1000 all terms except n = 1 of the sum in formula (4) can be neglected and thus $\rho(m;1) = \rho_{Bose} + \rho_{Fermi} = \rho(m)$ can be used in the form (9). The integral can be calculated numerically. For very large m, the asymptotic form (8) of the mass spectrum and of the Hankel function $K_2(m/T)$ can be used; we then obtain

$$\ln Z(V,T) = \frac{VT}{2\pi^2}\left\{ \sum_{n=1}^{\infty} \frac{1}{n^2} \int_0^{1000} \rho(m;n)\, m^2 K_2\left(n\frac{m}{T}\right) dm + \int_{1000}^{M_0} \frac{a\, e^{m/T_0}}{(m_0^2+m^2)^{5/4}}\, m^2 K_2\left(\frac{m}{T}\right) dm + a\sqrt{\frac{\pi T}{2}} \int_{M_0}^{\infty} \frac{dm}{m}\, e^{-m\left(\frac{1}{T}-\frac{1}{T_0}\right)} \right\} \qquad (11)$$

M_0 is chosen according to the required precision, 20,000 for example. The first two parts are found by numerical computation, the last is the exponential integral

$$a\sqrt{\frac{\pi T}{2}}\, E_1\left(\frac{M_0}{T_0}\cdot\frac{T_0-T}{T}\right) .$$

For $T \to T_0$ the first two parts become a finite constant, whilst the last term diverges like

$$\ln\left(\frac{T_0}{M_c} \cdot \frac{T}{T_0 - T}\right)$$

We introduce the variable $t = T/T_0$; then

$$E_1\left(\frac{M_c}{T_c} \cdot \frac{T_c - T}{T}\right) = -\ln\left(1/t - 1\right) + \text{terms finite for } t \to 1 .$$

Thus the whole can be written

$$\ln Z(V,T) = \frac{V}{V_0} \alpha_0 \left[H(t) - \ln\left(1/t - 1\right)\right]$$

$$t = T/T_0 \; ; \; V_0 = \frac{4\pi}{3} m_\pi^{-3} = 1.59 \times 10^{-6} \text{ MeV}^{-3} \tag{12}$$

$$\alpha_0 = aV_0 (T_0/2\pi)^{3/2} \cong 5.38 \text{ (dimensionless)} ,$$

where $H(t)$ is finite for $t \to 1$. Having thus separated out the diverging part, we may approximate the numerically given function

$$H(t) \equiv \frac{V_0}{\alpha_0 V} \ln Z(V,T) + \ln\left(1/t - 1\right) \tag{13}$$

by some simple function of t. We thus found a good approximation for $H(t)$ in the interval between $0.5 \leq t \leq 1$ with $\sim 0.4\%$ maximal and $\sim 0.1\%$ average error. It reads:

$$H(t) = 1.8479 - 3.7433\ t - \frac{0.034225}{t - 1.065} \tag{14}$$

This, inserted into formula (12), yields an approximation for ln Z that can be easily differentiated or integrated. All decimal places of formula (14) must be used, as there is large cancellation of terms among each other and with the logarithm. The maximum error in ln Z is $\sim$ 12% (at t = 0.5) and then the error drops rapidly (0.8% at t = 0.7; 0.1 at t = 0.96). This approximation must not be used below t = 0.5, but below t = 0.5 the hadronic black-body radiation becomes anyway uninteresting [see Figs. 9 and 12].

4.2 The total partition function including weak and electromagnetic forces

For t $\geq$ 0.5 the electron mass is negligible; therefore we have as contributing zero-mass particles γ, ν, e with weights 2 for γ, 4 for ν (ν_e, $\bar{\nu}_e$; ν_μ, $\bar{\nu}_\mu$) and 4 for e ($e^\pm$, two spins). Furthermore γ is a boson, the others are fermions. Thus in this case

$$\rho_{\gamma\nu e}(m;n) = \left[2 - 8\cdot(-1)^n\right]\cdot \delta(m) \tag{15}$$

With this mass spectrum inserted into formula (4) we obtain

$$\ln Z_{\gamma\nu e}(V,T) = \frac{V T_0^3}{10}\pi^2 t^3 \tag{16}$$

which, of course, could have been copied from any textbook. One might doubt whether neutrinos really participate in the equilibrium, because they escape so rapidly from the interaction region except when the density is extraordinarily high. But the point is that just when they are created, they have already the right equilibrium momentum distribution -- whether they escape soon afterwards or not. However, even if they do, this will occasion only a slight change in the numerical factor in front of t^3 in formula (16); and since this whole contribution becomes negligible for $t \to 1$, where $\ln Z_{\text{hadron}}$ diverges, we need not discuss this question any further.

The muon mass is about 100 MeV; thus it cannot be assumed to be equal to zero. Putting $\rho_\mu(m,n) = -(-)^n \cdot 4\delta(m - m_\mu)$ we find from formula (4):

$$\ln Z_\mu(V,T) = \frac{2m_\mu^2 V T_0}{\pi^2}\, t \left[\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} K_2\left(n \frac{m_\mu}{T_0} \cdot \frac{1}{t}\right)\right] \tag{17}$$

This has been evaluated numerically with a relative error of $< 10^{-4}$. [We may remark here that one can now check whether putting $m_e = 0$ is a good approximation to formula (17) with m_e inserted there; the error is less than 1%.] Of course one expects that $\ln Z_\mu$ will be roughly proportional to t^3, similar to $\ln Z_{\gamma\nu e}$. The following third-order polynomial is an approximation to Eq. (17) with $< 10^{-4}$ relative error:

$$\begin{aligned}\ln Z_\mu (V_0,T) = {} & 0.03923791 - 0.2392164\, t \\ & -0.0188072\, t^2 + 2.509236\, t^3 \ .\end{aligned} \tag{18}$$

The leading term is indeed t^3. Note that this is for $V = V_0$; multiplication by V/V_0 gives $\ln Z_\mu (V,T)$ -- as always. The total partition function of our universal fireball is then the sum of formulae (12), (16), and (18). In order to see what the relative contributions are, one may look at Fig. 12, where the pressure is shown. Since $P = (T/V) \ln Z$, $\ln Z (V,T)$ can be inferred from these figures by multiplying by $V/(tT_0)$ (which is in $0.5 \leq t \leq 1$ almost a constant) so that Fig. 12 indeed shows also the behaviour of the various partition functions.

With $t \to 1$ we obtain from formulae (12), (16), and (18) the asymptotic total partition function for V_0:

$$\begin{aligned}\ln Z_{\text{total}} (V_0, t \to 1) &= \\ &= \alpha_0 \left[H(1) - \ln(1/t - 1)\right] + \ln Z_{\gamma\nu e}(t \to 1) + \ln Z_\mu (t \to 1) \\ &= \alpha_0 \left[-1.37 - \ln(1/t - 1)\right] + V_0 \frac{\pi^2 T_0^3}{10} + 2.29 \ .\end{aligned} \tag{19}$$

Figure 12 shows that this is a good approximation for $T/T_0 > 0.95$.

4.3 The energy as function of T/T_0

The expectation value of the energy in a volume V is given by

$$E(V,T) = T^2 \frac{\partial \ln Z}{\partial T} = T_0 t^2 \frac{\partial \ln Z}{\partial t} \tag{20}$$

We obtain immediately an analytic approximation by simply differentiating the corresponding expressions for ln Z. We have calculated $E(V_0T)$; for any volume V one obtains E(V,T) by multiplication by V/V_0.

As the differentiation is trivial we do not write down the full formula, but for convenience give an asymptotic one for $t \to 1$. It reads

$$E(V_0, t\to 1) \Rightarrow T_0\alpha_0 \left[\langle \frac{dH}{dt} \rangle + \frac{1}{1-t}\right] + T_0 \frac{d}{dt} \ln Z_{\gamma\nu e\mu}(t\to 1)$$

$$= T_0\alpha_0 \left[-6.5 + \frac{1}{1-t}\right] + 4.245 \times 10^3 \tag{21}$$

$\langle dH/dt \rangle$ is a suitable average to make (21) a reasonable approximation also in the non-asymptotic region.

Figures 9, 10, 11 show the various contributions to the energy contained in V_0. The asymptotic formula happens to be a rather good approximation over the whole range. The hadronic contribution starts to be negligible below t = 0.5; this is the reason why we did not try to make our analytic approximation to H(t) valid over a larger range (which would have implied more terms).

The hadronic part of the energy wins at about t = 0.81 and, of course, diverges for $t \to 1$ where the γ, ν, e, μ contribution stays constant and tends to become negligible (Figs. 10 and 11). If in Fig. 9 one draws a horizontal line at $E/10^2 = 10$ (about the nucleon mass) it intersects the total energy curve at $t \simeq 0.64$, the γ, ν, e, μ curve at $t \simeq 0.69$, and the hadronic curve at $t \approx 0.76$. That is: at $t \approx 0.64$ the total energy density is about equal to the nucleon density, but about 70% of it is due to γ, ν, e (and a little μ); at $t \approx 0.69$ the γ, ν, e, μ contribution alone already reaches nuclear density, and the hadron contribution is still only ½ of it. Note that the energy scale is different in the three figures.

4.4 The pressure as function of T/T_0; equation of state

We start by considering the pressure as a function of T. The relation

$$P(T) = \frac{T}{V} \ln Z = \frac{T_0}{V_0} t \ln Z(V_0, t) \tag{22}$$

can be written immediately in analytic form using Eqs. (12), (16), (18), and the asymptotic expression is found from Eq. (19) by multiplication by T_0/V_0; it diverges logarithmically for $t \to 1$. As the last term in Eq. (19) is of order 1 and T_0/V_0 of order 10^8 we represent $P/10^8$ in Fig. 12. Again the total and partial pressures are shown. The hadronic partial pressure wins at $t = 0.955$, whilst the partial hadronic energy already did so at $t = 0.83$. The explanation is that at a given energy density there are a few (heavy) hadrons and many zero mass γ, ν, e; since the pressure is, at a given temperature, proportional to the number density, it is obvious that the partial pressure of many zero-mass particles can be larger than that of a few heavy ones even long after the energy density is mainly due to the heavy particles [at $t = 0.955$ about 80% of the energy density is due to hadrons (Fig. 10), whilst their partial pressure just equals that of γ, ν, e, μ].

The pressure in this theory is, of course, not a function of the volume V but only of T -- therefore Eq. (22) can be considered as the equation of state of our universal fireball.

It is, however, useful and instructive to express P also as a function of the energy density. As is well known, in ordinary electromagnetic black-body radiation the pressure equals ⅓ times the energy density. From formulae (20), (16), and (22) we see that this is equally true if not only γ but also ν and e are taken into account. But already for the μ this is no longer so, because this property stems from the fact that for zero-mass particles the partition function is proportional to exactly T^3, whereas for non-zero mass other powers of T enter. As long as the number of masses contributing to the spectrum is finite, that is: as long as $\int_0^\infty \rho(m)\, dm < \infty$, then neither is the temperature limited (it is not limited even for any spectrum growing less than exponentially). Furthermore, since $\int_0^\infty \rho\, dm < \infty$ implies that ρ decreases, there exists always a temperature T such that all masses, which really contribute, are small compared to T. In that case we have nearly the mass zero case, and for still larger T the partition function will have a leading T^3 dependence:

if $$\int_0^\infty \rho(m)\, dm < \infty$$

then $$\lim_{T\to\infty} \ln Z(V,T) = \text{const} \cdot V \cdot T^3 \tag{23}$$

and $$P \to \frac{1}{3}\,\frac{E}{V}$$

But as long as this limiting situation is not yet reached, the pressure will always be <u>smaller</u> than $\frac{1}{3}$ E/V because then the masses of the particles are not negligible; they count in the energy balance and therefore the number of particles is -- for given total energy -- smaller than it would be for zero masses. Since the pressure is proportional to the number of particles, it must be smaller than $\frac{1}{3}$ E/V. This qualitative argument explains why in our theory -- where $\int_0^\infty \rho(m)\, dm = \infty$ and where the temperature is limited by T_0, so that there are always masses >> T -- why, then, the pressure is considerably smaller than $\frac{1}{3}$ E/V; in fact, as we shall see now, it is proportional to ln (E/V) in the limit $E/V \to \infty$, $T \to T_0$.

Indeed, let us take the asymptotic form for E/V, Eq. (21), and neglect the constant contributions, thus

$$\varepsilon \equiv \frac{E}{V} \underset{t\to 1}{\Longrightarrow} \frac{T_0 \alpha_0}{V_0} \cdot \frac{1}{1-t} \tag{24}$$

Now we solve for 1 - t and insert this in the asymptotic formula for P [Eq. (19) multiplied by T_0/V_0]; here the constant contributions must not be neglected, because the divergence is only logarithmic. Result:

$$P(\varepsilon \to \infty) \Rightarrow \frac{T_0 \alpha_0}{V_0}\left[H(1) + \ln\left(\varepsilon \frac{V_0}{\alpha_0 T_0}\right)\right] + P_{\gamma\nu e\mu}(\varepsilon \to \infty)$$

$$= \left\{5.415\left[-1.37 + \ln\left(\frac{\varepsilon}{5.415 \times 10^8}\right)\right] + 8.773\right\} \times 10^8 \tag{25}$$

where the last constant is due to γ, ν, e, μ. We thus have the important result:

for large energy density ($T \to T_0$; $\varepsilon \to \infty$)
the pressure is proportional to ln ε. (26)

Since the logarithm varies slowly, one might doubt whether this asymptotic formula (25) is anywhere a numerically tolerable approximation. To find this out, we have calculated $P(\varepsilon)$ by numerically solving $E(T)/V$ for T and inserting $T(\varepsilon)$ in $P(T)$. The results are displayed in Figs. 13, 14 and 15 where we show $P/10^8$ as function of the energy density with different scale of the energy density in the different figures: the value $\varepsilon/10^8 \gtrsim 5$ in Fig. 13 corresponds to $\varepsilon \gtrsim m_p/V_0$ or to $T \gtrsim 0.64\ T_0$; the values $\varepsilon/10^9 \gtrsim 5$ in Fig. 14 and $\varepsilon/10^{10} \gtrsim 5$ in Fig. 15 correspond then to about 10 and 100 times the nuclear density, respectively. The asymptotic formula (curves A, dotted) gives never really very wrong results, but it starts to become a rather good approximation only at about 100 times the nuclear density ($T \gtrsim 0.99\ T_0$). However, as Fig. 15 shows, the partial pressure due to γ, ν, e, μ, though it becomes constant for large energy density, is "never" negligible compared to the total pressure, just because the latter diverges only $\sim$ ln ε: even at an energy density 1000 times larger than that of the nucleon, the γ, ν, e, μ partial pressure is still $\sim$ ¼ of the total one (Fig. 15).

5. SOME REMARKS ABOUT THE INTERNAL STRUCTURE OF A FIREBALL

When $T \to T_0$, the kinetic energy per particle cannot increase any more, and the only way to increase the total energy density is to create more and heavier particles. Since P is proportional to the particle number and diverges logarithmically, it follows that for $T \to T_0$ the particle number itself must diverge logarithmically, in fact it must be proportional to $\ln Z$. This divergence is, however, much too weak to account for the divergence of the energy density, which is proportional to $(T_0 - T)^{-1}$. Slow divergence of the particle number N and strong divergence of the energy (with constant kinetic energy per particle) means then that the average mass of the particles must itself diverge [see Eqs. (37)]: the particles present are mostly highly excited fireballs themselves! This result is not so surprising, because it is almost explicitly contained in our postulate (P) which is the corner-stone of this theory. Yet one might wonder how this is managed when, as we mentioned earlier, the production rate for particles of mass m is proportional to $\exp(-m/T)$. If $T \to T_0$, then clearly the production rate for a particle (fireball) of mass m tends to a constant, and to a terribly small one for large mass! How then is it possible that the average mass of the created particles and their total number should diverge? The reason is again the exponential mass spectrum $\rho(m) \sim \exp(m/T_0)$: for any given m the production rate stays (almost) constant when $T \to T_0$, but as there are so many different ones -- infinitely many -- and each one increases its production rate very little -- infinitely little -- all of them together can yield not only a diverging total number but also a diverging average mass.

The mechanism is always the same: the exponentially decreasing Boltzmann factor, which comes from the asymptotic form of the Hankel function, is almost compensated by the exponentially increasing mass spectrum; thus we have, in general, integrals of the type (with A, ..., a set of parameters)

$$I(A,\dots,T) = F(A,\dots,T) + \int_{M_0}^{\infty} f(A,\dots,m,T)\, e^{-m\left(\frac{1}{T}-\frac{1}{T_0}\right)}\, dm \qquad (27)$$

where for $T \to T_0$ the function $F(A, \dots, T) \to F(A, \dots, T_0)$ is well behaved and $f(A, \dots, m,T)$ varies less than exponentially in m and is finite for $T = T_0$. It depends then on the structure of this $f(A, \dots, m,T)$ whether for $T \to T_0$ the integral over m converges or not; and if not, how it diverges. For $T < T_0$ it always converges; all the quantities $I(A, \dots, T)$ would converge for any T in ordinary thermodynamical theories (i.e. with non-exponential mass spectrum).

In the following we shall list a few such quantities $I(A, \dots, T)$ and also a few differential ones -- all without proofs [which partly are found in the papers[5] and partly are unpublished; but all are straightforward].

5.1 Energy spectra of particles (Planck's law)

From formula (4) it follows that the mean occupation numbers are

$$\bar{\nu}_{\alpha\gamma} = x_{\alpha\gamma} \frac{\partial}{\partial x_{\alpha\gamma}} \ln Z = \frac{x_{\alpha\gamma}}{1 \pm x_{\alpha\gamma}} \tag{28}$$

which by going over to the continuum and replacing the label γ by m, becomes the momentum spectrum

$$f_m(p,T)\,dp = \frac{V z_m}{2\pi^2} \frac{p^2\,dp}{e^{\varepsilon/T} \pm 1} \qquad \begin{bmatrix} + \text{ fermions} \\ - \text{ bosons} \end{bmatrix}$$

$$\varepsilon = \sqrt{p^2 + m^2} \tag{29}$$

$z_m = (2I+1)(2J+1)$ spin-isospin multiplicity for the particle in question

[Note: here one sees clearly that our chemical potentials are zero in the relativistic and $-m$ in the non-relativistic case, because in ordinary thermodynamics with either given particle number or given chemical potential μ_0 one would have in the denominator an expression $\exp\left[(\varepsilon_{kin} - \mu_0)/T\right] \pm 1$, with $\varepsilon_{kin} = p^2/2m$. We, however, use $\varepsilon = \sqrt{p^2 + m^2} = \varepsilon_{kin}^{(rel)} + m$, the ordinary formula written with relativistic kinetic energy would read $\exp\left[(\varepsilon - m - \mu_0)/T\right] \pm 1$ and ours is $\exp(\varepsilon/T) \pm 1$; thus $m + \mu_0 = 0$.]

5.2 The average number of particles with mass m

By integrating formula (29) over p one obtains

$$\bar{\nu}(m,T) \cong \frac{V z_m}{2\pi^2} m^2 T K_2\left(\frac{m}{T}\right) \underset{m \gg T}{\Longrightarrow} V z_m \left(\frac{mT}{2\pi}\right)^{3/2} e^{-\frac{m}{T}} \tag{30}$$

This is the average number of particles of mass m in a volume V if these particles can be created freely (e.g. pions); if they are bound to be produced in pairs, $\bar{\nu}^2$ is the number of such pairs (e.g. nucleon - antinucleon pairs).

5.3 Average momentum

From formula (29) we find

$$\langle p\rangle \cong 2\sqrt{\frac{2mT}{\pi}}\;\frac{K_{5/2}\left(\frac{m}{T}\right)}{K_2\left(\frac{m}{T}\right)} \;\underset{m\gg T}{\Longrightarrow}\; 2\sqrt{\frac{2mT}{\pi}} \tag{31}$$

5.4 Total particle number; particle number distribution

The average number of created particles with mass in the interval {m, dm} is

$$\bar{n}(m,T)\,dm = \bar{\nu}(m,T)\quad \rho(m)\,dm \;,$$

and the average total number N of particles becomes

$$N = \int_0^\infty \bar{n}(m,T)\,dm = \int f_m(p,T)\,\rho(m)\,dp\,dm \approx \ln Z(V,T) \tag{32}$$

It can be shown that the particle number distribution is almost exactly a Poisson distribution:

$$W(n) = e^{-N}\,\frac{N^n}{n!}\;; \qquad N = \ln Z(V,T) \tag{33}$$

With $P = (T/V)\ \ln Z$ we find thus

$$PV = NT \tag{34}$$

which formally looks like the equation of state of an ideal gas — only here N is a function of T in contradistinction to the ideal gas where it is a given constant.

5.5 Average mass and energy of the particles

The mean energy per particle is

$$\varepsilon_1 = \frac{\int \varepsilon\, f_m(p,T)\,\rho(m)\,dp\,dm}{\int f_m(p,T)\,\rho(m)\,dp\,dm} \equiv \frac{E}{N} \tag{35}$$

If in f_m (p,T) one neglects the ±1 in the denominator (which in the integrals is permitted), then

$$\varepsilon_1 = -\frac{d}{d\frac{1}{T}} \log \int f_m(p,T)\,\rho(m)\,dp\,dm = \qquad (36)$$

$$\frac{1}{N}\int\left(\frac{3}{2}T + m\right)\bar{\nu}(m,T)\,\rho(m)\,dm = \frac{3}{2}T + \bar{m}(T)$$

Here $\bar{m}(T)$ is the average mass of the particles present at temperature T. Thus the average mass can be expressed in either way

$$\bar{m}(T) = \frac{\int m\, f_m(p,T)\,\rho(m)\,dp\,dm}{\int f_m(p,T)\,\rho(m)\,dp\,dm} \qquad (37a)$$

$$\bar{m}(T) = \frac{E}{N} - \frac{3}{2}T \qquad (37b)$$

where the first can be calculated exactly and the second is a very good approximation (±1 neglected). Here one sees that $\bar{m}$ will diverge just in the way to match the strong divergence of E with the weak one of N.

5.6 The partial gas of particles with mass $m \geq \mu \gg T$

Consider the partial gas of all particles with mass greater than μ, where μ is so large compared with T_0 that

$$\rho(m) = \frac{a}{m^{5/2}}\, e^{m/T_0} \qquad (38a)$$

$$K_m\left(\frac{m}{T}\right) = \sqrt{\frac{\pi T}{2m}}\; e^{-m/T} \qquad (38b)$$

can be considered as exact relations. In that case all the following are also exact relations:

$$\ln Z(V,T) = aV\left(\frac{T}{2\pi}\right)^{3/2} E_1\left(\frac{\mu}{T_0}\cdot\frac{T_0-T}{T}\right)$$

$$N = \ln Z$$

$$f_m(p,T) = \frac{Vz_m}{2\pi^2}\, p^2 e^{-\sqrt{p^2+m^2}/T}$$

$$\bar{\nu}(m,T) = Vz_m\left(\frac{mT}{2\pi}\right)^{3/2} e^{-m/T}$$

$$\bar{n}(m,T) = aV\left(\frac{T}{2\pi}\right)^{3/2}\frac{1}{m}\exp\left(\frac{m}{T_0}\cdot\frac{T_0-T}{T}\right) \tag{39}$$

$$E = aV\left(\frac{T}{2\pi}\right)^{3/2}\left\{\frac{3}{2}T E_1\left(\frac{\mu}{T_0}\cdot\frac{T_0-T}{T}\right) + \frac{TT_0}{T_0-T}\, e^{-\frac{\mu}{T_0}\cdot\frac{T_0-T}{T}}\right\}$$

$$\bar{m}(T) = \frac{T_0T}{T_0-T}\;\frac{\exp\left(-\frac{\mu}{T_0}\cdot\frac{T_0-T}{T}\right)}{E_1\left(\frac{\mu}{T_0}\cdot\frac{T_0-T}{T}\right)}$$

Since the average mass diverges like $(T_0 - T)^{-1}$, it is obvious that for T near enough to T_0 the partial gas with masses $\geq \mu$ will be dominating; that means that all formulae (40) become exact in that limit. Therefore these formulae illustrate the internal structure of the universal fireball.

5.7 Energy fluctuations

Perhaps the most important property of the universal fireball is that the local energy fluctuations do not vanish even for $\varepsilon \to \infty$. The relative energy fluctuations are

$$\frac{\Delta E}{E} = \left(\frac{\overline{E^2} - \overline{E}^2}{\overline{E}^2}\right)^{1/2} = \left(\frac{T^2}{E^2}\frac{dE}{dT}\right)^{1/2}$$

$$\overline{E} \equiv E = T^2 \frac{\partial \ln Z}{\partial T} \tag{40}$$

Now in our theory $E = V \cdot f(T)$ and therefore

$$\frac{\Delta E}{E} = \frac{g(T)}{\sqrt{V}} = \frac{h(T)}{\sqrt{E}} \tag{41}$$

where f, g, h are functions of T alone. Thus as long as we keep T fixed and consider larger and larger partial volumes V inside our universal fireball, the energy fluctuations in these volumes become negligible like $E^{-1/2}$, as one would expect. If, however, one considers a fixed volume and increases T, then $\Delta E/E$ in this volume is proportional to some function g(T), which in the limit $T \to T_0$ need not and indeed does not vanish in our theory.

As this seems to be an important point, let us first consider a pure γ, ν, e black-body radiation. From formulae (16) and (41) it follows that

$$E(V,T)_{\gamma\nu e} = V\sigma T^4$$

$$\sigma = \frac{3\pi^2}{10} = \text{Stefan-Boltzmann constant of } \gamma\nu e\text{-gas.} \tag{42}$$

Hence with $\varepsilon = E/V$ = energy density

$$\frac{\Delta E}{E} = \sqrt{\frac{4}{\sigma V}}\, T^{-3/2} = \sqrt{\frac{4}{\sigma V}} \left(\frac{\varepsilon}{\sigma}\right)^{-3/8} \tag{43}$$

That is: for the $\gamma\nu e$ gas, $\Delta E/E$ vanishes for fixed volume like $\varepsilon^{-3/8}$ -- almost like $\varepsilon^{-1/2}$, which latter would be considered to be the normal behaviour.

Now back to our theory: in the asymptotic case $T \to T_0$ we can use formula (21) to calculate $\Delta E/E$ according to formula (41). The result is for $V = V_0 = (4\pi/3)\, m_\pi^{-3}$:

$$\left(\frac{\Delta E}{E}\right)_{total} \underset{T\to T_0}{\Longrightarrow} \frac{1}{\sqrt{\alpha_0}} \cong 0.43 \tag{44}$$

and as we know it is $\sim 1/\sqrt{V}$, we obtain it for any other volume by multiplication by $\sqrt{V_0/V}$. This is a tremendous energy fluctuation inside volumes of the order of a nucleon volume. It means that if the average energy density is about, say, 1000 times the nucleon density, then, looking around, we will see a considerable fraction of volumes V_0 containing only a few hundred, and another considerable fraction of them containing several thousand nucleon masses!

Whilst for lower energy densities the relative fluctuations even increase (they do so also in ordinary thermodynamics), their absolute effect becomes less and less important. Numerical results are displayed in Figs. 16 and 17.

I feel formula (44) to be a very important result, because it leads to a conjecture (which I have not yet been able to prove or to disprove, but which seems plausible):

> if the total energy contained in an elementary volume V_0 can fluctuate so strongly, then it is to be expected that the baryon quantum number attached to V_0 may also strongly fluctuate (between positive and negative values).

The qualitative argument goes like this:

* the particle number fluctuations vanish for $T \to T_0$ like $\Delta N/N = N^{-1/2}$ [see formula (34)]; therefore they cannot be responsible for the large energy fluctuations;

* indeed, the particle number increases only logarithmically and therefore is always "small" but we know that the average mass $\bar{m}(T)$ of the "few" particles present in V_0 diverges strongly [see formula (39)] and is responsible for almost all of the divergence of the energy;

* we therefore expect that the strong energy fluctuation is due mostly to a strong fluctuation of the mass values of the particles present in V_0 (we shall prove that below).

Now, if that is so, then it is quite possible that also the baryon number in V_0 may strongly fluctuate, because nothing prevents the creation of fireballs with large baryon quantum number B (as large as m/m_p); and since, then, the baryon quantum number of a fireball will be the larger, the larger is its mass [for equidistribution it would be $\approx m/(m_p\sqrt{3})$], we must expect that the established large mass fluctuations may be accompanied by somewhat less spectacular but still considerable fluctuations of the total baryon quantum number of V_0.

This conjectured mechanism could provide for the strong local variations of the baryon quantum number, which is required in the first few instants of the universe in order to explain why not all hadronic matter has annihilated and why galaxies exist [8]).

It should be stressed, however, that large variations of B over distances >> 1 fermi are not very likely, because the baryon number conservation acts rather locally [see the discussion of the fourth point (mass distribution of $K\bar{K}$ pairs, Fig. 5) in Section 3].

It remains to show that the mass-fluctuations are indeed very large. We start with

$$\Delta m^2 = \overline{m^2} - \overline{m}^2 = \overline{m^2} - \left(\frac{E}{N}\right)^2 \qquad (45)$$

Here formula (38b) has been used with $\frac{3}{2}T \to \frac{3}{2}T_0$ neglected. To calculate $\overline{m^2}$ we use formulae (39):

$$\overline{m^2} = \frac{\int m^2 \bar{n}(m,T)\,dm}{\int \bar{n}(m,T)\,dm} = \frac{1}{N}\int m^2 \bar{n}(m,T)\,dm \tag{46}$$

For $T \to T_0$ we have

$$\bar{n}(m,T) = \frac{\text{const}}{m} e^{-m\left(\frac{1}{T}-\frac{1}{T_0}\right)} \tag{47}$$

so that

$$\overline{m^2} = \frac{1}{N}\frac{d^2}{d\frac{1}{T}^2}\int \bar{n}(m,T)\,dm = \frac{1}{N}\frac{d^2}{d\frac{1}{T}^2}N \tag{48}$$

But $N = \ln Z$ and

$$E = -\frac{d}{d\frac{1}{T}}\ln Z \;;\quad \frac{d^2N}{d\frac{1}{T}^2} = -\frac{dE}{d\frac{1}{T}} = \Delta E^2 \qquad \text{[see (40)]}$$

thus

$$\overline{m^2} = \frac{\Delta E^2}{N} \;;\quad \Delta m^2 = \overline{m^2} - \bar{m}^2 = \frac{\Delta E^2}{N} - \bar{m}^2 \tag{49}$$

Now we know that for $T \to T_0$ we have $\Delta E^2 = E^2/\alpha_0$ so that

$$\Delta m^2 = \frac{N}{\alpha_0}\cdot\frac{E^2}{N^2} - \bar{m}^2 = \frac{N}{\alpha_0}\cdot\bar{m}^2 - \bar{m}^2 \tag{50}$$

Thus for large N the second term can be neglected and we obtain the asymptotic formula

$$\Delta m^2 \Longrightarrow \bar{m}^2 \cdot \frac{N}{\alpha_0} \tag{51}$$

Since formula (50) and the second formula (49) are only different in notation, we can also in formula (49) drop the $-\bar{m}^2$ and obtain the particularly illuminating relation

$$\Delta E^2 \cong N \cdot \Delta m^2 \tag{52}$$

which says that in the limit $T \to T_0$ the energy fluctuations are entirely due to the fluctuations of the mass values of those particles present in V_0.

6. CONCLUSION; OPEN QUESTIONS

We have described a "universal fireball", namely a straightforward extrapolation to a macroscopic scale of objects that are very likely to exist in high-energy physics. Such a universal fireball might well be what our universe once has been in its very first moments. We have only described its thermodynamical properties and some structural ones, but have not made any attempt to combine them with general relativity to yield a theory of expansion and cooling. The three main features are the following.

a) The temperature was never higher than $T_0 \approx 160$ MeV, and during expansion from infinite to finite energy density it stays very near to this value for a long while; when the total energy density has dropped to nuclear density, T is still 103 MeV. Thereafter, with further expansion, it drops much faster.

b) The total pressure (including γ, ν, e, μ) is, for $T \to T_0$ and $\varepsilon \to \infty$, not proportional to $\frac{1}{3}\varepsilon$ as in conventional theories, but only to $\ln \varepsilon$.

c) Inside small volumes of the order of a nucleon volume V_0, the energy fluctuations are roughly of the order of $\frac{1}{2}$ times the total energy contained in V_0 and remain so in the limit $E/V_0 \to \infty$ ($T \to T_0$). These fluctuations are due to fluctuations of the masses of the particles inside V_0. It is likely that they are accompanied by smaller but still substantial fluctuations of the baryonic quantum number of V_0. If this could be proved to be true, it would be a very important clue to the understanding of how galaxies are formed[8].

Here we have not touched at all on a completely different problem, which probably also will suffer remarkable changes after coming up against the conventional treatment: that is, the behaviour of cold superdense matter, where the hadronic mass spectrum weakens the effect of the Pauli principle. This has already been realized long ago, but so far nobody seems to have published results using an exponential spectrum[9]. Unfortunately the problem is technically more difficult than the one dealt with in this paper.

* * *

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I apologize to all astrophysicists who may have written papers related to the questions dealt with here, and whom I did not quote (by simple ignorance) in the list of references.

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All computations of this paper (finding approximations for ln Z, evaluating $\ln Z_\mu$ and all other functions as well as functional inversion) were done on CERN's interactive computing system GAMMA.

Figures 9 to 17 are faithful off-line plots of the curves shown on the display tube during GAMMA-operation. The GAMMA system has become a valuable tool of CERN thanks to the effort and devotion of C.E. Vandoni.

APPENDIX

Units transformed
{CGS} versus $\{\hbar = c = k = 1, \text{MeV}\}$

$$
\begin{aligned}
1\ \text{g} &= 5.612 \times 10^{26}\ \text{MeV} \\
1\ \text{cm} &= 5.068 \times 10^{10}\ \text{MeV}^{-1} \\
1\ \text{sec} &= 1.519 \times 10^{21}\ \text{MeV}^{-1} \\
1\ \text{MeV} &= 1.782 \times 10^{-27}\ \text{g} \\
&= 5.068 \times 10^{10}\ \text{cm}^{-1} \\
&= 1.519 \times 10^{21}\ \text{sec}^{-1}
\end{aligned}
$$

check: $\frac{1\ \text{MeV}}{1\ \text{MeV}} = 1 = \frac{1.591}{5.068} \times 10^{11}\ \frac{\text{cm}}{\text{sec}} = 2.998 \times 10^{10}\ \frac{\text{cm}}{\text{sec}} = c$

Derived quantities:

Density	$1\ \text{cm}^{-3}$	$= 7.682 \times 10^{-33}\ \text{MeV}^3$	
	$1\ \text{g cm}^{-3}$	$= 4.311 \times 10^{-6}\ \text{MeV}^4$	
Velocity	$1\ \text{cm sec}^{-1}$	$= 3.336 \times 10^{-11}$	
Acceleration	$1\ \text{cm sec}^{-2}$	$= 2.196 \times 10^{-32}\ \text{MeV}$	
Force	$1\ \text{dyn}$	$= 1.232 \times 10^{-5}\ \text{MeV}^2$	
Energy	$1\ \text{erg}$	$= 6.242 \times 10^{5}\ \text{MeV}$	$= 624\ \text{GeV}$
Temperature	$1\,^\circ\text{K}$	$= 8.616 \times 10^{-11}\ \text{MeV}$	
	$1\ \text{MeV}$	$= 1.1606 \times 10^{10}\ ^\circ\text{K}$	
	T_0	$= 160\ \text{MeV} = 1.857 \times 10^{12}\ ^\circ\text{K}$	

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"Remarks about matter condensation in the early universe and a theory of quasars", Preprint "Laboratoire de Physique Théorique et Hautes Energies 91-Orsay, France", No. 69/33 (May 1969).
In Omnès' theory the mechanism which generates the necessary inhomogeneity of the baryon-number distribution is very different from (in fact incompatible with) the one proposed here, but the subsequent further separation of matter from antimatter (which is not treated in our paper) might be similar in both cases.

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Calculations concerning superdense stars using our exponential mass spectrum have been started now by Drs. E. Hilf and Rein at the University of Würzburg, Germany (Physikalisches Institut); private communication by E. Hilf.

Figure captions

Fig. 1 : The smoothed experimental mass spectrum as it developed from October 1964 until January 1967 (various dotted lines), and the function $\rho(m) = a(m_0^2 + m^2)^{-5/4} \exp(m/T_0)$ that has the asymptotic form required by the thermodynamical model.

Fig. 2 : The mean transverse momentum $\langle p_\perp \rangle$ as function of the mass of the particle and the temperature of the place from where it is emitted. T may be considered roughly as the mean value of the temperature over the interaction region. Experimental values (12-30 GeV/c) should lie near $T \approx 128$ MeV; antiprotons come from the hot central regions, protons also from the cold peripherical regions of the collision, hence $T_{\bar{p}} > T_p$. Some cosmic-ray data coming from very high primary energy have been drawn in. They approach the limiting curve ($T = 160$ MeV). [The references quoted as 27 are given in (II) from which this figure is taken.]

Fig. 3 : Comparison of the total charged multiplicities (obtained by integrating our spectra) with experimental values. This figure is taken from (II): "our spectra" are calculated there; the references 21 and 22 are listed there, too.

Fig. 4 : Reduced (in the sense explained in III) production rates for heavy pairs. For details see (III).

Fig. 5 : Experimental mass distribution $f(M^2)$ of $K\bar{K}$ pairs from $p\bar{p}$ annihilation at 1.2 GeV/c compared to two thermodynamical distributions obtained under the assumptions that K and $\bar{K}$ originate from the same or from distant locations, respectively. Details and references are found in (II) from which this figure is taken;

Fig. 6 : The π^- lab. spectrum ($p_0 = 30$ GeV/c) with some typical experimental errors drawn in. Our curves result from a common "one essential parameter" - fit together with K^- and p. This figure is taken from (II); references are given there.

Fig. 7 : p lab. spectrum (p_0 = 30 GeV/c) with some typical errors drawn. Our curves result from a common fit to p and π^+ (only 30 GeV/c data) with three "essential parameters". This figure is taken from (II); references are given there.

Fig. 8 : The energy density $\varepsilon(M,T)$ if the hadronic mass spectrum $\rho(m)$ grows like exp (m/T_0) only up to $m \approx M$ and then tends to zero. T_0 becomes exhibited for $M \gtrsim 10$ GeV; the presently (1969) known mass spectrum ($M \approx 1$ GeV) cannot explain the distinguished role which $T_0 \approx 160$ MeV plays in high-energy physics. While all results presented here could be obtained from the assumption $M \gtrsim 10$ to 20 GeV, only $M = \infty$ (our model) leads to a self-consistent picture.

Fig. 9 : The energy/10^2 in one elementary volume V_0. The various curves denote:

A: exact total energy (hadrons + γ, ν, e, μ);

B: asymptotic formula (21);

C: partial energy of hadrons;

D: partial energy of γ, ν, e, μ.

In this energy scale m_p lies near to 10.

Fig. 10 : The energy/10^3 in one elementary volume V_0. The various curves denote:

A: exact total energy (hadrons + γ, ν, e, μ);

B: (dotted) asymptotic formula (21);

C: partial energy of hadrons;

D: partial energy of γ, ν, e, μ.

In this energy scale m_p lies near to 1.

Fig. 11 : The energy/10^4 in one elementary volume V_0. Here the exact (A), the asymptotic (B) [formula (21)] and the hadronic contribution (C) become equal and the non-hadronic contribution (D) is negligible. In this energy scale 10 corresponds to 100 m_p.

Fig. 12 : The pressure/10^8 as function of T/T_0. The various curves denote:

A: asymptotic formula [multiply Eq. (19) by T_0/V_0];

B: exact total pressure (hadrons + γ, ν, e, μ);

C: partial pressure of γ, ν, e, μ;

D: partial pressure of hadrons.

Above $T/T_0 = 0.95$ the asymptotic formula becomes good, below $T/T_0 = 0.5$ the hadron contribution may be neglected.

Fig. 13 : Pressure/10^8 as function of energy density/10^8. The curves denote:

A: asymptotic formula (25);

B: total pressure;

C: partial pressure of γ, ν, e, μ;

m_p/V_0 lies near to 5 in this scale for the energy density.

Fig. 14 : Pressure/10^8 as function of energy density/10^9. The curves denote:

A: asymptotic formula (25);

B: total pressure;

C: partial pressure of γ, ν, e, μ.

In this scale of energy density 10 corresponds to about 20 times m_p/V_0.

Fig. 15 : Pressure/10^8 as function of energy density/10^{10}. The curves denote:

A $\cong$ B total pressure [asymptotic formula (25) valid];

C: partial pressure of γ, ν, e, μ.

In this scale of energy density 10 corresponds to about 200 times m_p/V_0.

Fig. 16 : Relative energy fluctuations in V_0 as function of T/T_0. The curves denote:

A: total $\Delta E/E$;

B: $\Delta E_{\gamma,\nu,e,\mu}/E$;

C: $\Delta E_{hadron}/E$.

Curves A and C tend to $\alpha_0^{-1/2}$ for $T \to T_0$ (dotted line C).

Fig. 17 : Relative energy fluctuations in V_0 as function of energy/10^3 in V_0. The curves denote:

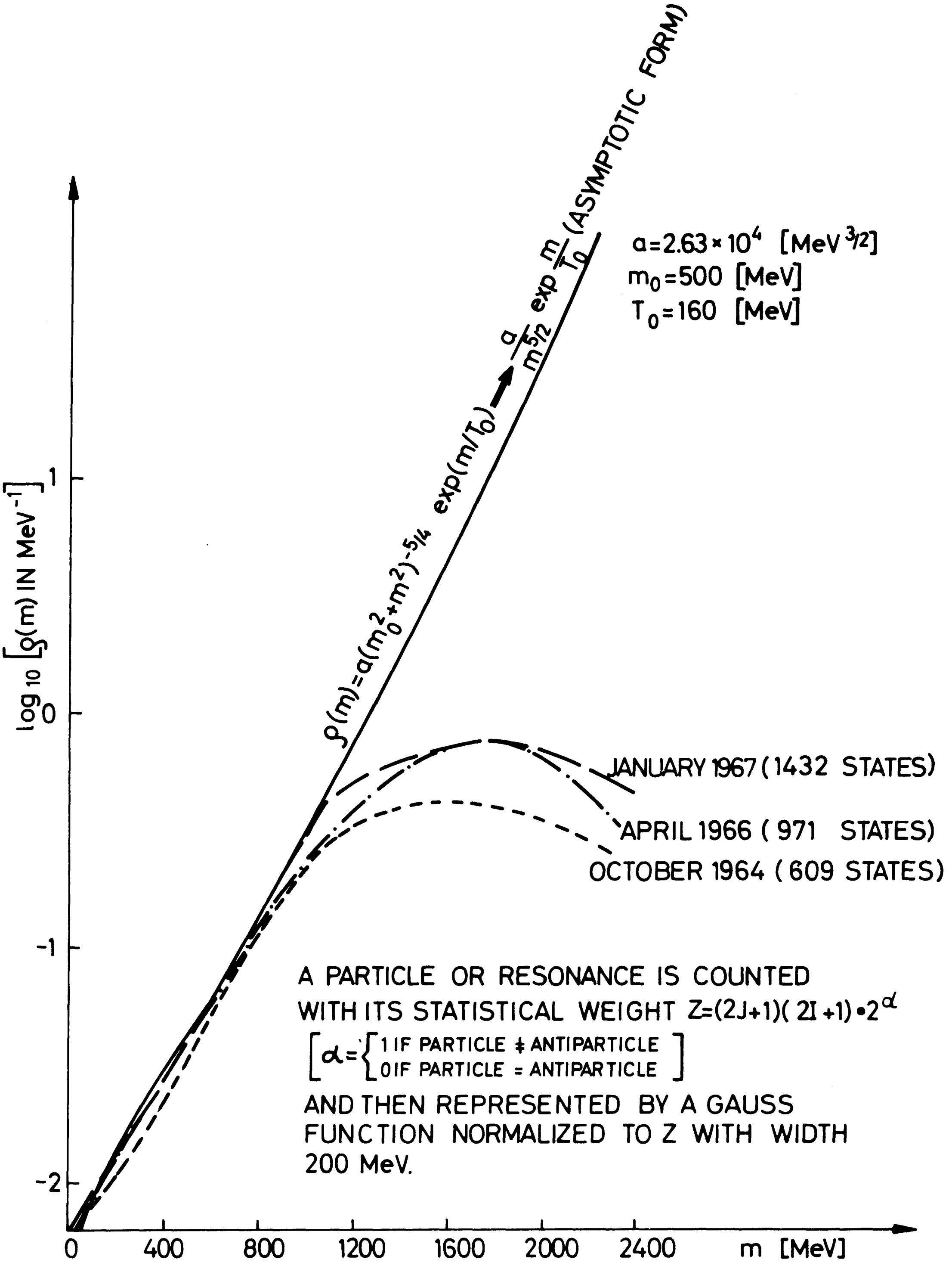
A: total $\Delta E/E$;

B: asymptotic value $\alpha_0^{-1/2}$;

C: $\Delta E_{hadron}/E$;

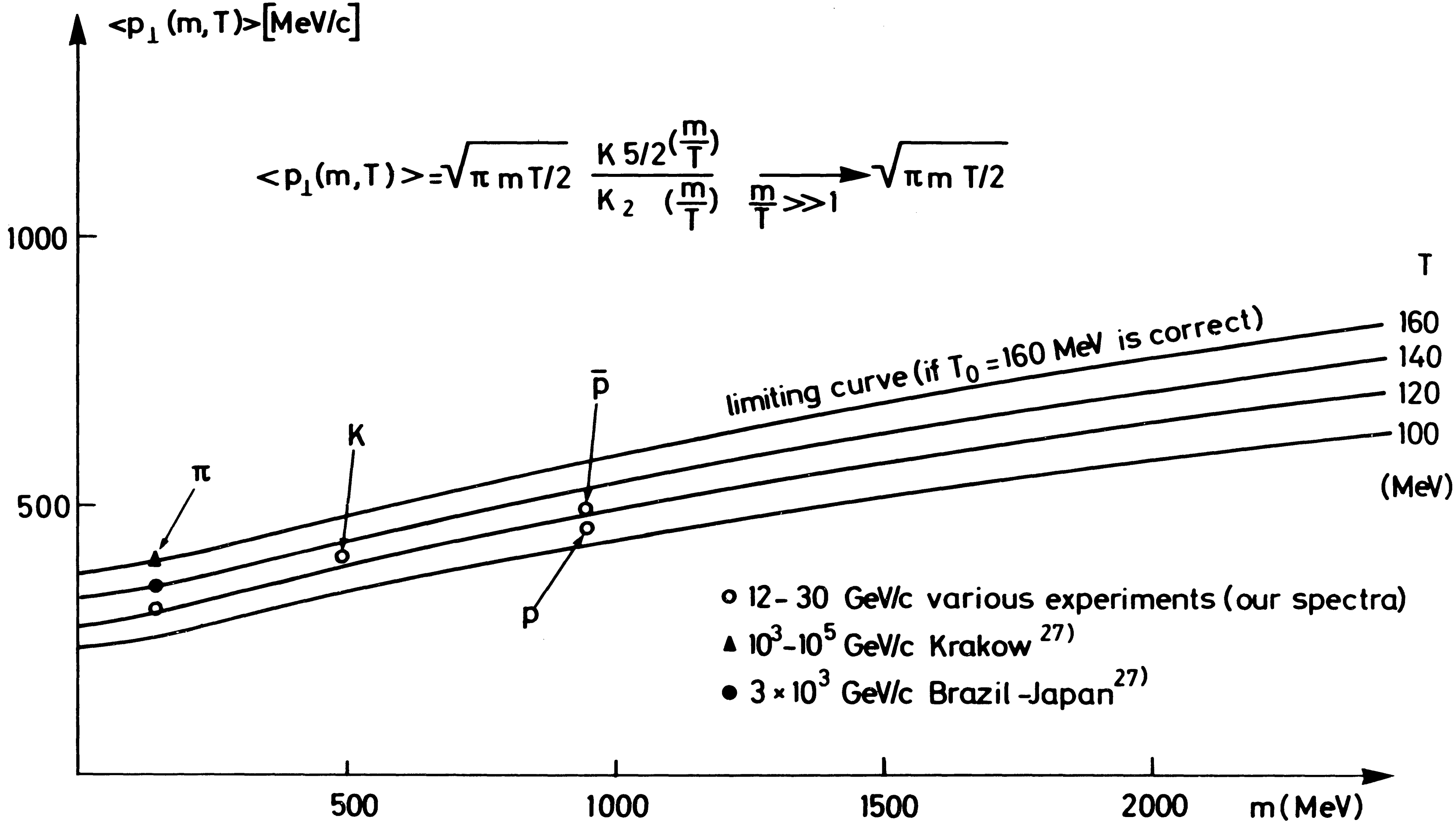
D: $\Delta E_{\gamma,\nu,e,\mu}/E$.

In this energy scale m_p lies near to 1.



ρ(m)=a(m₀²+m²)^{-5/4} exp(m/T₀) → a/m^{5/2} exp m/T₀ (ASYMPTOTIC FORM)
a=2.63×10⁴ [MeV 3/2]
m₀=500 [MeV]
T₀=160 [MeV]
log₁₀ [ρ(m) IN MeV⁻¹]
1
0
-1
-2
JANUARY 1967 (1432 STATES)
APRIL 1966 (971 STATES)
OCTOBER 1964 (609 STATES)
A PARTICLE OR RESONANCE IS COUNTED
WITH ITS STATISTICAL WEIGHT Z=(2J+1)(2I+1)•2^α
[α= {1 IF PARTICLE ≠ ANTIPARTICLE; 0 IF PARTICLE = ANTIPARTICLE}]
AND THEN REPRESENTED BY A GAUSS
FUNCTION NORMALIZED TO Z WITH WIDTH
200 MeV.
0
400
800
1200
1600
2000
2400
m [MeV]

Fig. 1



<p⊥(m,T)>[MeV/c]
<p⊥(m,T)> = √(π m T/2) K5/2(m/T) / K2(m/T) → (m/T >> 1) √(π m T/2)
1000
500
T
160
140
120
100
(MeV)
limiting curve (if T0 = 160 MeV is correct)
π
K
p̄
p
o 12-30 GeV/c various experiments (our spectra)
▲ 10^3-10^5 GeV/c Krakow 27)
● 3 × 10^3 GeV/c Brazil-Japan 27)
500
1000
1500
2000
m(MeV)

Fig. 2

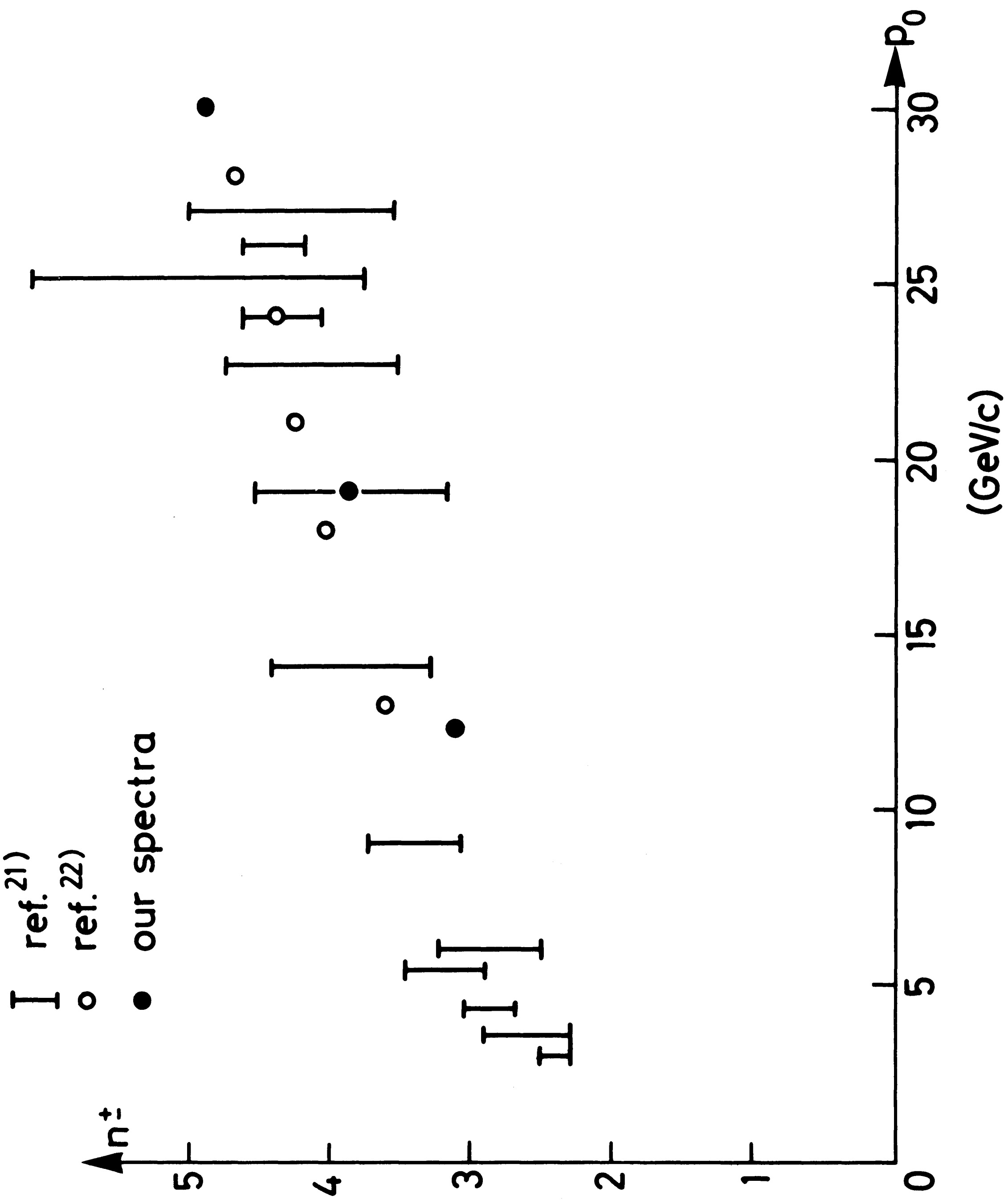


Fig. 3

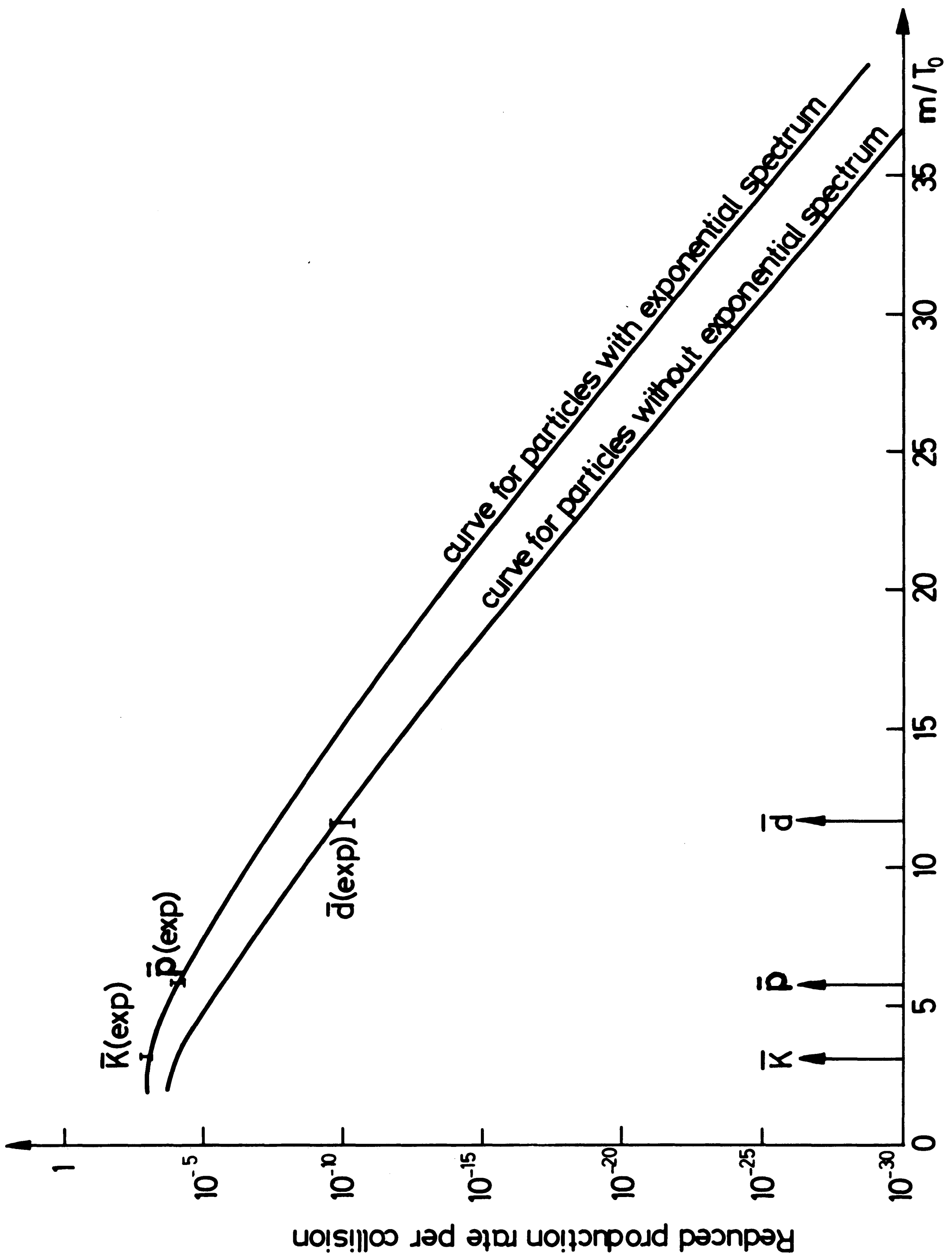

curve for particles with exponential spectrum
curve for particles without exponential spectrum
K̄(exp)
p̄(exp)
d̄(exp)
K̄
p̄
d̄
0
5
10
15
20
25
30
35
m/T₀
1
10⁻⁵
10⁻¹⁰
10⁻¹⁵
10⁻²⁰
10⁻²⁵
10⁻³⁰
Reduced production rate per collision

Fig. 4

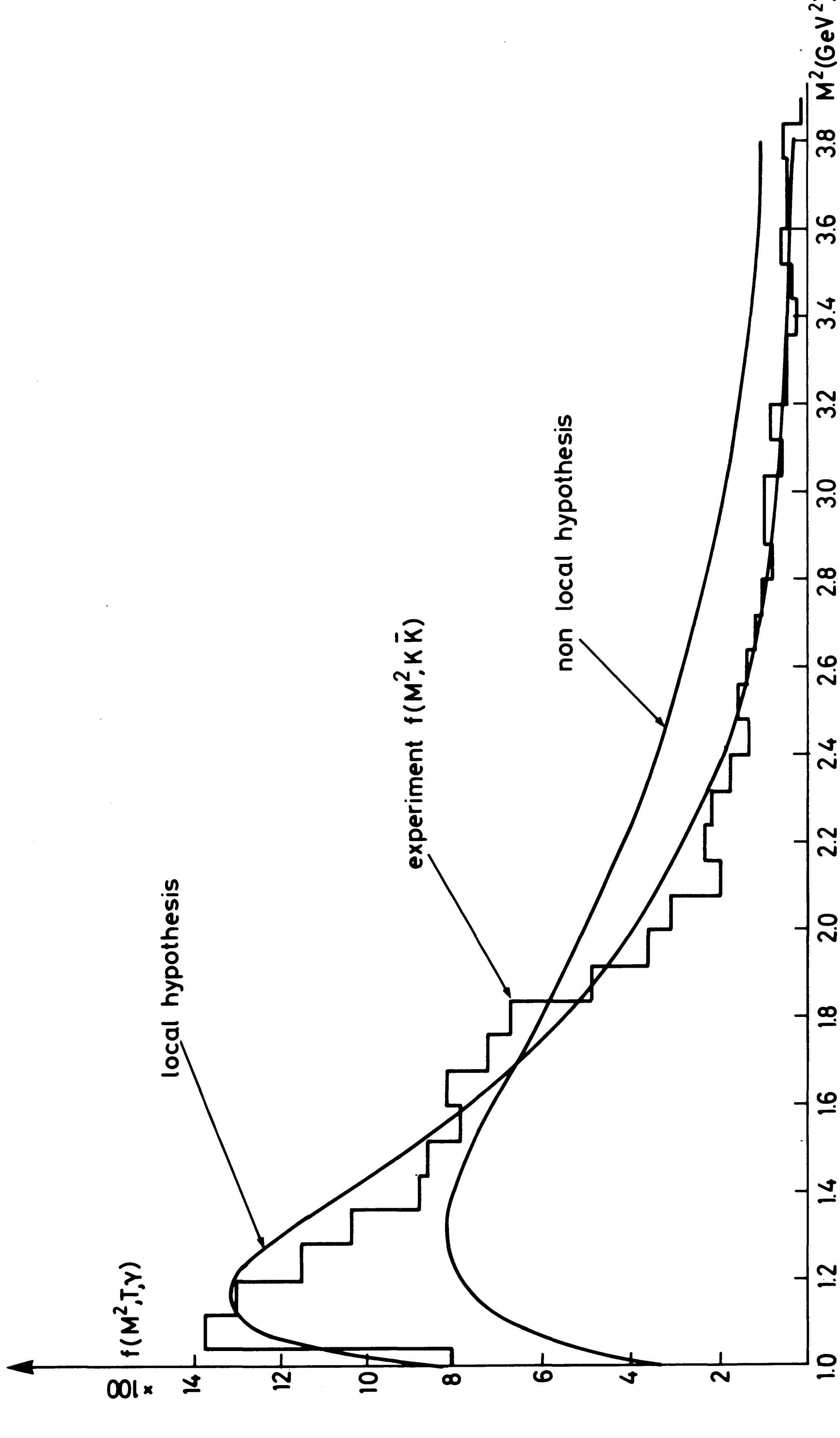
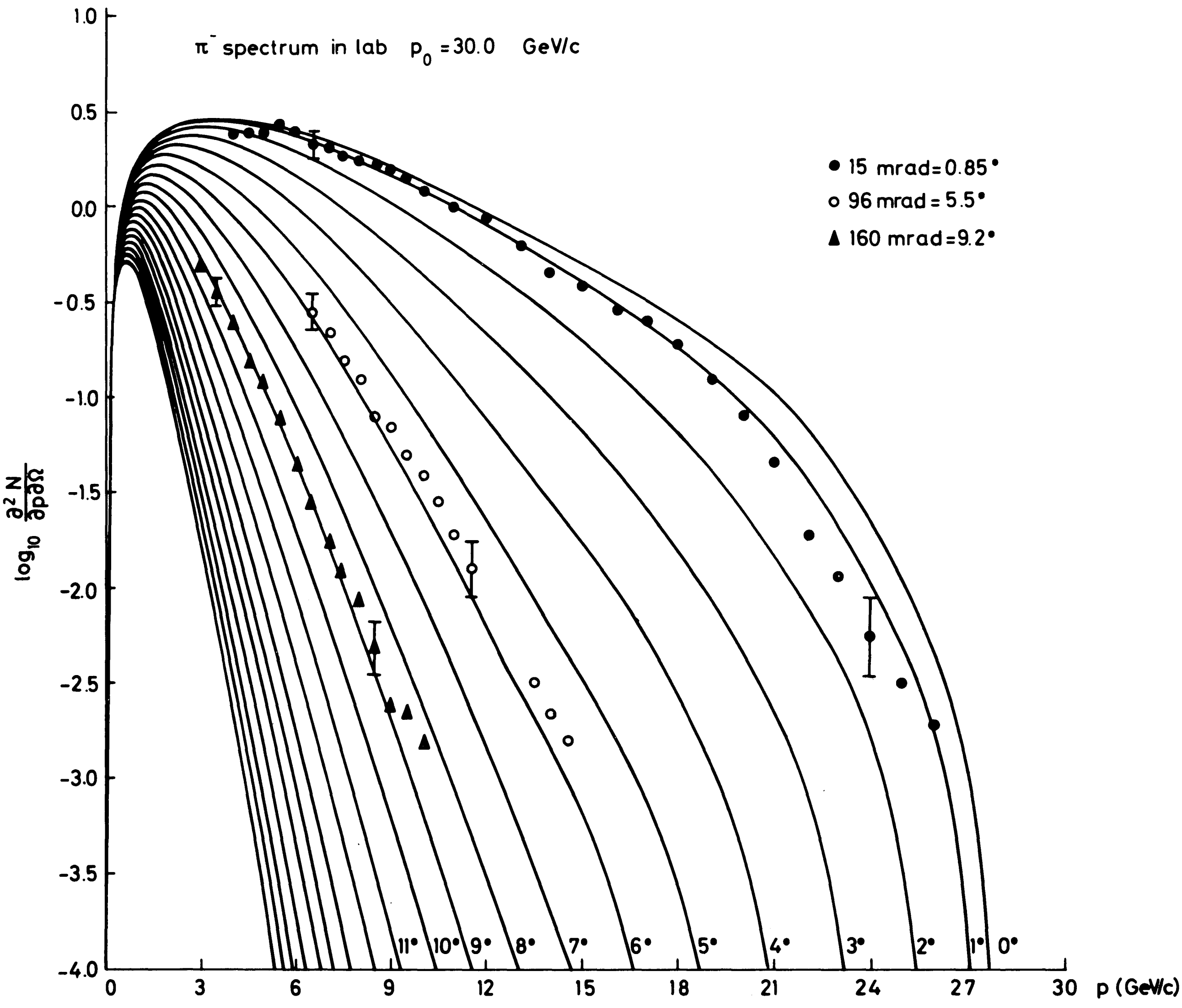
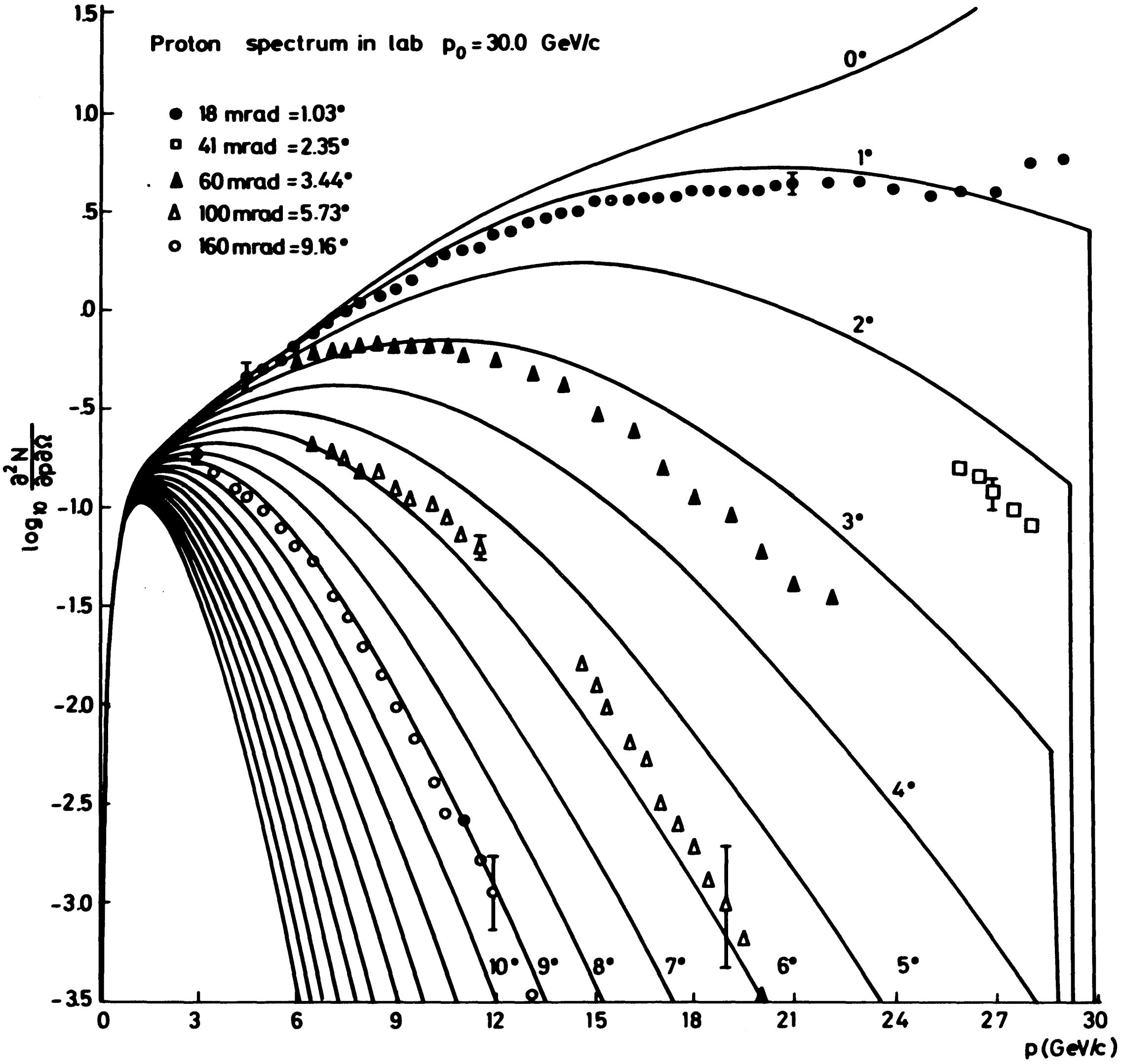

f(M²,T,Y)
x 10
local hypothesis
experiment f(M², K K̄)
non local hypothesis
M²(GeV²)
14
12
10
8
6
4
2
1.0
1.2
1.4
1.6
1.8
2.0
2.2
2.4
2.6
2.8
3.0
3.2
3.4
3.6
3.8

Fig. 5



π⁻ spectrum in lab p_0 = 30.0 GeV/c
● 15 mrad = 0.85°
○ 96 mrad = 5.5°
▲ 160 mrad = 9.2°
$\log_{10} \frac{\partial^2 N}{\partial p \partial \Omega}$
p (GeV/c)
1.0
0.5
0.0
-0.5
-1.0
-1.5
-2.0
-2.5
-3.0
-3.5
-4.0
0
3
6
9
12
15
18
21
24
27
30
11°
10°
9°
8°
7°
6°
5°
4°
3°
2°
1°
0°

Fig. 6



Proton spectrum in lab $p_0 = 30.0$ GeV/c
18 mrad = 1.03°
41 mrad = 2.35°
60 mrad = 3.44°
100 mrad = 5.73°
160 mrad = 9.16°
0°
1°
2°
3°
4°
5°
6°
7°
8°
9°
10°
$\log_{10} \frac{\partial^2 N}{\partial p \partial \Omega}$
1.5
1.0
.5
.0
-.5
-1.0
-1.5
-2.0
-2.5
-3.0
-3.5
0
3
6
9
12
15
18
21
24
27
30
p(GeV/c)

Fig. 7

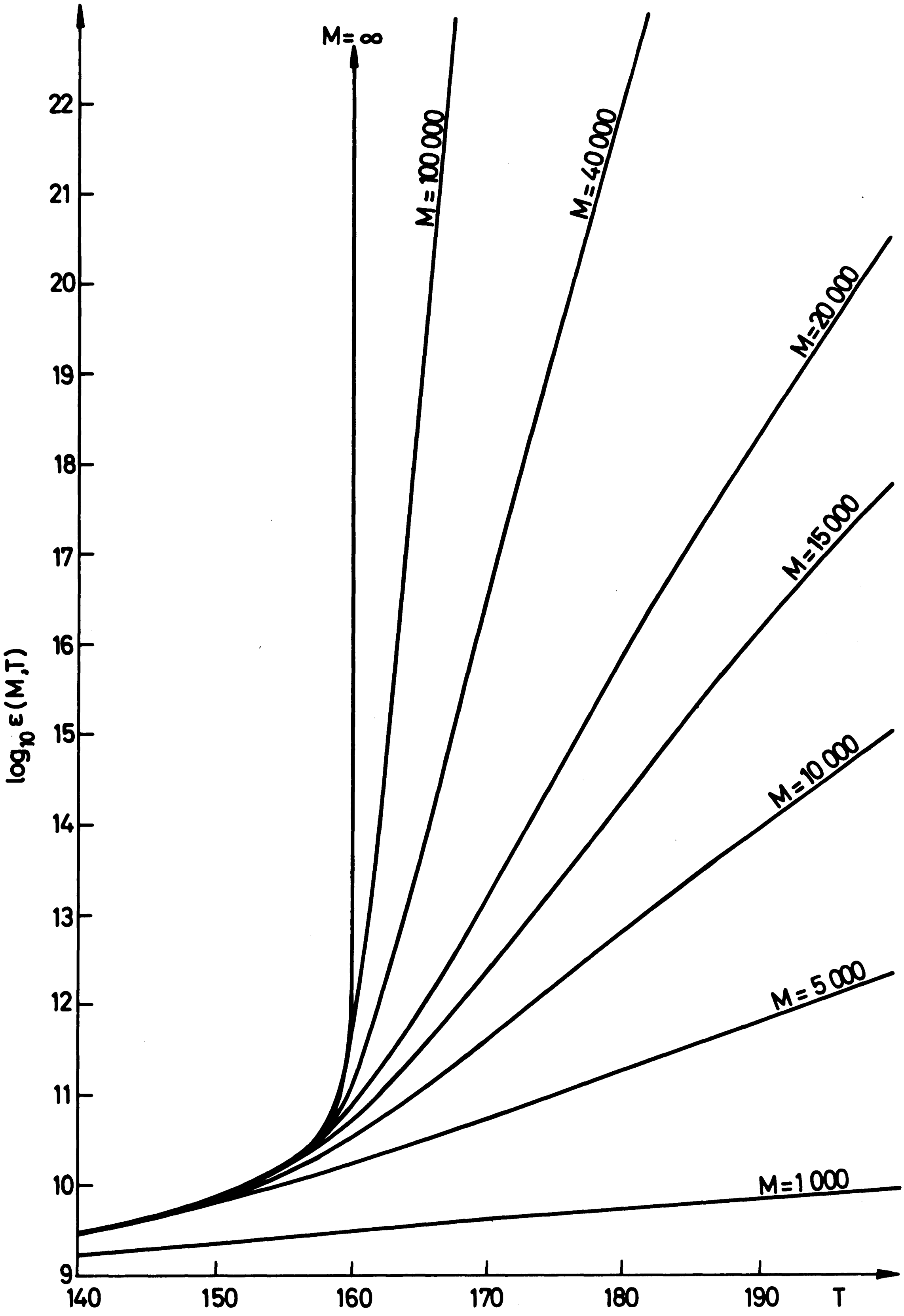

M=∞
M=100 000
M=40 000
M=20 000
M=15 000
M=10 000
M=5 000
M=1 000
22
21
20
19
18
17
16
15
14
13
12
11
10
9
log10 ε(M,T)
140
150
160
170
180
190
T

Fig. 8

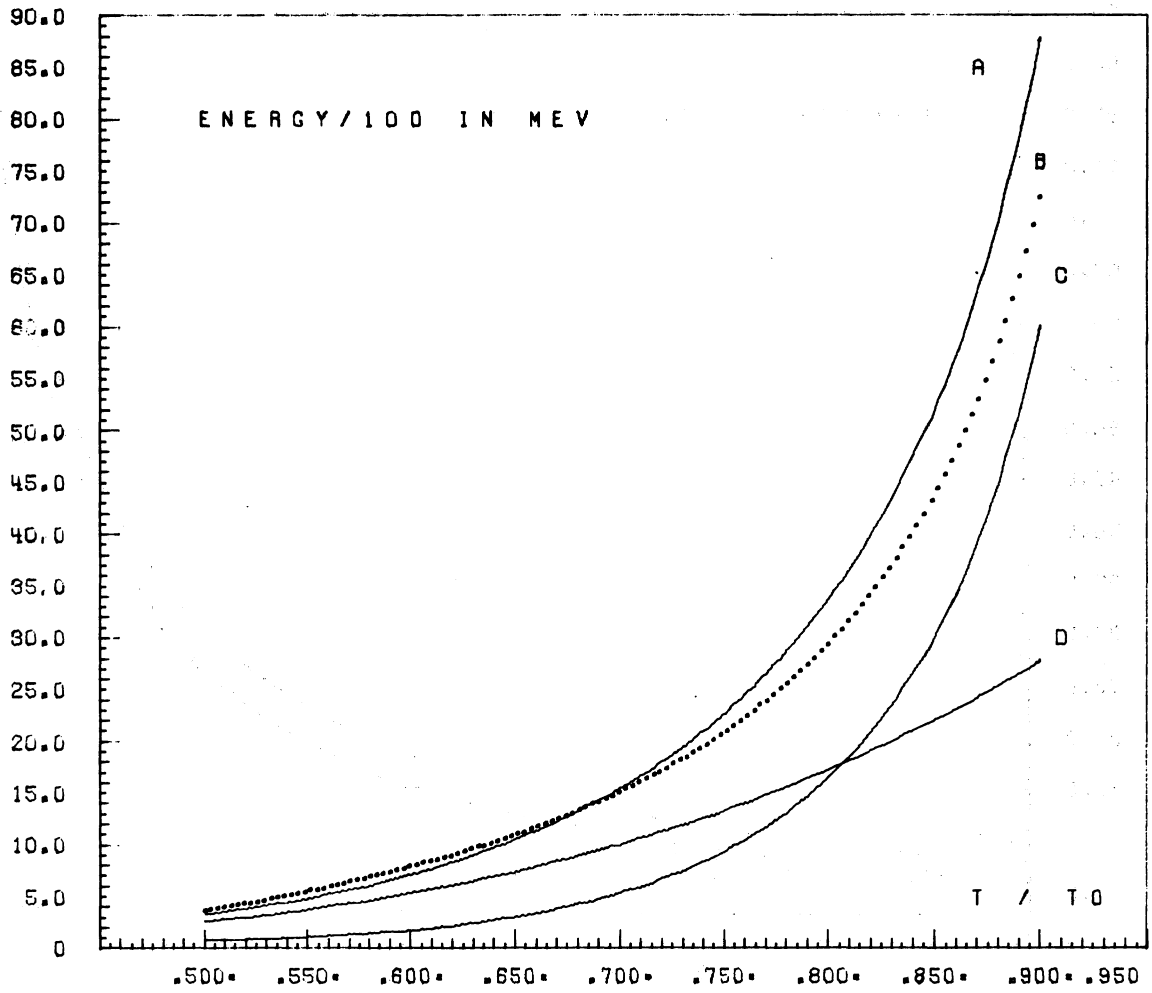

ENERGY/100 IN MEV
90.0
85.0
80.0
75.0
70.0
65.0
60.0
55.0
50.0
45.0
40.0
35.0
30.0
25.0
20.0
15.0
10.0
5.0
0
A
B
C
D
T / T0
.500
.550
.600
.650
.700
.750
.800
.850
.900
.950

Fig. 9

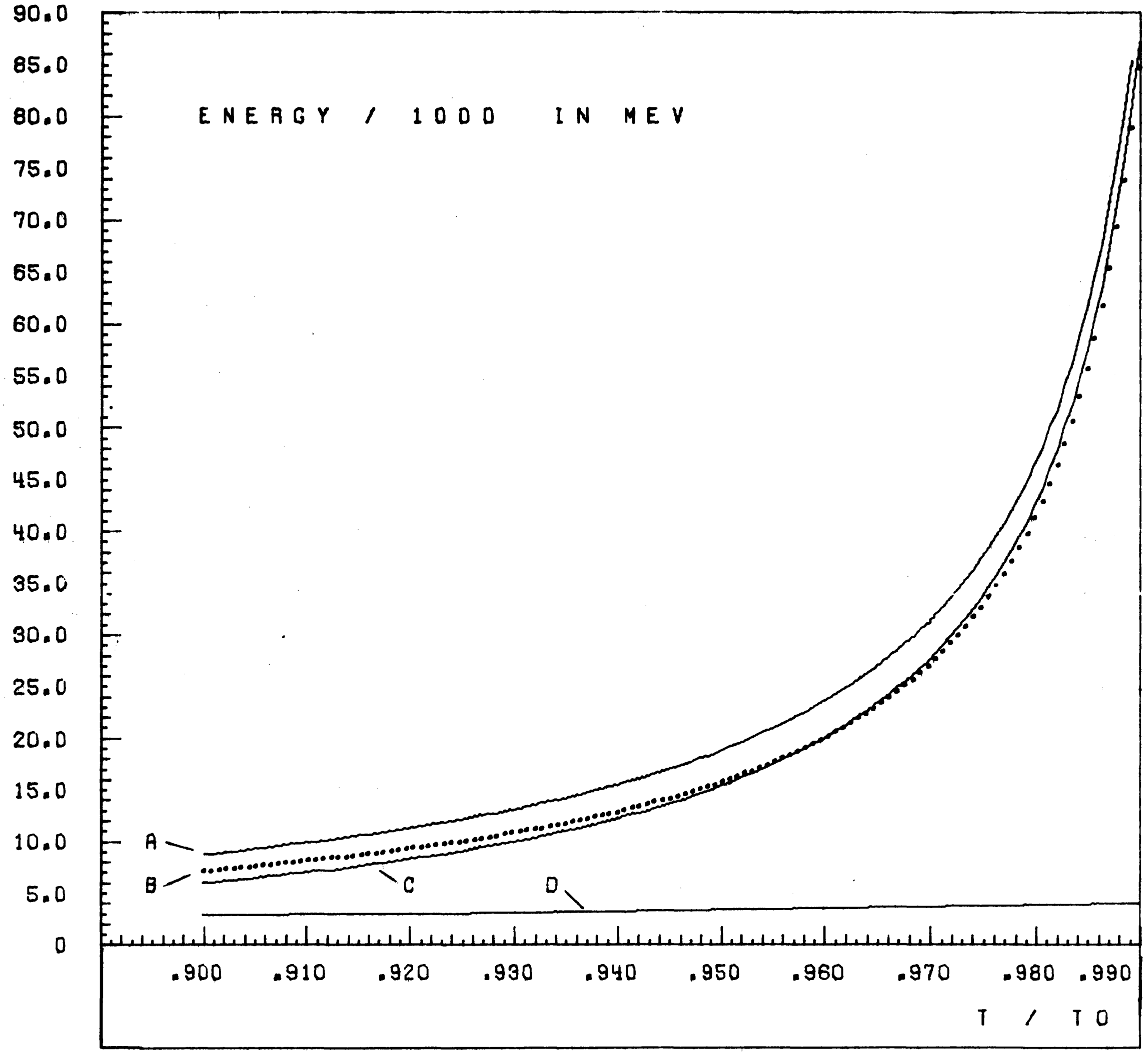
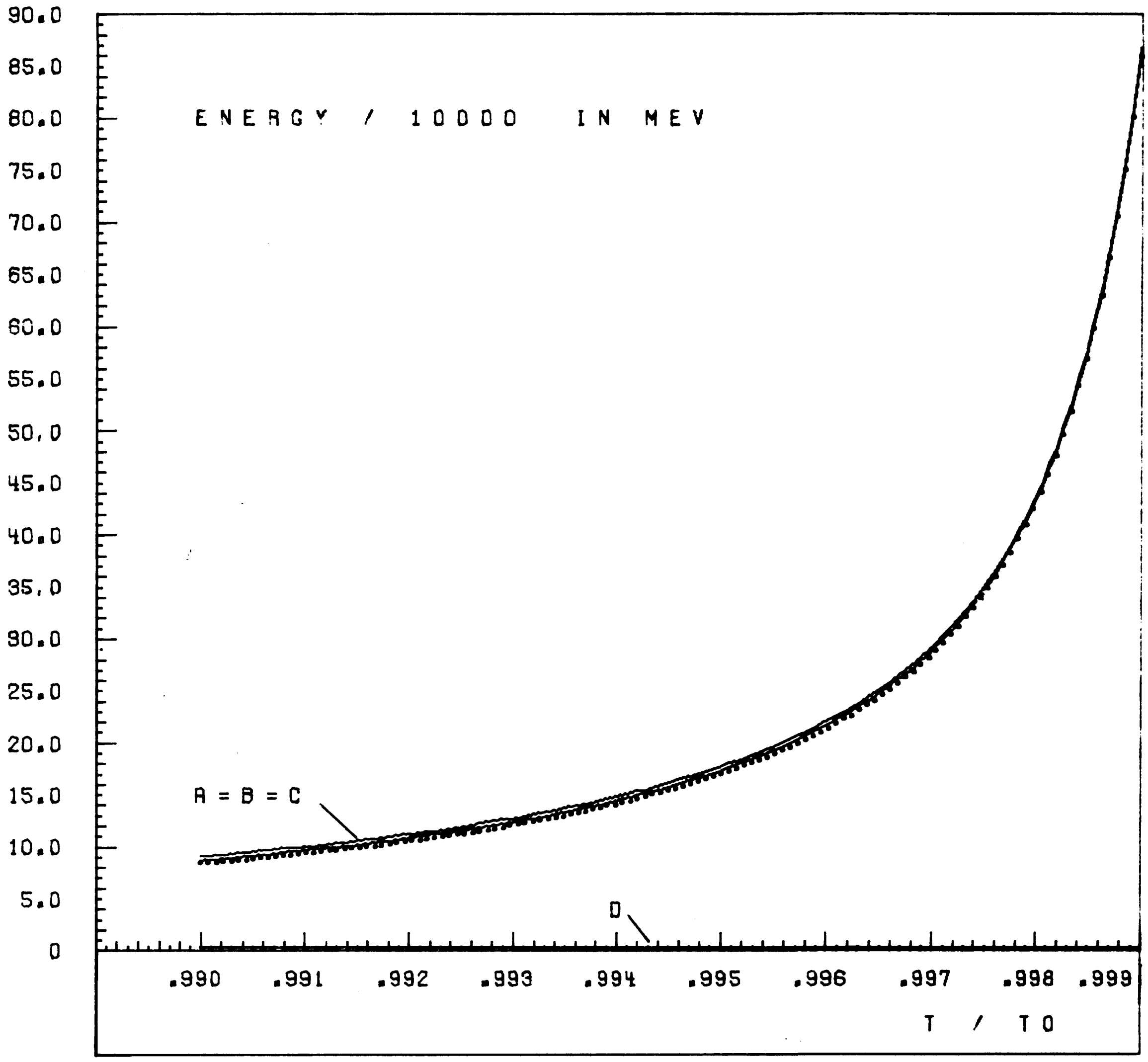

90.0
85.0
80.0
75.0
70.0
65.0
60.0
55.0
50.0
45.0
40.0
35.0
30.0
25.0
20.0
15.0
10.0
5.0
0
ENERGY / 1000 IN MEV
A
B
C
D
.900
.910
.920
.930
.940
.950
.960
.970
.980
.990
T / TO

Fig. 10



ENERGY / 10000 IN MEV
90.0
85.0
80.0
75.0
70.0
65.0
60.0
55.0
50.0
45.0
40.0
35.0
30.0
25.0
20.0
15.0
10.0
5.0
0
A = B = C
D
.990
.991
.992
.993
.994
.995
.996
.997
.998
.999
T / T0

Fig. 11

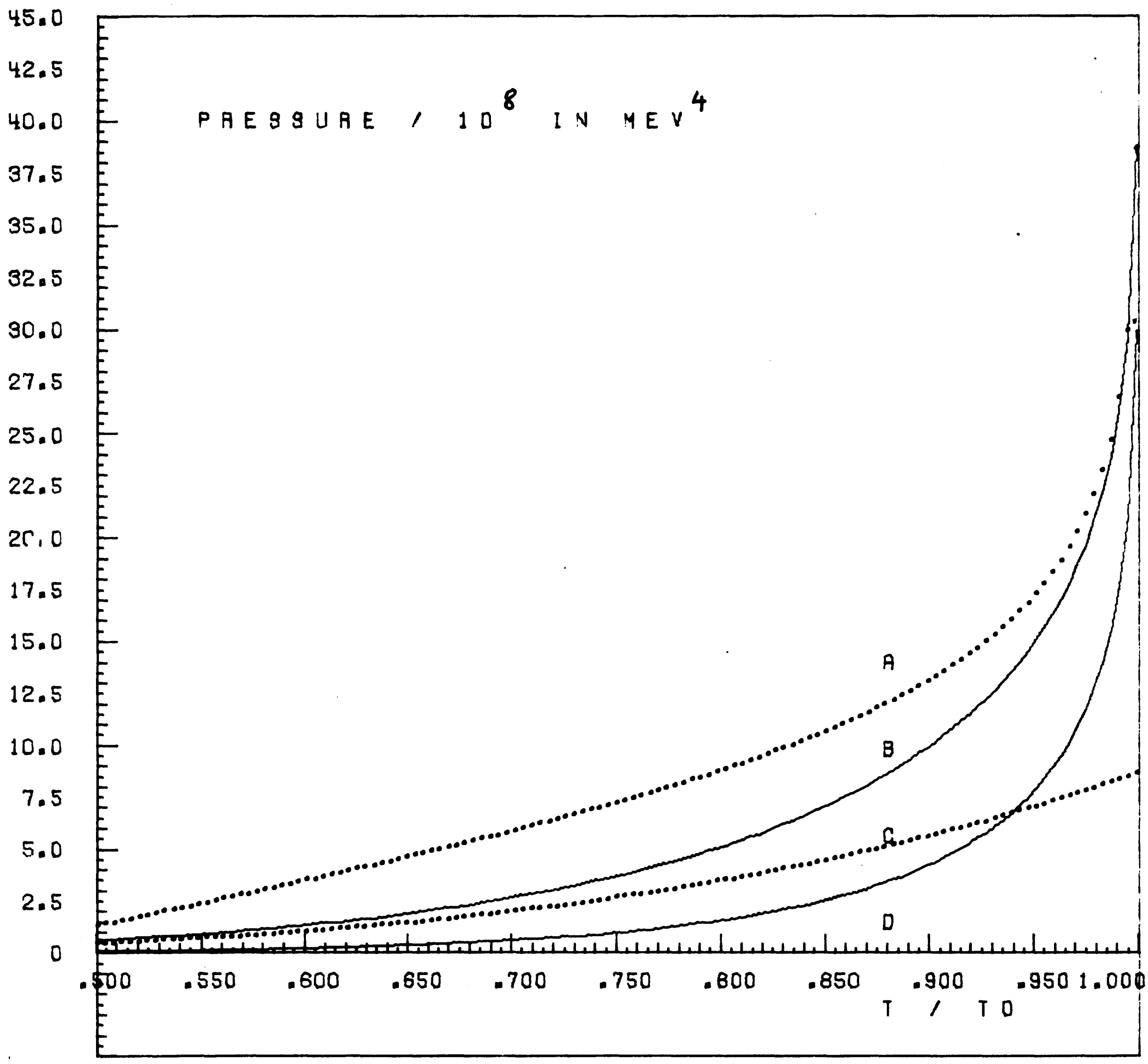

PRESSURE / 10^8 IN MEV^4
45.0
42.5
40.0
37.5
35.0
32.5
30.0
27.5
25.0
22.5
20.0
17.5
15.0
12.5
10.0
7.5
5.0
2.5
0
A
B
C
D
.500
.550
.600
.650
.700
.750
.800
.850
.900
.950
1.000
T / T0

Fig. 12

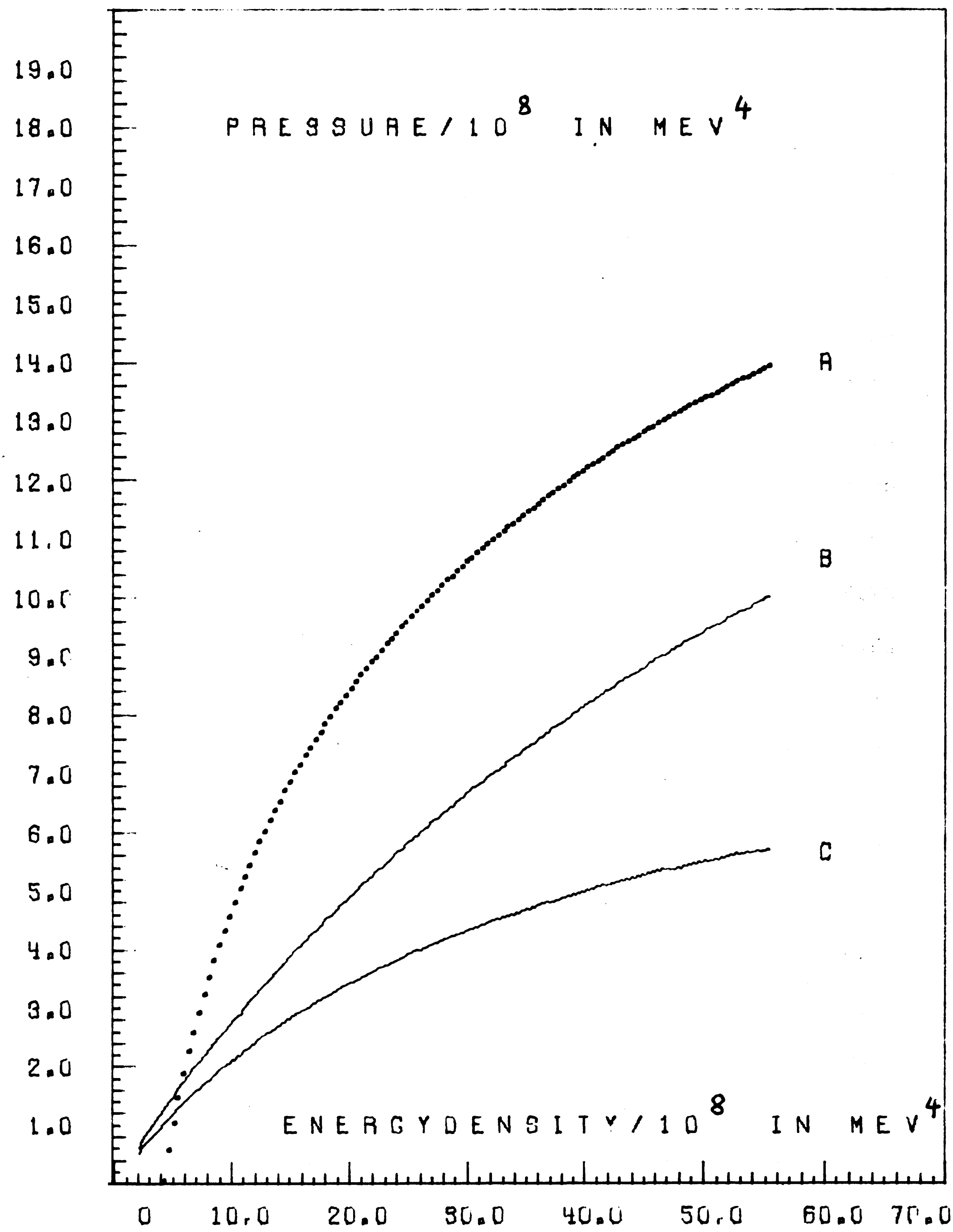

PRESSURE/10^8 IN MEV^4
19.0
18.0
17.0
16.0
15.0
14.0
13.0
12.0
11.0
10.0
9.0
8.0
7.0
6.0
5.0
4.0
3.0
2.0
1.0
A
B
C
ENERGYDENSITY/10^8 IN MEV^4
0
10.0
20.0
30.0
40.0
50.0
60.0
70.0

Fig. 13

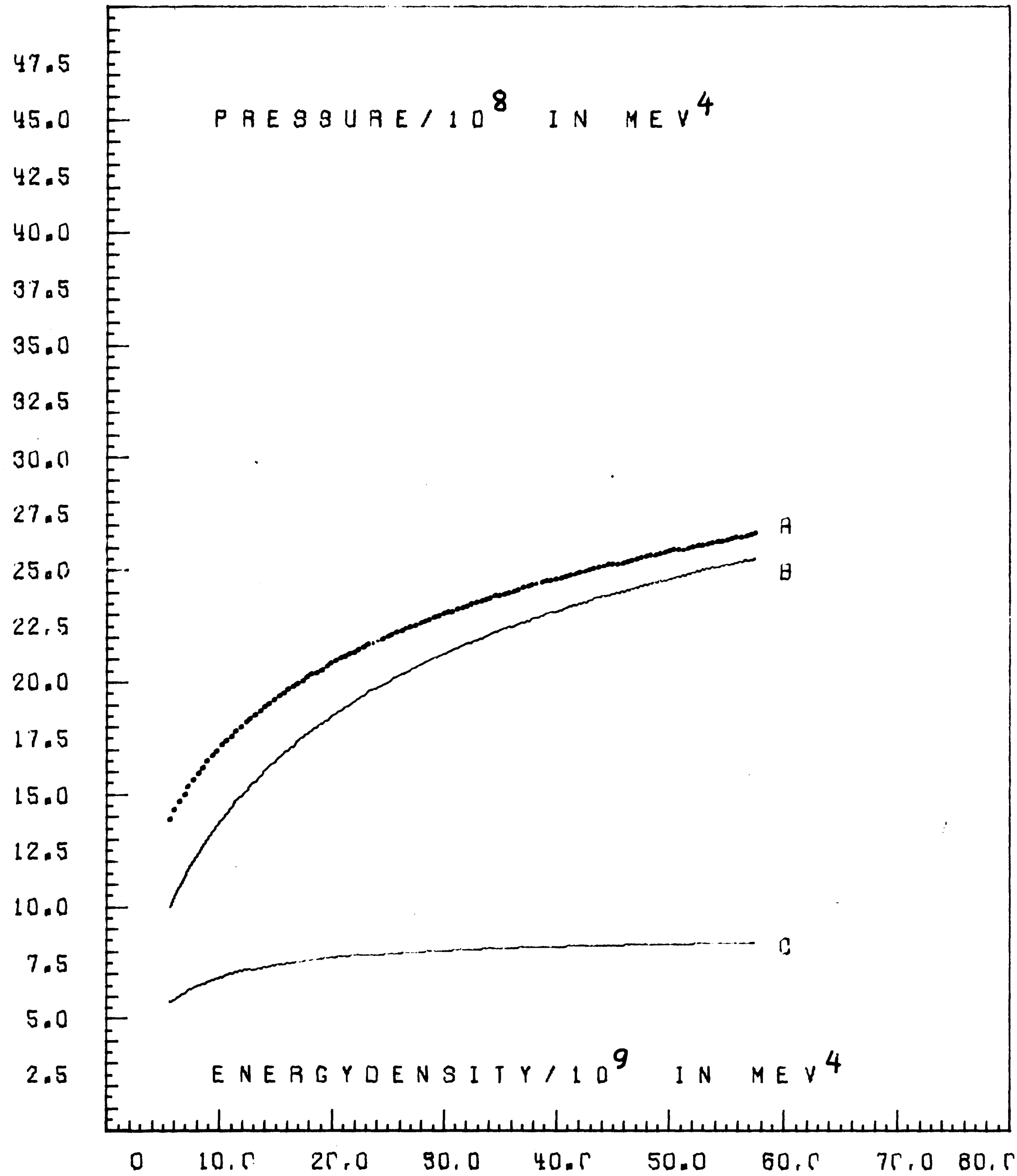

PRESSURE/10^8 IN MEV^4
ENERGYDENSITY/10^9 IN MEV^4
A
B
C
47.5
45.0
42.5
40.0
37.5
35.0
32.5
30.0
27.5
25.0
22.5
20.0
17.5
15.0
12.5
10.0
7.5
5.0
2.5
0
10.0
20.0
30.0
40.0
50.0
60.0
70.0
80.0

Fig. 14

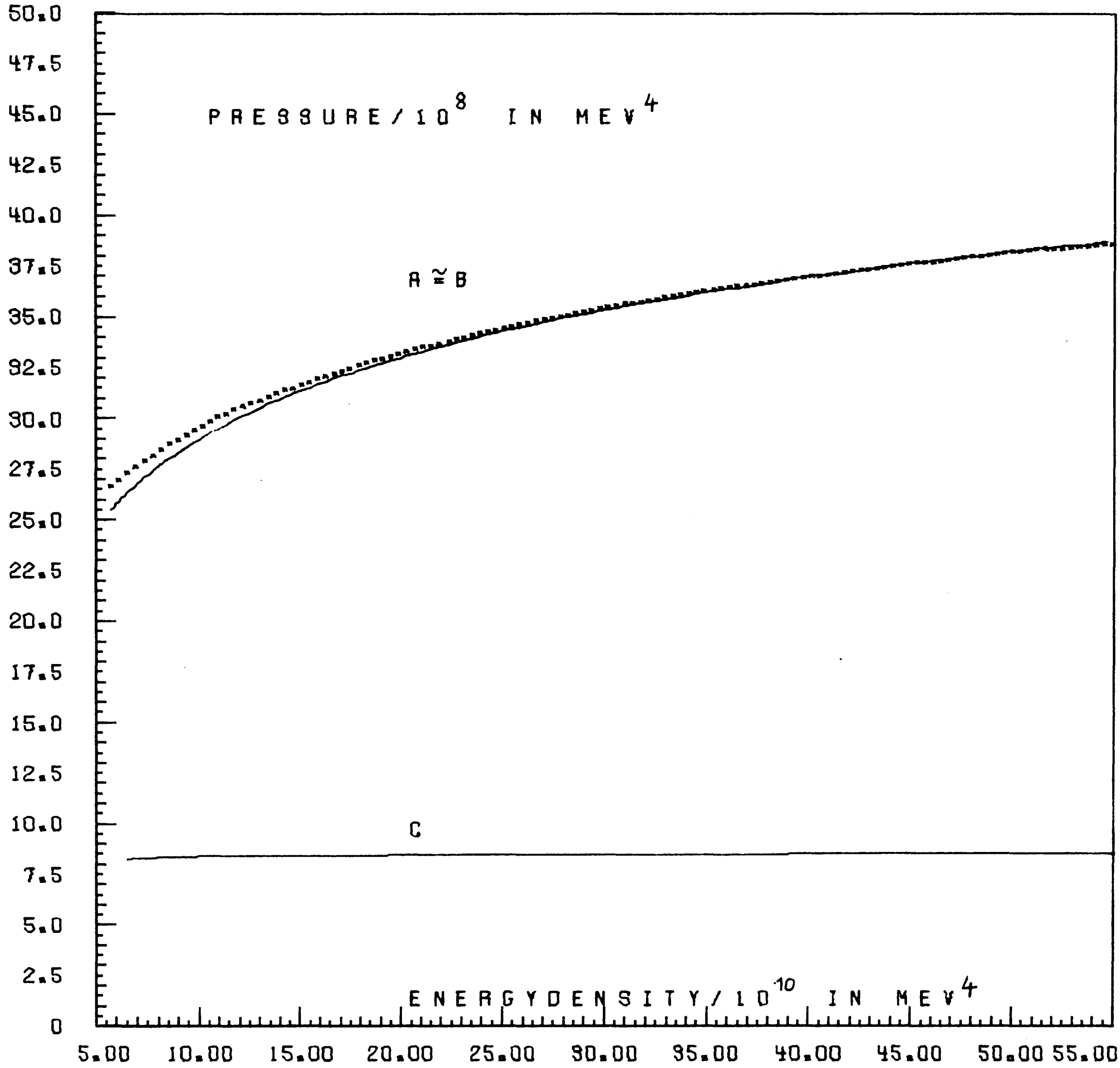

50.0
47.5
45.0
42.5
40.0
37.5
35.0
32.5
30.0
27.5
25.0
22.5
20.0
17.5
15.0
12.5
10.0
7.5
5.0
2.5
0
PRESSURE/10^8 IN MEV^4
A ≅ B
C
ENERGYDENSITY/10^10 IN MEV^4
5.00
10.00
15.00
20.00
25.00
30.00
35.00
40.00
45.00
50.00
55.00

Fig. 15

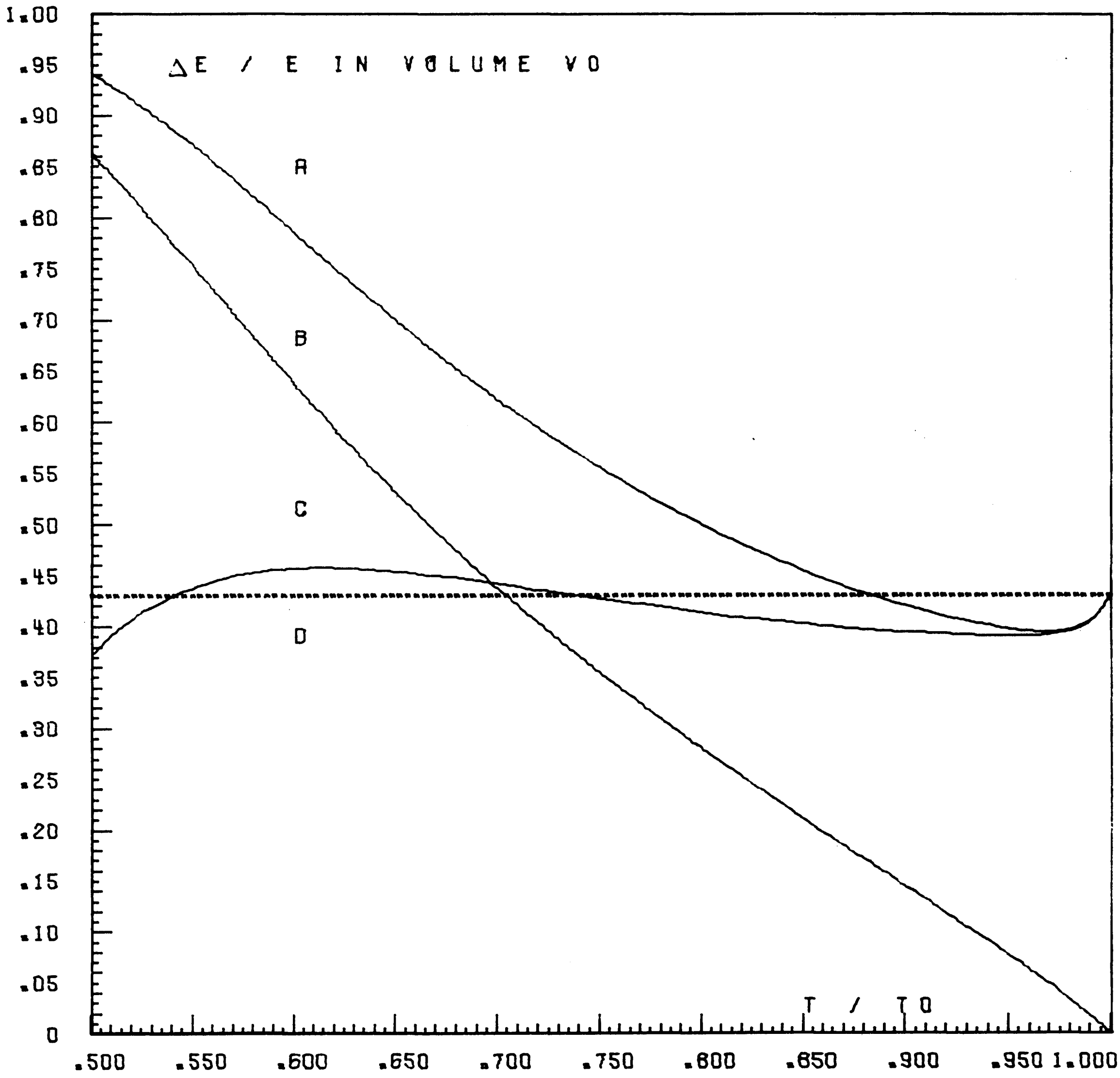
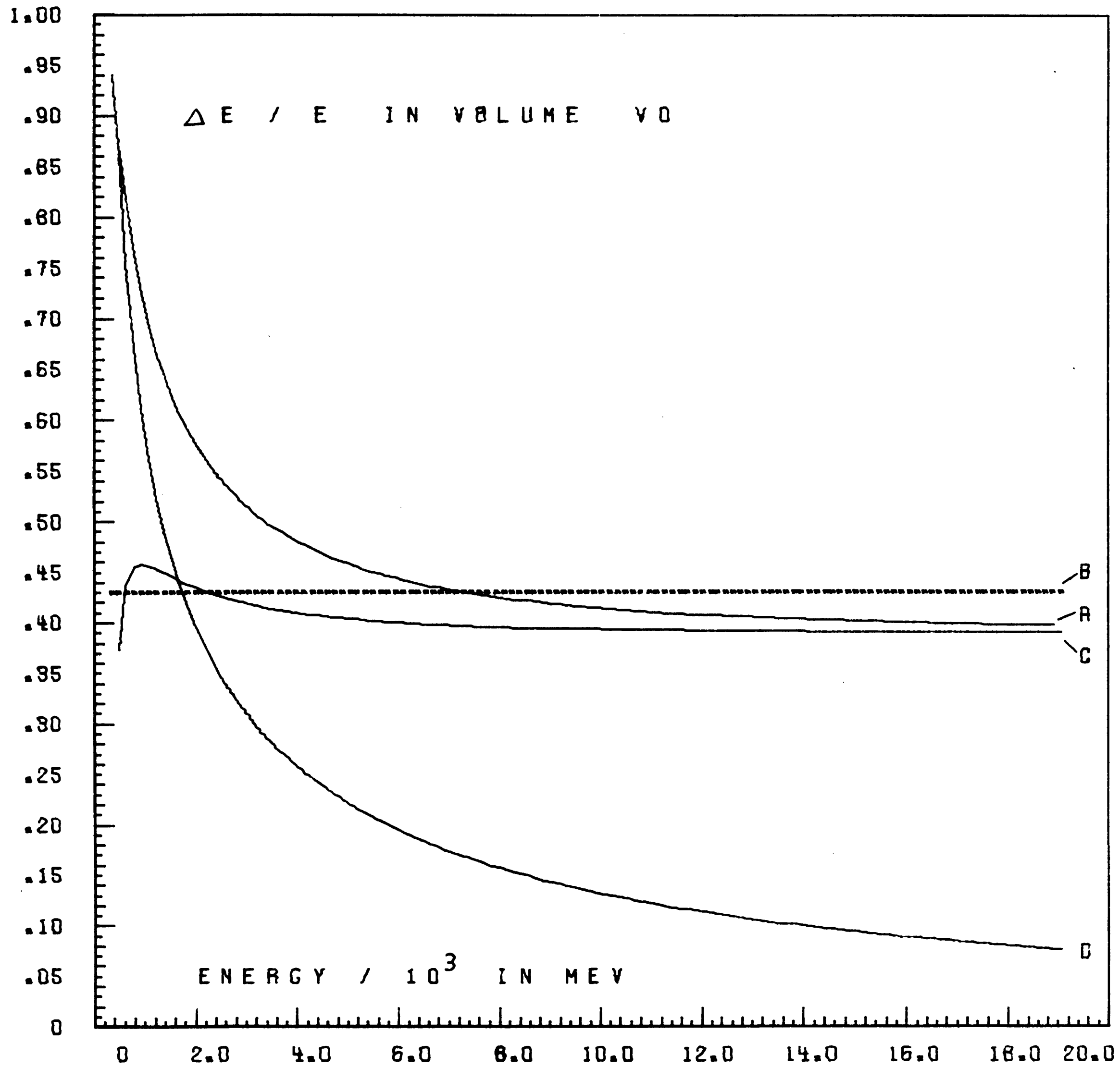

ΔE / E IN VOLUME VO
A
B
C
D
T / TO
1.00
.95
.90
.85
.80
.75
.70
.65
.60
.55
.50
.45
.40
.35
.30
.25
.20
.15
.10
.05
0
.500
.550
.600
.650
.700
.750
.800
.850
.900
.950
1.000

Fig. 16



△ E / E IN VOLUME V0
ENERGY / 10³ IN MEV
1.00
.95
.90
.85
.80
.75
.70
.65
.60
.55
.50
.45
.40
.35
.30
.25
.20
.15
.10
.05
0
0
2.0
4.0
6.0
8.0
10.0
12.0
14.0
16.0
18.0
20.0
B
A
C
D

Fig. 17