Università degli Studi di Roma Tre

Department of Physics "E. Amaldi"
Doctorate in Physics - XX Cycle

Doctorate Thesis

Università ROMA TRE – Université de la Mediterranée
***Co–tutorship* Agreement**

Complete Loop Quantum Gravity Graviton Propagator

by

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To my family,
Giulia
and anyone who has supported
or beared me in this journey

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Introduction

The XXth century has begun changing our understanding of the physical laws of the Universe. The advancement of human's technology has brought experimental data inexplicable in terms of the conceptual framework of Newtonian laws that has dominated for centuries. The change has been drastic: a twofold revolution of the physical and philosophical conception of the world. The microscopic observations of nuclear and subnuclear physics have been explained with Quantum Mechanics (QM) evolved in Quantum Fields Theory (QFT) and, on the other side, the large scale phenomena of the Universe have been explained with General Relativity (GR). QM and GR are the two conceptual pillars on which modern physics is built. The empirical success of the two theories has been enormous during the last century and so far there are not observed data in contradiction with them. However, QM and GR have destroyed the coherent picture of the world provided by Newtonian mechanics: each has been formulated in terms of assumptions contradicted by the other theory. On the one hand QM requires a static spatial background and an absolute time flow, when GR describes spacetime as a single dynamical entity; moreover GR is a classical deterministic theory when Quantum Mechanics is probabilistic and teach us that any dynamical field is quantized. Both theories work extremely well at opposite scales but the revolution they have started is clearly incomplete [1] unless we want to accept that Nature has opposite foundations in the quantum and in the cosmological realm.

The search for a theory which merges GR and QM in a whole coherent picture is the search for a theory of Quantum Gravity (QG). At the present stage we have not such a theory. The essential difficulty is that the theoretical framework is not at all helped by experimental measurement. The reason is simply that the effects of quantum gravity are supposed to become predominant at the Planck scale that is far out of reach of any technological apparatus of humanity. Hence, there is currently no way to test the validity of any theoretical framework by direct experiments, like for example trough the accelerators. To build this theory we have only the two pillars; what are the core lessons of QFT and GR?

- We have learned from GR two indications on reality: First, the world is relational; only events independent from the coordinate are meaningful; physics must be described by generally covariant theories. Second, the gravitational field *is* the geometry of spacetime. The spacetime geometry is fully dynamical: The gravitational field defines the geometry on top of which its own degrees of freedom and those of matter fields propagate. GR is not a theory of fields moving on a curved background geometry; GR is a theory of fields moving on top of each other[2]. The gravitational field is the spacetime field.

- We have learned from QFT that all dynamical fields are quantized. A quantum field

is made of quanta propagating the interactions, and has a probabilistic dynamics, that allows quantum superposition of different states.

If we merge the two lessons we might expect at small scales a "quantum spacetime" formed by "quanta of space" evolving probabilistically, and allowing "quantum superposition of spaces" in a theory fully background independent. The problem of QG is then to give a precise mathematical and physical meaning to the notion of "quantum spacetime".

The quest for QG is then mainly theoretical but it is needed if we want answer to fundamental physical questions. In particular classical GR predicts the existence of singularities such as those dealing with black hole physics and cosmology. Near spacetime singularities the classical description of the gravitational degrees of freedom simply breaks down. Questions related to the fate of singularities in black holes or in cosmological situations or those related with information paradoxes can only be answered with a theory of QG.

There are essentially two research programs that can be considered a candidate theory of QG: String Theory (ST) (at the present stage existing as a perturbative theory) and Loop Quantum Gravity (LQG) (in its canonical and covariant versions) . The first postulate that the particles are not pointlike but extended objects and builds up the theory on the ground of usual QFT trying to unify all the interactions; doing so it is not able, at the present stage, to implement background independence. The second, without the aim of unifying all the interactions, try to merge QM and GR taking seriously the lessons of GR and in particular its essential feature: generally covariance or diffeomorphisms invariance, simply translatable in background independence.

This thesis is in the framework of LQG.

LQG (there are a lot of reviews but the basic ones are Carlo Rovelli's book [2], Thomas Thiemann book [3], Ashtekar [4] and Perez [5] ones) is based on the canonical quantization program formulated by Dirac and its covariant version is based on the path integral approach developed by Feynman; the theory has a big predictive power at the Planck scale, in fact its main achievements are the construction of a kinematical Hilbert space [6, 7], the derivation of a discrete spectrum for the geometrical area and volume operators [8, 9, 10], the explanation of the black-hole entropy (see [4] and references therein) and a theory of Loop Quantum Cosmology ([11] see also [12]) but it is not yet able to make contact with the low energy world. The directions in which lacks are greater are in fact the limit of low energy of the theory, a way of recover classical solutions of GR and the possibility to calculate scattering amplitudes [13].

This thesis is a step in the attempt to try to fill this gap.

In particular we face the problem of building the graviton propagator in a background independent formalism. This can be considered as the basic building block in the construction of scattering amplitudes that are observable, comparable quantities. Moreover the semi–classical limit (where pure quantum effects are negligible) of these quantity can be compared with the one calculated in the conventional QFT; the positive or negative result of this comparison would be a strong theoretical confirmation of the theory.

The problem of defining a graviton propagator

The first possible way explored for the construction of QG is the usual QFT approach applied to perturbative GR that leads to a non–renormalizable theory; we can build the propagator of this theory but the quantum corrections bring infinities; neverthless its leading order is well known and it can be taken as a reference point for a candidate theory of QG.

Let us focus for a while on non–renormalizable perturbative GR.

To define the QFT we *need* a notion of non dynamical background on the top of which physics happens; one proceed splitting the degrees of freedom of the gravitational field in terms of a fixed background geometry $\eta_{\mu\nu}$ for $\mu, \nu = 1 \cdots 4$ and dynamical metric fluctuations $h_{\mu\nu}$. Explicitly, one writes the spacetime metric as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}. \tag{1}$$

apart from the infinities that arise in such treatments, this kind of theory requires the introduction of a *classical* background. The first point about this splitting is that it has no intrinsic meaning in GR. The first of the core teachings of GR is that the Nature is generally covariant; any physical phenomenon has to be invariant under diffeomorphism transformations. If we apply this lesson here, what do we get?

As noted in [5] for a generic space time metric $g_{\mu\nu}$ we can write

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} = \tilde{\eta}_{\mu\nu} + \tilde{h}_{\mu\nu}, \tag{2}$$

where $\eta_{\mu\nu}$ and $\tilde{\eta}_{\mu\nu}$ can be characterized by different background light-cone structures of the underlying spacetime $(M, g_{\mu\nu})$; this is a priori dangerous because η and $\tilde{\eta}$ may carry different notions of causality and the QFT *needs* a fixed notion of causality. The second point is that (1) make sense in classical GR when one considers perturbations of a fixed background $\eta_{\mu\nu}$ but in QG one has to deal with arbitrary superpositions of spacetimes; in a QG theory the above splitting can be meaningful only if we interpret η as a semi-classical states "peaked", around the classical geometry $\eta_{\mu\nu}$ with small fluctuations on it. How can a similar splitting have meaning in the full quantum regime?

If we want to take GR seriously we have to bring diff-invariance in the quantum realm and this can't be done splitting and quantizing; the best that we can do is quantize non–perturbatively, implementing the diff invariance of GR at quantum level, and then reproduce (1) in an appropriate semiclassical regime, at that point we can compare the resulting theory with the perturbative QG disregarding the quantum corrections.

In this thesis we try to realize this project starting from LQG: In fact, it is still not known if the various LQG and spinfoam models proposed contain semi–classical states that would reproduce Einstein's gravity in some limit. Neither it is known how to do a perturbative expansion that would allow calculations of scattering amplitudes between excitations of these semi–classical states. We would like to find an expression which reproduces the usual graviton propagator, once only small excitations around a flat background metric are considered.

The thesis is organized as follows

- In the first chapter we introduce the main ideas of LQG (following [5]) that will define the kinematical level of the theory; LQG defines the kinematical Hilbert space of the theory: the spinnetwork base. Spinnetworks define quantized 3–d metrics and their diff–invariant

versions define quantized 3d-geometries. The chapter will also provide the basic grasping operators needed to reproduce the graviton excitations of a flat background.

- In the second chapter we describe the covariant approach that starts where LQG find the biggest difficulties i.e in the definition of the dynamics; covariant methods deal more easily with interactions and dynamics: it thus seems natural to choose the formalism of Spinfoams Models (SM) to study the semi–classical limit. With the help of this chapter we will able to deal with the quantum dynamics of the quantized space.

- In the third chapter we will introduce Rovelli's formulation of background–independent n–point function (introduced in [14, 15, 16]) and we will construct some of the graviton propagator components as they emerge from these works

- In the forth and the fifth chapter based on original work [17, 18] we complete the LQG calculation of all its components; in doing so we discover difficulties that open the way to modification of the principal SFM for QG needed to make contact with the known perturbative graviton propagator.

At the end of this construction will end up with a propagator calculated in the LQG formalism that *coincides* with the well known propagator of a spin2 massless particle; we will get the graviton propagator from LQG.

Chapter 1

Introduction to Loop Quantum Gravity

In this chapter we briefly summarize the construction of LQG. We introduce the formulation of classical GR in terms of ADM variables and then we switch to the triad formulation as a step to the introduction of the Ashtekar variables. These variables allow for a description of GR in terms of Yang-Mills fields. The construction of the theory proceeds with the application of the canonical Dirac quantization program of constrained systems. We end up with a set of three quantum constraints: the solution of the first brings to a kinematical Hilbert space of states gauge invariant called *spinnetworks*, the implementation of the second to their $Diff$ invariant version called *s-knots*. This complete the kinematics of the theory. The third constraint, corresponding to the implementation of the dynamics in the quantum theory is less under control but leads to the introduction of the spinfoam models. All over the chapter, focusing on the geometrical meaning of the theory's variables, we discover one of the most relevant feature of LQG: the emerging of a discretized quantum space. This chapter is based on the reference [5] to which we refer the reader for more details.

1.1 Canonical formulation of GR in ADM variables

The action of general relativity in metric variables is given by the Einstein-Hilbert action

$$S[g_{\mu\nu}] = \frac{1}{2\kappa} \int dx^4 \, \sqrt{-g} R, \tag{1.1}$$

where $\kappa = 8\pi G/c^3 = 8\pi \ell_p^2/\hbar$, g is the determinant of the metric $g_{\mu\nu}$ and R is the Ricci scalar.

The starting point for the Hamiltonian formulation in term of ADM variables [19] is the introduction of a spacetime foliation in terms of space-like three dimensional surfaces Σ. For simplicity we assume Σ without boundaries. The ten components of $g_{\mu\nu}$ are replaced by the six components of the induced Riemannian metric $q_{ab}(a, b = 1, 2, 3)$ of Σ plus the three components of the shift vector N_a and the lapse function N.

$$g_{\mu\nu} \xrightarrow[ADM]{} \{q_{ab}, N_a, N\} \tag{1.2}$$

In terms of these variables, after performing the standard Legendre transformation, the action of general relativity becomes

$$S = \frac{1}{\kappa}\int dt \int_\Sigma d^3x\{\dot{q}_{ab}\pi^{ab} + \dot{N}P + \dot{N}^a P_a - [\lambda P + \lambda^a P_a + N^a V_a + NC]\} \tag{1.3}$$

where

$$\pi^{ab} = \kappa\frac{\delta S}{\delta q_{ab}} = \sqrt{\det(q)}[q^{ac}q^{bd} - q^{ab}q^{cd}]K_{cd} \tag{1.4}$$

is the momenta conjugate to q_{ab} related to the extrinsic curvature K_{ab} of Σ and P, P_a are the momenta conjugate to N and N^a.

The functions $C(q_{ab}, \pi^{ab}), V_a(q_{ab}, \pi^{ab})$ explicitly given by

$$C(q_{ab}, \pi^{ab}) = -(q^{1/2}[R^{(3)} - q^{-1}\pi_{cd}\pi^{cd} + 1/2q^{-1}\pi^2]) \tag{1.5}$$

and

$$V^b(q_{ab}, \pi^{ab}) = -2\nabla^{(3)}{}_a(q^{-1/2}\pi^{ab}) \tag{1.6}$$

are called the *Hamiltonian (or Scalar) and Spatial Diffeomorphism (or Vector) constraint*. In the previous expressions $\pi = \pi^{ab}q_{ab}$, $\nabla^{(3)}{}_a$ is the covariant derivative compatible with the metric q_{ab}, q is the determinant of the space metric and $R^{(3)}$ is the Ricci tensor of q_{ab}. Finally λ, λ^a are Lagrange multipliers: their presence is due to the singularity of the the Lagrangean (1.1), i.e we cannot solve all the velocities in terms of momenta and therefore we must use Dirac's procedure [20] for the Legendre transform of singular Lagrangeans. In this case the singularity structure is such that the variations of the Lagrange multipliers give

$$P = P_a = 0 \tag{1.7}$$

The equations of motion with respect to the Hamiltonian (i.e. $\dot{F} := \{H, F\}$ for any functional F of the canonical coordinates)

$$H = \int d^3x[\lambda P + \lambda^a P_a + N^a V_a + NC] \tag{1.8}$$

for N, N^a reveal that N, N^a are themselves Lagrange multipliers, i.e. completely unspecified functions (proportional to λ, λ^a) while the equations of motion for P, P_a give $\dot{P} = -C$, $\dot{P}_a = -V_a$. Since P, P_a are supposed to vanish, this requires

$$\begin{aligned} V_a &= 0 \\ C &= 0 \end{aligned} \tag{1.9}$$

This implies that *the Hamiltonian is constrained to vanish in GR.*

Now the equations of motion for q_{ab}, π^{ab} imply the *Dirac algebra*

$$\begin{aligned} \{V(\vec{N}), V(\vec{N}')\} &= \kappa V(\mathcal{L}_{\vec{N}}\vec{N}') \\ \{V(\vec{N}), C(N)\} &= \kappa C(\mathcal{L}_{\vec{N}}N) \\ \{C(N), C(N')\} &= \kappa V(q^{-1}(NdN' - N'dN)) \end{aligned} \tag{1.10}$$

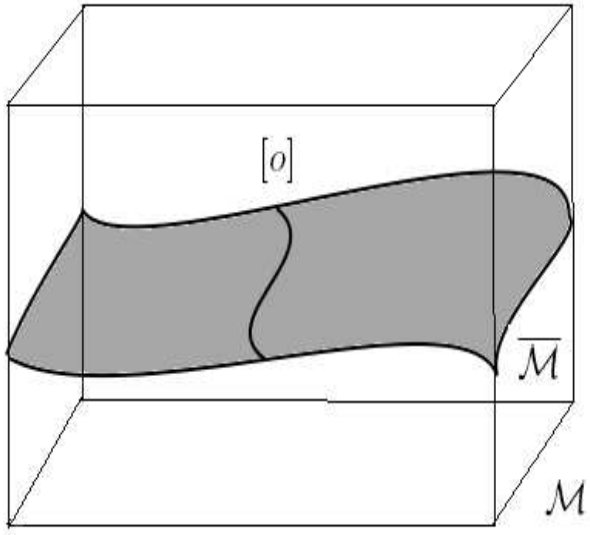


Figure 1.1: Constraint hypersurface $\overline{\mathcal{M}}$ and gauge orbit $[o]$ of $o \in \overline{\mathcal{M}}$ in $\mathcal{M}$

.

where $C(N) = \int d^3x NC$ and $V(\vec{N}) = \int d^3x N^a V_a$ are the smeared constraints on test functions . These equations tell us that the condition $C = V_a = 0$ i*s preserved under evolution* i.e the Hamiltonian and vector constraint form a *first class constraint algebra.*

We have seen that the variables P and P_a drops out completely from the analysis and N, N^a are Lagrange multipliers so we can rewrite the action (1.3) as

$$S = \frac{1}{\kappa} \int dt \int_\Sigma d^3x \, \{\dot{q}_{ab}\pi^{ab} - [N^a V_a + NH]\} \tag{1.11}$$

with the understanding that N, N^a are now completely arbitrary functions which parameterize the freedom in choosing the foliation. The symplectic structure can be read off from the previous equations, namely

$$\left\{\pi^{ab}(x), q_{cd}(y)\right\} = 2\kappa \; \delta^a_{(c}\delta^b_{d)}\delta(x,y), \qquad \left\{\pi^{ab}(x), \pi^{cd}(y)\right\} = \{q_{ab}(x), q_{cd}(y)\} = 0 \tag{1.12}$$

Since the Hamiltonian of GR depends on the completely unspecified functions N, N^a, the motions that it generates in the phase space $\mathcal{M}$ coordinatized by (π^{ab}, q_{ab}) subject to the Poisson brackets (1.12) are to be considered as pure gauge transformations.

We can summarize the gauge formulation of GR in figure 1.1: The constraints $C = V_a = 0$ define a constraint hypersurface $\overline{\mathcal{M}}$ within the full phase space $\mathcal{M}$. The gauge motions are defined on all of $\mathcal{M}$ but they have the feature that they leave the constraint hypersurface invariant, and thus the orbit of a point o in the hypersurface under gauge transformations will be a curve or gauge orbit $[o]$ entirely within it. The set of these curves defines the so-called reduced phase space and Dirac observables restricted to $\overline{\mathcal{M}}$ depend only on these orbits.

The counting of degrees of freedom proceeds as follows: we start with twelve phase space coordinates q_{ab}, π^{ab}. The four constraints C, V_a can be solved to eliminate four of those and there are still identifications under four independent sets of motions among the remaining eight variables leaving us with only four Dirac observables. The corresponding so-called reduced phase space has therefore precisely the two configuration degrees of freedom of general relativity.

1.1.1 The triad formulation

Now we shift to the triad formulation that will lead to the connection variables introduced first by Ashtekar [21] and generalized by Immirzi [22] and Barbero [23],[24]. We introduce:

A triad (a set of three 1-forms defining a frame at each point in Σ) in terms of which the metric q_{ab} becomes

$$q_{ab} = e^i_a e^j_b \delta_{ij}, \tag{1.13}$$

with $i, j = 1, 2, 3$.

The densitized triad

$$E^a_i := \frac{1}{2}\epsilon^{abc}\epsilon_{ijk}e^j_b e^k_c. \tag{1.14}$$

related to the inverse metric q^{ab} by

$$qq^{ab} = E^a_i E^b_j \delta^{ij} \tag{1.15}$$

and the following quantity

$$K^i_a := \frac{1}{\sqrt{\det(E)}} K_{ab} E^b_j \delta^{ij} \tag{1.16}$$

. We can rewrite (1.11) in terms of these new variables. In fact the canonical term in (1.11) can be rewritten as

$$\pi^{ab}\dot{q}_{ab} = -\pi_{ab}\dot{q}^{ab} = 2E^a_i \dot{K}^i_a \tag{1.17}$$

and also the constraints (1.5),(1.6) can be rewritten in terms of the new quantities

$$V^a(q_{ab}, \pi^{ab}) \ , \ C(q_{ab}, \pi^{ab}) \xrightarrow[triad]{} V^a(E^a_i, K^i_a) \ , \ C(E^a_i, K^i_a) \tag{1.18}$$

The triad formulation is then a shift from the variables q_{ab}, π^{ab} to E^a_i, K^i_a

$$q_{ab}, \pi^{ab} \xrightarrow[triad]{} E^a_i, K^i_a \tag{1.19}$$

It is immediate to see that the new variables are certainly redundant, in fact we are using the nine E^a_i to describe the six components of q^{ab}. The redundancy has a clear geometrical interpretation: the extra three degrees of freedom in the triad correspond to the possibility of choosing different local frames e^i_a by local $SO(3)$ rotations acting in the internal indices $i = 1, 2, 3$. There is in fact an additional constraint in terms of the new variables that makes this redundancy manifest. The missing constraint comes from (1.16): we overlooked the fact

that $K_{ab} = K_{ba}$. By inverting the definitions (1.14) and (1.16) in order to write K_{ab} in terms of E^a_i and K^i_a the condition $K_{[ab]} = 0$ reduces to

$$G_i(E^a_j, K^j_a) := \epsilon_{ijk} E^{aj} K^k_a = 0. \tag{1.20}$$

Therefore to completely reformulate the theory in triad variables we must include this additional constraint to (1.11) that becomes

$$\begin{aligned} &S[E^a_j, K^j_a, N_a, N, N^j] = \\ &\frac{1}{\kappa}\int dt \int_\Sigma dx^3 \ \left[E^a_i \dot{K}^i_a - N_b V^b(E^a_j, K^j_a) - N S(E^a_j, K^j_a) - N^i G_i(E^a_j, K^j_a)\right], \end{aligned} \tag{1.21}$$

where the explicit expressions for the constraints are given in [25]. The symplectic structure now becomes

$$\left\{E^a_j(x), K^i_b(y)\right\} = \kappa\, \delta^a_b \delta^i_j \delta(x,y), \qquad \left\{E^a_j(x), E^b_i(y)\right\} = \left\{K^j_a(x), K^i_b(y)\right\} = 0 \tag{1.22}$$

1.1.2 New variables: the Ashtekar-Barbero connection variables

Now we make a new change of variables introducing the Ashtekar-Barbero[23, 26, 27] connection variables defined by A^i_a given by

$$A^i_a = \Gamma^i_a + \gamma K^i_a, \tag{1.23}$$

where γ is a real number different from 0 called the Immirzi parameter[28] and Γ^i_a is the spin connection solution of Cartan's structure equations

$$\partial_{[a} e^i_{b]} + \epsilon^i{}_{jk} \Gamma^j_{[a} e^k_{b]} = 0 \tag{1.24}$$

that defines the notion of covariant derivative compatible with the triad.

The spin connection is an $so(3)$ connection that transforms in the standard inhomogeneous way under local $SO(3)$ transformations. The new variable A^i_a is also an $so(3)$ connection because adding a quantity that transforms as a vector (the densitized triad (1.14) transforms in the vector representation of $SO(3)$ under redefinition of the triad (1.13) and consequently, so does its conjugate momentum K^i_a) to a connection gives a new connection .

This new variable is conjugate to E^i_a. The new Poisson brackets are

$$\left\{E^a_j(x), A^i_b(y)\right\} = \kappa\, \gamma \delta^a_b \delta^i_j \delta(x,y), \qquad \left\{E^a_j(x), E^b_i(y)\right\} = \left\{A^j_a(x), A^i_b(y)\right\} = 0. \tag{1.25}$$

Using the connection variables the action becomes

$$\begin{aligned} &S[E^a_j, A^j_a, N_a, N, N^j] = \\ &\frac{1}{\kappa}\int dt \int_\Sigma dx^3 \left[E^a_i \dot{A}^a_i - N^b V_b(E^a_j, A^j_a) - N C(E^a_j, A^j_a) - N^i G_i(E^a_j, A^j_a)\right], \end{aligned} \tag{1.26}$$

where the constraints are explicitly given by:

$$V_b(E^a_j, A^j_a) = E^a_j F_{ab} - (1+\gamma^2) K^i_a G_i \tag{1.27}$$

$$C(E^a_j, A^j_a) = \frac{E^a_i E^b_j}{\sqrt{\det(E)}} \left(\epsilon^{ij}{}_k F^k_{ab} - 2(1+\gamma^2) K^i_{[a} K^j_{b]} \right) \tag{1.28}$$

$$G_i(E^a_j, A^j_a) = D_a E^a_i, \tag{1.29}$$

where $F_{ab} = \partial_a A^i_b - \partial_b A^i_a + \epsilon^i{}_{jk} A^j_a A^k_b$ is the curvature of the connection A^i_a and $D_a E^a_i = \partial_a E^a_i + \epsilon_{ij}{}^k A^j_a E^a_k$ is the covariant divergence of the densitized triad.

What is the advantage of this change of variables? At classical level we haven't gained so much: Instead of twelve variables q_{ab}, π^{ab} we now have eighteen variables A^i_a, E^a_i.

However, considering the phase space coordinatized by (A^j_a, E^b_j), with Poisson brackets (1.25) and constraints G_j, C, V_a, it can be shown [21, 29] that solving only the constraint $G_j = 0$ and determining the Dirac observables with respect to it leads us back to the ADM phase space with constraints C, V_a.

The virtue of this extended phase space is that *canonical GR can be formulated in the language of a canonical gauge theory* where the gauge field is given by the connection A^i_a and its conjugate momentum is the electric field E^b_j. The constraint (1.29) coincides with the standard Gauss law of Yang-Mills theory (e.g. $\vec{\nabla} \cdot \vec{E} = 0$ in electromagnetism).

In fact if we ignore (1.27) and (1.28) the phase space variables (A^i_a, E^b_j) together with the Gauss law (1.29) characterize the physical phase space of an $SU(2)$ Yang-Mills (YM) theory. We can switch from $SO(3)$ to $SU(2)$ because the constraint structure does not distinguish between them (both groups have the same Lie algebra) and $SU(2)$ is the gauge group if we want to include fermionic matter[30, 31, 32]. Now the advantage becomes clear: the close similarity between canonical GR and YM theory opens the road to the use of the techniques that are very natural in the context of YM theory.

Gauge transformations

Now let us analyze the structure of the gauge transformations generated by the constraints (1.27),(1.28), and (1.29). The Gauss law (1.29) generates local $SU(2)$ transformations as in the case of YM theory. Explicitly, if we define the smeared version of (1.29) as

$$G(\alpha) = \int_\Sigma dx^3 \ \alpha^i G_i(A^i_a, E^a_i) = \int_\Sigma dx^3 \alpha^i D_a E^a_i, \tag{1.30}$$

a direct calculation implies

$$\delta_G A^i_a = \left\{ A^i_a, G(\alpha) \right\} = -D_a \alpha^i \quad \text{and} \quad \delta_G E^a_i = \{ E^a_i, G(\alpha) \} = [E, \alpha]_i \,. \tag{1.31}$$

If we write $A_a = A^i_a \tau_i \in su(2)$ and $E^a = E^a_i \tau^i \in su(2)$, where τ_i are generators of $SU(2)$, we can write the finite version of the previous transformation

$$A'_a = g A_a g^{-1} + g \partial_a g^{-1} \quad \text{and} \quad E^{a\prime} = g E^a g^{-1}, \tag{1.32}$$

which is the standard gauge transformation of the connection and the electric field in YM theory.

The vector constraint (1.27) generates three dimensional diffeomorphisms of Σ. This is clear from the action of the smeared constraint

$$V(N^a) = \int_\Sigma dx^3 \ N^a V_a(A^i_a, E^a_i) \tag{1.33}$$

on the canonical variables

$$\delta_V A_a^i = \left\{A_a^i, V(N^a)\right\} = \mathcal{L}_N A_a^i \quad \text{and} \quad \delta_V E_i^a = \{E_i^a, V(N^a)\} = \mathcal{L}_N E_i^a, \tag{1.34}$$

where $\mathcal{L}_N$ denotes the Lie derivative in the N^a direction. The exponentiation of these infinitesimal transformations leads to the action of finite diffeomorphisms on Σ.

Finally, the scalar constraint (1.28) generates coordinate time evolution (up to space diffeomorphisms and local $SU(2)$ transformations). The total Hamiltonian $H[\alpha, N^a, N]$ of general relativity can be written as

$$H(\alpha, N^a, N) = G(\alpha) + V(N^a) + C(N), \tag{1.35}$$

where

$$C(N) = \int_\Sigma dx^3 \; NC(A_a^i, E_i^a). \tag{1.36}$$

Hamilton's equations of motion are therefore

$$\dot{A}_a^i = \left\{A_a^i, H(\alpha, N^a, N)\right\} = \left\{A_a^i, C(N)\right\} + \left\{A_a^i, G(\alpha)\right\} + \left\{A_a^i, V(N^a)\right\}, \tag{1.37}$$

and

$$\dot{E}_i^a = \{E_i^a, H(\alpha, N^a, N)\} = \{E_i^a, C(N)\} + \{E_i^a, G(\alpha)\} + \{E_i^a, V(N^a)\}. \tag{1.38}$$

The previous equations define the action of $C(N)$ up to infinitesimal $SU(2)$ and diffeomorphism transformations given by the last two terms and the values of α and N^a respectively. In general relativity coordinate time evolution does not have any physical meaning. It is analogous to a $U(1)$ gauge transformation in QED.

Ashtekar variables

The Ashtekar variables are defined for the choice $\gamma = i$ in (1.23)

$$A_a^i = \Gamma_a^i + iK_a^i \tag{1.39}$$

in this case the connection is complex[25] (i.e. $A_a \in sl(2, \mathbb{C})$) and to be sure not to double the degrees of freedom involved and recover real GR we must add the reality condition

$$A_a^i + \bar{A}_a^i = \Gamma_a^i(E). \tag{1.40}$$

These variables allow a self-dual formulation of GR: The connection obtained for this choice of the Immirzi parameter is simply related to a spacetime connection (and this is not the case for the real connection (1.23) that cannot be obtained as the pullback to Σ of a spacetime connection[33]). In fact it can be shown that A_a is the pullback of ω_μ^{+IJ} $(I, J = 1, \cdots 4)$ where

$$\omega_\mu^{+IJ} = \frac{1}{2}(\omega_\mu^{IJ} - \frac{i}{2}\epsilon^{IJ}{}_{KL}\omega_\mu^{KL}) \tag{1.41}$$

is the self dual part of a Lorentz $(SO(3,1))$ connection ω_μ^{IJ}. With the choice (1.39) the constraints (1.27) and (1.28) become

$$V_b^{SD} = E_j^a F_{ab} \tag{1.42}$$

$$S^{SD} = \frac{E_i^a E_j^b}{\sqrt{\det(E)}}\ \epsilon^{ij}{}_k F_{ab}^k \tag{1.43}$$

$$G_i^{SD} = D_a E_i^a, \tag{1.44}$$

where SD stands for self dual. The gauge group—generated by the (complexified) Gauss constraint—is in this case $SL(2,\mathbb{C})$.

Loop quantum gravity was initially formulated in terms of these variables. However, there are technical difficulties in defining the quantum theory when the connection is valued in the Lie algebra of a non compact group.

In spacetimes with Euclidean signature, the self-dual formulation does not suffer any drawbacks; in fact the Euclidean spin connection ω^{IJ} is an $SO(4)$ connection instead of an $SO(3,1)$ connection and the selfdual connection A is real.

From now on we restrict our attention to the real Ashtekar-Barbero variables.

1.1.3 Geometrical properties of the new variables

Holonomy

The geometric interpretation of the connection A_a^i, defined in (1.23), is standard. The connection provides a definition of *parallel transport* of $SU(2)$ spinors on the space manifold Σ. The natural object is the $SU(2)$ element defining parallel transport along a path $\gamma \subset \Sigma$ is called *holonomy* $h_\gamma[A]$,

Given a one dimensional oriented path

$$\gamma : [0,1] \subset \mathbb{R} \to \Sigma \tag{1.45}$$

$$s \to x^\mu(s) \tag{1.46}$$

the holonomy of the connection A along the path γ is given by the solution $h_\gamma[A,s]$ of the ordinary differential equation

$$h_\gamma[A,0] = \mathbb{1} \tag{1.47}$$

$$h_\gamma[A] = h_\gamma[A,1] \tag{1.48}$$

$$\frac{d}{ds} h_\gamma[A,s] + \dot{x}^\mu(s) A_\mu h_\gamma[A,s] = 0 \tag{1.49}$$

The formal solution of the previous equation is

$$h_\gamma[A] = P \exp - \int_\gamma A, \tag{1.50}$$

where P stands for path ordered and is defined by the series expansion

$$h_\gamma[A] = \sum_{n=0}^{\infty} \int_0^1 ds_1 \int_0^{s_1} ds_2 \ \cdots \int_0^{s_{n-1}} ds_n\ \dot{x}^{\mu_1}(s_1) \cdots \dot{x}^{\mu_n}(s_n)\ A_{\mu_1}(s_1) \cdots A_{\mu_n}(s_n), \tag{1.51}$$

. Let us list some important properties of the holonomy:

1. The definition of $h_\gamma[A]$ is independent of the parametrization of the path γ.

2. The holonomy of a path given by a single point is the identity, given two oriented paths γ_1 and γ_2 such that the end point of γ_1 coincides with the starting point of γ_2 so that we can define $\gamma = \gamma_1 \cdot \gamma_2$ in the standard way then we have

$$h_\gamma[A] = h_{\gamma_1}[A] h_{\gamma_2}[A], \tag{1.52}$$

where the multiplication on the right is the $SU(2)$ multiplication. We also have that

$$h_{\gamma^{-1}}[A] = h_\gamma^{-1}[A]. \tag{1.53}$$

3. The holonomy has a very simple behavior under gauge transformations. It is easy to check from (1.32) that under a gauge transformation generated by the Gauss constraint, the holonomy transforms as

$$h'_e[A] = g(x(1))\ h_\gamma[A]\ g^{-1}(x(0)). \tag{1.54}$$

4. The holonomy transforms in a very simple way under the action of diffeomorphisms (transformations generated by the vector constraint (1.27)). Given $\phi \in \mathrm{Diff}(\Sigma)$ we have

$$h_\gamma[\phi^* A] = h_{\phi^{-1}(e)}[A], \tag{1.55}$$

where $\phi^* A$ denotes the action of ϕ on the connection. In other words, transforming the connection with a diffeomorphism is equivalent to simply 'moving' the path with ϕ^{-1}.

Geometrically the holonomy $h_\gamma[A]$ is a functional of the connection that provides a rule for the parallel transport of $SU(2)$. If we think of it as a functional of the path e it is clear that it captures all the information of the field A_a^i.

The electric field flux

The densitized triad—or electric field—E_i^a also has a simple geometrical meaning. E_i^a encodes the full background independent Riemannian geometry of Σ as is clear from (1.15). Therefore, any geometrical quantity in space can be written as a functional of E_i^a.

The area in particular $A_\mathcal{S}[E_i^a]$ of a 2d surface $\mathcal{S} \subset \Sigma$ will play a major role in the following. Given a two dimensional surface $\mathcal{S} : \sigma = (\sigma^1, \sigma^2) \to x^a(\sigma)$ embedded in the 3d surface Σ—with normal

$$n_a = \frac{\partial x^b}{\partial \sigma^1} \frac{\partial x^c}{\partial \sigma^2} \epsilon_{abc} \tag{1.56}$$

where σ^1 and σ^2 are local coordinates on S—its area is given by

$$A_\mathcal{S}[q^{ab}] = \int_\mathcal{S} \sqrt{h}\ d\sigma^1 d\sigma^2, \tag{1.57}$$

where $h = \det(h_{ab})$ is the determinant of the metric $h_{ab} = q_{ab} - n^{-2} n_a n_b$ induced on S by q^{ab}. From equation (1.15) it follows that $\det(q^{ab}) = \det(E_i^a)$. Let us contract (1.15) with $n_a n_b$, namely

$$q q^{ab} n_a n_b = E_i^a E_j^b \delta^{ij} n_a n_b. \tag{1.58}$$

Now observe that $q^{nn} = q^{ab}n_a n_b$ is the nn-matrix element of the inverse of q_{ab}. Through the well known formula for components of the inverse matrix we have that

$$q^{nn} = \frac{\det(q_{ab} - n^{-2}n_a n_b)}{\det(q_{ab})} = \frac{h}{q}. \tag{1.59}$$

But $q_{ab} - n^{-2}n_a n_b$ is precisely the induced metric h_{ab}. Replacing q^{nn} back into (1.58) we conclude that

$$h = E^a_i E^b_j \delta^{ij} n_a n_b. \tag{1.60}$$

Finally we can write the area of S as an explicit functional of E^a_i:

$$A_{\mathcal{S}}[E^a_i] = \int_{\mathcal{S}} \sqrt{E^a_i E^b_j \delta^{ij} n_a n_b}\; d\sigma^1 d\sigma^2. \tag{1.61}$$

If we define the projection of the electric field in the direction normal to the surface $\mathcal{S}$ as

$$E_i(\sigma) = E^a_i(\vec{x}(\sigma))n_a \tag{1.62}$$

The definition of the Area (1.61) becomes

$$A_{\mathcal{S}} = \int_{\mathcal{S}} d^2\sigma |E(\sigma)| \tag{1.63}$$

Thus the area of a surface is the norm of the electric field flux trough the surface.

This property can be nicely summarized in the following sentence:

In gravity "'*the lenght of the electric field is the area*"'

1.2 The Dirac program applied to the non perturbative quantization of GR

Now we proceed with the quantization following the Dirac program[20, 34] to quantize generally covariant systems, applying it to the Asktekar gravity formulation; the realization of this program bring us to LQG.

Let us focus on the program steps:

- Find a representation of the phase space variables of the theory as operators in an auxiliary or kinematical Hilbert space $\mathcal{H}_{kin}$ satisfying the standard commutation relations, i.e., $\{\ ,\ \} \to -i/\hbar[\ ,\ ]$.

  The kinematical Hilbert space of LQG consists of a set of functionals of the connection $\psi[A]$ which are square integrable with respect to a suitable (gauge invariant and diffeomorphism invariant) measure $d\mu_{AL}[A]$ (called Ashtekar-Lewandowski measure[35]). The kinematical inner product is given by

$$< \psi, \phi >= \mu_{AL}[\overline{\psi}\phi] = \int d\mu_{AL}[A]\ \overline{\psi}[A]\phi[A]. \tag{1.64}$$

- Promote the constraints to (self-adjoint) operators in $\mathcal{H}_{kin}$. In the case of gravity we must quantize the seven constraints (1.27),(1.28), and (1.29)

$$G_i(A,E), V_a(A,E), C(A,E) \xrightarrow[quantization]{} \hat{G}_i(A,E), \hat{V}_a(A,E), \hat{C}(A,E) \tag{1.65}$$

- Characterize the space of solutions of the constraints and define the corresponding inner product that defines a notion of physical probability. This defines the so-called physical Hilbert space $\mathcal{H}_{phys}$. In the case of gravity the solution of the Gauss and space diffeomorphism constraints has been successfully completed, instead the space of solutions of quantum scalar constraint $\hat{C}$ remains an open issue in LQG. The definition of the physical inner product too is still an open issue but it leads to the definition of the spinfoam models.

- Find a (complete) set of gauge invariant observables, i.e., operators commuting with the constraints. This step is rather difficult in fact already in classical gravity the construction of gauge independent quantities is a subtle issue and we refer the reader to Rovelli's book[2] for an appropriate treatment

1.3 Loop Quantum Gravity

1.3.1 Definition of the kinematical Hilbert space

Now we face the first point of the program formally described in the previous section, defining the vector space of functionals of the connection and a notion of scalar product to equip it with a Hilbert space structure and define in this way $\mathcal{H}_{kin}$.

The Cyl algebra

Now we define the algebra of kinematical observables as the algebra of the *cylindrical functions* of generalized connections denoted Cyl. A generalized connection is an assignment of holonomies $h_\gamma \in SU(2)$ to any path $\gamma \subset \Sigma$.

Consider an ordered oriented graph Γ, given by an ordered collection of L *links* γ_l, i.e piecewise smooth oriented curves embedded in M and meeting only at their endpoints, called *nodes*, if at all. Now we can assign group elements to each link γ_l taking the holonomy h_{γ_l} and consequently assigning an element of $SU(2)^L$ to the graph.

Given a complex-valued function $f : SU(2)^L \to \mathbb{C}$ a couple (Γ, f) defines the functional of A

$$\Psi_{\Gamma,f}[A] := f(h_{\gamma_1}[A], h_{\gamma_2}[A], \cdots h_{\gamma_L}[A]), \tag{1.66}$$

This functional (called *cylindrical function*) is an element of the set Cyl_Γ of the cylindrical function defined on a fixed graph: if we consider the union of the set of functions of generalized connections defined on all the graphs $\Gamma \subset \Sigma$

$$\mathrm{Cyl} = \cup_\Gamma \mathrm{Cyl}_\Gamma, \tag{1.67}$$

we get a subset of the space of smooth functions on the space of connections, on which it is possible to define consistently an inner product [35, 36], and then complete the space of linear

combinations of cylindrical functions in the norm induced by this inner product. This is the basic algebra on which the definition of the kinematical Hilbert space $\mathcal{H}_{kin}$ is constructed.

A little remark about the *ordered oriented* graph, when we deal with cylindrical functions is needed: changing the ordering or the orientation of a graph is just the same as changing the order of the arguments of the function f, or replacing arguments with their inverse.

The Ashtekar-Lewandowski representation of Cyl

The cylindrical functions introduced are suitable states in $\mathcal{H}_{kin}$. To define $\mathcal{H}_{kin}$ and provide a representation of the algebra Cyl we need a measure in the space of generalized connections to give a meaning to the formal expression (1.64) and thus obtain a definition of the kinematical inner product. In order to do that we introduce a positive normalized state (state in the algebraic QFT sense) μ_{AL} on the ($C^\star$-algebra) Cyl as follows. Given a cylindrical function $\Psi_{\Gamma,f}[A] \in$ Cyl we define the $\mu_{AL}(\Psi_{\Gamma,f})$ as

$$\mu_{AL}(\Psi_{\Gamma,f}) = \int \prod_{\gamma_l \subset \Gamma} dh_{\gamma_l} \; f(h_{\gamma_1}, h_{\gamma_2}, \cdots h_{\gamma_L}), \tag{1.68}$$

where $h_{\gamma_l} \in SU(2)$ and dh_{γ_l} is the normalized Haar measure of $SU(2)$ defined by the following properties:

$$\int_{SU(2)} dg = 1, \quad \text{and} \quad dg = d(\alpha g) = d(g\alpha) = dg^{-1} \quad \forall \alpha \in SU(2).$$

. The state μ_{AL} is called the Ashtekar-Lewandowski measure[35]. The measure μ_{AL} is clearly normalized as $\mu_{AL}(1) = 1$ and positive

$$\mu_{AL}(\overline{\Psi_{\Gamma,f}}\Psi_{\Gamma,f}) = \int \prod_{\gamma_l \subset \Gamma} dh_{\gamma_l} \; \overline{f(h_{\gamma_1}, h_{\gamma_2}, \cdots, h_{\gamma_L})} f(h_{\gamma_1}, h_{\gamma_2}, \cdots, h_{\gamma_L}) \geq 0. \tag{1.69}$$

Using the properties of μ_{AL} we introduce the inner product between functionals defined with the same ordered oriented graph Γ

$$\begin{aligned} < \Psi_{\Gamma,f} | \Psi_{\Gamma,g} >: &= \mu_{AL}(\overline{\Psi_{\Gamma,f}}\Psi_{\Gamma,g}) = \\ &= \int \prod_{\gamma_l \subset \Gamma} dh_{\gamma_l} \; \overline{f(h_{\gamma_1}, \cdots, h_{\gamma_L})} g(h_{\gamma_1}, \cdots, h_{\gamma_L}), \end{aligned} \tag{1.70}$$

where the cylindrical functions become wave functionals of the connection corresponding to kinematical states

$$\Psi_{\Gamma,f}[A] =< A | \Psi_{\Gamma,f} >= f(h_{\gamma_1}, \cdots h_{\gamma_L}) \tag{1.71}$$

The extension of the inner product (1.70) to different orderings or orientation is obvious; the extension to functional defined on different graphs is straightforward, observing that the same functional can be defined by different couples $(\Gamma, f), (\Gamma', f')$, it is then enough to consider a new graph $\Gamma_\cup = \Gamma' \cup \Gamma''$ to transform a scalar product between functionals defined on different graph Γ' and Γ'' in an expression of the kind (1.70)

$$\langle \Psi_{\Gamma',f'} | \Psi_{\Gamma'',g''} \rangle = \langle \Psi_{\Gamma_\cup,f} | \Psi_{\Gamma_\cup,g} \rangle \tag{1.72}$$

The previous equation is the rigorous definition of (1.64). The measure μ_{AL}—through the GNS construction[37]—gives a faithful representation of the algebra of cylindrical functions (i.e., (1.69) is zero if and only if $\Psi_{\gamma,f}[A] = 0$). The kinematical Hilbert space $\mathcal{H}_{kin}$ is the Cauchy completion of the space of cylindrical functions Cyl in the Ashtekar-Lewandowski measure. In other words, in addition to cylindrical functions we add to $\mathcal{H}_{kin}$ the limits of all the Cauchy convergent sequences in the μ_{AL} norm. The operators depending only on the connection act simply by multiplication in the Ashtekar-Lewandowski representation. This completes the definition of the kinematical Hilbert space $\mathcal{H}_{kin}$.

An orthonormal basis of $\mathcal{H}_{kin}$.

The key ingredient to find a basis of $\mathcal{H}_{kin}$ is the Peter-Weyl theorem[38]. It states that a basis on the Hilbert space of functions $f \in \mathcal{L}^2[SU(2)]$ is given by the matrix elements of the unitary irreducible representations of the group and thus every function can be expanded in the following way

$$f(g) = \sum_j \sqrt{2j+1}\ f_j^{mm'} D^j_{mm'}(g), \tag{1.73}$$

where

$$f_j^{mm'} = \sqrt{2j+1} \int_{SU(2)} dg\ \ D^j_{m'm}(g^{-1}) f(g), \tag{1.74}$$

and dg is the Haar measure of $SU(2)$. This defines the harmonic analysis on $SU(2)$. The completeness relation

$$\delta(gh^{-1}) = \sum_j (2j+1) D^j_{mm'}(g) D^j_{m'm}(h^{-1}) = \sum_j (2j+1) \mathrm{Tr}[D^j(gh^{-1})], \tag{1.75}$$

follows. The previous equations imply the orthogonality relation for unitary representations of $SU(2)$

$$\int_{SU(2)} dg\ N^j_{m'm}(g) N^{j'}_{q'q}(g) = \delta_{jj'}\delta_{mq}\delta_{m'q'}, \tag{1.76}$$

where we have introduce the normalized representation matrices $N^j_{mn} := \sqrt{2j+1} D^j_{mn}$;

In our case the group elements are the holonomies $h_{\gamma_l}[A]$ of the connection; the basis representation matrix is, in ket notation,

$$N^j_{mn}(h_{\gamma_l}[A]) := \langle A | j, m, n \rangle \tag{1.77}$$

Given an arbitrary cylindrical function $\psi_{\Gamma,f}[A] \in \mathrm{Cyl}$ we can use the Peter-Weyl theorem and write

$$\begin{aligned} \Psi_{\Gamma,f}[A] &= f(h_{\gamma_1}[A], h_{\gamma_2}[A], \cdots h_{\gamma_L}[A]) = \\ &= \sum_{j_1 \cdots j_L} f^{m_1 \cdots m_L, n_1 \cdots n_L}_{j_1 \cdots j_L} N^{j_1}_{m_1 n_1}(h_{\gamma_1}[A]) \cdots N^{j_L}_{m_L n_L}(h_{\gamma_L}[A]) \end{aligned} \tag{1.78}$$

where the coefficients $f_{j_1\cdots j_L}^{m_1\cdots m_L,n_1\cdots n_L}$ are given by the kinematical scalar product (1.70) of the cylindrical function with the tensor product of irreducible representations

$$f_{j_1\cdots j_L}^{m_1\cdots m_L,n_1\cdots n_L} =< N_{m_1n_1}^{j_1}\cdots N_{m_Ln_L}^{j_L}|\Psi_{\Gamma,f}>, \tag{1.79}$$

. Fixed an ordered oriented graph Γ we have then a basis

$$|\Gamma, j_l, {}_{n_l}, {}_{m_l}\rangle = |\Gamma, j_1, \cdots, j_L, {}_{n_1}, \cdots, {}_{n_L}, {}_{m_1}, \cdots, {}_{m_L}\rangle \tag{1.80}$$

obtained by tensoring the base (1.77) on each of the L links l of the graph Γ

$$\langle A|\Gamma, j_{N_l}, {}_{n_l}, {}_{m_l}\rangle = N_{m_1n_1}^{j_1}(h_{\gamma_1}[A])\cdots N_{m_Ln_L}^{j_L}(h_{\gamma_L}[A]) \tag{1.81}$$

From equation (1.78) we have that (1.81) is a complete orthonormal basis of the Hilbert space $\mathcal{H}_{kin}^{\Gamma}$: this is the space of cylindrical functional restricted to the fixed graph Γ and thus is not yet a basis for $\mathcal{H}_{kin}$ because the same vector appears in $\mathcal{H}_{kin}^{\Gamma}$ and $\mathcal{H}_{kin}^{\Gamma'}$ if $\Gamma \subset \Gamma'$. To eliminate this redundancy is enough to think that all the paths that are in Γ' but not in Γ are represented in $\mathcal{H}_{kin}^{\Gamma}$ by the trivial representation: it is then enough to restrict the states (1.81) to the values $j_l > 0$ to get the proper graph subspace $\mathcal{H}_{kin}^{\Gamma}$. All the proper subspace $\mathcal{H}_{kin}^{\Gamma}$ are orthogonal each other and they span $\mathcal{H}_{kin}$

1.3.2 Quantum constraints

We have constructed the kinematical space $\mathcal{H}_{kin}$ of arbitrary wave functionals $\Psi[A]$. Now we have to implement the classical constraints promoted to operators in the quantum theory.

$$\hat{G}_i(A,E)|\Psi\rangle = 0 \tag{1.82}$$
$$\hat{V}_{ai}(A,E)|\Psi\rangle = 0 \tag{1.83}$$
$$\hat{C}(A,E)|\Psi\rangle = 0 \tag{1.84}$$

The constraint are the generators of the gauge transformations of the theory; at quantum level they correspond to the request of the gauge invariances of the states. The first equation requires the invariance of the states under local SU(2); the second invariance under 3d *Diff*. The third one is equivalent to invariance under "time" reparametrization or simply it codes the dynamic and is called Wheeler-De Witt equation. The solution of the theory, calling $\mathcal{H}_{kin}^{G}$ the space of states invariant under local $SU(2)$, $\mathcal{H}_{kin}^{Diff}$ the space of states invariant under local $SU(2)$ and *Diff* and finally $\mathcal{H}_{phys}$ the solution of the three (1.82),(1.83) and (1.84), consists of the following three sequences of Hilbert spaces

$$\mathcal{H}_{kin} \xrightarrow{SU(2)} \mathcal{H}_{kin}^{G} \xrightarrow{Diff} \mathcal{H}_{kin}^{Diff} \xrightarrow{C} \mathcal{H}_{phys} \tag{1.85}$$

where the three steps correspond to the implementation of the three constraints that the wave functional must satisfy.

1.3.3 Solutions of the Gauss constraint: $\mathcal{H}^G_{kin}$ and spin network states

We are now interested in the solutions of the quantum Gauss constraint; the first three of quantum Einstein's equations. These solutions are characterized by the states in $\mathcal{H}_{kin}$ that are $SU(2)$ gauge invariant. These solutions define a new Hilbert space that we call $\mathcal{H}^G_{kin}$. Now we will show how these are in fact a complete set of orthogonal solutions of the Gauss constraint, i.e., a basis of $\mathcal{H}^G_{kin}$.

The action of the Gauss constraint produces the local SU(2) gauge transformation on the connection (1.32) that induces on the holonomies the transformation (1.54).

Denoting U_g the operator generating a local gauge transformation $g(x) \in SU(2)$ its action can be defined directly on the elements of the basis of $\mathcal{H}_{kin}$ starting from its action on (1.77)

$$U_g N^j_{mn}[h_\gamma] = N^j_{mn}[g_f h_\gamma g_i^{-1}] \tag{1.86}$$

where $g_i = g(x_i)$ is the value of $g(x)$ at the initial point x_i of the path γ and $g_f = g(x_f)$ its value in the final point x_f. The generalization to an arbitrary base element of $\mathcal{H}_{kin}$ is then

$$U_g \prod_{l=1}^{L} N^{j_l}_{m_l n_l}[h_{\gamma_l}] = \prod_{l=1}^{L} N^{j_l}_{m_l n_l}[g_{f_l} h_{\gamma_l} g_{i_l}^{-1}]. \tag{1.87}$$

where i_l and f_l are the points where the link γ_l begins and ends. Using the obvious fact that

$$N^j_{mn}(gh[A]) = R^j_{mq}(g)\ N^j_{qn}(h[A]) \tag{1.88}$$

we can rewrite the U_g action (1.87) in the ket notation (1.80) as

$$U_g|\Gamma, j_l, {}_{n_l}, {}_{m_l}\rangle = R^{j_1}_{n_1 n'_1}(g_{f_1}) R^{j_1}_{m'_1 m_1}(g_{i_1}^{-1}) \cdots R^{j_1}_{n_N n'_N}(g_{f_N}) R^{j_N}_{m'_N m_N}(g_{i_N}^{-1}) |\Gamma, j_l, {}_{n'_l}, {}_{m'_l}\rangle \tag{1.89}$$

The complete base of $\mathcal{H}_{kin}$ transforms under gauge transformation according to (1.89).

Now we construct an orthonormal basis of states that are $SU(2)$ gauge invariant functionals of the connection called *spin network states* [39, 40, 41, 42].

Spinnetwork states

We call *nodes* the end points of the oriented curves γ_l in Γ. We assume without loss of generality that Γ is formed by a set of curves that overlap only at nodes. Γ is a graph immersed in the manifold i.e a collection of nodes n, which are points in Σ, joined by "'links"' that are curves in Σ. Given an ordered oriented graph Γ let j_l be an assigment of irreducible representations different from the trivial one to each link l. And i_n an assigment of an invariant tensor called *intertwiner* (see Appendix A) to each node n;

The intertwiner at the node n is an invariant vector in the tensor product of representations labelling the links converging at this node (see Figure 1.2).

The triplet $S = (\Gamma, j_l, i_n)$ is called a *spin-network* embedded in Σ; a choice of j_l and i_n is called a *coloring* of the links and of the nodes.

Now take a spinnetwork $S = (\Gamma, j_l, i_n)$ with L links and N nodes ; if we consider the state $|\Gamma, j_l, {}_{n_l}, {}_{m_l}\rangle$ defined on Γ it has exactly L indices n_l and L indices m_l. The N intertwiners

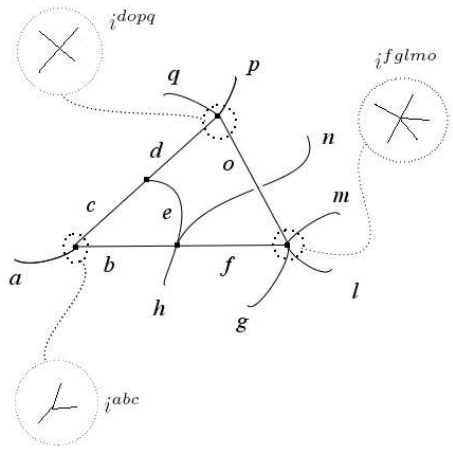


Figure 1.2: Schematic representation of the construction of a spin network. To each link we associate an irreducible representation. To each node we associate an invariant vector in the tensor product of irreducible representations converging at the node. In the picture we have in evidence an example of a 3-valent, a 4-valent and a 5-valent node. In the first case we associate to the node an intertwiner i^{abc} (unique), in the second i^{dopq} and in the last i^{fglmo} in the tensor product of the representation $a \otimes b \otimes c$ and so on.

i_n have exaclty a set of indices dual to these; the contraction of the basis elements with the intertwiners defines the spinnetwork state

$$|S\rangle = \sum_{n_l, m_l} i_1^{n_1 \cdots n_{a_1} m_1 \cdots m_{b_1}} \; i_2^{n_{a_1+1} \cdots n_{a_2} m_{b_1+1} \cdots m_{b_2}} \; \cdots \\ i_N^{n_{a_{N-1}+1} \cdots n_L m_{b_{N-1}+1} \cdots m_L} |\Gamma, j_l, {}_{n_l}, {}_{m_l}\rangle \tag{1.90}$$

The pattern of contraction is dictated by the topology of the graph; in particular we can distinguish between in-going and out-going paths; in the notation in fact we have a_1 links outgoing from the node 1 and b_1 links ingoing in that node and so on.

As functional of the connection the spinnetwork state (1.90) is

$$\Psi_S[A] = \langle A|S\rangle \equiv \bigotimes_l N^{j_l}(h_{\gamma_l}[A]) \cdot \bigotimes_n i_n \tag{1.91}$$

where the dot indicates contraction between dual spaces. It is immediate to see the gauge invariance of the state (1.91); it follows directly from the transformation properties (1.89) and the intertwiners invariance. The set of spinnetworks states

$$|S\rangle = |\Gamma, j_l, i_n\rangle \tag{1.92}$$

form an orthonormal base of $\mathcal{H}^G_{kin}$ as immediate consequence of the fact that $|\Gamma, j_l, {}_{n_l}, {}_{m_l}\rangle$ form a base in $\mathcal{H}_{kin}$ and the intertwiners definition. A final remark is the fact that the choice of

the base is not unique; in fact it depends in the choice of the base in each intertwiners space (Appendix A) on each node. Note also that in (1.92) the label Γ runs over unoriented and unordered graphs, but a choice of colorings imply an orientation and an ordering.

1.3.4 Solutions of the diffeomorphism constraint: $\mathcal{H}_{kin}^{Diff}$ and abstract spin networks

It is time to look at the second of the three quantum constraints, the vector constraint (1.27). The diffeomorphism constraint is more difficult to treat than the Gauss constraint because, while $\mathcal{H}_{kin}$ contains a subset of G-invariant states (spinnetwork states), the diffeomorphisms move the graph on the manifold and change the state. In fact it is immediate to write the action of the operator U_ϕ representing a diffeomorphism $\phi \in Diff(\Sigma)$, acting in the dense subset of cylindrical functions $\mathrm{Cyl} \subset \mathcal{H}_{kin}$ looking at the equation (1.55). Given $\psi_{\Gamma,f} \in \mathrm{Cyl}$ as in (1.66) we have

$$U_\phi \psi_{\Gamma,f}[A] = \psi_{\phi^{-1}\Gamma,f}[A], \tag{1.93}$$

Diffeomorphisms act on elements of Cyl (such as spin networks) by modifying the underlying graph. Notice that U_ϕ is unitary according to the definition (1.70). The spinnetwork states $|S\rangle$ in fact are not gauge invariant because

$$U_\phi |\Gamma, j_l, i_n\rangle = |\phi^{-1}\Gamma, j_l, i_n\rangle \tag{1.94}$$

and even more because we have to pay attention to orderings and orientations, in fact they can be changed by a diffeomorphism even without changing the graph: we call G_Γ the finite discrete subgroup of maps g_k acting on the space $\mathcal{H}_{kin}^\Gamma$ that changes ordering and orientation. However, because the orbits of the diffeomorphisms are not compact, diffeomorphism invariant states are not contained in the original $\mathcal{H}_{kin}$. In relation to $\mathcal{H}_{kin}$, they have to be regarded as distributional states [35]. General solutions to the vector constraint must be identified in some larger space, namely the algebraic dual $\mathrm{Cyl}^\star$ of Cyl (the space of linear forms on Cyl). As vector spaces we have the relation $\Phi \subset \mathcal{H}_{kin} \subset \Phi^\star$, usually called the Gelfand triple. In the case of LQG diffeomorphism the Gelfand triple of interest is $\mathrm{Cyl} \subset \mathcal{H}_{kin} \subset \mathrm{Cyl}^\star$.

An element $\Phi \in \mathrm{Cyl}^\star$ is defined by $[\Phi](\Psi) := \langle \Phi, \Psi \rangle$ for all $\Psi \in \mathrm{Cyl}$. The requirement of diff invariance makes sense in $\mathrm{Cyl}^\star$ because the action of the Diff group is well defined by duality

$$[U_\phi \Phi](\Psi) = [\Phi](U_{\phi^{-1}} \Psi) \tag{1.95}$$

and therefore a diff invariant state is such that

$$[U_\phi \Phi](\Psi) = [\Phi](\Psi) \tag{1.96}$$

The space $\mathcal{H}_{kin}^{Diff}$ is the space of such states.

The technique used to construct the diff invarinat states is the group averaging procedure [43] which consists in averaging the elements of Cyl with respect to the action of the diffeomorphism group of Σ. The averaging is obtained via a rigging map [43]

$$\eta_{\mathrm{Diff}} : \mathrm{Cyl} \to \mathrm{Cyl}^\star \tag{1.97}$$

defined by

$$[\eta_{\rm Diff}(\Psi_\Gamma)](\Psi'_{\Gamma'}) = \sum_{\phi\in Diff} \langle U_\phi \Psi_\Gamma | \Psi'_{\Gamma'}\rangle \tag{1.98}$$

where the scalar product is in the Ashtekar-Lewandowski measure. This sum is finite. Both Ψ and Ψ' in fact can be expanded in linear combinations of spinnetwork states and if a diffeomorphism changes the graph of a Ψ_S then it take it to an orthogonal state; if it leaves the graph unchanged, there are only two possibilities; or it leaves the state invariant, or changes the orientation or the ordering but these are only discrete operations that contribute at most with a discrete number of terms in the sum (1.98).

Because of the diffeomorphism invariance of the scalar product, the state $[\eta_{\rm Diff}(\Psi_\Gamma)]$ is invariant under the action of $Diff(\Sigma)$:

$$[\eta_{\rm Diff}(\Psi_\Gamma)](U_\phi \Psi'_{\Gamma'}) = [\eta_{\rm Diff}(\Psi_\Gamma)](\Psi'_{\Gamma'}) \tag{1.99}$$

We have thus obtained a general solution to the vector constraint. The image of the rigging map $\eta_{\rm diff}$ in Cyl provides a complete solution space $\eta_{\rm diff}({\rm Cyl}) = {\rm Cyl}^\star_{\rm Diff}$ which can be equipped with a scalar product

$$\langle \eta_{\rm diff}(\Psi) | \eta_{\rm diff}(\Phi)\rangle_{\rm Diff} = [\eta_{\rm diff}(\Phi)](\Psi) \tag{1.100}$$

Finally, we can define the general solution to both the Gauss and the diffeomorphism constraints by simply restricting the pre-image of the map η to gauge invariant spin network states $|S\rangle$. Equivalently $\mathcal{H}^{Diff}_{kin}$ is defined by The full kinematics of four dimensional quantum gravity are therefore solved by defining the kinematical Hilbert space $\mathcal{H}^{Diff}_{kin}$ of solutions as the completion of the normed vector space

$$\eta_{diff}({\rm Cyl}\cap \mathcal{H}^{\rm G}_{\rm kin}) \tag{1.101}$$

in the norm induced by the scalar product (1.100).

Knots and s-knot states

We can now understand the structure of $\mathcal{H}^{Diff}_{kin}$ looking at the scalar product (1.100) resctricted to the spinnetwork states basis. A diffeomorphism can act on $|S\rangle$ only in two ways: sending it to an orthogonal state if ϕ changes the graph, or sending it to a state with differnt orientation or ordering. This second kind of diffeomorphisms g_k form the finite discrete subgroup G_Γ. The situation can then be resumed by

$$\langle \eta_{\rm diff}(\Psi_S) | \eta_{\rm diff}(\Psi'_S)\rangle_{\rm Diff} = \begin{cases} 0 & \text{if } \Gamma \neq \phi\Gamma' \\ \sum_k \langle U_{g_k}\Psi_S | \Psi'_S\rangle & \text{if; } \Gamma = \phi\Gamma' \end{cases} \tag{1.102}$$

So we see that two spin networks define orthogonal states in $\mathcal{H}^{Diff}_{kin}$ if the corresponding graphs belong to different equivalence classes under diffeomorphism. An equivalence class K of unoriented graphs is called *knot*. The basis states in $\mathcal{H}^{Diff}_{kin}$ are therefore firstly labeled knots K. They then differ by their colouring of links and nodes. We can therefore define the *s-knots states* $|s>=|K,c\rangle$ where c stands for coloring. The key property of s-knots is that they form a discrete set. Therefore $\mathcal{H}^{Diff}_{kin}$ admits a discrete orthonormal basis $|s>=|K,c\rangle$.

1.3.5 Geometric operators: quantization of the triad

We have constructed the kinematical quantum state space of LQG. Now we look for the operators that will lead us to th physical interpretation of the quantum gravity states. The two basic fields of the canonical theory are the connection A_a^i and its momentum E_i^a. In the quantum theory these basic variables acts as multiplicative and derivative operators

$$A_a^i \ \Psi[A] = A_a^i \ \Psi[A] \tag{1.103}$$

$$\frac{1}{\kappa\gamma} E_i^a \ \Psi[A] = -i\hbar \frac{\delta}{\delta A_a^i} \ \Psi[A] \tag{1.104}$$

Where γ is the Immirzi parameter. Both these operators are not well defined in $\mathcal{H}_{kin}$. To overcome this problem is enough to shift the attention to the *observable* variables of our theory: the holonomies and the fluxes.

The holonomy $h_\gamma[A]$ is in fact well defined in $\mathcal{H}_{kin}$, its action is

$$(h_\gamma[A]\Psi)[A] = h_\gamma[A]\Psi[A] \tag{1.105}$$

i.e the action of cylindrical function is well defined as a multiplicative operator in $\mathcal{H}_{kin}$.

The second phase space functions that we are interested in are the flux-like variables associated to closed two dimensional surfaces $\mathcal{S}$. These variables are naturally introduced analyzing the effect of the functional derivative of the holonomy

$$\begin{aligned} \frac{\delta}{\delta A_c^i} h_\gamma[A] &= \frac{\delta}{\delta A_c^i(x)} \left(\mathrm{P} \ \exp \int ds \ \dot{x}^d(s) A_d^k \ \tau_k \right) = \\ &= \int ds \ \dot{x}^c(s) \delta^{(3)}(x(s) - x) h_{\gamma_1}[A] \tau_i h_{\gamma_2}[A], \end{aligned} \tag{1.106}$$

where $h_{\gamma_1}[A]$ and $h_{\gamma_2}[A]$ are the holonomy along the two segments separated by the point x and τ_i is an $SU(2)$ generator. An important aspect of this formula is that the distribution on the right hand side is only two dimensional; it is then natural search for an operator smearing of E^a in two dimensions. Given a two dimensional surface $\mathcal{S} : \sigma = (\sigma^1, \sigma^2) \to x^a(\sigma)$ with normal 1-form

$$n_a = \frac{\partial x^b}{\partial \sigma^1} \frac{\partial x^c}{\partial \sigma^2} \epsilon_{abc} \tag{1.107}$$

we can define the following operator

$$\widehat{E}_i(\mathcal{S}) = \int_{\mathcal{S}} d\sigma^1 d\sigma^2 n_a \widehat{E_i^a} = -i\hbar\kappa\gamma \int_{\mathcal{S}} d\sigma^1 d\sigma^2 \frac{\partial x^a}{\partial \sigma^1} \frac{\partial x^b}{\partial \sigma^2} \ \frac{\delta}{\delta A_c^i} \epsilon_{abc}, \tag{1.108}$$

that is an operator valued distribution to be smeared on functions f_i with values on the Lie algebra of $SU(2)$. The previous expression corresponds to the natural generalization of the notion of electric flux operator in electromagnetism.

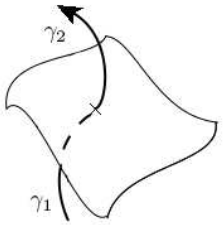
The grasping operator

We are the ready to see the action of (1.108) on the holonomies. Assuming that the surface is such that the end points of γ do not lie on the surface and γ intersects at most once the surface we have

$$\begin{aligned}&\widehat{E}_i(\mathcal{S})h_\gamma[A] = \\ &= -i8\pi\ell_p^2\gamma \int_{\mathcal{S}}\int_\gamma d\sigma^1 d\sigma^2 ds \frac{\partial x^a}{\partial\sigma^1}\frac{\partial x^b}{\partial\sigma^2}\frac{\partial x^c}{\partial s}\epsilon_{abc}\ \delta^{(3)}(x(\sigma),x(s))\ h_{\gamma_1}[A]\tau_i h_{\gamma_2}[A]. \quad (1.109)\end{aligned}$$

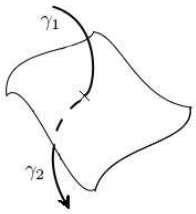
The integral vanishes unless the surface and the curve intersect in one point and using the definition of the delta function and computing the integral we obtain, depending on the relative orientations of the curve and the surface:

$$\widehat{E}_i(\mathcal{S})h_\gamma[A] = -i8\pi\ell_p^2\gamma\ h_{\gamma_1}[A]\tau_i h_{\gamma_2}[A] \quad , \quad (1.110)$$



and

$$\widehat{E}_i(\mathcal{S})h_\gamma[A] = i8\pi\ell_p^2\gamma\ h_{\gamma_1}[A]\tau_i h_{\gamma_2}[A] \quad , \quad (1.111)$$



and $\widehat{E}_i(\mathcal{S})h_\gamma[A] = 0$ when γ is tangential to $\mathcal{S}$ or $\gamma\cap\mathcal{S} = 0$. The action of $\widehat{E}_i(\mathcal{S})$ on holonomies is just to insert the matrix $i8\pi\ell_p^2\gamma\tau_i$ at the intersection point. We say that the operator $\widehat{E}_i(\mathcal{S})$ *grasps* [6, 7, 8, 9] the curve γ.

From its action on the holonomy, and using $SU(2)$ representation theory, one can easily obtain the action of $\widehat{E}_i(\mathcal{S})$ to holonomies in arbitrary j representations

$$\widehat{E}_i(\mathcal{S})D^j(h_\gamma[A]) = \pm i8\pi\ell_p^2\gamma\ D^j(h_{\gamma_1}[A])\ {}^{(j)}\tau_i\ D^j(h_{\gamma_2}[A]) \quad (1.112)$$

where ${}^{(j)}\tau_i$ is the $SU(2)$ generator in the spin j representation. From the previous formula is immediate the action on the spin network states and thus to any state in $\mathcal{H}_{kin}$. Using (1.70) one can also verify that $\widehat{E}_i(\mathcal{S})$ is self-adjoint. The operators $\widehat{E}_i(\mathcal{S})$ for all surfaces $\mathcal{S}$ contain all the information of the quantum Riemannian geometry of Σ. In terms of the operators $\widehat{E}_i(\mathcal{S})$ we can construct any geometric operator.

Quantization of the area

We focus on the area operator of a two-dimensional surface[8, 9] (see also [44] for the complete spectrum in degenerate cases) $S \subset \Sigma$ which classically depends on the triad E^a_i

as in (1.61). We introduce a decomposition of $\mathcal{S}$ in N two-cells S_N such that $\mathcal{S} = \bigcup_N \mathcal{S}_N$, that becomes smaller as $N \to \infty$ and write the integral defining the area as the limit of the Riemann sum,

$$A_{\mathcal{S}} = \lim_{N\to\infty} A^N_{\mathcal{S}} \tag{1.113}$$

where

$$A^N_{\mathcal{S}} = \sum_{I=1}^{N} \sqrt{E_i(\mathcal{S}_I)E^i(\mathcal{S}_I)} \tag{1.114}$$

where N is the number of cells, and $E_i(\mathcal{S}_I)$ corresponds to the flux of E^a_i through the I-th cell. We see that the fundamental object in the area formula is $E_i(\mathcal{S}_I)E^i(\mathcal{S}_I)$. To calculate the action of the quantum area operator we shift from the classical quantities to their quantum analogs acting in $\mathcal{H}_{kin}$, simply replacing the classical $E_i(\mathcal{S}_I)$ with $\widehat{E}_i(\mathcal{S}_I)$ according to (1.108) and thus we look for the action of $\widehat{E}_i(\mathcal{S})\widehat{E}^i(\mathcal{S})$ on a spinnetwork state $|S\rangle$. Its action is immediatly given by two grasping operators (1.112) in the same point if the spinnetwork intersect the surface only in a point. If the intersecting link is in the j representation we have

$$\begin{aligned}\widehat{E}_i(\mathcal{S})\widehat{E}^i(\mathcal{S})N^j_{mn}(h_\gamma[A]) =& i^2(8\pi\ell_p^2\gamma)^2 N^j(h_{\gamma_1}[A])^{(j)}\tau_i{}^{(j)}\tau^i N^j(h_{\gamma_2}[A]) = \\ =& (8\pi\ell_p^2\gamma)^2(j(j+1))N^j_{mn}(h_\gamma[A])\end{aligned} \tag{1.115}$$

where in the last equality we have used the fact that $-{}^{(j)}\tau_i{}^{(j)}\tau^i = j(j+1)\times\mathbb{I}$ is the $SU(2)$ Casimir. The previous equation simply imply that

$$\widehat{E}_i(\mathcal{S})\widehat{E}^i(\mathcal{S})|S\rangle = (8\pi\ell_p^2\gamma)^2(j(j+1))|S\rangle \tag{1.116}$$

the double grasping operator is *diagonal* on such spinnetwork base. If now we consider the quantum analog of the expression (1.113), the quantum area operator becomes

$$\widehat{A}_{\mathcal{S}} = \lim_{N\to\infty} \widehat{A}^N_{\mathcal{S}}, \tag{1.117}$$

. Now to calculate the previous expression it is enough to sum the contribution of the square of the electric flux trough the elementary cell $\mathcal{S}_I$ summing terms of the kind (1.116). This is possible because increasing N and shrinking $\mathcal{S}_I$ the cellular decomposition becomes such that in the limit $N \to \infty$ each $\mathcal{S}_I$ is punctured at most at a single point p by a link. The sum over I reduces to a sum over the intersections between Γ and $\mathcal{S}$ for large N. The area operator is then

$$\widehat{A}_{\mathcal{S}}|S> = 8\pi\ell_p^2\gamma \sum_{p\in(\mathcal{S}\cup\Gamma)} \sqrt{j_p(j_p+1)}|S> . \tag{1.118}$$

where j_p is the representation of the link that intersects $\mathcal{S}$ at p. See Figure 1.3.

Spin network states are the eigenstates of the quantum area operator and the spectrum is discrete The operator is $SU(2)$ gauge invariant by construction and also self adjoint. The remaining important case is when a spin network node is on $\mathcal{S}_I$. A careful analysis shows that the action is still diagonal in this case[9]. The spectrum of the area operator depends on the value of the Immirzi parameter γ (introduced in (1.23)). This is a general property of geometric operators.

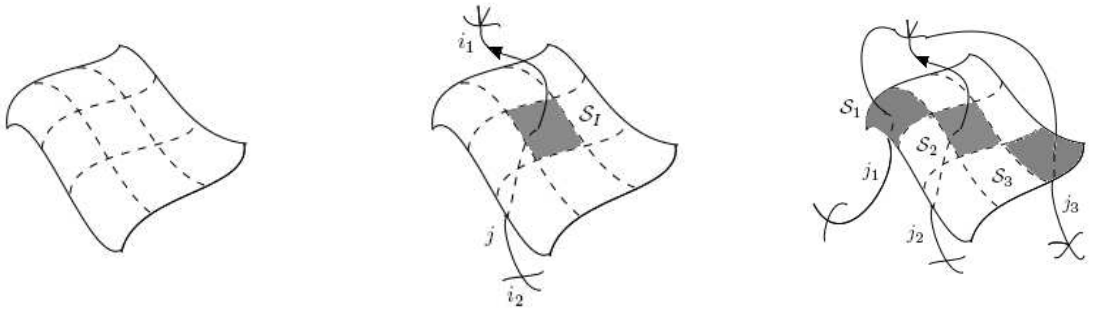


Figure 1.3: On the left: The regularization of (1.117) is defined so that the 2-cells are punctured by only one edge: in this case there is only one intersection that produces a value j of the area operator of $\mathcal{S}$; On the right: a generic spin–network puncturing the surface $\mathcal{S}$ in different points; the area operator get a contribution from each of the puncturing links .

Quantization of the volume

Another crucial geometrical operator that plays a key role in the physical interpretation of the quantum states of the gravitational field is the operator $\hat{V}_R[E]$ corresponding to the volume of a spacial region $R \subset \Sigma$. The volume of a three dimensional region $R \subset \Sigma$ is classically given by

$$V_R = \int_R \sqrt{q}\; d^3x, \tag{1.119}$$

Using (1.15) we conclude that

$$q = |\det(E)| = \left| \frac{1}{3!} \epsilon_{abc} E^a_i E^b_j E^c_j \epsilon^{ijk} \right|. \tag{1.120}$$

and therefore the volume can be expressed in terms of the densitized triad operator as

$$V_R = \int_R \sqrt{\left| \frac{1}{3!} \epsilon_{abc} E^a_i E^b_j E^c_j \epsilon^{ijk} \right|}\; d^3x \tag{1.121}$$

The corresponding operator can be constructed by promoting the momenta to derivation operator valued distributions. After regularization we rewrite the previous integral as the limit of Riemann sums defined in terms of a decomposition of R in terms of three-cells. Then we quantize the regularized version[8, 10, 45] using the grasping (or flux) operators. The final result gives a volume operator well defined and diffeomorphism covariant on $\mathcal{H}_{kin}$.

We underline some general properties of the volume operator. The volume operators acts on the spinnetwork nodes and the node must be at least 4-valent to have a non-vanishing volume. It depends on the associated intertwiner. The volume is a self-adjoint non negative operator and the volume of a region R is given by a sum of terms one for each node of the spinnetwork $|S\rangle = |\Gamma, j_l, i_1, \cdots i_N\rangle$ inside R; we get an expression of the kind

$$\widehat{V_R}|\Gamma, j_l, i_1, \cdots i_N\rangle = (16\pi \frac{\hbar G}{c^3})^{\frac{3}{2}} \sum_{n \in S \cap R} \mathcal{V}_{i_n}{}^{i'_n}\; |\Gamma, j_l, i_1, \cdots i'_n, \cdots i_N\rangle \tag{1.122}$$

where the numerical matrix $\mathcal{V}_{i_n}^{\ i'_n}$ can be numerically calculated from recoupling theory. It gives a *discrete* spectrum. The eigenvalue problem is not solved in terms of explicit closed formulas (there are however special cases [45, 46, 47, 48]), we refer to [49] for an analytical and numerical analysis of its spectrum.

1.3.6 The LQG physical picture

Geometric interpretation of spin network states

Assembling the results of the previous sections we see how the physical picture described by the spinnetwork states emerges. The volume operator takes contributions for each node of S inside a 3d region R. Therefore each node represents a quantum of volume. We can interpret a spinntework $|S\rangle$ with N nodes as an ensemble of N quanta of volume or grains of space located "'around"' the node with a quantized volume. The chunks of space are separated by surfaces, but the area of surfaces is given by the quantum operator $\widehat{A}_{\mathcal{S}}$ that has non vanishing discrete contribution from the spinnetwork links that puncture it.

The natural interpretation is the following: two chunks of space are contiguous if the corresponding nodes are connected by a link; in fact contiguous means separated by a surface whose quantum information is carried by the link representation. *Links of spin networks carry quanta of area while nodes carry quanta of volume.* A spinntework $|S\rangle = |\Gamma, j_l, i_n\rangle$ can be seen as a discrete quantized 3d *metric*: the graph Γ determines the adjacency relations (what is connected to what) of chunks of space whose volume is encoded in i_n, separated by surfaces whose area in contained in j_l. In other words the graph Γ can be viewed as the dual of a cellular decomposition of real space with a volume on each cell. This spin networks interpretation is still background dependent: the spinnetworks in fact still carry information about their embedding in the spatial manifold Σ. In fact spinnetworks states are solutions only of the Gauss constraint; their extension to solutions of $Diff$ constraint reveals the full background independent character of the LQG states.

Physical interpretation of s-knot states

The s-knot states $|s\rangle = |K, c\rangle$, represents the diffeomorphism equivalence class to which the spin network graph Γ belongs. In going from the spin network state $|S\rangle$ to the s-knot state $|s\rangle$, we preserve the information in $|S\rangle$ except for his location in Σ. This is the quantum analog the fact that physically distinguishable solutions of the classical Einstein equations are not fields, but equivalence classes of fields under diffeomorphisms. It reflects the core relational framework of general relativity. In GR we distinguish between a metric $g_{\mu\nu}$ and a geometry $[g]$ that is an equivalence class of metric under diffeomorphisms. The physical interpretation is then that the states $|s\rangle$ represent quantized geometry, formed by atoms of space that don't live in the manifold: they are localized with respect to one other and their spatial relation is coded only in the combinatorial adjacency structure of the links. Accordingly, the s-knot states are not quantum excitations *in* space, they are quantum excitation *of* space. An s-knot does not reside in the space: the s-knot itself defines the space. Moreover they carry the geometrical information necessary to construct the geometry of the space in which we decide to embed them. We can say that an s-knot is a purely algebraic kinematical quantum state of

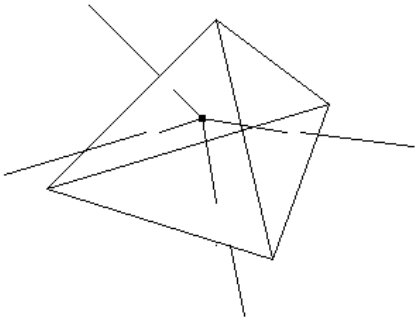


Figure 1.4: The dual 2-skeleton of a tetrahedron

the gravitational field in which the vertices give volume and the edges give areas to the space in which we embed it.

1.3.7 Quantum tetrahedron in 3d

So far we have considered only smooth embedding for the spin networks, but the picture above works also in other contexts, such as the case of spin networks embedded in a triangulated manifold. We summarize here how we can describe the quantum geometry of a tetrahedron in 3 dimensions as given by a spin network (for more details see [50, 51]) independently from the LQG approach that however perfectly match with it. This description will enter in the context of spinfoam models and in our calculation of the graviton propagator. Consider a compact, oriented, triangulated 3-manifold Δ, and the complex Δ^* dual to it, so having one node for each tetrahedron in Δ and one link for each face (triangle) (see Fig 1.4).

Considering a single tetrahedron in a 3d reference system $\mathbb{R}^3$; its geometry is uniquely determined by the assignment of its 4 vertexes. The same geometry can be determined by a set of 4 bivectors E_i (i.e. elements of $\wedge^2\mathbb{R}^3$ obtained taking the wedge product of the displacement vectors of the vertexes) normal to each of the 4 triangles satisfying the *closure constraint*

$$E_0 + E_1 + E_2 + E_3 = 0 \tag{1.123}$$

where the last constraint simply says that the triangles close to form a tetrahedron.

The quantum picture proceed as follows. Each bivector corresponds uniquely to an angular momentum operator (in 3 dimensions), so an element of $SU(2)$ (using the isomorphism between $\wedge^2\mathbb{R}^3$ and $so(3)$), and we can consider the Hilbert space of states of a quantum bivector as given by $\mathcal{H} = \oplus j$, where j indicates the spin-j representation space of $SU(2)$. Considering the tensor product of 4 copies of this Hilbert space, we have the following operators acting on it:

$$\hat{E}_0^I = J^I \otimes 1 \otimes 1 \otimes 1$$
$$\hat{E}_1^I = 1 \otimes J^I \otimes 1 \otimes 1 \quad (1.124)$$
$$\hat{E}_2^I = 1 \otimes 1 \otimes J^I \otimes 1 \quad (1.125)$$
$$\hat{E}_3^I = 1 \otimes 1 \otimes 1 \otimes J^I \quad (1.126)$$

with $I = 1, 2, 3$, and the closure constraint is given by $\sum_i \hat{E}_i^I \psi = 0 \quad \forall \psi \in \mathcal{H}^{\otimes 4}$. But now the closure constraint indicates nothing but the invariance under $SU(2)$ of the state ψ so that *the Hilbert space of a quantum tetrahedron* is given by:

$$\mathcal{T} = \oplus_{j_i} Inv(j_0 \otimes j_1 \otimes j_2 \otimes j_3) \quad (1.127)$$

or in other words the sum, for all the possible irreducible representations assigned to the triangles in the tetrahedron, of all the possible invariant tensors of them, i.e. all the possible intertwiners.

Moreover we can define 4 area operators $\hat{A}_i = \sqrt{\hat{E}_i \hat{E}_i}$ and a volume operator $\hat{V} = \sqrt{| \epsilon_{IJK} \hat{E}_1^I \hat{E}_2^J \hat{E}_3^K |}$, and find that all are diagonal on $Inv(j_0 \otimes j_1 \otimes j_2 \otimes j_3)$ (the area having eigenvalue $\sqrt{j_i(j_i+1)}$).

But now we can think of a spin network living in the dual complex Δ^* of the triangulation, and so with one 4-valent node, labelled with an intertwiner, inside each tetrahedron, and one link, labelled with an irreducible representation of $SU(2)$, intersecting exactly one triangle of the tetrahedron. We then immediately recognize that this spin network completely characterizes a state of the quantum tetrahedron, so a state in $\mathcal{T}$, and gives volume to it and areas to its faces (also matching the results from LQG).

1.3.8 The scalar constraint and the dynamics

The constraints solved so far generate kinematical gauge transformations in the sense that they operate at "fixed time".

The full quantum mechanical structure of spacetime is contained in the kernel of the Hamiltonian constraint (1.28). The implementation of the Hamiltonian constraint in LQG is still rather far from being as clean and complete as the two other constraints. Nevertheless, a series of results and properties of the hamiltonian operator have been established thanks to Thiemann's works [52, 53, 54, 29]. Here we present briefly a summary of its general properties.

The hamiltonian constraint (1.28) is the sum of two terms: the first one that we call C^E (because it defines Euclidean GR) and a second term involving extrinsic curvature terms. There is no quantisation possible because the hamiltonian can not be expressed simply in terms of the basic variables of the theory, due to the presence of terms containing extrinsic curvatures and inverses of the volume. However thank to Thiemann strategy [52],[53], these difficulties can be avoided by the use of the Poisson structure on the phase space to prove identities relating the undesired terms to Poisson brackets involving only the connection and the volume V of the spatial slice.

Here we sketch the quantization of the term C^E for Euclidean GR following[2]

Its smeared version $C^E(N)$ is

$$C^E(N) = \int_\Sigma dx^3 \; N \frac{E_i^a E_j^b}{\sqrt{\det(E)}} \epsilon^{ij}{}_k F_{ab}^k, \tag{1.128}$$

but, using the identity [55]

$$\frac{E_i^b E_j^c}{\sqrt{\det(E)}} \epsilon^{ijk} \epsilon_{abc} = \frac{4}{\kappa\gamma} \left\{ A_a^k, V \right\}. \tag{1.129}$$

where V is the volume (1.121), the Euclidean constraint can be rewritten as

$$S^E(N) = \int_\Sigma dx^3 \; N \; \epsilon^{abc} \delta_{ij} F_{ab}^i \left\{ A_c^j, V \right\}, \tag{1.130}$$

We can now quantize this constraint promoting the argument of the Poisson brackets to operators and the Poisson brackets to commutators. One needs the volume operator $\widehat{V}$ (section 1.3.5) and the quantization of the connection A and curvature F that can be performed defining these quantities as limits of holonomy operators of small paths [53, 29].

Given a point x and a tangent vector u at x denoting with $\gamma_{x,u}$ the path of ϵ length starting in x and tangent to u we have

$$h_{\gamma_{x,u}}[A] = 1 + \epsilon u^a A_a + \mathcal{O}(\epsilon^2) \tag{1.131}$$

and in the same way given two tangent vector u, v an infinitesimal triangular loop $\alpha_{x,uv}$ of area $\frac{\epsilon^2}{2}$ with a vertex in x and two sides tangent to u and v, the curvature tensor can be regularized observing that

$$h_{\alpha_{x,uv}}[A] = 1 + \epsilon^2 u^a v^b F_{ab}^i \tau_i + \mathcal{O}(\epsilon^3). \tag{1.132}$$

Now we can replace the expression (1.130) with its Riemann sum over cells R_I of coordinate volume ϵ^3 and express it in terms of holonomies

$$\begin{aligned} C^E(N) \quad &= \lim_{\epsilon\to 0} \sum_I \; N_I \epsilon^3 \; \epsilon^{abc} \mathrm{Tr}\left[F_{ab}(A) \left\{ A_c, V_{R_I} \right\} \right] = \\ &= \lim_{\epsilon\to 0} \sum_I \; N_I \; \epsilon^{abc} \mathrm{Tr}\left[h^{-1}_{\gamma_{x^I,u_c}}[A] h_{\alpha_{x^I,u_a u_b}}[A] \left\{ V_{R_I}, h_{\gamma_{x^I,u_c}}[A] \right\} \right] \end{aligned} \tag{1.133}$$

where u_1, u_2, u_3 are any three tangent vectors with triple product equal to 1 and x^I is an arbitrary point in the region R^I. Remarkably the dependence on the cell size ϵ in this way disappears.

Inside the cell R_I of volume V_{R_I} we have

1. An infinitesimal closed loop of coordinate area ϵ^2 $\alpha_{x^I,u_a u_b}$ in the $u^a u^b$-plane
2. An edge $\gamma^I_{x^I,u_c}$ of coordinate length ϵ dual to the u^c-plane

The expression (1.133) is now ready to became an operator quantizing the holonomies and the volume obtaining operators well defined in $\mathcal{H}_{kin}$. Formally we have

$$\widehat{C}^E(N) = \lim_{\epsilon\to 0} C_\epsilon(N) = \lim_{\epsilon\to 0} \sum_I \; N_I \; \epsilon^{abc} \mathrm{Tr}\left(\hat{h}^{-1}_{\gamma_{x^I,u_c}}[A]\hat{h}_{\alpha_{x^I,u_a u_b}}[A]\left[\widehat{V}_{R_I}, \hat{h}_{\gamma_{x^I,u_c}}[A]\right]\right). \quad (1.134)$$

Now in order to have a rigorous definition of $\widehat{C}^E(N)$ we need to complete the definition of its regularized version $C_\epsilon(N)$ choosing the paths and the decomposition in cells in such a way that the previous limit exists and gives a quantum operator covariant under diffeomorphisms and invariant under internal gauge.

The first key observation is that $\widehat{C}^E_\epsilon(N)$ *acts only on spin network nodes* due to the presence of the volume operator (1.122). In fact the volume operator vanishes on three-valent nodes and the holonomy can at most increase the valence of the node by one. This means that only the regions R_n with a node inside give a contribution

$$\widehat{C}_\epsilon(N)|S\rangle = \sum_{n\in\Gamma} N_n \widehat{C}^n_\epsilon \; |S\rangle, \quad (1.135)$$

where $\widehat{C}^n_\epsilon$ acts only on the node $n \subset \Gamma$ and N_n is the value of the lapse $N(x)$ at the node. Observing the commutator in (1.134) is immediate to see that the only possibility of having a non trivial commutator is that the link γ_{x^I,u_c} touch the node. *The position of the point x^I, irrelevant in the classical theory, now is fixed in the node n.* Then action is then non trivial only on nodes n at least three-valent, with x^I fixed in the node n; the natural choice for the vectors u^a is then to be tangent to the three links l,l' and l'' emerging from the node (see Figure 1.5) and the ϵ^{abc} gives a sum of permutations on the three links; we end up with an expression

$$\widehat{C}^n_\epsilon \; |S\rangle = \sum_{l,l',l''} \epsilon^{l\,l'\,l''} \mathrm{Tr}\left(\hat{h}^{-1}_{\gamma_{x_n,u_{l''}}}[A]\hat{h}_{\alpha_{x_n,u_l u_{l'}}}[A]\left[\widehat{V}_{R_n}, \hat{h}_{\gamma_{x_n,u_{l''}}}[A]\right]\right)|S\rangle \quad (1.136)$$

In general the previous limit, as usually happens in QFT, when operators products are involved, does not exist for spinntework states but remarkably *it exists for diff-invariant states* revealing how the diff invariance can cure the short scale behavior problems of QFT.

The action of $\widehat{C}_\epsilon(N)$ on diff invariant states has to be defined by duality

$$[\widehat{C}^\dagger\Phi](|S\rangle) \equiv [\Phi](\widehat{C}|S\rangle) \quad (1.137)$$

this means that we can consider the regualarized operator in $\mathcal{H}^{Diff}_{kin}$ and take the limit there.

$$[\widehat{C}^\dagger\Phi](|S\rangle) = \lim_{\epsilon\to 0}[\Phi](\widehat{C}_\epsilon|S\rangle) \quad (1.138)$$

i.e a limit of sequence of numbers and not a sequence of Hilbert space vectors.

If $\Phi \in \mathcal{H}^{Diff}_{kin}$ the limit exist! in fact in general the operator $\widehat{C}_\epsilon$ acts on the state $|S\rangle$ in two possible ways:

1. Modifying the graph with the operators $h_{\gamma_{x_n,l}}$ and $h_{\alpha_{x_n,l,l'}}$:
   - $h_{\gamma_{x_n,l}}$ superimpose a path of length ϵ to the link l of Γ;

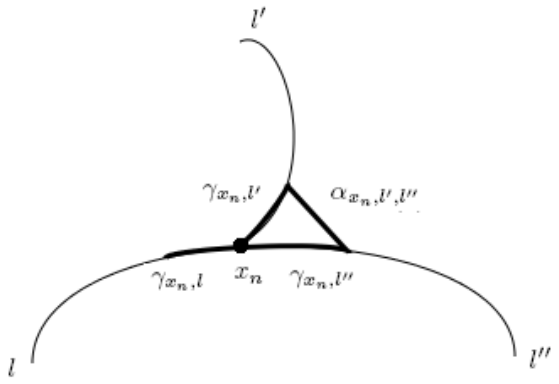


Figure 1.5: The three paths $\gamma_{x_n,l}$ $\gamma_{x_n,l'}$ $\gamma_{x_n,l''}$ and the loop $\alpha_{x_n,l',l''}$ given by $\gamma_{x_n,l'}$, $\gamma_{x_n,l''}$ and a third link external to Γ

- $h_{\alpha x_n,l,l'}$ superimpose a triangle with sides of ϵ length on Γ; two of them along the links l and l' and the third one not in Γ

See the Figure 1.5

Now acting on a spinnetwork $|S\rangle$ with $\widehat{C}_\epsilon$ for two different values of ϵ gives two different states, but they are *in the same diffeomorphisms equivalence class*! In fact in the two cases we are only adding two different triangle to the graph that under a diffeomorphism are the same object. The only limitation for ϵ to leave the state in the same diff-class is to be smaller than the value $\bar{\epsilon}$ such that the new link cross one other link or node of Γ.

Therefore for $\epsilon < \bar{\epsilon}$ the term in the round parantesis in (1.138) is always in the same equivalence class shrinking ϵ; this means that if $\Phi \in \mathcal{H}^{Diff}_{kin}$ it is invariant under diffeomorphism and the argument of the limit becomes constant. The equation (1.138) is then simply

$$[\widehat{C}^\dagger \Phi](|S\rangle) = \lim_{\epsilon \to 0} [\Phi](\widehat{C}_\epsilon |S\rangle) = [\Phi](\widehat{C_{reg}}|S\rangle) \tag{1.139}$$

where

$$\widehat{C_{reg}}|S\rangle = \sum_{n\in\Gamma} N_n \sum_{l,l',l''} \epsilon^{l\,l'\,l''} \mathrm{Tr}\left(\hat{h}^{-1}_{\gamma_{x_n,u_l}}[A] \hat{h}_{\alpha_{x_n,u_{l'}u_{l''}}}[A] \left[\widehat{V}_{R_n}, \hat{h}_{\gamma_{x_n,u_l}}[A]\right]\right) \tag{1.140}$$

is the regularized operator with the size ϵ of the regulator small enough to not cross other links or nodes in Γ. The operator is then finite! We are at the core of diff-invariance formalism of LQG: The coordinate $\vec{x}$ have not physical relevance; the physical location of things is only relational. The excitations of the theory are quantized. The union of these two features leaves no more "'space"' for infinite short distance limit. Making the regulator smaller than the minimal Planck scale does not change nothing simply because there is nothing under this scale.

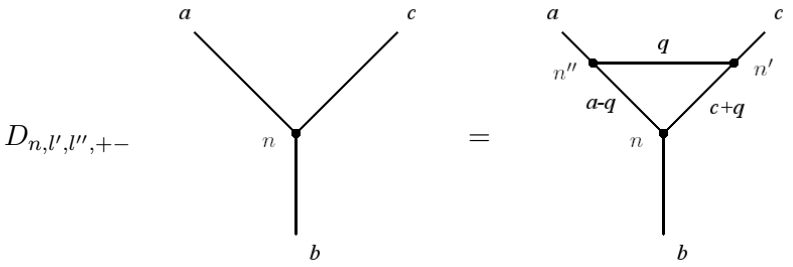


Figure 1.6: Example of the action of $D_{n,l',l'',\pm\pm}$. It acts on three valents nodes creating two new nodes and adding a link in q representation that connects these two nodes. It also changes the colorings of the grasped links and the intertwiner at n

General properties and difficulties

We don't enter in the details of the calculation of the matrix elements of (1.140) that depends on the action of the volume operator and from local features at the node. We can however summarize the fundamental properties of the operator (1.140).

$\widehat{C}_{reg}$ acts by creating new links around nodes, thanks to the action of the infinitesimal loop operators representing the regularized curvature, with coefficient depending on the particular quantization procedure. In general acting on an s-knot state $\widehat{C}_{reg}$ gives a sum of terms

- one for each node n in the state
- for each node $\widehat{C}_{reg}$ gives a sum of terms one for each triple of links emerging from the node n
- for each triple one term for each permutation of the three links
- each of these terms acts (see Figure 1.6)
  1. creating two new nodes n' and n'' at a finite distance from n on l' and l''
  2. create a new link (called *arc*) of spin q (depending on the quantization procedure considered) connecting n' and n''
  3. changes the colorings of the links connecting n and n' and n and n'' respectively (the color are the original one increased or decreased by q) changes the intertwiner at the node n; the new intertwiner is between the new representations of the adjacent links

If we call $D_{n,l',l'',\pm\pm}$ an operator acting like in the last three points then

$$\widehat{C}|S\rangle = \sum_{n\in S} N_n \sum_{l,l',l''} \sum_{\alpha',\alpha''=\pm} C_{n,l',l'',\alpha',\alpha''} D_{n,l',l'',\alpha'\alpha''}|S\rangle \tag{1.141}$$

where $C_{n,l',l'',\alpha',\alpha''}$ is a finite matrix in the space of intertwiners at the node n(see [56] for an explicit example of its computation). The version of quantum scalar constraint whose action

is (1.141) is not self adjoint. One can introduce self adjoint definitions which contain a term that creates arc and another one that destroys them (this is one of the ambiguities of the theory).

This complete the picture depicted from the hamiltonian constraint; it opens the way to inclusion of matter. A well defined quantization of the scalar constraint including Yang-Mills fields, scalar fields and fermions has been put forward by Thiemann[57].

The successful definition of the quantum scalar constraint operator including the cases with realistic matter couplings is a remarkable achievement of loop quantum gravity.

There is however a large degree of ambiguity on the definition of the quantum scalar constraint. The nature of solutions or the dynamics seems to depend critically on these ambiguities. For instance it is possible to arrive at a completely consistent quantization by essentially replacing the holonomies in (1.134), defined in the fundamental representation of $SU(2)$, by the corresponding quantities evaluated on an arbitrary representation [58]. In the applications of the theory to simple systems such as in loop quantum cosmology this is known to have an important physical effect[59]. Ambiguities are also present in the way in which the paths defining the holonomies that regularize the connection A_a^i and the curvature $F_{ab}^i(A)$ in (1.134) are chosen. See for instance [60] for an alternative to Thiemann's prescription and a discussion of the degree of ambiguity involved. There are also factor ordering ambiguities, which is evident from (1.134).

We end up with a wide class of consistent Hamiltonian operator but a yet unresolved issue is whether any of these theories is rich enough to reproduce general relativity in the classical continuum limit [61].

The dynamics of the theory depends crucially on these quantization ambiguities and there is, up to know, no criterion available to distinguish between all these operators. See [60],[62] for discussions on the subject. Some exact solutions to the scalar constraint are known for some specific quantizations [29], [60] but no physical vector space and no physical scalar product have been built so far even if some hope has recently been raised by the Master constraint program [63].

These concerns have opened different research programs in the hope of finding alternatives and some guiding principles that would lead to a clearer understanding of the physics behind the scalar constraint. One of them is the Spinfoam approach or the spinfoam representation of LQG [64], motivated to a large extent by the hope of solving the issue of ambiguities from a covariant perspective as well as by the search of a systematic definition of the physical scalar product.

Chapter 2

Spinfoam models

Classical Mechanics can be formulated in Hamiltonian (canonical) or Lagrangian (covariant) formulation. The same is true for Quantum mechanics where we have a canonical (Hilbert spaces, commutation relations, operators) and a covariant (sum over paths) framework. The same is possible with LQG.

In the previous chapter we have presented the canonical or Hamiltonian formulation of LQG. Now we concentrate on the covariant or path integral quantization of the same theory. Such path integrals formulation are called *spinfoam models* [2, 51, 65, 66, 67, 68]. The aim of the spinfoam approach is to provide an explicit tool to compute transition amplitudes in quantum gravity. These are expressed as a sum over paths. A path is a spinfoam representing a spacetime history which can be though of as a worldsurface swept by a spin network. Spinfoams are background independent combinatorial objects, and do not need a spacetime to live in. A spinfoam itself represents a quantum spacetime, in the same sense in which a spin network represents an atom of quantised space. They can be regarded as implementing the physical scalar product in canonical background independent theories.

The chapter is organized as follows;

After having introduced the general ideas underling this approach, we study the pure BF theory (topological theory) as a tool to discretize gravity. The idea is to use the lack of local degrees of freedom of such theories to calculate the path integral with lattice regularization techniques without needing a continuum limit. Then we study how gravity can be recasted in term of a constrained BF theory using its Plebanski formulation [69]. This formulation lead to the Barret-Crane (BC) model [70] , the path integral formulation for four dimensional quantum gravity most widely studied in the literature. Then we concentrate on the Group Field Theory (GFT) approach [71, 72] as a tool to restore the presence of local degrees of freedom giving full background independent models. In particular we derive two kind of BC model from GFT the GFT/A and the GFT/B (see [2]).

2.1 The idea

The application of Feynman's path integral to quantum gravity has been considered long ago by Misner and extensively studied by Hawking, Hartle and others [73, 74]. The idea is simple; given a 4d-manifold $\mathcal{M}$ with boundaries Σ_1 and Σ_2, and denoting by G the space of

metrics on $\mathcal{M}$ the transition amplitude between g on Σ_1 and g' on Σ_2 is formally

$$W[g,g'] = \int D[g_{\mu\nu}(x)] e^{iS_{GR}[g]} \tag{2.1}$$

where the integration on the right is performed over all space-time metrics up to 4- diffeomorphisms $g_{\mu\nu}(x) \in G/Diff(\mathcal{M})$ with fixed boundary values up to 3-diffeomorphisms g, and g', respectively. There are many difficulties associated with (2.1). The main issue is the definition of the measure $D[g_{\mu\nu}(x)]$; the non-perturbative one are unknow and the perturbative one lead to non-renormalizable divergencies. In addition, there is the problem of having to deal with the space $G/Diff(\mathcal{M})$, i.e., how to characterize the diffeomorphism invariant information in the metric. This gauge problem is also present in the definition of the boundary data and also there is no well defined notion of kinematical state $|g_{ab}\rangle$ in standard metric variables.

The situations changes in the framework of LQG. In fact the key ingredient of this theory, the spinnetwork basis (1.3.3) is able to deal with all the previous problems. First the notion of quantum state of 3-geometry is rigorously defined in terms of spin-network (s–knots) states and they carry the diff-invariant information of the Riemannian structure of Σ. These states as we have seen are intrinsically discrete (colored graphs on Σ).

Analizing a simple analogy we can see how the discreteness can give a possible solution to the functional measure problem. If we think to a one dimensional harmonic oscillator when we deal with the probability of measuring x' given x

$$W(x,t,x',t') = \langle x|e^{-iH_o(t-t')}|x'\rangle \tag{2.2}$$

the x appearing in this expression are not the classical variables but rather the label of the corresponding eigenvalues. The difference is irrelevant if we use continuum variables like the position; but it becomes important if the variable x has a discrete spectrum. If we consider for example the transition amplitude between the energies of the system $W(E,t,E',t')$: in the case of an harmonic oscillator with a small perturbation, the expression $W(E,t,E',t')$ will only make sense for the discrete energy states admitted by the quantum theory

$$W(E,t,E',t') = \langle E|e^{-iH_0}|E'\rangle \longrightarrow W(E_n,t,E_{n'},t') = \langle E_n|e^{-iH_0}|E_{n'}\rangle \tag{2.3}$$

In this case in fact the expression $W(x,t,x',t')$ has meaning only for discrete values of the labels admitted by the quantum theory, not for the classical one (continuous).

The same consideration can then be applied to (2.1) once that we have identified the eigenstates of the 3-geometry with the spinnetwork states $|s\rangle$; $|s\rangle$ are discrete states and the transition amplitude will make sense only between the possible spinnetworks that code the geometrical information contained in the the initial and final three geometry; we are then naturally lead to substitute the expression (2.1) with the transition amplitude $W(s,s')$ between initial and final spinnetworks states

$$W[g,g'] = \int D[g_{\mu\nu}(x)] e^{iS_{GR}[g]} \longrightarrow W(s,s') \tag{2.4}$$

The expression $W(s,s')$ leads to a sum over path formulation. This has been shown in [75, 76, 77] but we can proceed with formal expressions to indicate heuristically how this may happen. If we write $W(s,s')$ making use of the "'projection"' operator on physical states

$$W(s,s') = \langle s|P|s'\rangle_{\mathcal{K}} \tag{2.5}$$

where P is the "Projector' on the Kernel of the Hamiltonian operator and can formally be written, assuming that it has non negative spectrum, as

$$P = \lim_{t\to\infty} e^{-Ht} \tag{2.6}$$

since $H(x)$ is a function of the spatial coordinates $\vec{x}$, P is

$$P = \lim_{t\to\infty} \prod_x e^{-H(x)t} = \lim_{t\to\infty} e^{-t\int H(x)d^3x} \tag{2.7}$$

and if it is possible to define the propagation generated by H in a 4d invariant manner we can get rid of the limit and write

$$W(s,s') = \langle s|e^{-\int_0^1 dt \int H(x)d^3x}|s'\rangle_{\mathcal{K}} \tag{2.8}$$

at this point we can proceed exactly like in usual quantum mechanics inserting identity resolutions $|s\rangle\langle s| = 1$

$$W(s,s') = \lim_{N\to\infty} \sum_{s_1\dots s_N} \langle s|e^{-\int_0^1 dt \int H(x)d^3x}|s_N\rangle_{\mathcal{K}}\ \langle s_N|e^{-\int_0^1 dt \int H(x)d^3x}|s_{N-1}\rangle_{\mathcal{K}}\ \cdots\ \langle s_1|e^{-\int_0^1 dt \int H(x)d^3x}|s'\rangle_{\mathcal{K}} \tag{2.9}$$

and expanding the exponentials for small dt (first order) we produce, at fixed N, terms equivalent to histories with index N. The expression (2.5) is then a sum over spinnetworks sequences $\sigma = (s, s_1, \cdots, s_N, s')$ of amplitudes $A(\sigma)$

$$W(s,s') = \sum_\sigma A(\sigma) \tag{2.10}$$

where $A(\sigma)$ codes the dynamic of the theory.

$$A(\sigma) = \prod_v A_v(\sigma) \tag{2.11}$$

In this formula v denotes the steps of the history and A_v is determined by the terms

$$\langle s_n|e^{-\int_0^1 dt \int H(x)d^3x}|s_{n-1}\rangle_{\mathcal{K}} \tag{2.12}$$

the previous expression can be expanded for small dt and, using the action of the Hamiltonian operator (1.3.8) on the spinnetwork nodes, and gives a sum of terms non vanishing only if s_n and s_{n+1} differ at each node by the action of H. In this way the integral over fields is transformed in a *sum* over histories; the admitted paths depend on the action of H that also determines the amplitude of each step; the amplitude of the history is then the product of the step's amplitudes. An history of spinnetworks $\sigma = (s, s_1, \cdots, s_N, s')$ is called a spinfoam.

Table 2.1: Terminology

	0d	1d	2d	3d	4d
Spin networks:	*node,*	*link;*			
Spinfoams:	*vertex,*	*edge,*	*face;*		
Triangulation:	*point,*	*segment,*	*triangle,*	*tetrahedron,*	*four-simplex.*

2.2 The picture

There is a natural picture that can illustrate the spinfoam as a path integral version of LQG, given by a series of transitions through different spin-network states representing states of 3-geometries. As we have seen spinnetworks represent a 3d space with the geometrical information coded in the links and nodes representations. We can imagine a 4d space (space-time) with a "time" coordinate: if we let evolve the spinnetwork along this direction, the links become surfaces "faces" (f) and the nodes become lines "edges" (e). The spinfoam is then given by an history of spinnetworks in which at each step the H operator generate new links; in the evolving picture the image is then an edge that brunch out in other edges in a point that we call "vertex" v see Figure 2.1 . In this way the worldsheet of the original spinnetwork

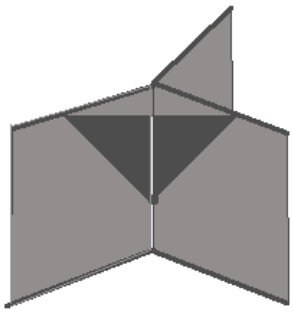


Figure 2.1: A vertex of a spifoam: the evolution along a "time" line of the Hamiltonian operator action on a spinnetwork node.

is a collection of faces meeting at edges with vertexes as boundary: a two complex $\mathcal{J}$. We resume the terminology in the Table 2.1.

Like the spinnet $s = (\Gamma, j_l, i_n)$ is defined not only by its graph Γ but also by the colorings, with irreducible representation j_l associated to links and intertwiners i_n associated to nodes, its evolution, the spinfoam $\sigma = (\mathcal{J}, j_f, j_e)$ is a two complex $\mathcal{J}$ colored with the irreducible representation j_f associated to faces and intertwiners j_e associated to edges. An important re-

mark is that the boundaries of the spin-foam are spinnetworks. The background-independent character of spin foams is then manifest. The 2-complex can be thought of as representing chunk of 'space-time' while the boundary graphs as representing chunk of 'space'. The geometrical information in contrast with the standard concept of a lattice is totally encoded in colorings which represent the degrees of freedom of the gravitational field. In the Figure 2.2 we can see an example of spinfoam with two vertex corresponding to a two steps history.

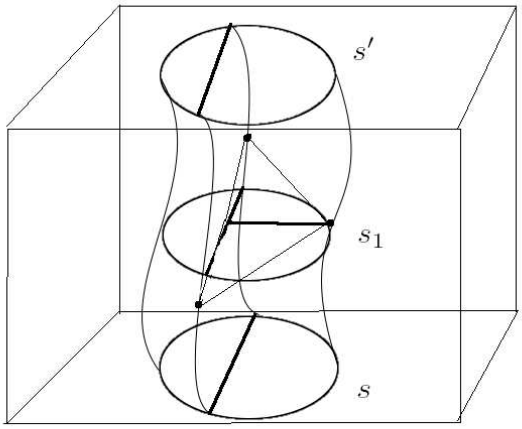


Figure 2.2: An example of a two step history of a spinetwork state: the initial spinnet s evolve in a spinnet s_1 and then to the final spinnet s' under the two steps action of an Hamiltonian operator. Nodes and links in the spin network evolve into 1-dimensional edges and faces.. The geometric degrees of freedom are encoded in the labeling with irreducible representations and intertwiners.

2.3 General definition

In general a spinfoam model is defined by the partition function

$$Z = \sum_{\mathcal{J}} w(\mathcal{J}(\sigma)) \sum_{j_f, i_e} \prod_f A_f(j_f) \prod_e A_e(j_f, i_e) \prod_v A_v(j_f, i_e) \tag{2.13}$$

where A_f, A_e, A_v are the amplitudes associated to respectively the faces, the edges and the vertices and $w(\Gamma)$ is the weight associated to the combinatorial structure of the two-complex $\mathcal{J}$. A model is completely determined by a choice of

1. a set of 2-complexes $\mathcal{J}$ and the corresponding weight $w(\mathcal{J})$

2. a set of representations and intertwiners j_f, i_e

3. a face amplitude $A_f(j_f)$, an edge amplitude $A_e(j_f, i_e)$ and a vertex amplitude $A_v(j_f, i_e)$

Generally speaking the first two choices depends on the kinematics, the last one on the dynamic. An expression of the kind (2.13) can be taken as a general definition of a background independent QFT formalism. The partition function (2.13) is defined for closed spinfoams. We can rewrite it the compact notation indicating with $\sigma = (\mathcal{J}, j_f, i_e)$

$$Z = \sum_{\sigma} w(\mathcal{J}(\sigma)) \sum_{j_f, j_e} \prod_f A_f(j_f) \prod_e A_e(j_f, i_e) \prod_v A_v(j_f, i_e) \tag{2.14}$$

We are now ready to make contact with (2.10). The boundary of a spinfoam is by definition a spinnetwork. We indicate the spinfoam σ with boundary s as $\partial\sigma = s$. If the boundary spinnetwork is composed by two connected components s and s' we have $\partial\sigma = s \cup s'$. The equation (2.14) can then naturally be extended to the sum over spinfoam with fiexd boundary s and s';

$$W(s, s') = \sum_{\partial\sigma = s \cup s'} w(\mathcal{J}(\sigma)) \sum_{j_f, i_e} \prod_f A_f(j_f) \prod_e A_e(j_f, j_e) \prod_v A_v(j_f, j_e) \tag{2.15}$$

and we can interpret this expression as a *sum over path definition of the transition amplitude between states of the quantum states of the gravitational field.* The previous definition extends naturally to the case of a boundary with a single spinnetwork s

$$W(s) = \sum_{\partial\sigma = s} w(\mathcal{J}(\sigma)) \sum_{j_f, i_e} \prod_f A_f(j_f) \prod_e A_e(j_f, j_e) \prod_v A_v(j_f, j_e) \tag{2.16}$$

The expressions (2.16) and (2.15) bridge between LQG and the spinfoam formalism. Notice however that in the spinfoam formalism, even if there is uncertainty in the definition of the right model, transition amplitudes can be computed, in a certain perturbative expansion. In the canonical formalism, on the other hand, even disregarding the uncertainties in the definition of the hamiltonian operator, we are not yet able to compute transition amplitudes. Indeed, it is important to stress that even if the general structures underlying covariant spinfoam models match nicely with the Hamiltonian theory, the precise relation between the two formalisms is not yet entirely clear. What is remarkable about the expressions (2.1) and (2.3) is that many very different approaches to non–perturbative quantum gravity have converged precisely to this formula, and perhaps an expression of this sort can be taken as a general definition of a background–independent covariant formalism for quantum field theories. Also, these formulas emerge in the context of topological quantum field theories (TQFTs)[78, 79]. 2d and 3d GR are topological theories. For the role of TQFTs for QG, see [80].

Among the many approaches leading to (2.1), there is one that links it with the quantum version of a classical theory.

The starting point is always pure BF theory. The idea is to use the lack of local degrees of freedom of such theories to calculate the path integral with lattice regularization techniques without needing a continuum limit. The calculations are then performed using the representation theory of the associated symmetry group.

To do so, we fix an oriented triangulation Δ of the spacetime manifold. In 3d this is made out of points p, segments s, triangles t, tetrahedra τ. In 4d we have the same elements but also 4-simplices χ. We need also an operation that allows the identification of the $n-1$ dimensional boundary of an n dimensional object in Δ. It is convenient to work with the dual of the triangulation Δ^* constructed with a one-to-one correspondence between ndimensional objects of Δ and 3-n dimensional objects of Δ^*. In 3d to obtain Δ^*, we place a vertex inside each tetrahedron τ; if two tetrahedra bound the same triangle t we connect the two vertices vith an edge e. For each segment of Δ we have a face f of the dual Δ^*: for each point we have a 3d region in the dual bounded by the faces dual to the segments bounded by the point. In 4d we place a vertex in each 4-simplex and so on. The 2-skeleton of Δ^* (collection of vertices, edges and faces) is a 2-complex $\mathcal{J}$. We summarize the construction in the Table 2.2

Table 2.2: Relation between a triangulation Δ and its dual Δ^*, in 3d and in 4d. In parenthesis: adjacent elements. In italic, the two-complex.

Δ_3	Δ_3^*	
tetrahedron	*vertex*	(4 edges,6 faces)
triangle	*edge*	(3 faces)
segment	*face*	
point	3d region	

Δ_4	Δ_4^*	
4-simplex	*vertex*	(5 edg, 10 fac)
tetrahedron	*edge*	(4 faces)
triangle	*face*	
segment	3d region	
point	4d region	

In the following, we restrict to spinfoams defined on a single 2-complex which is the 2--skeleton of a fixed triangulation Δ of the spacetime manifold. This restriction immediately opens a problem in theories with local degrees of freedom. GR and its couplings with matter, is invariant under diffeomorphisms of $\mathcal{M}$. The introduction of a fixed triangulation on $\mathcal{M}$ breaks the invariance under diffeomorphisms, and it is necessary to find a procedure to restore it. The first option that comes to mind, namely refining the triangulation (or the 2complex), as one does for instance in Lattice Gauge Theory (LGT), is not available. In LGT there is a parameter, the lattice spacing, that plays the role of an ultraviolet cut-off, which can be sent to zero to remove the regularization. The theory does not have ultraviolet divergences simply because there are no degrees of freedom beyond the Planck scale l_p , a physical cut- off which emerges dynamically. Therefore, there is no possibility to rescale the triangulation to refine it. Indeed, there are procedures to refine the triangulation simply using its topological properties (see for instance [81]); they work fine in the context of TQFTs, but are hard to reconcile with local degrees of freedom. Leaving alone for the moment the option of refining the triangulation, the immediate alternative seems to be the sum over triangulations. This would restore the infinite degrees of freedom of GR, and its original diffeomorphism invariance. Notice that in (2.14) is already present a sum over 2complexes; if on the one hand this leads to a sum over triangulations of a fixed topology, on the other hand it also includes a sum over topology. The meaning of this sum remains unclear, especially in the 4d case, where the classification of manifold is less understood. We work on a fixed triangulation. However in the next sections we will see that exists a candidate framework for this study, called the group field theory (GFT) formalism.

In the next section, we show how the discretization procedure described above allows us to link a classical theory with a quantum expression like (2.1). We consider BF theory, a topological field theory with no local degrees of freedom, for two reasons: on the one hand, its spinfoam quantization is straightforward, which allows us to introduce our tools in a simple way; on the other hand, the action for GR, in all spacetime dimensions, can be written starting from the BF action [82]. Consequently, the quantization of BF theory provides a good starting point for the quantization of GR.

2.4 BF theory

Consider a n–dimensional smooth manifold $\mathcal{M}$ representing spacetime, and a Lie group $\mathcal{G}$ (called "the structure group") whose Lie algebra $\mathfrak{g}$ is equipped with the Killing form Tr. The fundamental fields of BF theory are a connection A on a principal bundle $P(\mathcal{M},\mathcal{G})$, and a $\mathfrak{g}$-valued (n-2)-form B. The theory is defined by the action

$$S(A,B)=\int_{\mathcal{M}} TrB\wedge F(A) \tag{2.17}$$

where $F=dA+A\wedge A$ is the curvature of A. The B field has the physical dimensions of an action, while the connection and the curvature are dimensionless. The equations of motions are

$$F(A)=0 \qquad d_A B=0 \tag{2.18}$$

They imply that the connection A is flat, and the field B is covariantly constant in any dimension. The action is invariant under diffeomorphisms acting on spacetime indexes, and $SU(2)$ transformations acting on the algebra indices. Under these transformations, all solutions of the equations of motion are equivalent. See for instance [68] for more details. Therefore, once the topology of the bundle is fixed, there is a single (up to symmetries) classical solution. Consequently, the degrees of freedom of the theory cannot be local, but only global. BF is an example of topological theory.

To quantize the theory we have to give a meaning to the formal expression

$$Z=\int \mathcal{D}B\,\mathcal{D}A\; e^{\frac{i}{\hbar}\int TrB\wedge F} \tag{2.19}$$

that has to be regularized and gauge fixed. For the continuum analysis see [83]. We discretize the manifold with a finite triangulation Δ on $\mathcal{M}$.

On $\mathcal{J}$ (the 2–complex dual to Δ) , we introduce the algebra variables $B^I_f\in\mathfrak{g}$, associated to the faces, and the group variables $U_e\in\mathcal{G}$, associated to the edges. We define then the quantity:

$$U_f:=\mathcal{P}\prod_{e\in\partial f} U_e \tag{2.20}$$

In the last expression $\mathcal{P}$ means that the product over all the edges bounding the face f is oriented. The orientation is induced from the orientation of f . In terms of these variables, we define the action

$$S[B^I_f,U_e]=\hbar\sum_f Tr[B_f U_f] \tag{2.21}$$

The action (2.21) defines a theory on Δ, with variables B_f^I, U_e . To show that it is an approximation to the continuum theory with action (2.17), consider an embedding $i : \Delta \to \mathcal{M}$, which allows us to think of Δ as a cellular decomposition of $\mathcal{M}$. We call l_s^μ the vector tangent to the segment s; using i, we have $l_s^\mu \approx \int_s dx^\mu$.

The embedding i pushes forward to an embedding for the dual triangulation $\mathcal{J}$, and we analogously define the edge vector $l_e^\mu \approx \int_e dx^\mu$. We now make the following identifications between the new (discrete) variables and the previous (smooth) ones:

- The auxiliary field B is an (n-2)form; as such, it is naturally discretized on an (n-2)-complex, which is dual to a face:

$$B^I(x) \to B_f^I := \frac{1}{\hbar} B^I(l_{s_1} \wedge \ldots \wedge l_{s_{n-2}}) \approx \frac{1}{\hbar} \int B^I \tag{2.22}$$

  With the rescaling by $\hbar$ the variable B_f^I is purely algebraic.

- The connection is on the continuum geometrically interpreted as an infinitesimal parallel transport. On a discretised setting, this property is properly taken over by a group element representing the minimal parallel transport:

$$A_\mu^I(x) \to U_e := e^{iA_\mu^I l_e^\mu} \sim e^{i\int_e A^I} \tag{2.23}$$

  If we took a regular hypercubical lattice with lattice spacing a, this definition coincides with the one adopted in LGT, i.e. $U_e := e^{iaA_e}$.

We claim that for each configuration of the continuous fields we can find an embedding i such that the difference between (2.17) and (2.21) is arbitrarily small, if the variables in (2.21) are interpreted via (2.22) and (2.23). To see it, we apply the Stokes' theorem to the connection around a closed face f (for non-abelian groups, the theorem holds up to corrections in the area of the face):

$$\sum_{e \in \partial f} A_\mu l_e^\mu \sim \int_{\partial f} A \approx \int_f F(A) \tag{2.24}$$

As a consequence of the equality above, the definition (2.23) gives to the group variables defined in (2.20) the interpretation of holonomies,

$$U_f \sim e^{i\int_f F(A)} \tag{2.25}$$

When the embedding is sufficiently refined, and the coordinate areas consequently small, we can expand the group elements around the algebra, $U_f \simeq 1 + i\int_f F(A)$. Recalling that for $T \in su(N)$ we have $TrT = 0$, we see that (2.21) reduces to (2.17). Notice that we are indeed only using the 2-skeleton $\mathcal{J}$ of Δ^* . This is the setup for the discretized theory, which we use to regularize (2.19). On $\mathcal{J}$, (2.19) is realized as

$$Z_{BF}[\mathcal{G}] = \prod_f \int_{\mathfrak{g}} dB_f \prod_e \int_{\mathcal{G}} dU_e \, e^{i\sum_f TrB_f U_f} = \prod_e \int_{\mathcal{G}} dU_e \prod_f \delta(U_f) \tag{2.26}$$

Here we assumed that the integration over each algebra variable B_f gives a δ function on the group. This can be explicitly shown for $U(1)$ and $SU(2)$. A general argument can be given

using Kirillov's integral formula [84] for the character of a Lie group, see [83]. Since we have a $\delta(U_f)$ for each face in $\mathcal{J}$, the condition $U_f = 1$ is implemented all over $\mathcal{J}$. Comparing with (2.25), we see that this is the condition that imposes the equations of motion $F = 0$ in the discretised setting. To put (2.26) into the form (2.14), we proceed as follows: we character expand the δ-functions, by means of the Plancherel formula

$$\delta(U_f) = \sum_j \dim j \; \chi^j(U_f) \tag{2.27}$$

Here $\chi^j(U)$ is the character of U in the irrep j, namely the projection of U on the conjugacy class of $\mathcal{G}$. The holonomies U_f around the dual faces depend on the fundamental variables U_e as in (2.20). It follows that $\chi^j(U_f) = Tr \prod_{e\in\partial f} D^j(U_e)$ and we obtain

$$Z(\Delta) = \sum_{\{j_f\}} \int \prod_{e\in\mathcal{J}_\Delta} dU_e \prod_{f\in\mathcal{J}_\Delta} \dim j_f \; \mathrm{Tr}\left[D^{j_f}(U_{e_1^f} \dots U_{e_n^f})\right], \tag{2.28}$$

Where $U_{e_i^f}$ are n holonomies of edges that bound the face f. Now, the advantage of $\mathcal{J}$ is that the valency of its vertices is fixed: there are always $n + 1$ edges touching one vertex. Consequently, the number of faces sharing one edge is also fixed, to be n. This simple dependence allows us to perform the integrals over the group variables U_e , using the formula for the orthogonality of the characters:

$$\int_{\mathcal{G}} dU D^{j_1}(U_e)^{l_1}_{m_1} \cdots D^{j_n}(U_e)^{l_n}_{m_n} = \sum_i P^{(i)}_{m_1\dots m_n} \overline{P^{(i)l_1\dots l_n}}. \tag{2.29}$$

Here the sum is over an orthonormal basis $P^{(1)}, \cdots, P^{(k)}$ of projectors,

$$P^{(i)} : j_1 \otimes \cdots \otimes j_n \to \mathbb{C} \tag{2.30}$$

onto the trivial representation. Each projector $P^{(i)}$ corresponds to a compatible intertwiner i_e for the edge e in the sum over colours of (2.13). Compatible here means compatible with the representations assigned to the faces incident to e. Using (2.29) in (2.26), we obtain an expression of the type

$$Z_B F[\mathcal{G}] = \sum_{j_f, i_e} \prod_f \dim j_f \prod_v A_v(j_f, i_e) \tag{2.31}$$

The explicit form depends on the structure group $\mathcal{G}$ and the spacetime dimension n, which determine the structure of (2.29). The partition function (2.31) is of the type (2.13) on a single 2-complex, with $A_f = \dim j_f$ and $A_e = 1$. An important remark follows. In the continuum, the partition function of BF can be interpreted as the volume of the space of flat connections modulo gauge transformations. However, the quantisation is much more involved, especially concerning gauge–fixing and the presence of anomalies. Using it as a starting point for quantising GR would be at least controversial. However, we see that in the discretised setting used in spinfoams (and which is ultimately justified by the discrete picture of spacetime emerging from LQG), the quantisation is quite simpler, and in particular gauge-fixing much more under control (see [85, 86]).

Now we restrict our attention to $Spin(4)$ BF theory that will be an useful tool for the quantization of gravity.

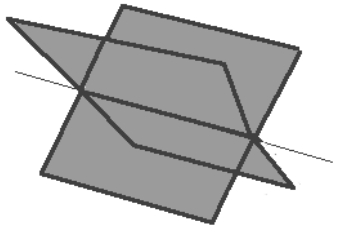


Figure 2.3: In a 4d triangulation an edge bounds 4 faces

2.4.1 Quantum $SO(4)$ BF theory

Classical $SO(4)$ BF theory is defined by the action (2.17) where $B^{IJ}_{\mu\nu}$ is a $Spin(4)$ Lie-algebra valued 2-form and A^{IJ}_{μ} is a connection on a $Spin(4)$ principal bundle over $\mathcal{M}$.

We concentrate on the triangulation Δ of $\mathcal{M}$.

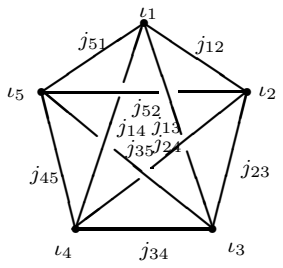
The field B is associated with Lie algebra elements B_t assigned to triangles $t \in \Delta$. In four dimensions triangles $t \in \Delta$ are dual to faces $f \in \mathcal{J}_\Delta$. This one-to-one correspondence allows us to work with discrete B associated to faces (B_f) or to triangles (B_t). B_t can be interpreted as the 'smearing' of the continuous 2-form B on triangles in Δ appling the definition (2.22). The connection A is discretized by the assignment of group elements $U_e \in Spin(4)$ to edges $e \in \mathcal{J}_\Delta$. The expression (2.28) in this case can be easilly performed noting that in the triangulation Δ of a 4-dimensional manifold, the edges $e \in \mathcal{J}_\Delta$ bound precisely four different faces (see Figure 2.3); ; therefore, the U_e's in (2.28) appear in four different traces. The formula (2.29) in this case reads

$$P^4_{inv} := \int dU_e \; D^{j_1}(U_e) \otimes D^{j_2}(U_e) \otimes \cdots \otimes D^{j_4}(U_e) = \sum_{\iota} C^{\iota}_{j_1 j_2 \cdots j_4} \; C^{*_\iota}_{j_1 j_2 \cdots j_4}, \tag{2.32}$$

where P^4_{inv} is the projector onto $\mathrm{Inv}[j_1 \otimes j_2 \otimes \cdots \otimes j_n]$ and on the r.h.s we have expressed the projector in terms of normalized $SO(4)$ intertwiners (Appendix D); the index ι label the orthonormal basis in the space of intertwiners, so we have a sum over the intertwiners for each edge in addition to the sum over the representation of the faces.

Finally, the last step is simply the contraction of the intertwiners indexes: from the integration over one edge we get the two invariant tensors of (2.32), and its indexes are contracted with the ones coming from the other edge at this vertex. Now the point is that we have 5 edges (dual to the 5 tetrahedra) for each vertex; so we have to contract for each ι five 4-valent intertwiner. The resulting partition function is

$$Z_{BFSO(4)}(\Delta) = \sum_{\{j_f\}} \sum_{\{\iota_e\}} \prod_{f \in \mathcal{J}_\Delta} \dim j_f \prod_{v \in \mathcal{J}_\Delta}$$



, (2.33)

where the pentagonal diagram representing the vertex amplitude denotes the pattern of contraction of the $SO(4)$ 4-intertwiners $C^{\iota}_{j_1 j_2 j_3 j_4}$ reproduces the structure of the two-skeleton of a four simplex. The vertex amplitude corresponds to the flat evaluation of the spin network state defined by the pentagonal diagram in (2.33), a $15j$-symbol. Vertices $v \in \mathcal{J}_\Delta$ are in one-to-one correspondence to 4-simplexes in the triangulation Δ. The sum is over the assignment $\{j_f\}$ of irrep to faces and $\{\iota_e\}$ of intertwiners to edges coming from (2.32).

An important remark about gluing: Degrees of freedom communicate through the lattice connection on the boundary. One can compute amplitudes of pieces of $\mathcal{J}_\Delta$ (at fixed boundary data) and then obtain the full $\mathcal{J}_\Delta$ amplitude by gluing the pieces together and integrating out the mutual boundary connections along common boundaries. The boundary of a portion of $\mathcal{J}_\Delta$ is a graph. The boundary value is an assignment of group elements to its links. The amplitude is a function of the boundary connection, i.e., an element of Cyl. In the case of a cellular 2–complex there is a maximal splitting corresponding to cutting out a neighborhood around each vertex. If the discretization is based on the dual of a triangulation these elementary building blocks are all alike and denoted *atoms* (or vertexes in the Feynmann language of GFT see below). Such an atom in four dimensions is represented in Figure 2.4.

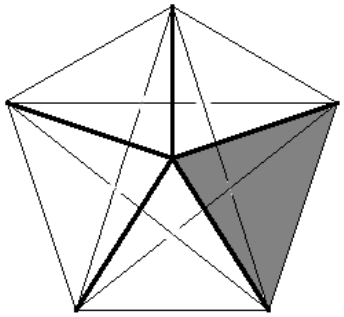


Figure 2.4: A fundamental *atom* is defined by the intersection of a dual vertex in $\mathcal{J}_\Delta$ (corresponding to a 4-simplex in Δ) with a 3-sphere. The thick lines represent the internal edges while the thin lines the intersections of the internal faces with the boundary. They define the boundary graph denoted Γ_5 below. One of the faces has been emphasized.

The atom amplitude depends on the boundary data given by the value of the holonomies on the ten links of the pentagonal boundary graph Γ_5 shown in the figure. This amplitude can be represented by a function

$$\mathcal{V}(\alpha_{ij}) \quad \text{for} \quad \alpha_{ij} \in G \quad \text{and} \quad i \neq j = 1, \ldots, 5 \tag{2.34}$$

where α_{ij} represents the boundary lattice connection along the link ij in Figure 2.4. Gauge invariance ($V(\alpha_{ij}) = V(g_i \alpha_{ij} g_j^{-1})$) implies that the function can be spanned in terms of spin networks functions $\Psi_{\Gamma_5, j_{ij}, \iota_i}(\alpha_{ij})$ based on the pentagonal graph Γ_5, namely

$$\mathcal{V}(\alpha_{ij}) = \sum_{j_{ij}} \sum_{\iota_i} \tilde{\mathcal{V}}(j_{ij}, \iota_i)\ \Psi_{\Gamma_5, j_{ij}, \iota_i}(\alpha_{ij}) \tag{2.35}$$

where $\tilde{\mathcal{V}}(j_{ij}, \iota_i)$ is the atom amplitude in 'momentum' space depending on ten spins j_{ij} labeling the faces and five intertwiners ι_i labeling the edges. Gluing the atoms together the integral

over common boundaries is replaced by the sum over common values of spin labels and intertwiners and we end up with (2.33).

The state sum (2.33) is generically divergent. A regularized version defined in terms of $SU_q(2) \times SU_q(2)$ was introduced by Crane and Yetter [87, 88]. The partition function (2.33) is topologically invariant and the spin foam path integral is discretization independent.

What is the utility of having quantized the 4-d BF theory? The answer will become clear in th next section: Classical GR in its tetrad (or 4-beins) formulation can be recasted as a constrained BF theory.

2.5 4-d gravity as a constrained BF theory

The action for General Relativity in the first order formalism is the so-called Palatini action:

$$S[e,\omega] \,=\, \int_{\mathcal{M}} *e \,\wedge\, e \,\wedge\, F(\omega) \,=\, \int_{\mathcal{M}} \epsilon_{IJKL}\, e^I \,\wedge\, e^J \,\wedge\, F^{KL}(\omega), \tag{2.36}$$

where the field variables in the action are: a 1-form tetrad field $e^I = e^I_\mu dx^\mu$ with internal $\mathbb{R}^{3,1}$ index I (in $\mathbb{R}^4$ in the Riemannian case), giving the spacetime metric as $g_{\mu\nu} = \eta_{IJ} e^I_\mu \otimes e^J_\nu$, and a 1-form Lorentz connection $\omega^{IJ} = \omega^{IJ}_\mu dx^\mu$ with values in the Lie algebra of the Lorentz group $so(3,1)$ ($so(4)$ in the Riemannian case) in the adjoint representation. $F^{IJ}(\omega)$ is the two-form curvature of ω. The corresponding equations of motion are:

$$De^I = de^I + \omega^I{}_J \wedge e^J = 0. \tag{2.37}$$

expressing the compatibility of the tetrad field and the connection, i.e. the fact that the connection we have is a torsion free, and

$$e \wedge F(\omega) = 0 \tag{2.38}$$

the Einstein equations describing the dynamics of the spacetime geometry. The action is diffeomorphism invariant and invariant under the internal gauge group (Lorentz or SO(4)).

This action is classically a subsector of a more general classical action for the gravitational field: the Plebanski action [69, 89], which describes gravity as a constrained topological theory.

The Plebanski action is a a BF-type action, in the sense that it gives gravity as a constrained BF theory, with quadratic constraints on the B field. More precisely the action is given by:

$$S \,=\, S(\omega, B, \phi) \,=\, \int_{\mathcal{M}} \left[B^{IJ} \,\wedge\, F_{IJ}(\omega) \,-\, \frac{1}{2} \phi_{IJKL}\, B^{KL} \,\wedge\, B^{IJ} \right] \tag{2.39}$$

where ω is a connection 1-form valued in $so(3,1)$ ($so(4)$), $\omega = \omega^{IJ}_\mu J_{IJ} dx^\mu$, J_{IJ} are the generators of $so(3,1)$ ($so(4)$), $F = D\omega$ is the corresponding two-form curvature and B is a 2-form also valued in $so(4)$ ($so(3,1)$), $B = B^{IJ}_{\mu\nu} J_{IJ} dx^\mu \wedge dx^\nu$, and ϕ_{IJKL} is a Lagrange multiplier, with symmetries $\phi_{IJKL} = \phi_{[IJ][KL]} = \phi_{[KL][IJ]}$, satisfying $\phi_{IJKL} \epsilon^{IJKL} = 0$.

The equations of motion are:

$$dB + [\omega, B] = 0 \tag{2.40}$$
$$F^{IJ}(\omega) = \phi^{IJKL} B_{KL} \tag{2.41}$$
$$B^{IJ} \wedge B^{KL} = e\, \epsilon^{IJKL} \tag{2.42}$$

where $e = \frac{1}{4!}\epsilon_{IJKL} B^{IJ} \wedge B^{KL}$.

When $e \neq 0$, i.e. for non-degenerate metric configurations, the constraint (2.42) called Plebanski constraint is equivalent to [90, 91]

$$\epsilon_{IJKL} B^{IJ}_{\mu\nu} B^{KL}_{\rho\sigma} = *B_{\mu\nu} \cdot B_{\rho\sigma} = e\epsilon_{\mu\nu\rho\sigma} \tag{2.43}$$

This system of constraint can be decomposed in three parts:

$$a) \quad {}^*B_{\mu\nu} \cdot B_{\mu\nu} = 0\ , \tag{2.44}$$
$$b) \quad {}^*B_{\mu\nu} \cdot B_{\mu\rho} = 0\ , \tag{2.45}$$
$$c) \quad {}^*B_{\mu\nu} \cdot B_{\rho\sigma} = \pm e\ . \tag{2.46}$$

where the indices $\mu\nu\rho\sigma$ are all different, and the sign in the last equation is determined by the sign of their permutation. They are called the *simplicity* constraints. GR can be written as an $SO(3,1)$ ($SO(4)$) BF theory whose B field satisfies the simplicity constraints (2.44), (2.45) and (2.46).

This is the form of the constraint that we are going to use and that leads to the Barrett-Crane spin foam model. In other words, (2.42) is satisfied if and only if there exists a real tetrad field $e^I = e^I_\mu dx^\mu$ so that one of the following equations holds:

$$I) \qquad B^{IJ} = \pm\, e^I \wedge e^J \tag{2.47}$$
$$II) \qquad B^{IJ} = \pm \frac{1}{2}\, \epsilon^{IJ}{}_{KL} e^K \wedge e^L = \pm * (e \wedge e). \tag{2.48}$$

Restricting the field B to be always in the sector II^+ (which is always possible classically), the action becomes:

$$S = \int_{\mathcal{M}} \epsilon_{IJKL}\, e^I \wedge e^J \wedge F^{KL} \tag{2.49}$$

which is the action for General Relativity in the first order Palatini formalism.

Also, the other sector, differing by a global change of sign only, is classically equivalent to this, while the other two, related by Hodge duality to the “geometric" ones, corresponds to “pathological geometries".

The imposition of a constraint reduces a topological theory to a theory with local degrees of freedom. This can be easily understood thinking that in BF theory the B field is a Lagrange multiplier that impose $F = 0$; nevertheless the constraint (2.42) on the constraint B relax its role giving to the resulting theory more degrees of freedom of the original one.

The possibility of this restriction at the classical level was shown in [91], where it is proven that initial data in the gravity sector do not evolve into any of the others provided that the tetrad field remains non-degenerate. Therefore we see that this action defines a

diffeomorphism and Lorentz invariant theory, a subsector of which describes classical General Relativity in the 1st order formalism, as we had anticipated. This is basically all, as long as the classical level is concerned.

The two theories, however, are different at the quantum level, since in the quantum theory one cannot avoid interference between different sectors. In fact in a partition function for the Plebanski action we have to integrate over all the possible values of the B field, so considering all the 4 sectors. Another way to see it is the existence in the Plebanski action of a $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry $B \to -B$, $B \to *B$ responsible for this interference. This is discussed in [90, 92]. This adds to the usual subtlety in dealing with a 1st order action, where degenerate configurations of the metric field cannot be easily excluded, instead of a 2nd order action (metric or ADM formalism) where one integrates over non-degenerate configurations only.

2.5.1 Discretization

Before turning to the quantum theory based on the Plebanski action, we have to discuss in detail the description of simplicial geometry based on it, since it is the real starting point of the spin foam quantization. In fact, a quantization of gravity along such lines should start by identifying suitable variables corresponding to the B and ω variables of the Plebanski action, and then the correct translation at the quantum level of the above constraints on the B field, leading to a realization of the path integral (2.33).

The connection field is naturally discretized along the edges of the dual complex, by integrating the 1-form connection along them $\omega^{IJ} = \int_e \omega^{IJ}(x)$, so that we obtain a holonomy U_e (with values in $SO(3,1)$ or $SO(4)$) associated to each edge; in this way, again, the curvature is obtained by choosing a closed path of edges, in particular for each dual face we have a product of group elements $U_f = \prod_{e \subset \partial f} U_e$ of the group elements g_e associated with the links of the boundary of the dual face f, and it is thus associated with the dual face itself. This is in turn dual to the triangles of the triangulation Δ, so we have the simplicial curvature associated to them, as it is common in 4-dimensional simplicial gravity. The logarithm of U_f gives a Lie algebra element Ω_t, the proper discretization of the curvature field of the Plebanski action.

The crucial point is however the discretization of the B field, since this is what marks the difference between gravity and BF theory. Being a 2-form, the B field is naturally discretized along the triangles in the triangulation obtaining a Lie algebra element associated to each triangle, thus to each face of the dual complex, via

$$B_t^{IJ} = \int_t B_{\mu\nu}^{IJ}(x) dx^\mu \wedge dx^\nu \tag{2.50}$$

. It is crucial to note that in this discretization, the sign of the bivector reflects the orientation of the triangle to which it is associated.

With this discretization, the constraints on the B field become constraints on the bivectors $B^{IJ} \in SO(3,1)(SO(4))$ associated to the various triangles.

The constraint term in the action is discretized analogously, by integrating over pairs of triangles, to get: $\phi_{IJKL} B_t^{IJ} B_{t'}^{KL}$.

Therefore, the discrete action we get is:

$$S(B,\omega) \,=\, \sum_t \mathrm{B}_t\,\Omega_t \,+\, \sum_{t,t'} \phi_{IJKL}\, B_t^{IJ}\, B_{t'}^{KL}. \tag{2.51}$$

Let us now analyze the constraints on the bi–vectors more closely.

Consider first a tetrahedron τ, whose boundary is made of four triangles t, we have:

$$0 = \int_\tau dB^{IJ}(x) = \int_{\partial\tau} B^{IJ}(x) = \sum_t \int_t B^{IJ}(x) = \sum_t B_t^{IJ} = 0 \tag{2.52}$$

In other words, the four bivectors associated to the same tetrahedron sum to zero, as a result of the tetrahedron being enclosed by the corresponding four triangles.

Let focus on the constraint (2.43): $e = \frac{1}{4!}\epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma} B^{IJ}_{\mu\nu} B^{KL}_{\rho\sigma} \neq 0$ is the spacetime volume element after the imposition of the constraints, so its being non zero implies that the bivector field is non degenerate.

Integrating this expression of the constraints over pairs of (triangles) we obtain [90]:

$$V_{t,t'} \,=\, \int_{x\in t, y\in t'} e\,\epsilon_{abcd}\, dx^a \wedge dx^b \wedge x^c \wedge dx^d \,=\, \epsilon_{IJKL}\, B_t^{IJ}\, B_{t'}^{KL}, \tag{2.53}$$

where $V_{t,t'}$ is obviously the 4-volume spanned by the two triangles t and t'.

This formula clearly implies two constraints on the bivectors associated to the triangles of the simplicial manifold, corresponding to the two cases in which the 4-volume spanned is zero: each bivector associated to a triangle t satisfies the discrete (2.44)

$$\epsilon_{IJKL}\, B_t^{IJ}\, B_t^{KL} = 0 \tag{2.54}$$

which corresponds geometrically to the requirement of the bivector being formed as a vector product of two (edge) vectors, i.e. of being a "simple" bivector; if we decompose the bivector into its selfdual and anti-selfdual part, this constraint also imposes *the equality* of these two parts.

For two triangles t and t' sharing an edge, thus belonging to the same tetrahedron τ, the corresponding bivectors must satisfy (2.45)

$$\epsilon_{IJKL}\, B_t^{IJ}\, B_{t'}^{KL} = 0 \tag{2.55}$$

this equation together with the previous constraint, implies that

$$\epsilon_{IJKL}\, B_{t+t'}^{IJ}\, B_{t+t'}^{KL} = 0 \tag{2.56}$$

, i.e. that the bivector $B^{KL}_{t+t'} = B^{KL}_t + B^{KL}_{t'}$ sum of the two bivectors corresponding to the two triangles t and t' sharing an edge is also a *simple* bivector.

The geometric meaning of this constraint is to impose that the triangles intersect pairwise in lines in $\mathbb{R}^{3,1}(\mathbb{R}^4)$, i.e. that they pairwise span 3-dimensional subspaces of $\mathbb{R}^{3,1}(\mathbb{R}^4)$ [70, 93, 94] (this is a strong condition in 4 dimensions, where two surfaces intersect in a point only).

At the continuum level, the Plebanski constraints on the bivector field B make it a geometric field, i.e. put it in correspondence with a tetrad field and thus with a spacetime metric,

so allow for a description of spacetime geometry in terms of this bivector field; at the discrete level, when the bivectors are assigned to the triangles, the constraints allow for a description of simplicial geometry in terms of them.

However, the same ambiguity that we have seen at the continuum level for the solutions of the Plebanski constraints exists at this discretized level, since there are again four sectors of solutions to these constraints corresponding to the bivectors 1) B^{IJ}, 2) $-B^{IJ}$, 3) $*B^{IJ} = \epsilon^{IJ}_{KL}B^{KL}$, 4) $-*B^{IJ}$. Again, the cases 1) and 2) correspond to well-defined simplicial geometries, differing only by a change in orientation, while the cases 3) and 4) are pathological cases with no geometric interpretation.

Resuming, the geometry of a 4-simplex, and thus the geometry of a full simplicial complex (where one glues 4-simplices along common tetrahedra imposing that the bivector data on the common tetrahedra match), is determined by a set of bivectors associated to the triangles in the complex and satisfying the following requirements:

- the bivectors change sign when the orientation of the triangles is changed (orientation constraint);
- the bivectors are "simple", i.e. they satisfy $B_t \cdot *B_t = 0$ (simplicity constraint);
- the bivectors associated to neighbouring triangles sum to simple bivectors, i.e. $B_t \cdot *B_{t'} = 0$ if t and t' share an edge (decomposition constraint);
- the four bivectors associated to the faces of a tetrahedron sum to zero (closure constraint).

These constraints, together with their quantum counterparts, were given in [51, 70, 93, 94], while their relation with the Plebanski formulation of gravity was shown in [90, 91]. We stress again that this description of simplicial geometry in terms of bivectors, with the associated constraints, holds in both the Lorentzian and Riemannian case, although in the Lorentzian case the definition and use of self–dual and anti–selfdual components for the bivectors is less straightforward since it implies a complexification procedure.

From now on we restrict to the Riemannian theory

2.6 The Barrett-Crane model

The Barrett-Crane model is one of the most extensively studied spin foam models for quantum gravity. In this thesis we concentrate on the definition of the model in the Riemannian sector.

The Barrett-Crane model can be viewed as a spin foam quantization of $SO(4)$ Plebanski's formulation of GR.

2.6.1 Quantum constraints

As we have seen in the previous section at classical level the BF theory constrained by the Plebanski constraints gives GR. Imposing the Plebanski constraint to a quantum BF theory would then give quantum GR. There are many ways to impose the constraints at quantum level.

We can proceed in two ways:

1. discretize the constraints and impose them directly on the BF spinfoam sum
2. use the geometrical interpretation of the B field looking at the Plebanski constraint as the requirement that the B field is an infinitesimal area element of a discretized quantum geometry.

The Barret Crane model in the original definition is made using the second approach and makes use of the concept of the 'quantum tetrahedron' [51, 93]. In this context, the analog of the B's is given by the bivectors associated with the triangles of a classical tetrahedron. A Hilbert space is defined using geometric quantization and the classical triangle bivectors are promoted to operators. This Hilbert space is reduced by implementation of the quantum version of the geometric constraints satisfied by a classical tetrahedron, to the Hilbert space of the so-called 'quantum tetrahedron'. These constraints are precisely of the form (2.43). The last step of this construction is the definition of an appropriate quantum amplitudes to be associated to the 4-simplices to be used as building blocks for the partition function.

We present the two approaches because they are complementary; in the first one we can derive everything directly from the path integral, in the second one the geometrical interpretation of the variables is more transparent and the idea of a quantum tetrahedron emerges as an independent physical system. We can then reproduce the same situation of QFT: we can construct the theory from the classical field theory and derive the existence of the particles or the other way around we can start from the existence of the particles, define the quantum theory of a particle and then build the many particle states and then their interaction: we end up with the same theory.

2.6.2 Spin(4)

Both approach make use of the $Spin(4)$ group; the double covering of SO(4). Here we review briefly some properties of this group.

The group $SO(4)$ is locally isomorphic to the product of two subgroups, each locally isomorphic to $SU(2)$: $SO(4) \sim (SU(2)_+ \times SU(2)_-)/\mathbb{Z}_2$. That is, we can write each $g \in SO(4)$ in the form $g = (g_+, g_-)$ where $g_+ \in SU(2)_+$ and $g_- \in SU(2)_-$ and $gg' = (g_+ g'_+, g_- g'_-)$. The algebra $so(4)$ is then the linear sum of two commuting $su(2)$ algebras. Explicitly if J^{IJ} are the generators of $so(4)$ we can define the selfdual and anti-selfdual generators

$$J_\pm := *J \pm J, \tag{2.57}$$

that satisfy $J_\pm = \pm * J_\pm$. Then it is immediate to see that $[J_+, J_-] = 0$. The $J_\pm$ are the generators of two commmuting subalgebras $su(2)_+$ and $su(2)_-$ of $so(4)$, both isomorphic to $su(2)$.

$SO(4)$ has two scalar Casimirs: the scalar Casimir

$$C = \frac{1}{4} J \cdot J \tag{2.58}$$

and the pseudo-scalar Casimir

$$\tilde{C} = \frac{1}{4} {}^*J \cdot J \tag{2.59}$$

.

We can always choose a basis in $su(2)_+$ and in $su(2)_-$ to calculate the previous Casimirs. For this, we can choose a unit norm vector n in R^4, and three other vectors $v_i, i = 1, 2, 3$ forming, together with n, an orthonormal basis, for instance $v_i^I = \delta_i^I$, and define

$$J_\pm^i = \frac{1}{2}({}^*J \pm J)_{IJ}\ v_i^I n^J \tag{2.60}$$

The $su(2)$ structure is then easy to see, since $[J_\pm^i, J_\pm^j] = \epsilon^i{}_{jk} J_\pm^k$. In particular, if we choose $n = (0, 0, 0, 1)$, and $v_i^I = \delta_i^I$, we have

$$J_\pm^i = -\frac{1}{4}\epsilon^i{}_{jk} J^{jk} \pm \frac{1}{2} J^{i0}. \tag{2.61}$$

with the previous choice we can evaluate the two Casimirs

$$C = J_+^i J_+^i + J_-^i J_-^i = j_+(j_+ + 1) + j_-(j_- + 1) \tag{2.62}$$

$$\tilde{C} = J_+^i J_+^i - J_-^i J_-^i = j_+(j_+ + 1) - j_-(j_- + 1) \tag{2.63}$$

This result is independent from the choice of n. They are,respectively, the sum and the difference of the quadratic Casimirs of $su(2)_+$ and $su(2)_-$.

The representations are then labelled by two half integers (j_+, j_-). The representations of $SO(4)$ form the subset for which $j_+ + j_-$ is integer. The representations satisfying $j_+ = j_-$ are called *simple*.

2.6.3 Imposing the Plebansky constraint in the BF path integral

In this approach the strategy is to impose the constraint directly on the BF spin foam sum, (2.28), after the B-integration has been performed. We follow [66] in this derivation. Formally we associate the discrete B_f to the differential operator $-i\frac{\partial}{\partial U_f}$ in (2.26):

$$B_f \to -i\frac{\partial}{\partial U_f} \tag{2.64}$$

More precisely, the observation is that the $Spin(4)$ left invariant vector field $-i\mathcal{X}^{IJ}(U) := U^\mu{}_\nu J^{IJ}{}_\sigma{}^\nu \frac{\partial}{\partial U^\mu{}_\sigma}$ acts as a quantum B^{IJ} on (2.26) since

$$\begin{aligned} &-i\mathcal{X}^{IJ}(U)\left(e^{i\mathrm{Tr}[BU]}\right)|_{U\sim 1} = U^\mu{}_\nu J^{IJ}{}_\sigma{}^\nu \frac{\partial}{\partial U^\mu{}_\sigma} e^{i\mathrm{Tr}[BU]}|_{U\sim 1} = \\ &= \mathrm{Tr}[UJ^{IJ}B] e^{i\mathrm{Tr}[BU]}|_{U\sim 1} \sim B^{IJ} e^{i\mathrm{Tr}[BU]}, \end{aligned} \tag{2.65}$$

where J^{IJ} are elements of an orthonormal basis in the $so(4)$ Lie-algebra. The evaluation at $U = 1$ is motivated by the fact that configurations in the BF partition function (2.26) have support on flat connections. This approximation is made in order to motivate our definition but it plays no role in the implementation of the constraints.

The constraints (2.43) are quadratic in the B's and so we have to worry about cross terms; the nontrivial case corresponds to:

$$\begin{aligned}
&\epsilon_{IJKL}\mathcal{X}^{IJ}(U)\mathcal{X}^{KL}(U)\left(e^{i\mathrm{Tr}[BU]}\right)|_{U\sim 1}\\
&= -\epsilon_{IJKL}\left(\mathrm{Tr}[J^{IJ}UB]\mathrm{Tr}[J^{KL}UB]e^{i\mathrm{Tr}[BU]} + i\mathrm{Tr}[J^{IJ}J^{KL}UB]e^{i\mathrm{Tr}[BU]}\right)|_{U\sim 1}\\
&\sim \epsilon_{IJKL}B^{IJ}B^{KL}e^{i\mathrm{Tr}[BU]},
\end{aligned} \tag{2.66}$$

where the second term on the second line can be dropped using that $\epsilon_{IJKL}X^{IJ}X^{KL} \propto 1$ (one of the two $SO(4)$ Casimir operators) and $U \sim 1$. Therefore, we define the B_f field associated to a face at the level of equation (2.26) as the appropriate left invariant vector field $-i\mathcal{X}^{IJ}(U_f)$ acting on the corresponding discrete holonomy U_f, namely

$$B_f^{IJ} \to -i\mathcal{X}^{IJ}(U_f). \tag{2.67}$$

Gauge invariance of the BF partition function implies that for every tetrahedron

$$\sum_{t\in\tau} B_t^{IJ} = 0 \tag{2.68}$$

where $t \in \tau$ denotes the triangles in the corresponding tetrahedron.

In order to implement (2.54 and 2.55) we concentrate on a single 4-simplex amplitude, i.e., the fundamental *atom* of the simplicial decomposition. Once the constrained 4-simplex amplitude is constructed any spin foam amplitude can be obtained by gluing atoms together along faces by integration over common boundary data.

The amplitude of the fundamental *atom* (the amplitude of the vertex v) is a (cylindrical) function depending on the boundary values of the connection on the boundary graph Γ_5. We denote as $h_{ij} \in Spin(4)$ ($i \neq j$, $i,j = 1\cdots 5$ and $h_{ij} = h_{ji}^{-1}$) the corresponding 10 boundary variables (associated to thin boundary edges in Figure 2.4) and $g_i \in Spin(4)$ ($i = 1,\cdots,5$) the internal connection (corresponding to the thick edges in Figure 2.4). According to (2.26) the 4-simplex BF amplitude $A_v^{4dBF}(h_{ij})$ is given by

$$A_v^{4dBF}(h_{ij})_v = \int \prod_i dg_i \prod_{i<j} \delta(g_i h_{ij} g_j), \tag{2.69}$$

where $U_{ij} = g_i h_{ij} g_j$ is the holonomy around the triangular face $0ij$. With the definition of the B fields given in (2.67) the constrained amplitude, $A^{4dBF_{con}}(h_{ij})$, formally becomes

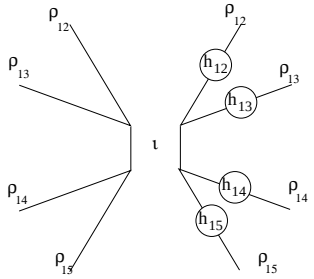
$$A_v^{4dBF_{con}}(h_{ij}) = \int \prod_i dg_i \delta\left[\mathrm{Constraints}(-i\mathcal{X}(U_{ij}))\right] \prod_{i<j} \delta(g_i h_{ij} g_j). \tag{2.70}$$

It is easy to verify, using an equation analogous to (2.65) and the invariance of ϵ_{IJKL}, that one can define the B's by simply acting with the left invariant vector fields on the boundary connection h_{ij}. Therefore, the previous equation is equivalent to

$$A_v^{4dBF_{con}}(h_{ij}) = \delta\left[\mathrm{Constraints}(-i\mathcal{X}(h_{ij}))\right] \int \prod_i dg_i \prod_{i<j} \delta(g_i h_{ij} g_j), \tag{2.71}$$

where we have taken the delta function out of the integral. The quantity on which the formal delta distribution acts is simply $A_v^{4dBF}(h_{ij})$. The amplitude $A_v^{4dBF}(h_{ij})$ can be expressed in a more convenient way if we expand the delta functions in modes as in (2.28) and then integrate over the internal connection g_i. The integration is analogous to the one in (2.32), for example integration over g_1 yields

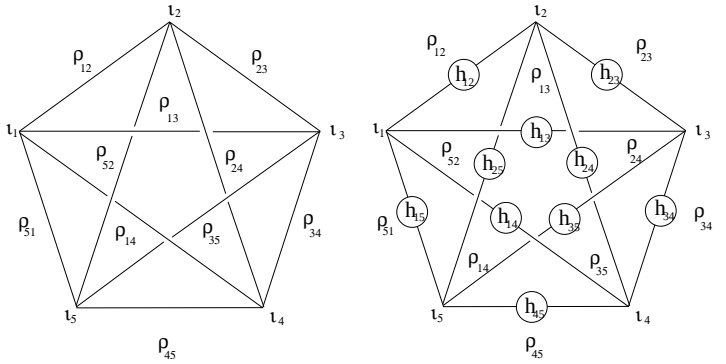
$$P^4_{inv}\, D^{12}(h_{12})\otimes D^{13}(h_{13})\otimes D^{14}(h_{14})\otimes D^{15}(h_{15}) = \sum_{\iota} \dim \iota$$



, (2.72)

where on the RHS we have chosen a particular basis (pairing) of 4-intertwiners to span the projector P^4_{inv} into $\mathrm{Inv}\,[\rho_{12}\cdots\rho_{15}]$ (ρ_{ij} are $SO(4)$ representations). The 4-intertwiners are explicitly given as contractions of normalized intertwiners (see Appendix D). The circles represent the corresponding D^j-representation matrices evaluated on the corresponding boundary connection h. The 4-simplex amplitude becomes

$$A_v^{4dBF}(h_{ij}) = \sum_{\rho_{ij},\iota_i} \prod_{i<j} \dim \rho_{ij}$$



. (2.73)

In the previous equation 4-valent nodes denote normalized 4-intertwiners and the tree decomposition is left implicit (the factors $\dim\iota$ in (2.72) have been absorbed into the notation). The term on the left is a $15j$-symbol (of $SO(4)$) as in (2.33) while the term on the right is the trace of five 4-intertwiners with the respective boundary connection insertions.

The 4-simplex amplitude for the constrained spin foam model is then defined as the restriction of $A_v^{4dBF}(h_{ij})$ imposed by the quantum version of the constraints (2.54) and (2.55), defined by the following set of differential equations

$$\hat{C}_{ij,\ ik}\ A_v^{4dBF_{con}}(h_{ij}) = \epsilon_{IJKL}\mathcal{X}^{IJ}(h_{ij})\mathcal{X}^{KL}(h_{ik})\ A_v^{4dBF_{con}}(h_{ij}) = 0 \quad \forall\ j,k, \tag{2.74}$$

and where the index $i = 1,\cdots,5$ is held fixed and $\hat{C}_{ij,\ ik}$ denotes the corresponding constraint operator.

There are six independent constraints (2.74) for each value of $i = 1,\cdots,5$. If we consider all the equations for the 4-simplex amplitude then some of them are redundant due to (2.68).

We have 20 independent relations. For a given i in (2.74) (i.e., a given tetrahedron) and for $j = k$ the equation becomes

$$\begin{aligned}\epsilon_{IJKL}\mathcal{X}^{IJ}(h_{ij})\mathcal{X}^{KL}(h_{ij})\ A_v^{4dBF_{con}}(h_{ij}) = \\ = \delta_{ik}\left[J^i(h_{ij}^+)J^k(h_{ij}^+) - J^i(h_{ij}^-)J^k(h_{ij}^-)\right]\ A_v^{4dBF_{con}}(h_{ij}^+, h_{ij}^-) = \\ = \left[j_{ij}^+(j_{ij}^+ + 1) - j_{ij}^-(j_{ij}^- + 1)\right] A_v^{4dBF_{con}}(h_{ij}) = 0,\end{aligned} \tag{2.75}$$

where we have used that: $Spin(4) = SU(2)^+ \times SU(2)^-$ so that for $h \in Spin(4)$, $h^+, h^- \in SU(2)$ denote its right and left components and irreducible representations can be expressed as $\rho = j^+ \otimes j^-$ for $j^+, j^- \in \mathrm{Irrep}[SU(2)]$, and the left invariant vector field

$$\mathcal{X}^{IJ}(h) = P_i^{+IJ}J^i(h^-) + P_i^{-IJ}J^i(h^+), \tag{2.76}$$

for ${\epsilon_{IJ}}^{KL}P^{\pm IJ} = \pm P^{\pm KL}$ and J^i's being left invariant vector fields on the corresponding left and right $SU(2)$ copies of $Spin(4)$. The previous constraints are solved by requiring the corresponding representations $\rho^{SO(4)}$ to be simple, i.e.,

$$\rho_{ij}^{SO(4)} = j_{ij} \otimes j_{ij}^* \quad \text{or} \quad \rho^{SO(4)} = j_{ij} \otimes j_{ij}. \tag{2.77}$$

This ambiguity is analogous to the classical one in (2.47),(2.48). We take $\rho_{ij} = j_{ij} \otimes j_{ij}^*$ in correspondence to the choice ${}^*(e \wedge e)$ that produces the gravity sector [93] . This solves 10 of the 20 equations. The next non-trivial conditions imposed by (2.74) is when $j \neq k$. In this case we have

$$\begin{aligned}&2\epsilon_{IJKL}\mathcal{X}^{IJ}(h_{ij})\mathcal{X}^{KL}(h_{ik})\ A_v^{4dBF_{con}}(h_{ij}) \\ &= \epsilon_{IJKL}\left(\mathcal{X}^{IJ}(h_{ij}) + \mathcal{X}^{IJ}(h_{ik})\right)\left(\mathcal{X}^{KL}(h_{ij}) + \mathcal{X}^{KL}(h_{ik})\right) A_v^{4dBF_{con}}(h_{ij}) \\ &= \left[\iota^+(\iota^+ + 1) - \iota^-(\iota^- + 1)\right] A_v^{4dBF_{con}}(h_{ij}) \\ &= 0,\end{aligned} \tag{2.78}$$

where in the second line we used the fact that we have already solved (2.75). In the third line we have used the gauge invariance (or the analog of (2.68) for the 3-valent node in the tree decomposition that pairs the representation $j_{ij}^{SO(4)}$ with the $j_{ik}^{SO(4)}$) in order to express the sum of invariant vector fields as the invariant vector field acting on the virtual link labeled by ι. This choice of tree decomposition in the case $ij = 12$ and $ik = 13$ is the one used in equation (2.72). The solution is clearly $\iota^{SO(4)} = \iota \otimes \iota^*$.

What happens now to any of the two remaining conditions (only one is independent), for example, $\hat{C}_{ij',\ ik'}$ for $k \neq k'$, $j \neq j'$ and $j' \neq k'$? At first sight it looks like this equations cannot be (generically) satisfied because an intertwiner that has simple ι in one tree decomposition does not have only simple ι' components in a different tree decomposition as a consequence of the recoupling identity (Appendix D). There is however a linear combination of intertwiners found by Barrett and Crane in [70] which is simple in any tree decomposition, namely

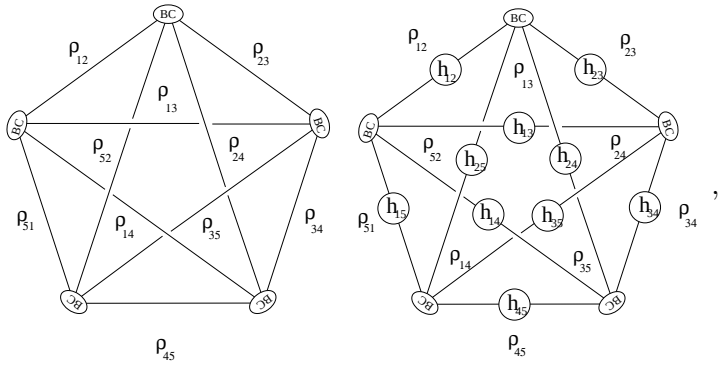
$$|i_{BC}\rangle = \sum_{simple\ \iota} C^{\iota}_{\rho_1,\cdots,\rho_4}. \tag{2.79}$$

$C^{\iota}_{\rho_1,\cdots,\rho_4}$ is a normalized 4-intertwiner and the summation is over simple $\iota^{SO(4)}$ (i.e. $\iota^{SO(4)} = \iota \otimes \iota^*$) and the $\rho_i^{SO(4)}$ are also simple ($\rho_i^{SO(4)} = j_i \otimes j_i^*$ for $i = 1..4$). This is clearly a solution to all the constraints and has been shown to be the unique one (up to an overall factor) by Reisenberger in [95]. $|i_{BC}\rangle$ defines the so-called Barrett-Crane intertwiner. Now the projector P^4_{inv} in (2.72)—the building block of the BF amplitude—can be written as

$$P^4_{inv} = |i_{BC}\rangle \langle i_{BC}| \quad + \quad orthogonal \;\; terms, \tag{2.80}$$

using the standard Gram-Schmidt construction of a basis in $\mathrm{Inv}\,[\rho_1 \cdots \rho_4]$. In other words P^4_{inv} is the sum of 1-dimensional projector to the solutions of the constraints (2.74) plus the orthogonal complement. The solution to (2.74) is then unique (up to scaling) and can be written as

$$A_v^{4dBF_{con}}(h_{ij}) = \sum_{\rho_{ij},\iota_i} \prod_{i<j} \dim \rho_{ij}$$

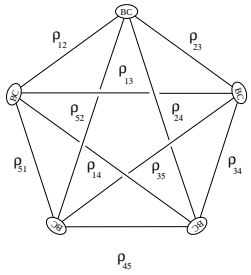


, (2.81)

where the $\rho_{ij}^{SO(4)} = j_{ij} \otimes j_{ij}^*$ and we graphically represent $|i_{BC}\rangle$ by  .

The amplitude of an arbitrary simplicial complex is computed by putting together 4-simplexes with consistent boundary connections and gluing them by integration over boundary data. If we do that we obtain the Barrett-Crane state sum on a fixed triangulation

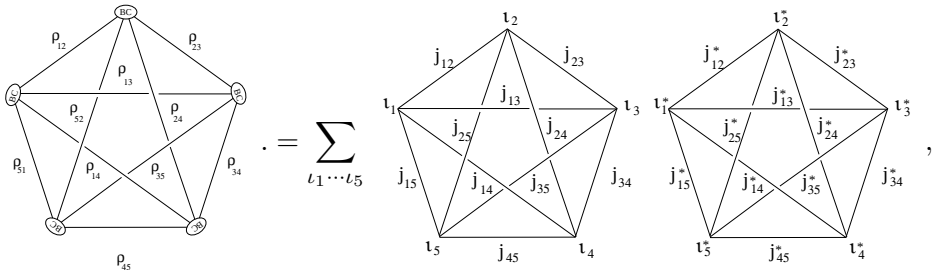
$$Z_{BC}(\mathcal{J}_\Delta) = \sum_{\{j_f\}} \prod_{f\in\mathcal{J}_\Delta} (2j_f+1)^2 \prod_{e\in\mathcal{J}_\Delta} A_e \prod_{v\in\mathcal{J}_\Delta}$$



, (2.82)

where we have made the replacement $\dim \rho^{SO(4)} = (2j+1)^2$ for $\rho^{SO(4)} = j \otimes j^*$. The vertex amplitude depends on the ten representations labeling the ten faces in a 4-simplex and it is referred to as $10j$-symbol. Using the definition (2.79) of the Barrett-Crane intertwiner,  ,

the $10j$-symbol can be written explicitly in terms of $15j$-symbols as



$$= \sum_{\iota_1 \cdots \iota_5} \qquad , \tag{2.83}$$

where we represent the normalized $SO(4)$-intertwiners $C^{\iota}_{\rho_1 \cdots \rho_4}$ in (2.79) in terms of the corresponding pair of left-right $SU(2)$-intertwiners.

A_e denotes a possible edge amplitude which is not determined by this argument but is determined by the GFT approach.

Constraints that involve different tetrahedra in a given 4-simplex—corresponding to (2.53)—are automatically satisfied as operator equations on the Barrett-Crane solutions. This can be checked using (2.74) and (2.68).

2.6.4 Geometrical interpretation and quantum tetrahedron

Now we look for the second approach leading to the BC model; based on the geometrical interpretation of the discretized B fields on a fixed triangulation.

The first step is to turn the bivectors associated to the triangles into operators. To this end, we make use of the isomorphism between the space of bivectors $\wedge^2\mathbb{R}^4$ and $so(4)$ identifying the bivectors associated with the triangles with the generators of the algebra:

$$B_t^{IJ} \to J^{IJ}(\rho_t) \tag{2.84}$$

Then we proceed turning these variables into operators by associating to the different triangles t an irreducible representation ρ_t of the group and the corresponding representation space, so that the the generators of the algebra act on it (as derivative operators). We choose unitary representations.

In the Riemannian case we use the group $Spin(4)$.

- We can then define the Hilbert space of a quantum $SO(4)$ bivector to be

$$\mathcal{H}_B = \oplus_{j_+,j_-} \mathcal{H}^{(j_+,j_-)} \tag{2.85}$$

  in the Riemannian case. However, this is not yet the Hilbert space of a quantum geometric triangle, simply because a set of bivectors does not describe a simplicial geometry unless it satisfies constraints equal to the Plebanski one for the discretized B fields. The task is then to translate these constraints in the language of group representation theory into the quantum domain.

  Of the constraints given for the bivectors, only the first two (2.52),(2.54) refer to a triangle alone, and these are then enough to characterize a quantum triangle.

The most important constraint is however the simplicity constraint, that forces the bivectors to be simple, i..e formed as wedge product of two edge vectors. Classically this was expressed by: $B_t \cdot *B_t = 0$. Using the quantization map given above and substituting the bivectors with the Lie algebra elements in the representation ρ_t for the triangle t, we obtain the condition:

$$J(\rho_t) \cdot *J(\rho_t) = \epsilon_{IJKL} J^{IJ}(\rho_t) J^{KL}(\rho_t) = \tilde{C}(\rho_t) = 0, \tag{2.86}$$

The condition is then of having vanishing psuedoscalar Casimir i.e the condition is to have simple representations.

- The simplicity constraint forces the selfdual and anti-selfdual parts to be equal, and we are restricted to considering only representations of the type (j, j).

  The Hilbert space of a **quantum triangle** is given by:

  $$\mathcal{H}_t = \oplus_j \mathcal{H}^{(j,j)} \tag{2.87}$$

  In this way the geometrical meaning of the parameters labelling the representations becomes clear: Classicaly the area of a triangle is given in terms of the associated bivector by:

  $$A^2 = B_t \cdot B_t \tag{2.88}$$

  that in the quantum case translates into

  $$A^2 = J^{IJ}(\rho_t) J_{IJ}(\rho_t) = C(\rho_t) \tag{2.89}$$

  , i.e. the area operator is the scalar Casimir and it is diagonal on each representation space associated to the triangle. Its eigenvalues are then in the Riemannian case, for simple representations

  $$A^2 = C((j, j)) = 2j(j + 1) \tag{2.90}$$

  The representation label characterizes the quantum area of the triangle to which that representation is associated.

States of the quantum theory are assigned to 3-dimensional hypersurfaces embedded in the 4-dimensional spacetime, and are thus formed by tetrahedra glued along common triangles. We then have to define a state associated to each tetrahedron in the triangulation, and then define a tensor product of these states to obtain any given state of the theory. In turn, the Hilbert space of quantum states for the tetrahedron has to be obtained from the Hilbert space of its triangles, since these are the basic building blocks at our disposal corresponding to the basic variables of the theory.

Each tetrahedron is formed by gluing 4 triangles along common edges, and *this gluing is naturally represented by the tensor* product of the corresponding representation spaces; $Spin(4)$ decomposes into irreducible representations which do not necessarily satisfy the simplicity constraint.

- The third of the constraints (composition) on bivectors translates at the quantum level as the requirement that only simple representations appear in this decomposition.

  Therefore we are considering for each tetrahedron with given representations assigned to its triangles a tensor in the tensor product: $\mathcal{H}^1 \otimes \mathcal{H}^2 \otimes \mathcal{H}^3 \otimes \mathcal{H}^4$ of the four representation

spaces for its faces, with the condition that the tensored spaces decompose pairwise into vector spaces for simple representations only. We have still to impose the closure constraint; this is an expression of the invariance under the gauge group of the tensor we assign to the tetrahedron, thus in order to fulfill the closure cosntraint we have to associate to the tetrahedron an invariant tensor, i.e. an intertwiner between the four simple representations associated to its faces $\iota^{\rho_1\rho_2\rho_3\rho_4} : \mathcal{H}^1 \otimes \mathcal{H}^2 \otimes \mathcal{H}^3 \otimes \mathcal{H}^4 \to \mathbb{C}$.

Therefore the Hilbert space of a **quantum tetrahedron** is given by:

$$\mathcal{H}_{tet} \;=\; Inv\left(\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3 \otimes \mathcal{H}_4\right), \tag{2.91}$$

with $\mathcal{H}_i$ being the Hilbert space for the i-th triangle defined above. Each state in this Hilbert space turns out to be the BC intertwiner defined in the previous section [70, 94].

This construction was rigorously perfomed in [93] in the Riemannian case, using geometric quantization methods. In any case, this invariant tensor represents as we said the state of a tetrahedron whose faces are labelled by the given representations; it can be represented graphically as in Figure 2.5.

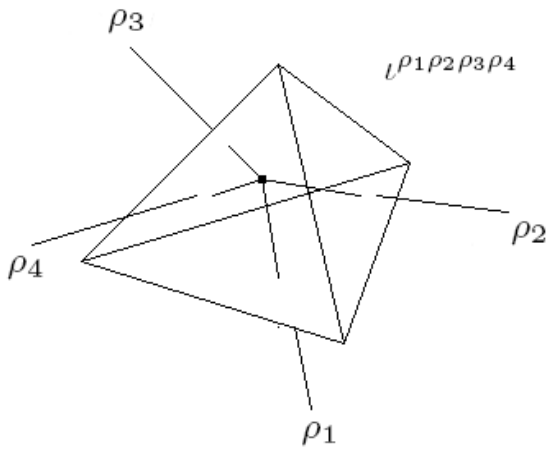


Figure 2.5: A vertex corresponding to a quantum tetrahedron in 4d, with links labelled by the representations (and state labels) associated to its four boundary triangles.

The result is that a quantum tetrahedron is characterized uniquely by 4 parameters, i.e. the 4 irreducible simple representations of $so(4)$ assigned to the 4 triangles in it, which in turn are interpretable as the (oriented) areas of the triangles.

Even if rigorous, this result is geometrically rather puzzling, since the geometry of a tetrahedron is classically determined by its 6 edge lengths, so imposing only the values of the 4 triangle areas should leave 2 degrees of freedom, i.e. a 2-dimensional moduli space of tetrahedra with given triangle areas. For example, this is what would happen in 3 dimensions, where we have to specify 6 parameters also at the quantum level. So why does a tetrahedron have fewer degrees of freedom in 4 dimensions than in 3 dimensions, at the quantum level, so that its quantum geometry is characterized by only 4 parameters? The answer was given in [93]. The essential difference between the 3-dimensional and 4-dimensional cases is represented

by the simplicity constraints that have to be imposed on the bivectors in 4 dimensions. At the quantum level these additional constraints reduce significantly the number of degrees of freedom for the tetrahedron, as can be shown using geometric quantization [93], leaving us at the end with a 1-dimensional state space for each assignment of simple irreps to the faces of the tetrahedron, i.e. with a unique quantum state up to normalization, as we have just seen above. Then the question is: what is the classical geometry corresponding to this state? In addition to the four triangle areas operators, there are two other operators that can be characterized just in terms of the representations assigned to the triangles: one can consider the parallelograms with vertices at the midpoints of the edges of a tetrahedron and their areas, and these are given by the representations entering in the decomposition of the tensor product of representations labelling the neighbouring triangles. Analyzing the commutation relations of the quantum operators corresponding to the triangle and parallelogram area operators, it turns out (see [51, 93]) that while the 4 triangle area operators commute with each other and with the parallelogram areas operators (among which only two are independent), the last ones have non-vanishing commutators among themselves. This implies that we are free to specify 4 labels for the 4 faces of the tetrahedron, giving 4 triangle areas, and then only one additional parameter, corresponding to one of the parallelogram areas, so that only 5 parameters determine the state of the tetrahedron itself, the other one being completely randomized, because of the uncertainty principle.

Consequently, we can say that a quantum tetrahedron does not have a unique metric geometry, since there are geometrical quantities whose value cannot be determined even if the system is in a well-defined quantum state. In the context of the Barrett-Crane spin foam model, this means that a complete characterization of two glued 4-simplices at the quantum level does not imply that we can have all the informations about the geometry of the tetrahedron they share. This is an example of the kind of quantum uncertainty relations that we can expect to find in a quantum gravity theory.

A generic quantum gravity state is to be associated to a 3-dimensional hypersurface in spacetime, and this will be triangulated by several tetrahedra glued along common triangles; therefore a generic state will live in the tensor product of the Hilbert spaces of the tetrahedra of the hypersurface, and in terms of the Barrett-Crane intertwiners it will be given by a product of one intertwiner $\iota^{\rho_1\rho_2\rho_3\rho_4}_{BC\, m_1 m_2 m_3 m_4}$ for each tetrahedron with a sum over the parameters ("angular momentum projections") labelling the particular triangle state for the common triangles (the triangles along which the tetrahedra are glued). The resulting object will be a function of the simple representations labelling the triangles, intertwined by the Barrett-Crane intertwiner to ensure gauge invariance, and it will be given by a graph which has representations of $Spin(4)$ labelling its links and the Barrett-crane intertwiner at its nodes; in other words, it will be given by a (simple) spin network (see Figure 2.6).

We are left with a last ingredient of a quantum geometry still to be determined: the quantum amplitude for a 4-simplex χ, interpreted as an elementary change in the quantum geometry and thus encoding the dynamics of the theory, and representing the fundamental building block for the partition function and the transition amplitudes of the theory.

This amplitude is to be constructed out of the tensors associated to the tetrahedra in the 4-simplex, so that it immediately fulfills the conditions necessary to describe the geometry of the simplicial manifold, at both the classical and quantum level, and has to be of course invariant under the gauge group of "'spacetime"' $Spin(4)$. The natural choice is to obtain a $\mathbb{C}$-

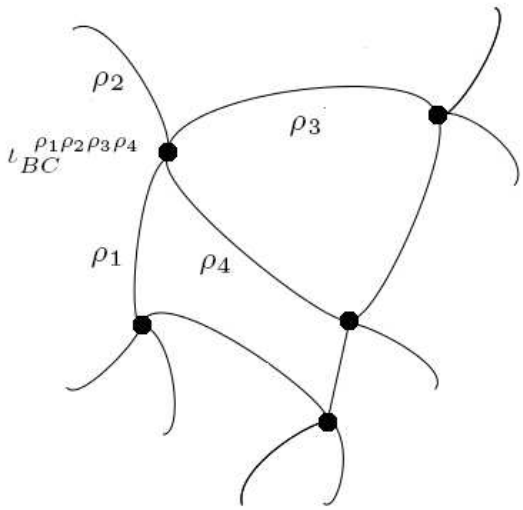


Figure 2.6: An example of a (simple) spin network in 4d, with three vertices (to which the Barrett-Crane intertwiners are associated), and both open and closed links.We have only 4-valent nodes with BC intertwiners

number for each 4-simplex, a function of the 10 representations labelling its triangles, by fully contracting the tensors associated to its five tetrahedra pairwise summing over the parameters associated to the common triangles and respecting the symmetries of the 4-simplex, so:

$$A_\chi \,:\, \otimes_i Inv\left(\mathcal{H}^{1_i} \otimes \mathcal{H}^{2_i} \otimes \mathcal{H}^{3_i} \otimes \mathcal{H}^{4_i}\right) \,\rightarrow\, \mathbb{C}. \tag{2.92}$$

The amplitude for a quantum 4-simplex is thus, in the Riemannian case:

$$A_\chi \,=\, \mathcal{B}_{BC} \,=\, \iota_{BC\, m_1 m_2 m_3 m_4}^{\rho_1\rho_2\rho_3\rho_4}\, \iota_{BC\, m_4 m_5 m_6 m_7}^{\rho_4\rho_5\rho_6\rho_7}\, \iota_{BC\, m_7 m_3 m_8 m_9}^{\rho_7\rho_3\rho_8\rho_9}\, \iota_{BC\, m_9 m_6 m_2 m_{10}}^{\rho_9\rho_6\rho_2\rho_{10}}\, \iota_{BC\, m_{10} m_8 m_5 m_1}^{\rho_{10}\rho_8\rho_5\rho_1}, \tag{2.93}$$

We thus obtain as amplitude the $10j$-symbol (2.83).

This amplitude is for fixed representations associated to the triangles, i.e. for fixed triangle areas; the full amplitude involves a sum over these representations with the above amplitude as a weight for each configuration. In general, the partition function of the quantum theory describing the quantum geometry of a simplicial complex made of a certain number of 4-simplices, will be given by a product of these 4-simplex amplitudes, one for each 4-simplex in the triangulation, and possibly additional weights for the other elements of it, triangles and tetrahedra, with a sum over all the representations assigned to triangles. We can then envisage a model of the form:

$$Z(\mathcal{M},\Delta)_{BC} \,=\, \sum_\rho \prod_{triangles} A_{tr} \prod_{tetrahedra} A_{tet} \prod_{4-simpl} \mathcal{B}_{BC}, \tag{2.94}$$

or, if one sees all the data as assigned to the 2–complex Δ^* dual to the triangulation Δ (with faces f dual to triangles, edges e dual to tetrahedra, and vertices v dual to 4-simplices):

$$Z(\mathcal{M},\Delta^*)_{BC} \,=\, \sum_\rho \prod_f A_f \prod_e A_e \prod_v \mathcal{B}_{BC}^v, \tag{2.95}$$

clearly with the general structure of a spin foam model.

The last aspect that needs to be implemented is sum over triangulations; the restriction to a fixed triangulation Δ of spacetime has to be lifted, since in this non-topological case it represents a restriction of the dynamical degrees of freedom of the quantum spacetime, and a suitably defined sum over triangulations has to be implemented. In the next sections we will see how this can be performed with GFT.

2.6.5 An integral expression for the $10j$-symbol

The vertex (2.83) admits an integral representation [96] that will be useful in the following.

The basic observation is that equation (2.79) has precisely the form (2.32) if we write the $Spin(4)$ intertwiners as tensor products of $SU(2)$ ones. Therefore,

$$|i_{BC}\rangle = \int_{SU(2)} du\ D^{j_1}(u) \otimes D^{j_2}(u) \otimes D^{j_3}(u) \otimes D^{j_4}(u), \tag{2.96}$$

where $D^j(u)$ are $SU(2)$ representation matrices in the representation j. Each one of the five pairs of intertwiners in (2.83) can be obtained as an integral (2.96) over $SU(2)$. Each of the ten representation matrices $D^{j_{ik}}$ $(i \neq k = 1 \cdots 5)$ appears in two integrals corresponding to the intertwiners at the node i and j respectively. Contracting the matrix indexes according to (2.83) these two representation matrices combine into a trace $\mathrm{Tr}\left[D^{j_{ik}}(u_i u_k^{-1})\right]$ $(u_i \in SU(2))$. Parameterizing $SU(2)$ with spherical coordinates on S^3

$$\mathrm{Tr}\left[D^{j_{ik}}(u_i u_k^{-1})\right] = \frac{\sin(2j_{ik}+1)\psi_{ik}}{\sin(\psi_{ik})} := (2j_{ik}+1)K_{j_{ik}}(y_i, y_k), \tag{2.97}$$

where ψ_{ik} is the azimuthal angle between the points y_i, y_k on the sphere corresponding to u_i and u_k respectively. We have also introduced the definition of the kernel $K_{j_{ik}}(y_i, y_k)$ in terms of which the Barret-Crane vertex amplitude (2.83) becomes

$$A_v(j_{ik}) = \int_{(S^3)^5} \prod_{i=1}^{5} dy_i \ \prod_{i<k} (2j_{ik}+1) K_{j_{ik}}(y_i, y_k). \tag{2.98}$$

Each of the five integration variables in S^3 can be regarded as a unit vector in $\mathbb{R}^4$. They are interpreted as unit normal vectors to the 3-dimensional hyperplanes spanned by the corresponding five tetrahedra. The angles ψ_{ik} is defined by $\cos\psi_{ik} = y_i \cdot y_k$ and corresponds to the exterior angle between two hyperplanes (analogous to the dihedral angles of Regge calculus). These normals determine a 4-simplex in $\mathbb{R}^4$ up to translations and scaling [96].

2.6.6 The asymptotics for the vertex amplitude

The large spin behavior of the spin foam amplitudes provides information about the low 'energy' or semi-classical limit of the model [97] in the naive sense $\hbar \to \infty$ while geometric quantities such as the area are held fixed. Evidence showing a connection between the asymptotics of the Barrett-Crane vertex and the action of general relativity was found by Crane and Yetter in [98].

A computation of the asymptotic (large j) expression of the Barrett-Crane vertex amplitude for non-degenerate configurations was obtained by Barrett and Williams in [99]. They computed $A_v(j_{ik})$ for large j_{ik} by looking at the stationary phase approximation of the oscillatory integral (2.98). The large spin behavior of the vertex amplitude is given by

$$A_v(j_{ik}) \sim \sum_{\sigma} P(\sigma) \cos\left[S_{Regge}(\sigma) + \kappa\frac{\pi}{4}\right] + D \tag{2.99}$$

where the sum is over geometric 4-simplexes σ whose face areas are fixed by the spins. The action in the argument of the cosine corresponds to Regge action which in four dimensions is defined by $S_{Regge}(\sigma) = \sum_{i<k} A_{ik}\ \psi_{ik}(\sigma)$ where A_{ik} is the area of the ik-triangle. $P(\sigma)$ is a normalization factor which does not oscillate with the spins. D is the contribution of degenerate configurations, i.e. those for which some of the hyperplane normals defined above coincide. However [100] Baez, Christensen and Egan show that the term D is in fact dominant in the previous equation, i.e. the leading order terms are contained in the set of degenerate configurations! This has been later confirmed by the results of Freidel and Louapre [101] and Barrett and Steele [102].

2.7 Group field theory (GFT) formulation

In the GFT formalism, the 2-complex emerges as the Feynman diagram of dual field theory formulation defined over a group manifold, generalizing matrix models [103, 104, 105]. In 3d [106] gives the Ponzano-Regge-Turaev-Viro (spin foam) formulation of 3d quantum gravity (still a topological theory), and in 4 dimensions [107], gives the Crane-Yetter spin foam formulation of 4d BF theory [108, 109](All topological theories). The idea is to write a field theory over a group manifold (GFT) whose Feynman amplitudes yield the spinfoam partition functions defined on the cellular decompositions dual to the Feynman diagrams of the field theory. In particular, in [110] it was shown that for each spinfoam amplitude on a 2-complex $\mathcal{J}$, one can always find a GFT having that amplitude as its Feynman amplitude associated with the diagram $\mathcal{J}$. The GFT formalism generalizes a single spinfoam amplitude to a sum over 2-complexes, thus restoring the infinite degrees of freedom. This approach has recently been rethought as a third quantised version of gravity, including a sum over topologies [71, 111]. The first derivation of the Barrett-Crane model from a field theory over a group manifold was given in [112], and an alternative one was proposed in [113]. We concentrate on GFT formulations of the BC model.

2.8 General formalism

The fundamental idea of this approach is that one can represent a tetrahedron in 4-dimensions, to which a state of the quantum theory can be associated, by a function of 4 group variables (where the group G is the Lorentz group sl in the Lorentzian case and $Spin(4)$ in the Riemannian) to be thought of as associated to its four triangular faces. The action is then chosen in such a way to mimic the combinatorial structure of a simplicial manifold with:

- an interaction term given by a product of five fields since a 4-simplex has five tetrahedra and a 4-simplex can be interpreted as an elementary interaction of tetrahedra.

- a kinetic term quadratic in the fields, as in usual field theories.

With this framework we obtain group field theory actions giving rise to the topological models corresponding to BF theory in different dimensions.

In the case of gravity the situation is more complicated since we have to impose the constraints that reduce BF theory to gravity, as conditions on the fields in the action and this can be done in different ways; this ambiguity is what originates the different models, all understandable as different versions of the Barrett-Crane model.

In the case we are interested in, the group is $SO(4)$ and the (scalar) field is a real function over $SO(4)^4$, i.e. a function of 4 group elements $\phi(g_1, g_2, g_3, g_4)$ with $g_i \in SO(4)$. The field is a (scalar) function of four group elements: $\phi(g_1, g_2, g_3, g_4)$, that can be simply written as $\phi(g_i)$. We may require different symmetry properties on this field with respect to permutations of its four arguments, i.e. we may require:

$$\phi(g_1, g_2, g_3, g_4) = \phi(g_{\pi(1)}, g_{\pi(2)}, g_{\pi(3)}, g_{\pi(4)}) \tag{2.100}$$

, where π is a generic permutation of (1234) or an even permutation only, etc. Different choices give rise to different models. We then define the following projector operators:

$$P_G\, \phi(g_1, g_2, g_3, g_4) \;=\; \int_{SO(4)} d\gamma\, \phi(g_1\gamma, g_2\gamma, g_3\gamma, g_4\gamma) \tag{2.101}$$

and

$$P_H\, \phi(g_1, g_2, g_3, g_4) \;=\; \int_{SO(3)} dh_1 ... dh_4\, \phi(g_1 h_1, g_2 h_2, g_3 h_3, g_4 h_4) \tag{2.102}$$

(all the integrals here and in the following are in the normalized Haar measure).

- P_G imposes on the field the invariance under the action of $SO(4)$ on its 4 arguments that can be seen as gauge invariance or the implementation of the closure constraint of Sections (2.5-2.6);

- P_H projects the field over the subspace of fields that are constant on the orbits of $SO(3)$ in $SO(4)$; if we expand the field in modes, i.e. in terms of a sum over the irreducible representations of $SO(4)$, the projection restricts the representations to the ones in which there is a vector invariant under $SO(3)$, i.e. the simple representations (simplicity constraint of Section 2.6).

We consider an action of the general form

$$\begin{aligned} S[\phi] \;=& \frac{1}{2} \int dg_1 ... dg_4\; [\phi(g_1, g_2, g_3, g_4)]^2 \;+ \\ &+ \frac{\lambda}{5!} \int dg_1 ... dg_{10}\, \phi(g_1, g_2, g_3, g_4)\, \phi(g_4, g_5, g_6, g_7)\, \phi(g_7, g_3, g_8, g_9)\, \phi(g_9, g_6, g_2, g_{10})\, \phi(g_{10}, g_8, g_5, g_1) \end{aligned} \tag{2.103}$$

given by a quadratic kinetic term without derivatives and a potential term of fifth order, again with no derivative, with a coupling constant λ.

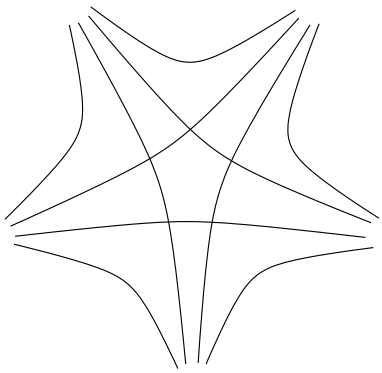


Figure 2.7: The 4-simplex structure of the interaction vertex.

The potential has the structure of a 4-simplex: we can represent each of the five fields as a node with 4 legs (one for each g_i) and connect the legs with the same argument: the result is the two–skeleton of a four simplex see Figure 2.7

The integral over the group elements can be thought of as representing the triangles of a simplicial complex; more precisely:

$$S[\phi] = \frac{1}{2}\int dg_i d\tilde{g}_i \phi(g_i)\mathcal{K}(g_i,\tilde{g}_i)\phi(\tilde{g}_i) + \frac{\lambda}{5}\int dg_{ij}\mathcal{V}(g_{ij})\phi(g_{1j})\phi(g_{2j})\phi(g_{3j})\phi(g_{4j})\phi(g_{5j}) \quad (2.104)$$

with $\phi(g_{1j}) = \phi(g_{12}, g_{13}, g_{14}, g_{15})$ and so on. $\mathcal{K}$ and $\mathcal{V}$ are the kinetic operator (whose inverse, in the space of gauge invariant fields, is the propagator of the theory) and the vertex (or potential) operator respectively. Alternatively, one can expand the field in modes (expansion in the "'momentum"' space), i.e. apply the rules of the harmonic analysis of functions on groups, and re write the action in terms of these modes, so that a conjugate expression of the kinetic and potential operators has to be used. The modes of the field will be functions of four group representations, instead of the four group elements.

The partition function of the theory is given, as in the usual quantum field theory literature, by an integral over the field values of the exponential of the action, and can be re-expressed in terms of its Feynman graphs (as an expansions in powers of λ) as:

$$Z = \int \mathcal{D}\phi\, e^{-S[\phi]} = \sum_{\Gamma} \frac{\lambda^{v[\Gamma]}}{5!^v\, v!\, sym[\Gamma]}\, Z[\Gamma], \quad (2.105)$$

where $v[\Gamma]$ and $sym[\Gamma]$ are the number of vertices and the order of symmetries (number of automorphisms) of the Feynman graph Γ. So we are interested in finding the form of the amplitude for a generic Feynman graph, i.e. the Feynman rules of the theory.

We will see that this amplitude for each Feynman graph is exactly the partition function for fixed triangulation that we have derived from a lattice gauge theoretic approach above, for a triangulation to be put in correspondence with a given interaction graph of the group field theory. In order to understand how this correspondence is to be found, let us analyze more closely the construction of the Feynman graphs. The propagator (see Figure 2.8) can

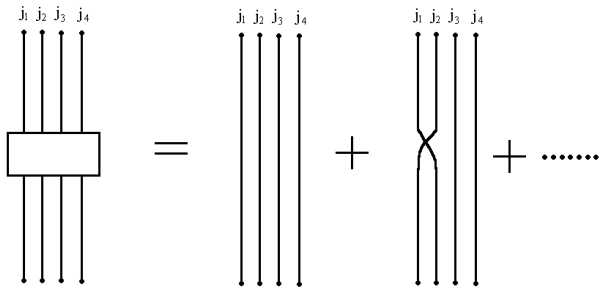


Figure 2.8: The propagator of the theory; each of the four strands carries a (simple) representation of the group and the box stands for a symmetrization of the four arguments, i.e. for a sum over given permutations of the ordering of the arguments

be represented by four straight parallel lines, representing the four arguments of the field or of its modes, and has a group variable or a group representation) at the two ends of each line, while the vertex operator (2.136) (see Figure 2.9) has the combinatorial structure (given by the way it pairs the arguments of the fields) of a 4-simplex, with 5 vertices and 4 lines coming out of each of them (five tetrahedra-propagators with four triangles-lines each), again with a group variable or group representation at the two ends of each line. Everywhere at the open ends of propagators and vertices are the four group variables that are the arguments of the field. All the possible Feynman graphs are obtained (see Figure 2.10) connecting a number of vertices with the propagators, constructing in this way what is called a "fat graph" for each possible permutation of the lines in the propagator.

Each of the strands of a propagator is connected to a strand in one of the five "open sites" of the vertex, in such a way that the orientations in vertices and propagators match. Moreover, because of the symmetry of the field, each propagator really corresponds to many different terms given by the different ways in which its four carried indices can be permutated. Which are these possible ways depends of course by the symmetries under permutations we have imposed on the field, so that all possible permutations appear in the propagator, or only all the even ones, etc.

Now, each strand of the fat graph can go through several propagators and several vertices, and at the end closes on itself, and consequently forms a cycle. This cycle is labelled (in "momentum space") by the representation assigned to the strand that contains it. The abstract object formed by these cycles, the edges and the vertices is a 2–complex, moreover, since each face (cycle) is labelled by a representation of G, this is a labelled 2–complex, i.e. a spin foam. So we see that there is a 1-1 correspondence between the Feynman graphs of our field theory and spin foams. This means that the sum in (2.105) can be interpreted as a sum over spin foams (labelled 2–complexes), the amplitude $Z(\Gamma)$ as an amplitude for each spin foam, and the partition function itself defines a spin foam model. In turn, each 2–complex can be thought of as dual to a triangulation, i.e. a simplicial complex. In general it is not true that a simplicial complex is determined by the dual 2-skeleton, but it is true in this case since the complex is obtained by gluing of codimension-1 faces of simplices. Knowing this, we can interpret the vertices as referring to 4-simplices, and the propagators to the tetrahedra

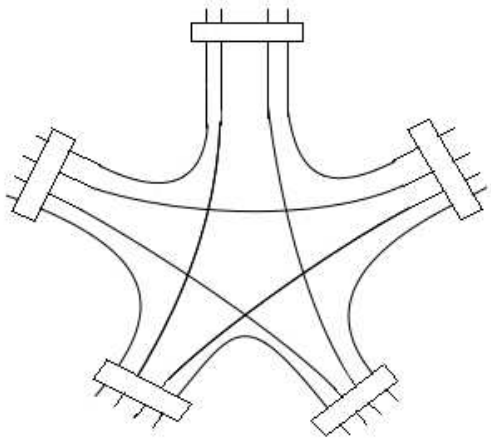


Figure 2.9: The vertex of the theory; it has the combinatorial structure of a 4-simplex and has a (simple) representation of the group at each open end

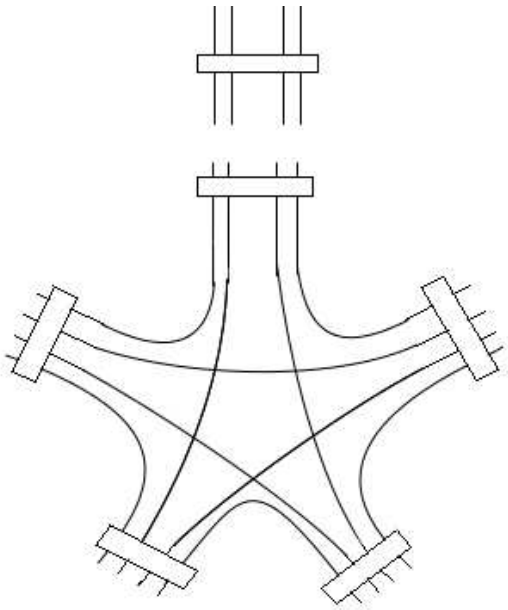


Figure 2.10: The construction of a Feynman graph of the theory, connecting a propagator with a vertex.

they share.

Therefore, the group field theory generates a sum over topologies as well as a sum over different triangulations of the same topology, so gives rise to a fully background independent theory of spacetime.

2.9 Riemannian group field theories

We present now different versions of the Riemannian Barrett-Crane model, existing in the literature, and as obtained by group field theory, and then give the most general structure of this kind of model. These different versions arise because of the non-trivial interplay between the two projectors P_G and P_G.

The starting point is the field $\phi(g_1, g_2, g_3, g_4)$. One can then impose on it the projector P_H making it a field over four copies of the homogeneous space G/H and then impose gauge invariance by means of the projector P_G, so that the basic object of the theory is the field: $P_G P_H \phi(g_1, g_2, g_3, g_4)$. We recall that the projector P_H imposes invariance under a given subgroup H of G, and restricts the representations involved in these models to be the simple representations with respect to this subgroup, while the projector P_G imposes gauge invariance (invariance under the full group G). Alternatively, one can decide to work with gauge invariant fields from the beginning, having as basic object the field $P_G\phi(g_1, g_2, g_3, g_4)$. On this field one can then impose further projectors. Another choice then is involved: we can decide to impose more projectors and in different combinations in the kinetic and in the potential terms in the action. We will see that the models existing in the literature all work with gauge invariant fields as basic objects, but differ in the way they impose further projectors in the kinetic and potential terms. These difference are more manifest in the conjugate representation of the fields in terms of representations of the group, i.e. using their mode expansion.

Introducing the shorthand notation for the action

$$S[\phi] = \int \phi^2 + \frac{\lambda}{5!} \int \phi^5 \tag{2.106}$$

We have three possible choices for the action leading to the three models TOCY [107], BCB[112], BCC[113, 114], in the terminology of [2]:

GFT/TOCY

$$S[\phi] = \frac{1}{2} \int (P_G\phi)^2 + \frac{\lambda}{5!} \int (P_G\phi)^5 \tag{2.107}$$

GFT/B

$$S[\phi] = \frac{1}{2} \int (P_G P_H\phi)^2 + \frac{\lambda}{5!} \int (P_G P_H\phi)^5 \tag{2.108}$$

GFT/C

$$S[\phi] = \frac{1}{2} \int (P_G\phi)^2 + \frac{\lambda}{5!} \int (P_G P_H\phi)^5 \tag{2.109}$$

or

$$S[\phi] = \frac{1}{2} \int (P_G\phi)^2 + \frac{\lambda}{5!} \int (P_G P_H P_G\phi)^5 \tag{2.110}$$

in this case the two forms of the action define the same theory, since the extra P_G in the second expression can be always absorbed into the P_G of some propagator when computing an amplitude. The second form of the action simplifies the analysis of the theory in momentum space.

2.9.1 GFT/B

We impose the combination of projectors P_G and P_H in the kinetic and interaction term, using the combination $P_H P_G$, the action is:

$$\begin{aligned} S[\phi] \;=\; & \frac{1}{2}\int dg_1...dg_4\;\left[P_H P_G\phi(g_1,g_2,g_3,g_4)\right]^2 \;+ \\ +\quad & \frac{\lambda}{5!}\int dg_1...dg_{10}\;\left[P_H P_G\phi(g_1,g_2,g_3,g_4)\right]\left[P_H P_G\phi(g_4,g_5,g_6,g_7)\right] \\ \times\quad & \left[P_H P_G\phi(g_7,g_3,g_8,g_9)\right]\left[P_H P_G\phi(g_9,g_6,g_2,g_{10})\right]\left[P_H P_G\phi(g_{10},g_8,g_5,g_1)\right], \end{aligned} \tag{2.111}$$

The removal of the combination of projectors $P_G P_H$ would lead to a model directly analogous to the Boulatov model, called Ooguri model, and corresponding to 4-dimensional BF theory. In this case one can see that the kinetic operator of the theory is given, in coordinate space, by:

$$\mathcal{K}(g_i,\tilde{g}_i) \;=\; \sum_\sigma \int dh_i d\gamma d\tilde{\gamma}\;\prod_i \delta\left(g_i\,\gamma\,h_i\,\tilde{\gamma}\,\tilde{g}^{-1}_{\sigma(i)}\right), \tag{2.112}$$

while the vertex is:

$$\mathcal{V}(g_{ij}) \;=\; \frac{1}{5!}\int d\beta_i d\tilde{\beta}_i dh_{ij}\;\prod_{i<j}\delta\left(g^{-1}_{ji}\tilde{\beta}_i h_{ij}\beta_i^{-1}\beta_j h_{ji}\tilde{\beta}_j^{-1}g_{ij}\right). \tag{2.113}$$

2.9.2 Mode expansion

Consider a square integrable function $\phi(g)$ over $SO(4)$, invariant under the right action of $SO(3)$. Using Peter-Weyl theorem, one can expand it in the matrix elements $D^j_{\alpha\beta}(g)$ of the irreducible representations j

$$\phi(g) = \sum_j\;\phi^j_{\alpha\beta}\;D^j_{\alpha\beta}(g) \tag{2.114}$$

The indices α,β label basis vectors in the corresponding representation space, and the sum over repeated indices is understood. In other words, we choose a basis in the representation space such that the metric in this basis is just the Kronecker delta, which is the standard choice in representation theory literature. The requirement of invariance under the right SO(3) action can be written using the projector P_H as

$$P_H\phi(g) \;\;=\;\; \int\limits_{SO(3)} dh\;\phi(gh). \tag{2.115}$$

Expanding this into the modes, we have

$$P_H\phi(g) = \int\limits_{SO(3)} dh\ \phi(gh) = \sum_j \int\limits_{SO(3)} dh\ \phi^j_{\alpha\beta}\ D^j_{\alpha\gamma}(g)\ D^j_{\gamma\beta}(h). \tag{2.116}$$

The integral in the last term projects the β index over a (normalized) $SO(3)$ invariant vector, which we denote by w_β using (D.9). As this exists only in simple representations (and is unique, see the Appendix C), the sum over representations Λ reduces to a sum over the simple representations N only, and thus

$$\phi(g) = \sum_N\ \phi^N_\alpha\ D^{(N)}_{\alpha\beta}(g)\ w_\beta. \tag{2.117}$$

The quantities $H^N_\alpha(g) = D^{(N)}_{\alpha\beta}(g)w_\beta$ are invariant under the (right) action of $SO(3)$, and, thus, can be thought of as functions on the three-sphere: $H^N_\alpha(g) = H^N_\alpha(x)$. In fact, they form an orthogonal basis for S^3 spherical harmonics.

Since ϕ is real, we have

$$\sum_N\ \overline{\phi^N_\alpha}\ \overline{D^{(N)}_{\alpha\beta}(g)}\ \overline{w_\beta} = \sum_N\ \phi^N_\alpha\ D^{(N)}_{\alpha\beta}(g)\ w_\beta. \tag{2.118}$$

The invariant vectors w_β are real (Appendix D), and the matrix elements can also be chosen to be real. Thus, the reality condition simply requires ϕ^N_α to be real.

It is immediate to generalize the previous expansions to the field $\phi(g_1, g_2, g_3, g_4)$:

$$P_H\phi(g_1, \dots, g_4) = \sum_{N_1 \dots N_4}\ \phi^{N_1 \dots N_4}_{\alpha_1 \dots \alpha_4}\ D^{(N_1)}_{\alpha_1\beta_1}(g_1) \dots D^{(N_4)}_{\alpha_4\beta_4}(g_4)\ w_{\beta_1} \dots w_{\beta_4}. \tag{2.119}$$

Note that the symmetry of $\phi(g_1, \dots, g_4)$ under permutation of its four arguments implies that

$$\phi^{N_1 \dots N_4}_{\alpha_1 \dots \alpha_4} = \phi^{N_\sigma(1) \dots N_\sigma(4)}_{\alpha_\sigma(1) \dots \alpha_\sigma(4)}, \tag{2.120}$$

where σ is any permutation of $\{1, 2, 3, 4\}$.

To find the effect of the $SO(4)$ invariance property on the modes, it is enough to apply the P_G projector

Substituting the mode expantion (2.119) into the definition of ϕ

$$P_G P_H\phi(g_1 \dots g_4) \quad = \int\limits_{SO(4)} dg \quad P_H\phi(g_1 g \dots g_4 g). \tag{2.121}$$

one obtains:

$$P_G P_H\phi(g_1, \dots, g_4) = \sum_{N_1 \dots N_4} \sum_\Lambda \left(\phi^{N_1 \dots N_4}_{\alpha_1 \dots \alpha_4}\ D^{(N_1)}_{\alpha_1\beta_1}(g_1) \dots D^{(N_4)}_{\alpha_4\beta_4}(g_4)\ C^{N_1 \dots N_4\,\Lambda}_{\beta_1 \dots \beta_4}\right)\ \left(C^{N_1 \dots N_4\,\Lambda}_{\gamma_1 \dots \gamma_4} w_{\gamma_1} \dots w_{\gamma_4}\right). \tag{2.122}$$

The quantity in the second parenthesis is the scalar product of two $SO(3)$ invariant vectors in the representation Λ. Since invariant vectors exist only in simple representations, this

quantity is non-vanishing only when Λ is a simple representations N. In this case its value is given by (D.10)

$$C^{N_1\dots N_4\, N}_{\gamma_1\dots\gamma_4} w_{\gamma_1}\dots w_{\gamma_4} = \frac{1}{\sqrt{\dim_{N_1}\dots\dim_{N_4}}} \tag{2.123}$$

This suggests to redefine the mode expansion in term of the new fields

$$\phi_{\mathcal{B}}{}^{N_1\dots N_4}_{\alpha_1\dots\alpha_4} = \phi^{N_1\dots N_4}_{\alpha_1\dots\alpha_4}\sqrt{\dim_{N_1}\dots\dim_{N_4}} \tag{2.124}$$

Substituting this into the mode expansion, we get

$$\phi(g_1,\dots,g_4) = \sum_{N_1\dots N_4} \phi_{\mathcal{B}}{}^{N_1\dots N_4}_{\alpha_1\dots\alpha_4}\; D^{(N_1)}_{\alpha_1\beta_1}(g_1)\dots D^{(N_4)}_{\alpha_4\beta_4}(g_4)\; S^{N_1\dots N_4}_{\beta_1\dots\beta_4}. \tag{2.125}$$

Here $S^{N_1\dots N_4}_{\beta_1\dots\beta_4}$ is the *normalized* Barrett-Crane intertwiner

$$S^{N_1\dots N_4}_{\beta_1\dots\beta_4} := \frac{\sum_N C^{N_1\dots N_4\, N}_{\beta_1\dots\beta_4}}{\sqrt{\sum_N C^{N_1\dots N_4\ N}_{\beta_1\dots\beta_4} C^{N_1\dots N_4\ N}_{\beta_1\dots\beta_4}}} \quad . \tag{2.126}$$

The normalization, given by the denominator in the above expression, is in the scalar product of $K_{\vec{N}_e}$. Since the quantities $C^{N_1\dots N_4\ N}_{\beta_1\dots\beta_4}$ are normalized, the denominator is the square root of the dimension of the subspace of $K_{\vec{N}_e}$ spanned by the intertwiners having an intermediate simple representation (See Appendix D).

Now is easy to find the mode expansion of the action. Using the result for the integral of the product of two matrix elements, the kinetic term becomes

$$\mathcal{K} = \frac{1}{2}\int\prod_{i=1}^{4} dg_i \quad (P_G P_H \phi(g_1,\dots,g_4))^2 = \frac{1}{2}\sum_{N_1\dots N_4} \phi_{\mathcal{B}}{}^{N_1\dots N_4}_{\alpha_1\dots\alpha_4}\; \phi_{\mathcal{B}}{}^{N_1\dots N_4}_{\alpha_1\dots\alpha_4}. \tag{2.127}$$

and the potential term gives

$$\begin{aligned}\mathcal{V} &= \frac{\lambda}{5!}\int\prod_{i=1}^{10} dg_i \;\; \phi(g_1,g_2,g_3,g_4)\;\phi(g_4,g_5,g_6,g_7)\;\phi(g_7,g_3,g_8,g_9)\;\phi(g_9,g_6,g_2,g_{10})\;\phi(g_{10},g_8,g_5,g_1)\\ &= \frac{\lambda}{5!}\sum_{N_1\dots N_{10}}\sum_{\Lambda_1\dots\Lambda_5} \phi_{\mathcal{B}}{}^{N_1N_2N_3N_4}_{\alpha_1\alpha_2\alpha_3\alpha_4}\phi_{\mathcal{B}}{}^{N_4N_5N_6N_7}_{\alpha_4\alpha_5\alpha_6\alpha_7}\phi_{\mathcal{B}}{}^{N_7N_3N_8N_9}_{\alpha_7\alpha_3\alpha_8\alpha_9}\phi_{\mathcal{B}}{}^{N_9N_6N_2N_{10}}_{\alpha_9\alpha_6\alpha_2\alpha_{10}}\phi_{\mathcal{B}}{}^{N_{10}N_8N_5N_1}_{\alpha_{10}\alpha_8\alpha_5\alpha_1} \qquad \mathcal{B}_{N_1,\dots,N_{10}}\ .\end{aligned} \tag{2.128}$$

Here $\mathcal{B}_{N_1,\dots,N_{10}}$ is the Barrett-Crane vertex-amplitude, which is a $(15-j)$-symbol with Barrett-Crane intertwiners or a 10j symbol

$$\mathcal{B}_{N_1,\dots,N_{10}} := S^{N_1N_2N_3N_4}_{\alpha_1\alpha_2\alpha_3\alpha_4}\; S^{N_4N_5N_6N_7}_{\alpha_4\alpha_5\alpha_6\alpha_7}\; S^{N_7N_3N_8N_9}_{\alpha_7\alpha_3\alpha_8\alpha_9}\; S^{N_9N_6N_2N_{10}}_{\alpha_9\alpha_6\alpha_2\alpha_{10}}\; S^{N_{10}N_8N_5N_1}_{\alpha_{10}\alpha_8\alpha_5\alpha_1}. \tag{2.129}$$

We are ready to look at the Feynman rules: it is immediate to extract the form of the propagator from the kinetic term but in any case it can be computed from the partition function simply computing gaussian integrals:

The partition function in this case is given by

$$Z = \int \left[D\phi_{\mathcal{B}}{}^{N_1\dots N_4}_{\alpha_1\dots\alpha_4}\right] \;\; e^{-S[\phi_{\mathcal{B}}]}. \tag{2.130}$$

The propagator is

$$P_{\mathcal{B}}{}^{N_1\dots N_4,\;\; N'_1\dots N'_4}_{\alpha_1\dots\alpha_4,\;\;\; \alpha'_1\dots\alpha'_4} =: < \phi_{\mathcal{B}}{}^{N_1\dots N_4}_{\alpha_1\dots\alpha_4}, \phi_{\mathcal{B}}{}^{N'_1\dots N'_4}_{\alpha'_1\dots\alpha'_4} > = \;\; \frac{1}{4!}\sum_{\sigma} \delta^{N_1 N'_{\sigma(1)}} \dots \delta^{N_4 N'_{\sigma(4)}} \;\; \delta^{\alpha_1\alpha'_{\sigma(1)}} \dots \delta^{\alpha_4\alpha'_{\sigma(4)}} \tag{2.131}$$

The vertex is given by:

$$< \phi_{\mathcal{B}}{}^{N_1N_2N_3N_4}_{\alpha_1\alpha_2\alpha_3\alpha_4} \dots \phi_{\mathcal{B}}{}^{N_{10}N_8N_5N_1}_{\alpha'_{10}\alpha'_8\alpha'_5\alpha'_1} > \;\; = \;\; \lambda \;\; \delta^{\alpha_1\alpha'_1} \dots \delta^{\alpha_{10}\alpha'_{10}} \;\; \mathcal{B}_{N_1\dots N_{10}} \;\; , \tag{2.132}$$

The Feynman graphs we gets are all possible "4-strand" 5-valent graphs, where a "4-strand graph" is a graph whose edges are collections of four strands, and whose vertices are the ones shown in Fig. 2.7.

Each strand of the propagator can be connected to a single strand in each of the five "openings" of the vertex. Orientations in the vertex and in the propagators should match (this can always be achieved by changing a representation to its conjugate). Each strand of the 4-strand graph, goes through several vertices and several propagators, and then closes to itself, forming a cycle. Note that a particular strand can go through a particular edge of the 4-strand graph more than once. Cycles get labeled by the simple representations of the indices. For each graph, the abstract set formed by the vertices, the edges, and the cycles forms a 2–complex, in which the faces are the cycles. The labeling of the cycles by simple representations of $SO(4)$ determines a coloring of the faces by spins. Thus, we obtain a colored 2-simplex, namely a spin foam.

In the case of the above constructed GFT/B, there are no labeling of edges, and we gets an additional contribution from the summing of the $\alpha' s$ indices of the Kroneker deltas $\delta^{\alpha\alpha'}$ around each cycle. This gives a factor $\dim_{N_f}$ for every face f. Thus, faces f are labeled by simple representations N_f and contribute a factor $\dim_{N_f}$. Vertices contribute a factor λ times $\mathcal{B}$, where $\mathcal{B}$ is the Barrett-Crane symbol, which depends on the ten simple representations of the faces adjacent to the vertex.

$$Z_{BCB}[\Gamma] = \sum_{\{N_f\}} \;\; \prod_f \dim_{N_f} \;\; \prod_v A(\vec{N}_v, i_{BC}). \tag{2.133}$$

This is precisely the Barrett-Crane amplitude for a triangulation Δ whose dual 2-skeleton is Γ.

2.9.3 GFT/C[113]

This second model consists in working with gauge invariant fields $P_G\phi$, and impose the combination of projectors $P_G P_H$ in the interaction term of the action only.

The action is then:

$$\begin{aligned} S[\phi] =& \frac{1}{2}\int dg_1...dg_4\ [P_G\phi(g_1,g_2,g_3,g_4)]^2 + \\ &+\frac{\lambda}{5!}\int dg_1...dg_{10}\ [P_GP_HP_G\phi(g_1,g_2,g_3,g_4)]\ [P_GP_HP_G\phi(g_4,g_5,g_6,g_7)] \\ &[P_GP_HP_G\phi(g_7,g_3,g_8,g_9)]\ [P_GP_HP_G\phi(g_9,g_6,g_2,g_{10})]\ [P_GP_HP_G\phi(g_{10},g_8,g_5,g_1)] \end{aligned} . \tag{2.134}$$

The kinetic term in "coordinate space" is given by:

$$\mathcal{K}(g_i,\tilde{g}_i) = \sum_\sigma \int d\gamma \prod_i \delta\left(g_i\,\gamma\,\tilde{g}^{-1}_{\sigma(i)}\right), \tag{2.135}$$

which corresponds to a projector in to the space of gauge invariant fields, and is such that it is equal to its inverse, so that the propagator is $\mathcal{K}$ itself. In the formula above the product is over the four arguments of the field and the sum is over the possible even permutations of them.

The vertex operator is:

$$\mathcal{V}(g_{ij}) = \frac{1}{5!}\int d\beta_i d\tilde{\beta}_i dh_{ij} \prod_{i<j}\delta\left(g^{-1}_{ji}\tilde{\beta}_i h_{ij}\beta_i^{-1}\beta_j h_{ji}\tilde{\beta}_j^{-1}g_{ij}\right) \tag{2.136}$$

where β and $\tilde{\beta}$ are $SO(4)$ integration variables, and $h_{ij}\in SO(3)$. Again the product is over the arguments of the fields entering in the interaction term.

The amplitude for a Feynman graph is then given by a product of interaction terms $\mathcal{V}(g_{ij})$, one for each vertex of the graph, connected by propagators $\mathcal{P}(g_i,\tilde{g}_i)=\mathcal{K}$, one for each edge of the graph, with an integral over the original group elements common to both kinetic and potential terms. Doing this for a generic Feynman graph, using the 2–complex corresponding to it, and recalling the correspondence between this and a simplicial complex, it is easy to check that this model gives a slightly different amplitude for the Barrett-Crane model BCB.

Let us look at the the momentum formulation of the model

We expand as before $\phi(g)$ in modes in terms of matrices $D^J_{\alpha\beta}(g)$ of the irreducible representations J of $SO(4)$; now however the basic fields are $P_G\phi(g_1,g_2,g_3,g_4)$ that become

$$\begin{aligned} P_G\phi(g_1,g_2,g_3,g_4) &= \int_{SO(4)} dg\ \phi(g_1g,g_2g,g_3g,g_4g) = \\ &= \sum_{J_1,J_2,J_3,J_4}\phi^{J_1\cdots J_4}_{\alpha_1\beta_1\cdots\alpha_4\beta_4}\int_{SO(4)} dg\ D^{J_1}_{\alpha_1\beta_1}(g_1g)...D^{J_4}_{\alpha_4\beta_4}(g_4g) = \\ &= \sum_{J_1,J_2,J_3,J_4}\phi^{J_1\cdots J_4}_{\alpha_1\beta_1\cdots\alpha_4\beta_4}\,D^{J_1}_{\alpha_1\gamma_1}(g_1)...D^{J_4}_{\alpha_4\gamma_4}(g_4)\int_{SO(4)} dg\, D^{J_1}_{\gamma_1\beta_1}(g)...D^{J_4}_{\gamma_4\beta_4}(g) = \\ &= \sum_{J_1,J_2,J_3,J_4}\phi^{J_1\cdots J_4}_{\alpha_1\beta_1\cdots\alpha_4\beta_4}\,D^{J_1}_{\alpha_1\gamma_1}(g_1)...D^{J_4}_{\alpha_4\gamma_4}(g_4)\,C^{J_1J_2J_3J_4\Lambda}_{\gamma_1\gamma_2\gamma_3\gamma_4}\,C^{J_1J_2J_3J_4\Lambda}_{\beta_1\beta_2\beta_3\beta_4}. \end{aligned} \tag{2.137}$$

we proceed redefining the field components as:

$$\Phi^{J_1\cdots J_4\Lambda}_{\beta_1\cdots\beta_4} \equiv \frac{\phi^{J_1\cdots J_4}_{\alpha_1\beta_1\cdots\alpha_4\beta_4}\,C^{J_1\cdots J_4\Lambda}_{\alpha_1\cdots\alpha_4}}{(\Delta_{J_1}\cdots\Delta_{J_4})^{\frac{3}{2}}}, \tag{2.138}$$

where the C's are $SO(4)$ intertwiners (Appendix D) and Δ_J denotes the dimension of the irreducible representation of order J, and the factor $(\Delta_{J_1}\dots\Delta_{J_4})^{\frac{3}{2}}$ has been chosen to simplify the expression of the interaction vertex computed below. This choice of field yields the mode expansion:

$$P_G\phi(g_1,\dots,g_4) = \sum_{J_1\dots J_4,\Lambda} (\Delta_{J_1}\dots\Delta_{J_4})^{\frac{3}{2}}\ \Phi^{J_1\dots J_4,\Lambda}_{\alpha_1\dots\alpha_4}\ D^{J_1}_{\alpha_1\gamma_1}(g_1)\dots D^{J_4}_{\alpha_4\gamma_4}(g_4)\ C^{J_1\cdots J_4\Lambda}_{\gamma_1\cdots\gamma_4}. \quad (2.139)$$

Using (D.1) and the orthonormality of the intertwiners, the kinetic term in (2.134) becomes

$$\mathcal{K} = \sum_{J_1\dots J_4,\Lambda} \Phi^{J_1\dots J_4,\Lambda}_{\alpha_1\dots\alpha_4}\Phi^{J_1\dots J_4,\Lambda}_{\mu_1\dots\mu_4}(\Delta_{J_1}\dots\Delta_{J_4})^2\ \delta_{\alpha_1\mu_1}\dots\delta_{\alpha_4\mu_4}. \quad (2.140)$$

We can directly read the propagator of the theory from this expression

$$\mathcal{P}_{\alpha_1\mu_1\dots\alpha_4\mu_4} = \frac{\delta_{\alpha_1\mu_1}\dots\delta_{\alpha_4\mu_4}}{(\Delta_{J_1}\dots\Delta_{J_4})^2}. \quad (2.141)$$

In order to write the potential term we need to express $P_GP_HP_G\phi$ in terms of irreducible representations. Starting with (2.139) and using equations (D.8) and (D.9) we obtain

$$\begin{aligned} P_GP_HP_G\phi \ &= \sum_{J_1\dots J_4,\Lambda} (\Delta_{J_1}\dots\Delta_{J_4})^{\frac{3}{2}}\ \Phi^{\alpha_1\dots\alpha_4}_{J_1\dots J_4,\Lambda}\ D^{(J_1)\gamma_1}_{\alpha_1}(g_1)\dots D^{(J_4)\gamma_4}_{\alpha_4}(g_4) \quad (2.142)\\ &\sum_N C^{J_1\cdots J_4N}_{\gamma_1\cdots\gamma_4}C^{J_1\cdots J_4N}_{\beta_1\cdots\beta_4}\ w^{\beta_1}\dots w^{\beta_4}w^{\mu_1}\dots w^{\mu_4}C^{J_1\cdots J_4\Lambda}_{\mu_1\cdots\mu_4}. \end{aligned}$$

Applying equation (D.10) and (D.11) we obtain

$$P_GP_HP_G\phi \ = \sum_{N_1\dots N_4,\Lambda} \sqrt{\Delta_{N_1}\dots\Delta_{N_4}}\ \Phi^{\alpha_1\dots\alpha_4}_{N_1\dots N_4,\Lambda}D^{(N_1)\gamma_1}_{\alpha_1}(g_1)\dots D^{(N_4)\gamma_4}_{\alpha_4}(g_4)\ B_{\gamma_1\dots\gamma_4}, \quad (2.143)$$

where the sum is now over simple representations only, and $B_{\gamma_1\dots\gamma_4}$ denotes the Barrett-Crane intertwiner. Using the previous equation the potential term in (2.134) becomes

$$\frac{1}{5!}\sum_{N_1\dots N_{10}}\sum_{\Lambda_1\dots\Lambda_5} \Phi^{\alpha_1\alpha_2\alpha_3\alpha_4}_{N_1N_2N_3N_4,\Lambda_1}\Phi^{\alpha_4\alpha_5\alpha_6\alpha_7}_{N_4N_5N_6N_7,\Lambda_2}\Phi^{\alpha_7\alpha_3\alpha_8\alpha_9}_{N_7N_3N_8N_9,\Lambda_3}\Phi^{\alpha_9\alpha_6\alpha_2\alpha_{10}}_{N_9N_6N_2N_{10},\Lambda_4}\Phi^{\alpha_{10}\alpha_8\alpha_5\alpha_1}_{N_{10}N_8N_5N_1,\Lambda_5}\ \mathcal{B}_{N_1,\dots,N_{10}}, \quad (2.144)$$

where $\mathcal{B}_{N_1,\dots,N_{10}}$ corresponds to a $15j$-symbol constructed with Barrett-Crane intertwiners which corresponds to the Barrett-Crane vertex amplitude [70]. Explicitly,

$$\mathcal{B}_{N_1,\dots,N_{10}} := B^{N_1N_2N_3N_4}_{\alpha_1\alpha_2\alpha_3\alpha_4}\ B^{N_4N_5N_6N_7}_{\alpha_4\alpha_5\alpha_6\alpha_7}\ B^{N_7N_3N_8N_9}_{\alpha_7\alpha_3\alpha_8\alpha_9}\ B^{N_9N_6N_2N_{10}}_{\alpha_9\alpha_6\alpha_2\alpha_{10}}\ B^{N_{10}N_8N_5N_1}_{\alpha_{10}\alpha_8\alpha_5\alpha_1}. \quad (2.145)$$

Thus, the potential part of the action in the new model is given by (2.144) as in the Barrett-Crane model. Notice, however, that there is an extra sum over Λ in (2.144), absent in Barrett-Crane. The propagator (2.141) of the theory in momentum space is rescaled with respect to the Barrett-Crane propagator ($\mathcal{P}_{\alpha_1\mu_1\dots\alpha_4\mu_4} = (\Delta_1\dots\Delta_4)^{-2}\mathcal{P}^{(BC)}_{\alpha_1\mu_1\dots\alpha_4\mu_4}$). As a consequence of this rescaling, and of the extra sum over Λ, there is a non-trivial amplitude associated to edges in the spin foam. Each edge contributes to the amplitude as

$$A_e = \frac{\Delta_{N_1,\dots,N_4}}{(\Delta_{N_1}\dots\Delta_{N_4})^2}, \quad (2.146)$$

where N_1 to N_4 are the colors of the four faces meeting at the given edge, and $\Delta_{N_1,\ldots,N_4}$ is the dimension of the space of the interwiners between the representations $N_1,\ldots,N_4$. In conclusion, the amplitude of a Feynman diagram Γ is given by

$$Z_{BCC}(\Gamma) \;=\; \sum_N \prod_f \Delta_{N_f} \prod_e A_e \prod_v \mathcal{B}_{N_1\ldots N_{10}}. \tag{2.147}$$

where the sum is over simple representations N of $SO(4)$, and $\mathcal{B}$ denotes the Barrett-Crane vertex amplitude.

Equivalently, as every edge connects two vertices, we can absorb the edge amplitude in the vertex amplitude and write Z with a vertex amplitude

$$A_v = \frac{\prod_i \Delta^{1/2}_{N_{i1},\ldots,N_{i4}}}{(\Delta_{N_1}\ldots\Delta_{N_{10}})^2}\; \mathcal{B}_{N_1\ldots N_{10}}, \tag{2.148}$$

where $N_1\ldots N_{10}$ are the ten colors of the ten faces adjacent to the vertex v, and $N_{i1}\ldots N_{i4}, i = 1\ldots 5$ are the four colors of the four faces adjacent to i'th edge adjacent to the vertex v.

We close this section with a comment. Unlikely the Barrett-Crane model, in the mode expansion of the model presented here, the field depends on five representations (four external and one intertwiner), which can be seen precisely as the quantum numbers of a "first quantized" geometry of a tetrahedron.

Chapter 3

Graviton propagator in LQG

In this chapter we put together the lessons learned from the previous chapters and focus on the main subject of this thesis; the construction of the graviton propagator from LQG. It is an open problem in quantum gravity. The difficulty is that general covariance makes conventional n-point functions ill–defined in the absence of a background. A strategy for addressing this problem has been suggested in [14]; the idea is to study the boundary amplitude, namely the functional integral over a finite spacetime region, seen as a function of the boundary value of the field [115]. In conventional quantum field theory, this boundary amplitude is well–defined (see [116, 117]) and codes the physical information of the theory; so does in quantum gravity, but in a fully background–independent manner [118]. A generally covariant definition of n-point functions can then be based on the idea that the distance between physical points –arguments of the n-point function– is determined by the state of the gravitational field on the boundary of the spacetime region considered. This strategy was first implemented in the letter [15], where some components of the graviton propagator were computed to the first order in the expansion parameter λ of the GFT associated to the dynamics. For an implementation of these ideas in 3d, see [119, 120].

Here we present the extension of the calculation presented in [15], appeared in [16]. In this paper the authors have calculated the diagonal components of the (connected) two-point function, starting from full non-perturbative quantum general relativity, in an appropriate large distance limit. They found that only a few components of the boundary states contribute to low order on λ. This reduces the model to a 4d generalization of the "nutshell" 3d model studied in [121]. The associated boundary amplitude can be read as the creation, interaction and annihilation of few "atoms of space", in the sense in which Feynman diagrams in conventional quantum field theory expansion can be viewed as creation, interaction and annihilation of particles. Using a natural gaussian form of the vacuum state, peaked on the intrinsic *as well as the* extrinsic geometry of the boundary, they derived the expression for the diagonal components of the graviton propagator.

At large distance, this agrees with the conventional graviton propagator!!

In the works [14, 15, 16] the authors have showed that a technique for computing particle scattering amplitudes in background–independent theories can be developed. (The viability of the notion of particle in a finite region is discussed in [122]. For the general relativistic formulation of quantum mechanics underlying this calculation, see [2]. On the relation between graviton propagator and 3-geometries transition amplitudes in the conventional perturbative

expansion, see [123].)

We consider riemaniann general relativity without matter and we use basic LQG results presented in Chapter 1 defining the dynamics by means of the spinfoam technique presented in Chapter 2. The specific model we use is the theory GFT/B, in the terminology of [2], defined using group field theory methods. On the definition of spin network states in group field theory formulation of spin foam models, see [124] and [2]. The result extends immediately also to the theory GFT/C. The first [112] (see Section (2.9.1)) is favored by a number of arguments recently put forward [125, 126]. The second [113],[114] (Section 2.9.3) is characterized by particularly good finiteness properties [127].

The physical correctness of these theories has been questioned because in the large distance limit their interaction vertex ($10j$ symbol, or Barrett-Crane vertex amplitude [70]) has been shown to include –beside the "good" term approximating the exponential of the Einstein-Hilbert action [99]– also two "bad" terms: an exponential with opposite sign, giving the cosine of Regge action [99] (analogous to the cosine in the Ponzano–Regge model) and a dominant term that depends on the existence of degenerate four-simplices [100, 101, 102]. We show here that only the "good" term contributes to the propagator. The others are suppressed by the rapidly oscillating phase in the vacuum state that peaks the state on its correct extrinsic geometry. Thus, the physical state selects the "forward" propagating [128] component of the transition amplitude. This phenomenon was anticipated in [129].

3.1 The generally covariant description of 2-points functions

In this section we describe a generally covariant definition of n–point functions; it can be based on the idea that the distance between physical points —arguments of the n–point function— is determined by the state of the gravitational field on the boundary of the space-time region considered.

We begin by illustrating the quantities and some techniques that will be used in Quantum Gravity within a simple context.

3.1.1 A single degree of freedom

Consider the two-point function of a single harmonic oscillator with mass m and angular frequency ω. This is given by

$$G_0(t_1, t_2) = \langle 0|x(t_1)x(t_2)|0\rangle = \langle 0|x\ e^{-\frac{i}{\hbar}H(t_1-t_2)}\ x|0\rangle \tag{3.1}$$

where $|0\rangle$ is the vacuum state, $x(t)$ is the Heisenberg position operator at time t and H the hamiltonian. We write a subscript $_0$ in $G_0(t_1, t_2)$ to remind us that this is an expectation value computed on the vacuum state. Later we will also consider similar expectation values computed on other classes of states, as for instance in

$$G_\psi(t_1, t_2) = \langle \psi|x(t_1)x(t_2)|\psi\rangle. \tag{3.2}$$

Elementary creation and annihilation operator techniques give

$$G_0(t_1, t_2) = \frac{\hbar}{2m\omega}\ e^{-\frac{3}{2}i\omega(t_1-t_2)}. \tag{3.3}$$

In the Schrödinger picture (Appendix K.2), the r.h.s. of (3.1) reads

$$G_0(t_1,t_2) = \int dx_1 dx_2 \; \overline{\psi_0(x_1)} \; x_1 \; W[x_1,x_2;t_1,t_2] \; x_2 \; \psi_0(x_2) \tag{3.4}$$

where $\psi_0(x) = \langle x|0\rangle$ is the vacuum state and $W[x_1,x_2;t_1,t_2]$ is the propagator, namely the matrix element of the evolution operator

$$W[x_1,x_2;t_1,t_2] = \langle x_1|e^{-iH(t_1-t_2)}|x_2\rangle. \tag{3.5}$$

Recalling that

$$\psi_0(x) = \sqrt[4]{\frac{m\omega}{\pi\hbar}}\, e^{-\frac{m\omega}{2\hbar}x^2} \tag{3.6}$$

and (see for instance [2], page 168 and *errata*)

$$W(x_1,x_2;T) = \sqrt{\frac{m\omega}{2\pi i\hbar \sin\omega T}}\; e^{i\frac{m\omega}{2\hbar}\frac{\left(x_1^2+x_2^2\right)\cos\omega T - 2x_1x_2}{\sin\omega T}} \tag{3.7}$$

are two gaussian expressions, we obtain the two-point function (3.1) as the second momentum of a gaussian

$$G_0(t_1,t_2) = \frac{m\omega}{\pi\hbar}\sqrt{\frac{1}{2i\sin\omega T}} \int dx_1 dx_2 \; x_1x_2 \; e^{i\frac{m\omega}{2\hbar}\frac{\left(x_1^2+x_2^2\right)\cos\omega T - 2x_1x_2}{\sin\omega T}} \; e^{-\frac{m\omega}{\hbar}x^2}, \tag{3.8}$$

where the gaussian is the product of a "bulk" gaussian term and a "boundary" gaussian term. The evaluation of the integral in (3.8) is a simple matter of gaussian integrations and gives

$$G_0(t_1,t_2) = \frac{m\omega}{\pi\hbar}\sqrt{\frac{1}{2i\sin\omega(t_1-t_2)}}\,\frac{2\pi}{\sqrt{\det A}}A^{-1}_{12} \tag{3.9}$$

in terms of the inverse of the covariance matrix of the gaussian

$$A = \frac{m\omega}{\hbar}\begin{pmatrix} 1-i\frac{\cos\omega(t_1-t_2)}{\sin\omega(t_1-t_2)} & \frac{i}{\sin\omega(t_1-t_2)} \\ \frac{i}{\sin\omega(t_1-t_2)} & 1-i\frac{\cos(t_1-t_2)}{\sin\omega(t_1-t_2)} \end{pmatrix} = \frac{-im\omega}{\hbar\sin\omega(t_1-t_2)}\begin{pmatrix} e^{i\omega(t_1-t_2)} & -1 \\ -1 & e^{i\omega(t_1-t_2)} \end{pmatrix}. \tag{3.10}$$

The matrix A is easy to invert and (3.9) gives precisely (3.3).

This is kind of general feature of these functions:

In the two point function both dynamics and vacuum state are gaussian expressions, that mix in a new one, giving the propagator as the second momentum of this new gaussian.

3.1.2 Harmonic oscillator with the path integral

The two-point function (3.1) can also be written as the (analytic continuation of the euclidean version of) the functional integral

$$G_0(t_1,t_2) = \int Dx(t) \; x(t_1)x(t_2) \; e^{i\int_{-\infty}^{\infty} L(x,dx/dt)}. \tag{3.11}$$

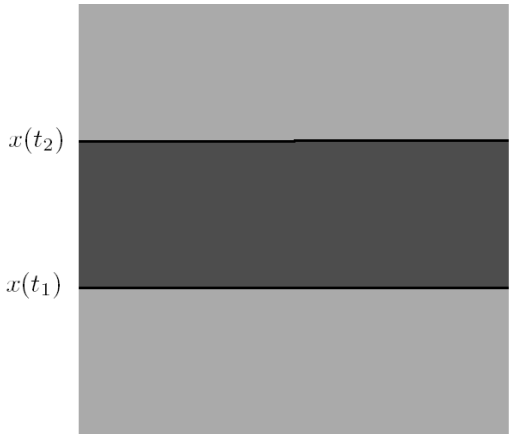


Figure 3.1: The infinite integration variables are divided in those inside and outside the region $(x(t_1), x(t_2))$. The functional integral restricted to the interior region (dark) is the bulk term and gives the propagation kernel. The functional integral restricted to the outside region (clear) is the boundary term and gives the vacuum state.

where L is the harmonic oscillator lagrangian, and the measure is appropriately normalized. Let us break the (infinite number of) integration variables $x(t)$ in various groups: those where t is less, equal or larger than, respectively, t_1 and t_2 (like in Figure 3.1) . Using this, and writing the integration variable $x(t_1)$ as x_1 and the integration variable $x(t_2)$ as x_2, we can rewrite (3.11) as

$$G_0(t_1, t_2) = \int dx_1 dx_2 \;\; \overline{\psi_0(x_1)}\; x_1 \; W[x_1, x_2; t_1, t_2] \; x_2 \; \psi_0(x_2) \tag{3.12}$$

where

$$W[x_1, x_2; t_1, t_2] = \int_{x(t_2)=x_2}^{x(t_1)=x_1} Dx(t) \; e^{i \int_{t_2}^{t_1} L(x, dx/dt)} \tag{3.13}$$

is the functional integral restricted to the open interval (t_1, t_2) integrated over the paths that start at x_2 and end at x_1; while

$$\psi_0(x) = \int_{x(-\infty)=0}^{x(t_1)=x} Dx(t) \; e^{i \int_{-\infty}^{t_1} L(x, dx/dt)} \tag{3.14}$$

is the functional integral restricted to the interval $(-\infty, t_1)$. As well known, in the euclidean theory this gives the vacuum state. Thus, we recover again the form (3.4) of the two-point function, with the additional information that the "bulk" propagator term can be viewed as the result of the functional integral in the interior of the (t_1, t_2) interval, while the "boundary" term can be viewed as the result of the functional integral in the exterior. In this language the specification of the particular state $|0\rangle$ on which the expectation value of $x(t_1)x(t_2)$ is computed, is coded in the boundary behavior of the functional integration variable at infinity: $x(t) \to 0$ for $t \to \pm\infty$.

The normalization of the functional measure in (3.11) is determined by

$$1 = \int Dx(t) \; e^{i \int_{-\infty}^{\infty} L(x, dx/dt)}. \tag{3.15}$$

Breaking this functional integral in the same manner as the above one gives

$$1 = \int dx_1 dx_2 \;\; \overline{\psi_0(x_1)}\; W[x_1, x_2; t_1, t_2]\; \psi_0(x_2) \tag{3.16}$$

or equivalently

$$1 = \langle 0 | e^{-\frac{i}{\hbar} H(t_1 - t_2)} | 0 \rangle. \tag{3.17}$$

Let us comment on the interpretation of (3.11) and (3.16), since analogues of these equation will play a major role below. Observe that (3.11) can be written in the form

$$G_0(t_1, t_2) = \langle W_{t_1,t_2} \mid \hat{x}_1 \hat{x}_2 \Psi_0 \rangle, \tag{3.18}$$

in terms of states and operators living in the Hilbert space $\mathcal{K}_{t_1,t_2} = \mathcal{H}^*_{t_1} \times \mathcal{H}_{t_2}$ (the tensor product of the space of states at time t_1 and the space of states at time t_2) formed by functions $\psi(x_1, x_2)$. (See Section 5.1.4 of [2] for details on $\mathcal{K}_{t_1,t_2}$.) Using the relativistic formulation of quantum mechanics developed in [2], this expression can be directly re-interpreted as follows.

- (i) The "boundary state" $\Psi_0(x_1, x_2) = \overline{\psi_0(x_1)}\psi_0(x_2)$ represents the joint boundary configuration of the system at the two times t_1 and t_2, if no excitation of the oscillator is present; it describes the joint outcome of a measurement at t_1 and a measurement at t_2, both of them detecting no excitations.

- (ii) The two operators $\hat{x}_1$ and $\hat{x}_2$ create a ("incoming") excitation at $t = t_2$ and a ("outgoing") excitation at $t = t_1$; thus the state $\hat{x}_1 \hat{x}_2 \Psi_0$ can be interpreted as a boundary state representing the joint outcome of a measurement at t_1 and a measurement at t_2, both of them detecting a single excitation.

- (iii) The bra $W_{t_1,t_2}(x_1, x_2) = W[x_1, x_2; t_1, t_2]$ is the linear functional coding the dynamics, whose action on the two-excitation state associates it an amplitude, which can be compared with other similar amplitudes.

For instance, observe that

$$\langle W_{t_1,t_2} \mid \hat{x}_2 \; \Psi_{t_1,t_2} \rangle = 0; \tag{3.19}$$

that is, the probability amplitude of measuring a single excitation at t_2 and no excitation at t_1 is zero. Finally, the normalization condition (3.16) reads

$$1 = \langle W_{t_1,t_2} \; | \Psi_0 \rangle; \tag{3.20}$$

which requires that the boundary state Ψ_0 is a solution of the dynamics, in the sense that its projection on t_1 is precisely the time evolution of its projection to t_2. As we shall see below, this condition generalizes to the case of interest for general relativity. We call (3.20) the "Wheeler-deWitt" (WdW) condition. This condition satisfied by the boundary state should not be confused with the normalization condition,

$$1 = \langle \Psi_0 | \Psi_0 \rangle, \tag{3.21}$$

which is also true, and which follows immediately from the fact that $|0\rangle$ is normalized in $\mathcal{H}_t$.

In general, given a state $\Psi \in \mathcal{K}_{t_1,t_2}$, the equations

$$\langle W_{t_1,t_2} \, |\Psi\rangle = 1; \tag{3.22}$$

and

$$\langle \Psi|\Psi\rangle = 1, \tag{3.23}$$

are equivalent to the full quantum dynamics, in the following sense. If the state is of the form $\Psi = \bar{\psi}_{\rm f} \otimes \psi_{\rm i}$, then (3.22) and (3.23) imply that

$$\psi_{\rm f} = e^{-iHt} \, \psi_{\rm i}. \tag{3.24}$$

The boundary state as a coherent state

In ordinary QM a coherent (semiclassical) state

$$\psi_{\mathbf{q}}(x) \sim e^{-\frac{\alpha}{2}(x-q)^2 + \frac{i}{\hbar}px} \tag{3.25}$$

where $\mathbf{q} = (q, p)$ is peaked on mean values q and p of position and momentum of the classical trajectory. In particular, the vacuum state of the harmonic oscillator is the coherent state peaked on the values $q = 0$ and $p = 0$, with $\alpha = m\omega/\hbar$. Thus we can write $\psi_0 = \psi_{(q=0,p=0)}$. In the same manner, the boundary state $\Psi_0 = \overline{\psi_0(x_1)}\psi_0(x_2)$ can be viewed as a coherent *boundary* state, associated with the values $q_1 = 0$ and $p_1 = 0$ at t_1 and $q_2 = 0$ and $p_2 = 0$ at t_2.

We can write a *generic* coherent boundary state as

$$\Psi_{q_1,p_1,q_2,p_2}(x_1, x_2) = \overline{\psi_{(q_1,p_1)}(x_1)} \, \psi_{(q_2,p_2)}(x_2). \tag{3.26}$$

A special case of these coherent boundary states is obtained when (q_1, p_1) are the classical evolution at time $t_1 - t_2$ of the initial conditions (q_2, p_2). That is, when in the $t_1 - t_2$ interval there exists a solution $q(t), p(t)$ of the classical equations of motion precisely bounded by q_1, p_1, q_2, p_2, namely such that $q_1 = q(t_1), p_1 = p(t_1)$ and $q_2 = q(t_2), p_2 = p(t_2)$. If such a classical solution exists, we say that the quadruplet (q_1, p_1, q_2, p_2) is *physical*. As well known the harmonic oscillator dynamics gives in this case $e^{-iH(t_1-t_2)}\Psi_{q_2,p_2} = \Psi_{q_1,p_1}$, or

$$\langle W_{t_1,t_2} \, |\Psi_{q_1,p_1,q_2,p_2}\rangle = 1. \tag{3.27}$$

That is, it satisfies the WdW condition (3.22). In this case, we denote the semiclassical boundary state a *physical* semiclassical boundary states.

The vacuum boundary state Ψ_0 is a particular case of this: it is the physical semiclassical boundary state determined by the classical solution $q(t) = 0$ of the equations of motion, which is the one with minimal energy. Given a physical boundary state, we can consider a two-point function describing the propagation of a quantum excitation "over" the semiclassical trajectory $q(t), p(t)$ as

$$G_{q_1,p_1,q_2,p_2}(t_1, t_2) = \langle \psi_{(q_1,p_1)}|x(t_1)x(t_2)|\psi_{(q_2,p_2)}\rangle = \langle W_{t_1,t_2} \mid \hat{x}_1\hat{x}_2 \, \Psi_{q_1,p_1,q_2,p_2}\rangle. \tag{3.28}$$

This quantity will pay a considerable role below. Indeed, the main idea here is to compute quantum–gravity n-point functions using states that describe the boundary value of the gravitatonal field on given boundary surfaces.

There is an interesting phenomenon regarding the *phases* of the boundary state $\Psi_{q_1,p_1,q_2,p_2}(x_1,x_2)$ and of the propagator $W_{t_1,t_2}(x_1,x_2)$ that should be noticed. If p_1 and p_2 are different from zero, they give rise to a phase factor $e^{-\frac{i}{\hbar}(p_1x_1-p_2x_2)}$, in the boundary state. In turn, it is easy to see that $W_{t_1,t_2}(x_1,x_2)$ contains precisely the inverse of this same phase factor, when expanded around (q_1,q_2). In fact, the phase of the propagator is the classical Hamilton function $S_{t_1,t_2}(x_1,x_2)$ (the value of the action, as a function of the boundary values [2]). Expanding the Hamilton function around q_1 and q_2 gives to first order

$$S_{t_1,t_2}(x_1,x_2) = S_{t_1,t_2}(q_1,q_2) + \frac{\partial S}{\partial x_1}(x_1-q_1) + \frac{\partial S}{\partial x_2}(x_2-q_2), \tag{3.29}$$

but

$$\frac{\partial S}{\partial x_1} = p_1 \quad \text{and} \quad \frac{\partial S}{\partial x_2} = -p_2. \tag{3.30}$$

Giving a phase factor $e^{\frac{i}{\hbar}(p_1x_1-ip_2x_2)}$, which is precisely the inverse of the one in the boundary state. In the Schrödinger representation of (3.28), the gaussian factor in the boundary state peaks the integration around (q_1,q_2); in this region, we have that *the phase of the boundary state is determined by the classical value of the momentum, and is cancelled by a corresponding phase factor in the propagator* W. In particular, the rapidly oscillating phase in the boundary state fails to suppress the integral precisely because it is compensated by a corresponding rapidly oscillating phase in W. This, of course, is nothing that the realization, in this language, of the well–known emergence of classical trajectories from the constructive coherence of the quantum amplitudes. This phenomenon, noted in [15] in the context of quantum gravity, plays a major role below.

3.1.3 Field theory

Let us now go over to field theory. The two-point function (or particle propagator) is defined by the (analytic continuation of the euclidean version of the) path integral ($\hbar = 1$ from now on)

$$G_0(x,y) = \langle 0|\phi(x)\phi(y)|0\rangle = \langle 0|\phi(\vec{x})\ e^{-iH(x_0-y_0)}\ \phi(\vec{y})|0\rangle = \int D\phi(x)\ \phi(x)\phi(y)\ e^{iS[\phi]}, \tag{3.31}$$

where the normalization of the measure is determined by

$$1 = \int D\phi(x)\ \ e^{iS[\phi]} \tag{3.32}$$

and the $_0$ subscript reminds that these are expectation values of products of field operators in the particular state $|0\rangle$. These equations generalize (3.11) and (3.15) to field theory.[1] As

[1] A well-known source of confusion is of course given by the fact that in the case of a free particle the propagator (3.5) coincides with the 2-point function of the free field theory.

before, we can break the integration variables of the path integral in various groups. For instance, in the values of the field in the five spacetime–regions identified by t being less, equal or larger than, respectively, x_0 and y_0. This gives a Schrödinger representation of the two-point function of the form (see Appendix K.2)

$$G_0(x,y) = \int D\varphi_1 D\varphi_2 \; \overline{\psi_0(\varphi_1)} \; \varphi_1(\vec{x}) \; W[\varphi_1,\varphi_2;(x_0-y_0)] \; \varphi_2(\vec{y}) \; \psi_0(\varphi_2). \tag{3.33}$$

where φ_1 is the three-dimensional field at time t_1, and φ_2 is the three-dimensional field at time t_2. For a free field, the field propagator (or propagation kernel)

$$W(\varphi_1,\varphi_2;T) = \langle\varphi_1|e^{-iHT}|\varphi_2\rangle. \tag{3.34}$$

and the boundary vacuum state are gaussian expression in the boundary field $\varphi = (\varphi_1,\varphi_2)$. These expressions, and the functional integral (3.33), are explicitly computed in [117]. In a free theory, the boundary vacuum state can be written as a physical semiclassical state peaked on vanishing field and momentum π, as in (3.26):

$$\Psi_0(\varphi_1,\varphi_2) \equiv \Psi_{\varphi_1=0,\pi_1=0,\varphi_2=0,\pi_2=0}(\varphi_1,\varphi_2) = \overline{\psi_0(\varphi_1)} \; \psi_0(\varphi_2). \tag{3.35}$$

Notice that the momentum $\pi = \frac{d\varphi_1}{dt}$ is the derivative of the classical field normal to Σ.

More interesting for what follows, we can choose a compact finite region $\mathcal{R}$ in spacetime, bounded by a closed 3d surface Σ, such that the two points x and y lie on Σ. Then we can separate the integration variables in (3.31) into those inside $\mathcal{R}$, those on Σ and those outside $\mathcal{R}$, and thus write the two-point function (3.31) in the form

$$G_0(x,y) = \int D\varphi \; \varphi(x) \; \varphi(y) \; W[\varphi;\Sigma] \; \Psi_0(\varphi), \tag{3.36}$$

where φ is the field on Σ,

$$W[\varphi;\Sigma] \; = \int_{\partial\phi=\varphi} D\phi_R \; e^{-iS_{\mathcal{R}}[\phi_R]} \tag{3.37}$$

is the functional integral restricted to the region $\mathcal{R}$, and integrated over the interior fields ϕ_R bounded by the given boundary field φ. The boundary state $\Psi_0(\varphi)$ is given by the integral restricted to the outside region, $\overline{\mathcal{R}}$ (see Figure 3.2). The boundary conditions on the functional integration variable

$$\phi_{\overline{\mathcal{R}}}(x) \to 0, \qquad \text{for} \qquad |x| \to \infty \tag{3.38}$$

determine the vacuum state. In a free theory, this is still a gaussian expression in φ, but the covariance matrix is non–trivial and is determined by the shape of Σ. The state Ψ_0 can nevertheless be still viewed as a semiclassical boundary state associated to the compact boundary, peaked on the value $\varphi = 0$ of the field and the value $\pi = 0$ of a (generalized) momentum (the derivative of the field normal to the surface) [2]. Equation (3.36) will be our main tool in the following.

In analogy with (3.18), equation (3.36) can be written in the form

$$G_0(x,y) = \langle W_\Sigma \mid \hat{\varphi}(x) \, \hat{\varphi}(y) \; \Psi_0\rangle. \tag{3.39}$$

in terms of states and operators living in a boundary Hilbert space $\mathcal{K}_\Sigma$ associated with the 3d surface Σ. In terms of the relativistic formulation of quantum mechanics developed in [2], this expression can be interpreted as follows.

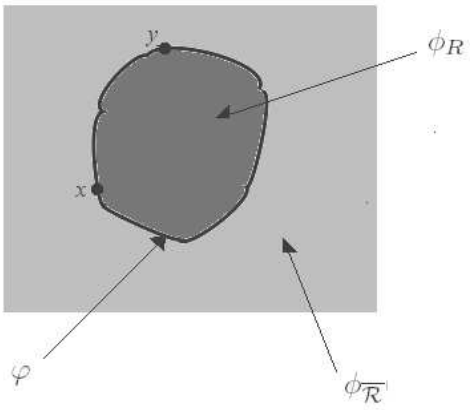


Figure 3.2: We take a closed 3d surface Σ such that x and y, the arguments of the 2 point function lie on this surface. In the picture the functional integral restricted to the interior fields ϕ_R (dark region), gives the propagation kernel. The integral restricted to the outside fields $\phi_{\mathcal{R}}$ gives the vacuum state.The field φ on Σ creates in the points x and y the ingoing and outgoing exctitations.

- (i) The "boundary state" Ψ_0 represents the boundary configuration of a quantum field on a surface Σ, when no particles are present; it represents the joint outcome of measurements on the entire surface Σ, showing no presence of particles.
- (ii) The two operators $\hat{\varphi}(x)$ $\hat{\varphi}(y)$ create a ("incoming") particle at y and a ("outgoing") particle at x; so that the boundary state $\varphi(x)\,\varphi(y)\,\Psi_0$ represents the joint outcome of measurements on Σ, detecting a ("incoming") particle at y and a ("outgoing") particle at x.
- (iii) Finally, the bra W_Σ is the linear functional coding the dynamics, whose action on the two-particle boundary state associates it an amplitude, which can be compared with other analogous amplitudes. The normalization condition for the measure, equation (3.32), becomes the WdW condition

$$1 = \langle W_\Sigma \,|\Psi_0\rangle, \tag{3.40}$$

  which singles out the physical boundary states.

Finally, as before, let $\mathbf{q} = (q, p)$ be a given couple of boundary values of the field φ and its generalized momentum on Σ. If there exists a classical solution ϕ of the equations of motion whose restriction to Σ is q and whose normal derivative to Σ is p, then we say that $\mathbf{q} = (q, p)$ are *physical* boundary data. Let $\Psi_{\mathbf{q}}$ be a boundary state in $\mathcal{K}_\Sigma$ peaked on these values: schematically

$$\Psi_{\mathbf{q}}(\varphi) \sim e^{-\int(\varphi - q)^2 + i\int p\phi}. \tag{3.41}$$

If $\mathbf{q} = (q, p)$ are physical boundary data, we say that $\Psi_{\mathbf{q}}$ is a *physical* semiclassical state. In this case, we can consider the two-point function

$$G_{\mathbf{q}}(x, y) = \langle W_\Sigma \mid \hat{\varphi}(x)\, \hat{\varphi}(y)\ \Psi_{\mathbf{q}} \rangle \tag{3.42}$$

describing the propagation of a quantum, from y to x, over the classical field configuration ϕ giving the boundary data $\mathbf{q} = (q, p)$. In the Schrödinger representation of this expression, there is a cancellation of the phase of the boundary state $\Psi_{\mathbf{q}}$ with the phase of the propagation kernel W_Σ, analogous to the one we have seen in the case of a single degree of freedom.

3.1.4 Quantum gravity

Let us formally write (3.36) for pure general relativity, ignoring for the moment problems such as the definition of the integration measure, or ultraviolet divergences. Given a surface Σ, we can choose a generalized temporal gauge in which the degrees of freedom of gravity are expressed by the 3-metric γ induced on Σ, with components $\gamma_{ab}(x)$ $a, b = 1, 2, 3$. That is, if the surface is locally given by $x^4 = 0$, we gauge fix the 4d gravitational metric field $g_{\mu\nu}(x)$ by $g_{44} = 1, g_{40} = 0$, and $\gamma_{ab} = g_{ab}$. Then the graviton two-point function (3.36) reads in this gauge

$$G_0^{abcd}(x, y) = \int [D\gamma]\ h^{ab}(x)\ h^{cd}(y)\ W[\gamma; \Sigma]\ \Psi_0(\gamma), \tag{3.43}$$

where $h^{ab}(x) = \gamma^{ab}(x) - \delta^{ab}$. As observed for instance in [118], if we assume that $W[\gamma; \Sigma]$ is given by a functional integration on the bulk, as in (3.37), where measure and action are generally covariant, then we have immediately that $W[\gamma; \Sigma]$ is independent from (smooth deformations of) Σ. Hence, at fixed topology of Σ (say, the surface of a 3-sphere), we have $W[\gamma; \Sigma] = W[\gamma]$, that is

$$G_0^{abcd}(x, y) = \int [D\gamma]\ h^{ab}(x)\ h^{cd}(y)\ W[\gamma]\ \Psi_0(\gamma). \tag{3.44}$$

What is the interpretation of the boundary state $\Psi_0(\gamma)$ in a general covariant theory? In the case of the harmonic oscillator, the vacuum state $|0\rangle$ is the state that minimizes the energy. In the case of a free theory on a background, in addition, it is the sole Poincaré invariant state. In both cases the vacuum state can also be obtained from a functional integral by fixing the behavior of the fields at infinity. But in background–independent quantum gravity, there is no energy to minimize and no global Poincaré invariance. Furthermore, there is no background metric with respect to which to demand the gravitational field to vanish at infinity. In fact, it is well known that the unicity and the very definition of the vacuum state is highly problematic in nonperturbative quantum gravity (see for instance [2]), a phenomenon that begins to manifest itself already in QFT on a curved background. Thus, in quantum gravity there is a multiplicity of possible states that we can consider as boundary states, instead of a single preferred one.

Linearized quantum gravity gives us a crucial hint, and provides us with a straightforward way to *interpret* semiclassical boundary states. Indeed, consider linearized quantum gravity, namely the well–defined theory of a noninteracting spin–2 graviton field $h_{\mu\nu}(x)$ on a flat spacetime with background metric $g^0_{\mu\nu}$. This theory has a preferred vacuum state $|0\rangle$. Now,

choose a boundary surface Σ and denote $\mathbf{q} = (q, p)$ its three-geometry, formed by the 3-metric q_{ab} and extrinsic curvature field p^{ab}, induced on Σ by the flat background metric of spacetime. The vacuum state defines a gaussian boundary state on Σ, peaked around $h = 0$. We can schematically write this state as $\Psi_\Sigma(h) \sim e^{-\int h^2}$. (In the conventional case in which Σ is formed by two parallel hyper-planes, the explicit form of this state is given in [123].)

Now, on Σ there are two metrics: the metric q induced by the background spacetime metric, and the metric $\gamma = q + h$, induced by the true physical metric $g_{\mu\nu} = g^0_{\mu\nu} + h_{\mu\nu}$, which is the sum of the background metric and the dynamical linearized gravitational field. Therefore the vacuum functional $\Psi_0(h)$ defines a functional $\Psi_{\mathbf{q}}(\gamma)$ *of the physical metric* γ *of* Σ as follows

$$\Psi_{\mathbf{q}}(\gamma) = \Psi_{\mathbf{q}}(q + h) \equiv \Psi_0(h). \tag{3.45}$$

Schematically

$$\Psi_{\mathbf{q}}(\gamma) = \Psi_0(h) = \Psi_0(\gamma - q) \sim e^{-\int (\gamma - q)^2}. \tag{3.46}$$

A bit more precisely, as was pointed out in [15], we must also take into account a phase term, generated by the fact that the normal derivative of the induced metric does not vanish (q changes if we deform Σ). This gives, again very schematically

$$\Psi_{\mathbf{q}}(\gamma) \sim e^{-\int (\gamma - q)^2 + i \int p\gamma} \tag{3.47}$$

as in (3.41). Recall indeed that in general relativity the intrinsic and extrinsic geometry play the role of canonical variable and conjugate variable. As pointed out in [15], a semiclassical boundary state must be peaked on both quantities, as coherent states of the harmonic oscillator are equally peaked on q and p. The functional $\Psi_{\mathbf{q}}$ of the metric can immediately be interpreted as a boundary state of quantum gravity, as determined by the linearized theory.

Observe that *it depends on the background geometry of* Σ*, because* q *and* p *do*: the form of this state is determined by the location of Σ with respect to the background metric of space. Therefore (when seen as a function of the true metric γ) there are different possible boundary states in the linearized theory, depending on where is the boundary surface. Equivalently, there are different boundary states depending on what is the mean boundary geometry q on Σ.

Now, in full quantum gravity we must expect, accordingly, to have many possible distinct semiclassical boundary states $\Psi_{\mathbf{q}}(\gamma)$ that are peaked on distinct 3-geometries $\mathbf{q} = (q, p)$. In the background-independent theory they cannot be *anymore* interpreted as determined by the location of Σ with respect to the background (because there is no background!). But they can *still* be interpreted as determined by the mean boundary geometry $\mathbf{q}$ on Σ.

Their interpretation is therefore immediate: *they represent coherent semiclassical states of the boundary geometry.* The multiplicity of the possible locations of Σ with respect to the background geometry in the background-dependent theory, translates into a multiplicity of possible coherent boundary states in the background-independent formalism.

In fact, this conclusion follows immediately from the core physical assumption of general relativity: the identification of the gravitational field with the spacetime metric. A coherent boundary state of the gravitational field is peaked, in particular, on a given classical value of the metric. In the background-dependent picture, this can be interpreted as information about the location of Σ in spacetime. In a background-independent picture, there is no location in

spacetime: the geometrical properties of anything is solely determined by the local value of the gravitational field. In a background-independent theory, the dependence on a boundary geometry is not in the location of Σ with respect to a background geometry, but rather in the boundary state of the gravitation field on the surface Σ itself.

Having understood this, it is clear that the two-point function of a background-independent theory can be *defined* as a function of the mean boundary geometry, instead of a function of the background metric. If $\mathbf{q} = (q, p)$ is a given geometry of a closed surface Σ with the topology of a 3-sphere, and $\Psi_{\mathbf{q}}$ is a coherent state peaked on this geometry, consider the expression

$$G^{abcd}_{\mathbf{q}}(x, y) = \int [D\gamma]\ h^{ab}(x)\ h^{cd}(y)\ W[\gamma]\ \Psi_{\mathbf{q}}(\gamma). \tag{3.48}$$

At first sight, this expression appears to be meaningless. The r.h.s. is completely independent from the location of Σ on the spacetime manifold. What is then the meaning of the 4d coordinates x and y in the l.h.s.? In fact, this is nothing than the usual well–known problem of the conventional definition of n-point functions in generally covariant theories: if action and measure are generally covariant, equation (3.31) is independent from x and y (as long as $x \neq y$); because a diffeomorphism on the integration variable can change x and y, leaving all the rest invariant. We seem to have hit the usual stumbling block that makes n-point functions useless in generally covariant theories.

3.1.5 New definition of generally covariant 2-point functions

In fact, we have not, because the very dependence of $G^{abcd}_{\mathbf{q}}(x, y)$ on $\mathbf{q}$ provides the obvious solution to this problem: let us *define* a "generally covariant 2-point function" $\mathbf{G}^{abcd}_{\mathbf{q}}(x, y)$ as follows. Given a three-manifold S_3 with the topology of a 3-sphere, equipped with given fields $\mathbf{q} = (q_{ab}(\mathbf{x}), p^{ab}(\mathbf{y}))$, and given two points $\mathbf{x}$ and $\mathbf{y}$ *on this metric manifold*, we define

$$\mathbf{G}^{abcd}_{\mathbf{q}}(\mathbf{x}, \mathbf{y}) = \int [D\gamma]\ h^{ab}(\mathbf{x})\ h^{cd}(\mathbf{y})\ W[\gamma]\ \Psi_{\mathbf{q}}(\gamma). \tag{3.49}$$

The difference between (3.48) and (3.49) is that in the first expression x and y are coordinates in the background 4d *differential* manifold, while in the second $\mathbf{x}$ and $\mathbf{y}$ are points in the 3d *metric* manifold (S_3, q). It is clear that with this definition the dependence of the 2-point function on $\mathbf{x}$ and $\mathbf{y}$ is non trivial: metric relations between $\mathbf{x}$ and $\mathbf{y}$ are determined by $\mathbf{q}$. In particular, a 3d active diffeomorphism on the integration variable g changes $\mathbf{x}$ and $\mathbf{y}$, but also $\mathbf{q}$, leaving the metric relations between $\mathbf{x}$ and $\mathbf{y}$ invariant.

The physically interesting case is when $\mathbf{q} = (q, p)$ are a set of *physical* boundary conditions. Since we are considering here pure general relativity without matter, this means that there exists a Ricci flat spacetime with 4d metric g and an imbedding $\Sigma : S_3 \to M$, such that g induces the three metric q and the extrinsic curvature p on S_3. In this case, the semiclassical boundary state $\Psi_{\mathbf{q}}$ is a physical state. Measure and boundary states must be normalized in such a way that

$$\int [D\gamma]\ W[\gamma]\ \Psi_{\mathbf{q}}(\gamma) = 1. \tag{3.50}$$

Then the two point function (3.49) is a non-trivial and invariant function of the physical 4d distance

$$L = d_g(\Sigma(\mathbf{x}), \Sigma(\mathbf{y})). \tag{3.51}$$

It is clear that if g is the flat metric this function must reduce immediately to the conventional 2-point function of the linearized theory, in the appropriate large distance limit.

This is the definition of a generally covariant two-point function proposed in [14], which we use here.

Finally, the physical interpretation of (3.49) is transparent: it defines an amplitude associated to a *joint* set of measurements performed on a surface Σ bounding a finite spacetime region, where the measurements include:

- (i) the average geometry of Σ itself, namely the physical distance between detectors, the time lapse between measurements, and so on;
- (ii) the detection of a ("outgoing") particle (a graviton) at x and the detection of a ("incoming") particle (a graviton) at y.

The two kinds of measurements, that are considered of different nature in non-generally-relativistic physics, are on equal footing in general relativistic physics (see [2], pg. 152-153). In generally covariant quantum field theory, the single boundary state $\hat{h}^{ab}(x)\hat{h}^{cd}(y)\Psi_{\mathbf{q}}$ codes the two.

Notice that the quantum geometry in the *interior* of the region $\mathcal{R}$ is free to fluctuate. In fact, W can be interpreted as the sum over all interior 4-geometries. What is determined is a boundary geometry as measured by the physical apparatus that surrounds a potential interaction region.

Equation (3.49) can be realized concretely in LQG by identifying

- (i) the boundary Hilbert space associated to Σ with the (separable [130]) Hilbert space spanned by the (abstract) spin network states s, namely the s-knot states;this transform the sum over geometries in a sum over s-knots

$$\int [D\gamma] \to \sum_{s} \tag{3.52}$$

- (ii) the linearized gravitational field operators $\hat{h}^{ab}(x)$ and $\hat{h}^{cd}(y)$ with the corresponding LQG operators;
- (iii) the boundary state $\Psi_{\mathbf{q}}$ with a suitable spin network functional $\Psi_{\mathbf{q}}[s]$ peaked on the geometry q;
- (iv) the boundary functional $W[s]$, representing the functional integral on the interior geometries bounded by the boundary geometry s, with the $W[s]$ defined by a spin foam model. This, indeed, is given by a sum over interior spinfoams, interpreted as quantized geometries.

This gives the expression

$$\mathbf{G}^{abcd}_{\mathbf{q}}(\mathbf{x},\mathbf{y}) = \sum_{s} W[s]\ \hat{h}^{ab}(\mathbf{x})\ \hat{h}^{cd}(\mathbf{y})\ \Psi_{\mathbf{q}}[s]. \tag{3.53}$$

which we analyze in rest of the chapter.

The WdW condition reads

$$1 = \sum_s W[s]\ \Psi_{\mathbf{q}}[s]. \tag{3.54}$$

Using these two equations together, we can write

$$\mathbf{G}^{abcd}_{\mathbf{q}}(\mathbf{x},\mathbf{y}) = \frac{\sum_s W[s]\ \hat{h}^{ab}(\mathbf{x})\ \hat{h}^{cd}(\mathbf{y})\ \Psi_{\mathbf{q}}[s]}{\sum_s W[s]\ \Psi_{\mathbf{q}}[s]}, \tag{3.55}$$

a form that allows us to disregard the overall normalization of W and $\Psi_{\mathbf{q}}$. We analyze these ingredients in detail in the next section.

3.2 Graviton propagator: definition and ingredients

Equation (3.53) is well-defined if we choose a dynamical model giving $W[s]$, a boundary state $\Psi_{\mathbf{q}}[s]$ and a form for the operator $\hat{h}^{ab}(x)$. In the exploratory spirit of [14], we make here some tentative choices for these ingredient. In particular, we choose the boundary functional $W[s]$ defined by the group field theory GFT/B (Section 2.9.1). We consider here only some lowest order terms in the expansion of $W[s]$ in the GFT coupling constant λ. Furthermore, we consider only the first order in a large distance expansion. Our aim is to recover the 2-point function of the linearized theory, namely the graviton propagator, in this limit.

3.2.1 The boundary functional $W[s]$

We use the definition of $W[s]$ in the context of the spinfoam theory GFT/B, referring to Section 2.9.1. We expand the action in modes

$$S[\phi] = S_{\text{kin}}[\phi] + \frac{\lambda}{5!}\ S_{\text{int}}[\phi]. \tag{3.56}$$

expanding the field ϕ in modes introducing the shorthand notation $\phi^{j_1\dots j_4}_{\alpha_1\dots\alpha_4\, i} = \phi^{j_n}_{\alpha_n i}$. where the indices j_n, $n = 1,...,4$ label simple $SO(4)$ irreducible representations, α_n labels the components of vectors in the representation j_n and i labels an orthonormal basis of intertwiners on the tensor product of the four representations j_n. We choose a basis in which one of the basis elements is the Barrett-Crane intertwiner i_{BC}. Expanded in terms of these modes, the kinetic term of the action is

$$S_{\text{kin}} = \frac{1}{2} \sum_{\alpha_n, j_n, i} \phi^{j_n}_{\alpha_n i}\ \phi^{j_n}_{\alpha_n i} \tag{3.57}$$

The interaction term is (eq. (9.74) of [2])

$$S_{\text{int}} = \sum_{\alpha_{nm}, j_{nm}, i_n} \left(\prod_m\ \phi^{j_{nm}}_{\alpha_{nm}, i_n}\right)\ \mathcal{A}(j_{nm}, i_n). \tag{3.58}$$

Here the notation is as follows. The indices n and m run from 1 to 5, with $n \neq m$. $N_{nm} \equiv N_{mn}$ and $\phi^{j_{1m}}_{\alpha_{1m}, i_1} = \phi^{j_{12} j_{13} j_{14} j_{15}}_{\alpha_{12}\alpha_{13}\alpha_{14}\alpha_{15}, i_1}$ and so on. $\mathcal{A}(j_{nm}, i_n)$ is the Barrett-Crane vertex amplitude (2.83). This is

$$\mathcal{A}(j_{nm}, i_n) = \left(\prod_n \delta_{i_n i_{\text{BC}}}\right) \mathcal{B}(j_{nm}), \tag{3.59}$$

where $\mathcal{B}(j_{nm}) \equiv \mathcal{A}(j_{nm}, i_{\rm BC})$ is the $10j$ symbol. In the following we use also the formal notation $\int \phi^2 \equiv S_{\rm kin}[\phi]$ and $\int \phi^5 \equiv S_{\rm int}[\phi]$.

$SO(4)$-invariant observables of the theory are computed as the expectation values

$$W[s] = \frac{1}{Z} \int \mathrm{D}\phi \;\; f_s(\phi) \;\; \mathrm{e}^{-\int \phi^2 - \frac{\lambda}{5!}\int \phi^5} \tag{3.60}$$

where the normalization Z is the functional integral without $f_s(\phi)$, and $f_s(\phi)$ is the function of the field determined by the spin network $s = (\Gamma, j_l, i_n)$. Recall that a spin network is a graph Γ formed by nodes n connected by links l, colored with representations j_l associated to the links and intertwiners i_n associated to the nodes. We note l_{nm} a link connecting the nodes n and m, and $j_{nm} \equiv j_{mn}$ the corresponding color. The spin network function is defined in terms of the modes introduced above by

$$f_s(\phi) = \sum_{\alpha_{nm}} \prod_n \phi^{j_{nm}}_{\alpha_{nm} i_n}. \tag{3.61}$$

Here n runs over the nodes and, for each n, the index m runs over the four nodes that bound the four links l_{nm} joining at n. Notice that each index $\alpha_{nm} \equiv \alpha_{mn}$ appears exactly twice in the sum, and are thus contracted.

Fixed a spin network s, (3.60) can be treated by a perturbative expansion in λ, which leads to a sum over Feynman diagrams. Expanding both numerator and denominator, we have

$$\begin{aligned} W[s] \;\; = \;\; & \frac{1}{Z_0} \int \mathrm{D}\phi \, f_s(\phi) \, \mathrm{e}^{-\int \phi^2} - \\ & + \frac{1}{Z_0} \frac{\lambda}{5!} \left[\int \mathrm{D}\phi \, f_s(\phi) \left(\int \phi^5 \right) \mathrm{e}^{-\int \phi^2} - \frac{\int \mathrm{D}\phi \, \left(\int \phi^5 \right) \mathrm{e}^{-\int \phi^2}}{Z_0} \int \mathrm{D}\phi \, f_s(\phi) \, \mathrm{e}^{-\int \phi^2} \right] + \\ & + \frac{1}{Z_0} \frac{\lambda^2}{2(5!)^2} \left[\int \mathrm{D}\phi \, f_s(\phi) \, \left(\int \phi^5 \right)^2 \mathrm{e}^{-\int \phi^2} + \dots \right], \end{aligned} \tag{3.62}$$

where $Z_0 = \int \mathrm{D}\phi \, \mathrm{e}^{-\int \phi^2}$. As usual in QFT, the normalization Z gives rise to all vacuum–vacuum transition amplitudes, and it role is to eliminate disconnected diagrams.

Recall that this Feynman sum can be expressed as a sum over all connected spinfoams $\sigma = (\Sigma, j_f, i_e)$ bounded by the spin network s. A spinfoam is a two-complex Σ, namely an ensemble of faces f bounded by edges e, in turn bounded by vertices v, colored with representations j_f associated to the faces and intertwiners i_e associated to the edges.

The boundary of a spinfoam $\sigma = (\Sigma, j_f, i_e)$ is a spin network $s = (\Gamma, j_l, i_n)$, where the graph Γ is the boundary of the two-complex Σ, $j_l = f_f$ anytime the link l of the spin network bounds a face f of the spinfoam and $i_n = i_e$ anytime the node n of the spin network bounds an edge e of the spinfoam. See the Table 2.1 for a summary of the terminology.

The amplitudes can be reconstructed from the following Feynman rules; the propagator

$$\mathcal{P}^{j_n \;\; j'_n}_{\alpha_n i \;\; \alpha'_n i'} = \delta_{i,i'} \sum_{\pi(n)} \prod_n \delta_{j_n, j'_{\pi(n)}} \, \delta_{\alpha_n \alpha'_{\pi(n)}} \tag{3.63}$$

where $\pi(n)$ are the permutations of the four numbers $n = 1, 2, 3, 4$; and the vertex amplitude

$$\mathcal{V}_{j_{nm}}^{\alpha_{nm} i_n} = \Big(\prod_n \delta_{i_n i_{\rm BC}}\Big)\Big(\prod_{n\neq m} \delta_{\alpha_{nm}\alpha_{mn}}\Big)\, \mathcal{B}(j_{nm}), \tag{3.64}$$

where the index $n = 1, ..., 5$ labels the five legs of the five-valent vertex; while the index $m \neq n$ labels the four indices on each leg.

A Feynman graph has vertices v and propagators that we call "edges" and denote e. A spinfoams σ is obtained from a Feynman graph by: (i) selecting one term in each sum over representations and one term in each sum over permutations (eq. (3.63)), in the sum that gives the amplitude of the graph; (ii) contracting the closed sequences of $\delta_{\alpha_n\alpha_m}$ in the propagators, vertices and boundary spin-network function; and (iii) associating a *face* f, colored by the corresponding representation j_f, to each such sequence of propagators and boundary links. See [2] for more details. We obtain in this manner the amplitude

$$W[s] = \frac{1}{Z} \sum_{\sigma, \partial\sigma = s} \prod_{f\in\sigma} \dim(j_f) \prod_{v\in\sigma} \lambda \mathcal{B}(j^v_{nm}) \left(\prod_{n\in s} \langle i_n | i_{\rm BC}\rangle\right). \tag{3.65}$$

Here σ are spinfoams with vertices v dual to a four–simplex, bounded the spin network s. f are the faces of σ; the spins j^v_{nm} label the representations associated to the ten faces adjacent to the vertex v, where $n \neq m = 1, ..., 5$; $\dim(j)$ is the dimension of the representation j. The colors of a faces f of σ bounded by a link l of s is restricted to match the color of the link: $j_f = j_l$. The expression is written for arbitrary boundary spin-network intertwiners i_n: the scalar product is in the intertwiner space and derives from the fact that the vertex amplitude projects on the sole Barrett-Crane intertwiner.

The sum (3.65) can be written as a power series in λ

$$W[s] = \sum_{k=0}^{\infty} \lambda^k\, W_k[s] \tag{3.66}$$

with

$$W_k[s] = \frac{1}{Z} \sum_{\sigma^k, \partial\sigma^k = s} \prod_{f\in\sigma} \dim(j_f) \prod_{v\in\sigma} \mathcal{B}(j^v_{nm}) \left(\prod_{n\in s} \langle i_n | i_{\rm BC}\rangle\right), \tag{3.67}$$

where σ^k is a spinfoam with k vertices.

Finally, recall that the last expression can be interpreted as the quantum gravity boundary amplitude associated to the boundary state defined by the spin network s [2]. The individual spin foams σ appearing in the sum can be interpreted as (discretized) spacetimes bounded by a 3-geometry determined by s. That is, (3.65) can be interpreted as a concrete definition of the formal functional integral

$$\Psi[q] = \int_{\partial g = q} Dg\; e^{iS_{\rm GR}[g]} \tag{3.68}$$

where q is a 3-geometry and the integral of the exponent of the general relativity action is over the 4-geometries g bounded by q. Indeed, (3.65) can also be derived from a discretization of a suitable formulation of this functional integral. We now turn to the physical interpretation of this boundary 3-geometry.

Table 3.1: Relation between a triangulation and its dual, in the and 4d bulk and in its 3d boundary. In parenthesis: adjacent elements. In italic, the two-complex and the spin-network's graph. The spinfoam is $\sigma = (\Delta_4^*, j_f, i_e)$. The spin network is $s = (\Delta_3^*, j_l, i_n)$.

Δ_4	Δ_4^*		coloring
4-simplex	*vertex*	(5 edg, 10 fac)	
tetrahedron	*edge*	(4 faces)	i_e
triangle	*face*		j_f
segment			
point			

Δ_3	Δ_3^*		coloring
tetrahedron	*node*	(4 links)	$i_n = i_e$
triangle	*link*		$j_l = j_f$
segment			
point			

3.2.2 Relation with geometry

In order to select a physically relevant boundary state $\Psi_{\mathbf{q}}[s]$, we need a geometrical interpretation of the boundary spin networks s. To this aim, recall that the spinfoam model can be obtained from a discretization of general relativity on a triangulated spacetime. The discretization can be obtained as follows.

We associate an R^4 vector $e^I_{\rm s}$ to each segment s of the triangulation. The relation with the gravitational field can be thought as follows. Introduce 4d coordinates x^μ and represent the gravitational field by means of the one-form tetrad field $e^I(x) = e^I_\mu(x)dx^\mu$ (related to Einstein's metric by $g_{\mu\nu}(x) = e^I_\mu(x)e_{I\mu}(x)$). Assuming that the triangulation is fine enough for this field to be approximately constant on a tetrahedron, with constant value e^I_μ, associate the 4d vector $e^I_{\rm s} = e^I_\mu \Delta x^\mu_{\rm s}$ to the segment s, where $\Delta x^\mu_{\rm s}$ is the coordinate difference between the initial and final extremes of s. Next, to each triangle t of the triangulation, associate the bivector (that is, the object with two antisymmetric indices)

$$B^{IJ}_t = e^I_{\rm s} e^J_{\rm s'} - e^J_{\rm s} e^I_{\rm s'}, \tag{3.69}$$

where s and s$'$ are two sides of the triangle. (As far as orientation is kept consistent, the choice of the sides does not affect the definition of B^{IJ}_t). B^{IJ}_t is a discretization of the Plebanski two-form $B^{IJ} = e^I \wedge e^J$. The quantum theory is then formally obtained by choosing the quantities B^{IJ}_t as basic variables, and identifying them with $SO(4)$ generators J^{IJ}_t associated to each triangle of the triangulation, or, equivalently, to each face of the corresponding dual spinfoam. (For a compairaison with Regge calculus, see [131].)

The geometry is then easily reconstructed using the $SO(4)$ Casimirs. In particular, the peculiar form (3.69) implies immediately that

$$\epsilon_{IJKL} B^{IJ}_t B^{KL}_{t'} = 0 \tag{3.70}$$

any time $t = t'$ or t and t' share an edge. Accordingly, the pseudo–scalar Casimir $\tilde{C} = \epsilon_{IJKL} J^{IJ}_t J^{KL}_t = 0$ is required to vanish. This determines the restriction to the simple representations, which are precisely the ones for which $\tilde{C}$ vanishes.

The scalar Casimir $C = \frac{1}{2} J^{IJ}_t J_{tIJ} = \frac{1}{2} B^{IJ}_t B_{tIJ}$, on the other hand, is easily recognized, using again (3.69), as the square of the *area* A_t of the triangle t. Indeed, calling $\alpha_{\rm ss'}$ the angle

between s and s′, we have:

$$\begin{aligned} C &= \frac{1}{2}B_t^{IJ}B_{tIJ} = \frac{1}{2}(e_{\rm s}^I e_{\rm s'}^J - e_{\rm s}^J e_{\rm s'}^I)(e_{{\rm s}I}e_{{\rm s'}J} - e_{{\rm s},J}e_{{\rm s'}I}) \\ &= e_{\rm s}\cdot e_{\rm s}\ e_{\rm s'}\cdot e_{\rm s'} - (e_{\rm s}\cdot e_{\rm s'})^2 = |e_{\rm s}|^2|\ e_{\rm s'}|^2\ (1-\cos^2\alpha_{\rm ss'}) \\ &= (|e_{\rm s}|\ |e_{\rm s'}|\ \sin\alpha_{\rm ss'})^2 = A_t^2. \end{aligned} \tag{3.71}$$

For simple representations, the value of C is $j(j+1)$. The quantization of the geometrical area, with $j(j+1)$ eigenvalues is of course a key result of LQG, reappearing here in the context of the spinfoam models. It is the LQG result that assures us that we can interpret it as a physical discretization and not an artifact of the triangulation of spacetime.

An explanation about units is needed. B_t^{IJ} has units of a length square, hence C has units $[L]^4$. In the quantum theory, B_t^{IJ} is identified with J_t^{IJ} and C has discrete eigenvalues. The identification requires evidently a scale to be fixed. This scale determines the Planck constant. A posteriori, we can simply reconstruct the correct scale by using again LQG, where the area eigenvalues are

$$A_j = \frac{8\pi\gamma\hbar G}{c^3}\sqrt{j(j+1)} \tag{3.72}$$

where γ is the Immirzi parameter, which we fix to unit below, together with the speed of light c. This fixes the scale of the discretization (that is, it fixes the "size" of the compact $SO(4)$ group in physical units).

Next, consider two triangles sharing a side. Say the triangle t has two sides: the segments $\rm s_1$ and $\rm s_2$ while the triangle t' has two sides $\rm s_1$ and $\rm s_3$. Consider the action of the $SO(4)$ generators on the tensor product of the representation spaces associated to the two (faces dual to the two) triangles. This is given by the operators $J_{tt'}^{IJ} = J_t^{IJ} + J_{t'}^{IJ}$ (we omit the tensor with the identity operator in the notation)). Equation (3.70), for $t \neq t'$ implies, with simple algebra, that the pseudo–scalar Casimir $\tilde{C}_{tt'} = \epsilon_{IJKL}J_{tt'}^{IJ}J_{tt'}^{KL}$ vanishes as well. This implies that the tensor product of the two representations associated with the triangles t and t' is –again– only allowed to contain simple representations. Let t and t' be two of the four triangles of a given tetrahedron. In the dual picture, they correspond to two faces joining along an edge e of the spinfoam. Then $\tilde{C}_{tt'}$ is the pseudo–scalar Casimir of the virtual link that defines the intertwiner associated to this edge, under the pairing that pairs t and t'. The vanishing of $\tilde{C}_{tt'}$ implies that this virtual link, as well, is labeled by a simple representation. In the model we are considering all internal edges are labeled by the Barrett-Crane intertwiner, whose key property is precisely that it is a linear combination of virtual links with simple representations for any possible pairing of the four adjacent faces, thus consistently with $\tilde{C}_{tt'} = 0$. This is in fact *la raison d'être* of the Barrett–Crane intertwiner.

Let us now consider the boundary s of the spinfoam σ. A face f that cuts the boundary, labelled by a simple representation j_f, defines a link l of the boundary spin network s, equally colored with a representation $j_l = j_f$. As we have seen, the quantity $j_f(j_f+1)$ is to be interpreted as the area of the triangle dual to the face f. This triangle lies on the boundary and is cut by the link l.

Notice that we have precisely the LQG result that the area of a triangle is determined by the spin associated to the link of the spin network that cuts it. We can therefore identify in a natural way the boundary spin networks with the spin network states of canonical LQG.

Recall that in LQG a basis of states of the quantum geometry of a 3d surface is labelled by abstract spin networks s. Since our aim here is not to fix the details of the physically correct quantum theory of gravity, but only to develop a general relativistic quantum formalism, we will do so in the following, disregarding some open issues raised by this identification (see below).

The interpretation of the intertwiners at the boundaries is more delicate. Consider an edge e of σ that cuts the boundary at a node n of s. The node n, or the edge e are dual to a tetrahedron sitting on a boundary. Let t and t' be two faces of this tetrahedron, and say, as above, that the triangle t has two sides s_1 and s_2 while the triangle t' has two sides s_1 and s_3. Consider now the scalar Casimir $C_{tt'} = J^{IJ}_{tt'} J_{tt'\,IJ}$ on the tensor product of the representation spaces of the two triangles. Straightforward algebra shows that

$$C_{tt'} = |C_t| + |C_{t'}| + 2\ \vec{n}_t \cdot \vec{n}_{t'}. \tag{3.73}$$

where $n^I_t = \epsilon^I{}_{JKL} B^{JK}_t t^L$ is and t^L is the normalized vector normal to t and t' (that is, to $\mathrm{s}_1, \mathrm{s}_2$ and s_3). Finally, $\vec{n}_t \cdot \vec{n}_{t'} = A_t A_{t'} \cos\alpha_{tt'}$, where $\alpha_{tt'}$ is the dihedral angle between t and t'. This provides the interpretation of the color of a virtual link in the intertwiner associated to the node, in the corresponding decomposition: if the virtual link of this intertwiner is simple, with spin $j_{tt'}$, we have

$$j_{tt'}(j_{tt'}+1) = A_t^2 + A_{t'}^2 + A_t A_{t'} \cos\alpha_{tt'}. \tag{3.74}$$

That is, the color of the virtual link is a quantum number determining the dihedral angle $\cos\alpha_{tt'}$ between the triangles t and t'; or, in the dual picture, the angle between the two corresponding links that join at n.

Once more, this result is exactly the same in 3d LQG. In this case, to each link is associated an $SU(2)$ generator $J^i, i = 1, 2, 3$, that can be identified with the $SU(2)$ valued two-form E^i integrated on the dual triangle. The color of the link is the quantum number of the $SU(2)$ Casimir $\mathbf{C} = (J^i_t + J^i_{t'})(J_{t\,i} + J_{t'\,i})$. Expanding, we have $c = |J_t|^2 + |J_{t'}|^2 + 2J^i_t J_{t'\,i}$ or

$$\mathbf{j}_{tt'}(\mathbf{j}_{tt'}+1) = A_t^2 + A_{t'}^2 + A_t A_{t'} \cos\alpha_{tt'}. \tag{3.75}$$

where $\mathbf{j}_{tt'}$ is the quantum number labelling the eigenspaces of $\mathbf{C}$. We are therefore lead to identify the intertwiner $i_{j_{tt'}}$ in the boundary spin network, with the intertwiner $i_{\mathbf{j}_{tt'}}$ in the LQG spinnetwork states, since they represent the same physical quantity.

In fact, there is a key difference between (3.74) and (3.75). In (3.74), $j_{tt'}$ is the quantum number labelling a simple $SO(4)$ representation (recall $SO(4)$ irreducibles are labelled by pairs of spins, which are equal for simple representations); while in (3.75), $\mathbf{j}_{tt'}$ is the single spin labelling an $SU(2)$ representation. Some potential difficulties raised by this difference are discussed in Appendix E. As argued in the Appendix, if we disregard these difficulties and we identify the intertwiner $i_{j_{tt'}}$ with the LQG intertwiner $i_{\mathbf{j}_{tt'}}$, we obtain simply and consistently

$$\langle i_{\mathbf{j}_{tt'}} | i_{\mathrm{BC}} \rangle = (2j_{tt'}+1) = \dim(j_{tt'}). \tag{3.76}$$

The details of this interpretation as we will see in the next chapter will lead to some concrete problems in the calculation of the not diagonal terms of the propagator with the BC vertex.

This completes the geometrical interpretation of all quantities appearing in the spinfoam model.

3.2.3 Graviton operator

The next ingredient we need is the graviton field operator. This is the fluctuation of the metric operator over the flat metric. At every point of the surface Σ we chose a local frame in which the surface is locally stationary: three coordinates x^a with $a = 1, 2, 3$ coordinatize Σ locally, and the metric is in the "temporal" gauge: $g_{44} = 1, g_{4a} = 0$. To the first relevant order, we define $h^{ab}(\vec{x}) = g^{ab}(\vec{x}) - \delta^{ab}$. It is convenient to consider here the fluctuation of the *densitized* metric operator

$$\tilde{h}^{ab}(\vec{x}) = (\det g)g^{ab}(\vec{x}) - \delta^{ab} = E^{ai}(\vec{x})E^{bi}(\vec{x}) - \delta^{ab}. \tag{3.77}$$

In the linear theory, the propagators of the two agree because of the trace-free condition. To determine its action, we can equally use the geometrical interpretation discussed above, or, directly, LQG. We study the action of this operator on a boundary spin network state:

$$E^{ai}(\vec{x})E^{bi}(\vec{x})|s\rangle. \tag{3.78}$$

Let us identify the point $\vec{x}$ with one of the nodes n of the boundary spin network s. Equivalently, with (the center of) one of the tetrahedra of the triangulation. Four links emerge from this vertex. Say these are $e_I, I = 1, 2, 3, 4$. They are dual to the faces of the corresponding tetrahedron. Let n^I_a be the oriented normal to this face, defined as the vector product of two sides. Then $E(n)^{Ii} = E^{ai}(\vec{x})n^I_a$ can be identified with the action of the an $SU(2)$ generator J^i on the edge e_I. We have then immediately that the diagonal terms define diagonal operators

$$E^{Ii}(n)E^I_i(n)|s\rangle = (8\pi\hbar G)^2\ j_a(j_a + 1)|s\rangle \tag{3.79}$$

where j_a is the spin of the link in the direction a. The non–diagonal terms, will be considered in the next chapters.

3.2.4 The boundary vacuum state

As discussed in Section 3.1.4, the propagator will depend on a geometry $\mathbf{q}$ of the boundary surface Σ. Let us begin by choosing this 3d geometry. Let $\mathbf{q}$ be isomorphic to the intrinsic and extrinsic geometry of the boundary $\Sigma_{\mathbf{q}}$ of a 4d (metric) ball in Euclidean R^4 with given radius, much larger than the Planck length. We want to construct the state $\Psi_{\mathbf{q}}[s]$. (On the vacuum states in LQG, see [7, 132, 133, 134, 135, 136].) Below we shall only need the value of $\Psi_{\mathbf{q}}[s]$ for the spinnetworks $s = (\Gamma, j_l, i_n)$ defined on graphs Γ which are dual to 3d triangulations Δ. We identify each such Δ with a fixed triangulation of $\Sigma_{\mathbf{q}}$.

We assume here for simplicity that, for each graph, $\Psi_{\mathbf{q}}[s]$ is given by a function of the spins of s which is non-vanishing only on a single intertwiner on each node, which prjects on the $i_{\rm BC}$ intertwiner under (3.76). Note that this condition has to be relaxed if we want to deal with the non diagonal components of the propagator.

The area A_l of the triangle t_l of Δ, dual to the link l, determines background values $j_l{}^{(0)}$ of the spins j_l, via

$$A_l = 8\pi\hbar G\ \sqrt{j_l{}^{(0)}(j_l{}^{(0)} + 1)}. \tag{3.80}$$

We take these background values large with respect to the Planck length, and we will later consider only the dominant terms in $1/j_l{}^{(0)}$.

We want a state $\Psi_{\mathbf{q}}[s] = \Psi_{\mathbf{q}}(\Gamma, \mathbf{j})$, where $\mathbf{j} = \{j_l\}$, to be peaked on these background values. The simplest possibility is to choose a Gaussian peaked on these values, for every graph Γ

$$\Psi_{\mathbf{q}}[s] = C_\Gamma \ \exp\left\{-\frac{1}{2}\sum_{ll'} \alpha_{ll'} \ \frac{j_l - j_l^{(0)}}{(j_l^{(0)})^{\frac{k}{2}}} \ \frac{j_{l'} - j_{l'}^{(0)}}{(j_{l'}^{(0)})^{\frac{k}{2}}} + i\sum_l \Phi_l^{(0)}\, j_l\right\} \tag{3.81}$$

where l runs on links of s, $\alpha_{ll'}$ is a given numerical matrix, $k \in (0,2)$ (see below), and C_Γ is a graph–dependent normalization factor for the gaussian.

The phase factors in (3.81) play an important role [15]. As we know from elementary quantum mechanics, the phase of a semiclassical state determines where the state is peaked in the conjugate variables, here the variables conjugate to the spins j_l. Recall the form of the Regge action for one simplex, $S_{\text{Regge}} = \sum_l \Phi_l(j_l) j_l$, where $\Phi_l(j_l)$ are the dihedral angles at the triangles[2], which are function of the areas themselves and recall that $\partial S_{\text{Regge}}/\partial j_l = \Phi_l$. It is then easy to see that these dihedral angles are precisely the variables conjugate to the spins. Notice that they code the extrinsic geometry of the boundary surface, and in GR the extrinsic curvature is indeed the variable conjugate to the 3-metric. Thus, $\Phi_l^{(0)}$ are determined by the dihedral angles of the triangulation Δ.

Concerning the quadratic term in (3.81) we have put the $(1/j_l^{(0)})^{k/2}$ factors in evidence because we want a semiclassical state for which the relative uncertainties of area and angle become small when all the areas are large, namely in the large distance limit in which all the spins $j_l^{(0)}$ are of the order of a large j_L. That is, we demand that

$$\frac{\Delta A}{A} \to 0 \quad \text{and} \quad \frac{\Delta \Phi}{\Phi} \to 0, \quad \text{when} \quad j_l^{(0)} \sim j_L \to \infty. \tag{3.82}$$

Assuming that the matrix elements $\alpha_{(l)(l')} \sim \alpha$ do not scale with j_L, the fluctuations determined by the gaussian state (3.81) are of the order

$$\Delta j \sim \frac{j_L^{k/2}}{\sqrt{\alpha}}, \qquad\qquad \Delta\Phi \sim \frac{\sqrt{\alpha}}{j_L^{k/2}}. \tag{3.83}$$

Therefore, since angles do not scale,

$$\frac{\Delta A}{A} \sim \frac{\Delta j}{j} \sim \frac{j_L^{k/2-1}}{\sqrt{\alpha}}, \qquad\qquad \Delta\Phi \sim \frac{\sqrt{\alpha}}{j_L^{k/2}}. \tag{3.84}$$

(3.82) and (3.84) restricts to $k \in (0,2)$. From now on, we choose $k = 1$. That is

$$\Psi_{\mathbf{q}}[s] = C_\Gamma \ \exp\left\{-\frac{1}{2}\sum_{ll'} \alpha_{ll'} \ \frac{j_l - j_l^{(0)}}{\sqrt{j_l^{(0)}}} \ \frac{j_{l'} - j_{l'}^{(0)}}{\sqrt{j_{l'}^{(0)}}} + i\sum_l \Phi_l^{(0)}\, j_l\right\}. \tag{3.85}$$

[2]These are angles between the normals to the tetrahedra, and should not be confused with the angles between the normals to the faces, which are related to the intertwiners, as we discussed in Section 3.2.2

The need for this dependence on the scale of the background of the covariance matrix of the vacuum state was been pointed out in the 3d context [119] and by John Baez in the 4d case, following numerical investigation by Dan Christensen and Greg Egan.

A strong constraint on the graph–dependent constants C_Γ and matrix $\alpha_{ll'}$ is given by the WdW condition (3.54), which requires the state to satisfy the dynamics. The physical interpretation of the matrix $\alpha_{ll'}$ is rather obvious: it reflects the vacuum correlations, and is the analog of the covariance matrix in the exponent of the vacuum functional in the conventional Schrödinger representation of quantum field theory.

3.2.5 The $10j$ symbol and its derivatives

Baez, Christensen and Egan have performed in [100] a detailed numerical analysis, which has lead them to conjecture that if we rescale all spins by a factor λ, then for large λ the $10j$ symbol can be expressed as a sum of two terms,

$$B(j_{ij}) = \sum_\sigma P(\sigma)\ \cos\left[S_{\text{Regge}}(\sigma) + k\frac{\pi}{4}\right] + D(j_{ik}). \tag{3.86}$$

$P(\sigma)$ is a slowly varying factor, that grows as $\lambda^{-9/2}$ when scaling the spins by λ. To understand this formula, consider a 4-simplex in R^4, with triangles t_{ij} having areas $A_{ij} = \sqrt{j_{ij}(j_{ij}+1)}$. In general, there may be several distinct 4-simplices with triangles having these areas: let's label the distinct 4-simplices with a discrete label σ. Each triangle t_{ij} separates two boundary tetrahedra τ_i and τ_j of the 4-simplex. Each tetrahedron τ_i defines a normal vector n_i, normalized and normal to all sides of the tetrahedron. The angle Φ_{ij} between the normals n_i and n_j is the dihedral angle between the tetrahedra τ_i and τ_j. (The triangles t_{ij} are in one-to-one correspondence with the links l of the boundary spin network, hence the notation Φ_{ij} is consistent with the notation Φ_l used above.) For a fixed σ, we can compute the dihedral angles Φ_{ij} as a function $\Phi_{ij}(j_{ij})$, of the areas A_{ij}, hence of the 10 spins j_{ij}. The Regge action associated to the 10 spins is

$$S_{\text{Regge}}(\sigma) = \sum_{ij} j_{ij}\ \Phi_{ij}(j_{ij}). \tag{3.87}$$

It is characterized by the fact that

$$\frac{\partial S_{\text{Regge}}(\sigma)}{\partial j_{ij}} = \Phi_{ij}(j_{ij}); \tag{3.88}$$

that is, the derivative with respect to the j_{ij} in the angles does not contribute to the total derivative (this is the discrete analog to the fact that when we vary the Einstein-Hilbert action with respect to the metric, the metric variation of the Christoffel symbols does not contribute.)

The form (3.86) for the $10j$ symbol has later been confirmed by detailed analytical calculations by Barrett and Steele [102] and by Freidel and Louapre [101]. As first noted in [99], the first term in (3.86) is very good news for quantum gravity: it indicates that the $10j$ symbols are indeed related to 4d general relativity.

On the other hand, to understand the origin of the second term $D(j_{ik})$ in (3.86), recall that the $10j$ symbols can be expressed (Section 2.6.5) in the form

$$B(j_{ij}) = \int_{(S^3)^5} dy_i \prod_{i<j} \frac{\sin((2j_{ij}+1)\Phi_{ij})}{\sin(\Phi_{ij})}. \tag{3.89}$$

Here Φ_{ij} is the angle between the two unit vectors y_i and y_j. The large j behavior of this expression has been evaluated in [102] and [101] using a stationary phase approximation. It turns out that the integration variables y_i admit a very interesting geometrical interpretation as the normals to the tetrahedra $y_i = n_i$. The stationary points of the integral can be interpreted as different geometrical configurations. Some stationary points are given by non degenerate tetrahedra. These yield the Regge term in (3.86). But there are also contributions to the integral coming when two of the y_i are parallel, or more in general when the linear span of the five y_i is of dimension smaller than four. These degenerate contributions yield the $D(j_{ik})$ term.

The bad news is that this degenerate term strongly dominates for large j. This fact casts a thick shadow of doubt over hope that a Barrett-Crane spinfoam model could yield the correct general relativistic dynamics. In fact, the discovery of the degenerate contributions is one of the sources of a recent decrease in interest in these models. However, light comes back into the shadow, in consideration of the results of the previous section. In fact, observe that what enters in the expression we have found in the previous section is not the $10j$ symbol itself, but rather a second derivative of the $10j$ symbol, because of the field insertions in (3.55). The fact that the degenerate term $D(j_{ij})$ dominates over the Regge term at large j does not imply that its second derivative dominates as well.

A hint that this hope may be correct comes from the following naive argument. Degenerate contributions arise when the denominator in the integrand of (3.89) vanishes. The formal second derivative with respect to j_{ij}, considered as a continuous variable, is

$$\frac{\partial^2 B(j_{ij})}{\partial j_{kl}^2} = -4 \int_{(S^3)^5} dy_i \; \Phi_{kl} \prod_{i<j} \frac{\sin((2j_{ij}+1)\Phi_{ij})}{\sin(\Phi_{ij})}, \tag{3.90}$$

which decreases the order of the divergence in $\Phi_{ij} = 0$. A better version of this argument is the following. The diagonal term of the discrete second derivative is $B(j+1,...) - 2B(j,...) + B(j-1,...)$. Using the trigonometric identity:

$$\sin((a+2)t) - 2\sin(at) + \sin((a-2)t) = -4\sin(t)^2 \sin(at) \tag{3.91}$$

we see that the effect of taking the diagonal discrete second derivative is to multiply the relevant kernel by $-4\sin(\Phi)^2$. This should then eliminate all the fully degenerate points from the integral. (The elimination of the fully degenerate points does not necessarily mean that the only remaining contribution is the Regge one, since, as shown in [102], there is a complex zoology of other degenerate contributions to the integral)

3.3 Order zero

Let us begin by evaluating the general covariant 2-point function to order zero in λ. To this order

$$W_0[s] = Z_0^{-1} \int \mathrm{D}\phi \, f_s(\phi) \, \mathrm{e}^{-\int \phi^2}. \tag{3.92}$$

The Wick expansion of this integral gives non–vanishing contributions for all s with an even number of nodes. Since there are no vertices, each of these contributions is simply given by products of face contributions, namely products of dimensions of representations. The 2-point function (3.53) reads

$$\mathbf{G}^{abcd}_{\mathbf{q}}(\mathbf{x},\mathbf{y}) = \sum_s W_0[s] \; \hat{h}^{ab}(\mathbf{x}) \hat{h}^{cd}(\mathbf{y}) \Psi_{\mathbf{q}}[s]. \tag{3.93}$$

Inserting (3.85), we have

$$\mathbf{G}^{abcd}_{\mathbf{q}}(\mathbf{x},\mathbf{y}) = \sum_s W_0[s] \; \hat{h}^{ab}(\mathbf{x}) \hat{h}^{cd}(\mathbf{y}) \; C_\Gamma \exp\left\{ -\frac{1}{2} \sum_{ll'} \alpha_{ll'} \; \frac{j_l - j_l^{(0)}}{\sqrt{j_l^{(0)}}} \; \frac{j_{l'} - j_{l'}^{(0)}}{\sqrt{j_{l'}^{(0)}}} + i \sum_l \Phi_l^{(0)} \, j_l \right\}. \tag{3.94}$$

We are interested in this expression for large $j_l^{(0)}$. In this regime, the gaussian effectively restricts the sum over (a large region of) spins of order $j_l^{(0)}$. Over this region, the phase factor fluctuates widely, and suppresses the sum, unless it is compensated by a similar phase factor. But $W_0[s]$ contains only powers of j_l's, and cannot provide this compensation. Hence we do not expect a contribution of zero order to the sum. The only exception can be the null spin network $s = \emptyset$, which gives $W[\emptyset] = 1$ because of the normalization. Hence, to order zero

$$\mathbf{G}^{abcd}_{\mathbf{q}}(\mathbf{x},\mathbf{y}) = W_0[\emptyset] \; \hat{h}^{ab}(\mathbf{x}) \hat{h}^{cd}(\mathbf{y}) \; C_\emptyset. \tag{3.95}$$

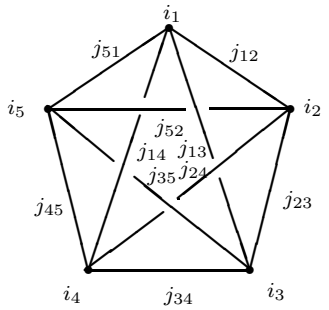
But is reasonable to assume that the semiclassical boundary state on a macroscopic geometry $\mathbf{q}$ has vanishing component on $s = \emptyset$, whose interpretation is that of a quantum state without any volume. Hence we take $C_\emptyset = 0$, and we conclude that the 2-point function has no zero order component in λ.

This result has a compelling geometrical interpretation. The sum over spinfoams can be interpreted as a sum over 4-geometries. The boundary state $\Psi_{\mathbf{q}}[s]$ describes a boundary geometry which has a nontrivial extrinsic curvature, described by the phase of the state. In the large distance limit, we expect semiclassical histories to dominate the path integral. These must be close to a classical solution of the equations of motion, fitting the boundary data. Because of the extrinsic curvature of the boundary data, it is necessary that the internal geometry has non vanishing 4-volume. A round soccer ball must have volume inside. But the four-volume of a spinfoam is given by its vertices, which are dual to the four-simplices of the triangulation. Absence of vertices means absence of four-volume. It is therefore to be expected that the zero order contribution, which has no vertices, and therefore zero volume, is suppressed by the phases of the boundary state, representing the extrinsic curvature.

Let us therefore go over to to the first order in λ.

3.4 First order: the 4d nutshell

Consider a spinfoam σ, dual to a single 4-simplex. Its the boundary spinnetwork has five nodes, connected by 10 links, forming the five-valent graph Γ_5. That is

$s =$ 

(3.96)

The boundary function $f_s(\phi)$ determined by this spin network is

$$f_s(\phi) = \sum_{\alpha_{nm}} \phi^{\alpha_{12}\alpha_{13}\alpha_{14}i_1}_{j_{12}j_{13}j_{14}j_{15}} \; \phi^{\alpha_{21}\alpha_{23}\alpha_{24}i_2}_{j_{21}j_{23}j_{24}j_{25}} \; \phi^{\alpha_{31}\alpha_{32}\alpha_{34}i_3}_{j_{31}j_{32}j_{34}j_{35}} \; \phi^{\alpha_{12}\alpha_{13}\alpha_{14}i_4}_{j_{41}j_{42}j_{43}j_{45}} \; \phi^{\alpha_{51}\alpha_{52}\alpha_{53}i_5}_{j_{51}j_{52}j_{53}j_{54}} \tag{3.97}$$

where $\alpha_{nm} \equiv \alpha_{mn}$. This can be compactly written as

$$f_s(\phi) = \sum_{\alpha_{nm}} \prod_{n=1,5} \phi^{\alpha_{nm} i_n}_{j_{nm}} \tag{3.98}$$

This is a monomial of order five in the field, and is an observable in the group field theory. Its expectation value is given by (3.60). We consider the perturbative expansion (3.63). At order λ, the only term remaining is

$$W[s] = \frac{\lambda}{5!} \int \mathrm{D}\phi \, f_s(\phi) \left(\int \phi^5 \right) \mathrm{e}^{-\int \phi^2}. \tag{3.99}$$

The Wick expansion of this integral gives one vertex v and five propagators

$$W'[s] = \frac{\lambda}{5!} \Big(\prod_{n=1,5} \mathcal{P}^{j_{nm} \;\; j'_{nm}}_{\alpha_{nm} i_n \; \alpha'_{nm} i'_n} \Big) \mathcal{V}^{j'_{nm}}_{\alpha'_{nm} i'_n}, \tag{3.100}$$

where repeated representation indices are summed over. This expression still contains many terms due to the summation over the permutations in the propagators. Recall that we can give the geometrical meaning of a face to each closed sequence of deltas in this expression. Each face contributes with a factor equal to the dimension of the representation.

The dominant term for large representations is therefore the one with the largest number of surfaces. A short reflection will convince the reader that this is the term in which the surfaces correspond precisely at the faces of the dual of a four-simplex. That is, the dominant term of $W[s]$ to order λ is

$$W[s] = \frac{\lambda}{5!} \left(\prod_n \langle i_n | i_{\mathrm{BC}} \rangle \right) \left(\prod_{n<m} \dim(j_{nm}) \right) \mathcal{B}(j^{\mathrm{u}}_{nm}). \tag{3.101}$$

Since we have chosen boundary states peaked on an intertwiner that projects on $i_{\rm BC}$, this reduces to

$$W[s] = \frac{\lambda}{5!}\left(\prod_{n<m} \dim(j_{nm})\right) \mathcal{B}(j^{\rm u}_{nm}). \tag{3.102}$$

This is the dominant term of the connected component of the amplitude for the boundary spin network considered, in the limit of large representations. This is the expression we will use within equation (3.53).

The value of $\Psi_{\mathbf{q}}[s]$ on the spin-networks $s = (\Gamma_5, j_{nm})$ (here $n, m = 1, ..., 5$) can be determined by triangulating $\Sigma_{\mathbf{q}}$ with the 3d triangulation formed by the boundary of a *regular* four–simplex of side L. The area of the triangles is $A_L = \sqrt{3}L^2/4$. Then (3.80) implies that $j^{(0)}_{nm} = j_L$ where $8\pi\hbar G\sqrt{j_L(j_L+1)} = A_L$. In the large L limit we take $j_L = 8\pi\hbar G A_L$. The dihedral angles $\Phi^{(0)}_{nm} = \Phi$ of a regular tetrahedron are given by $\cos(\Phi) = -1/4$. Therefore (3.85) becomes

$$\Psi_{\mathbf{q}}[s] = C_5 \ \exp\left\{-\frac{1}{2j_L}\sum_{(nm)(pq)} \alpha_{(nm)(pq)}\ (j_{nm}-j_L)(j_{pq}-j_L) + i\Phi\sum_{(n,m)} j_{nm}\right\}. \tag{3.103}$$

To respect the symmetry of the sphere, the covariance matrix $\alpha_{(nm)(pq)}$ of the gaussian can depend only on three numbers

$$\alpha_{(nm)(pq)} = \alpha_1\ a_{(nm)(pq)} + \alpha_2\ \delta_{(nm)(pq)} + \alpha_3\ b_{(nm)(pq)} \tag{3.104}$$

where $\delta_{(nm)(pq)} = 1$ if $(nm) = (pq)$, $a_{(nm)(pq)} = 1$ if just two indices are the same, and $b_{(nm)(pq)} = 1$ if all four indices are different, and in all other cases these quantities vanish. We will use this notation, namely $\alpha_{(12)(13)} = \alpha_1,\ \alpha_{(12)(12)} = \alpha_2,\ \alpha_{(12)(34)} = \alpha_3$ repeatedly.

The component of the state (3.85) that matters at first order in λ is thus completely determined up to the three numbers $\alpha_1, \alpha_2, \alpha_3$, and the constant C_5. This amounts to select a vacuum state which is a coherent state peaked both on the background values of the spins (the intrinsic geometry of the boundary surface), *and* on the background values of the angles (the intrinsic geometry of the boundary surface). See [121] for a similar construction in 3d.

For clarity, let us stress that we are *not* assuming that the boundary state has components *only* on the five-valent graph considered. What we are saying is that only this component of the boundary state enters the expansion to first order in λ that we are considering.

3.4.1 First order graviton propagator

We have now all the elements needed to compute the expression (3.53). Inserting (3.102), (3.103) and (3.79) in (3.53) we obtain a completely well–defined expression for the propagator. As a first step towards the analysis of the resulting expression, we choose the points $\mathbf{x}$ and $\mathbf{y}$ to be two distinct nodes of the boundary spinnetwork. Equivalently, these can be thought as points located, say, in the centers of the corresponding dual tetrahedra: in the theory, of course, position is not determined with better precision that the individual "atoms of spaces" described by the individual tetrahedra. We consider the ten by ten matrix $\widetilde{\mathbf{G}}(L)$ formed by the "diagonal" components of the propagator

$$\widetilde{\mathbf{G}}(L)_{(ij)(kl)} \equiv \mathbf{G}^{abcd}_{\mathbf{q}}(x,y)\ n^{(ij)}_a n^{(ij)}_b\ n^{(kl)}_c n^{(kl)}_d, \tag{3.105}$$

where $n_a^{(ij)}$ is the normal to the triangle t_{ij}. Since all ten triangles have the same background area, $|n^{(ij)}| = |n| = 8\pi\hbar G j_L$ for large j_L, we can write $\mathbf{G}(L) \equiv \widetilde{\mathbf{G}}(L)/|n|^4 = \widetilde{\mathbf{G}}(L)/(8\pi\hbar G j_L)^4$. By symmetry

$$\mathbf{G}(L)_{(ij)(kl)} = \mathbf{G}_1(L)\ a_{(ij)(kl)} + \mathbf{G}_2(L)\ \delta_{(ij)(kl)} + \mathbf{G}_3(L)\ b_{(ij)(kl)}. \tag{3.106}$$

This expression depends of course on L, which determines the distance between the points considered and the angles between the directions considered.

Before computing this quantity in the background independent theory, let us compute it in conventional linearized quantum general relativity, for later comparison. There are two ways of making the comparison. One is to compare the propagator obtained in the background independent calculation with the linearized–theory propagator $G^{abcd}_{\text{linearized}}(x,y)$, where x and y are in the *center* of the corresponding tetrahedron. The other is to compare it with the quantity obtained by *integrating* $G^{abcd}_{\text{linearized}}(x,y)$ in x and y, over the entire tetrahedron that they represent. The difference is a numerical factor that is not relevant for us, and we choose here the first option. In a flat background metric, two points in the center of adjacent tetrahedra, in a surface with the boundary geometry chosen, are at a distance $|x-y|_q = L/4$. If the four indices i,j,k,l are all distinct, it is easy to see that $n^{(ij)}$ and $n^{(kl)}$ are orthogonal; then the propagator is easily computed to be

$$G^{\text{linearized}}_{(ij)(kl)}(L) = i\frac{8\pi\hbar G}{4\pi^2}\frac{1}{|x-y|_q^2} = i\frac{32\hbar G}{\pi L^2} \tag{3.107}$$

On the other hand, the components $G^{\text{linearized}}_{(ij)(ij)}$ and $G^{\text{linearized}}_{(ij)(ik)}$ are vacuum expectation values at fixed "time": the first is the fluctuation of the area square of a triangle, and the second is the vacuum correlation between the fluctuations of the area squares of two adjacent triangles in the same tetrahedron. These are also proportional to L^{-2}. We can therefore write

$$G^{\text{linearized}}(L) = \frac{32\hbar G}{\pi L^2}\ \mathcal{W} \tag{3.108}$$

where $\mathcal{W}$ is a numerical matrix, with the same symmetry structure as in (3.106). For instance, $\mathcal{W}_{(12)(34)} = i$ as in (3.107), while the others projections are easily obtained from the linear theory.

We now compute the matrix $\mathbf{G}(L)$ in the full theory. Since this is a diagonal term in the propagator, we can use (3.79) and (3.53) reads

$$\mathbf{G}(L)_{(ij)(kl)} = \frac{1}{8\pi\hbar G j_L^4}\sum_s W[s]\ ((8\pi\hbar G)^2 j_{ij}(j_{ij}+1) - |n|^2)((8\pi\hbar G)^2 j_{kl}(j_{kl}+1) - |\tilde{n}|^2)\Psi_q[s]. \tag{3.109}$$

The terms $|n|^2$ come from the background δ^{ab} and are equal to the square of the area of the face, namely to $(8\pi\hbar G j_L)^2$, for large j. Inserting (3.102) and (3.103) we have, to first order in λ

$$\begin{aligned}\mathbf{G}(L)_{(ij)(kl)} \quad = \quad & \frac{\lambda}{5!}\frac{1}{j_L^4}\sum_{j_{nm}}\left(\prod_{n<m}\dim(j_{nm})\right)\ (j_{ij}(j_{ij}+1) - j_L^2)\ (j_{kl}(j_{kl}+1) - j_L^2) \\ & \mathcal{B}(j_{nm})\ \ C_5\ \exp\left\{-\frac{1}{2j_L}\alpha_{(nm)(pq)}(j_{nm}-j_L)(j_{pq}-j_L) + i\Phi\sum_{n<m} j_{nm}\right\}\end{aligned} \tag{3.110}$$

where we have used the Einstein convention in the exponent. Since we have assumed that j_L is large and the vacuum exponential peaks the sum around j_L, we can discard the $+1$ in the parenthesis. We expand the summand in the fluctuations $\delta j_{ij} = (j_{ij} - j_L)$, and keep only the lowest term, assuming that the gaussian suppress the higher terms. This gives

$$\begin{aligned}\mathbf{G}(L)_{(ij)(kl)} &= \frac{\lambda}{5!}\frac{4}{j_L^2}\sum_{j_{nm}}\left(\prod_{n<m}\dim(j_{nm})\right)\ \delta j_{nm}\,\delta_{pq} \\ &\qquad \mathcal{B}(j_{nm})\ \ C_5\ \exp\left\{-\frac{1}{2j_L}\alpha_{(nm)(pq)}\,\delta j_{nm}\,\delta_{pq} + i\Phi\sum_{n<m} j_{nm}\right\}. \quad (3.111)\end{aligned}$$

We assume that the $\dim j$ terms vary slowly over the range where the gaussian is peaked, and can be considered constant. Let us absorb C_5 and these constants in a factor $\mathcal{N}_5$.

$$\mathbf{G}(L)_{(ij)(kl)} = \mathcal{N}_5\frac{4}{j_L^2}\sum_{j_{nm}}\delta j_{nm}\,\delta_{pq}\ \mathcal{B}(j_{nm})\ \exp\left\{-\frac{1}{2j_L}\alpha_{(nm)(pq)}\,\delta j_{nm}\,\delta_{pq} + i\Phi\sum_{n<m} j_{nm}\right\}. \quad (3.112)$$

We change summation variable from the spins to the fluctuation of the spins

$$\mathbf{G}(L)_{(ij)(kl)} = \mathcal{N}_5\frac{4}{j_L^2}\sum_{\delta j_{nm}}\delta j_{ij}\ \ \delta j_{kl}\ \mathcal{B}(j_L+\delta j_{nm})\ e^{-\frac{1}{2j_L}\alpha_{(nm)(pq)}\delta j_{nm}\delta j_{pq} + i\Phi\sum_{nm} j_{nm}}. \quad (3.113)$$

The sum can be approximated with a gaussian integral. The rapidly oscillating term $\exp i\Phi\sum_{nm} j_{nm}$ tends to suppress the sum. To evaluate it, we need the explicit form of $\mathcal{B}(j_L+\delta j_{nm})$ in the large j regime, discussed above. Since the sum (3.113) is peaked around $j_{nm} = j_L$, let us expand the $10j$ symbol around this point. To second order around $j_{nm} = j_L$, the Regge action reads

$$S_{\text{Regge}}(j_{nm}) = \Phi\sum_{nm} j_{nm} + \frac{1}{2}G_{(mn)(pq)}\delta j_{mn}\delta j_{pq}, \quad (3.114)$$

where, introducing the "discrete derivative" $\frac{\partial f(j)}{\partial j} \equiv\ f(j+1/2) - f(j)$, we have defined

$$G_{(mn)(pq)} = \left.\frac{\partial\Phi_{mn}(j_{rs})}{\partial j_{pq}}\right|_{j_{rs}=j_L}. \quad (3.115)$$

Thus, around $j_{nm} = j_L$, (3.86) gives

$$B(j_{nm}) = P_{\tau_R}\left[e^{i(\Phi\sum_{nm} j_{nm} + \frac{1}{2}G_{(nm)(pq)}\delta j_{nm}\delta j_{pq} + \frac{\pi}{4})} + e^{-i(\Phi\sum_{nm} j_{nm} + \frac{1}{2}G_{(nm)(pq)}\delta j_{nm}\delta j_{pq} + \frac{\pi}{4})}\right] + D(j_{nm}) \quad (3.116)$$

where τ_R is the regular four simplex (for which $k_{\tau_R} = 1$), which is the only non-degenerate four-simplex with these areas [100].

The key observation is now the fact that the rapidly oscillating $\exp\{i\Phi\sum_{nm} j_{nm}\}$ *term in the second term of this expression cancels with the rapidly oscillating term in* (3.113). Therefore the second term of the last expression contributes in a non-negligible way to the sum (3.113). The first term is suppressed (by the rapidly oscillating factor $\exp 2i\Phi\sum_{nm} j_{nm}$) and it is reasonable to expect that so is the degenerate term $D(j_{nm})$ because this corresponds

to 4-simplices with different angles, and should be dominated by different frequencies. This is the cancellation of the phases that we mentioned in Section 3.1. Therefore (3.113) becomes, keeping only the first term

$$\mathbf{G}(L)_{(ij)(kl)} = \mathcal{N}_5' \frac{4}{j_L^2} \sum_{\delta j_{nm}} \delta j_{ij} \quad \delta j_{kl} \; e^{-\frac{i}{2} G_{(nm)(pq)} \delta j_{nm} \delta j_{pq}} e^{-\frac{1}{2j_L} \alpha_{(nm)(pq)} \delta j_{nm} \delta j_{pq}}, \tag{3.117}$$

where we have absorbed some constant factors in $\mathcal{N}_5'$. This factor, which contain the constant C_5 of the state, is determined by the WdW condition (3.54). This is simply given by the same expression without the field operator insertions, namely

$$1 = \mathcal{N}_5' \sum_{\delta j_{nm}} \; e^{-\frac{i}{2} G_{(nm)(pq)} \delta j_{nm} \delta j_{pq}} e^{-\frac{1}{2j_L} \alpha_{(nm)(pq)} \delta j_{nm} \delta j_{pq}}. \tag{3.118}$$

Thus using the form (3.55) for the two-point function, we have

$$\mathbf{G}(L)_{(ij)(kl)} = \frac{4}{j_L^2} \frac{\sum_{\delta j_{nm}} \delta j_{ij} \quad \delta j_{kl} \; e^{\left(-\frac{i}{2} G_{(nm)(pq)} - \frac{1}{2j_L} \alpha_{(nm)(pq)}\right) \delta j_{nm} \delta j_{pq}}}{\sum_{\delta j_{nm}} \; e^{\left(-\frac{i}{2} G_{(nm)(pq)} - \frac{1}{2j_L} \alpha_{(nm)(pq)}\right) \delta j_{nm} \delta j_{pq}}}. \tag{3.119}$$

This expression for the 2-point function has the same structure as equation (3.10). Approximating the sum by a gaussian integral gives

$$\mathbf{G}(L) = \frac{4}{j_L^2} \left(j_L^{-1} \alpha + i G \right)^{-1}. \tag{3.120}$$

We only need to evaluate the derivatives (3.115) of the angles with respect to the spins. 4d geometry gives (see the Appendix A of [16])

$$G_{(nm)(pq)} = 8\pi\hbar G \; \frac{\partial \Phi_{nm}}{\partial A_{pq}} = \frac{8\pi\hbar G}{\sqrt{5} L^2} \left(\frac{7}{2} a_{(nm)(pq)} - 9\delta_{(nm)(pq)} - 4 b_{(nm)(pq)} \right) \equiv \frac{8\pi\hbar G}{L^2} \; \mathcal{K}_{(nm)(pq)}. \tag{3.121}$$

The ten by ten matrix $\mathcal{K}$ has purely numerical entries. From the relation between areas and spins, we have $j_L = \sqrt{3} L^2 / (32\pi\hbar G)$. The j_L factor that combines with the one in front of α in (3.120) to give the crucial overall $1/L^2$ dependence of the propagator. Finally (3.120) reads

$$\mathbf{G}(L) = \frac{32\pi\hbar G}{\sqrt{3}/4 \; L^2} \left(\alpha + i\sqrt{3}/4 \; \mathcal{K} \right)^{-1}. \tag{3.122}$$

This is the value (3.107-3.108) of the propagator computed from the linearized theory, with the correct $1/|x-y|^2$ spacetime dependence. The three numerical coefficients of the matrix α are completely determined by $\alpha = 4\pi^2/\sqrt{3} \; \mathcal{W}^{-1} - i\sqrt{3}/4 \; \mathcal{K}$.

We see that the first order term (but the same is true for the second order, see [16]) in the expansion in λ of the diagonal components of the graviton propagator, in a large distance regime, calculated starting from a background–independent formulation of quantum gravity has the $1/|x-y|^2$ dependence on the distance expected from the linearized quantum theory. Moreover it has the expected dependence on the physical constants, and the numerical proportionality constants can be fixed as a condition on the semiclassical boundary state. The main tool we have used is the definition of general covariant n-point function, given in (3.49). Now we are ready to look at the not diagonal terms to obtain the complete propagator from LQG

Chapter 4

The complete LQG propagator: Difficulties with the Barrett-Crane vertex

In this chapter we present the result of the paper [17]. We want to extend the result of the previous chapter based on [15] (at first order) and [16] (to higher order) starting from the background-independent theory and using a suitable expansion. In fact even if the result of the previous chapter has been extended to the three-dimensional theory in [119] and other improvements of the ingredients to use in the calculation have been studied (an improved form of the boundary states has been considered in [137]; and the exploration of some Planck-length corrections to the propagator of the linear theory has begun in [120] see also [139, 138]. Nobody has yet verified if Rovelli' strategy to compute background independent n-point function is able to reproduce the entire tensorial structure of the graviton propagator. In this chapter we complete the calculation computing the nondiagonal terms of $G^{abcd}(x, y)$, those where $a \neq b$ or $c \neq d$, and therefore derive the full tensorial structure of the propagator. The nondiagonal terms are important because they involve the *intertwiners* of the spin networks. Avoiding the complications given by the intertwiners' algebra was indeed the rational behind the relative simplicity of the diagonal terms.

The dependence of the vertex from the intertwiners is a crucial aspect of the definition of the quantum dynamics. The particular version of the dynamics used in [15] and [16], indeed, is defined by the Barrett-Crane (BC) vertex [70], where the dependence on the intertwiners is trivial. This is an aspect of the BC dynamics that has long been seen as suspicious (see for instance [140]); and it is directly tested here.

We find that under our assumptions the BC vertex *fails* to give the correct tensorial structure of the propagator in the large-distance limit. We argue that this result is general, and cannot be easily corrected, say by a different boundary state. This result is of interest for a number of reasons. First, it indicates that the propagator calculations are nontrivial; in particular they are not governed just by dimensional analysis, as one might have worried, and they do test the dynamics of the theory. Second, it reinforces the expectation that the BC model fails to yield classical GR in the long-distance limit. Finally, and more importantly, it opens the possibility of studying the conditions that an alternative vertex must satisfy, in

order to yield the correct long-distance [18] behavior. This analysis will be presented in the next chapter .

The BC model exists in a number of variants presented in the thesis[112, 113, 114] (see also [141]); the results presented here are valid for all of them. Alternative models have been considered, see for instance [142]. The paper [17] has motivated the search for a vertex amplitude that modifies the BC amplitude; recently new vertexes, which addresses precisely the problems that we find here, have been proposed [143, 144, 145, 146, 147], see also [148].

This chapter is organized as follows. In the first section we formulate the problem and we compute the action of the field operators on the intertwiner spaces. This calculation is a technical result with an interest in itself. Here we will use only part of this result, the rest will be relevant in the next chapter. We proceed discussing the form of the boundary state needed to describe a semiclassical geometry to the desired approximation. Then we face the main calculation and we discuss the interpretation of our result.

The calculation of this chapter are made using the Appendix B were all basic equations of the recoupling theory can be found. This Appendix corrects some imprecisions in previous formularies. We work entirely in the euclidean theory.

4.1 The propagator in LQG

We want to compute

$$\mathbf{G}^{abcd}_{\mathbf{q}}(x,y) = \langle W|h^{ab}(x)\ h^{cd}(y)|\Psi_{\mathbf{q}}\rangle \tag{4.1}$$

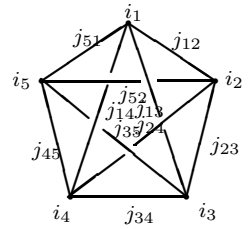
to first order in λ. Here $\Psi_{\mathbf{q}}$ is a state peaked on $\mathbf{q}$, which is the (intrinsic and extrinsic) 3d geometry of the boundary of a spherical 4-ball of radius L in R^4, x and y are two points in this geometry, ab are tangent indices at x and cd tangent indices at y in this geometry. That is, $\mathbf{G}^{abcd}_{\mathbf{q}}(x,y)$ is a quantity that transforms covariantly under 3d diffeomorphisms acting *conjointly* on x, y, on the indices $abcd$, *and* on $\mathbf{q}$. $h^{ab}(x)$ is the fluctuation of the gravitational field over the euclidean metric. W is the boundary functional, that defines the dynamics; it is assumed to be given here by a Barrett-Crane GFT with coupling constant λ. We work here to first order in λ. Following [144], if we identify the terms generated by the GFT perturbative expansion with the ones obtained form a Regge-like lattice triangulation of GR, then we can interpret the expansion in λ as a cut-off in the degrees of freedom. More precisely, it corresponds to neglecting wavelengths much smaller than L. Degrees of freedom of wavelength larger than L do not matter, since we take a large L limit. We normalize here $\Psi_{\mathbf{q}}$ by $\langle W|\Psi_{\mathbf{q}}\rangle = 1$.

Consider the s-knot (abstract spin network) basis $|s\rangle = |\Gamma, \mathbf{j}, \mathbf{i}\rangle$, where Γ is an abstract graph, $n, m...$ label the nodes of Γ, $\mathbf{j} = \{j_{mn}\}$ are the spins and $\mathbf{i} = \{i_n\}$ the intertwiners of a spin network with graph Γ. Insert a resolution of the identity in (4.1)

$$\mathbf{G}^{abcd}_{\mathbf{q}}(x,y) = \sum_s \langle W|s\rangle\langle s|h^{ab}(x)\ h^{cd}(y)|\Psi_{\mathbf{q}}\rangle. \tag{4.2}$$

In the previous chapter we have seen (3.102) that at first order in λ, $\langle W|s\rangle = W[s] = W[\Gamma, \mathbf{j}, \mathbf{i}]$ where the dominant term in the large-spin limit contains only the contribution coming from

Γ being the pentagonal graph, that is, for the s-knot

$s =$ 

. (4.3)

In this case, and from now on, we have five intertwiners $\mathbf{i} = \{i_n\}$, labeled by $n, m, ... = 1, ..., 5$ and ten spins $\mathbf{j} = \{j_{mn}\}$. We use equally the indices $i, j, k, ... = 1, ..., 5$ to indicate the nodes. Since the operators $h^{ab}(x)$ do not change the graph (they are operators acting on the spin and intertwiners variables $\mathbf{j}, \mathbf{i}$)

$$\mathbf{G}^{abcd}_{\mathbf{q}}(x, y) = \sum_{\mathbf{j},\mathbf{i}} W(\mathbf{j}, \mathbf{i})\ h^{ab}(x)\, h^{cd}(y)\, \Psi(\mathbf{j}, \mathbf{i}) \tag{4.4}$$

where $W(\mathbf{j}, \mathbf{i}) = W[\Gamma_5, \mathbf{j}, \mathbf{i}]$ and $\Psi(\mathbf{j}, \mathbf{i}) = \Psi_{\mathbf{q}}[\Gamma_5, \mathbf{j}, \mathbf{i}] = \langle \Gamma_5, \mathbf{j}, \mathbf{i} | \Psi_{\mathbf{q}} \rangle$, and the sum is over the fifteen variables $(\mathbf{j}, \mathbf{i}) = (j_{nm}, i_n)$. (We use the physicists notation $h^{cd}(y)\, \Psi(\mathbf{j}, \mathbf{i})$ for $[h^{cd}(y)\, \Psi](\mathbf{j}, \mathbf{i})$.)

Following [16], we choose the form of $\Psi(\mathbf{j}, \mathbf{i})$ by identifying Γ_5 with the dual skeleton of a regular triangulation of the three-sphere. Each node $n = 1, ..., 5$ corresponds to a tetrahedron $t_1 ....t_5$ and we choose the points x and y to be the centers x_n and x_m of the two tetrahedra t_n and t_m. We consider

$$\mathbf{G}^{ij,kl}_{\mathbf{q}\,n,m} := \mathbf{G}^{abcd}_{\mathbf{q}}(x_n, x_m)\ n^{(ni)}_a n^{(nj)}_b\ n^{(mk)}_c n^{(ml)}_d, \tag{4.5}$$

where $n^{(ij)}_a$ is the one–form normal to the triangle that bounds the tetrahedra t_i and t_j. From now on, we assume $n \neq m$. Since $h^{ab} = g^{ab} - \delta^{ab} = E^{ai} E^b_i - \delta^{ab}$, this is given by

$$\begin{aligned} \mathbf{G}^{ij,kl}_{\mathbf{q}\,n,m} &= \langle W | \big(E^{(ni)}_n \cdot E^{(nj)}_n - n^{(ni)} \cdot n^{(nj)}\big)\big(E^{(mk)}_m \cdot E^{(ml)}_m - n^{(mk)} \cdot n^{(ml)}\big) | \Psi_{\mathbf{q}} \rangle \\ &= \sum_{\mathbf{j},\mathbf{i}} W(\mathbf{j}, \mathbf{i}) \big(E^{(ni)}_n \cdot E^{(nj)}_n - n^{(ni)} \cdot n^{(nj)}\big)\big(E^{(mk)}_m \cdot E^{(ml)}_m - n^{(mk)} \cdot n^{(ml)}\big) \Psi(\mathbf{j}, \mathbf{i}). \end{aligned} \tag{4.6}$$

where, $E^{(ml)}_n = E^a(\vec{x}) n^{(ml)}_a$ is valued in the $su(2)$ algebra and, with abuse of notation, the scalar product between the triad fields indicates the product in the $su(2)$ algebra (in the internal space); while the scalar product among the one forms $n^{(ij)}$ is the one defined by the background metric δ^{ab}. In the rest of the chapter, we compute the right hand side of (4.6).

4.1.1 Linearity conditions

Before proceedings to the actual computation of (4.6), let us pause to consider the following question. The four normal one-forms of a tetrahedron sum up to zero. Thus

$$\sum_{i \neq n} n^{(ni)}_a = 0. \tag{4.7}$$

This determines a set of linear conditions that must be satisfied by $\mathbf{G}^{ij,kl}_{\mathbf{q}\,n,m}$. In fact, from the last equation it follows immediately that

$$\sum_{i\neq n} \mathbf{G}^{ij,kl}_{\mathbf{q}\,n,m} = 0. \tag{4.8}$$

(The existence of conditions of this kind, of course, is necessary, since the four one forms $n_a^{(ni)}$ (for fixed n) span a three-dimensional space, namely the space tangent to the boundary surface at x_n, and therefore the quantities $\mathbf{G}^{ij,kl}_{\mathbf{q}\,n,m}$ are determined by the restriction of the bi-tensor $\mathbf{G}^{abcd}_{\mathbf{q}}$ to these tangent spaces.) How is it possible that the linear conditions (4.8) are satisfied by the expression (4.6)?

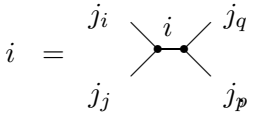
The answer is interesting. The operator $E^{(ni)}_n \cdot E^{(nj)}_n$ acts on the space of the intertwiners of the node n. This is the $SU(2)$ invariant part of the tensor product of the four $SU(2)$ irreducible representations determined by the four spins j_{ni}. In particular, $E^{(ni)}_n$ is the generator of $SU(2)$ rotations in the representation j_{ni}. Therefore

$$J = \sum_{i\neq n} E^{(ni)}_n \tag{4.9}$$

is the generator of $SU(2)$ rotations in the tensor product of these representations. But the intertwiners space is precisely the $SU(2)$ invariant part of the tensor product. Therefore $J = 0$ on the intertwiner space. Inserting this in (4.6), equation (4.8) follows immediately. Therefore the linearity conditions between the projections of the propagator in the space tangent to the boundary surface are implemented by the $SU(2)$ invariance at the nodes.

4.1.2 Operators

We begin by computing the action of the field operator $E^{(ni)}_n \cdot E^{(nj)}_n$ on the state. This operator acts on the intertwiner space at the node n. It acts as a "double grasping" [6, 7, 8, 9, 136] operator (Section 1.3.5) that inserts a virtual link (in the spin-one representation) at the node, connecting the links labelled ni and nj. The state of each node n ($n = 1, .., 5$) is determined by five quantum numbers: the four spins j_{nj} ($n \neq j$, $j = 1, .., 5$) that label the links adjacent to the node and a quantum number i_n of the virtual link that specifies the value of the intertwiner. In this section we study the action of this operator on a single node n; hence we drop for clarity the index n and write the intertwiner quantum number as i, the adjacent spins as j_i, j_j, j_p, j_q, and the operator as $E^{(i)} \cdot E^{(j)}$. We use the graphic notation of $SU(2)$ recoupling theory to compute the action of the operators on the spin network states (see [2]). The basics of this notation are given in Appendix B and the details of the derivation of the action of the operator are given in Appendix G. Choose a given pairing at the node, say $(i, j)(p, q)$ (and fix the orientation, say clockwise, of each of the two trivalent vertices). We represent the node in the form

$i \;=$ 

(4.10)

where we use the same notation i for the intertwiner and the spin of the virtual link that determines it. This basis diagonalizes the operator $E^{(i)} \cdot E^{(j)}$, but not the operators $E^{(i)} \cdot E^{(q)}$

and $E^{(i)} \cdot E^{(p)}$. We consider the action of these three "doublegrasping" operator on this basis. The simplest is the action of $E^{(i)} \cdot E^{(i)}$. Using the formulas in Appendix G we have easily

$$E^{(i)} \cdot E^{(i)} \left| \begin{matrix} j_i & & j_q \\ & i & \\ j_j & & j_p \end{matrix} \right\rangle = -(N^i)^2 \left| \begin{matrix} j_i & 1 & j_q \\ & i & \\ j_j & & j_p \end{matrix} \right\rangle = C^{ii} \left| \begin{matrix} j_i & & j_q \\ & i & \\ j_j & & j_p \end{matrix} \right\rangle, \tag{4.11}$$

where

$$C^{ii} = C^2(j_i). \tag{4.12}$$

with $C^2(a) = a(a+1)$ is the Casimir of the representation a. Just slightly more complicated is the action of $E^{(i)} \cdot E^{(j)}$

$$E^{(i)} \cdot E^{(j)} \left| \begin{matrix} j_i & & j_q \\ & i & \\ j_j & & j_p \end{matrix} \right\rangle = -N^i N^j \left| \begin{matrix} j_i & & j_q \\ 1 & i & \\ j_j & & j_p \end{matrix} \right\rangle = D^{ij} \left| \begin{matrix} j_i & & j_q \\ & i & \\ j_j & & j_p \end{matrix} \right\rangle, \tag{4.13}$$

where

$$D^{ij} = \frac{C^2(i) - C^2(j_i) - C^2(j_j)}{2}. \tag{4.14}$$

In these two cases the action of the operator is diagonal. If, instead, the grasped links are *not* paired together, the action of the operator is not diagonal in this basis. In this case, the recoupling theory in the Appendix gives

$$\begin{aligned} E^{(i)} \cdot E^{(q)} \left| \begin{matrix} j_i & & j_q \\ & i & \\ j_j & & j_p \end{matrix} \right\rangle &= -N^i N^q \left| \begin{matrix} j_i & 1 & j_q \\ & i & \\ j_j & & j_p \end{matrix} \right\rangle = \\ = X^{iq} \left| \begin{matrix} j_i & & j_q \\ & i & \\ j_j & & j_p \end{matrix} \right\rangle &- Y^{iq} \left| \begin{matrix} j_i & & j_q \\ & i-1 & \\ j_j & & j_p \end{matrix} \right\rangle - Z^{iq} \left| \begin{matrix} j_i & & j_q \\ & i+1 & \\ j_j & & j_p \end{matrix} \right\rangle, \end{aligned} \tag{4.15}$$

where

$$X^{iq} = -\frac{\left(C^2(i) + C^2(j_i) - C^2(j_j)\right)\left(C^2(i) + C^2(j_q) - C^2(j_p)\right)}{4\,C^2(i)}, \tag{4.16}$$

$$\begin{aligned} Y^{iq} = -\frac{1}{4i\dim(i)} &\sqrt{(j^i + j^j + i + 1)(j^i - j^j + i)(-j^i + j^j + i)(j^i + j^j - i + 1)} \cdot \\ &\cdot \sqrt{(j^p + j^q + i + 1)(j^p - j^q + i)(-j^p + j^q + i)(j^p + j^q - i + 1)}, \end{aligned} \tag{4.17}$$

$$\begin{aligned} Z^{iq} = -\frac{1}{4(i+1)\dim(i)} &\sqrt{(j^i + j^j + i + 2)(j^i - j^j + i + 1)(-j^i + j^j + i + 1)(j^i + j^j - i)} \cdot \\ &\cdot \sqrt{(j^p + j^q + i + 2)(j^p - j^q + i + 1)(-j^p + j^q + i + 1)(j^p + j^q - i)}. \end{aligned} \tag{4.18}$$

The last possibility is

$$E^{(i)} \cdot E^{(p)} \left| \begin{matrix} j_i & & j_q \\ & i & \\ j_j & & j_p \end{matrix} \right\rangle = -N^i N^p \left| \begin{matrix} j_i & & j_q \\ & i & \\ j_j & & j_p \end{matrix} \right\rangle =$$
$$= X^{ip} \left| \begin{matrix} j_i & & j_q \\ & i & \\ j_j & & j_p \end{matrix} \right\rangle + Y^{ip} \left| \begin{matrix} j_i & & j_q \\ & i-1 & \\ j_j & & j_p \end{matrix} \right\rangle + Z^{ip} \left| \begin{matrix} j_i & & j_q \\ & i+1 & \\ j_j & & j_p \end{matrix} \right\rangle . \tag{4.19}$$

Note that X^{ip} is exactly X^{iq} with p and q switched and $Y^{ip} = Y^{iq}$, $Z^{ip} = Z^{iq}$.

Finally, we have to take care of the orientation. As shown in the Appendix, the sign of the non diagonal terms is influenced by the orientations: in the planar representation that we are using, there is a $+$ sign if the added link intersect the virtual one and a -1 otherwise.

Summarizing, in a different notation and reinserting explicitly the index $_n$ of the node, we have the following action of the EE operators. If the grasped links are paired together we have the diagonal action

$$E^{(ni)} \cdot E^{(nj)} |\Gamma_5, \mathbf{j}, i_1, .., i_n, .., i_5\rangle = S_n^{ij} \; |\Gamma_5, \mathbf{j}, i_1, .., i_n, .., i_5\rangle, \tag{4.20}$$

where

$$S_n^{ij} = \begin{cases} C^{ii} = C^2(j_{ni}) & \text{if } i = j, \\ D^{ij} = \frac{C^2(i_n) - C^2(j_{ni}) - C^2(j_{nj})}{2} & \text{if } i \neq j. \end{cases} \tag{4.21}$$

If the grasped links are not paired together, we have the non-diagonal action

$$E^{(ni)} \cdot E^{(nq)} |\Gamma_5, \mathbf{j}, i_1, .., i_n, .., i_5\rangle =$$
$$= \begin{cases} X_n^{iq} \; |\Gamma_5, \mathbf{j}, i_1, .., i_n, .., i_5\rangle + Y_n^{iq} \; |\Gamma_5, \mathbf{j}, i_1, .., i_n - 1, .., i_5\rangle & \\ \qquad + Z_n^{iq} \; |\Gamma_5, \mathbf{j}, i_1, .., i_n + 1, .., i_5\rangle & \text{if } i \text{ opposite to } q, \\ X_n^{iq} \; |\Gamma_5, \mathbf{j}, i_1, .., i, .., i_5\rangle - Y_n^{iq} \; |\Gamma_5, \mathbf{j}, i_1, .., i_n - 1, .., i_5\rangle & \\ \qquad - Z_n^{iq} \; |\Gamma_5, \mathbf{j}, i_1, .., i_n + 1, .., i_5\rangle & \text{otherwise.} \end{cases} \tag{4.22}$$

This completes the calculation of the action of the gravitational field operators.

4.2 The boundary state

The boundary state utilized in the previous chapter [16] was assumed to have a gaussian dependence on the spins, and to be peaked on a particular intertwiner. This intertwiner was assumed to project trivially onto the BC intertwiner of the BC vertex. This was a simplifying assumption permitting to avoid dealing with the intertwiners, motivated by the fact that intertwiners play no role for the diagonal terms. However, it was also pointed out in [16] that this procedure is not well defined, because of the mismatch between $SO(4)$ linearity and

$SU(2)$ linearity (see the discussion in the Appendix E). Here we face the problem squarely, and consider the intertwiner dependence of the boundary state explicitly.

A natural generalization of the gaussian state used in [16], whith a well-defined and non-trivial intertwiner dependence, is the state

$$\begin{aligned}\Phi(\mathbf{j},\mathbf{i}) = C\exp\left\{-\frac{1}{2j_0}\sum_{(ij)(mr)}\alpha_{(ij)(mr)}\ (j_{ij}-j_0)(j_{mr}-j_0) + i\Phi\sum_{(ij)} j_{ij}\right\}\cdot \\ \cdot\exp\left\{-\sum_n\left(\frac{(i_n-i_0)^2}{4\sigma}+\sum_{p\neq n}\phi(j_{np}-j_0)(i_n-i_0)+i\chi(i_n-i_0)\right)\right\}\end{aligned} \tag{4.23}$$

The first line of this equation is precisely the spin dependence of the state used in the previous chapter. The second line contains a gaussian dependence on the intertwiner variables. More precisely, it includes a diagonal gaussian term, a nondiagonal gaussian spin-intertwiner term, and a phase factor. We do not include non-diagonal intertwiner-intertwiner terms here. These will be considered in the companion paper.

Let us fix some of the constants appearing in (4.23), by requiring the state to be peaked on the expected geometry. The constant j_0 determines the background area A_0 of the faces, via $C(j_{nm}) = A_{nm}$. As in [16], we leave j_0 free to determine the overall scale. The constant Φ determines the background values of the angles between the normals to the tetrahedra. As in [16], we fix them to those of a regular four-simplex, namely $\cos\Phi = -1/4$.

The constant i_0 is the background value of the intertwiner variable. As shown in [16], the spin of the virtual link i_n is the quantum number of the *angle* between the normals of two triangles. More precisely, the Casimir $C(i_n)$ of the representation i_n is the operator corresponding to the classical quantity

$$C^2(i_n) = A_{ni} + A_{nj} + 2\ \vec{n}^{(ni)}\cdot\vec{n}^{(nj)}, \tag{4.24}$$

where i and j are the paired links at the node n and A_{ni} is the area of the triangle dual to the link (ni). The scalar product of the normals to the triangles can therefore be related to the Casimirs of spins and intertwiners:

$$n^{(ni)}\cdot n^{(nj)} = \frac{C(i_n) - C(j_{ni}) - C(j_{nj})}{2}. \tag{4.25}$$

For each node, the state must therefore be peaked on a value i_0 such that

$$i_0(i_0+1) = A_0 + A_0 + 2A_0A_0\cos\theta_{ij}, \tag{4.26}$$

where $\cos\theta_{ij}$ is the 3d dihedral angle between the faces of the tetrahedron. For the regular 4-symplex, in the large distance limit we have $A_{ij} = j_0$, $\cos\theta_{ij} = -\frac{1}{3}$, which gives

$$i_0 = \frac{2}{\sqrt{3}}\ j_0. \tag{4.27}$$

This fixes i_0. Notice that in [16] equation (4.24) refers to the Casimir of an $SO(4)$ simple representation and follows from the quantization of the Plebanski 2 form $B^{IJ} = e^I \wedge e^J$

associated with the discretized geometry. Exactly the same result follows from equation (4.13) directly from *LQG*.

Fixing i_0 in this manner determines only the mean value of the angle θ_{ij} between the two triangles that are paired together in the chosen pairing. What about the mean value of the angles between faces that are not paired together, such as θ_{iq}? It is shown in [149] that a state of the form $e^{(i-i_0)^2/\sigma}$ is peaked on $\theta_{iq} = 0$, which is not what we want; but the mean value of θ_{iq}, can be modified by adding a phase to the state. This is the analog of the fact that a phase changes the mean value of the momentum of the wave packet of a non relativistic particle, without affecting the mean value of the position. In particular, it was shown in [149] that by choosing the phase and the width of the Gaussian to be

$$\chi = \frac{\pi}{2}, \qquad \sigma = \frac{j_0}{3}, \tag{4.28}$$

we obtain a state whose mean value and variance for all angles is the same.

Let us therefore adopt here these values. Still, the present situation is more complicated than the case considered in [149], because the tetrahedron considered there had fixed *and equal* values of the external spins; while here the spins can take arbitrary values around the peak symmetric configuration $j_{nm} = j_0$. As a consequence, when repeating the calculation in [149], one finds additional spin-intertwiner gaussian terms. These, however can be corrected by fixing the spin-intertwiner gaussian terms in (4.23). A detailed calculation (see below), shows indeed that in the large j_0 limit, the state (4.23) transforms under change of pairing into a state with the same intertwiner mean value and the same variance σ, provided we also choose

$$\phi = -i\frac{3}{4j_0}, \tag{4.29}$$

which we assume from now on. With these values and introducing the difference variables $\delta i_n = i_n - i_0$ and $\delta j_{mr} = j_{mr} - j_0$ the wave functional, given in (4.23), reads

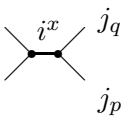
$$\Phi(\mathbf{j},\mathbf{i}) \;= C\; e^{-\frac{1}{2j_0}\sum \alpha_{(ij)(mr)}\delta j_{ij}\delta j_{mr} + i\Phi\sum_{ij}\delta j_{ij}} e^{-\sum_n\left(\frac{3(\delta i_n)^2}{4j_0} - i\left(\sum_a \frac{3}{4j_0}\delta j_{an} - \frac{\pi}{2}\right)\delta i_n\right)}. \tag{4.30}$$

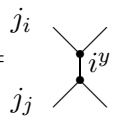
This state, however, presents a problem, which we discuss in the next section.

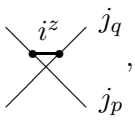
4.2.1 Pairing independence

It is natural to require that the state respects the symmetries of the problem. A moment of reflection shows that the state (4.30) does not. The reason is that the variables i_n are the spin of the virtual links *in one specific pairing*, and this breaks the symmetry of the four-simplex. The phases and variances chosen assure that the mean values are the desired ones, hence symmetric; but an explicit calculation confirms that the relative fluctuations of the angle variables determined by the state (4.30) depend on the pair chosen.

To correct the problem, recall that there are three natural bases in each intertwiner space, determined by the three possible pairings of these links. Denote them as follows.

$i^x =$ 

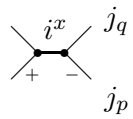
$i^y =$ 

$i^z =$ 

, (4.31)

where we conventionally denote $i^x \equiv i$ the basis in the pairing chosen as reference. These bases diagonalize the three non commuting operators $E^{(i)} \cdot E^{(j)}$, $E^{(i)} \cdot E^{(q)}$ and $E^{(i)} \cdot E^{(p)}$, respectively. Furthermore a spin-network state is specified by the orientation of the three-valent nodes [41]; we fix this orientation by giving an ordering to the links. We write for instance

$$i^{x(+,-)} = \tag{4.32}$$



where the plus sign + (-) means anticlockwise (clockwise) ordering of the links in the two nodes. A complete basis in the space of the spin networks on Γ_5 is specified giving the pairing and the orientation at each node. In order to label the different bases, introduce at each node n a variable m_n that takes the values $m_n = x, y, z$, namely that ranges over the three possible pairings at the node. Similarly, introduce a variables $o_n = \{(++),(+-),(-+),(--)\}$ that labels the possible orientations. To correct the pairing dependence of the state (4.23), let us first rewrite it in the notation

$$|\Phi_{\mathbf{q}}\rangle_{x++} = \sum_{\mathbf{j},\mathbf{i}^{x++}} \Phi[\mathbf{j}, \mathbf{i}^{x++}] \; |\mathbf{j}, \mathbf{i}^{x++}\rangle. \tag{4.33}$$

where the suffix $_{x++}$ to the ket emphasizes the fact that the state has been defined with the chosen pairing and orientation at each node. We can now consider a new state obtained by summing (4.33) over all choices of pairings and orientations. That is, we change the definition of the boundary state to

$$|\Psi_{\mathbf{q}}\rangle = \sum_{m_n, o_n} |\Phi_{\mathbf{q}}\rangle_{m_n o_n}. \tag{4.34}$$

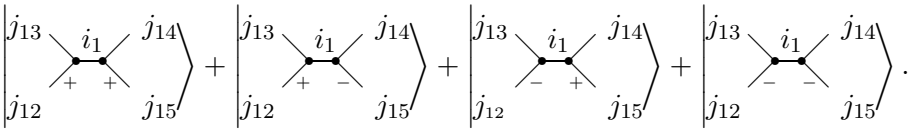
where $\sum_{m_n,o_n} = \sum_{m_1 \ldots m_5} \; \sum_{o_1 \ldots o_5}$ and

$$|\Phi_{\mathbf{q}}\rangle_{m_n o_n} = \sum_{\mathbf{j},\mathbf{i}^{m_n o_n}} \Phi[\mathbf{j}, \mathbf{i}^{m_n o_n}] \; |\mathbf{j}, \mathbf{i}^{m_n o_n}\rangle. \tag{4.35}$$

namely $|\Phi_{\mathbf{q}}\rangle_{m_n o_n}$ is the same as the state $|\Phi_{\mathbf{q}}\rangle_{x++}$, but defined with a different choice of pairing at each node.

Since (by assumption) (4.23) does not depend on the orientation, the sum over the orientation of the node (say) 1, in (4.34) reduces to a term proportional to

$$\sum_o |\mathbf{j}, i_1^o, i_2, i_3, i_4, i_5\rangle \sim \tag{4.36}$$



As shown in the Appendix B, the change in orientation of a vertex produces the sign $(-1)^{a+b+c}$,

where a, b, c are the three addiacent spins. Hence

$$\begin{aligned}\sum_o |\mathbf{j}, i_1^o, i_2, i_3, i_4, i_5\rangle \sim \\ = \left(1 + (-1)^{j_{14}+j_{15}+i_1} + (-1)^{j_{12}+j_{13}+i_1} + (-1)^{j_{12}+j_{13}+j_{14}+j_{15}+2\,i_1}\right) |\mathbf{j}, i_1^{++}, i_2, i_3, i_4, i_5\rangle \\ = \begin{cases} 4\,|\mathbf{j}, i_1^{++}, i_2, i_3, i_4, i_5,\rangle & \text{if } \left(\ j_{12}+j_{13}+i_1^{m_1} = 2n_1 \quad \text{and} \quad j_{14}+j_{15}+i_1^{m_1} = 2n_2\ \right), \\ 0 & \text{otherwise.}\end{cases}\end{aligned} \tag{4.37}$$

We can therefore trade the sum over orientations in (4.34) with a condition on the spins summed over: at all trivalent vertices, the sum of the two external spins and the virtual spin, must be an even integer. (The factor 4 is absorbed in the normalization factor C.) With this understanding, we drop the sum over orientations in (4.34), which now reads

$$|\Psi_{\mathbf{q}}\rangle = \sum_{m_n} |\Phi_{\mathbf{q}}\rangle_{m_n}, \tag{4.38}$$

where all orientations are fixed. This state can of course also be expressed in terms of a single basis

$$|\Psi_{\mathbf{q}}\rangle = \sum_{\mathbf{j},\mathbf{i}} \Psi_{\mathbf{q}}(\mathbf{j},\mathbf{i})\ |\mathbf{j},\mathbf{i}\rangle, \tag{4.39}$$

where we have returned to the notation $i_n = i_n^{x,++}$. Its components are

$$\Psi(\mathbf{j},\mathbf{i}) = \ \langle \mathbf{j},\mathbf{i}|\Psi_{\mathbf{q}}\rangle = \sum_{m_n} \Phi(\mathbf{j},\mathbf{i}^{m_n})\ \langle \mathbf{j},\mathbf{i}|\mathbf{j},\mathbf{i}^{m_n}\rangle. \tag{4.40}$$

The matrices of the change of basis $\langle \mathbf{j},\mathbf{i}|\mathbf{j},\mathbf{i}^{m_n}\rangle$ are (products of five) 6-j Wigner-symbols, as given by standard recoupling theory.

The state (4.38) is the boundary state we shall use. The complication of the sum over pairings is less serious than what could seem at first sight, due to a key technicality that we prove in the next section: the components of (4.38) become effectively orthogonal in the large distance limit.

Orthogonality of the terms in different bases in the large j_0 limit

Suppose we want to compute the norm of the boundary state, in the limit of large j_0. From (4.38), this is given by

$$|\Psi|^2 = \sum_{m_n} \sum_{m'_n} {}_{m_n}\langle \Phi_{\mathbf{q}}|\Phi_{\mathbf{q}}\rangle_{m'_n} \tag{4.41}$$

We now show that in the large j_0 limit the non-diagonal terms of this sum (those with $m_n \neq m'_n$) vanish. Consider one of these terms, say

$$I = {}_{m_n}\langle \Phi_{\mathbf{q}}|\Phi_{\mathbf{q}}\rangle_{m'_n} = \sum_{\mathbf{j}\mathbf{i}^{m_n}} \sum_{\mathbf{j}'\mathbf{i}^{m'_n}} \overline{\Phi(\mathbf{j},\mathbf{i}^{m_n})}\ \Phi[\mathbf{j}',\mathbf{i}^{m'_n}]\ \langle \mathbf{j}\mathbf{i}^{m_n}|\mathbf{j}'\mathbf{i}^{m'_n}\rangle \tag{4.42}$$

where, say, $m_n = (x, x, x, x, x)$ and $m'_n = (y, x, x, x, x)$. The scalar product is diagonal in the spins $\mathbf{j}$ and is given by 6-j symbol in the intertwiners quantum numbers. Hence

$$I = \sum_{\mathbf{j}} \sum_{\mathbf{i}^{m_n}} \sum_{\mathbf{i}^{m'_n}} \overline{\Phi(\mathbf{j}, \mathbf{i}^{m_n})}\ \Phi[\mathbf{j}, \mathbf{i}^{m'_n}]\ \langle i_1^x | i_1^y \rangle, \tag{4.43}$$

where (see Appendix H),

$$\langle i_1^x | i_1^y \rangle = (-1)^{j_{13}+j_{14}+i_1^x+i_1^y}\ \sqrt{d_{i_1^x} d_{i_1^y}}\ \left\{ \begin{array}{ccc} j_{12} & j_{13} & i_1^x \\ j_{15} & j_{14} & i_1^y \end{array} \right\}. \tag{4.44}$$

In the large j_0 limit, this sum can be approximated by an integral, as in [16]. Both the spin and the interwiner sums become gaussian integrals, peaked respectively on j_0 and i_0. The range of the sum over intertwiners is finite for finite j_0, because of the Clebsh Gordan conditions at the two trivalent node; but this range is much larger than the width of the Gaussian in the limit, and therefore the integral over the intertwiner variables too can be taken over the entire real line. In the limit, the 6-j symbol has the asymptotic value [101]

$$\left\{ \begin{array}{ccc} j_{12} & j_{13} & i_1^x \\ j_{15} & j_{14} & i_1^y \end{array} \right\} \approx\ \frac{e^{i(S_R+\frac{\pi}{4})} + e^{-i(S_R+\frac{\pi}{4})}}{\sqrt{12\pi V}}, \tag{4.45}$$

where S_R is the Regge action of a tetrahedron with side length determined by the spins of the 6j symbol, and V is its volume. Changing the sum into an integration and using this, we have

$$I = \int d\mathbf{j} \int d\mathbf{i} \int di_1^y\ \overline{\Phi(\mathbf{j}, \mathbf{i})}\ \Phi(\mathbf{j}, \mathbf{i}^{m'_n})\ (-1)^{j_{13}+j_{14}+i_1^x+i_1^y}\ \frac{e^{i(S_R+\frac{\pi}{4})} + e^{-i(S_R+\frac{\pi}{4})}}{\sqrt{12\pi V}}. \tag{4.46}$$

Inserting the explicit form of the state (4.30) gives

$$\begin{aligned} I = \int d\mathbf{j} \int d\mathbf{i}\ \ & e^{-\frac{1}{j_0}\sum \alpha_{(ij)(mr)}\delta j_{ij}\delta j_{mr} - \sum_{n\neq 1}\frac{3(\delta i_n)^2}{2j_0} - \frac{3(\delta i_1)^2}{4j_0} - i\left(\sum_a \frac{3}{4j_0}\delta j_{an} - \frac{\pi}{2}\right)\delta i_1^x} \\ & \cdot \int di_1^y\ e^{-\frac{3(\delta i_1^y)^2}{4j_0}} e^{i\left(\sum_a \frac{3}{4j_0}\delta j_{an} - \frac{\pi}{2}\right)\delta i_1^y}\ \frac{e^{i(S_R+\pi\delta i_1^y+\frac{\pi}{4})} + e^{-i(S_R-\pi\delta i_1^y+\frac{\pi}{4})}}{\sqrt{12\pi V}}. \end{aligned} \tag{4.47}$$

In the limit, only the first terms in the expansion of the Regge action around the maximum of the peak of the Gaussian matter. We thus Taylor expand the Regge action in its six entries j_{1n}, i_1^x, i_1^y around the background values j_0 and i_0.

$$S_j[j_{na}] = \left.\frac{\partial S_R}{\partial j_{1n}}\right|_{j_0,i_0} \delta j_{1n} + \left.\frac{\partial S_R}{\partial i_1^x}\right|_{j_0,i_0} \delta i_1^x + \left.\frac{\partial S_R}{\partial i_1^y}\right|_{j_0,i_0} \delta i_1^y + \text{higher order terms}. \tag{4.48}$$

The key point now is that the first of these terms is a rapidly oscillating phase factor in the j_{1n} variable. The Gaussian j_{1n} integration in (4.47) is suppressed by this phase factor. More precisely, the integral is like a Fourier transform in the j_{1n} variable, of a gaussian centered around a large value of j_0 with variance proportional $\sqrt{j_0}$; this Fourier transform is then a gaussian with variance $1/\sqrt{j_0}$, which goes to zero in the $j_0 \to \infty$ limit. *QED.*

Change of basis

For later convenience, let us also give here the expression of the state (4.30) under the transformation induced by the change of basis associated to a change of pairing. Say we change from the basis i^y to the basis i^x in the node $n = 1$. Then directly from (4.40) we have

$$\begin{aligned}\Phi'_{\mathbf{q}}[\mathbf{j}, i_1^x, i_2...i_5] = \quad & e^{-\frac{1}{2j_0}\sum \alpha_{(ij)(mr)}\delta j_{ij}\delta j_{mr}+i\sum \Phi\delta j_{ij}} e^{-\sum_{n\neq 1}\left(\frac{3(\delta i_n)^2}{4j_0}-i\left(\sum_a \frac{3}{4j_0}\delta j_{an}-\frac{\pi}{2}\right)\delta i_n\right)} \\ & \cdot \sum_{i_1^y} e^{-\left(\frac{3(\delta i_1^y)^2}{4j_0}-i\left(\sum_a \frac{3}{4j_0}\,\delta j_{a1}-\frac{\pi}{2}\right)\delta i_1^y\right)} (-1)^{j_{13}+j_{14}+i_1^x+i_1^y}\sqrt{d_{i_1^x}d_{i_1^y}}\begin{Bmatrix} j_{12} & j_{13} & i_1^x \\ j_{15} & j_{14} & i_1^y \end{Bmatrix}\end{aligned} \tag{4.49}$$

where, we recall, the sum over intertwiners is under the condition (4.37) that that gives $(-1)^{j_{13}+j_{14}+i_1^x} = 1$ We can evaluate the sum in the large j_0 limit by approximating it again with an integral. Inserting the asymptotic expansion of the $6j$ symbol, we have

$$\begin{aligned}\Phi'_{\mathbf{q}}(\mathbf{j}, i_1^x, i_2...i_5) = \quad & e^{-\frac{1}{2j_0}\sum \alpha_{(ij)(mr)}\delta j_{ij}\delta j_{mr}+i\sum \Phi\delta j_{ij}} e^{-\sum_{n\neq 1}\left(\frac{3(\delta i_n)^2}{4j_0}-i\left(\sum_a \frac{3}{4j_0}\delta j_{an}-\frac{\pi}{2}\right)\delta i_n\right)} e^{i\pi i_0} \\ & \cdot \int d\delta i_1^y e^{-\left(\frac{3(\delta i_1^y)^2}{4j_0}-i\left(\sum_a \frac{3}{4j_0}\,\delta j_{a1}-\frac{\pi}{2}\right)\delta i_1^y\right)}\sqrt{d_{i_1^x}d_{i_1^y}}\,\frac{e^{i(S_R+\pi\delta i_1^y+\frac{\pi}{4})}+e^{-i(S_R-\pi\delta i_1^y+\frac{\pi}{4})}}{\sqrt{12\pi V}}.\end{aligned} \tag{4.50}$$

This can be computed expanding the Regge action to second order around j_0 and i_0. As shown in the Appendix J, the result is

$$\Phi'_{\mathbf{q}}(\mathbf{j}, i_1^x, i_2, ..., i_5) = \Phi(\mathbf{j}, i_1^x, i_2, ..., i_5) N_1\; e^{-iS[j_{1a}]} e^{-2i\left(\sum_a \frac{3}{4j_0}\,\delta j^{a1}\right)\delta i_1^x}, \tag{4.51}$$

where N_1 is a normalization constant with $|N_1|^2 = 1$, and $S[j_{1a}]$ is the expansion of the Regge Action linked to the tetrahedron associated with the {6j} symbol (4.45) up to the second order *only in the link variables*, that is

$$S[j_{na}] = \left.\frac{\partial S_R}{\partial j_{1n}}\right|_{j_0,i_0}\delta j_{1n} + \left.\frac{\partial^2 S_R}{\partial j_{1n}\partial j_{1n'}}\right|_{j_0,i_0}\delta j_{1n}\delta j_{1n'} + \frac{1}{2}\left.\frac{\partial^2 S_R}{\partial^2 j_{1n}}\right|_{j_0,i_0}(\delta j_{1n})^2. \tag{4.52}$$

This result follows from the choice (4.28) and (4.29) of the parameters in (4.23). In particular, the value $\chi_n = \frac{\pi}{2}$ makes the intertwiner phase equal, with opposite sign, to the term $\exp -i\left(\left.\frac{\partial S_R}{\partial i_1^y}\right|_{j_0,i_0}\delta i_1^y - \pi\delta i_1^y\right)$, namely the term in the expansion of the Regge action S_R linear in the variable δi_1^y. This selects one of the two exponentials in the asymptotic expansion (4.45), while the rapidly oscillating phase factor in the variables δi_1^y cancels the other.

The same calculation gives the $i^z \to i^x$ change of variable

$$\Phi''_{\mathbf{q}}(\mathbf{j}, i_1^x, i_2, ..., i_5) = \Phi(\mathbf{j}, i_1^x, i_2, ..., i_5) N_1\; e^{-iS'[j_{1a}]} e^{-2i\left(\sum_a \frac{3}{4j_0}\,\delta j^{a1}\right)\delta i_1^x}, \tag{4.53}$$

with the same constant N_1 as above. The only differences between (4.51) and (4.53) is that the arguments of the 6-j symbol enter with a different order, so that $S'(j_{12}, j_{13}, j_{14}, j_{15},) = S(j_{12}, j_{13}, j_{15}, j_{14})$.

Using these results, we can explicitly rewrite the state (4.38) in our preferred basis. We obtain easily

$$|\Psi_{\mathbf{q}}\rangle = 4^5 \sum_{\mathbf{j},\mathbf{i}} \Phi(\mathbf{j},\mathbf{i}) \prod_{n=1}^{5} G[\delta j_{na}, \delta i_n] \, |\mathbf{j},\mathbf{i}\rangle \,, \tag{4.54}$$

where

$$G[\delta j_{na}, \delta i_n] = \left(1 + N_1 e^{-2i\left(\sum_a \frac{3}{4j_0}\,\delta j^{an}\right)\delta i_n^x} \left(e^{-iS[j_{na}]} + e^{-iS'[j_{na}]}\right)\right). \tag{4.55}$$

4.2.2 Mean values and variances

With these preliminary completed, we can now check that mean values and relative fluctuations of areas and angles have the right behavior in the large scale limit. With the notation

$$\langle O\rangle := \frac{\langle\Psi_{\mathbf{q}}|\, O\, |\Psi_{\mathbf{q}}\rangle}{\langle\Psi_{\mathbf{q}}|\Psi_{\mathbf{q}}\rangle} \quad \text{and} \quad \Delta O = \sqrt{\langle O^2\rangle - \langle O\rangle^2} \tag{4.56}$$

we demand

$$\langle j_{ni}\rangle = j_0 \quad \text{and} \quad \frac{\Delta j_{ni}}{\langle j_{ni}\rangle} \to 0 \quad \text{when } j_0 \to \infty, \tag{4.57}$$

as in [16], as well as

$$\langle i_n^{m_n}\rangle = i_0 \quad \text{and} \quad \frac{\Delta i_n^{m_n}}{\langle i_n^{m_n}\rangle} \to 0 \quad \text{when } j_0 \to \infty. \tag{4.58}$$

Notice that we demand this for all m_n, namely for each node *in each pairing*.

It is easy to show that the state (4.38) satysfies (4.57). Because of the vanishing of the interference terms proven above, in large j_0 limit the mean values reduce to the average of the mean values on each diagonal term.

$$\begin{aligned}
\langle j_{ni}\rangle &\approx \frac{\sum_{m_n}\sum_{\mathbf{j}}\sum_{\mathbf{i}_n^{m_n}} j_{ni}\,|\Phi[\mathbf{j}\ \mathbf{i}_n^{m_n}]|^2}{\sum_{m_n}\sum_{\mathbf{j}}\sum_{\mathbf{i}_n^{m_n}}|\Phi[\mathbf{j}\ \mathbf{i}_n^{m_n}]|^2} \\
&\approx \frac{\sum_{m_n}\int d\delta\mathbf{j}\, d\delta\mathbf{i}_n^{m_n}\, j_{ni}\; e^{-\frac{1}{j_0}\sum \alpha_{(ij)(mr)}\delta j_{ij}\delta j_{mr}} e^{-\sum_n \frac{3(\delta i_n^{m_n})^2}{2j_0}}}{\sum_{m_n}\int d\delta\mathbf{j}\, d\delta\mathbf{i}_n^{m_n}\, e^{-\frac{1}{j_0}\sum \alpha_{(ij)(mr)}\delta j^{ij}\delta j_{mr}} e^{-\sum_n \frac{3(\delta i_n^{m_n})^2}{2j_0}}} = j_0.
\end{aligned} \tag{4.59}$$

The calculation of the variance and mean value in the intertwiner variable is a bit more complicated. It is convenient to express the state in the pairing of the relevant variable using (4.51) and (4.53). With this, we have

$$\begin{aligned}
\langle i_1^x\rangle &\approx \frac{\sum_{m_n\neq m_1}\sum_{\mathbf{j}}\sum_{\mathbf{i}_{n\neq 1}^{m_n}}\sum_{i_1^x} i_1^x \left(|\Phi_{\mathbf{q}}|^2 + |\Phi'_{\mathbf{q}}|^2 + |\Phi''_{\mathbf{q}}|^2\right)}{\sum_{m_n}\sum_{\mathbf{j}}\sum_{\mathbf{i}_n^{m_n}}|\Phi_{\mathbf{q}}|^2} \\
&\approx 3\,\frac{\sum_{m_n\neq m_1}\sum_{\mathbf{j}}\sum_{\mathbf{i}_{n\neq 1}^{m_n}}\sum_{i_1^x} i_1^x\,|\Phi_{\mathbf{q}}|^2}{\sum_{m_n}\sum_{\mathbf{j}}\sum_{\mathbf{i}_n^{m_n}}|\Phi_{\mathbf{q}}|^2} = \frac{\sum_{m_n\neq m_1}\sum_{\mathbf{j}}\sum_{\mathbf{i}_{n\neq 1}^{m_n}}\sum_{i_1^x} i_1^x\,|\Phi_{\mathbf{q}}|^2}{\sum_{m_n\neq m_1}\sum_{\mathbf{j}}\sum_{\mathbf{i}_{n\neq 1}^{m_n}}\sum_{i_1^x}|\Phi_{\mathbf{q}}|^2} = i_0,
\end{aligned} \tag{4.60}$$

where we have used the (4.51) and (4.53) and the fact that the constant N_1 in these expression satisfies $|N_1|^2 = 1$. The same procedure can be used to compute the variance and check that (4.58) is satisfied.

4.3 Calculation of the propagator

We are now ready to compute all components of the propagator (4.6). Consider this quantity for a fixed value of m, n, i, j, k, l. Because of the sum in (4.55), the propagator can be written in the form:

$$\mathbf{G}^{ij,kl}_{\mathbf{q}\,n,m} = 4^5 \sum_{\mathbf{j}} \sum_{\mathbf{i}_n} \Phi(\mathbf{j},\mathbf{i}) \prod_{n=1}^{5} G[\delta j_{na}, \delta i_n] \cdot \langle W | \big(E^{(ni)}_n \cdot E^{(nj)}_n - n^{(ni)} \cdot n^{(nj)}\big)\big(E^{(mk)}_m \cdot E^{(ml)}_m - n^{(mk)} \cdot n^{(ml)}\big) \, |\mathbf{j}, \mathbf{i}_n\rangle \,, \tag{4.61}$$

For a given value of m, n, i, j, k, l, we now can fix the reference choice of pairing so that (ij) (if different) are paired at the node n and (kl) (if different) are paired at the node m. With this choice of basis the action of the operators is diagonal, and we have

$$\mathbf{G}^{ij,kl}_{\mathbf{q}\,n,m} = 4^5 \sum_{\mathbf{j}} \sum_{\mathbf{i}_n} \Phi(\mathbf{j},\mathbf{i}) \prod_{n=1}^{5} G[\delta j_{na}, \delta i_n] \big(D^{ij}_n - n^{(ni)} \cdot n^{(nj)}\big)\big(D^{kl}_m - n^{(mk)} \cdot n^{(ml)}\big) \langle W | \mathbf{j}, \mathbf{i}\rangle \tag{4.62}$$

We use the same form of the the Barret-Crane vertex as in [14, 15]. This is given by

$$\langle W | \mathbf{j}, \mathbf{i}\rangle := W(\mathbf{j},\mathbf{i}) = W(\mathbf{j}) \prod_n \langle i_{BC} | i_n\rangle = W(\mathbf{j}) \prod_n (2i_n + 1), \tag{4.63}$$

where $W(\mathbf{j})$ is the Barrett-Crane vertex, which a functions of the ten spins alone. In the large distance limit, $\prod_n (2i_n + 1) = 2i_0^5$, hence

$$W(\mathbf{j},\mathbf{i}) = 2i_0^5 \, W(\mathbf{j}). \tag{4.64}$$

Using this, (4.62) becomes

$$\mathbf{G}^{ij,kl}_{\mathbf{q}\,n,m} = \sum_{\mathbf{j}} W(\mathbf{j}) \sum_{\mathbf{i}^x_n} \Phi(\mathbf{j},\mathbf{i}) \prod_{n=1}^{5} G[\delta j_{na}, \delta i_n] \big(D^{ij}_n - n^{(ni)} \cdot n^{(nj)}\big)\big(D^{kl}_m - n^{(mk)} \cdot n^{(ml)}\big) \tag{4.65}$$

where we have absorbed numerical factors and i_0^5 in the normalization of the state. Each factor $G[\delta j_{na}, \delta i_n]$ in this expression has the form $(1 + Ne^{iS} + Ne^{iS'})$. The terms with the exponents contain rapidly oscillating phases in the spin variables, which again suppress the integral in the large j_0 limit. Therefore we can drop these factors.

The value of the eigenvalues D^{ij}_n is given in (4.21). The value of the product of normals is given in (4.25). Using these, we have

$$D^{ij}_n - n^{(ni)} \cdot n^{(nj)} = \frac{(C(i_n) - C(i_0)) - (C(j^{(ni)}) - C(j_0)) - (C(j^{(nj)}) - C(j_0))}{2}. \tag{4.66}$$

Expanding up to second order around the background values j_0 and i_0

$$C(j_j) - C(j_0) = (\delta j_j)^2 + 2\delta j_j j_0 + \delta j_j, \tag{4.67}$$

we obtain, in the large j_0 limit

$$D_n^{ij} - n^{(ni)} \cdot n^{(nj)} = \delta i_n \, i_0 - \delta j_j j_0 - \delta j_{nk} j_0. \tag{4.68}$$

Inserting this in (4.65) we have

$$\mathbf{G}^{ij,kl}_{\mathbf{q}\,n,m} = j_0^2 \sum_{\mathbf{j}} W(\mathbf{j}) \sum_{\mathbf{i}_n^x} \left(\frac{2}{\sqrt{3}} \, \delta i_n - \delta j_{ni} - \delta j_{nk} \right) \left(\frac{2}{\sqrt{3}} \, \delta i_m - \delta j_{mk} - \delta j_{ml} \right) \Phi(\mathbf{j}, \mathbf{i}) \tag{4.69}$$

In the case in which two of the indices of the propagator are parallel, say $i = j$, this reduces easily to

$$\mathbf{G}^{ii,kl}_{\mathbf{q}\,n,m} = 2j_0^2 \sum_{\mathbf{j}} W(\mathbf{j}) \sum_{\mathbf{i}_n^x} \delta j_{ni} \left(\frac{2}{\sqrt{3}} \delta i_m - \delta j_{mk} - \delta j_{ml} \right) \Phi(\mathbf{j}, \mathbf{i}). \tag{4.70}$$

While if $i = j$ and $k = l$ we recover the diagonal terms,

$$\mathbf{G}^{ii,kk}_{\mathbf{q}\,n,m} = 4j_0^2 \sum_{\mathbf{j}} W(\mathbf{j}) \sum_{\mathbf{i}_n^x} \delta j_{ni} \delta j_{mk} \; \Phi(\mathbf{j}, \mathbf{i}). \tag{4.71}$$

We can now evaluate (4.69). Inserting the explicit form of the state gives

$$\begin{aligned} \mathbf{G}^{ij,kl}_{\mathbf{q}\,n,m} = C j_0^2 \sum_{\delta\mathbf{j},\delta\mathbf{i}} W(\mathbf{j}) \left(\frac{2}{\sqrt{3}} \, \delta i_n - \delta j_{ni} - \delta j_{nk} \right) \left(\frac{2}{\sqrt{3}} \, \delta i_m - \delta j_{mk} - \delta j_{ml} \right) \\ \cdot \; e^{-\frac{1}{2j_0} \sum \alpha_{(ij)(mr)} \delta j_{ij} \delta j_{mr} + i \sum \Phi \delta j_{ij}} e^{-\sum_n \left(\frac{3(\delta i_n)^2}{4j_0} - i \left(\sum_a \frac{3}{4j_0} \, \delta j_{an} + \frac{\pi}{2} \right) \delta i_n \right)}. \end{aligned} \tag{4.72}$$

Using the asymptotic expression for the BC vertex, we can proceed like in [14] and [15]. The rapidly oscillating phase term in the spins selects one of the the factors of this expansion, giving

$$\begin{aligned} \mathbf{G}^{ij,kl}_{\mathbf{q}\,n,m} = \mathcal{N} j_0^2 \sum_{\delta j^{(ab)},\, \delta i_\alpha} \prod_{a<b} \dim(j^{(ab)}) \left(\frac{2}{\sqrt{3}} \, \delta i_n - \delta j_{ni} - \delta j_{nk} \right) \left(\frac{2}{\sqrt{3}} \, \delta i_m - \delta j_{mk} - \delta j_{ml} \right) \\ \cdot \; e^{-\frac{1}{2j_0} (\alpha + i G j_0)_{(ij)(mn)} \; \delta j_{ij} \delta j_{mn}} e^{-\sum_n \left(\frac{3(\delta i_n)^2}{4j_0} - i \left(\sum_a \frac{3}{4j_0} \, \delta j^{an} + \frac{\pi}{2} \right) \delta i_n \right)}, \end{aligned} \tag{4.73}$$

where the phase factor $i\Phi \sum_{pq} j_{pq}$ in (4.72) has been absorbed by the corresponding phase factor in the asymptotic expansion of the $10j$ symbol $W(\mathbf{j})$ (see Section 3.2.5), as in [15, 16]. Here G is the matrix of the second derivatives of the Regge action (see [15, 16]) and should not be confused with the G used in the Appendix. Finally,

$$\begin{aligned} \mathbf{G}^{ij,kl}_{\mathbf{q}\,n,m} = \mathcal{N}' j_0^2 \sum_{\delta j^{(ab)},\, \delta i_\alpha} \left(\frac{2}{\sqrt{3}} \, \delta i_n - \delta j_{ni} - \delta j_{nk} \right) \left(\frac{2}{\sqrt{3}} \, \delta i_m - \delta j_{mk} - \delta j_{ml} \right) \cdot \\ \cdot \; e^{-\frac{1}{2j_0} (\alpha + i G j_0)_{(ij)(mn)} \; \delta j_{ij} \delta j_{mn}} e^{-\sum_n \left(\frac{3(\delta i_n)^2}{4j_0} - i \left(\sum_a \frac{3}{4j_0} \, \delta j_{an} + \frac{\pi}{2} \right) \delta i_n \right)}. \end{aligned} \tag{4.74}$$

We can rearrange this expression introducing the 15 components vector $\delta I^\alpha = (\delta j^{ab}, \delta i_n)$ and $\Theta^\alpha = (0, \chi_{i_n})$ and the 15×15 correlation matrix

$$M = \begin{pmatrix} \mathrm{A}_{10\times 10} & \mathrm{C}_{10\times 5} \\ \mathrm{C}^T_{5\times 10} & \mathrm{S}_{5\times 5} \end{pmatrix}, \tag{4.75}$$

where $A_{ab\,cd} = \frac{1}{2}(\alpha + iGj_0)_{ab\,cd}$ is a 10×10 matrix and $S_{nm} = I_{nm}\frac{3}{4}$ is a diagonal 5×5 matrix and C is a 10×5 matrix and C^T is its transpose, and evaluate it approximating the sum with an integral

$$\mathbf{G}^{ij,kl}_{\mathbf{q}\,n,m} = \mathcal{N}' j_0^2 \int d\delta I^\alpha \left(\frac{2}{\sqrt{3}}\,\delta i_n - \delta j_{ni} - \delta j_j\right)\left(\frac{2}{\sqrt{3}}\,\delta i_m - \delta j_{mk} - \delta j_{ml}\right) e^{-\frac{M_{\alpha\beta}}{j_0}\delta I^\alpha \delta I^\beta}\; e^{i\Theta_\alpha \delta I^\alpha}. \tag{4.76}$$

The matrix M is invertible and independent from j_0. Direct calculation using (M.5) gives a sum of terms of the kind

$$\frac{e^{-j_0 \Theta M^{-1}\Theta}}{\sqrt{\det M}}\left(j_0^3 M^{-1}_{\alpha\beta} - j_0^4 M^{-1}_{\alpha\gamma}\Theta^\gamma M^{-1}_{\beta\delta}\Theta^\delta\right). \tag{4.77}$$

These terms go to zero fast in the $j_0 \to \infty$ limit, and therefore do not match the expected large distance behavior of the propagator.

One could hope to circumvent the problem behaviour thanks to the normalization factor. Including this explicitly we have

$$\tilde{\mathbf{G}}^{ij,kl}_{\mathbf{q}\,n,m} = \frac{\langle W|\left(E^{(ni)}_n \cdot E^{(nj)}_n - n^{(ni)}\cdot n^{(nj)}\right)\left(E^{(mk)}_m \cdot E^{(ml)}_m - n^{(mk)}\cdot n^{(ml)}\right)|\Psi_{\mathbf{q}}\rangle}{\langle W|\Psi_{\mathbf{q}}\rangle}. \tag{4.78}$$

The denominator gives

$$\langle W|\Psi_{\mathbf{q}}\rangle = \frac{e^{-j_0\Theta M^{-1}\Theta}}{\sqrt{\det M}}. \tag{4.79}$$

Terms of the kind (4.77) are still pathological, since they give

$$\left(\frac{M^{-1}_{\alpha\beta}}{j_0} - M^{-1}_{\alpha\gamma}\Theta^\gamma M^{-1}_{\beta\delta}\Theta^\delta\right) \tag{4.80}$$

in the limit. In conclusion, the calculation presented does not appear to give the correct low energy propagator.

4.4 Conclusions

The calculation presented above is based on a number of assumptions on the form of the boundary state. Could the negative result that we have obtained be simply the result of these assumptions being too strict, or otherwise wrong? Could, in particular, a different boundary state give the correct low energy behavior? Although we do not have a real proof, we do not think that this is the case. The original aim of the research program motivating

the paper [17] was to find such a state; the negative result we report here has initially come as a disappointment, and we have fought against it at long. We have eventually got to the conclusion that the problem is more substantial, and is related to the BC vertex itself, at least as it is used in the present approach. There are several indications pointing to this conclusion.

First, the trivial intertwiner dependence of the Barrett-Crane structure clashes with the intertwiner dependence of the boundary state that is needed to have a good semiclassical behavior. Since the variables associated to the angles between faces do not commute with one another, the boundary state cannot be sharp on a classical configuration. In order for a state peaked on a given angle to be also peaked on the other non-commuting angles, the state must have a phase dependence from intertwiners and spin variables. Following the general structure of quantum mechanics, one then expect the transition amplitude matching between coherent states to include a phase factor exactly balancing those phases. This is the case for instance for the free propagator of a non-relativistic quantum particles, as well as for the phases associated to the angles between tetrahedra in the calculation illustrated in [14, 15]. However, no such phase factor appears in the BC vertex. In particular, the phase factor $i\frac{\pi}{2}\sum_p i_p$ present in the boundary state (necessary to have the complete symmetry of the state) is not matched by a corresponding factor in the vertex amplitude. This factor gives the rapidly oscillating term that suppresses the sum.

Second, as already mentioned, there is in fact a structural difficulty, already pointed out in [14, 15], with the definition (4.63) of the amplitude, and we think that this difficulty is at the roots of the problem. Let us illustrate this difficulty in detail.

There are two possible interpretations of equation (4.63). The first is that this is true is one particular basis, namely for $i_n = i^x_n$. Let us discard this possibility, which would imply that the BC the vertex itself would depend on a specific choice of pairing. The second is that it is (simultaneously) true in all possible bases, that is

$$\langle W|\mathbf{j},\mathbf{i}^{m_n}\rangle = W(\mathbf{j})\prod_n(2i^{m_n}+1) \tag{4.81}$$

for any choice of pairing, namely for any choice of m_n. This is indeed the definition of the vertex that we have implicitly used. However, defined in this way, the vertex $\langle W|$ is not a *linear* functional on the state space. This is immediately evident by expressing, say $\langle i^y_1|$ on the $\langle i^x_1|$ basis.

We can say this in other words. The Barrett-Crane intertwiner is defined as a sum of simple $SO(4)$ intertwiners, that we can write as

$$i_{\rm BC} = \sum_{i^x}(2i^x+1)|i^x,i^x\rangle = \sum_{i^y}(2i^y+1)|i^y,i^y\rangle$$

$$= \sum_{i^x}(2i^x+1)$$   $= \sum_{i^y}(2i^y+1)$   (4.82)

Hence

$$\langle i_{BC}|i^m,i^m\rangle = (2i^m+1) \tag{4.83}$$

whatever is m. Since the simple SO(4) intertwiner $|i^x,i^x\rangle$ diagonalizes the same geometrical

quantity as the $SO(3)$ intertwiner $|i^x\rangle$, it is tempting to physically identify the two and write

$$\langle i_{BC}|i^m\rangle = (2i^m + 1). \tag{4.84}$$

But there is no state $\langle i_{BC}|$ in the the SO(3) intertwiner space that has this property. In other words, there is a mismatch between the linear structures of SO(4) and SO(3) in building up the theory that we have used.

In the next chapter, we show that, perhaps surprisingly, a vertex with a suitable asymptotic behavior can overcame all these difficulties.

Chapter 5

The complete LQG propagator: II. Asymptotic behavior of the vertex

In the previous chapter we have shown that there are difficulties in obtaining the correct LQG graviton propagator from the dynamics defined by the Barrett-Crane vertex amplitude and that this vertex *fails* to give the correct tensorial structure of the graviton propagator in the large-distance limit. The natural question is whether this is an intrinsic difficulty of the background-independent loop and spinfoam formalism, or whether it is a specific difficulty of the BC vertex.

The natural question is whether this is an intrinsic difficulty of the background-independent loop and spinfoam formalism, or whether it is a specific difficulty of the BC vertex. In this chapter based on the paper [18] we show that the answer is the second. We do so by explicitly exhibiting a vertex amplitude W that yields the correct propagator in the large distance limit. We have no claim that this vertex amplitude is physically correct. In fact, it is a rather artificial object, chosen by simply taking the asymptotic form of the BC vertex, and correcting the detail for which the BC vertex fails to work. Thus, W has at best an interest in the asymptotic region. But its existence shows that the background-independent loop and spinfoam formalism, *can* yield the full tensorial structure of the perturbative n-point functions.

Furthermore, the properties of W give some indications on the asymptotic that the dynamics can have, if it has to yield the correct low energy limit. The detail of the BC vertex that needs to be corrected turns out to be a *phase* in the intertwiner variables. A posteriori, the need for this phase appears pretty obvious on physical grounds, as we shall discuss in detail. This might provide a useful indication for selecting a definition of the dynamics alternative to the one provided by the BC vertex. While the BC vertex is defined by the $SO(4)$ Wigner 10j symbol, an alternative vertex given by the square of an $SU(2)$ Wigner 15j symbol has been introduced recently [143, 144]. This vertex can be derived also using coherent states techniques, and can be extended to the Lorentzian case and to arbitrary values of the Immirzi parameter [145, 146, 147, 148, 150, 151]. It would be very interesting to see whether the asymptotics of this vertex exhibit the phase dependence that we find here to be required for the low energy limit.

In the first section we introduce the vertex W and we give a simple explanation of the reason why the additional phase is needed. In the rest of the chapter we prove that W

yields the correct full tensorial structure of the propagator. In developing this calculation we have stumbled upon an unexpected result that indicates that the state used in [17] is too symmetric. This does not affect the results of [17], but forces us to reconsider the definition of the state. We discuss this issue in detail and give the appropriate boundary state. Then we compute the propagator, and we compare it with the one computed in linearized quantum general relativity.

5.1 The vertex and its phase

Following [14, 15, 16], the graviton propagator can be computed in a background independent context as the scalar product

$$\mathbf{G}^{ij,kl}_{\mathbf{q}\,n,m} = \langle W|\big(E^{(i)}_n \cdot E^{(j)}_n - n^{(i)}_n \cdot n^{(j)}_n\big)\big(E^{(k)}_m \cdot E^{(l)}_m - n^{(k)}_m \cdot n^{(l)}_m\big)|\Psi_{\mathbf{q}}\rangle. \tag{5.1}$$

Here $\langle W|$ is the boundary functional, which can be intuitively understood as the path integral of the Einstein-Hilbert action on a finite spacetime region $\mathcal{R}$, with given boundary configuration. The indices $i, j, k, l, m, n, ...$ run over the values $1, ..., 5$ and label the tetrahedra of a 4-simplex. The operator $E^{(i)}_n$ (denoted $E^{(ni)}_n$ in [17]) is the triad operator at the points n, contracted with (test) one-forms $n^{(i)}_n$ (denoted n^{ni} in [17]) at the same point. $|\Psi_{\mathbf{q}}\rangle$ is a state on the boundary of $\mathcal{R}$, picked on a given classical boundary (extinsic and extrinsic) geometry $\mathbf{q}$.

Fixing such a boundary geometry is equivalent to fixing a background metric g in the interior, where g is the solution of the Einstein equations with boundary data $\mathbf{q}$. The existence of such a background metric is part of the definition of the propagator, which is a measure of fluctuations around a given background. Criticisms to the approach of [14, 15, 16] have been raised on the ground that a propagator makes no sense in a background independent context, because it is a quantity that depends on a background geometry. These criticisms follow from a misunderstanding of this point. The information about the background over which the propagator is defined is in the boundary state, via $\mathbf{q}$.

We are interested in the value of (5.1) to first order in the GFT expansion parameter λ, and in the limit in which the boundary surface (whose size is determined by $\mathbf{q}$) is large. On the physical interpretation of this limit, see [143]. To first order, the leading contribution to W has support only on spin networks with a 4-simplex graph. If $\mathbf{j} = (j_{nm})$ and $\mathbf{i} = (i_n)$ are, respectively, the ten spins and the five intertwiners that color this graph, then in this approximation (5.1) reads

$$\mathbf{G}^{ij,kl}_{\mathbf{q}\,n,m} = \sum_{\mathbf{j},\mathbf{i}} W(\mathbf{j},\mathbf{i})\big(E^{(i)}_n \cdot E^{(j)}_n - n^{(i)}_n \cdot n^{(j)}_n\big)\big(E^{(k)}_m \cdot E^{(l)}_m - n^{(k)}_m \cdot n^{(l)}_m\big)\Psi(\mathbf{j},\mathbf{i}). \tag{5.2}$$

To this order, W is just determined by the amplitude of a single vertex. In [14, 16, 17], (a suitable adjustment of) the BC vertex was chosen for W. The propagator depends only on the asymptotic behavior of the vertex. This has the structure [99, 100, 101, 102]

$$W_{BC}(\mathbf{j}) \sim e^{\frac{i}{2}(\delta\mathbf{j}G\delta\mathbf{j})}e^{i\Phi\cdot\delta\mathbf{j}} + e^{-\frac{i}{2}(\delta\mathbf{j}G\delta\mathbf{j})}e^{-i\Phi\cdot\delta\mathbf{j}}, \tag{5.3}$$

where G is the 10×10 matrix given by the second derivatives of the 4d Regge action around the symmetric state, $\delta\mathbf{j}$ is the difference between the ten spins $\mathbf{j}$ and their background value

j_0, and Φ is a 10d vector with all equal components, which were shown in [14, 16] to precisely match those determined by the background extrinsic curvature. The diagonal components of the propagator determined by (5.1) turn out to be correct at first [14] and second [16] order, but the nondiagonal components fail to do so [17].

Here we make a different choice for W. We choose a vertex W with an asymptotic form that includes a gaussian intertwiner-intertwiner and spin-intertwiner dependence, and –most crucially– a phase dependence on the intertwiner variables. To write this, introduce a 15d vector $\delta\mathbf{I} = (\delta\mathbf{j}, \delta\mathbf{i})$, where $\delta\mathbf{i}$ is the difference between the five intertwiners $\mathbf{i}$ and their background value i_0. Explicitly, $\delta I_\alpha = (\delta j_{nm}, \delta i_n) = (j_{nm} - j_0, i_n - i_0)$, where $\alpha = (nm, n)$. And consider the state

$$W(\mathbf{j}, \mathbf{i}) = e^{\frac{i}{2}(\delta\mathbf{I}G\delta\mathbf{I})} e^{i\phi\cdot\delta\mathbf{I}} + e^{-\frac{i}{2}(\delta\mathbf{I}G\delta\mathbf{I})} e^{-i\phi\cdot\delta\mathbf{I}}. \tag{5.4}$$

Here G is now a 15×15 matrix and $\phi = (\phi_{nm}, \phi_n)$ is a 15d vector. Its 10 spin components ϕ_{nm} just reproduce the spin phase dependence of (5.3); while its five intertwiner components are equal and we fix them to have value

$$\phi_n = \frac{\pi}{2}. \tag{5.5}$$

This phase dependence is the crucial detail that makes the calculation work.

Let us illustrate upfront the reason why this additional phase cures the problems that appeared with the BC vertex. The boundary state must have an intertwiner dependence, in order to have the correct semiclassical value of the mean values of the angles between the faces of the boundary tetrahedra. The mean value of an intertwiner variable i_n –namely of the virtual link of the intertwiner in a given pairing– must have a certain value i_0. For this, it is sufficient, say, that the state be a gaussian around i_0. However, in quantum geometry the different angles of a tertrahedron do not commute [50, 93, 153]. Therefore a state with a behavior like $exp\{-(i_n - i_0)^2\}$ will be peaked on the virtual spin i_n in one pairing, but *it will not be peaked in the virtual spin in a different pairing.* Therefore, the other angles of the tetrahedron will not be peaked on the correct semiclassical value. We can of course write a gaussian which is peaked on a variable as well as on another, non-commuting, variable. For instance, a standard Schrödinger wave packet $\psi(x) = exp\{-\frac{(x-x_0)^2}{2}\sigma + ip_0x\}$ is peaked on position as well as momentum. But in order to do so, we must have a phase dependence on the x. Similarly, the boundary state needs a phase dependence on the intertwiner variable i_n, in order to be peaked on all angles. As shown in [149], the correct value for this is $exp\{i\frac{\pi}{2}i_n\}$. Now, the general mechanism through which the dynamical kernel reproduces the semiclassical dynamics in quantum mechanics is the cancellation of the phases between the propagation kernel and the boundary state. If this does not happens, the rapidly oscillating phases suppresses the amplitude. For instance, in the non-relativistic quantum mechanics of a free particle, the propagation kernel $K(x, y)$ in a time t has a phase dependence from small fluctuations $\delta x = x - x_0$ and $\delta y = y - y_0$ of the form

$$K(x_0 + \delta x, y_0 + \delta y) = \langle x_0 + \delta x | e^{-\frac{i}{\hbar}\frac{p^2}{2m}t} | y_0 + \delta y\rangle \sim C\ e^{-ip_0\delta x}\ e^{ip_0\delta y}. \tag{5.6}$$

where $p_0 = m(y_0 - x_0)/t$. This phase precisely cancels the phase of an initial and final wave packets ψ_i and ψ_f centered on x_0 and y_0, if these have the correct momentum. That is

$$\langle\psi_f | e^{-\frac{i}{\hbar}Ht} | \psi_i\rangle = \int dx \int dy\ e^{-\frac{(x-x_0)^2}{2\sigma} - \frac{i}{\hbar}p_f x}\ K(x, y)\ e^{-\frac{(y-y_0)^2}{2\sigma} + \frac{i}{\hbar}p_i y} \tag{5.7}$$

is suppressed by the oscillating phases unless $p_i = p_f = p_0$. This is the standard mechanism through which quantum theory reproduces the (semi-)classical behavior. In quantum gravity, it is reasonable to expect the same to happen if we have to recover the Einstein equations in the semiclassical limit. That is, the propagation kernel W, must have a phase dependence that matches the one in a semiclassical boundary state. This is precisely the role of the phase $exp\{i\frac{\pi}{2}i_n\}$ that we have included in (5.4).

In the rest of the paper we show that a vertex amplitude that has the phase dependence as above can reproduce the tensorial structure of the graviton propagator. First, however, we must improve the definition of the vertex given above, and correct a problem with the definition of the state in [17].

5.2 Boundary state and symmetry

Following [14, 16], we consider a boundary state defined as a gaussian wave packet, centered on the values determined by the background geometry $\mathbf{q}$. Here

$$\Phi_{\mathbf{q}}(\mathbf{j},\mathbf{i}) = C\ e^{-\frac{1}{2j_0}(\delta\mathbf{I}A\delta\mathbf{I})+i\phi\cdot\delta\mathbf{I}}. \tag{5.8}$$

Where A is a 15×15 matrix and the normalization factor C is determined by $\langle W|\Phi_{\mathbf{q}}\rangle = 1$. The spin phase coefficients are fixed by the background extrinsic geometry [14]. The intertwiner phase coefficients are fixed by requirement that the state remain peaked after a change of pairing to the value $\phi_n = \pi/2$. [17, 149]

At each node n we have three possible pairings, that we denote as x_n, y_n and z_n. For instance, at the node 5, let $x_5 = \{(12)(34)\}, y_5 = \{(13)(24)\}, z_5 = \{(14)(23)\}$, and denote $i_{x_5} = i_{\{(12)(34)\}}$ the intertwiner in the pairing x_5, and so on. The vertex (5.4) and the state (5.8) are written in terms of the intertwiner variable i_n, which is the virtual link of the node n *in one chosen pairing*. Because of this, the definition of these states depend on the pairing chosen. It follows that the vertex and the state do not have the full symmetry of the 4-simplex. The corresponding propagator turn out not to be invariant under $SO(4)$, as it should in the euclidean theory. In [17], a simple strategy was adopted in order to overcome this difficulty: sum over the three pairings at each of the five nodes. The state was defined as

$$|\Psi_{\mathbf{q}}\rangle = \sum_{m_n} \sum_{\mathbf{j},\mathbf{i}_{m_n}} \Phi_{\mathbf{q}}(\mathbf{j},\mathbf{i}_{m_n})\ |\mathbf{j},\mathbf{i}_{m_n}\rangle, \tag{5.9}$$

where $m_n = x, y, z$ for each node n. This sum implements the full symmetry of the 4-simplex. Summing over the three bases removes the basis dependence.

In developing the calculations presented in the present paper, at first we adopted this same strategy. To our surprise, nothing worked, and something quite funny happened: the dependence on the intertwiner variables i_n misteriously cancelled out in all components of the propagator!

The solution of the puzzle was to realize that to sum over the three basis with a correlation matrix A does implement the symmetry of the 4-simplex, but not just this symmetry. It implements a larger symmetry, that has the effect of cancelling the intertwiner dependence. Geometrically, this additional symmetry can be viewed as an *independent* rotation of each of the five tetrahedra forming the boundary of the 4-simplex.

To understand what happens, consider for instance the correlation $\langle j_{12} i_{x_5}\rangle$ between the spin j_{12} which is the quantum number of the area of a triangle, and the intertwiner i_{x_5}, which is the quantum number of the angle θ_{12} between the faces 2 and 3 of the tetrahedron 5. More precisely, i_{x_5} is the eigenvalue of the quantity $A_2^2 + A_3^2 + A_2 A_3 \cos(\theta_{12})$, where A_i is the area of the face i of the tetrahedron 5. Now, if the state is summed over pairings, then it does not distinguish pairings, hence

$$\langle j_{12} i_{x_5}\rangle = \frac{1}{3}\left(\langle j_{12} i_{x_5}\rangle + \langle j_{12} i_{y_5}\rangle + \langle j_{12} i_{z_5}\rangle\right). \tag{5.10}$$

That is

$$\langle j_{12} i_{x_5}\rangle = \frac{1}{3}\langle j_{12}\left(3A_1^2 + A_2^2 + A_3^2 + A_4^2 + A_1 A_2\cos(\theta_{12}) + A_1 A_3\cos(\theta_{13}) + A_1 A_4\cos(\theta_{14})\right)\rangle. \tag{5.11}$$

But let n_i, $i = 1, ..., 4$ be the normal to the face i of the tetrahedron 5, with length $|n_i| = A_i$. The closure relation reads

$$\sum_{i=1,4} n_i = 0. \tag{5.12}$$

Taking the scalar product with n_1 gives

$$A_1^2 + A_1 A_2\cos(\theta_{12}) + A_1 A_3\cos(\theta_{13}) + A_1 A_4\cos(\theta_{14}) = 0. \tag{5.13}$$

It follows from this equation and (5.11) that

$$\langle j_{12} i_{x_5}\rangle = \frac{1}{3}\langle j_{12}(2A_1^2 + A_2^2 + A_3^2 + A_4^2)\rangle = \frac{1}{3}(2\langle j_{12} j_{15}\rangle + \langle j_{12} j_{25}\rangle + \langle j_{12} j_{35}\rangle + \langle j_{12} j_{45}\rangle). \tag{5.14}$$

That is, the spin-intertwiner correlations are just functions of the spin-spin correlations for a state with this symmetry! The intertwiner dependence drops out! This means that the propagator is completely unaffected from the correlations involving the intertwiners. It then turns out that the sole spin-spin correlations in the state are not sufficient to give the full tensorial structure of the propagator.

The solution of the difficulty is just to choose a boundary state and a kernel W that do not have the extra symmetry. The simplest possibility is to choose an abitrary pairing, and then to symmetrize *only* under the symmetries of the four-simplex. These are generated by the 5! permutations σ of the five vertices of the four-simplex. A permutation $\sigma : \{1,2,3,4,5\} \to \{\sigma(1), \sigma(2), \sigma(3), \sigma(4), \sigma(5)\}$ acts naturally on the boundary states

$$\sigma|j_{nm}, i_{x_n}\rangle = |j_{\sigma(n)\sigma(m)}, i_{\sigma(x_n)}\rangle \tag{5.15}$$

where the action $\sigma(x_n)$ of the permutation on a node is defined by

$$\sigma(\{(ab)(cd)_n\}) = \{(\sigma(a)\sigma(b))(\sigma(c)\sigma(d))_{\sigma(n)}\} \tag{5.16}$$

and can therefore change the original pairing at the node.

We therefore define the boundary state by replacing (5.9) with

$$|\Psi_{\mathbf{q}}\rangle = \sum_{\sigma} \sigma|\Phi_{\mathbf{q}}\rangle = \sum_{\sigma}\sum_{\mathbf{j},\mathbf{i}} \Phi_{\mathbf{q}}(\mathbf{j},\mathbf{i})\ \ \sigma|\mathbf{j},\mathbf{i}\rangle. \tag{5.17}$$

This modification of the boundary state does not affect the conclusions of the paper [17]. Similarly, we pose

$$|W\rangle = \sum_{\sigma} \sum_{\mathbf{j},\mathbf{i}} W(\mathbf{j},\mathbf{i}) \;\; \sigma|\mathbf{j},\mathbf{i}\rangle. \tag{5.18}$$

Before beginning the actual calculation of the propagator, consider what happens by contracting the vertex amplitude with the boundary state. We have the double sum over permutations

$$\langle W|\Psi\rangle = \sum_{\sigma\sigma'} \Big(\sum_{\mathbf{jij'i'}} \overline{W(\mathbf{j},\mathbf{i})}\Phi(\mathbf{j'},\mathbf{i'}) \; \langle\sigma(\mathbf{j},\mathbf{i})|\sigma'(\mathbf{j'},\mathbf{i'})\rangle \Big). \tag{5.19}$$

The scalar product is

$$\langle \mathbf{j},\mathbf{i}|\mathbf{j'},\mathbf{i'}\rangle = \delta_{\mathbf{j},\mathbf{j'}} \prod_{n} \langle i_n|i'_n\rangle, \tag{5.20}$$

where $\langle i_n|i'_n\rangle$ is δ_{i_n,i'_n} if the two intertwiners are written in the same basis, and is the matrix of the change of basis, namely a $6j$-symbol, otherwise. Now, it was observed in [17] that if one of these $6j$-symbols enters in a sum like (5.19) then the sum is suppressed in the large j_0 limit, because the $6j$-symbol contains a rapidly oscillating factor which is not compensated. Hence, in this limit we can effectively rewrite (5.19) in the form

$$\langle W|\Psi\rangle = \sum_{\sigma\sigma'} \Big(\sum_{\mathbf{jij'i'}} \overline{W(\mathbf{j},\mathbf{i})}\Phi(\mathbf{j'},\mathbf{i'}) \; \delta_{\sigma\mathbf{j},\sigma'\mathbf{j'}} \; \delta_{\sigma\mathbf{i},\sigma'\mathbf{i'}} \Big), \tag{5.21}$$

where the second delta vanishes unless the two intertwiners have the same value and are written in the same basis. Up to accidental symmetry factors that we absorb in the state, we can then rewrite the scalar product in the form

$$\langle W|\Psi\rangle = \sum_{\sigma} \Big(\sum_{\mathbf{ji}} \overline{W(\mathbf{j},\mathbf{i})}\Phi(\mathbf{j},\mathbf{i}) \Big) = 5! \sum_{\mathbf{ji}} \overline{W(\mathbf{j},\mathbf{i})}\Phi(\mathbf{j},\mathbf{i}). \tag{5.22}$$

We shall see that a similar simplification happens in the calculation of the matrix elements of the propagator.

5.3 The propagator

Let us begin by recalling the action of the grasping operators. This was computed in [17], to which we refer for the notation. Consider the operators acting on a node n. The diagonal action is simply

$$E^{(i)}_n \cdot E^{(i)}_n \, |\Phi_{\mathbf{q}}\rangle = C^{ni} \, |\Phi_{\mathbf{q}}\rangle \tag{5.23}$$

where C^{ni} is the Casimir of the representation associated to the link ni. The non-diagonal action depends on the pairing at the node n. We have three cases, depending on the three possible pairings. These are as follows. Say the node n is in the pairing $(ij),(ef)$, with positive orientation at the two trivalent vertices (i_n,i,j) and (i_n,e,f). Then we have the diagonal double grasping

$$E^{(i)}_n \cdot E^{(j)}_n \, |\Phi_{\mathbf{q}}\rangle = \sum_{\mathbf{j},\mathbf{i}} D^{ij}_n \;\; \Phi(\mathbf{j},\mathbf{i}) \;\; |\mathbf{j},\mathbf{i}\rangle \, . \tag{5.24}$$

while the two possible non-diagonal graspings give

$$E_n^{(i)} \cdot E_n^{(e)} \, |\Phi_{\mathbf{q}}\rangle = \sum_{\mathbf{j},\mathbf{i}} \Phi(\mathbf{j},\mathbf{i}) \left(X_n^{ie} \ |\mathbf{j},\mathbf{i}\rangle - Y_n^{ie} \ \left|\mathbf{j},(i_n-1),\mathbf{i}'\right\rangle - Z_n^{ie} \ \left|\mathbf{j},(i_n+1),\mathbf{i}'\right\rangle \right) \tag{5.25}$$

and

$$E_n^{(i)} \cdot E_n^{(f)} \, |\Phi_{\mathbf{q}}\rangle = \sum_{\mathbf{j},\mathbf{i}} \Phi(\mathbf{j},\mathbf{i}) \left(X_n^{if} \ |\mathbf{j},\mathbf{i}\rangle + Y_n^{if} \ \left|\mathbf{j},(i_n-1)\mathbf{i}'\right\rangle + Z_n^{if} \ \left|\mathbf{j},(i_n+1)\mathbf{i}'\right\rangle \right). \tag{5.26}$$

and so on cyclically. The quantities D_n^{ij}, X_n^{ij}, Y_n^{ij} and Z_n^{ij} are defined in [17]. Here $\mathbf{i}'$ indicates the four intertwiners different from i_n.

Inserting the expressions (5.17) and (5.18) in the expression (5.1) for the propagator, gives the double sum over permutations

$$\mathbf{G}_{\mathbf{q}\,n,m}^{ij,kl} = \sum_{\sigma'\sigma} \Big[\sum_{\mathbf{j},\mathbf{i}} \overline{W(\sigma'(\mathbf{j}),\sigma'(\mathbf{i}))} \big(E_n^{(i)} \cdot E_n^{(j)} - n_n^{(i)} \cdot n_n^{(j)}\big) \big(E_m^{(k)} \cdot E_m^{(l)} - n_m^{(k)} \cdot n_m^{(l)}\big) \Phi(\sigma(\mathbf{j}),\sigma(\mathbf{i})) \Big]. \tag{5.27}$$

The E operators do not change the spin, and the argument at the end of the last section can be repeated. This time, however, the residual sum over permutations remains, because the operators are not invariant under it

$$\mathbf{G}_{\mathbf{q}\,n,m}^{ij,kl} = \sum_{\sigma} \left(\sum_{\mathbf{j},\mathbf{i}} \overline{W(\sigma(\mathbf{j}),\sigma(\mathbf{i}))} \big(E_n^{(i)} \cdot E_n^{(j)} - n_n^{(i)} \cdot n_n^{(j)}\big) \big(E_m^{(k)} \cdot E_m^{(l)} - n_m^{(k)} \cdot n_m^{(l)}\big) \Phi(\sigma(\mathbf{j}),\sigma(\mathbf{i})) \right). \tag{5.28}$$

By changing variables, we can move the symmetrization to the operators, hence writing

$$\mathbf{G}_{\mathbf{q}\,n,m}^{ij,kl} = \sum_{\sigma} \tilde{\mathbf{G}}_{\mathbf{q}\,\sigma(n),\sigma(m)}^{\sigma(i)\sigma(j),\sigma(k)\sigma(l)} \tag{5.29}$$

where

$$\tilde{\mathbf{G}}_{\mathbf{q}\,n,m}^{ij,kl} = \sum_{\mathbf{j},\mathbf{i}} \overline{W(\mathbf{j},\mathbf{i})} \big(E_n^{(i)} \cdot E_n^{(j)} - n_n^{(i)} \cdot n_n^{(j)}\big) \big(E_m^{(k)} \cdot E_m^{(l)} - n_m^{(k)} \cdot n_m^{(l)}\big) \Phi(\mathbf{j},\mathbf{i}). \tag{5.30}$$

In other words, we can first compute the propagator with unsymmetrized states and vertex, and then symmetrize the propagator.

We can now begin the actual calculation of the various terms of the propagator. It is usefull to distinguish three cases: the diagonal–diagonal components $\tilde{\mathbf{G}}_{\mathbf{q}\,n,m}^{ii,kk}$; the diagonal–non-diagonal components $\tilde{\mathbf{G}}_{\mathbf{q}\,n,m}^{ii,kl}$; and the non-diagonal–non-diagonal components $\tilde{\mathbf{G}}_{\mathbf{q}\,n,m}^{ij,kl}$, where again different indices are distinct. Let us considered the three cases separately.

In the diagonal–diagonal case, from the expression of the last section, we have

$$\tilde{\mathbf{G}}_{\mathbf{q}\,n,m}^{ii,kk} = \sum_{\mathbf{ji}} \ \overline{W(\mathbf{j},\mathbf{i})} \ (C^{ni} - |n_n^{(i)}|^2)(C^{nk} - |n_m^{(k)}|^2) \ \Phi(\mathbf{j},\mathbf{i}) \tag{5.31}$$

As we have seen in [17] the background geometry determines the background link j^0

$$|n_n^{(i)}|^2 = C^2(j^0) = j^0(j^0+1) \tag{5.32}$$

and

$$C^{ni} = C^2(j^{ni}). \tag{5.33}$$

In the large j^0 limit we have at leading order

$$C^{ni} - |n_n^{(i)}|^2 \approx 2j^0 \delta j^{ni} \tag{5.34}$$

the propagator components are then

$$\tilde{\mathbf{G}}^{ii,kk}_{\mathbf{q}\,n,m} = 4j_0^2 \sum_{\mathbf{j},\mathbf{i}} \overline{W(\mathbf{j},\mathbf{i})}\ \delta j^{ni}\, \delta j^{mk}\ \Phi(\mathbf{j},\mathbf{i}) \tag{5.35}$$

The sum over permutations is now trivial. It only gives a 5! factor that cancels with the same factor in the normalization. We can therefore drop the tilde from (5.35).

In the diagonal–non-diagonal case, from (5.30) we have

$$\tilde{\mathbf{G}}^{ij,kk}_{\mathbf{q}\,n,m} = \sum_{\mathbf{j},\mathbf{i}} W(\mathbf{j},\mathbf{i})\big(E_n^{(i)} \cdot E_n^{(j)} - n_n^{(i)} \cdot n_n^{(j)}\big)\big(E_m^{(k)} \cdot E_m^{(k)} - |n_m^{(k)}|\big)\Phi(\mathbf{j},\mathbf{i}) \tag{5.36}$$

now the second operator is diagonal and gives (5.34) at leading order; the action of the first operator instead gives only one of the three terms (5.24), (5.25), (5.26) depending on how the two links ni and nj are paired at the node n. The possible results (at leading order) are

$$\tilde{\mathbf{G}}^{ij,kk}_{\mathbf{q}\,n,m} = \sum_{\mathbf{j},\mathbf{i}} W(\mathbf{j},\mathbf{i}) 2j_0 \delta j^{(mk)} \left(D_n^{(ij)} + \frac{j_0^2}{3}\right) \Phi(\mathbf{j},\mathbf{i}) \tag{5.37}$$

if the two links are paired. The second term in the parenthesis comes from the fact that the background normals are fixed by the background geometry. In the large j^0 limit

$$n_n^{(i)} \cdot n_n^{(nj)} \approx -\frac{1}{3}(j_0)^2. \tag{5.38}$$

And

$$\tilde{\mathbf{G}}^{ij,kk}_{\mathbf{q}\,n,m} = \sum_{\mathbf{j},\mathbf{i}} \left(\overline{W(\mathbf{j},\mathbf{i})}\left(X_n^{ij} + \frac{j_0^2}{3}\right) - \overline{W(\mathbf{j},\mathbf{i}',i_n-1)}\ Y_n^{ij} - \overline{W(\mathbf{j},\mathbf{i}',i_n+1)}\ Z_n^{ij}\right) 2j_0 \delta j^{mk} \Phi(\mathbf{j},\mathbf{i}), \tag{5.39}$$

or

$$\tilde{\mathbf{G}}^{ij,kk}_{\mathbf{q}\,n,m} = \sum_{\mathbf{j},\mathbf{i}} \left(\overline{W(\mathbf{j},\mathbf{i})}\left(X_n^{ij} + \frac{j_0^2}{3}\right) + \overline{W(\mathbf{j},\mathbf{i}',i_n-1)}\ Y_n^{ij} + \overline{W(\mathbf{j},\mathbf{i}',i_n+1)}\ Z_n^{ij}\right) 2j_0 \delta j^{mk} \Phi(\mathbf{j},\mathbf{i}), \tag{5.40}$$

according to orientation, if they are not paired.

In (5.39) and (5.40) the term in Y and Z cancel at the leading order for the following reason. First, recall from [17] that Y and Z are equal at leading order. The difference between the Y-term and the Z-term is then only given by the ± 1 in the argument of W. But the dependence of W on i_n is of the form $e^{i\frac{\pi}{2}i_n}$. Hence (up to subleading terms in the large j_0 limit)

$$W(\mathbf{j},\mathbf{i}',i_n+1) = -W(\mathbf{j},\mathbf{i}',i_n-1) \tag{5.41}$$

The different between the two terms is just a sign and they cancel. Thus we have

$$\tilde{\mathbf{G}}^{ij,kk}_{\mathbf{q}\,n,m} = \sum_{\mathbf{j},\mathbf{i}} \overline{W(\mathbf{j},\mathbf{i})} \left(X^{ij}_n + \frac{j_0^2}{3} \right) 2j_0 \delta j^{mk} \Phi(\mathbf{j},\mathbf{i}), \tag{5.42}$$

anytime ni and nj are not paired.

In the large distance limit we have (5.38) and

$$D^{ij}_n - n^{(i)}_n \cdot n^{(j)}_n = \frac{C^2(i_n) - C^2(j^{(ni)}) - C^2(j^{(nj)})}{2} + \frac{1}{3}(j_0)^2. \tag{5.43}$$

Introduce the fluctuations variables $\delta j_{nj} = j_{nj} - j_0$, and $\delta i_n = i_n - i_0$ and expand around the background values j^0 and i^0. In the large j_0 limit (which is also large i_0). The dominant term of the (5.43) is

$$D^{ij}_n - n^{(ni)} \cdot n^{(nj)} = \delta i_n \; i_0 - \delta j^{ni} j_0 - \delta j^{nj} j_0. \tag{5.44}$$

Similarly, using the results of [17], the X terms are approximated substituting $C^2(j) \approx j^2$ and keeping the dominant terms

$$X^{ij}_n = -\frac{1}{4} \left((i_0)^2 + 2j_0 \; \delta j^{ni} + 2j_0 \; \delta j^{nj} - 2j_0 \; \delta j^{nf} - 2j_0 \; \delta j^{ne} + 2i_0 \; \delta i_n \right) \tag{5.45}$$

where nf and ne indicate the other two links of the node n. Recalling that $i_0 = \frac{2}{\sqrt{3}} j_0$, we have that the first term of the sum cancels the norm of the n, leaving

$$X^{ij}_n + \frac{j_0^2}{3} = -\frac{1}{4} \left(2j_0 \; \delta j^{ni} + 2j_0 \; \delta j^{nj} - 2j_0 \; \delta j^{nf} - 2j_0 \; \delta j^{ne} + 2i_0 \; \delta i_n \right) \tag{5.46}$$

In conclusion, we have for the paired case

$$\tilde{\mathbf{G}}^{ij,kk}_{\mathbf{q}\,n,m} = 2j_0^2 \sum_{\mathbf{j},\mathbf{i}} \overline{W(\mathbf{j},\mathbf{i})} \left(\frac{2}{\sqrt{3}} \delta i_n - \delta j^{ni} - \delta j^{nj} \right) \delta j^{mk} \; \Phi(\mathbf{j},\mathbf{i}), \tag{5.47}$$

and for the unpaired one

$$\tilde{\mathbf{G}}^{ij,kk}_{\mathbf{q}\,n,m} = j_0^2 \sum_{\mathbf{j},\mathbf{i}} \overline{W(\mathbf{j},\mathbf{i})} \left(-\delta j^{ni} - \delta j^{nj} + \delta j^{nf} + \delta j^{ne} - \frac{2}{\sqrt{3}} \delta i_n \right) \delta j^{mk} \; \Phi(\mathbf{j},\mathbf{i}). \tag{5.48}$$

Finally, the non-diagonal–non-diagonal case is

$$\tilde{\mathbf{G}}^{ij,kl}_{\mathbf{q}\,n,m} = \langle W | \left(E^{(i)}_n \cdot E^{(j)}_n - n^{(i)}_n \cdot n^{(j)}_n \right) \left(E^{(k)}_m \cdot E^{(l)}_m - n^{(k)}_m \cdot n^{(l)}_m \right) | \Phi \rangle \, . \tag{5.49}$$

The calculations are clearly the same as above.

The final result is

$$\tilde{\mathbf{G}}^{ij,kl}_{\mathbf{q}\,n,m} = j_0^2 \sum_{\mathbf{j},\mathbf{i}} \overline{W(\mathbf{j},\mathbf{i})} K^{ij}_n K^{kl}_m \; \Phi(\mathbf{j},\mathbf{i}), \tag{5.50}$$

where

$$K^{ij}_n = \frac{2}{\sqrt{3}} \delta i_n - \delta j^{ni} - \delta j^{nj} \tag{5.51}$$

if ni and nj are paired at n and

$$K_n^{ij} = \frac{1}{2}\left(-\delta j^{ni} - \delta j^{nj} + \delta j^{nf} + \delta j^{ne} - \frac{2}{\sqrt{3}}\delta i_n\right) \tag{5.52}$$

if they are not; while

$$K_n^{ii} = 2\delta j^{ni}. \tag{5.53}$$

Both the state coefficients $\Phi(\mathbf{j},\mathbf{i})$ and the vertex coefficients $W(\mathbf{j},\mathbf{i})$ are given by a gaussian in δI_α. The phases in the boundary state cancels with the phase of one of the two terms of W, while the other term is suppressed for large j_0. Thus, (5.50) reads

$$\tilde{\mathbf{G}}^{ij,kl}_{\mathbf{q}\,n,m} = j_0^2 \sum_{\mathbf{j},\mathbf{i}} e^{-\frac{1}{2j_0} M_{\alpha\beta}\delta I_\alpha \delta I_\beta} K_n^{ij} K_m^{kl}, \tag{5.54}$$

where $M = A + ij_0 G$. As in [17], we approximate the sum by a Gaussian integral with quadratic insertions. The result of the integral is easily expressed in terms of the matrix M^{-1} obtained inverting the 15×15 covariance matrix M, in the 10 spin variables δj_{nm} and the five intertwiner variables δi_n.

The symmetries of the matrix M^{-1} are the same as the symmetries of M, and are dictated by the symmetries of the problem. Which ones are these symmetries? At first sight, one is tempted to say that M^{-1} must respect the symmetries of the 4-simplex, and therefore it must be invariant under any permutation of the five vertices n. Therefore therefore it can have only seven independent components:

$$\begin{aligned} M^{-1}_{(ij)(ij)} = c_2, \quad M^{-1}_{(ij)(ik)} = c_1, \quad M^{-1}_{(ij)(kl)} = c_3, \\ M^{-1}_{ii} = c_4, \quad M^{-1}_{ij} = c_5, \quad M^{-1}_{(ij)i} = c_6 \quad M^{-1}_{(ij)k} = c_7. \end{aligned} \tag{5.55}$$

where different indices are distinct. The ratio for this being for instance that M_{11} must be equal to M_{22} because of the symmetry under the exchange of the vertex 1 and the vertex 2. However, this argument is incorrect.

The reason is that the vertex function and the state function are written as a function of intertwiner variables i_n which are tied to a given choice of pairing at each node. Specifically, we have chosen the pairing $i_1^{(23)(45)}, i_2^{(34)(51)}, i_3^{(45)(12)}, i_4^{(51)(23)}, i_5^{(12)(34)}$. This choice breaks the symmetry under the permutations of the vertices, although this is not immediately evident. To see this, consider for instance the two matrix elements $M^{-1}_{(12)3}$ and $M^{-1}_{(12)4}$. According to (5.55), they should be equal (both be equal to c_7 by symmetry. But notice that 1 and 2 are paired at the node 3, while they are not paired at the node 4. Therefore the two are not equal under the symmetries of the paired 4-simplex. To see this more formally, let us indicate explicitly the pairing in which the intertwiner is written by writing $i_n^{(ij)(ef)}$ instead of i_n. Then we see that $M^{-1}_{(12)3}$ is of the form $M^{-1}_{(ij)i_n^{(ij)(kl)}}$ while $M^{-1}_{(12)4}$ is of the form $M^{-1}_{(ij)i_n^{(ik)(jl)}}$, which makes it obvious that a permutation $ijklm \to i'j'k'l'm'$ cannot transform one into the other, since it cannot undo the fact that the ij indices of the link are paired at the node. As a consequence, we must for instance replace the last entry of (5.55) by

$$M^{-1}_{(ij)i_n^{(ij)(kl)}} = c_7 \quad M^{-1}_{(ij)i_n^{(ik)(jl)}} = c_8. \tag{5.56}$$

and so on. Thus, the matrix M^{-1} may in general have a more complicated structure than (5.55).

Now, the details of this structure depend on the pairing chosen. In fact, there are five possible inequivalent ways of choosing the pairings at the nodes, which do not transform into one another under permutations. These are illustrated in Figure 5.1.

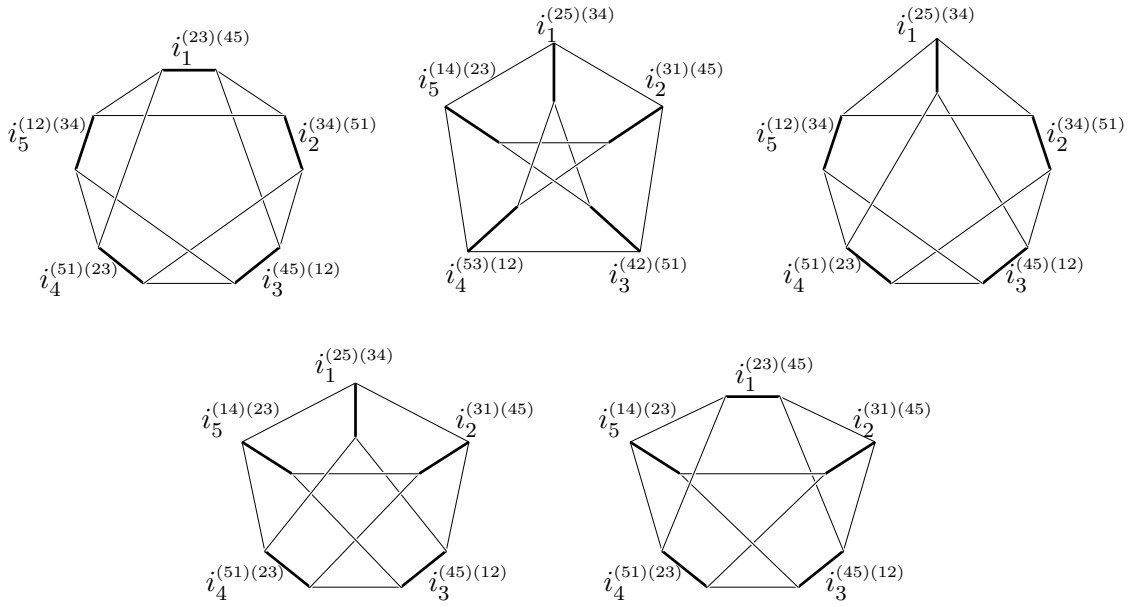


Figure 5.1: The five classes of pairings: from the upper left: (10), (5,5), (7,3), (6,4) and (4,3,3).

The fact that they cannot be transformed into one another by a permutation can be deduced from the following consideration. In each diagram of Figure 5.1, consider the sequences of links that can be followed without ever crossing an intertwiner. Observe that in the first case all links are clustered in a single cluster of length 10. In the second, they are clustered in two diagrams of lenght (5,5), and so on as indicated. Clearly a permutation cannot change the structure of these clusterings, and therefore these pairing choices cannot be transformed into one another under permutations. The five cases illustrated correspond to the five different 15j Wigner symbols illustrated in [154]. These five classes define therefore distinct possibilities for the definitionns of vertex and the state. As here we are not interested in generality, we have just picked one of these: the first case. Also, since we are not interested in the full generality of an arbitrary gaussian vertex and state, we just assume a particular form, compatible with the symmetries, for the matrix M^{-1}. Specifically, we assume that M^{-1} has the form (5.55) with the last entry replaced by (5.56). That is, we assume the state depends on (at least) eight

independent parameters that determine $\mathbf{c} = (c_1, ..., c_8)$. The symmetries of the 4-symplex equivalence class admit a greater number of free parameters, but we do not need the most general possible gaussian state for what follows. Assuming thus this form for M^{-1}, we can then proceed with the calculation of (5.54).

Each term of the normalized propagator is a sum of individual elements of the matrix M^{-1}. The overall dependence on j_0 is as in the diagonal case, and gives the expected inverse-square dependence. The normalization factor is

$$\mathcal{N}^{-1} = j_0^2 \int d(\delta I_\alpha)\ e^{-\frac{1}{2j_0} M_{\alpha\beta}\delta I_\alpha \delta I_\beta} \tag{5.57}$$

The diagonal-diagonal term gives

$$\tilde{\mathbf{G}}^{ii,kk}_{\mathbf{q}\,n,m} = \mathcal{N} j_0^2 \int d(\delta I_\alpha)\ e^{-\frac{1}{2j_0} M_{\alpha\beta}\delta I_\alpha \delta I_\beta} 2\delta j_{ni}\ 2\delta j_{mk} \ = \frac{4}{j^0} M^{-1}_{(ni)\,(mk)} = \begin{cases} \frac{4}{j^0} c_1 & \text{if i=k or i=m,} \\ \frac{4}{j^0} c_3 & \text{otherwise.} \end{cases} \tag{5.58}$$

In this case $\tilde{G}$ gives immediately G since the permutation does not mix c_1 and c_3 terms.

Proceeding in the same way for the other cases, we get for the diagonal–non-diagonal term the two cases

$$\tilde{\mathbf{G}}^{ij,kk}_{\mathbf{q}\,n,m} = \frac{1}{j^0}\big(-2M^{-1}_{(mk)\,(ni)} - 2M^{-1}_{(mk)\,(nj)} + \frac{4}{\sqrt{3}} M^{-1}_{(mk)\,n}\big) = \begin{cases} -\frac{4}{j^0}\big(c_1 - \frac{1}{\sqrt{3}} c_7\big) & \text{if i=k and j=m,} \\ -\frac{4}{j^0}\big(c_3 - \frac{1}{\sqrt{3}} c_7\big) & \text{if i}\neq\text{ k and j}\neq\text{ k,m,} \\ -\frac{2}{j^0}\big(c_1 + c_3 - \frac{2}{\sqrt{3}} c_8\big) & \text{otherwise.} \end{cases} \tag{5.59}$$

and

$$\tilde{\mathbf{G}}^{ij,kk}_{\mathbf{q}\,n,m} = \frac{1}{j^0}\big(- M^{-1}_{(mk)\,(ni)} - M^{-1}_{(mk)\,(nj)} + M^{-1}_{(mk)\,(np)} + M^{-1}_{(mk)\,(nq)} - \frac{2}{\sqrt{3}} M^{-1}_{(mk)\,(n)}\big) \tag{5.60}$$

$$= \begin{cases} \frac{2}{j^0}(-c_1 + c_3 - \frac{1}{\sqrt{3}} c_8) & \text{if i=k and j=m} \\ \frac{2}{j^0}(-c_3 + c_1 - \frac{1}{\sqrt{3}} c_8) & \text{if i}\neq\text{ k and j}\neq\text{ k,m} \\ -\frac{2}{\sqrt{3} j^0} c_8 \ \text{ or } \ -\frac{2}{\sqrt{3} j^0} c_7 & \text{otherwise} \end{cases} \tag{5.61}$$

depending on the pairing of the node n. For the non-diagonal–non-diagonal terms, we have the three possibilities: diagonal double grasping on the two nodes

$$\begin{aligned} \tilde{\mathbf{G}}^{ij,kl}_{\mathbf{q}\,n,m} = \frac{1}{j_0}\ \big(&\frac{4}{3} M^{-1}_{m\,n} - \frac{2}{\sqrt{3}}\big(M^{-1}_{n\,mk} + M^{-1}_{n\,ml} + M^{-1}_{m\,ni} + M^{-1}_{m\,nj}\big) \\ &+ M^{-1}_{ni\,mk} + M^{-1}_{ni\,ml} + M^{-1}_{nj\,mk} + M^{-1}_{nj\,ml}\big); \end{aligned} \tag{5.62}$$

diagonal double grasping on one node and non-diagonal on the other one

$$\begin{aligned} \tilde{\mathbf{G}}^{ij,kl}_{\mathbf{q}\,n,m} = \frac{1}{2j_0}\big(&- \frac{4}{3} M^{-1}_{n\,m} + \\ &- \frac{2}{\sqrt{3}} M^{-1}_{n\,mk} - \frac{2}{\sqrt{3}} M^{-1}_{n\,ml} + \frac{2}{\sqrt{3}} M^{-1}_{n\,mp} + \frac{2}{\sqrt{3}} M^{-1}_{n\,mq} + \\ &+ \frac{2}{\sqrt{3}} M^{-1}_{ni\,m} + M^{-1}_{ni\,mk} + M^{-1}_{ni\,ml} - M^{-1}_{ni\,mp} - M^{-1}_{ni\,mq} + \\ &+ \frac{2}{\sqrt{3}} M^{-1}_{nj\,m} + M^{-1}_{nj\,mk} + M^{-1}_{nj\,ml} - M^{-1}_{nj\,mp} - M^{-1}_{nj\,mq};\big), \end{aligned} \tag{5.63}$$

and non-diagonal on both nodes

$$\begin{aligned}\tilde{\mathbf{G}}^{ij,kl}_{\mathbf{q}\,n,m} = \frac{1}{4j_0}\Big(&\frac{4}{3}M^{-1}_{n\,m}+\\ &+\frac{2}{\sqrt{3}}M^{-1}_{n\,mk}+\frac{2}{\sqrt{3}}M^{-1}_{n\,ml}-\frac{2}{\sqrt{3}}M^{-1}_{n\,mp}-\frac{2}{\sqrt{3}}M^{-1}_{n\,mq}+\\ &+\frac{2}{\sqrt{3}}M^{-1}_{ni\,m}+M^{-1}_{ni\,mk}+M^{-1}_{ni\,ml}-M^{-1}_{ni\,mp}-M^{-1}_{ni\,mq}+\\ &+\frac{2}{\sqrt{3}}M^{-1}_{nj\,m}+M^{-1}_{nj\,mk}+M^{-1}_{nj\,ml}-M^{-1}_{nj\,mp}-M^{-1}_{nj\,mq}+\\ &-\frac{2}{\sqrt{3}}M^{-1}_{ne,m}-M^{-1}_{ne\,mk}-M^{-1}_{ne\,ml}+M^{-1}_{ne\,mp}+M^{-1}_{ne\,mq}+\\ &-\frac{2}{\sqrt{3}}M^{-1}_{nf,m}-M^{-1}_{nf\,mk}-M^{-1}_{nf\,ml}+M^{-1}_{nf\,mp}+M^{-1}_{nf\,mq}\Big)\end{aligned} \tag{5.64}$$

whose expression in terms of the c coefficients in turn depends on pairings. And so on. Notice that the only the six parameters c_1, c_2, c_3 and c_5, c_7, c_8 enter the components of the propagator. The other two, namely c_4 and c_6 do not, because we are only looking at the propagator between points on different tetrahedra.

The last step is to symmetrize the propagator under permutations. The only terms that change under permutations, at this point, are those due to the pairing. Hence, the only result of a sum over permutation is to combine the two coefficients c_7 and c_8, which are the only pairing dependent ones. For instance, a straightforward calculation gives the diagonal–non-diagonal term (which has the peculiarity of not depending on the pairing class)

$$\begin{aligned}\mathbf{G}^{ij,kk}_{\mathbf{q}\,n,m} &= \sum_{\sigma}\tilde{\mathbf{G}}^{\sigma i\sigma j,\sigma k\sigma k}_{\mathbf{q}\,\sigma n,\sigma m} \\ &= \begin{cases}\frac{1}{3j_0}\big[4(-c_1+c_3)-4c_1+\frac{4}{\sqrt{3}}(c_7-c_8)\big] & \text{if i=k and j=m,}\\ \frac{1}{3j_0}\big[4(-c_3+c_1)-4c_3+\frac{4}{\sqrt{3}}(c_7-c_8)\big] & \text{if i}\neq\text{ k and j}\neq\text{ k,m,}\\ \frac{1}{3j_0}\big[-2(c_1+c_3)-\frac{2}{\sqrt{3}}(c_7-c_8)\big] & \text{otherwise.}\end{cases}\end{aligned} \tag{5.65}$$

It is easy to see that the sum over permutation replaces all terms c_7 and c_8 with a term proportional to the linear combination $(c_7 - c_8)$. In conclusion, the propagator depends on the five parameters $c_1, c_2, c_3, c_5, (c_7 - c_8)$. Varying the parameters in the state we can span a five-dimensional space of tensors $\tilde{\mathbf{G}}^{ij,kl}_{\mathbf{q}\,n,m}$. In conclusion, $\mathbf{G}^{ii,kk}_{\mathbf{q}\,n,m}$ turns out to be a matrix with the symmetries of the 4-simplex, freely dependending on five arbitrary parameters. Can this give the same propagator as the linearized theory?

5.4 Comparison with the linearized theory

The number of components of $\mathbf{G}^{ij,kl}_{\mathbf{q}\,n,m}$ is large, and it may seem hard to believe that the five-parameters freedom in the state could be sufficient to recover the tensorial structure of the linearized propagator. However, there are two properties of the propagator that strongly constrain it. First, the symmetrization of the 4-simplex symmetries largely reduce the number

of indepedent components. Second, as proven in [17], the propagator satisfies the closure relation

$$\sum_i \mathbf{G}^{ij,kl}_{\mathbf{q}\,n,m} = 0. \tag{5.66}$$

Let us count the number of free parameters of an arbitrary tensor $\mathbf{G}^{ij,kl}_{\mathbf{q}\,n,m}$ satisfying these requirements. Using (5.66), we can always express a term in which any of the four indices i, j, k, l is equal to either n or m as sum of terms not of this kind. This reduces the independent terms to, say $\mathbf{G}^{ij,kl}_{\mathbf{q}\,1,2}$ where $i, j, k, l = 3, 4, 5$. A few pictures and a moment of reflection will convince the reader that the only independent ones of these are

$$\mathbf{G}^{ii,ii}_{\mathbf{q}\,n,m},\ \mathbf{G}^{ii,kk}_{\mathbf{q}\,n,m},\ \mathbf{G}^{ij,kk}_{\mathbf{q}\,n,m},\ \mathbf{G}^{ij,ij}_{\mathbf{q}\,n,m},\ \mathbf{G}^{ij,ik}_{\mathbf{q}\,n,m}. \tag{5.67}$$

All the other terms can be obtained from these by a permutation of the indices. Therefore a tensor with these symmetries depends only on *five* parameters. This implies that adjusting the five parameters in the state, we can match any such tensor, and in particular the propagator.

This can be checked by an explicit calculation of the propagator of the linearized theory in the harmonic gauge (on the compatibility of the radial and harmonic gauge, see [152]). The quantity $\mathbf{G}^{ij,kl}_{\mathbf{q}\,n,m}$ is the propagator projected in the directions normal to the faces of the tetrahedra. The 4d linearized graviton propagator is

$$G_{\mu\nu\rho\sigma} = \frac{1}{2L^2}\,(\delta_{\mu\rho}\delta_{\beta\gamma} + \delta_{\mu\sigma}\delta_{\beta\gamma} - \delta_{\mu\nu}\delta_{\rho\sigma}) \tag{5.68}$$

and its projection on the four linear dependent normals to the faces of each tetrahedron reads

$$G^{ij,kl}_{nm} \equiv G^{\mu\nu\rho\sigma}\ (n^{(i)}_n)_\mu (n^{(j)}_n)_\nu (n^{(k)}_m)_\rho (n^{(l)}_m)_\sigma \tag{5.69}$$

We need the explicit expressions of the normals; to this aim, fix the coordinate of a four simplex giving the 5 vertices of a 4-simplex fixing the 4d-vectors e^μ_I where μ is the 4d space index and $I (I = 1, .., 5)$ is the label of the vertex. The easiest way to deal with this 4d geometry is to introduce the bivectors $B^{\mu\nu}_{IJ}$

$$B^{\mu\nu}_{IJ} = e^\mu_K \wedge e^\nu_L + e^\mu_L \wedge e^\nu_M + e^\mu_M \wedge e^\nu_K \tag{5.70}$$

where the indices $IJKLM$ form an even permutation of $1, 2, 3, 4, 5$. If t_1 is the tetrahedron with vertexes e_2, e_3, e_4, e_5 and so on cyclically, the bivector $B^{\mu\nu}_{nm}$ will be the bivector normal to the triangle t_{nm} shared by the tetrahedra t_n and t_m. The normal n^m_n to this triangle, in the 3 surface determined by the tetrahedron t_n is $(n^m_n)^\nu = B^{\mu\nu}_{nm}(t_n)_\mu$, where $(t_n)_\mu$, is the normal to the tetrahedron. Using this, it is a tedious but straightforward exercise to compute the components of the projected linearized propagator. Writing the bimatrix $G^{ij,kl}_{linearized\,1,2} = (G^{kl})^{ij}$, where $ijkl = 3, 4, 5$ we have

$$(G^{kl})^{ij} \sim \frac{1}{512}\begin{pmatrix} \begin{pmatrix} -16 & 6 & 6 \\ 6 & -28 & 16 \\ 6 & 16 & -28 \end{pmatrix} & \begin{pmatrix} 6 & 4 & -7 \\ 4 & 6 & -7 \\ -7 & -7 & 16 \end{pmatrix} & \begin{pmatrix} 6 & -7 & 4 \\ -7 & 16 & -7 \\ 4 & -7 & 6 \end{pmatrix} \\ \begin{pmatrix} 6 & 4 & -7 \\ 4 & 6 & -7 \\ -7 & -7 & 16 \end{pmatrix} & \begin{pmatrix} -28 & 6 & 16 \\ 6 & -16 & 6 \\ 16 & 6 & -28 \end{pmatrix} & \begin{pmatrix} 16 & -7 & -7 \\ -7 & 6 & 4 \\ -7 & 4 & 6 \end{pmatrix} \\ \begin{pmatrix} 6 & -7 & 4 \\ -7 & 16 & -7 \\ 4 & -7 & 6 \end{pmatrix} & \begin{pmatrix} 16 & -7 & -7 \\ -7 & 6 & 4 \\ -7 & 4 & 6 \end{pmatrix} & \begin{pmatrix} -28 & 16 & 6 \\ 16 & -28 & 6 \\ 6 & 6 & -16 \end{pmatrix} \end{pmatrix} \tag{5.71}$$

which displays the equality of the various terms. The five different components have here values $(-16, 6, -28, -7, 4)/512$. A judicious choice of the parameters $c_1, c_2, c_3, c_5, (c_7 - c_8)$ can match these values.

5.5 Conclusion and perspectives

We have shown that a vertex with an appropriate asymptotic expansion, combined with a suitable boundary state, can yield the full tensorial structure of the propagator.

In doing so, we have also learned several lessons. The main lesson is that the non-commutativity of the angles requires a semiclassical state to have an oscillatory behavior in the intertwiners. In order to match this behavior, and approximate the semiclassical dynamics, the vertex must have a similar *oscillatory dependence on the intertwiners*. (This should not affect with possible finitness properties of the model [127].) The second lesson is that the symmetries of the boundary state must be considered with care, if we do not want to loose relevant dynamical information. Symmetrizing over the permutation of the vertices is a simple way of achieving a symmetric state without inserting additional unwanted symmetries. In doing so, however, one must take into account that a choice of pairing breaks the 4-simplex symmetry.

The most interesting open question, in our opinion, is whether other vertex amplitudes considered (such as [110][142]) and in particular the vertex amplitude recently studied in [143, 144] satisfies the requirements for yielding the correct full tensorial structure of the graviton propagator. In particular, whether there is an oscillation in the intertwiners. This issue can be addressed analytically, via a saddle point analysis of the asymptotic of the new vertex, or numerically, using the technology developed in [155]. Some preliminary numerical indications appear to be optimistic [156]. Also, we think that the role of the five inequivalent structures illustrated in Figure 5.1 deserve to be better understood.

Chapter 6

Conclusions and perspectives

This PhD thesis is addressed to try to make possible comparisons of LQG,SM with the usual theories with which we describe the world. The object studied is the graviton propagator.

The principal tools to calculate the scattering amplitudes are in fact the N point functions of usual QFT. However the standard N-point functions loose of meaning because of diff invariance proper of a background independent theory. An alternative strategy is then needed.

In this thesis we have presented and developed Rovelli strategy to overcome this problem. We have organized the work in a self-consistent way presenting in the first two chapters, all the basic technical an theoretical tools of LQG and SM, needed to calculate the propagator using Rovelli's technique.

We have then summarized, in the third chapter, the recent works of Rovelli, Modesto and others [14, 15, 16] that have opened a way for the definition of scattering amplitudes background independent consistent with the generally covariant description proper of LQG and SM. From the cited papers emerges the possibility of calculate N point functions looking at the boundary amplitudes, namely the functional integral over a finite region of space-time seen as a function of the boundary values of the field; this formulation, well defined in QFT can be extended to Quantum Gravity in a way completely background-independent including among the boundary fields the gravitational potential and observing that is this last one that determines the distance between the points arguments of n-point functions.

This strategy united to the use of SM to describe the dynamic, has made possible the calculation of some components of the 2 point function, that in appropriate limits, show the behavior of the usual graviton propagator, namely the one emerging from the quantization of a spin 2 massless field.

Starting from these encouraging results my work, has been turned to the reconstruction of the whole tensorial structure of the LQG propagator to get an object fully comparable with the usual perturbative one. The calculations have started using the same three key ingredients used for the diagonal terms: the graviton operators, the boundary state, coding the kinematics, and the vertex, coding the dynamics.

The construction of the non diagonal terms at first sight was only a calculation with fixed ingredients.

Step by step we have realized (forced by difficulties in using those ingredients) that new

ingredients were needed. We have stumbled on the necessity of think over the whole used theory.

The non diagonal terms in fact have called into play the dependence of the spinnetworks from the intertwiners. The non diagonal graviton operators in fact depend on intertwiners and geometrically they cause angles fluctuation of the triangulated manifold, while the diagonal operators cause only areas fluctuations. The presence of intertwiners, as kinematical variables, in turn, has called into play the dependence of the vertex from these variables. This is a crucial aspect of the definition of the quantum dynamics. In particular, the dynamics used to compute the diagonal terms was defined by the Barrett-Crane (BC) vertex, where the dependence on the intertwiners was trivial. This kind of dynamics in this contest is "frozen", in the sense that the tetrahedra of the triangulation can fluctuate only with respect to areas, but the angles are fixed.

The BC model has then a number of degrees of freedom lesser of what the kinematics requires.

In the paper [17] we have found that forcing the calculations with the BC vertex we don't get the correct tensorial structure of the propagator and also we loose its good behavior in the large-distance limit found for the diagonal terms. This result is of interest for many reasons; first, it indicates that the propagator calculations are nontrivial. In particular, they are not governed just by dimensional analysis and they do test the dynamics of the theory. Second, it reinforces the expectation that the BC model fails to yield classical GR in the long-distance limit. Moreover it has motivated the search for a vertex amplitude that modifies the BC amplitude: recently new vertexes, which address precisely the problems that we found, have been proposed [143, 144, 145, 146, 147, 148, 150, 151]. Finally, and more importantly, it opens the possibility of studying the conditions that an alternative vertex must satisfy in order to yield the correct long-distance behavior. We have found that these difficulties are not an intrinsic problem of the background-independent loop and spinfoam formalism, but instead, they are a specific difficulty of the BC vertex.

In the paper [18] we have given general indications on the asymptotic properties that the dynamics should have, if it has to yield the correct low energy limit. Recomputing the propagator using these assumption on the vertex, we end up with a propagator, calculated in the LQG formalism, that coincides with the well known propagator of a spin2 massless particle! This result allows to establish a close relation between LQG and the low energy world. What's more the underlyng theory is not plagued by non–renormalizability and it opens the way to calculate quantum corrections without infinities.

Future Directions

There is a lot of work to do starting from this thesis:

The first step is to understand whether other vertex amplitudes, in particular the vertex amplitude recently proposed in references [143, 144, 145, 146, 147, 148, 150, 151] satisfies the requirements for yielding the correct full tensorial structure of the graviton propagator. The paper [18] provides a clear test for new spin foam models: it is enough to check the asymptotics of any spin foam vertex to see if it is able to reproduce perturbative GR.

Next step will be the extension of these results to calculate the N point functions; with

similar objects in hands, it will finally be possible to see how from LQG can emerge the low energy physics. The calculations done show the Newton Law: exploring higher order correction, we could obtain modification of the gravitation potential and in general corrections to the usual perturbative predictions. There are many other limits to explore; all the calculations have been done in the large distance limit to reproduce the model of a graviton propagating on flat background; exploring the opposite limit means enter in the domain of pure quantum gravity effects. There are also the higher order terms of the expansion in Feynman graphs of the group field theory considered to study.

An important step to strengthen these results would be to clarify the relation between 4d SM and LQG, trying to extend the results of Noui and Perez [157] to the 4d case, starting from the new models. Moreover the natural evolution of this project will be the introduction of matter fields. We can think to extend the formalism tested without matter to include in this framework the results of Freidel and Livine [158]. They have shown as in three dimensions is possibile to derive rigorously the effective quantum field theory of a scalar self-gravitating field, starting from a SF model. The quantum properties of the gravitational field manifest itself in the effective QFT by a spacetime non-commutativity. This is a remarkable concrete realization of the idea that non-commutativity of spacetime emerges as an effect of quantum gravitational effects at the Planck scale.

If we could extend these results to the four dimensional case two directions of research would arise:

First, theoretically, we could search for the connection between the existing theories of Non–Commutative Geometry (NCG) and LQG.

Second, at phenomenological level, we could investigate the explicit consequences of this noncommutativity on the n-point functions of the scalar field and on eigenvalues of certain observables. These calculations could bring to a violation of the usual dispersion relations (this is in fact one of the most interesting results of NCG) and we could search for the phenomenological implications of this violation from this new perspective. In this case, astrophysical phenomena at cosmological distances, as Gamma Ray bursts, could represent the ideal framework to test QG [159].

Speculating a little bit these kind of calculations could bridge between QFT and LQG and bring the effects of the quantized spacetime directly inside Standard Model.

APPENDIX

Appendix A

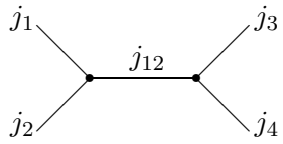
Intertwiners

Consider the direct decomposition of the tensor product of n $SU(2)$ irreps, $\bigotimes_{i=1}^{n} \mathcal{H}_{j_i} = \bigoplus_J \mathcal{H}_J$. Under the assumption $j_n \leq j_1 + \ldots + j_{n-1}$, the decomposition contains the invariant space $\mathcal{H}_0$, which we call $\mathcal{I}_{j_1 \ldots j_n}$. This is the space of the spherically symmetric vectors, and its elements are called intertwiners. Under normalisation, they are given by the generalised Wigner $3m$-coefficients.

The first non–trivial example is for $n = 4$.[1] The dimension d of $\mathcal{I}_{j_1 \ldots j_4}$ is, assuming $j_1 \leq \cdots \leq j_4$,

$$d = j_1 + j_2 - \frac{1}{2}|j_1 + j_4 - (j_2 + j_3)| + 1,$$

and the maximum reached for the symmetrical case $j_i \equiv j$, $\forall i$, where $d = 2j + 1$. A basis of $\mathcal{I}_{j_1 \ldots j_4}$ is usually written grouping the systems in couples. Choosing for instance the tree (12)(34), the independent intertwiners correspond to the $2j_{12}+1$ possible values of the coupled spin $\vec{J}_{12} = \vec{J}_1 + \vec{J}_2$. Introducing a "virtual" link, we might represent this choice as follows:

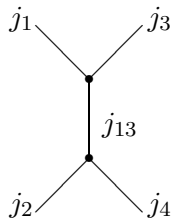


As a shorthand notation, we use $|j_{12}\rangle$ as the basis vectors of $\mathcal{I}_{j_1 \ldots j_4}$. The change of basis to a different tree decomposition, say (13)(24), is given by the recoupling relation, which uses the Wigner $6j$ symbol (see below),

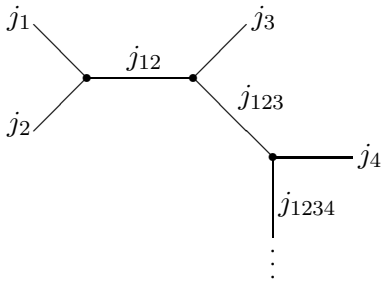
$$|j_{13}\rangle = (-1)^{\sum_i j_i} \sqrt{\dim j_{12}\, \dim j_{13}} \begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j_4 & j_{13} \end{Bmatrix} |j_{12}\rangle, \tag{A.1}$$

and it corresponds to the grouping

[1] For $n = 3$ all invariant spaces are 1-dimensional.



The case $n > 4$ can be treated as above, with the introduction of $n-3$ virtual links, each one labeled with irreps of the sums $J_1 + J_2$, $(J_1 + J_2) + J_3$, and so on:



Appendix B

Recoupling theory

We give here the definitions at the basis of recoupling theory and the graphical notation that is used in the text. Our main reference source is [161].

- *Wigner 3j-symbols.* These are represented by a 3-valent node, the three lines stand for the angular momenta wich are coupled by the 3j-symbol. We denote the anti-clockwise orientation with a + sign and the clockwise orientation with a sign -. in index notation $v^{\alpha\beta\gamma}$:

$$\begin{pmatrix} a & b & c \\ \alpha & \beta & \gamma \end{pmatrix} = \quad a\,\alpha \;\underline{\qquad}\; \begin{matrix} c\,\gamma \\ + \\ b\,\beta \end{matrix} \quad = \quad a\,\alpha \;\underline{\qquad}\; \begin{matrix} b\,\beta \\ - \\ c\,\gamma \end{matrix} \tag{B.1}$$

  The symmetry relation $v^{\alpha\beta\gamma} = (-1)^{a+b+c}\, v^{\alpha\gamma\beta}$

$$\begin{pmatrix} a & b & c \\ \alpha & \beta & \gamma \end{pmatrix} = (-1)^{a+b+c} \begin{pmatrix} a & c & b \\ \alpha & \gamma & \beta \end{pmatrix} \tag{B.2}$$

  implies

$$a \;\underline{\qquad}\; \begin{matrix} b \\ + \\ c \end{matrix} \quad = (-1)^{a+b+c} \quad a \;\underline{\qquad}\; \begin{matrix} b \\ - \\ c \end{matrix} \tag{B.3}$$

- *The Kroneker delta.*

$$\delta_{ab}\, \delta^{\alpha}_{\beta} = \quad a\alpha \;\underline{\qquad\qquad}\; b\beta. \tag{B.4}$$

- *Anti-symmetric or "metric" tensor.* (1-j symbol). In vector notation: ${}^{a}\epsilon_{\alpha\beta}$

$$\begin{pmatrix} a \\ \alpha\beta \end{pmatrix} = (-1)^{a+\alpha}\, \delta_{\alpha-\beta} \tag{B.5}$$

in graphical notation:

$$\delta_{ab} \begin{pmatrix} a \\ \alpha\beta \end{pmatrix} = \quad a\alpha \longleftarrow b\beta \tag{B.6}$$

the relations $\epsilon^{\alpha'\beta}\epsilon_{\alpha\beta} = \delta^{\alpha'}_{\alpha}$ and $\epsilon^{\alpha'\beta}\epsilon_{\beta\alpha} = -\delta^{\alpha'}_{\alpha}$, for the fundamental representation, read, for generic representations

$$\sum_{\beta} \begin{pmatrix} a \\ \alpha'\beta \end{pmatrix} \begin{pmatrix} a \\ \alpha\beta \end{pmatrix} = \delta^{\alpha'}_{\alpha} \tag{B.7}$$

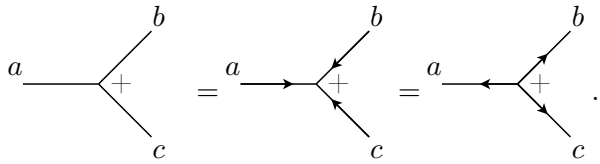
$$a\alpha \longrightarrow \longleftarrow a\alpha' = a\alpha \longrightarrow\!\!\!\!\!\!\longrightarrow a\alpha' \tag{B.8}$$

and

$$\sum_{\beta} \begin{pmatrix} a \\ \alpha'\beta \end{pmatrix} \begin{pmatrix} a \\ \beta\alpha \end{pmatrix} = (-1)^{2a}\, \delta^{\alpha'}_{\alpha} \tag{B.9}$$

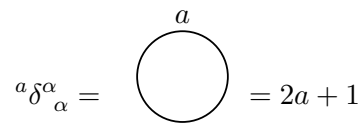
$$a\alpha \longrightarrow\!\!\!\!\!\!\longrightarrow a\alpha' = (-1)^{2a} \quad a\alpha \longrightarrow\!\!\!\!\!\!\longrightarrow a\alpha' \tag{B.10}$$

From the properties of the 3j symbols it follows: in vector notation: $v^{\alpha\beta\gamma} = v_{\alpha\beta\gamma}$; in graphical notation:



. (B.11)

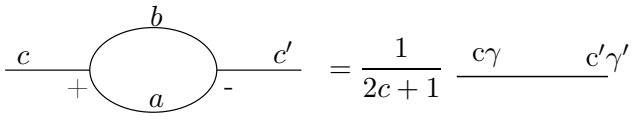
Trace of the identity



(B.12)

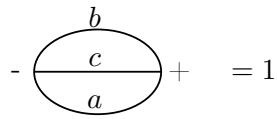
- *First orthogonality relation for 3j-symbols.*

$$\sum_{\alpha,\beta} \begin{pmatrix} a & b & c \\ \alpha & \beta & \gamma \end{pmatrix} \begin{pmatrix} a & b & c' \\ \alpha & \beta & \gamma' \end{pmatrix} = \frac{1}{2c+1}\, \delta_{cc'}\, \delta^{\gamma}_{\gamma'} \tag{B.13}$$



(B.14)

This implies

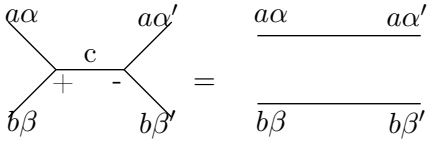


(B.15)

- *Second orthogonality relation.*

$$\sum_{c\gamma}(2c+1)\begin{pmatrix} a & b & c \\ \alpha & \beta & \gamma \end{pmatrix}\begin{pmatrix} a & b & c \\ \alpha' & \beta' & \gamma \end{pmatrix} = \delta^{\alpha}_{\alpha'}\,\delta^{\beta}_{\beta'} \tag{B.16}$$

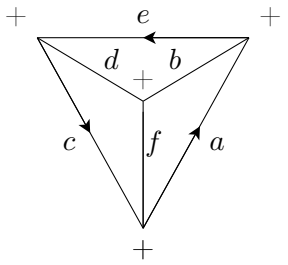
Graphically

$$\sum_{c}(2c+1)$$ 

$$= \quad \overline{a\alpha \qquad a\alpha'} \quad \overline{b\beta \qquad b\beta'} \tag{B.17}$$

- *6j symbol.*

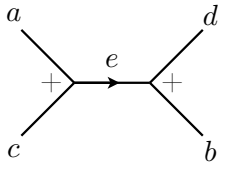
$$\left\{ \begin{matrix} a & b & e \\ d & c & f \end{matrix} \right\} = \sum_{\alpha\epsilon\gamma}(-1)^{a+e+c-\alpha-\epsilon-\gamma}\begin{pmatrix} a & f & c \\ \alpha & \phi & -\gamma \end{pmatrix}\begin{pmatrix} c & d & e \\ \gamma & \delta & -\epsilon \end{pmatrix}\begin{pmatrix} e & b & a \\ \epsilon & \beta & -\alpha \end{pmatrix}\begin{pmatrix} b & d & f \\ \beta & \delta & \phi \end{pmatrix}$$



(B.18)

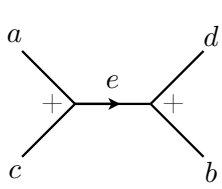
- *The 4j coefficient, or 4-valent node.*

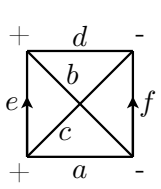
$$\begin{pmatrix} a & c & b & d \\ \alpha & \gamma & \beta & \delta \end{pmatrix} = \sum_{\epsilon}(-1)^{e-\epsilon}\begin{pmatrix} e & a & c \\ \epsilon & \alpha & \gamma \end{pmatrix}\begin{pmatrix} e & b & d \\ -\epsilon & \beta & \delta \end{pmatrix} \tag{B.19}$$

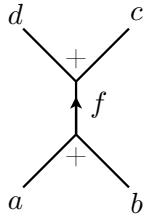


- *Recoupling theorem.*

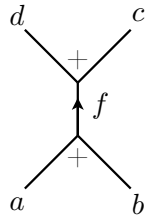
$$\begin{pmatrix} a & c & b & d \\ \alpha & \gamma & \beta & \delta \end{pmatrix} = \sum_{f}\dim f\,(-1)^{b+c+e+f}\left\{ \begin{matrix} a & b & f \\ d & c & e \end{matrix} \right\}\begin{pmatrix} a & b & c & d \\ \alpha & \beta & \gamma & \delta \end{pmatrix} \tag{B.20}$$



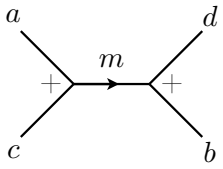
$= \sum_{f}\dim f$ 



- *Inverse transformation.*



$$= \sum_m \dim m \, (-1)^{b+c+f+m} \left\{ \begin{array}{ccc} a & c & m \\ d & b & f \end{array} \right\}$$



(B.21)

- *Orthogonality relation for the 6j symbols.*

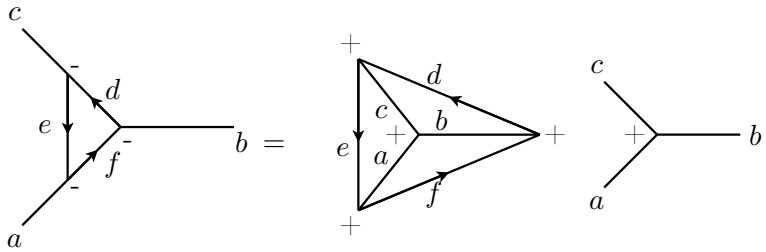
$$\sum_f \dim m \dim f \left\{ \begin{array}{ccc} a & b & f \\ d & c & e \end{array} \right\} \left\{ \begin{array}{ccc} a & c & m \\ d & b & f \end{array} \right\} = \delta_{em} \tag{B.22}$$

- *Biedenharn-Elliot identity.*

$$\begin{aligned} \sum_x \dim x \, (-1)^{a+b+c+d+e+f+g+h+i+x} \left\{ \begin{array}{ccc} e & f & x \\ b & a & i \end{array} \right\} \left\{ \begin{array}{ccc} a & b & x \\ c & d & h \end{array} \right\} \left\{ \begin{array}{ccc} d & c & x \\ f & e & g \end{array} \right\} \\ = \left\{ \begin{array}{ccc} g & h & i \\ a & e & d \end{array} \right\} \left\{ \begin{array}{ccc} g & h & i \\ b & f & c \end{array} \right\} \end{aligned} \tag{B.23}$$

- *The "basic rule".*

$$\begin{aligned} \sum_{\delta\epsilon\phi} (-1)^{d+e+f-\delta-\epsilon-\phi} \left(\begin{array}{ccc} d & e & c \\ -\delta & \epsilon & \gamma \end{array} \right) \left(\begin{array}{ccc} e & f & a \\ -\epsilon & \phi & \alpha \end{array} \right) \left(\begin{array}{ccc} f & d & b \\ -\phi & \delta & \beta \end{array} \right) \\ = \left\{ \begin{array}{ccc} a & b & c \\ d & e & f \end{array} \right\} \left(\begin{array}{ccc} a & b & c \\ \alpha & \beta & \gamma \end{array} \right) \end{aligned} \tag{B.24}$$



(B.25)

Appendix C

Some facts on SO(N) representation theory

We collect here some facts on the representation theory of $SO(D)$. We label finite dimensional irreducible representation of $SO(D)$ by their highest weight Λ. Λ is a vector of length $n = [D/2]$ ($[\cdot]$ is the integral part): $\Lambda = (N_1, \cdots, N_n)$, where N_i are integer and $N_1 \geq \cdots \geq N_n$. If we are interested in representations of $Spin(D)$ we let the N_i be half integers. The representation labeled by the highest weight $\Lambda = (N, 0, \cdots, 0)$ are called *simple* or *spherical*. Let $X_{ij}, 1 \leq i, j \leq D$ be a basis of the Lie algebra of $SO(D)$. The simple representation are the ones for which the "simplicity" relations

$$X_{[ij}X_{ij]} \cdot V_N = 0 \tag{C.1}$$

are satisfied. Here, $[\cdot]$ denotes the antisymmetrisation. The representation space V_N of a simple representation can be realized as a space of spherical harmonics, that is, harmonic homogeneous polynomial on R^D. Any L^2 function on the sphere can be uniquely decomposed in terms of these spherical harmonics

$$\mathcal{L}^2(S^{D-1}) = \oplus_{N=0}^{\infty} V_N. \tag{C.2}$$

In the case of $SO(4)$, since $Spin(4) = SU(2) \times SU(2)$, there is an alternate description of the representation as products of two representations j' and j'' of $SU(2)$. The relation with the highest weight presentation is given by

$$N_1 = j' + j'', \qquad N_2 = j' + j''. \tag{C.3}$$

the simple representation are therefore the representation in which $j' = j'' := j$. Thus, we can label simple representations with a half integer spin j. Notice that the integer "color" $N = 2j$ is also the (nonvanishing component of the) highest weight of the representation.

In the text we have used the following properties of the simple representations

- Let V_Λ be a representation of $SO(D)$, we say that $\omega \in V_\Lambda$ is a spherical vector if it is invariant under the action of $SO(D-1)$. Such a vector exists if and only if the representation is simple. In that case this vector is unique up to normalization.

- The space of intertwiner of three representations of $SO(4)$ is at most one dimensional.
- The representation of $SO(N)$ are real. This means that it is always possible to choose a basis of V_Λ such that the representation matrices are real. If we are interested by representation of half integer spin of $Spin(N)$ it is still true that Λ is equivalent to its complex conjugate or dual. However the isomorphism is non trivial.

Appendix D

SO(4) Intertwiners and their spaces

In this appendix we review some properties and definitions in the theory of irreducible representations of SO(4) (on this see [164]). We follow the conventions of [112].

Given $g \in SO(4)$ we denote by $D^{(\Lambda)}_{\alpha\beta}(g)$ the representation matrix corresponding to the irreducible representation of order Λ. Integration over $SO(4)$ or the $SO(3)$ subgroup H is performed with the normalized Haar measure of the group and the subgroup respectively. The integration of two representation matrices is given by

$$\int_{SO(4)} dg\ \overline{D^{(\Lambda)}_{\alpha\beta}(g)} D^{(\Lambda')}_{\alpha'\beta'}(g) = \frac{1}{\Delta_\Lambda} \delta^{\Lambda\Lambda'}\ \delta_{\alpha\alpha'}\ \delta_{\beta\beta'}, \tag{D.1}$$

where $\Delta(N)$ denotes the dimension of the representation. In the case of $SO(4)$ we can choose a basis in which matrices are orthogonal, and the bar can be dropped from the previous equation. The integral of the product of three group elements is

$$\int\limits_{SO(4)} dg\ D^{(N_1)}_{\alpha_1\beta_1}(g) D^{(N_2)}_{\alpha_2\beta_2}(g) D^{(\Lambda)}_{\alpha\beta}(g) = C^{N_1 N_2\, \Lambda}_{\alpha_1\alpha_2\alpha}\ \ C^{N_1 N_2\, \Lambda}_{\beta_1\beta_2\beta}. \tag{D.2}$$

Here $C^{N_1 N_2 \Lambda}_{\gamma_1\gamma_2\gamma}$ are normalized intertwiners (Wigner 3-j symbols) between three representations of SO(4); that is $C^{N_1 N_2\, \Lambda}_{\alpha_1\alpha_2\alpha}\ \ C^{N_1 N_2\, \Lambda}_{\alpha_1\alpha_2\alpha} = 1$ (the θ–graph of $SO(4)$ is normalized to 1). The intertwiner from the tensor product of two representations N_1, N_2 to a representation Λ, if it exists is unique.

Next, consider *four* representations $N_1 \dots N_4$. They are defined on the Hilbert spaces $H_1 \dots H_4$. Consider the tensor product $H_{N_1 \dots N_4} = H_1 \otimes \dots \otimes H_4$. This space decomposes into irreducibles. In particular, it contains the trivial representation, with a certain multiplicity m. We denote the m dimensional subspace of $H_{N_1 \dots N_4}$ formed by the trivial representations, that is, the $SO(4)$ invariant subspace of $H_{N_1 \dots N_4}$ as $K_{N_1 \dots N_4}$.

When the representations $N_1 \dots N_4$ are associated to the four edges adjacent to the edge e, we write $K_{N_1 \dots N_4}$ also as $K_{\vec{N}_e}$. The vectors in $K_{N_1 \dots N_4}$ are the "intertwiners" between the representations $N_1 \dots N_4$. They are $SO(4)$ invariant tensors with four indices, one in each representation H_i. We write them as $V^{N_1 \dots N_4}_{\gamma_1 \dots \gamma_4}$. An orthonormal basis in $K_{N_1 \dots N_4}$ can be obtained as follows. We pair the representations as $(N_1, N_2), (N_3, N_4)$. Then we define

$$C^{N_1 \dots N_4\, \Lambda}_{\gamma_1 \dots \gamma_4} = \sqrt{\dim_\Lambda}\ C^{N_1 N_2 \Lambda}_{\gamma_1\gamma_2\gamma}\ C^{N_3 N_4 \Lambda}_{\gamma_3\gamma_4\gamma}. \tag{D.3}$$

As Λ runs over the finite number of representations for which the $(3-j)$-symbols do not vanish, the $C^{N_1\dots N_4\,\Lambda}_{\gamma_1\dots\gamma_4}$ form an orthonormal basis of $K_{N_1\dots N_4}$. The factor $\sqrt{\dim_\Lambda}$ normalizes these vectors in $K_{N_1\dots N_4}$. Clearly, there are other bases of this kind, obtained by pairing the indices in a different manner. For example, we can we pair the indices as $(N_1,N_2),(N_3,N_4)$ and define the basis

$$\tilde{C}^{N_1N_2N_3N_4\,\Lambda}_{\gamma_1\gamma_2\gamma_3\gamma_4} \;=\; C^{N_1N_3N_2N_4\,\Lambda}_{\gamma_1\gamma_3\gamma_2\gamma_4}. \tag{D.4}$$

Since both the C's and the $\tilde{C}$'s are orthonormal bases, the transformation matrix M is immediately given by linear algebra

$$\tilde{C}^{N_1N_2N_3N_4\,\Lambda}_{\gamma_1\gamma_2\gamma_3\gamma_4} \;=\; \sum_{\Lambda'} M^{N_1N_2N_3N_4\,\Lambda}{}_{\Lambda'}\; C^{N_1N_2N_3N_4\,\Lambda'}_{\gamma_1\gamma_2\gamma_3\gamma_4} \tag{D.5}$$

$$\begin{aligned} M^{N_1N_2N_3N_4\,\Lambda}{}_{\Lambda'} &= C^{N_1N_2N_3N_4\,\Lambda}_{\gamma_1\gamma_2\gamma_3\gamma_4}\; \tilde{C}^{N_1N_2N_3N_4\,\Lambda'}_{\gamma_1\gamma_2\gamma_3\gamma_4} \\ &= C^{N_1N_2N_3N_4\,\Lambda}_{\gamma_1\gamma_2\gamma_3\gamma_4}\; C^{N_1N_3N_2N_4\,\Lambda'}_{\gamma_1\gamma_3\gamma_2\gamma_4}. \end{aligned} \tag{D.6}$$

In fact, $M^{N_1N_2N_3N_4\,\Lambda}{}_{\Lambda'}$ is a $6-j$ symbol for $SO(4)$. For a generic permutation σ of four elements, we have a basis

$${}^{\sigma}C^{N_1N_2N_3N_4\,\Lambda}_{\gamma_1\gamma_2\gamma_3\gamma_4} \;=\; C^{N_{\sigma(1)}N_{\sigma(2)}N_{\sigma(3)}N_{\sigma(4)}\,\Lambda}_{\gamma_{\sigma(1)}\gamma_{\sigma(2)}\gamma_{\sigma(3)}\gamma_{\sigma(4)}} \tag{D.7}$$

and a corresponding matrix of change of basis $M_\sigma{}^{N_1N_2N_3N_4\,\Lambda}{}_{\Lambda'}$.

Using this technology, the integral of the product of four group elements is simply a resolution of the identity in $K_{N_1\dots N_4}$ and can be written (for any choice of basis) as

$$\int\limits_{SO(4)} dg\; D^{(N_1)}_{\alpha_1\beta_1}(g)\dots D^{(N_4)}_{\alpha_4\beta_4}(g) = \sum_{\Lambda}\; C^{N_1\dots N_4\,\Lambda}_{\alpha_1\dots\alpha_4}\; C^{N_1\dots N_4\,\Lambda}_{\beta_1\dots\beta_4}. \tag{D.8}$$

Another important equation corresponds to the integration of one representation matrix over a sub-group $SO(3)\subset SO(4)$, namely

$$\int\limits_{H=SO(3)} dh\; D^{(N)}_{\alpha\beta}(h) = w^{(N)}_\alpha w^{(N)}_\beta, \tag{D.9}$$

where $w^{(N)}_\alpha$ represents the unit vector in the irreducible representation of order N left invariant by the action of the subgroup H ($w^{(N)}_\alpha$ is non vanishing only if N is simple). Equation (D.9) defines the projector into that one-dimensional vector space.

As mentioned, invariant vectors exist only in simple representations (see the Appendix of [112]). As a consequence the projection of the intertwiner $C^{N_1\dots N_4\,N}_{\gamma_1\dots\gamma_4} w_{\gamma_1}\dots w_{\gamma_4}$ vanishes unless all the N_i and N are simple. In this case its value (see the Appendix of [112]) is given by

$$C^{N_1\dots N_4\,N}_{\gamma_1\dots\gamma_4} w_{\gamma_1}\dots w_{\gamma_4} = \frac{1}{\sqrt{\Delta(N_1)\dots\Delta(N_4)}}, \tag{D.10}$$

Finally we give the definition of the Barrett-Crane intertwiner:

$$B^{N_1,N_2,N_3,N_4}_{\gamma_1\dots\gamma_4} \equiv \sum_N C^{N_1\dots N_4\,N}_{\gamma_1\dots\gamma_4}. \tag{D.11}$$

The previous is the un-normalized Barrett-Crane intertwiner as originally defined in [70].

Appendix E

Boundary intertwiners

At the end of Section 3.2.2 , we have mentioned the difference between the space of the simple $SO(4)$ intertwiners and the space of the $SU(2)$ intertwiners. The difference shows up in the linear structure of the states. Suppose we decide to pair the four faces of the tetrahedron differently, and to represent the intertwiner in terms of the virtual link $k = j_{tt''}$ for a different pairing. In both cases, the linear properties of the space of the intertwiner allows to express a virtual link as a linear combination of virtual links of a different pairing, but the linear structure is different. In fact, in the $SU(2)$ case, the recouping theorem gives (see for instance (A.65) of [2], also for the notation.)

$$i_{\mathbf{j}} = \quad = \sum_{\mathbf{k}} \left\{ \begin{array}{ccc} \mathbf{j}_1 & \mathbf{j}_2 & \mathbf{k} \\ \mathbf{j}_3 & \mathbf{j}_4 & \mathbf{j} \end{array} \right\} \quad = \sum_{\mathbf{k}} c_{\mathbf{k}}\, i_{\mathbf{k}} \tag{E.1}$$

On the other hand, in (3.74) the simple intertwiner labelled by j is in fact formed by a couple of intertwiners, one for the left and one for the right component of $SO(4)$, having the same spin. Namely

$$i_j \quad = \quad = \quad = \sum_{k,l} \left\{ \begin{array}{ccc} j_1 & j_2 & k \\ j_3 & j_4 & j \end{array} \right\} \left\{ \begin{array}{ccc} j_1 & j_2 & l \\ j_3 & j_4 & j \end{array} \right\} \tag{E.2}$$

where the first diagram represents the self-dual and the second diagram the anti–selfdual components of the representation. Therefore in the 4d case a simple link in one pairing is equal to a sum including non simple links (when $k \neq l$) links in another pairing. That is

$$i_j = i_{(j,j)} = \sum_{k,l} c_{kl}\; i_{(k,l)} \neq \sum_{k} c_k\; i_k \tag{E.3}$$

The reason of the discrepancy between the linear structures in (E.2) and (E.3) is not entirely clear too us. This discrepancy, on the other hand, does not affect the computations

in the theory or the interpretation of the boundary states in the model we are considering. The reason is that the only intertwiner appearing in the spinfoam sum is the Barrett-Crane intertwiner, which decomposes into simple virtual links for any pairing:

$$i_{\mathrm{BC}} \;=\; \sum_j \; (2j+1) \quad$$   $$\quad = \quad \sum_k \; (2k+1) \quad$$   (E.4)

Therefore if we identify the intertwiner $i_{j_{tt'}}$ with the LQG intertwiner $i_{\mathbf{j}_{tt'}}$, we obtain simply and consistently equation (3.76).

Appendix F

Analytic expressions for 6j symbols

From [161].

$$\left\{ \begin{array}{ccc} a & b & e \\ d & c & f \end{array} \right\} = (-1)^{a+b+c+d} \Delta(a,b,e)\Delta(a,c,f)\Delta(b,d,f)\Delta(c,d,e) \sum_z (-1)^z \frac{f(z)}{z!} \tag{F.1}$$

where

$$\Delta(a,b,c) = \sqrt{\frac{(a+b-c)!(a+c-b)!(b+c-a)!}{(a+b+c+1)!}} \tag{F.2}$$

and

$$f(z) = \frac{(a+b+c+d+1-z)!}{(e+f-a-d+z)!(e+f-b-c+z)!(a+b-e-z)!(c+d-e-z)!(a+c-f-z)!(b+d-e-f)!} \tag{F.3}$$

The sum is extended to all the positive integer z, such that no factorial has negative argument.

The definition (F.1) implies some restrictions on the arguments of the 6j:

In particular the $\Delta(a,b,c)$ restricts the arguments to satisfy the triangle inequalities

$$(a+b-c) \geq 0 \quad (a-b+c) \geq 0 \quad (-a+b+c) \geq 0 \tag{F.4}$$

and $a+b+c$ has to be an integer number.

The expression (F.1) reduces to the following simple expressions used in the calculation

$$\left\{ \begin{array}{ccc} a & a & 1 \\ b & b & e \end{array} \right\} = \frac{(-1)^{a+b+e+1}}{2} \frac{C^2(a) + C^2(b) - C^2(e)}{\sqrt{C^2(a)\dim(a) C^2(b)\dim(b)}} \tag{F.5}$$

$$\left\{ \begin{array}{ccc} e & e-1 & 1 \\ a & a & b \end{array} \right\} = \frac{(-1)^{a+b+e}}{2} \sqrt{\frac{(a+b+e+1)(a-b+e)(-a+b+e)(a+b-e+1)}{C^2(a)\dim(a)\, e\dim(e)\dim(e-1)}} \tag{F.6}$$

$$\left\{ \begin{array}{ccc} e & e+1 & 1 \\ a & a & b \end{array} \right\} = \frac{(-1)^{a+b+e+1}}{2} \sqrt{\frac{(a+b+e+2)(a-b+e+1)(-a+b+e+1)(a+b-e)}{C^2(a)\dim(a)\,(e+1)\dim(e)\dim(e+1)}} \tag{F.7}$$

The 6j symbol is invariant for interchange of any two columns, and also for interchange of the upper and lower arguments in each of any two columns:

$$\left\{ \begin{array}{ccc} a & b & e \\ d & c & f \end{array} \right\} = \left\{ \begin{array}{ccc} a & e & b \\ d & f & c \end{array} \right\} = \left\{ \begin{array}{ccc} e & a & b \\ f & d & c \end{array} \right\} = \left\{ \begin{array}{ccc} a & c & f \\ d & b & e \end{array} \right\} = \left\{ \begin{array}{ccc} d & c & e \\ a & b & f \end{array} \right\}, etc \quad \text{(F.8)}$$

We have also used the trivial facts

$$(-1)^a = (-1)^{-a} \quad \forall a \in \mathbb{Z}, \qquad (-1)^{2a} = 1 \quad \forall a \in \mathbb{Z}, \qquad (-1)^{3s} = (-1)^{-s} \quad \forall s \in \frac{\mathbb{Z}}{2}$$

in the calculations involving the 6j symbols

Appendix G

Grasping operators

The operator $E^a(\vec{x})n_a^{(ni)}$ is the "grasping operator" that acts on the spinnetwork's link dual to the triangle with normal $n_a^{(ni)}$. Let say that this link is in the j representation ; $E^a(\vec{x})n_a^{ni}$ will acts inserting an $SU(2)$ generator in the same representation [2] or equivalently, by inserting an intertwiner between the (j) rep and the rep 1, namely a 3j symbol not normalized:

$$E^{(ni)}(\vec{x})^{i\alpha}{}_{\beta} = i\,{}^{(j)}J^{i\alpha}{}_{\beta} = iN^j v^{i\alpha}{}_{\beta} \tag{G.1}$$

where ${}^{(j)}J^{i\alpha}{}_{\beta}$ is the $SU(2)$ generator in the j representation $(i = -1, 0, 1), (\alpha, \beta = -j, .., j)$, N^j is a normalization factor and $v^{i\alpha\beta}$ is the normalized 3j symbol. The action of the operator $E^{(ni)}$ is then determined by the representation of the links on which it acts; in the following we will call $E^{(j)}$ an operator acting on the link with rep j.

Graphically, with our conventions

$$E_n^{(j)} = i\,N^{(j)}\ \ j \xrightarrow[\text{-}]{1} j \tag{G.2}$$

(Note the arrow that reflect the lowered magnetic index).

To fix the normalization factor N^j is enough to square the expression (G.1), use (B.14)

$${}^{j}J^2{}^{\alpha}_{\beta} = C^2(j)\,{}^{j}I^{\alpha}{}_{\beta} = \frac{(N^j)^2}{\dim j}\,{}^{j}I^{\alpha}{}_{\beta} \tag{G.3}$$

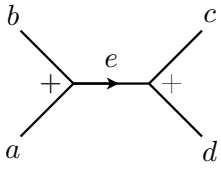
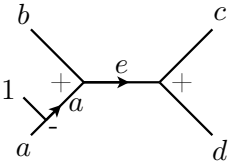
and take the trace of the previous equation (where ${}^{j}I^{\alpha}{}_{\beta}$ is the identity in the rep j), obtaining

$$N^j = \sqrt{j(j+1)\dim j} \tag{G.4}$$

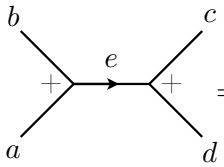
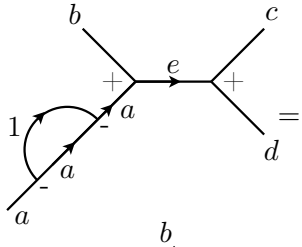
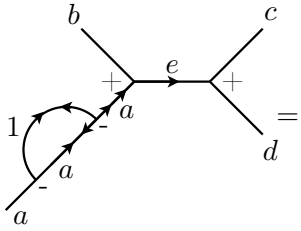
Our triangulated manifold consist of a 4-simplex made of 5 tetrahedron t_n, bounded by triangles t_{nm}. In the dual picture the 4 symplex is represented by the pentagonal net where the tetrahedra are the 4-valent nodes n, labeled by the intertwiners i_n in a given pairing, and the triangles are the links nm labeled by the spin numbers j^{nm}.

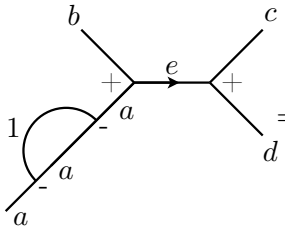
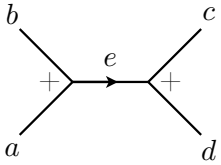
In our calculation we act with the operator $E^a(\vec{x})n_a^{(nl)}$ on the tetrahedron t_n in the direction $n_a^{(nl)}$ orthogonal to the triangle t_{nl}; in the dual picture we are then acting on the 4-valent nodes n and precisely on the link j^{ni}. To enlighten the notation, fixed a node n, we will call the four

possible colorings corresponding to the 4 directions ni with a, b, c, d where the letter indicate the representation of the links. Graphically the action of a single grasping operator operating on the link a for example is

$E_n^{(a)}$  $= iN^a$  (G.5)

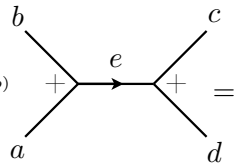
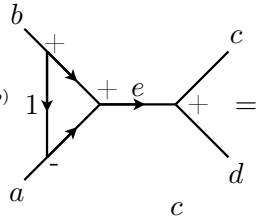
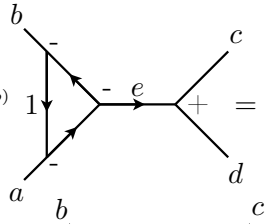
The action of our operators $E_n^{(ni)} \cdot E_n^{(nj)}$ on a node in a fixed pairing can then produce four different result depending on the two directions n^{ni},n^{nj}

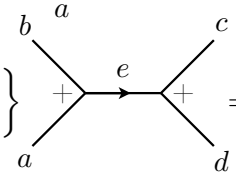
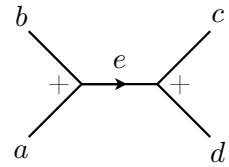
$E_n^{(a)} \cdot E_n^{(a)}$  $= -(N^a)^2$  $= -(N^a)^2$  $=$

$= (-1)^{2a+1}(N^a)^2$  $= C^2(a)$ 

(G.6)

where in the last equalities we have used the relation (B.11),(B.8),(B.10) to eliminate the arrows and the (B.3) to solve the loop using (B.14). The other possible case is

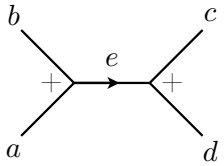
$E_n^{(a)} \cdot E_n^{(b)}$  $= -N^{(a)}N^{(b)}$  $= (-1)^{a+b+e}N^{(a)}N^{(b)}$  $=$

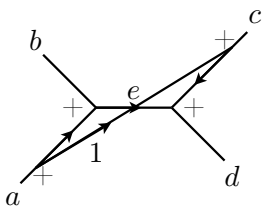
$= (-1)^{a+b+e}N^{(a)}N^{(b)} \left\{ \begin{array}{ccc} b & e & a \\ a & 1 & b \end{array} \right\}$  $= \dfrac{C^2(e) - C^2(a) - C^2(b)}{2}$ 

(G.7)

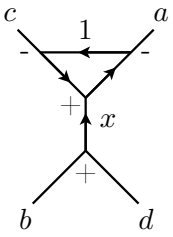
where we have changed the orientations of the 3-valent nodes to simplify the loop, using the basic identity (B.25),and used the symmetry properties of $6j$ symbols and its explicit expression (F.5)

The other possible action is

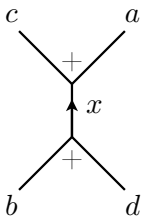
$$E_n^{(a)} \cdot E_n^{(c)}$$ 

$$= -N^{(a)}N^{(c)}$$ 

$$=$$

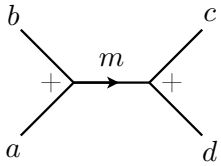
$$= N^{(a)}N^{(c)} \sum_x (-1)^{a+d+e+x} \dim x \begin{Bmatrix} b & d & x \\ c & a & e \end{Bmatrix}$$ 

$$=$$

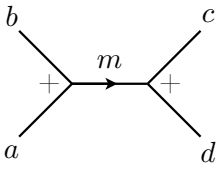
$$= N^{(a)}N^{(c)} \sum_x (-1)^{a+c+x}(-1)^{a+d+e+x} \dim x \begin{Bmatrix} b & d & x \\ c & a & e \end{Bmatrix} \begin{Bmatrix} c & a & x \\ a & c & 1 \end{Bmatrix}$$ 

$$=$$

$$= N^{(a)}N^{(c)} \sum_x (-1)^{a+c+x}(-1)^{a+d+e+x} \dim x \sum_m \dim m (-1)^{a+d+m+x} \cdot$$

$$\cdot \begin{Bmatrix} b & d & x \\ c & a & e \end{Bmatrix} \begin{Bmatrix} c & a & x \\ a & c & 1 \end{Bmatrix} \begin{Bmatrix} b & a & m \\ c & d & x \end{Bmatrix}$$ 

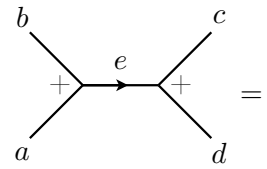
$$=$$

$$= -N^{(a)}N^{(c)}(-1)^{3d+a+b-c} \sum_m \dim m \begin{Bmatrix} e & m & 1 \\ a & a & b \end{Bmatrix} \begin{Bmatrix} e & m & 1 \\ c & c & d \end{Bmatrix}$$ 

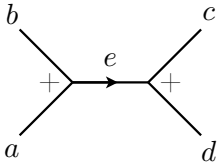
(G.8)

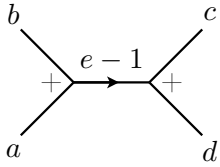
In the derivation of the result we have used, in order, the recoupling theorem (B.21) to change the pairing of the node, the basic rule (B.25) to solve the loop, the inverse transformation (B.21) to put the graph on the starting pairing and the Biedenharn-Elliot identity (B.23), having adjusted the sign factors, using the triangles inequalities of the 3j symbols defining the 6j. To analyze the result we have to look at the existence conditions of the {6j}(Appendix F)

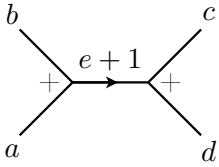
concluding that m can only take the values $e-1$, e, $e+1$, the final resul is then

$E_n^{(a)} \cdot E_n^{(c)}$ 

$=$

$= X_e^{ac}$ 

$+ Y_e^{ac}$ 

$+ Z_e^{ac}$ 

(G.9)

The form of the coefficient form is easily calculated inserting the explicit expression of the {6j} symbols given in Appendix F

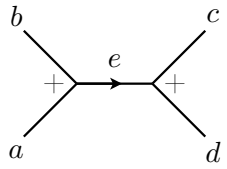
$$\begin{aligned} X_e^{ac} &= -N^{(a)}N^{(c)}(-1)^{3d+a+b-c}\dim(e)\left\{\begin{array}{ccc} e & e & 1 \\ a & a & b \end{array}\right\}\left\{\begin{array}{ccc} e & e & 1 \\ c & c & d \end{array}\right\} = \\ &= -\frac{(-1)^{2(a+b+e)}}{4}\frac{\left(C^2(b)-C^2(a)-C^2(e)\right)\left(C^2(d)-C^2(c)-C^2(e)\right)}{C^2(e)} \end{aligned} \tag{G.10}$$

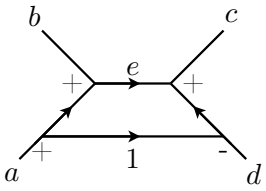
$$\begin{aligned} Y_e^{ac} &= -N^{(a)}N^{(c)}(-1)^{3d+a+b-c}\dim(e-1)\left\{\begin{array}{ccc} e & e-1 & 1 \\ a & a & b \end{array}\right\}\left\{\begin{array}{ccc} e & e-1 & 1 \\ c & c & d \end{array}\right\} = \\ &= -\frac{(-1)^{2(a+b+e)}}{4e\dim(e)}\sqrt{(a+b+e+1)(a-b+e)(-a+b+e)(a+b-e+1)}\cdot \\ &\cdot\sqrt{(c+d+e+1)(-c+d+e)(c-d+e)(c+d-e+1)} \end{aligned} \tag{G.11}$$

$$\begin{aligned} Z_e^{ac} &= -N^{(a)}N^{(c)}(-1)^{3d+a+b-c}\dim(e+1)\left\{\begin{array}{ccc} e & e+1 & 1 \\ a & a & b \end{array}\right\}\left\{\begin{array}{ccc} e & e+1 & 1 \\ c & c & d \end{array}\right\} = \\ &= -\frac{(-1)^{2(a+b+e+1)}}{4(e+1)\dim(e)}\sqrt{(a+b+e+2)(a-b+e+1)(-a+b+e+1)(a+b-e)} \\ &\cdot\sqrt{(c+d+e+2)(-c+d+e+1)(c-d+e+1)(c+d-e)} \end{aligned} \tag{G.12}$$

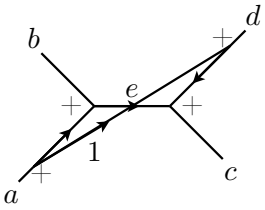
Note that by definition $(a+b+e)$ is an integer, so there aren't sign factors appearing in these expressions.

The last term is

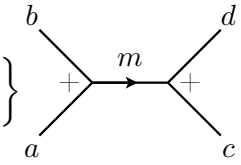
$E_n^{(a)} \cdot E_n^{(d)}$ 

$= -N^{(a)} N^{(d)}$ 

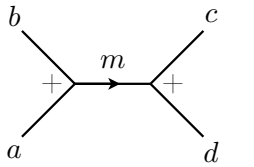
$=$

$= -N^{(a)} N^{(d)} (-1)^{c+d+e}$ 

$=$,

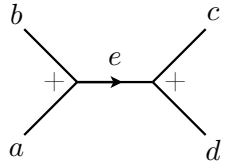
$$= -N^{(a)} N^{(d)} (-1)^{3c+a+b-d} (-1)^{c+d+e} \sum_m \dim m \begin{Bmatrix} e & m & 1 \\ a & a & b \end{Bmatrix} \begin{Bmatrix} e & m & 1 \\ d & d & c \end{Bmatrix}$$ 

$=$

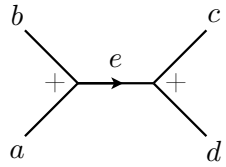
$$= -N^{(a)} N^{(d)} (-1)^{a+b+e} \sum_m (-1)^{c+d+m} \dim m \begin{Bmatrix} e & m & 1 \\ a & a & b \end{Bmatrix} \begin{Bmatrix} e & m & 1 \\ d & d & c \end{Bmatrix}$$ 

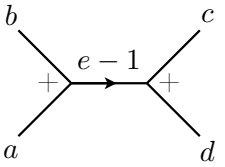
(G.13)

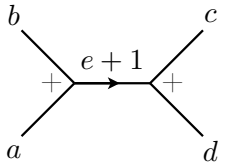
The result is obtained flipping the two link's c and d to recast the graph in the form (G.8), using the previous result and flipping back the graph in the summation. Keeping in mind that the product of {6j}appearing in the non diagonal terms is left unchanged by the change $c \to d$, the final result is then the same as (G.9) apart from the sign of the non-diagonal terms and the change $c \to d$ in the diagonal one

$E_n^{(a)} \cdot E_n^{(d)}$ 

$=$

$= X_e^{ad}$ 

$- Y_e^{ad}$ 

$- Z_e^{ad}$ 

(G.14)

where

$$\begin{aligned} X_e^{ad} &= -N^{(a)} N^{(d)} (-1)^{a+b+c+d+2e} \dim(e) \begin{Bmatrix} e & e & 1 \\ a & a & b \end{Bmatrix} \begin{Bmatrix} e & e & 1 \\ d & d & c \end{Bmatrix} = \\ &= -\frac{1}{4} \frac{\left(C^2(b) - C^2(a) - C^2(e)\right)\left(C^2(c) - C^2(d) - C^2(e)\right)}{C^2(e)} \end{aligned} \tag{G.15}$$

Note that by definition

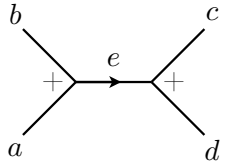
$$Y_e^{ac} = Y_e^{ad} \qquad Z_e^{ac} = Z_e^{ad} \tag{G.16}$$

The operators that we have calculated have to satisfy

$$E_n^{(a)} \cdot E_n^{(a)} + E_n^{(a)} \cdot E_n^{(b)} + E_n^{(a)} \cdot E_n^{(c)} + E_n^{(a)} \cdot E_n^{(d)} = 0 \tag{G.17}$$

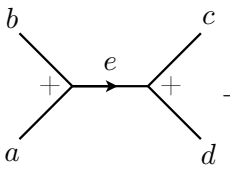
as a direct consequence of (4.7) which, at quantum level, implies that a four-valent node (by definition an intertwiner) is invariant under under the action of the group. A direct calculation on our four-valent node shows that this is indeed the case

$$\left(E_n^{(a)} \cdot E_n^{(a)} + E_n^{(a)} \cdot E_n^{(b)} + E_n^{(a)} \cdot E_n^{(c)} + E_n^{(a)} \cdot E_n^{(d)}\right)$$



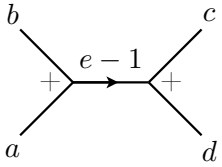
$=$

$$= \left(C^2(a) + \frac{C^2(e) - C^2(a) - C^2(b)}{2} + X_e^{ac} + X_e^{ad}\right)$$

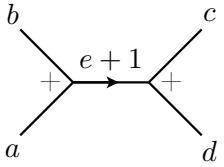


$+$

$$+ (Y_e^{ac} - Y_e^{ad})$$



$$+ (Z_e^{ac} - Z_e^{ad})$$



$= 0$ (G.18)

being 0 the coefficient of all the states.

Appendix H

Normalization of the spinnetwork states

Following [2], we define a spinnetwork $S = (\Gamma, j_l, i_n)$ as given by a graph Γ with a given orientation (or ordering of the links) with L links and N nodes, and by a representation j_l associated to each to each link and an intertwiner i_n to each node. As a functional of the connection, a spin network state is given by

$$\Psi_S[A] = \langle A|S\rangle \equiv \left(\otimes_l R^{j_l}(H[A,\gamma_l])\right) \cdot (\otimes_n i_n) \tag{H.1}$$

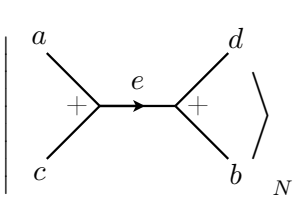
where the notation $\cdot$ indicates the contraction between dual spaces and $R^{j_l}(H[A,\gamma_l])$ is the j_l representation of the holonomy group element $H[A,\gamma_l]$ along the curve γ_l of the gravitation field connection A. In the paper we have used states normalized in such a way that

$$\left\langle S|S'\right\rangle = \delta_{S,S'}. \tag{H.2}$$

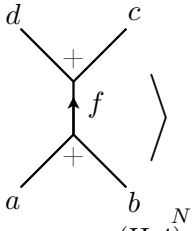
Following [162, 163] we can see that the scalar product reduces to the evaluation of the spinnetwork and that the definition of the spinnet state has to be properly normalized in order for (H.2) to be satisfied. Here we have used three-valent intertwiners (3j-Wigner symbols (B.1)) normalized to 1, so that the evaluation of the theta-graph gives 1: see (B.15). This means that the formula (8.7) of [162] defining a normalized spinnetwork state in our case reads

$$|S\rangle_N = \sqrt{\prod_{e\in\mathcal{E}} \dim j_e}\,|S\rangle, \tag{H.3}$$

where $\mathcal{E}$ is the set of real and virtual edges (intertwiner links of the decomposition of multivalent nodes). We can then see that the recoupling theorem (B.21) when applied to spinnetwork normalized state becomes



$$= \sum_f \sqrt{\dim e}\sqrt{\dim f}(-1)^{b+c+e+f} \left\{ \begin{array}{ccc} a & b & f \\ d & c & e \end{array} \right\}$$



(H.4)

Appendix I

Regge Action and its derivatives

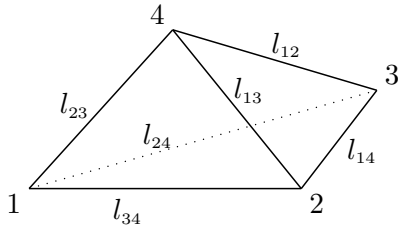
Following [160], we can write the asymptotic formula of a 6j symbol as

$$\left\{ \begin{array}{ccc} a & b & c \\ d & e & f \end{array} \right\} \approx \frac{1}{\sqrt{12\pi V}} \cos\left(S_R + \frac{\pi}{4}\right) \tag{I.1}$$

where

$$S_R = \sum_{i,j=1}^{4} l_{ij}\phi_{ij} \tag{I.2}$$

where S_R is the Regge action of the tetrahedron



(I.3)

associated to the 6j symbol, and $\phi_{ij} = \phi_{ji}$ $(i \neq j)$ are the dihedral angle at the edge l_{ij}. The edge lengths in terms of the 6j entries are: $l_{12} = a + \frac{1}{2}$, $l_{13} = b + \frac{1}{2}$, $l_{14} = c + \frac{1}{2}$, $l_{34} = d + \frac{1}{2}$, $l_{23} = b + \frac{1}{2}$ and $l_{hh} = 0$, $l_{hk} = l_{kh}$.

The dihedral angles can be expressed in terms of the volume and the areas of the tetrahedron

$$A_i A_j \sin\phi_{ij} = \frac{3}{2} l_{ij} V \tag{I.4}$$

where A_i is the area of the triangle opposite to the vertex i (A_i, A_j are the areas of the triangles that share the edge l_{ij}). We are interested in the expansion of the Regge action in the variables l_{ij}; we can express everything in term of the edge length expressing the volume and the areas using the formula

$$V_d^2 = \frac{(-1)^{d+1}}{2^d (d!)^2} \det C_d \tag{I.5}$$

where V_d is the volume of a simplex of dimension d and C_d is the Cayley matrix of dimension d; in particular given 6 edges for the tetrahedron or 3 for the triangle, with the following

Cayley matrix we can calculate all the quantities appearing in (I.4)

$$C_3 = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & l_1^2 & l_2^2 & l_3^2 \\ 1 & l_1^2 & 0 & l_4^2 & l_5^2 \\ 1 & l_2^2 & l_4^2 & 0 & l_6^2 \\ 1 & l_3^2 & l_5^2 & l_6^2 & 0 \end{pmatrix} \qquad C_2 = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & l_1^2 & l_2^2 \\ 1 & l_1^2 & 0 & l_3^2 \\ 1 & l_2^2 & l_3^2 & 0 \end{pmatrix} \tag{I.6}$$

We are interested in the asymptotic expansion of the 6j symbol that realizes the change of pairing at a given node; in the node 1 for example

$$\left\{ \begin{array}{ccc} j_{12} & j_{13} & i_1^x \\ j_{15} & j_{14} & i_1^y \end{array} \right\} \tag{I.7}$$

with link variables j_{1n} centered around j^0 and intertwiners variables i_1^{mn} centered around $i^0 = \frac{2}{\sqrt{3}} j^0$. Using the previous formula we can calculate the coefficients of the Regge action expansion linked to this symbol. The relevant derivatives for our calculation are (see also the Appendix of [120])

$$\left.\frac{\partial S_R^A}{\partial i_1^x}\right|_{j^0,i^0} = \left.\frac{\partial S_R^A}{\partial i_1^y}\right|_{j^0,i^0} = \frac{\pi}{2}, \tag{I.8}$$

$$\left.\frac{\partial^2 S_R^A}{\partial j_{1n} \partial i_1^x}\right|_{j^0,i^0} = \left.\frac{\partial^2 S_R^A}{\partial j_{1n} \partial i_1^y}\right|_{j^0,i^0} = \frac{3}{4j^0}, \tag{I.9}$$

$$\left.\frac{\partial^2 S_R^A}{\partial i_1^x \partial i_1^y}\right|_{j^0,i^0} = -\frac{\sqrt{3}}{j^0}, \tag{I.10}$$

$$\left.\frac{\partial^2 S_R^A}{\partial^2 i_1^x}\right|_{j^0,i^0} = \left.\frac{\partial^2 S_R^A}{\partial^2 i_1^y}\right|_{j^0,i^0} = -\frac{\sqrt{3}}{2j^0}. \tag{I.11}$$

Appendix J

Change of pairing on the boundary state

Here we show how one of the coefficients defined by (4.35) transforms under the change of basis determined by a different pairing. In particular, we show that with the choice of parameters in (4.23), equation (4.49) becomes (4.51). Under the change of basis,

$$\Phi'_{\mathbf{q}}[\mathbf{j}, i_1^x, i_2 ... i_5] = \sum_{i_1^y} \Phi_{\mathbf{q}}[\mathbf{j}, i_1^y, i_2 ... i_5] (-1)^{j_{13}+j_{14}+i_1^x+i_1^y} \sqrt{d_{i_1^x} d_{i_1^y}} \left\{ \begin{array}{ccc} j_{12} & j_{13} & i_1^x \\ j_{15} & j_{14} & i_1^y. \end{array} \right\} \tag{J.1}$$

With the choice of the boundary state defined by (4.23), this reads

$$\begin{aligned} \Phi'_{\mathbf{q}}[\mathbf{j}, i_1^x, i_2 ... i_5] = \quad & e^{-\frac{1}{2j_0}\sum \alpha_{(ij)(mr)} \delta j_{ij} \delta j_{mr} + i \sum \Phi \delta j_{ij}} e^{-\sum_{n\neq 1} \left(\frac{(\delta i_n^{m_n})^2}{4\sigma_{i^{m_n}}} + \sum_a \phi_{j_{na}\, i_n^{m_n}} \delta j^{an} \delta i_n^{m_n} + i \chi_{i_n^{m_n}} \delta i_n^{m_n} \right)} \\ & \cdot \sum_{i_1^y} e^{-\left(\frac{(\delta i_1^y)^2}{4\sigma_{i_1^y}} + \sum_a \phi_{j_{a1}\, i_1^y} \delta j^{a1} \delta i_1^y + i \chi_{i_1^y} \delta i_1^y \right)} (-1)^{j_{13}+j_{14}+i_1^x+i_1^y} \sqrt{d_{i_1^x} d_{i_1^y}} \left\{ \begin{array}{ccc} j_{12} & j_{13} & i_1^x \\ j_{15} & j_{14} & i_1^y \end{array} \right\}. \end{aligned} \tag{J.2}$$

Expanding the 6j symbol in the large-j limit, and applying the relation (4.37) we get

$$\begin{aligned} \Phi'_{\mathbf{q}}(\mathbf{j}, i_1^x, i_2, ..., i_5) = & e^{-\frac{1}{2j^0}\sum \alpha_{(ij)(mr)} \delta j^{ij} \delta j^{mr} + i \sum \Phi \delta j^{ij}} e^{-\sum_{n\neq 1} \left(\frac{(\delta i_n^{m_n})^2}{4\sigma_{i^{m_n}}} + \sum_a \phi_{j_{na}\, i_n^{m_n}} \delta j^{an} \delta i_n^{m_n} + i \chi_{i_n^{m_n}} \delta i_n^{m_n} \right)} \\ & \cdot \frac{e^{i\pi i_0}}{2} \int d\delta i_1^y e^{-\left(\frac{(\delta i_1^y)^2}{4\sigma_{i_1^y}} + \sum_a \phi_{j_{a1}\, i_1^y} \delta j^{a1} \delta i_1^y + i \chi_{i_1^y} \delta i_1^y \right)} \sqrt{d_{i_1^x} d_{i_1^y}} \, \frac{e^{i(S_R + \pi \delta i_1^y + \frac{\pi}{4})} + e^{-i(S_R - \pi \delta i_1^y + \frac{\pi}{4})}}{\sqrt{12\pi V}}. \end{aligned} \tag{J.3}$$

We expand the Regge action up to second order in all its 6 entries; the external link around j^0 and the intertwiners around i^0

$$\begin{aligned}S_R[j_{1n}, i_1^y, i_1^x] =& S_R[j^0, i^0] + \left.\frac{\partial S_R}{\partial j_{1n}}\right|_{j^0,i^0} \delta j_{1n} + \left.\frac{\partial S_R}{\partial i_1^x}\right|_{j^0,i^0} \delta i_1^x + \left.\frac{\partial S_R}{\partial i_1^y}\right|_{j^0,i^0} \delta i_1^y + \left.\frac{\partial^2 S_R}{\partial j_{1n}\partial j_{1n'}}\right|_{j^0,i^0} \delta j_{1n}\delta j_{1n'} + \\ &+ \left.\frac{\partial^2 S_R}{\partial j_{1n}\partial i_1^x}\right|_{j^0,i^0} \delta j_{1n}\delta i_1^x + \left.\frac{\partial^2 S_R}{\partial j_{1n}\partial i_1^y}\right|_{j^0,i^0} \delta j_{1n}\delta i_1^y + \left.\frac{\partial^2 S_R}{\partial i_1^x\partial i_1^y}\right|_{j^0,i^0} \delta i_1^x\delta i_1^y \\ &+ \frac{1}{2}\left.\frac{\partial^2 S_R}{\partial^2 j_{1n}}\right|_{j^0,i^0} (\delta j_{1n})^2 + \frac{1}{2}\left.\frac{\partial^2 S_R}{\partial^2 i_1^x}\right|_{j^0,i^0} (\delta i_1^x)^2 + \frac{1}{2}\left.\frac{\partial^2 S_R}{\partial^2 i_1^y}\right|_{j^0,i^0} (\delta i_1^y)^2 + ...\end{aligned} \tag{J.4}$$

In the background in which we are interested, $i^0 = \frac{2}{\sqrt{3}} j^0$ and $\left.\frac{\partial S_R}{\partial i_1^y}\right|_{j^0,i^0} = \left.\frac{\partial S_R}{\partial i_1^x}\right|_{j^0,i^0} = \frac{\pi}{2}$. The value $\chi = \left.\frac{\partial S_R}{\partial i_1^y}\right|_{j^0,i^0} = \frac{\pi}{2}$, yields a phase in the intertwiner variable $e^{-i\frac{\pi}{2}\delta i_1^y}$ that cancels one of the two rapidly-oscillating phase factor due to the linear term of the expansion of the Regge action. In particular the linear part in the intertwiner variable of the first exponential $e^{i(\left.\frac{\partial S_R}{\partial i_1^y}\right|_{j^0,i^0} + \pi)\delta i_1^y} = e^{i\frac{3\pi}{2}\delta i_1^y}$ combines with the boundary phase factor but the linear part of the second one $e^{-i(\left.\frac{\partial S_R}{\partial i_1^y}\right|_{j^0,i^0} - \pi)\delta i_1^y} = e^{i\frac{\pi}{2}\delta i_1^y}$ is canceled: for the same mechanism described in [16] only the second term in the summation (J.3) survives. Denoting $\tilde{S_R} = S_R - i\frac{\pi}{2}\delta i_1^y$, we have that (J.3) reduces to

$$\begin{aligned}\Phi'_{\mathbf{q}}(\mathbf{j}, i_1^x, i_2, ..., i_5) =& e^{-\frac{1}{2j^0}\sum \alpha_{(ij)(mr)}\delta j^{ij}\delta j^{mr} + i\sum \Phi\delta j^{ij}} e^{-\sum_{n\neq 1}\left(\frac{(\delta i_n^{m_n})^2}{4\sigma_{i^{m_n}}} + \sum_a \phi_{j_{na}\, i_n^{m_n}}\delta j^{an}\delta i_n^{m_n} + i\frac{\pi}{2}\delta i_n^{m_n}\right)} \cdot \\ &\cdot \frac{e^{i\pi i_0}}{2}\int d\delta i_1^y\, e^{-\left(\frac{(\delta i_1^y)^2}{4\sigma_{i_1^y}} + \sum_a \phi_{j_{1a}\, i_1^y}\,\delta j^{a1}\delta i_1^y + i\frac{\pi}{2}\delta i_1^y\right)} \sqrt{d_{i_1^x} d_{i_1^y}}\, \frac{e^{-i(\tilde{S_R} + \frac{\pi}{4})}}{\sqrt{12\pi V}}.\end{aligned} \tag{J.5}$$

From [120], we have that denoting $\mu = \sqrt{\frac{d_{i_1^x} d_{i_1^y}}{12\pi V}}$, the dominant term is $\mu[j^0]$. We take $\mu[j^0]$ out of the integration and evaluate the integral following [149]. To simplify the notation, rename the second derivative of the Regge action $G_{j^{na}, i_n^{m_n}} = \left.\frac{\partial^2 S_R}{\partial j_{na}\partial i_n^{m_n}}\right|_{j^0,i^0}$, $G_{i_n^{m_{n'}}, i_n^{m_n}} = \left.\frac{\partial^2 S_R}{\partial i_n^{m_{n'}}\partial i_n^{m_n}}\right|_{j^0,i^0}$ and indicate with $S[j_{na}]$ (4.52) the part of the Regge action that depends only on the boundary links involved in the $6j$ symbol considered and with no dependence from the intertwiners.

Substituting we get

$$\begin{aligned}\Phi'_{\mathbf{q}}(\mathbf{j}, i_1^x, i_2, ..., i_5) =& e^{-\frac{1}{2j^0}\sum \alpha_{(ij)(mr)}\delta j^{ij}\delta j^{mr}+i\sum \Phi\delta j^{ij}} e^{-\sum_{n\neq 1}\left(\frac{(\delta i_n^{m_n})^2}{4\sigma_{i^{m_n}}}+\sum_a \phi_{j_{na}\, i_n^{m_n}}\delta j^{an}\delta i_n^{m_n}+i\frac{\pi}{2}\delta i_n^{m_n}\right)} \\ & \cdot \frac{e^{i\pi i_0}}{2} e^{-i\frac{\pi}{4}}\mu[j^0] e^{-iS_R[j^0,i^0]} \\ & \cdot e^{-iS_j[j_{1a}]} e^{-i\frac{\pi}{2}\delta i_1^x} e^{-i\left(\sum_a G_{j_{1a}\, i_1^x}\,\delta j^{a1}\right)\delta i_1^x} e^{-\frac{i}{2}G_{i_1^x i_1^x}(\delta i_1^x)^2} \\ & \cdot \int d\delta i_1^y\, e^{-\frac{1}{2}(\frac{1}{2\sigma_{i_1^y}}+iG_{i_1^y i_1^y})(\delta i_1^y)^2} e^{-iG_{i_1^x\, i_1^y}\,\delta i_1^x \delta i_1^y} e^{-\left(\sum_a\left(\phi_{j_{1a}\, i_1^y}+iG_{j_{1a}\, i_1^y}\right)\delta\, j^{a1}\right)\delta i_1^y}\end{aligned} \tag{J.6}$$

The choice $\phi = -iG_{j_{1a}\, i_1^y} = -i\frac{3}{4j^0}$ eliminates the argument of last exponential. So that we fall into the same as calculation [149], and we can transform the gaussian in another gaussian with the same variance. Evaluating the integral we get

$$\begin{aligned}\Phi'_{\mathbf{q}}(\mathbf{j}, i_1^x, i_2, ..., i_5) =& e^{-\frac{1}{2j^0}\sum \alpha_{(ij)(mr)}\delta j^{ij}\delta j^{mr}+i\sum \Phi\delta j^{ij}} e^{-\sum_{n\neq 1}\left(\frac{(\delta i_n^{m_n})^2}{4\sigma_{i^{m_n}}}-i\left(\sum_a \frac{3}{4j^0}\delta j^{an}-\frac{\pi}{2}\right)\delta i_n^{m_n}\right)} \cdot \\ & \sqrt{\frac{\pi}{2(\frac{1}{2\sigma_{i_1^y}}+iG^A_{i_1^y i_1^y})}} \cdot e^{i\pi i_0} e^{-i\frac{\pi}{4}}\mu[j^0] e^{-iS^A[j^0,i^0]} \\ & e^{-iS_j^A[j_{1a}]} e^{-i\frac{\pi}{2}\delta i_1^x} e^{-i\left(\sum_a G_{j_{1a}\, i_1^x}\,\delta j^{a1}\right)\delta i_1^x} \\ & e^{-\frac{1}{2}\left(\frac{G^2_{i_1^x\, i_1^y}}{(\frac{1}{2\sigma_{i_1^y}}+iG^A_{i_1^y i_1^y})}+iG^A_{i_1^x i_1^x}\right)(\delta i_1^x)^2}\end{aligned} \tag{J.7}$$

The Gaussian in the last equation has variance

$$\sigma_{i_1^x} = \frac{1}{2}\left(\frac{G^2_{i_1^x\, i_1^y}}{(\frac{1}{2\sigma_{i_1^y}}+iG_{i_1^y i_1^y})}+iG_{i_1^x i_1^x}\right)^{-1} \tag{J.8}$$

as in [149]. Proceeding in the same way, we fix σ so that both $\sigma_{i_1^y}$ and $\sigma_{i_1^x}$ are real quantities. Remarkably the auxiliary tetrahedron described by S_R is isosceles and in this case $\sigma_{i_1^x} = \sigma_{i_1^y} = j^0/3$

The final form of the coefficient is then

$$\begin{aligned}\Phi'_{\mathbf{q}}(\mathbf{j}, i_1^x, i_2, ..., i_5) =& e^{-\frac{1}{2j^0}\sum \alpha_{(ij)(mr)}\delta j^{ij}\delta j^{mr}+i\sum \Phi\delta j^{ij}} e^{-\sum_{n\neq 1}\left(\frac{(\delta i_n^{m_n})^2}{4\sigma_{i^{m_n}}}-i\left(\sum_a \frac{3}{4j^0}\delta j^{an}-\frac{\pi}{2}\right)\delta i_n^{m_n}\right)} \cdot \\ & \cdot N_1 e^{-iS_j[j_{1a}]} e^{-\frac{1}{4}\frac{1}{\sigma_{i_1^x}}(\delta i_1^x)^2} e^{-i\left(\sum_a G_{j_{1a}\, i_1^x}\,\delta j^{a1}+\frac{\pi}{2}\right)\delta i_1^x}\end{aligned} \tag{J.9}$$

where

$$N_1 = \sqrt{\frac{\pi}{2(\frac{1}{2\sigma_{i_1^y}}+iG_{i_1^y i_1^y})}} e^{i\pi i_0} e^{-i\frac{\pi}{4}}\mu[j^0] e^{-iS_R[j^0,i^0]} \tag{J.10}$$

and we have the result (4.51).

Summarizing, the parameters (4.28) and (4.29) are determined by the requirement that the gaussian has the same shape in all bases.

Appendix K

Schrödinger representation and propagation Kernel

Here we briefly review the Schrödinger's representation of the Quantum Field Theory and use the extension of the Feynman's path integral formulation in this representation. This formalism is needed to make contact between conventional Quantum Field Theory and Spinfoam models. In fact one of the fundamental tools of these models is the propagation amplitude between spin networks, which is formally similar to the field–to–field propagator in Schrödinger's representation. We briefly review Feynman's path integral formulation of Quantum Mechanics and Schrödinger representation [165, 166, 167].

K.1 Feynman's Path Integral

The core of Feynman's path integral formulation of Quantum Mechanics consists in introducing a probability amplitude associated with every possible event. This amplitude is proportional to the classical action associated to the specific event considered. It is possible to associate an amplitude, called "kernel", with the overall event by adding together the amplitudes of each alternative way in which an event can be realized.

For example, in the case of the propagation of a particle from point a to point b the kernel W is given by the sum of the amplitude associated to every possible path in space and time,

$$W(x_2, t_2, x_1, t_1) \propto \sum_{\text{all path}\atop\text{from } x_1 \text{ to } x_2} \exp \frac{i}{\hbar} S[x(t)], \tag{K.1}$$

where $S[x(t)]$ is the classical action, calculated over the path $x(t)$ such that

$$x(t_1) = x_1, \qquad x(t_2) = x_2.$$

The absolute square of the overall amplitude is interpreted as the probability that the event will happen.

It is useful to make a brief comparison with the situation in Classical Mechanics. In the latter, the propagator of a particle from a to b is described by a unique path $\bar{x}(t)$, determined by the principle of least action. The action calculated on the classical path is also called

Hamilton function. In Quantum Mechanics, not just the particular path of extreme action contributes; all paths do. In the classical approximation, even a small change in the path, small on the classical scale, will correspond to huge changes in the action, when compared to $\hbar$. Contributions to the action for generic paths will average out, except for the classical path, according to the principle of the stationary phase. Actually trajectories differing from the classical path can still contribute as long as their action is within $\hbar$ from its extremal value. The classical trajectory is indefinite to this slight extent, and this rule serves as a measure of the limitations of the precision of the classically defined trajectory.

(K.1) can be formally rewritten as a path integral, that is, a functional integral over all paths joining x_1 and x_2:

$$W(x_1, t_1; x_2, t_2) = \int_{x_1}^{x_2} \mathcal{D}[x(t)] \ \exp \frac{i}{\hbar} S[x(t)]. \tag{K.2}$$

In Quantum Mechanics, the kernel W is solution of the Schrödinger equation, in either the variables a and b

$$i\hbar \frac{\partial}{\partial t_{1/2}} W(x_1, t_1; x_2, t_2) = H_{x_{1/x_2}} \ W(x_1, t_1; x_2, t_2).$$

Thus the knowledge of W relative to a system at a given time $\tilde{t}$ implies its knowledge at all subsequent times, which translates into a complete knowledge of the evolution of the system. The knowledge of W allows a complete description of the system and its evolution in time. Given the wave function of the system at the time 0, the kernel W allows to calculate the wave function at any subsequent time t

$$\psi(x, t) = \int dy \, W(x, t; y, 0) \, \psi(y, 0).$$

This can be made clear by considering a representation of the kernel W in the base of eigenstates $|x\rangle$ of the position operator. We have in fact

$$W(x_1, t_1, x_2, t_2) = \langle x_2 | \, e^{-iH(t_2 - t_1)} \, |x_1\rangle \, , \tag{K.3}$$

where H is the hamiltonian of the system. The representation (K.3) also allows to derive easily a representation of the kernel W in terms of eigenstates of the energy ϕ_n

$$\begin{aligned} W(x_1, t_1, x_2, t_2) &= \langle x_2 | \, e^{-iH(t_2 - t_1)} \, |x_1\rangle = \sum_n \langle x_2 | n \rangle \, \langle n | \, e^{-iH(t_2 - t_1)} \, |x_1\rangle = \\ &= \sum_n e^{-iE_n(t_2 - t_1)} \, \langle x_2 | n \rangle \, \langle n | x_1 \rangle = \sum_n e^{-iE_n(t_2 - t_1)} \phi_n(x_2) \overline{\phi_n(x_1)}, \end{aligned} \tag{K.4}$$

where $|n\rangle$ are the eigenkets and E_n the eigenvalues of H.

K.2 Schrödinger's Representation

In nonrelativistic Quantum Mechanics, the starting point is a hamiltonian operator which is canonically quantized by postulating commutation relations between position operators and

their conjugate momenta. Using Schrödinger's representation amounts to choosing the basis where the position operator X is diagonal:

$$X\,|x\rangle = x\,|x\rangle\,,$$

where $|x\rangle$ is the eigenstate with eigenvalue x. The coordinate representation of a state $|\psi\rangle$, that is, its projection on the eigenstates $|x\rangle$ is the corresponding wavefunction $\psi(x)$:

$$\psi(x) = \langle x|\psi\rangle\,,$$

$\psi(x)$ is the probability density of finding the particle in the position x. The Schrödinger equation becomes a differential equation whose solutions, the eigenfunctions of the hamiltonian differential operator, represent possible states of the system [168].

Now we transport this formalism in Quantum Field Theory, obtaining a description of Quantum Field Theory in terms of fields rather than particles.

In the Schrödinger's representation a basis for the Fock space is used where the time independent operator $\phi(\vec{x})$ is diagonal. Then in the space of states the following relation holds:

$$\hat{\phi}(\vec{x})\,|\phi\rangle = \phi(\vec{x})\,|\phi\rangle\,, \tag{K.5}$$

with the important difference that $\hat{\phi}(\vec{x})$ is an operator while $\phi(\vec{x})$ is a function. $|\phi\rangle$ are the eigenstates of the field operator with eigenvalues $\phi(\vec{x})$. Coordinate representations of state vectors or elements of Fock space are given by the projection of a state $|\Psi\rangle$ on the basis of eigenstates $|\phi\rangle$ of the field operator:

$$\Psi[\phi] = \langle\phi|\Psi\rangle\,;$$

where $\Psi[\phi]$ is a wave functional which determines the possible field configurations. $\Psi[\phi]$ is a functional in ψ and represents the probability amplitude for a field measure on the state $|\Psi\rangle$ to give the classical field $\phi(\vec{x})$. The situation is completely analogous to nonrelativistic Quantum Mechanics, where the scalar product $\langle x|\psi\rangle$ gives the probability amplitude for the particle in the generic state $|\psi\rangle$ to be found at position x.

Just like in Quantum Mechanics the states $|x\rangle$ are normalized to a $\delta(x-x')$, here the eigenstates $|\phi\rangle$ are normalized to a functional delta:

$$\langle\phi'|\phi\rangle = \prod_x \delta\big(\phi(x) - \phi'(x)\big); \tag{K.6}$$

i.e. the scalar product is nonzero only if the two configurations coincide everywhere. This formula contains an infinite product, which will often appear when dealing with the functional formalism, potentially rendering equations ill–defined. However, infinities pose no real obstacle, since it has been proved that the Schrödinger's representation is renormalizable, both in the case of static and time–dependent problems.

The scalar product between wave functionals is also an obvious extension:

$$\langle\phi_2|\phi_1\rangle = \int \prod_x \delta\phi(x)\,\langle\phi_2|\phi\rangle\,\langle\phi|\phi_1\rangle = \int \prod_x \delta\phi(x)\overline{\Psi_2(\phi)}\Psi_1(\phi).$$

In the Schrödinger's representation of nonrelativistic Quantum Mechanics, one uses a differential representation of the commutators by replacing the conjugate momenta with derivatives:

$$p \to -i\hbar \frac{\partial}{\partial x},$$

In Quantum Field Theory, the equal–time commutators are given a functional differential representation through similar steps:

$$\dot{\phi}(\vec{x}) = -i\hbar \frac{\partial}{\partial \phi(\vec{x})}$$

where $\phi(\vec{x})$ is the function defined in (K.5).

K.3 Propagation Kernel

Now we look at the propagation kernel between field configurations. In conventional Quantum Field Theory, $W[\varphi_1, t_1; \varphi_2, t_2]$ propagates the field ϕ from the field configuration $\phi|_{t=t_1} = \varphi_1$ defined on the spatial hyperplane at time t_1 to the field configuration $\phi|_{t=t_2} = \varphi_2$ defined on the spatial hyperplane at time t_2. It is an extension of the propagation kernel $W(x_1, t_1; x_2, t_2)$ propagating a particle from position x_1 at time t_1 to position x_2 at time t_2. I will now proceed to define it both in Minkowskian and Euclidean space; I will denote the propagation kernel in the Minkowskian and Euclidean case as W_M and W_E respectively, leaving the notation W for situations where both cases are concerned. In the Minkowskian case the propagation kernel can be defined by generalizing either (K.2)

$$W_M[\varphi_1, t_1; \varphi_2, t_2] = \int_{\phi|_{t=t_1}=\varphi_1}^{\phi|_{t=t_2}=\varphi_2} \mathcal{D}[\phi] \ \exp iS[\phi],$$

or (K.3)

$$W_M[\varphi_1, t_1; \varphi_2, t_2] = \langle \varphi_2 | \, e^{-iH(t_2-t_1)} \, | \varphi_1 \rangle \, . \tag{K.7}$$

From this last definition an extension of (K.4) easily follows, inserting sums on eigenstates of the energy,

$$W_M[\varphi_1, t_1; \varphi_2, t_2] = \sum_n \ e^{-iE_n(t_2-t_1)} \ \Psi_n[\varphi_2] \ \overline{\Psi_n[\varphi_1]}. \tag{K.8}$$

The kernel W is a field–to–field propagator,

$$\Psi[\varphi_2, t_2] = \int \mathcal{D}\varphi_1 \ W[\varphi_1, t_1; \varphi_2, t_2] \ \Psi[\varphi_1, t_1]. \tag{K.9}$$

The state space at time t_1, $\mathcal{H}_{t_1}$, is a Fock space, on which the hamiltonian H acts. The corresponding definitions of the propagation kernel in the Euclidean case are straightforward extensions of the above expressions to imaginary time.

The propagation kernel must obey the following properties:

i. Limit $T \to 0$:

$$\lim_{T \to 0} W[\varphi_1, 0; \varphi_2, T] = \prod_x \delta\big(\varphi_1(\vec{x}) - \varphi_2(\vec{x})\big),$$

where the functional delta must be interpreted as specified in (K.6).

ii. Convolution property:

$$W[\varphi_1, t_1; \varphi_3, t_3] = \int \mathcal{D}\varphi_2 \; W[\varphi_1, t_1; \varphi_2, t_2] \; W[\varphi_2, t_2; \varphi_3, t_3]. \tag{K.10}$$

K.4 Relation with the Vacuum State

The propagation kernel propagates the vacuum state functional Ψ_0 into itself

$$\Psi_0(\psi) = \int \mathcal{D}\phi \; W[\phi, 0; \psi, T] \; \Psi_0(\psi).$$

More importantly, the vacuum state functional can be calculated using the propagation kernel alone. Indeed, from the definition (K.8) of the propagation kernel

$$\lim_{T\to\infty} W[\varphi_1, 0; \varphi_2, T] = \lim_{T\to\infty} \sum_n \; e^{-iE_nT} \; \Psi_n(\varphi_2) \; \overline{\Psi_n(\varphi_1)} = \Psi_0(\varphi_2) \; \overline{\Psi_0(\varphi_1)};$$

this limit being valid in theory with a mass gap, that is, if E_0 and E_1 are separated by a finite amount. To obtain precisely the vacuum state it is necessary to set $\varphi_1 = 0$:

$$\lim_{T\to\infty} W[0, 0; \varphi, T] = \Psi_0(\varphi). \tag{K.11}$$

In the Euclidean case the limit $T \to \infty$ is straightforward, it can be also made rigorous with stationary phase arguments in the Minkowskian case [118, 169].

K.5 Relations with the N–point Functions

I now wish to clarify the relation between the propagation kernel and the ordinary particle propagator, also called Feynman propagator. The latter is defined as the two-point function

$$i\Delta_F(x_1 - x_2) = \langle 0 |\, T\big(\phi(x_1)\phi(x_2)\big)\, |0\rangle \,,$$

where the fields ϕ must be fundamental fields, i.e. fields which create the particle whose propagation the propagator describes and not composite operators of any kind [170].

In the Minkowskian case, the two–point function $\langle 0 |\, T\big(\phi(x_1)\phi(x_2)\big)\, |0\rangle$ can be expressed via the propagation kernel in the following way

$$\begin{aligned} &\langle 0 |\; T\big(\phi(x_1)\phi(x_2)\big) \; |0\rangle = \\ &\quad = \langle 0 |\; \phi(\vec{x}_2)\, e^{-i\,H\,(t_2 - t_1)}\, \phi(\vec{x}_1) \; |0\rangle \\ &\quad = \lim_{T\to\infty} W^{-1}[0, -T; 0, T] \int \mathcal{D}\phi_1 \, \mathcal{D}\phi_2 \; W[0, -T; \phi_1, t_1] \; \phi_1(\vec{x}_1) \; \cdot \\ &\qquad \cdot W[\phi_1, t_1; \phi_2, t_2] \; \phi_2(\vec{x}_2) \; W[\phi_2, t_2; 0, T] \\ &\quad = \int \mathcal{D}\phi_1 \, \mathcal{D}\phi_2 \; \Psi_0(\phi_1) \; \phi_1(\vec{x}_1)\, W[\phi_1, t_1; \phi_2, t_2] \; \phi_2(\vec{x}_2) \; \Psi_0(\phi_2), \end{aligned} \tag{K.12}$$

where $t_1 < t_2$. Since using the Minkowskian propagation kernel the equation involves a limit $T \to \infty$ it will be easier to perform the calculation in the Euclidean case.

With the only use of the propagation kernel it is possible to write down the two–point function, i.e. the particle propagator for the given theory. However, this result allows to do much more than that. Indeed, the quantities which allow to make contact with experiment, i.e. the scattering amplitudes like $\left\langle q_1^{\text{out}}, \ldots, q_n^{\text{out}} | p_1^{\text{in}}, \ldots, p_m^{\text{in}} \right\rangle$, can be rewritten in terms of n–point functions via the Lehmann–Symanzik–Zimmerman reduction formulas [171].

In turn, n–point functions can be rewritten in terms of two–point functions, i.e. propagators, thanks to Wick's theorem. This leads to the crucial result that the single tool of the propagation kernel allows to reconstruct scattering amplitudes via the propagators, i.e. it allows to reconstruct any quantity of physical interest that can be derived from Quantum Field Theory. Therefore the propagation kernel formulation allows to completely reformulate Quantum Field Theory.

Appendix L

Regular simplices

We collect here some simple geometrical formulas used in the text. An equilateral triangle of side L has area $A = \frac{\sqrt{3}}{4}L^2$. An equilateral tetrahedron of side L has volume $V_3 = \frac{1}{6\sqrt{2}}L^3$ and height $h = \sqrt{\frac{2}{3}}L$. The barycenter of the tetrahedron is at a distance $d = \frac{h}{4} = \frac{1}{2\sqrt{6}}L$ from a face.

A regular 4-simplex of side L has 4-volume $V_4 = \frac{\sqrt{5}}{96}L^4$. The dihedral angles Θ of the 4-simplex, defined as the angles between the outward normals to the tetrahedra, satisfy $\cos\Theta = -1/4$. The center of two tetrahedra are at a distance $D = \frac{L}{4}$ from one another and at a distance $R = \frac{1}{2\sqrt{10}}$ from the center of the 4-simplex.

Appendix M

Simple gaussian integrals used in the calculation

$$\int_{-\infty}^{+\infty} dx^D \exp -\frac{1}{2} x^a A_{ab} x^b = \frac{(2\pi)^{\frac{D}{2}}}{\sqrt{detA}}, \tag{M.1}$$

$$\int_{-\infty}^{+\infty} dx^D x^i x^j \exp -\frac{1}{2} x^a A_{ab} x^b = \frac{(2\pi)^{\frac{D}{2}}}{\sqrt{detA}}\ A^{-1}_{ij}, \tag{M.2}$$

$$\int_{-\infty}^{+\infty} dx^D \exp -\frac{1}{2} x^a A_{ab} x^b + i\theta_a x^a = \frac{(2\pi)^{\frac{D}{2}}}{\sqrt{detA}} \exp -\frac{1}{2} \theta^a A^{-1}_{ab} \theta^b, \tag{M.3}$$

$$\int_{-\infty}^{+\infty} dx^D x^i \exp -\frac{1}{2} x^a A_{ab} x^b + i\theta_a x^a = \frac{(2\pi)^{\frac{D}{2}}}{\sqrt{detA}}\ i\ A^{-1}_{ia}\theta^a\ \exp -\frac{1}{2} \theta^a A^{-1}_{ab} \theta^b, \tag{M.4}$$

$$\int_{-\infty}^{+\infty} dx^D x^i x^j \exp -\frac{1}{2} x^a A_{ab} x^b + i\theta_a x^a = \frac{(2\pi)^{\frac{D}{2}}}{\sqrt{detA}}\ \ (A^{-1}_{ia}\theta^a\, A^{-1}_{jb}\theta^b - A^{-1}_{ij})\ \exp -\frac{1}{2} \theta^a A^{-1}_{ab} \theta^b, \tag{M.5}$$

$$\int_{-\infty}^{+\infty} dx\ x^m \exp -\frac{1}{2} ax^2 + i\theta x = \sqrt{\frac{2\pi}{a}} (-i)^m\ \frac{\partial^m}{\partial \theta^m}\ \exp -\frac{1}{2a} \theta^2. \tag{M.6}$$

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