

ABSTRACT

Re-examination of Dirac and Wu-Yang Monopoles

In this thesis, we revisit the Dirac monopole and the Wu-Yang monopole. In both the cases, we showed that there was a singular magnetic field in addition to the monopole field. In the Dirac case this magnetic field singularity takes the form of the infinite string. While in the Wu-Yang case we find disk like singularity in the xy -plane. Both of these singular magnetic fields are shown to carry a field momentum and field angular momentum. In order to balance the field momentum we find both the Dirac monopole and the Wu-Yang monopole has to carry hidden mechanical momentum, implying that these singular features are real. We argue that this means that these features are real and that neither Dirac monopole and nor Wu-Yang monopole are true magnetic charges.

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RE-EXAMINATION OF DIRAC AND WU-YANG MONOPOLES

by

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The Wheel weaves...

INTRODUCTION

Electric charges are one of the fundamental properties of matter. The two types of electric charges, positive and negative, interact with each other, with like charges repelling and unlike charges attracting. The strength of this force is described by Coulomb's Law, which states that the force is proportional to the product of the charges and inversely proportional to the square of the distance between them. Throughout this thesis, we use cgs units as in reference [1] and we set the speed of light equal to 1 ($c = 1$).

A charge does not only influence other charges in its immediate vicinity; it also influences the space around it, creating an electric field. This field is a vector quantity, meaning it has both a magnitude and a direction. At any point in space, the electric field vector represents the force that a unit positive charge would experience if it were placed at that point. This effect is described by the equation [1]:

$$\mathbf{E} = \frac{q}{r^2} \hat{r} \quad (1)$$

In this equation, \vec{E} is the electric field, q is the charge creating the field, r is the distance from the charge to the point in space where the field is being evaluated, and \hat{r} is a unit vector pointing from the charge to that point. This unit vector is crucial because it gives the electric field its direction, highlighting that it is a vector field.

When we talk about the electric field, we inevitably come across the concept of electric potential. While the electric field describes the force experienced by a unit positive charge, the electric potential at a point in space is a measure of the work done to bring a unit positive charge from infinity to that point in the presence of the electric field.

Electric potential is represented by the symbol ϕ . Given a point charge q , the electric potential ϕ at a distance r from the charge is given by the equation:

$$\phi = \frac{q}{r} \quad (2)$$

This expression highlights the difference between the electric field and electric potential: while the electric field is a vector quantity depending on the square of the distance from the charge, the electric potential is a scalar quantity and inversely proportional to the distance from the charge.

$$\mathbf{E} = -\nabla\phi = -\nabla\frac{q}{r} = \frac{q}{r^2}\hat{r} \quad (3)$$

Gauss' Law is one of the four Maxwell's equations that together form the foundation of classical electrodynamics. These equations describe how electric and magnetic fields interact with each other and with electric charges and currents. Gauss' Law, in particular, relates the electric field surrounding a distribution of charges to the total charge enclosed.

In Gaussian units, Gauss' Law is written as [1]:

$$\nabla \cdot \mathbf{E} = 4\pi\rho \quad (4)$$

Here, $\nabla \cdot \mathbf{E}$ is the divergence of the electric field, and ρ is the electric charge density. For a point charge, $\rho(\vec{r}) = q\delta(\vec{r})$, where $\delta(\vec{r})$ is the Dirac delta function that is zero everywhere except at the location of the charge, where it is infinitely large such that its integral over all space is 1.

Substituting ρ into Gauss' Law, we get:

$$\nabla \cdot \mathbf{E} = \nabla \cdot \frac{q}{r^2}\hat{r} = 4\pi q\delta(\mathbf{r}) \quad (5)$$

This result confirms our earlier conclusion from Coulomb's Law: the electric field spreads out from a point charge in all directions, as the divergence of the field is non-zero only at the charge's location.

Now, let us turn our attention to the world of magnetism. Just as electric charges generate an electric field, we can speculate that magnetic charges (if they exist) would

generate a magnetic field. These hypothetical magnetic charges are known as magnetic monopoles.

Assuming magnetic monopoles exist, we can describe the magnetic field \mathbf{B} created by a magnetic monopole in a manner analogous to the electric field in equation(1). This magnetic field is of the form (see section 6.12 of reference [1]):

$$\mathbf{B} = \frac{g}{r^2} \hat{r} \quad (6)$$

Just like the electric field in equation (1), the magnetic field is radial and depends on the inverse square of the distance. Unlike the electric field, the magnetic field can be obtained by taking the curl of the vector potential, $\mathbf{B} = \nabla \times \mathbf{A}$. This is problematic since the divergence of a curl of a well behaved vector function is always zero, $\nabla \cdot (\nabla \times \mathbf{A}) = \mathbf{0}$. This fact would seem to preclude Gauss's law for magnetic fields as given below in (7). In this thesis, we will discuss how various approaches, such as the Dirac string approach or the Wu-Yang fiber bundle approach evade this restriction.

Finally, let us consider Gauss' Law for magnetic fields. Just as Gauss' Law for electric fields relates the electric field to the charge distribution that generates it, Gauss' Law for magnetic fields (if magnetic monopoles exist) would relate the magnetic field to the magnetic charge distribution. In Gaussian units, this would be written as:

$$\nabla \cdot \mathbf{B} = 4\pi\rho_m \quad (7)$$

Here, $\nabla \cdot \mathbf{B}$ is the divergence of the magnetic field, and ρ_m is the magnetic charge density. For a point magnetic charge, $\rho_m(\mathbf{r}) = g\delta(\mathbf{r})$. Substituting ρ_m into Gauss' Law for magnetic fields, we obtain:

$$\nabla \cdot \mathbf{B} = 4\pi g\delta(\mathbf{r}) \quad (8)$$

This result mirrors our earlier finding for the electric field: the magnetic field spreads out radially from a point magnetic charge in all directions and falls as the inverse square of the distance.

The other two Maxwell's equations are Faraday's law of electromagnetic induction and Ampère - Maxwell's law. They relate the curls of the electric and magnetic fields to the time derivatives of the magnetic and electric fields and electric current density respectively:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad (9)$$

$$\nabla \times \mathbf{B} = 4\pi\mathbf{J} + \frac{\partial \mathbf{E}}{\partial t} \quad (10)$$

In these equations, \mathbf{J} is the electric current density, and t is time. Faraday's law, (9), describes how a changing magnetic field induces an electric field. This is the principle behind the operation of generators and transformers. Ampère - Maxwell's law, (10), describes how electric currents and changing electric fields generate a magnetic field. If magnetic monopoles existed, we would need to add a term representing the magnetic current density \mathbf{J}_m , so that the would take the form

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} - 4\pi\mathbf{J}_m \quad (11)$$

(see section 6.12 of reference [1]).

Maxwell's equations with magnetic charge, (4), (7), (9), and (11), have an extra symmetry compared to Maxwell's equations with only electric charge. This symmetry is called dual symmetry and is given by the following dual transformations for the fields,

$$\mathbf{E} = \mathbf{E}' \cos(\theta) + \mathbf{B}' \sin(\theta) \quad (12)$$

$$\mathbf{B} = -\mathbf{E}' \sin(\theta) + \mathbf{B}' \cos(\theta), \quad (13)$$

and for the sources.

$$\rho_e = \rho'_e \cos(\theta) + \rho'_m \sin(\theta) \quad (14)$$

$$\rho_m = -\rho'_e \sin(\theta) + \rho'_m \cos(\theta) \quad (15)$$

$$\mathbf{J}_e = \mathbf{J}'_e \cos(\theta) + \mathbf{J}'_m \sin(\theta) \quad (16)$$

$$\mathbf{J}_m = -\mathbf{J}'_e \sin(\theta) + \mathbf{J}'_m \cos(\theta), \quad (17)$$

Under the dual symmetry Maxwell's equations are symmetric under the exchange of electric and magnetic quantities.

DIRAC STRING

Dirac String Formulation Review

We will briefly review the Dirac string method for magnetic charge. Magnetic charge, denoted as g , is defined in such a way that it generates a Coulomb magnetic field represented as $\mathbf{B} = \frac{g\mathbf{r}}{r^3}$. This leads to the conclusion that the divergence of \mathbf{B} originates from a delta function source, expressed as $\nabla \cdot \mathbf{B} = 4\pi g\delta(\mathbf{r})$. However, this outcome seems to contradict the relationship between the magnetic field and the vector potential, described by $\mathbf{B} = \nabla \times \mathbf{A}$. If \mathbf{A} is a well-behaved function then $\nabla \cdot \mathbf{B} = \nabla \cdot \nabla \times \mathbf{A} = 0 \neq 4\pi g\delta(\mathbf{r})$, which contradicts having a Coulomb magnetic field.

Dirac found the way around this contradiction by proposing the following Dirac string vector potentials, $\mathbf{A}_{\pm}(\mathbf{x})$ given by [2, 3]

$$\mathbf{A}_{\pm}(\mathbf{x}) = \frac{g}{r} \left(\frac{\pm 1 - \cos \theta}{\sin \theta} \right) \hat{\varphi} = \frac{g}{\rho} \left(\pm 1 - \frac{z}{\sqrt{\rho^2 + z^2}} \right) \hat{\varphi}. \quad (18)$$

The middle expression in equation (10), is in spherical polar coordinates where θ is the polar angle and r is the radial coordinate. The last expression in equation (10), is in cylindrical coordinates where ρ is the cylindrical radial coordinate, and z is the axial coordinate. It is worth noting that taking the curl of either \mathbf{A}_{\pm} does yield a Coulomb magnetic field, *i.e.* $\nabla \times \mathbf{A}_{\pm}(\mathbf{x}) = \frac{g\mathbf{r}}{r^3}$. However, these potentials exhibit singularities along the $+z$ -axis (for \mathbf{A}_{-}) or the $-z$ -axis (for \mathbf{A}_{+}), rendering them problematic. It is important to note that these potentials \mathbf{A}_{\pm} are related to each other through gauge transformations. Gauge transformations are mathematical operations that can change the form of the vector potentials without affecting the electric and magnetic fields. The specific gauge transformation relating \mathbf{A}_{+} and \mathbf{A}_{-} is given by

$$\mathbf{A}_{+}(\mathbf{x}) - \mathbf{A}_{-}(\mathbf{x}) = \nabla\alpha = \frac{2g}{\rho}\hat{\varphi}, \quad (19)$$

where the gauge function is given by $\alpha = 2g\phi$. To address the singularity issue, a common approach is to introduce a regularized vector potential. The regularized vector potential, denoted as \mathbf{A}_{\pm}^{reg} , is defined as [4]:

$$\mathbf{A}_{\pm}^{reg} = \frac{g\Theta(\rho - \epsilon)}{\rho} \left(\pm 1 - \frac{z}{\sqrt{\rho^2 + z^2 + \epsilon^2}} \right) \hat{\phi}, \quad (20)$$

where ϵ is an infinitesimal quantity. The Heaviside step function, denoted as $\Theta(x)$, is a mathematical function that outputs 1 for positive values of x and 0 for negative values of x . The Heaviside step function regularizes \mathbf{A}_{\pm}^{reg} by setting the vector potentials equal to zero in an infinitesimal cylinder of radius $\rho = \epsilon$, *i.e.* along the direction of the string singularities. Taking the curl of (20) and then letting ϵ approach zero yields:

$$\begin{aligned} \mathbf{B} &= \lim_{\epsilon \rightarrow 0} \nabla \times (\mathbf{A}_{\pm}^{reg}) = g \frac{\rho \hat{\rho} + z \hat{\mathbf{z}}}{(\rho^2 + z^2)^{3/2}} \pm 2g \frac{\delta(\rho)}{\rho} \Theta(\mp z) \hat{\mathbf{z}} \\ &= g \frac{\hat{\mathbf{r}}}{r^2} \pm 4\pi g \delta(x) \delta(y) \Theta(\mp z) \hat{\mathbf{z}} \end{aligned} \quad (21)$$

where we have utilized the relationship $\frac{\delta(\rho)}{2\pi\rho} = \delta(x)\delta(y)$ in going from the first line to the second line. In this expression, the first term on the right-hand side represents the Coulomb part of the magnetic field, while the second term corresponds to the string contribution. It is important to note that the Coulomb part gives rise to an outgoing magnetic flux of $4\pi g$, while the string part yields an incoming flux of $4\pi g$, resulting in a net flux of zero. As a consequence, $\nabla \cdot \mathbf{B} = 0$. One argument for the unphysical nature of the string part of (21) is that the two vector potentials $\mathbf{A}_+(\mathbf{x})$ and $\mathbf{A}_-(\mathbf{x})$ are related by the gauge transformation given in (19).

Useful reviews of magnetic charge that give a more extended discussion of the above results can be found in references [4, 5, 6, 7, 8, 9, 10].

One way to obtain the Dirac quantization condition is through the use of the Aharonov-Bohm (AB) effect [11]. The standard AB effect involves placing an infinite

solenoid between the slits of the double-slit experiment. Even though, the particles in this modified double slit experiment move through a region where the magnetic field is zero the vector potential is non-zero in this region. This non-zero vector potential leads to a shifting of the standard double slit interference pattern. This shift depends on the magnetic flux carried by the solenoid. One can replace the infinite solenoid in the AB setup by the Dirac string as in Fig.1. By requiring that the phase shift is some integer multiple of 2π one obtains the Dirac quantization condition. $qg = \frac{n\hbar}{2}$

In this Fig.1, we can visualize a double-slit AB experiment with a Dirac string positioned between the slits. Electric charges are emitted from a source at point A, traverse through two slits (1 and 2) situated on the screen at point B, and eventually get detected at point C. In a region with zero vector potential, the wave function can be expressed as $\Psi = \Psi_1 + \Psi_2$, where Ψ_1 and Ψ_2 are the wave functions of the charges passing through slits 1 and 2, respectively. Without the presence of the string, the wave functions combine coherently, resulting in a probability density at C given by $P = |\Psi_1 + \Psi_2|^2$ [4].

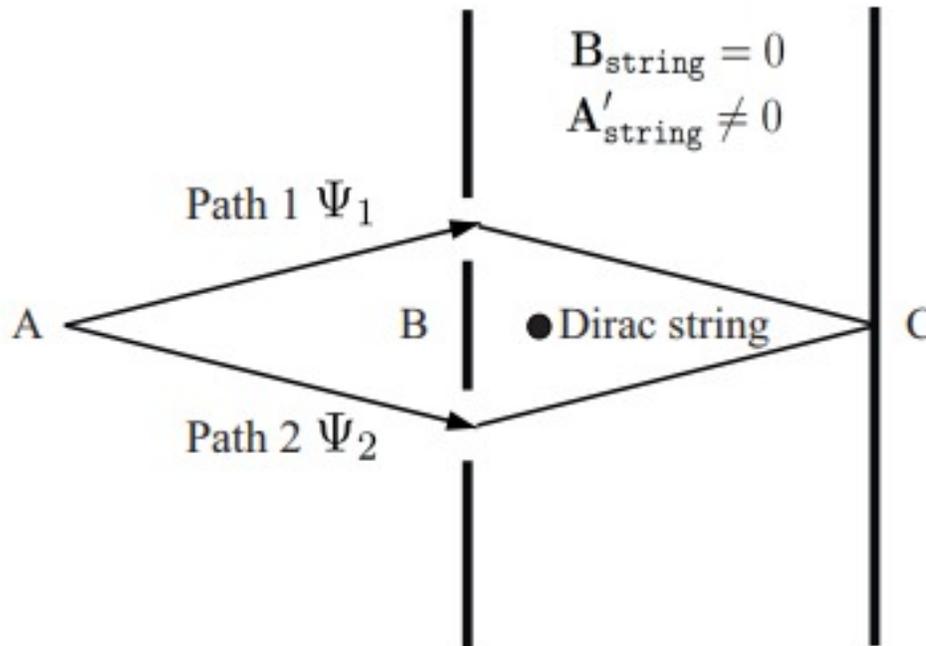


Figure 1. Double Slit Experiment

Since the Dirac string is inserted between the two slits, each wave function Ψ_1 and Ψ_2 acquires a phase due to the string potential $A_{\text{string}} \equiv A_s$. Therefore, the wave function of the charges is now given by:

$$\Psi = e^{i(q/\hbar) \int_1 A_s \cdot \rho d\phi \hat{\phi}} \Psi_1 + e^{i(q/\hbar) \int_2 A_s \cdot \rho d\phi \hat{\phi}} \Psi_2 \quad (22)$$

where $\int_1 A_s \cdot \rho d\phi \hat{\phi}$ represents the integral along path 1 and $\int_2 A_s \cdot \rho d\phi \hat{\phi}$ represents along the path 2. Factoring out the phase factor connected with path 1 then gives

$$\begin{aligned} \Psi &= (\Psi_1 + e^{i(q/\hbar) [\int_2 A_s \cdot \rho d\phi \hat{\phi} - \int_1 A_s \cdot \rho d\phi \hat{\phi}]} \Psi_2) (e^{i(q/\hbar) \int_1 A_s \cdot \rho d\phi \hat{\phi}}) \\ &= (\Psi_1 + e^{i(q/\hbar) [\oint A_s \cdot \rho d\phi \hat{\phi}]} \Psi_2) (e^{i(q/\hbar) \int_1 A_s \cdot \rho d\phi \hat{\phi}}) \end{aligned} \quad (23)$$

where $\oint A_s \cdot \rho d\phi \hat{\phi}$ is the closed loop integral for path 2 minus path 1. Carrying out the closed loop integral gives $4\pi g$. Putting all this together results in :

$$\Psi = (\Psi_1 + e^{i4\pi qg/(\hbar)} \Psi_2) e^{i(q/\hbar) \int_1 A_s \cdot \rho d\phi \hat{\phi}} \quad (24)$$

The probability density associated with (24) is $P = |\Psi|^2$, which using (24) becomes

$$P = |\Psi_1 + e^{i4\pi qg/(\hbar)} \Psi_2|^2 \quad (25)$$

We want this probability density to remain unaffected by the presence of the Dirac string. This leads to the requirement that the phase factor in (25) is equal to 1 *i.e.* $e^{i4\pi qg/(\hbar)} = 1$. This then leads to the Dirac quantisation condition $qg = \frac{n\hbar}{2}$. Under this condition, the probability density becomes $P = |\Psi_1 + \Psi_2|^2$, indicating that there would be no observable change in the interference pattern due to the Dirac string. In summary, the Dirac string is undetectable if the Dirac quantisation condition holds.

Dirac String Potential plus Charge q

While the Dirac quantization condition renders the Dirac string “invisible” in terms of its effect on the charge, q , we demonstrate in this section that the string does not become invisible with respect to the electric field generated by q . Initially, we position the magnetic charge at the origin, resulting in the magnetic field from (21), where we employ cylindrical coordinates with $\rho = \sqrt{x^2 + y^2}$ and z representing the Cartesian coordinates. We will show that the system of charge plus monopole carries both momentum and angular momentum in its electromagnetic fields. This has interesting consequences for both Dirac string formulation magnetic charge and Wu-Yang fiber bundle approach to magnetic charge.

Field Momentum of the Dirac string

We begin by placing the electric charge at position \mathbf{r}_0 . The electric field is denoted as $E = q\hat{r}'/r'^2$, where $\mathbf{r}' = \mathbf{r} - \mathbf{r}_0 = (x - x_0)\hat{x} + (y - y_0)\hat{y} + (z - z_0)\hat{z}$. Calculating the field momentum resulting from the Coulomb part of the magnetic field (i.e., $g\hat{r}/r^2$), we find

$$\mathbf{P}_{EM}^{point} = \frac{1}{4\pi} \int (\mathbf{E} \times \mathbf{B}_{Coulomb}) d^3x = \frac{qg}{4\pi} \int \left(\frac{\mathbf{r}'}{r'^3} \times \frac{\hat{\mathbf{r}}}{r^2} \right) d^3x \quad (26)$$

Using $\mathbf{r}' \times \hat{\mathbf{r}} = (\mathbf{r} - \mathbf{r}_0) \times \hat{\mathbf{r}} = -\mathbf{r}_0 \times \hat{\mathbf{r}}$ since $\mathbf{r} \times \hat{\mathbf{r}} = 0$. Thus, we obtain

$$\mathbf{P}_{point}^{EM} = -\frac{qg}{4\pi} \int \left(\frac{\mathbf{r}_0 \times \mathbf{r}}{r'^3 r^3} \right) d^3x \quad (27)$$

Without loss of generality, due to the spherical symmetry of the Coulomb magnetic field, we can take the location of the electric charge to be $\mathbf{r}_0 = (0, 0, z_0)$. With this we have $r' = \sqrt{x^2 + y^2 + (z - z_0)^2}$ and $r = \sqrt{x^2 + y^2 + z^2}$. Also, the cross product in (27) is

$$\mathbf{r}_0 \times \mathbf{r} = (z_0\hat{z}) \times (x\hat{x} + y\hat{y} + z\hat{z}) = z_0(x\hat{y} - y\hat{x}) \quad (28)$$

Inserting the result of (28) into (27), the integral becomes

$$\propto \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{x\hat{y} - y\hat{x}}{(x^2 + y^2 + (z - z_0)^2)^{3/2}(x^2 + y^2 + z^2)^{3/2}} dx dy dz \quad (29)$$

The dx integration of the \hat{y} component is proportional to

$$\int_{-\infty}^{\infty} \frac{x}{(x^2 + y^2 + (z - z_0)^2)^{3/2}(x^2 + y^2 + z^2)^{3/2}} dx = 0 \quad (30)$$

The integrand is anti - symmetric in x while the integration over dx is over an even range, thus by symmetry this integral is zero. Next, the dy integration of the \hat{x} component is proportional to

$$\int_{-\infty}^{\infty} \frac{-y}{(x^2 + y^2 + (z - z_0)^2)^{3/2}(x^2 + y^2 + z^2)^{3/2}} dy = 0 \quad (31)$$

Again, since the integrand is anti-symmetric in y while the integrand over dy is over an even range, thus the integral is zero. Combining all the results of (27), (28), (29), (30) and (31) shows that $\mathbf{P}_{\text{point}}^{\text{EM}} = 0$.

This outcome signifies a total point momentum of zero, as expected when the components of the system (the magnetic and electric charges) are stationary. Any non-zero field momentum in such a system would have implied either: (i) a violation of the special relativistic center-of-energy theorem or (ii) there is some hidden momentum carried by the sources responsible for generating the electric and magnetic fields [12, 13, 14].

Next, we examine the field momentum associated with the string component of the magnetic field in (21) (specifically, the term $\pm 4\pi g \delta(x)\delta(y)\Theta(\mp z)\hat{\mathbf{z}}$). We demonstrate that this string contribution carries a field momentum, which has been overlooked in previous research. This oversight is addressed in this study, leading to novel findings concerning the Dirac string formulation of magnetic charge. The string contribution to the field momentum is

$$\begin{aligned}
\mathbf{P}_{\text{string}}^{\text{EM}} &= \frac{1}{4\pi} \int q \frac{\hat{r}'}{r'^2} \times (\pm 4\pi g \delta(x) \delta(y) \Theta(\mp z) \hat{z}) d^3x \\
&= \pm gq \int_{-\infty}^{\infty} \Theta(\mp z) \frac{x_0 \hat{y} - y_0 \hat{x}}{(\rho_0^2 + (z - z_0)^2)^{3/2}} dz \\
&= gq \frac{r_0 \mp z_0}{r_0 \rho_0^2} (-y_0 \hat{x} + x_0 \hat{y})
\end{aligned} \tag{32}$$

The quantities $\rho_0^2 = x_0^2 + y_0^2$ and $r_0^2 = x_0^2 + y_0^2 + z_0^2 = \rho_0^2 + z_0^2$ are introduced. As a result, the system possesses a non-zero field momentum, given by $\mathbf{P}_{EM}^{\text{total}} = \mathbf{P}_{EM}^{\text{point}} + \mathbf{P}_{EM}^{\text{string}} \neq 0$. Since the electric and magnetic charges are at rest and lack mechanical momentum, we are faced with two possibilities: either the center-of-energy theorem from special relativity is violated or there exists hidden momentum within the system. If we choose to maintain the validity of the center-of-energy theorem, it implies that Dirac monopoles with Dirac strings are not permitted. Alternatively, we can explore the second possibility, which offers a resolution but necessitates acknowledging the reality of the string. For the presence of hidden momentum in a system, there must be an associated physical current, which indeed exists in connection with the string component of the magnetic field as expressed in equation (21). By substituting the magnetic field from equation (21) into the equation $\nabla \times \mathbf{B} = 4\pi \mathbf{J}$, we immediately obtain a current density

$$\mathbf{J} = \pm g \Theta(\mp z) [\delta(x) \delta'(y) \hat{\mathbf{x}} - \delta(y) \delta'(x) \hat{\mathbf{y}}] \tag{33}$$

The primes in the equations indicate that differentiation is performed with respect to the argument of the delta function. The current density mentioned arises solely from the string component of equation (21) because the Coulomb part of the magnetic field has a curl of zero. To facilitate further calculations, we have determined the current density \mathbf{J} using Cartesian coordinates.

There is a possibility that the current density described in equation (33) and the presence of the charge q could account for a “hidden” mechanical momentum. This hidden momentum could potentially balance the electromagnetic field momentum stated in (32), thereby preserving the center-of-energy theorem. To investigate this possibility, let us consider the scenario where the charge is located at $\mathbf{r}_0 = (x_0, 0, z_0)$, with y_0 being zero, and with the string along the $-z$ axis. For these parameters (32) gives

$$\mathbf{P}_{EM}^{string} = gq \frac{r_0 - z_0}{r_0 x_0} \hat{\mathbf{y}}. \quad (34)$$

The hidden mechanical momentum arising from the sources q and \mathbf{J} is [15, 16]

$$\mathbf{P}_{mech}^{hid} = - \int \phi \mathbf{J} d^3x \quad (35)$$

In the specific configuration where the charge q is located at $\mathbf{r}_0 = (x_0, 0, z_0)$, the potential can be expressed as $\phi = \frac{q}{\sqrt{(x-x_0)^2 + y^2 + (z-z_0)^2}}$. Assuming that the string lies along the negative z axis, resulting in $\mathbf{J} = +g\Theta(-z) [\delta(x)\delta'(y)\hat{\mathbf{x}} - \delta(y)\delta'(x)\hat{\mathbf{y}}]$, equation (35) takes the following form:

$$\mathbf{P}_{mech}^{hid} = -gq \int_{-\infty}^0 dz \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dx \frac{(\delta(x)\delta'(y)\hat{\mathbf{x}} - \delta(y)\delta'(x)\hat{\mathbf{y}})}{\sqrt{(x-x_0)^2 + y^2 + (z-z_0)^2}} \quad (36)$$

To account for the presence of the $\Theta(-z)$ function, suitable limits are applied during the integration with respect to z . In order to handle the terms involving $\delta'(y)$ and $\delta'(x)$, integration by parts is employed. More specifically, when integrating with respect to dy for the $\hat{\mathbf{x}}$ term, integration by parts is utilized, leading to the following outcome.

$$\frac{\delta(y)}{\sqrt{(x-x_0)^2 + y^2 + (z-z_0)^2}} \Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \frac{y\delta(y)}{((x-x_0)^2 + y^2 + (z-z_0)^2)^{3/2}} dy$$

The first term goes to zero as the function vanishes at extremely large values of its argument. Similarly, the second term becomes zero due to the presence of the $y\delta(y)$ term in

the integrated expression. As a result, there is no hidden mechanical momentum in the $\hat{\mathbf{x}}$ direction.

Moving forward, we continue by integrating the second term in equation (36). The integration with respect to dy is simple and involves substituting $y = 0$ into the rest of the integrand. Conversely, the integration with respect to dx is approached using the method of integration by parts, resulting in:

$$\frac{\delta(x)}{\sqrt{(x-x_0)^2 + (z-z_0)^2}} \Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \frac{(x-x_0)\delta(x)}{((x-x_0)^2 + (z-z_0)^2)^{3/2}} dx .$$

The additional term that arises from the integration by parts and accounts for the function evaluated at infinity results in zero, as the function approaches zero at both positive and negative infinity. After integrating with respect to x , the second term simplifies to $-\frac{x_0}{(x_0^2 + (z-z_0)^2)^{3/2}}$. By considering all the integrations and taking the appropriate signs into account, we can express (36) as follows:

$$\begin{aligned} \mathbf{P}_{mech}^{hid} &= -gqx_0 \int_{-\infty}^0 dz \frac{\hat{y}}{(x_0^2 + (z-z_0)^2)^{3/2}} \\ &= -gqx_0 \frac{1}{x_0^2 - z_0x_0\sqrt{x_0^2 + z_0^2}} \hat{y} \\ &= -gq \frac{r_0 - z_0}{r_0x_0} \hat{\mathbf{y}} \end{aligned} \tag{37}$$

The existence of this concealed mechanical momentum offsets the electromagnetic field momentum described in (32), resulting in $\mathbf{P}_{EM}^{string} + \mathbf{P}_{mech}^{hid} = 0$, which restores the center-of-energy theorem. However, this implies that the string must be acknowledged as a real physical entity, as it carries the mechanical momentum described in (37) to balance the field momentum from (32).

Field Angular Momentum

In this section, we explore the field angular momentum carried by a charge plus a Dirac string monopole configuration. We also discuss how this leads to another approach to the Dirac quantization condition. We start by considering the magnetic charge positioned at the origin, resulting in the \mathbf{B} -field expressed in (21). Additionally, we place the electric charge at the location \mathbf{r}_0 , which gives rise to the electric field $\mathbf{E} = q\frac{\hat{\mathbf{r}}'}{r'^2}$, where $\mathbf{r}' = \mathbf{r} - \mathbf{r}_0$. The angular momentum associated with the point component of the magnetic field (i.e., $g\frac{\hat{\mathbf{r}}}{r^2}$) can be determined using the well-known formula [1, 4, 5, 6, 7, 8].

$$\mathbf{L}_{EM}^{point} = \frac{1}{4\pi} \int \mathbf{r} \times (\mathbf{E} \times \mathbf{B}) d^3x = \frac{qg}{4\pi} \int r\hat{\mathbf{r}} \times \left(\frac{\mathbf{r}'}{r'^3} \times \frac{\hat{\mathbf{r}}}{r^2} \right) d^3x = -qg\hat{\mathbf{r}}_0 \quad (38)$$

By examining the magnitude of equation (38) and taking into account the quantum mechanical principle that angular momentum is quantized in integer increments of $\frac{\hbar}{2}$, we quickly arrive at the well-known Dirac condition: $qg = n\frac{\hbar}{2}$. This approach to the Dirac quantization condition, which revolves around field angular momentum, makes use of the electric field produced by q . In contrast, the other methods for deriving the Dirac condition, such as those mentioned earlier and the fiber bundle approach introduced by Wu and Yang [17, 18], involve imposing constraints on the wavefunction of q .

It is important to note that the string component of the magnetic field in equation (21) (specifically, the term $\pm 4\pi g\delta(x)\delta(y)\Theta(\mp z)\hat{\mathbf{z}}$) also contributes to the overall angular momentum of the field. This particular contribution to the angular momentum has been overlooked in previous studies. Taking into account this additional string component in the angular momentum of the field, we obtained :

$$\begin{aligned}
\mathbf{L}_{EM}^{string} &= \frac{1}{4\pi} \int \mathbf{r} \times \left(q \frac{\hat{\mathbf{r}}'}{r'^2} \times (\pm 4\pi g \delta(x) \delta(y) \Theta(\mp z) \hat{\mathbf{z}}) \right) d^3x \\
&= \mp gq \left(\frac{z_0 \mp \sqrt{\rho_0^2 + z_0^2}}{\rho_0} \right) \hat{\rho}_0 \\
&= \mp gq \left(\frac{\cos \theta_0 \mp 1}{\sin \theta_0} \right) \hat{\rho}_0
\end{aligned} \tag{39}$$

The position of the charge q is represented by $\mathbf{r}_0 = \rho_0 \hat{\rho}_0 + z_0 \hat{\mathbf{z}}$. In contrast to the point contribution \mathbf{L}_{EM}^{point} , the contribution from the string, \mathbf{L}_{EM}^{string} , is influenced by the orientation of the string ($\pm z$). As a result, it is gauge-dependent, meaning that the outcome varies depending on which of the two gauge-related potentials, \mathbf{A}_+^g or \mathbf{A}_-^g , is utilized.

Furthermore, unlike the point contribution \mathbf{L}_{EM}^{point} , the contribution of the string to the field angular momentum, \mathbf{L}_{EM}^{string} , and consequently the overall field angular momentum, depends on the angular coordinate θ_0 . By converting equation (38) into cylindrical coordinates (i.e., $\mathbf{L}_{EM}^{point} = -qg(\sin \theta_0 \hat{\rho}_0 + \cos \theta_0 \hat{\mathbf{z}})$) and combining it with equation (24), we derive the following expression :

$$\begin{aligned}
\mathbf{L}_{EM}^{total} = \mathbf{L}_{EM}^{point} + \mathbf{L}_{EM}^{string} &= -qg \cos \theta_0 \hat{\mathbf{z}} + qg(\mp 1 + \cos \theta_0) \cot \theta_0 \hat{\rho}_0 \\
&= -qg \left(\frac{z_0}{r_0} \right) \hat{\mathbf{z}} + qg \left(\mp 1 + \frac{z_0}{r_0} \right) \frac{z_0}{\rho_0^2} (x_0 \hat{\mathbf{x}} + y_0 \hat{\mathbf{y}})
\end{aligned} \tag{40}$$

In the second line of (40), we have expressed the result using Cartesian unit vectors and Cartesian coordinates for future reference. The magnitude of the total field angular momentum for the (-) case is :

$$|\mathbf{L}_{EM}^{total}| = qg \cos(\theta_0) \sec(\theta_0/2). \tag{41}$$

For the (+) case, the magnitude of the total field momentum is :

$$|\mathbf{L}_{EM}^{total}| = qg \cos(\theta_0) \csc(\theta_0/2). \quad (42)$$

Hence, the magnitude of \mathbf{L}_{EM}^{total} varies depending on the position of q due to the terms dependent on θ_0 . Consequently, the heuristic approach of angular momentum quantization to the Dirac quantization condition will not work.

WU YANG APPROACH

In this chapter, we will explore the Wu-Yang fiber bundle formulation of magnetic charge and show how this formulation leads to the Dirac quantization condition.

We began with a brief review of the Wu-Yang approach. From (18) the two vector potentials are singular along the entire negative/positive z-axis (for $\mathbf{A}_+(r)/\mathbf{A}_-(r)$). Wu and Yang dealt with these string singularities using a fiber bundle approach [17, 18]; \mathbf{A}_+ was the vector potential over the Northern hemisphere and \mathbf{A}_- was the vector potential over the Southern hemisphere. At the equator there was an overlap region where the vector potential was given by either \mathbf{A}_+ or \mathbf{A}_- . This avoided the string singularity in each region. One can write the Wu-Yang fiber bundle potential in a single expression as

$$\mathbf{A}_{WY}(\mathbf{x}) = \frac{g}{\rho} \left(+\Theta(+z + \epsilon) - \Theta(-z + \epsilon) - \frac{z}{\sqrt{\rho^2 + z^2}} \right) \hat{\phi}, \quad (43)$$

$\Theta(x)$ is the step function, which equals 1 when the argument is positive and equals 0 when the argument is negative and $0 < \epsilon \ll 1$ is an infinitesimal, positive constant. As a last comment we note that the two vector potentials $\mathbf{A}_+(\mathbf{x})$, $\mathbf{A}_-(\mathbf{x})$, are related by the gauge transformation which is given in (19).

The Dirac quantization condition (DQC) arises when we introduce an electric charge, denoted as 'q,' near a magnetic monopole. We observe that the wavefunction of 'q' in the presence of the monopole splits into two parts corresponding to the Northern and Southern hemispheres, referred to as $\Psi_+(\mathbf{x})$ and $\Psi_-(\mathbf{x})$ respectively. These wavefunctions are connected through a standard gauge transformation

$$\Psi_+(\mathbf{x}) = e^{iq\alpha/\hbar} \Psi_-(\mathbf{x}) = e^{2iqg\varphi/\hbar} \Psi_- \quad (44)$$

If the wavefunctions $\Psi_+(\mathbf{x})$ and $\Psi_-(\mathbf{x})$ are to match up as φ goes from 0 to 2π one needs $\frac{2qg}{\hbar} = n$, where n is an integer. This ensures that the factor $e^{2iqg\varphi/\hbar} = 1$ and $\Psi_+(\mathbf{x})$ and $\Psi_-(\mathbf{x})$, from (44), will match. The condition $\frac{2qg}{\hbar} = n \rightarrow qg = n\frac{\hbar}{2}$ is the DQC.

While the wavefunctions in the Northern and Southern hemispheres now match at the equator it is easy to see from (19) that there is a discontinuity in the vector potential at the equator. We show that this discontinuity results in a previously overlooked “disk” magnetic field at $z = 0$. This disk magnetic field is similar to the Dirac string, but instead of being confined to a string singularity, there is a disk discontinuity.

The magnetic field in the Wu-Yang approach is obtained by taking the curl of the Wu-Yang vector potential in (43) which has non-zero parts given by

$$\mathbf{B} = \nabla \times \mathbf{A}_{WY} = -\partial_z(A_{WY}^\varphi)\hat{\rho} + \frac{1}{\rho}\partial_\rho(\rho A_{WY}^\varphi)\hat{\mathbf{z}}. \quad (45)$$

Plugging the $-\frac{g}{\rho}\frac{z}{\sqrt{\rho^2+z^2}}\hat{\varphi}$ term from (43) into (45) gives $g\frac{\rho\hat{\rho}+z\hat{\mathbf{z}}}{(\rho^2+z^2)^{3/2}}$ which is just the Coulomb magnetic field in cylindrical coordinates. At first it might seem that this would be the only contribution to the curl of the Wu-Yang vector potential, but the $+\frac{g\Theta(+z+\epsilon)}{\rho}$ and $-\frac{g\Theta(-z+\epsilon)}{\rho}$ terms of $\mathbf{A}_{WY}(\mathbf{x})$ yield a non-zero result since $\lim_{\epsilon \rightarrow 0} \frac{d}{dz}(+\Theta(+z+\epsilon)) = \delta(z)$ and $\lim_{\epsilon \rightarrow 0} \frac{d}{dz}(-\Theta(-z+\epsilon)) = \delta(-z) = \delta(z)$ (the last step used the fact that the delta function is an even function). Putting this together gives a magnetic field coming from $\nabla \times \mathbf{A}_{WY}$ of the form

$$\mathbf{B} = g\frac{\rho\hat{\rho} + z\hat{\mathbf{z}}}{(\rho^2 + z^2)^{3/2}} - \frac{2g\delta(z)}{\rho}\hat{\rho}. \quad (46)$$

This second term in (46) has, to the best of our knowledge, been overlooked in all previous work on the Wu-Yang monopole. In the next section we obtain the delta-function term in (46) without explicitly using $\mathbf{A}_{WY}(\mathbf{x})$ from (43), but using boundary conditions of the vector potential, and magnetic field across the equator and the gauge condition (19). We now investigate how this added term changes many of the standard results for the Wu-Yang

approach to magnetic charge. These changes strongly point to the conclusion that Wu-Yang monopoles are not true magnetic charges.

We now briefly compare the Wu-Yang approach, as outlined above, with the Dirac string approach.

Comparison to Dirac String Formulation

In this subsection, we delve into the interpretation of the delta function term in (46) and establish its connection to the conventional Dirac string approach to monopoles.

The Dirac method of magnetic charge also introduced a delta function contribution to the magnetic field - see equation (21). However, in contrast to (46) where it extends throughout the entire z -plane, in the Dirac approach, this contribution is confined solely along the z -axis. To address the string singularity in the Dirac vector potential, one introduced a regularized vector potential as given in (20). By taking the curl of the regularized string vector potential one obtained the magnetic field given in (21)

Both the delta function terms in the Wu-Yang approach (46) and the Dirac approach (21) exhibit an inward flux of $-4\pi g$ towards the origin. Consequently, this inward flux gives rise to an outward flux of $+4\pi g$ originating from the Coulomb magnetic field terms found in equations (46) and (21). To demonstrate this flux behavior for both delta function terms, we can perform a surface area integral by considering a cylindrical surface with a height of $2H$ and a radius $\rho = R$ centered at the origin. This integral is denoted as $\oint \mathbf{B}_{\delta\text{-term}} \cdot d\mathbf{a}$. The sides of the cylindrical ‘‘box’’ contribute a surface integration given by

$$\int_{-H}^{+H} \int_0^{2\pi} - \left(\frac{2g\delta(z)}{R} \right) \hat{\rho} \cdot \hat{\rho} R d\varphi dz = -4\pi g. \quad (47)$$

The $d\varphi$ integration yields 2π and the dz integration results in unity. This gives an inward flux of $4\pi g$. The top and bottom surfaces are zero since $\hat{\rho} \cdot \hat{z} = 0$.

$$\int_0^R \int_0^{2\pi} - \left(\frac{2g\delta(z)}{R} \right) \hat{\rho} \cdot (\pm\hat{z}) \rho d\rho d\varphi = 0. \quad (48)$$

Thus, the total surface integral is the sum of (47) and (48) which yields the following result

$$-\oint \left(\frac{2g\delta(z)}{\rho} \hat{\rho} \right) \cdot d\mathbf{a} = -2\pi \int_{-H}^{+H} \delta(z) \frac{2g}{R} \hat{\rho} \cdot \hat{\rho} R dz = -4\pi g \quad (49)$$

The contribution to the flux comes solely from the curved sides of the cylindrical box.

By employing the same surface for integrating the Dirac delta function term for the Dirac string magnetic field given in (21), we found

$$\mp \oint \left(\frac{2g\delta(\rho)}{\rho} \Theta(\pm z) \hat{z} \right) \cdot d\mathbf{a} = \mp 2\pi \int_0^R \frac{2g\delta(\rho)}{\rho} \hat{z} \cdot (\pm \hat{z}) \rho d\rho = -4\pi g \quad (50)$$

Thus, the delta function term from the Dirac string magnetic field also yields an inward flux of $4\pi g$. Here, only the top and bottom surfaces of the cylindrical box contribute to the flux calculation. The $d\varphi$ integration, as before, yields 2π , and the $d\rho$ integration results in unity due to the delta function term. Consequently, this integration also leads to an inward flux of $4\pi g$. It is important to note that this inward flux is counterbalanced by an equivalent outward flux originating from the Coulomb component of the magnetic field.

For the Wu-Yang magnetic field, the necessity for the delta function term, represented by the second term in (46), becomes apparent when considering the discontinuity of the vector potential across the equator, as given by both equations (43) and (19). Generally, electromagnetic potentials do not exhibit discontinuities, unlike electric and magnetic fields that are known to be discontinuous across surface charges or currents. However, there are specific scenarios in which the potentials can display discontinuities. For instance, the scalar potential can be discontinuous across a surface dipole layer - see details in section 1.6 of [1]. Similarly, the observed discontinuity in the vector potential, as shown in (19), indicates the presence of a non-trivial surface current density.

To derive the condition for a discontinuous vector potential, we utilize the differential relationship $\mathbf{B} = \nabla \times \mathbf{A}$, which implies the integral relationship $\int \mathbf{B} \cdot d\mathbf{a} = \oint \mathbf{A} \cdot d\mathbf{l}$. For the closed line integral $\oint \mathbf{A} \cdot d\mathbf{l}$, we consider a closed path

consisting of two line elements just above and just below the equator, and two line elements in the z -direction. The line elements below and above the equator have the form $\pm R\Delta\varphi\hat{\varphi}$. The two line elements in the z -direction are of the form $\pm\Delta z\hat{z}$. This forms a curved rectangle with a length of $R\Delta\varphi$ and a height of Δz . The path is traversed counterclockwise when viewed from the origin, with the path above the equator in the $+\varphi$ direction, the path below the equator in the $-\varphi$ direction, and the enclosed area in the $-\hat{\rho}$ direction. The line elements $\Delta z\hat{z}$ do not contribute since $\mathbf{A}_{\pm} \cdot \hat{z} = 0$. However, the line elements above and below the z -plane do contribute since they are in the $\pm\hat{\varphi}$ directions,

$$\oint \mathbf{A} \cdot d\mathbf{l} = (\mathbf{A}_+(r) - \mathbf{A}_-(r)) \cdot \hat{\varphi} R\Delta\varphi = \frac{2g}{R} R\Delta\varphi = 2g\Delta\varphi \quad (51)$$

In the previous step, the gauge transformation from (19) was applied to obtain the final equality.

Now, let us focus on the area integral $\int \mathbf{B} \cdot d\mathbf{a}$ and demonstrate the necessity of the delta function term in (46) to ensure $\int \mathbf{B} \cdot d\mathbf{a} = \oint \mathbf{A} \cdot d\mathbf{l}$. The area element associated with the line integral $\oint \mathbf{A} \cdot d\mathbf{l}$ described earlier is given by $d\mathbf{a} = -R\Delta\varphi\Delta z\hat{\rho}$. Since the line integral was traversed counterclockwise, the area is in the negative $\hat{\rho}$ direction. The first term in (46), the Coulomb term, contributes $g\frac{1}{R^2}(R\Delta\varphi\Delta z) = -g\frac{\Delta\varphi\Delta z}{R}$ to the area integral. Here, $z = 0$ is taken for the first term in (46) since the upper and lower extent of the area is infinitesimally above and below the z -plane. However, when we take the limit as $\Delta z \rightarrow 0$, we observe that the contribution from the Coulomb part of (46) becomes zero.

On the other hand, the second term, the disk term, in (46) contributes $-\frac{2g}{R}\hat{\rho} \cdot (-\hat{\rho}R\Delta\varphi)\Delta z\delta(z) = 2g\Delta\varphi$ to the area integral. Notably, the delta function in z allows for a finite result for $\Delta z\delta(z)$ in the limit $\Delta z \rightarrow 0$, which is in contrast to the Coulomb part of the magnetic field. These findings yield $\int \mathbf{B} \cdot d\mathbf{a} = 2g\Delta\varphi$, which is equal to $\oint \mathbf{A} \cdot d\mathbf{l} = 2g\Delta\varphi$. Consequently, the inclusion of the additional disk magnetic field in (46) is necessary to satisfy the boundary condition for the vector potential, as given by (43) and (19).

Finally, we observed that the magnetic field from (46) leads to the surface current density using $\nabla \times \mathbf{B} = 4\pi\mathbf{J}$. The first, Coulomb term in (46) gives a curl of zero since it is radially symmetric. The second, disk term gives

$$\nabla \times \left(-\frac{2g\delta(z)}{\rho} \hat{\rho} \right) = -\frac{2g}{\rho} \frac{d(\delta(z))}{dz} \hat{\varphi} = -\frac{2g}{\rho} \delta'(z) \hat{\varphi}, \quad (52)$$

This result indicates the presence of a surface current density given by $\mathbf{J} = -\frac{g}{2\pi\rho} \delta'(z) \hat{\varphi}$. It is important to note that this current density, characterized by the $\delta'(z)$ term, is only well-defined within the context of an integral. This current density is a dipole current density.

Total Field Momentum

We will now demonstrate that the delta function term originating from the magnetic field in (46) implies a non - zero field momentum. This raises concerns about the consistency of the Wu-Yang approach to magnetic charge since a system with a non zero field momentum, but having no moving constituents, indicates one of two possibilities: (i) either there is “hidden” momentum within the system [12, 14], or (ii) there is a violation of the center of energy theorem [12, 13].

To calculate the field momentum, we integrate $\mathbf{E} \times \mathbf{B}^{\text{disk}}$. The Coulomb part of the magnetic field would contribute a term of the form $\mathbf{E} \times \frac{g\hat{\mathbf{r}}}{r^3}$; however, it is well-known that this term results in zero field momentum (although, as we will show in the next section, it does contribute to non-zero field angular momentum). By using the cylindrical symmetry and without the loss of generality, we placed the electric charge at $\mathbf{r}_0 = \rho_0\hat{\mathbf{x}} + z_0\hat{\mathbf{z}}$, aligning it along the x -axis. Consequently, the electric field is given by $\mathbf{E} = q\frac{\mathbf{r}'}{r'^3}$, where $\mathbf{r}' = \mathbf{r} - \mathbf{r}_0$. In cylindrical coordinates, $r' = (\rho^2 + \rho_0^2 - 2\rho\rho_0 \cos\varphi + (z - z_0)^2)^{\frac{1}{2}}$. By combining all these elements, we have:

$$\begin{aligned}
\mathbf{P}_{EM}^{disk} &= \frac{1}{4\pi} \int q \frac{\hat{\mathbf{r}}'}{r'^2} \times \left(-\frac{2g\delta(z)}{\rho} \hat{\boldsymbol{\rho}} \right) d^3x \\
&= -\frac{qg}{2\pi} \int_0^\infty \rho d\rho \int_0^{2\pi} d\varphi \frac{[\rho\hat{\boldsymbol{\rho}} - \rho_0\hat{\mathbf{x}} - z_0\hat{\mathbf{z}}]}{((\rho^2 + \rho_0^2 + z_0^2 - 2\rho\rho_0 \cos \varphi)^{3/2}} \times \frac{\hat{\boldsymbol{\rho}}}{\rho} \\
&= -\frac{qg}{2\pi} \int_0^\infty d\rho \int_0^{2\pi} d\varphi \left(\frac{z_0 \sin \varphi \hat{\mathbf{x}} - z_0 \cos \varphi \hat{\mathbf{y}} - \rho_0 \sin \varphi \hat{\mathbf{z}}}{(\rho^2 + \rho_0^2 + z_0^2 - 2\rho\rho_0 \cos \varphi)^{3/2}} \right).
\end{aligned} \tag{53}$$

In transitioning from the first to the second line, we integrated over dz using the delta function. The $d\varphi$ integration of the \hat{x} and \hat{z} components yields zero. Similarly, the integration of the \hat{y} component results in

$$\mathbf{P}_{EM}^{disk} = -\frac{qgz_0}{\rho_0 r_0} \hat{\mathbf{y}} \tag{54}$$

When the electric charge is in the z -plane *i.e.*, when $z_0 = 0$ the field momentum in (54) is zero. When the electric charge is on the z -axis *i.e.*, when $\rho_0 = 0$ it appears that the field momentum in (54) diverges. However, for this case we need to go back and reevaluate the integral in (53). Setting $\rho_0 = 0$ and redoing the integral in (53) yields $\mathbf{P}_{EM}^{disk}|_{\rho_0=0} = 0$.

Consequently, the field momentum is zero when the electric charge is situated in the z -plane or along the z -axis. However, it is non-zero in all other cases. This violates the center of energy theorem [12, 13] since both the electric charge and the magnetic point charge are stationary, and yet there seems to be some non-zero momentum associated with the electromagnetic field component of the system, as given in(54).

These kinds of violations of the center of energy theorem were originally discussed by Shockley and James in [14]. Coleman and Van Vleck [12] resolved this apparent violation of the center of energy theorem by showing that the systems considered by Shockley and James carried a “hidden” momentum in the charges and currents of the system. A thorough discussion of “hidden” momentum in electrodynamics can be found in the following pedagogical articles [15, 16].

The presence of non-zero field momentum in (54) poses a challenge for the Wu-Yang formulation of magnetic charge. The non - zero field momentum implies one of two outcomes:

(i) If $\mathbf{P}_{EM}^{\text{disk}} \neq 0$, it implies a violation of the center of energy theorem [12, 13].

(ii) Alternatively, there could be some “hidden” momentum to balance the non-zero field momentum. However, the existence of this hidden momentum suggests the presence of charge or current densities, indicating that the system is no longer purely a magnetic charge system.

Let us consider the second option which has “hidden” momentum to balance the electromagnetic field momentum from (54). The hidden momentum is carried by the charges and currents in the system which is given in (35).

The current density in (35) is given by (52) as $\mathbf{J} = -\frac{g}{2\pi\rho}\delta'(z)\hat{\varphi}$, and the potential for the charge q in (35) is given by $\phi = \frac{q}{\sqrt{(x-x_0)^2+y^2+(z-z_0)^2}}$. It's important to note that without loss of generality, due to the cylindrical symmetry of the magnetic field, we have placed the charge q at $\mathbf{r}_0 = (x_0, 0, z_0)$. At the end x_0 is equivalent to ρ_0 , the cylindrical radial distance.

The “hidden” momentum for the setup is

$$\mathbf{P}_{mech}^{hid} = \frac{gq}{2\pi} \int \frac{\delta'(z)(-\sin\varphi\hat{\mathbf{x}} + \cos\varphi\hat{\mathbf{y}})}{\rho\sqrt{\rho^2 + x_0^2 + (z - z_0)^2 - 2x_0\rho\cos\varphi}} d^3x . \quad (55)$$

For the $\hat{\mathbf{x}}$ -term in (55), the $d\varphi$ integration is of the form $\int_0^{2\pi} \frac{\sin\varphi}{\sqrt{A-B\cos\varphi}} d\varphi$, which is equal to zero. Thus there is no $\hat{\mathbf{x}}$ component to the hidden mechanical momentum which is consistent with (54). Next, we examine the $\hat{\mathbf{y}}$ -term by doing the dz -integration which

requires an integration by parts

$$\begin{aligned}
& \frac{gq \cos \varphi}{2\pi \rho} \hat{\mathbf{y}} \int_{-\infty}^{\infty} \frac{\delta'(z)}{\sqrt{K + (z - z_0)^2}} dz \\
&= \frac{gq \cos \varphi}{2\pi \rho} \hat{\mathbf{y}} \left[\frac{\delta(z)}{\sqrt{K + (z - z_0)^2}} \Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \frac{\delta(z)(z - z_0)}{(K + (z - z_0)^2)^{3/2}} \right] \\
&= \frac{gq \cos \varphi}{2\pi \rho} \left(\frac{-z_0}{K + z_0^2} \right) \hat{\mathbf{y}} .
\end{aligned} \tag{56}$$

After integration by parts, the surface term is zero and all the non- z dependent terms becomes ‘‘constant’’ $K = \rho^2 + x_0^2 - 2x_0\rho \cos \varphi$. Finally, the $d\rho$ integration gives

$$\begin{aligned}
& - \frac{gqz_0 \cos \varphi}{2\pi} \hat{\mathbf{y}} \int_0^{\infty} \frac{1}{\rho(\rho^2 + x_0^2 + z_0^2 - 2x_0\rho \cos \varphi)} \rho d\rho \\
&= - \frac{gqz_0 \cos \varphi}{2\pi} \hat{\mathbf{y}} \left(\frac{2(1 + x_0 \cos \varphi / r_0)}{x_0^2 + 2z_0^2 - x_0^2 \cos 2\varphi} \right) ,
\end{aligned} \tag{57}$$

with $r_0 = \sqrt{x_0^2 + z_0^2}$ - since we choose $y_0 = 0$. This effectively means $\rho_0 = x_0$ and thus $r_0 = \sqrt{\rho_0^2 + z_0^2} = \sqrt{x_0^2 + z_0^2}$. Finally, we carry out the $d\varphi$ integration which gives

$$\mathbf{P}_{mech}^{hid} = - \frac{gqz_0}{2\pi} \hat{\mathbf{y}} \int_0^{2\pi} \left(\frac{2 \cos \varphi (1 + x_0 \cos \varphi / r_0)}{x_0^2 + 2z_0^2 - x_0^2 \cos 2\varphi} \right) d\varphi = - \frac{gqz_0}{2\pi} \hat{\mathbf{y}} \left(\frac{-2\pi}{x_0 r_0} \right) \rightarrow \frac{gqz_0}{\rho_0 r_0} \hat{\mathbf{y}} \tag{58}$$

Adding the mechanical momentum from (58) with the field momentum from (54) gives the result $\mathbf{P}_{EM}^{disk} + \mathbf{P}_{mech}^{hid} = 0$. Thus, the ‘‘hidden’’ mechanical momentum balances the field momentum of (54), and saves the center of energy theorem. However, as for the Dirac string case the system is not a pure monopole but rather a monopole plus additional currents.

Total Field Angular Momentum

In this section, we explored the implications of the delta function disk term in (46) and its connection to the field angular momentum. Firstly, we found that a system consisting of both magnetic and electric charges possesses a field angular momentum due to the Coulomb magnetic and electric fields associated with each particle. To begin our

analysis, we position the magnetic charge at the origin, resulting in the magnetic field given in (46). Without loss of generality, we placed the electric charge at $\mathbf{r}_0 = \rho_0 \hat{\mathbf{x}} + z_0 \hat{\mathbf{z}}$, giving rise to an electric field of $\mathbf{E} = q \frac{\hat{\mathbf{r}}'}{r'^2}$ (where $\mathbf{r}' = \mathbf{r} - \mathbf{r}_0$).

Consequently, the field angular momentum arising from the point component of the magnetic field (i.e., $g \frac{\hat{\mathbf{r}}}{r^2}$) follows [4, 5, 6, 7, 8, 9, 19, 20, 21, 22, 23]

$$\mathbf{L}_{EM}^{\text{point}} = \frac{1}{4\pi} \int (\mathbf{r} \times (\mathbf{E} \times \mathbf{B}_{Coulomb})) d^3x = \frac{qg}{4\pi} \int \mathbf{r} \hat{\mathbf{r}} \times \left(\frac{\mathbf{r}'}{r'^3} \times \frac{\hat{\mathbf{r}}}{r^2} \right) d^3x = -qg \hat{\mathbf{r}}_0 \quad (59)$$

By taking the magnitude of the angular momentum from (59) and considering the quantum mechanical requirement that angular momentum must be quantized in integer multiples of $\frac{\hbar}{2}$, we can quickly arrive at the Dirac condition: $qg = n \frac{\hbar}{2}$ [19, 20, 21]. This approach to the Dirac quantization condition utilizes the electric field associated with q .

However, it is important to note that the disk component of the magnetic field in (46) (specifically, the term $-\frac{2g\delta(z)}{\rho} \hat{\rho}$) also contributes to the field angular momentum. Surprisingly, this contribution from the disk term has been overlooked in all previous research. In our current work, we demonstrate that this disk contribution significantly complicates the Wu-Yang approach to magnetic charge, ultimately rendering this formulation inconsistent. This critical issue, which we assert has been overlooked in previous studies, lies at the core of our investigation. The contribution of the disk term to the field angular momentum is as follows:

$$\begin{aligned} \mathbf{L}_{EM}^{disk} &= \frac{1}{4\pi} \int \mathbf{r} \times \left(q \frac{\hat{\mathbf{r}}'}{r'^2} \times \left(-\frac{2g\delta(z)}{\rho} \hat{\rho} \right) \right) d^3x \\ &= -\frac{qg}{2\pi} \int_0^\infty \rho d\rho \int_0^{2\pi} d\varphi \rho \hat{\rho} \times \left(\frac{[\rho \hat{\rho} - \rho_0 \hat{\mathbf{x}} - z_0 \hat{\mathbf{z}}]}{((\rho^2 + \rho_0^2 + z_0^2 - 2\rho\rho_0 \cos \varphi)^{3/2})} \times \frac{\hat{\rho}}{\rho} \right) \\ &= -\frac{qg}{2\pi} \int_0^\infty \rho d\rho \int_0^{2\pi} d\varphi \left(\frac{-\rho_0 \sin^2 \varphi \hat{\mathbf{x}} + \rho_0 \sin \varphi \cos \varphi \hat{\mathbf{y}} - z_0 \hat{\mathbf{z}}}{(\rho^2 + \rho_0^2 + z_0^2 - 2\rho\rho_0 \cos \varphi)^{3/2}} \right), \end{aligned} \quad (60)$$

Doing the φ integration of the $\hat{\mathbf{y}}$ component in (60) gives zero. Next performing the $d\rho$ integration for $\hat{\mathbf{x}}$ and $\hat{\mathbf{z}}$ components gives

$$\mathbf{L}_{EM}^{disk} = -\frac{qg}{2\pi} \int_0^{2\pi} d\varphi \left(\frac{-\rho_0 \sin^2 \varphi \hat{\mathbf{x}} - z_0 \hat{\mathbf{z}}}{\sqrt{\rho_0^2 + z_0^2} - \rho_0 \cos \varphi} \right) = qg \frac{\sqrt{\rho_0^2 + z_0^2}}{\rho_0} \hat{\mathbf{x}} \rightarrow qg \frac{r_0}{\rho_0} \hat{\rho}_0. \quad (61)$$

In the middle step we have carried out the $d\varphi$ integration, and in the last step we have converted from $\hat{\mathbf{x}}$ back to $\hat{\rho}_0 = \cos \varphi_0 \hat{\mathbf{x}} + \sin \varphi_0 \hat{\mathbf{y}}$, and the location of the electric charge is given by $r_0 = \sqrt{\rho_0^2 + z_0^2}$ and φ_0 .

If the electric charge is positioned on the z -axis (meaning ρ_0 tends towards zero), the result obtained in (61) might suggest that \mathbf{L}_{EM}^{disk} approaches infinity. However, this interpretation is incorrect, and we need to revert back to equation (60) and set ρ_0 equal to zero from that point onward. The integration process becomes much simpler in this scenario, and the final outcome when the electric charge is situated precisely on the z -axis is $\mathbf{L}_{EM}^{disk}|_{\rho_0=0} = qg \frac{z}{|z|} \hat{\mathbf{z}}$. Therefore, if the electric charge is positioned above the $z = 0$ plane, we have $\mathbf{L}_{EM}^{disk}|_{\rho_0=0} = +qg\hat{\mathbf{z}}$, and if the electric charge is located below the $z = 0$ plane, we have $\mathbf{L}_{EM}^{disk}|_{\rho_0=0} = -qg\hat{\mathbf{z}}$.

Our goal is to determine the total field angular momentum by combining the contributions from equations (59) and (61). To accomplish this, let us begin by converting (59) to cylindrical coordinates. This conversion results in $\mathbf{L}_{EM}^{point} = -\frac{qg}{r_0}(\rho_0 \hat{\rho}_0 + z_0 \hat{\mathbf{z}})$, where $r_0 = \sqrt{\rho_0^2 + z_0^2}$. Next, we can combine this expression with (61), leading to the following results

$$\mathbf{L}_{EM}^{total} = \mathbf{L}_{EM}^{point} + \mathbf{L}_{EM}^{disk} = qg \frac{z_0}{\rho_0} \left(\frac{\hat{z}_0}{\hat{\rho}_0} \hat{\rho}_0 - \hat{\mathbf{z}} \right) \quad (62)$$

The absolute value of the field angular momentum calculated in (62) is given by $|\mathbf{L}_{EM}^{total}| = qg \frac{z_0}{\rho_0}$. When the electric charge lies within the z -plane (meaning $z_0 = 0$), the field angular momentum becomes zero as the contributions from the point and disk cancel each other out. However, the scenario where the electric charge is positioned along the z -axis (with $\rho_0 = 0$) requires separate consideration.

As mentioned in the previous statement in (61), $\mathbf{L}_{EM}^{\text{disk}}|_{\rho_0=0} = +qg\hat{z}$ ($-qg\hat{z}$) when the electric charge is located above (below) the $z = 0$ plane. By referring to (59) and noting that $\hat{r}_0 = +\hat{z}$ ($-\hat{z}$) when the charge q is positioned above (below) the $z = 0$ plane, it can be observed that $\mathbf{L}_{EM}^{\text{point}}|_{\rho_0=0} = -qg\hat{z}$ ($+qg\hat{z}$) when the electric charge is located above (below) the $z = 0$ plane. Consequently, the contributions from the disk and point cancel each other out, resulting in no field angular momentum. This cancellation phenomenon also occurs when the electric charge lies within the z -plane ($z_0 = 0$), as shown in (62). Therefore, the magnitude of $\mathbf{L}_{EM}^{\text{total}}$ varies depending on the position of the charge q through the terms dependent on θ_0 and one cannot use the angular momentum quantization method [19, 20, 21] to derive the Dirac quantization condition.

SUMMARY AND CONCLUSION

In this thesis, we reviewed electrodynamics with both electric and magnetic charges. We pointed out some problems with formulating magnetic charges *i.e.* since the magnetic field is the curl of the vector potential this seem to imply that the divergence of the magnetic field would always be zero. Another problem with magnetic charge not directly discussed here is that isolated magnetic charges have never been observed experimentally. Despite these problems Maxwell's equations with magnetic charges exhibit an extra dual symmetry as given in equations (12)(13), for the fields and equations (14)(15)(16)(17) for the sources. This expanded symmetry connected with magnetic charges is one of the main reasons why researchers still consider magnetic charges.

In chapter 2 of the thesis, we explored the Dirac string approach to magnetic charge. This approach begins by the introduction of the Dirac string potential of (18). The Dirac string potential has a string singularity along either the $-z$ axis for \mathbf{A}_+ or the $+z$ axis for \mathbf{A}_- . The two vector potentials $\mathbf{A}_\pm(\mathbf{x})$ are related by the gauge transformation in (19). We next regularized the string singularities to obtain the regularized vector potentials in (20). By taking the curl of these regularized potentials, we derived expressions for the magnetic field, which included a Coulomb magnetic plus an infinite solenoid contribution. In order, for the infinite solenoid part of the magnetic field to be “invisible” one needed to impose the Dirac quantization condition, $qg = \frac{n\hbar}{2}$. There are many ways to obtain the DQC [1, 4, 5, 6, 7, 8, 9]. In this thesis, we showed how to obtain DQC using the vanishing of the Aharonov-Bohm effect connected with the infinite solenoid of the magnetic field in (21). Next, we placed an electric charge q in a regularized magnetic field of (21) and showed that that there was a non-zero field momentum coming from the string part of the magnetic field and the Coulomb electric field of the charge q . This non-zero field momentum is a new feature discovered by our analysis, and apparently spoils the center of energy theorem [12, 13]. However, we showed that there is a current density in (33) which carries a “hidden” mechanical momentum. This hidden mechanical momentum exactly balances the

non-zero field momentum and thus restores the center of energy theorem. However, the cost of this is that the Dirac string becomes a real physical quantity and thus the Dirac string approach does not give a pure Coulomb magnetic field but is a Coulomb magnetic field plus an infinite solenoid. In this case, the string part does not give the field angular momentum but it does contribute the angular momentum. The Coulomb part gives the field angular momentum which we already knew that but we found that there is also coming from the string part. However, this new thing spoils the DQC, but it still shows the field angular momentum from one part.

In chapter 3 of this thesis, we have shown that the Wu-Yang fiber bundle approach to a magnetic charge [18]. This approach begins by introducing the vector potentials along $-z$ axis for \mathbf{A}_+ over the northern hemisphere and $+z$ axis for \mathbf{A}_- over the southern hemisphere are related by the gauge transformation (19). Next, to avoid the string singularity in each region, we defined the Wu-Yang fiber bundle potential (43). Next, we placed an electric charge q near a magnetic monopole it raises the DQC and shows that the two wave functions are connected by the standard gauge transformation. These two wave functions match at the equator and found that there is a discontinuity in the presence of magnetic field at $z=0$. This magnetic field is same as the Dirac string *i.e.* there is a disk discontinuity. Now, by taking the curl of the Wu-Yang vector potential leads to a magnetic field which is a Coulomb magnetic field plus delta function term (45). Then, we compared Wu-Yang approach with the Dirac string approach. This result indicates the presence of a surface current density in (52). The delta function shows that there is a non-zero field momentum. It leads to the center of energy theorem and there is a “hidden” momentum which cancels the field angular momentum but the “hidden” mechanical momentum balances the field momentum. However, the Wu-Yang monopole is not a monopole but rather than a monopole plus a current density.

Overall, the important thing is that the Wu-Yang fiber bundle approach to a magnetic charge avoids the Dirac string approach for a magnetic charge. However, this

approach has a field momentum which is very similar to the Wu-Yang approach [24, 25]. But the field momentum spoils in the Wu-Yang approach for magnetic charge. However, The Dirac monopole and Wu-Yang monopoles are not true monopoles [26, 27] but these are the non-Abelian monopoles [28, 29].

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