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The Three Faces of $U(3)$

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Abstract: $U(n)$ is a semi-direct product group characterized by nontrivial homomorphisms mapping $U(1)$ into the automorphism group of $SU(n)$. For $U(3)$, there are three nontrivial homomorphisms that induce three separate defining representations. In a toy model of $U(3)$ Yang–Mills (endowed with a suitable inner product) coupled to massive fermions, this renders three distinct covariant derivatives acting on a single matter field. Employing a mod 3 permutation induced by a large gauge transformation acting on the defining representation vector space, the three covariant derivatives and one matter field can alternatively be expressed as a single covariant derivative acting on three distinct species of matter fields possessing the same $U(3)$ quantum numbers. One can interpret this as three species of matter fields in the defining representation.

Keywords: gauge symmetry; new gauge interactions; theories of flavour

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1. Introduction

In this note, we consider a toy model of $U(3)$ Yang–Mills coupled to massive fermionic matter fields. Off hand, it seems $U(3)$ is an untenable symmetry group for constructing a gauge field theory. After all, a tenant of standard gauge theory says the most general symmetry group must be a direct product of semi-simple and $U(1)$ groups (see, e.g., [1]).

From where comes the tenant? For a physically acceptable gauge field theory, one must start with a compact real group G and impose a positive-definite, Ad-invariant, real bilinear form on the gauge symmetry Lie algebra $\mathfrak{g}$. And it is well known that the Lie algebra of a compact real group decomposes into a direct sum of semi-simple $\mathfrak{s}_i$ and $\mathfrak{u}(1)_j$ factors $\bigoplus_{i,j} \mathfrak{s}_i \oplus \mathfrak{u}(1)_j$ if and only if the Killing form on $\mathfrak{g}$ is nondegenerate and hence negative-definite (see, e.g., [2]).

Meanwhile, $U(3)$ is not semi-simple, and its Killing form is degenerate. But a Killing inner product is only a sufficient condition for an acceptable gauge theory. It happens that $U(3)$ is a connected, compact real group. Being compact, it is endowed with at least one bi-invariant metric [3,4]. In fact, it is possible to formulate on $\mathfrak{u}(3)$ a two-parameter class of positive-definite, Ad-invariant, real bilinear forms. Hence, it is possible to construct a consistent gauge theory with $U(3)$ gauge symmetry without using the Killing inner product.

Notably, unlike $SU(3) \times U(1)$, where the gauge field associated with $U(1)$ completely decouples from the rest, all of the $U(3)$ gauge fields will mutually interact as a true $U(3)$ symmetry dictates. Indeed, we have $U(3) = SU(3) \rtimes U(1)$ as a semi-direct product, and an element $u_{(3)} \in U(3)$ can be factored as $u_{(3)} = s_{(3)}u_{(1)}$ with $u_{(1)} \in U(1)$ and $s_{(3)} \in SU(3)$. The semi-direct product $SU(3) \rtimes U(1)$ is characterized by a (not necessarily unique) homomorphism $\varphi : U(1) \to \mathrm{Aut}\, SU(3)$ where $\mathrm{Aut}\, SU(3)$ is the automorphism group of $SU(3)$ [5–7]. In particular, in the defining representation, said homomorphism induces a (not necessarily unique) representation $\varrho : U(1) \to L_B(\mathbb{C}^3)$ where $L_B(\mathbb{C}^3)$ denotes the set of linear bounded matrix operators on $\mathbb{C}^3$. Now, in the defining representation there are three *nontrivial* ways to represent the $U(1)$ factor in $L_B(\mathbb{C}^3)$, with $e^{i\theta}$ in one of the diagonal entries, 1 in the other two diagonal entries, and 0 in all off-diagonal entries. Then, an element of $U(3)$ represented in $L_B(\mathbb{C}^3)$ can be written $\rho_r(u_{(3)}) = \rho_r(su_{(3)})\rho_r(u_{(1)})$, where $\rho_r : U(3) \to L_B(\mathbb{C}^3)$ is an extension of ϱ_r and $r \in \{1,2,3\}$.

There is no reason to favor one particular representation over another, so when constructing a gauge field theory coupled to fermions in the defining representation, the most general Lagrangian contains the standard Yang–Mills term $\frac{1}{4}F \cdot F$ and fermion terms $\sum_r \overline{\Psi}\not{D}^{(r)}\Psi$ summed over the three representations ρ_r. Consider permuting some chosen basis of $\mathbb{C}^3$ with some unitary permutation matrix in $L_B(\mathbb{C}^3)$. There are two classes of such permutations: one class induces small gauge transformations and the other induces large gauge transformations. Of course, the small gauge transformations represent a redundant state description in the quantum version. In contrast, the large gauge transformations effect a genuine matter field re-characterization: they essentially add phases to permuted field components that exert their influence through nontrivial global/topological gauge field configurations. Accordingly, the $U(3)$ symmetry allows the fermion contribution $\sum_r \overline{\Psi}\not{D}^{(r)}\Psi$ to be rewritten with the covariant derivative in a single representation as $\sum_r \overline{\Psi}^{(r)}\not{D}\Psi^{(r)}$, where $\Psi^{(r)}$ are three *different* species of fermion matter fields, each species a $U(3)$ triplet characterized by three quantum numbers coming from the action of the Cartan subalgebra.

This is our main result: The most general $U(3)$ gauge invariant Lagrangian for fermions in a chosen defining representation includes precisely three species of matter fields relative to an imbedding $U(1) \hookrightarrow SU(3) \rtimes U(1)$. We make no claim here that $U(3)$ models QCD phenomenology, and the three types of matter fields coming from $U(3)$ may or may not be a phenomenological red herring. However, in § Section 3 we briefly discuss the feasibility of expanding the strong-force symmetry from $SU(3)$ to $U(3)$ within the Standard Model framework (the $U(1) \subset U(3)$ subgroup having nothing to do with electromagnetism) with the aim of encouraging further investigation. Our primary purpose though is to point out the viability of semi-direct product groups for gauge field theories in general and to highlight the emanating effect of multiple defining representations.

Of course, the occurrence of three generations in particle physics is still a mystery, and there have been attempts to explain the "three" using a variety of mechanisms. Most notable perhaps are preon models [8–11], super string models [12,13], and so-called 3-3-1 models [14,15]. But there are also models based on nonanomalous discrete R-symmetry [16], extra dimensions with anomaly cancellation [17], and the anthropic principle [18].

2. $U(3)$ Toy Model

2.1. The Inner Product

$U(3)$ is neither simple nor semi-simple, and its Killing form is only semi-definite. So the first order of business is to construct a suitable inner product on $\mathfrak{u}(3)$. We start with a well-known result:

Proposition 1. *The Killing form of* $U(3)$ *is given by* $K(\mathbf{X},\mathbf{Y}) = 6\mathrm{tr}(\mathbf{X}\cdot\mathbf{Y}) - 2\mathrm{tr}(\mathbf{X})\mathrm{tr}(\mathbf{Y})$ *and is negative semi-definite for all skew-Hermitian* $\mathbf{X},\mathbf{Y} \in \mathfrak{u}(3)$.

Proof. The Lie algebra brackets are $[\mathfrak{u}_{ab}\,,\,\mathfrak{u}_{cd}] = \delta_{bc}\mathfrak{u}_{ad} - \delta_{ad}\mathfrak{u}_{cb}$ where $\mathfrak{u}_{ab} \in \mathfrak{u}(3)$ are a chosen skew-Hermitian basis with $a,b,c,d \in \{1,2,3\}$. From these brackets, it follows that the adjoint map is given by $ad_{\mathbf{X}}(\mathfrak{u}_{ab}) = \sum_c x_{ca}\mathfrak{u}_{cb} - x_{bc}\mathfrak{u}_{ac}$ with $\mathbf{X} = \sum_{a,b} x_{ab}\mathfrak{u}_{ab}$ and $x_{ab} \in \mathbb{R}$. Hence,

$$\begin{aligned} ad_{\mathbf{X}} \circ ad_{\mathbf{Y}}(\mathfrak{u}_{ab}) &= \sum_{c,d}(x_{ca}y_{dc}\mathfrak{u}_{db} - x_{bc}y_{da}\mathfrak{u}_{dc} + x_{bc}y_{cd}\mathfrak{u}_{ad} - x_{ca}y_{bd}\mathfrak{u}_{cd}) \\ &= \sum_{c}(x_{ca}y_{ac} - x_{bc}y_{aa}\delta_{cb} + x_{bc}y_{cb} - x_{ca}y_{bb}\delta_{ac})\mathfrak{u}_{ab} \end{aligned} \quad (1)$$

implies

$$K(\mathbf{X},\mathbf{Y}) := \mathrm{tr}(ad_{\mathbf{X}} \circ ad_{\mathbf{Y}}) = \sum_{a,b,c} (x_{ca}y_{ac} - x_{bc}y_{aa}\delta_{cb} + x_{bc}y_{cb} - x_{ca}y_{bb}\delta_{ac})$$
$$= 3\mathrm{tr}(\mathbf{X}\cdot\mathbf{Y}) - \mathrm{tr}(\mathbf{X})\mathrm{tr}(\mathbf{Y}) + 3\mathrm{tr}(\mathbf{X}\cdot\mathbf{Y}) - \mathrm{tr}(\mathbf{X})\mathrm{tr}(\mathbf{Y})\,. \quad (2)$$

The center of $\mathfrak{u}(3)$ is $\mathrm{span}_{i\mathbb{R}}\{\mathbf{1}\}$, and it is easy to see that $K(i\mathbf{1},\mathbf{X}) = 0$ for all $\mathbf{X} \in \mathfrak{u}(3)$. Negativity follows from the skew-hermiticity of $\mathbf{X}, \mathbf{Y}$. □

This suggests to define a bilinear inner product on the Lie algebra $\mathfrak{u}(3)$ in the defining representation $\rho : U(3) \to L_B(\mathbb{C}^3)$ by

$$\langle \mathbf{\Lambda}_\alpha, \mathbf{\Lambda}_\beta \rangle_{\mathfrak{u}(3)} := -\frac{1}{6}\left[6\,\mathrm{tr}(\mathbf{\Lambda}_\alpha \mathbf{\Lambda}_\beta^\dagger) - 2\left(1 - \frac{g_1^2}{g_2^2}\right)\mathrm{tr}(\mathbf{\Lambda}_\alpha)\mathrm{tr}(\mathbf{\Lambda}_\beta^\dagger)\right] \quad (3)$$

where the basis elements $\{\mathbf{\Lambda}_\alpha\} = \{\rho'(\mathfrak{u}_{ab})\}$ are 3×3 skew-Hermitian matrices with $\alpha \in \{1,\ldots,9\}$ and the parameters $g_1, g_2 \in \mathbb{R}$ obey $0 < g_2^2 < g_1^2$. It is clearly positive-definite, *Ad*-invariant, and real. For a triangular decomposition of the basis $\{\mathbf{\Lambda}_\alpha\}$ denoted by $\{\mathbf{S}_a^\pm, \mathbf{H}_a\}$ with $a \in \{1,2,3\}$, the structure constants associated with the brackets $[\mathbf{S}_a^\pm, \mathbf{H}_a]$ differ from those associated with the Killing form. These structure constants, which are functions of (g_1, g_2), characterize quantum numbers of non-neutral gauge bosons, and eigenvalues of the (neutral) Cartan generators $\{\mathbf{H}_a\}$ in the defining representation characterize quantum numbers of matter fields.

2.2. Semidirect Structure of $U(3)$

Mathematically, it is fruitful to view $U(3)$ as an extension of a group $H \cong U(1)$ by a normal subgroup $N \cong SU(3) \triangleleft U(3)$. This is represented by the short exact sequence

$$1 \longrightarrow N \xrightarrow{f} U(3) \xrightarrow{\pi} H \longrightarrow 1\,. \quad (4)$$

If there exists an injective homomorphism $s : H \to U(3)$ such that $\pi \circ s = id_H$, then the extension is a semi-direct product $N \rtimes H$. In this case, $U(3)$ can be regarded as a principle bundle with base H, structure group N, and global section(s) $s : H \to U(3)$. A choice of section corresponds to a choice of coset representative. Then, $s(H) \cong U(1) \subset U(3)$ yields a unique decomposition $U(3) = SU(3)U(1)$ with $SU(3) \cap U(1) = \{id\}$, and s induces a homomorphism $\tilde{\varphi} : s(H) \cong U(1) \to \mathrm{Aut}\,N$. These observations are demonstrated by the following theorem.

Theorem 1 ([5–7])**.** *Let* $1 \longrightarrow N \xrightarrow{f} U(3) \xrightarrow{\pi} H \longrightarrow 1$ *be a short exact sequence equipped with an injective homomorphism* $s : H \to U(3)$ *such that* $\pi \circ s = id_H$*. Then, there exists a homomorphism* $\varphi : H \to \mathrm{Aut}\,N$ *and an isomorphism* $\theta : U(3) \to N \rtimes_\varphi H$*.*

Proof. For $h \in H$ and $n \in N$,

$$\pi(s(h)f(n)s(h^{-1})) = \pi \circ s(h)\,\pi \circ f(n)\,\pi \circ s(h^{-1}) = h\,id_H\,h^{-1}\,. \quad (5)$$

Since f is injective and $\mathrm{im}\,f = \ker\pi$, then $s(h)f(n)s(h^{-1}) = f(n')$ for some unique $n'(n,h) \in N$ that depends on (n,h). It is convenient to write $\varphi_h(\cdot) \equiv n'(\cdot,h)$ so that $\varphi_h : N \to N$. Note that $\varphi_h(id_N) = id_N$ for all $h \in H$ since s is a homomorphism.

Lemma 1. *The function* $\varphi_h \in \mathrm{Aut}\,N$*.*

First, $f(\varphi_{id_H}(n)) = f(n)$ *implies* $\varphi_{id_H}(n) = n$ *for all* $n \in N$*. Next, for* $n_1, n_2 \in N$*,*

$$s(h)f(n_1)f(n_2)s(h^{-1}) = s(h)f(n_1)s(h)^{-1}s(h)f(n_2)s(h^{-1}) = f(\varphi_h(n_1)\varphi_h(n_1)) \quad (6)$$

where we used s *is a homomorphism. On the other hand, from the definition of* φ_h*, we have* $s(h)f(n_1 n_2)s(h)^{-1} = f(\varphi_h(n_1 n_2))$*. Injective* f *then implies* $\varphi_h(n_1 n_2) = \varphi_h(n_1)\varphi_h(n_2)$*. This proves the lemma.*

Let $\varphi : H \to \mathrm{Aut}\, N$ by $h \mapsto \varphi_h$.

Lemma 2. *$\varphi : H \to \mathrm{Aut}\, N$ is a homomorphism.*
For $h_1, h_2 \in H$,

$$s(h_1)s(h_2)f(n)s(h_2)^{-1}s(h_1)^{-1} = s(h_1)f(\varphi_{h_2}(n))s(h_1)^{-1} = f(\varphi_{h_1}(\varphi_{h_2}(n))) \,. \tag{7}$$

On the other hand, $s(h_1)s(h_2)f(n)s(h_2)^{-1}s(h_1)^{-1} = s(h_1h_2)f(n)s(h_1h_2)^{-1} = f(\varphi_{h_1h_2}(n))$ since s is a homomorphism. Again, injective f implies $\varphi_{h_1h_2} = \varphi_{h_1} \circ \varphi_{h_2}$. This proves the lemma.

It follows that φ defines a group operation on $N \rtimes_\varphi H$ by $(n_1, h_1)(n_2, h_2) = (n_1\varphi_{h_1}(n_2), h_1h_2)$ if the inverse is defined by $(n,h)^{-1} := (\varphi_{h^{-1}}(n^{-1}), h^{-1})$ for all $(n,h) \in N \rtimes_\varphi H$.

Finally, let $\theta^{-1} : N \rtimes_\varphi H \to U(3)$ by $(n,h) \mapsto f(n)s(h)$. Then,

$$\begin{aligned} \theta^{-1}((n_1,h_1)(n_2,h_2)) &= \theta^{-1}(n_1\varphi_{h_1}(n_2), h_1h_2) \\ &= f(n_1)(s(h_1)f(n_2)s(h_1)^{-1})s(h_1)s(h_2) \\ &= f(n_1)s(h_1)f(n_2)s(h_2) \\ &= \theta^{-1}(n_1,h_1)\theta^{-1}(n_2,h_2) \,. \end{aligned} \tag{8}$$

Since the decomposition $U(3) = NH$ is unique (which we won't bother to prove), the homomorphism θ^{-1} is bijective. One can go on to show that the semi-direct product reduces to a direct product if and only if $H \lhd U(3)$; in which case N and H commute and φ is trivial. □

Observe the homomorphism $\tilde{\varphi} : s(H) \cong H \in N \rtimes_\varphi H \to \mathrm{Aut}\, N$ induced by s is given by

$$\begin{aligned} \tilde{\varphi}_{s(h)}(n, id_H) = s(h)(n, id_H)s(h^{-1}) &= [(id_N, h)(n, id_H)](id_N, h^{-1}) \\ &= [(\varphi_h(n), h)](id_N, h^{-1}) \\ &= (\varphi_h(n), id_H) \,. \end{aligned} \tag{9}$$

In this sense, $\tilde{\varphi}$ induced by the section s coincides with φ. It is important to note that there may be multiple homomorphisms φ and hence multiple sections s that render a semi-direct product. Physically, a nontrivial φ corresponds to a direct interaction between the gauge fields of the respective subgroups.

In particular, for the matrix group $U(3)$ as a semi-direct product, there exist three such nontrivial sections:

$$s : H \to \begin{cases} \mathrm{diag}(e^{i\omega}, 1, 1) \\ \mathrm{diag}(1, e^{i\omega}, 1) \\ \mathrm{diag}(1, 1, e^{i\omega}) \end{cases} \tag{10}$$

where $\omega \in \mathbb{R}$. Each section gives rise to a different conjugation of $SU(3)$ by $s(h)$, and each of these induces a different representation $\varrho_r : H \to L_B(\mathbb{C}^3)$ where $r \in \{1,2,3\}$. These can then be extended to three defining representations $\rho_r : U(3) \to L_B(\mathbb{C}^3)$.

2.3. Lagrangian Matter Field Term

Given the existence of a suitable inner product and three representations, constructing the model is rather elementary. The decisive step is to insist that all allowed defining representations be included in the Lagrangian.

Postulate 1. *The matter field portion of the Lagrangian of a gauge field theory must include all allowed defining representations.*

For our toy model of Yang–Mills coupled to a massive matter field in the defining representations, the bare gauge field kinetic term uses the chosen inner product $\frac{1}{4}\langle F_{\rm B}, F_{\rm B}\rangle_{\mathfrak{u}(3)}$ with $F \in \mathfrak{u}(3)$, and the bare matter field term will be $\sum_r i\overline{\Psi}_{\rm B} \not{D}^{(r)}_{\rm B}\Psi_{\rm B}$, where we have (unconventionally) included the bare mass parameter in the covariant derivative $\not{D}^{(r)}_{\rm B}$. In momentum space, the matrix representation of the covariant derivative is $[\not{D}^{(r)}_{\rm B}] = (\not{p} + m_{\rm B})[\mathbf{1}] + \not{A}^{\alpha}_{\rm B}[i\mathbf{\Lambda}^{(r)}_{\alpha}]$ with gauge fields A^{α}_{μ}, and $\{\mathbf{\Lambda}^{(r)}_{\alpha}\}$ a basis of $\mathfrak{u}(3)$ in the r-defining representation.

In the quantum version of this model, each covariant derivative $\not{D}^{(r)}$ will give rise to different vertex factors in the Feynman rules and hence *ostensibly different* renormalizations of the gauge fields, matter fields, and mass parameter. The renormalized matter field term is then $\sum_r i\overline{\Psi}_{\rm R}\not{D}^{(r)}_{\rm R}\Psi_{\rm R}$ where $[\not{D}^{(r)}_{\rm R}] = (\not{p} + m^{(r)}_{\rm R})[\mathbf{1}] + \not{A}^{\alpha}_{\rm R}[i\mathbf{\Lambda}^{(r)}_{\alpha}]$. In effect, through renormalization, the quantum theory distinguishes the classically isomorphic vector spaces carrying the defining representations. Notably, assuming different renormalizations for different r, the bare mass degeneracy among the defining representations will be lifted by the quantum version.

We can make use of the $U(3)$ symmetry to re-characterize the matter field Lagrangian. There exists a class of elements in $U(3)$ of the form

$$P(x) := \begin{pmatrix} 0 & 0 & e^{i\theta_1(x)} \\ e^{i\theta_2(x)} & 0 & 0 \\ 0 & e^{i\theta_3(x)} & 0 \end{pmatrix} \tag{11}$$

with $\theta_1(x), \theta_2(x), \theta_3(x) \in \mathbb{R}$. The adjoint action of $P(x)$ on the Lie algebra $\mathfrak{u}(3)$ leaves the normal subalgebra $\mathfrak{su}(3)$ invariant, but it cyclically permutes the generators of the $s(H)$ matrices

$$\mathrm{diag}(i\omega, 0, 0) \overset{Ad_P}{\to} \mathrm{diag}(0, i\omega, 0) \overset{Ad_P}{\to} \mathrm{diag}(0, 0, i\omega) \overset{Ad_P}{\to} \mathrm{diag}(i\omega, 0, 0)\,. \tag{12}$$

Similarly, $P^{-1}(x) = P^{\dagger}(x)$ permutes in the reverse direction. Crucially, $P^3 = e^{i(\theta_1+\theta_2+\theta_3)}$ $\mathrm{diag}(1,1,1)$. We claim that $\theta_1(x) + \theta_2(x) + \theta_3(x) = \pm(2n)\pi$ with $n \in \mathbb{N}$ induces small gauge transformations while $\theta_1(x) + \theta_2(x) + \theta_3(x) = \pm(2n+1)\pi$ induces large gauge transformations. The latter cannot be reached by a gauge transformation homotopic to the identity because $\det P = -1$. (To see this, use the identity in three dimensions $\det A = 1/6\big((\mathrm{tr}\,A)^3 - 3\mathrm{tr}\,A\,\mathrm{tr}\,(A^2) + 2\mathrm{tr}\,(A^3)\big)$ and put $A \to U(x)$ with $U(x) = \mathbf{1} + i\sigma^{\alpha}(x)$ $\mathbf{\Lambda}_{\alpha} + O(\sigma^2)$ an infinitesimal gauge transformation. To first order in σ, find that $\det U(x) >$ 0.) It then follows from $\mathrm{tr}\log P = i\pi(2k+1)$ that $\log P$ in this case involves a combination of Cartan generators (which are not present in the small permutation case) that contributes a multivalued $\mathrm{mod}\,3$ phase to matter field configurations, and it transforms between three physically distinct classes of gauge field configurations that survive gauge fixing in the quantized theory.

Given P, we have $\not{D}^{(2)} = P\not{D}^{(1)}P^{-1}$ and $\not{D}^{(3)} = P^2\not{D}^{(1)}P^{-2}$. Define the fields $\Psi^{(r)} := P^{r-1}\Psi$. Clearly, P cyclically permutes the components of Ψ up to phases. For large gauge transformations, which imply $P^3 = -\mathbf{1}$, we can write $\sum_r \overline{\Psi}_{\rm B}\not{D}^{(r)}_{\rm B}\Psi_{\rm B} = \sum_r \overline{\Psi}^{(r)}_{\rm B}\not{D}_{\rm B}\Psi^{(r)}_{\rm B}$. In the quantum version, the $SU(3)$-identical $\Psi^{(r)}$ are physically distinct fields with inequivalent renormalized masses (again, assuming different renormalizations for different r). Hence we claim.

Claim 1. *Given Postulate 1, matter fields with $U(3)$ gauge symmetry necessarily come in three species due to the existence of large gauge transformations that realize* $\mathrm{mod}\,3$ *permutations of the basis in a defining representation.*

This perspective can be turned around: One can view fermions in the defining representation as a single field, and different fermion species are just a manifestation of the three faces of $U(3)$.

3. Outlook

We have presented the simple $U(3)$ toy model in order to focus attention on $U(n) \simeq SU(n) \rtimes U(1)$ versus $SU(n) \times U(1)$ as a (classical) gauge field theory. This is particularly relevant for string theory phenomenology where $U(n)$ groups arise quite naturally in type I, IIA, and IIB compactifications of n stacked D-branes. But for practical purposes, one would like to know if phenomenological models incorporating $U(n)$ have any chance of being consistent, non-supersymmetric QFTs.

There are phenomenological reasons to suspect there might be some kind of non-electric charge-carrying gauge field(s) beyond the Standard Model. Along these lines, many models incorporate a "dark photon" that interacts with a hidden matter-field sector but may or may not interact with the Standard Model sector. The dark photon literature is quite extensive: For a review see [19] and references therein. The idea of appending a hypercolor symmetry group $SU(3)_H \times U(1)_H$ to the minimal supersymmetric $SU(5)_{GUT}$ is studied in [20–23]. The extra factor group resolves some shortcomings of the model, and it can be viewed as a $D3 - D7$ brane system in type IIB supergravity. A model of dark matter coming from an anomalous $U(1)$ gauge boson in type I, IIA, and IIB string compactifications is put forward by [24]. They use the trivial representations of $U(n)$ for $U(3) \times U(2) \times U(1) \times U(1)$. Similarly, a string completion of the 3-3-1 model that contains a novel seesaw mechanism is given by [15]. They use the trivial representations of $U(n)$ for $U(3) \times U(3) \times U(1) \times U(1)$ along with symmetry breaking down to $SU(3) \times SU(3) \times U(1)$, and they derive conditions for gauge and string anomaly cancellation. Note that the three defining representations displayed by our toy model might not survive the requisite $N = 1$ supersymmetry of the string theory models, but perhaps one could dispense with the Stuekelberg symmetry breaking mechanism owing to the viability of $U(n)$ as a gauge symmetry group. A genuine semi-direct product group $(SU(3)_C \times SU(2)_L) \rtimes U(1)_Y$ and anomaly cancellation were used by [25] to put constraints on matter field hypercharge. Lastly, a model of cosmic inflation due to a $U(1)$ gauge field coupled to a fermionic charge density was studied in [26]. An evident avenue for further research is to explore how viewing $U(n)$ as a semi-direct product might impact these various studies.

Apart from the above models, we propose $U(3) \times U(2)$ as a candidate symmetry group for physics beyond the Standard Model. Here the $U(3)$ symmetry is viewed as an extension of $SU(3)$ that commutes with the electroweak symmetry $U(2)$ which is spontaneously broken to electromagnetic $U(1)$ in the usual manner. This model is a rather economical extension of the Standard Model with only the gauge kinetic terms and fermion representations differing from the Standard Model Lagrangian. A full account of the model is beyond our present scope, but we will give a brief discussion.

Compare our toy model with the $U(3) \times U(2)$ extended Standard Model. The first difference one sees is the mass term in the toy model versus the Yukawa term for the Higgs coupling to massless fermions in the Standard Model. Notice that the argument for three physically distinct fields goes through just the same if the mass term is absent in the toy model. So the conclusion of three distinct fields holds also for $U(3) \times U(2)$, and the Yukawa term will be a sum over three distinct fields assuming different field renormalizations. Furthermore, since the $U(1) \subset U(3)$ has nothing to do with the $U(1)$ of electromagnetism, the presence of $U(3)$ affects only the QCD sector of the extended Standard Model and does not interfere with the electroweak force or symmetry breaking. The next difference to consider is the possibility of an anomaly associated with $U(1) \subset U(3)$. A detailed study of the full QFT model is required to reliably comment on this since it is not obvious how the group structure of $U(n)$ will affect anomaly considerations. But off hand it appears there would be no anomaly for the same reason that $SU(3)$ does not contribute anomalous currents in the Standard Model given a balance of color fermion and anti-fermion representations. Of course, it is possible the fermionic field content might require modification to ensure anomaly cancellation in the $U(2)$ context.

Assuming no anomaly, model building would branch into (i) symmetry breaking $U(3) \to SU(3)$ producing a new massive gauge boson or (ii) no symmetry breaking. For no symmetry breaking, it is natural to wonder if there could be realistic strong-force phenomenology coming from gauged $U(3)$. One might be sceptical, because long ago Fischbach et al. [27] proposed the symmetry group $SU(3)_C \times U(1)_B$ with $SU(3)_C$ being the color symmetry of QCD and $U(1)_B$ coupling to baryon number, but it was effectively falsified by experiment [28]. However, as we have stressed, the gauge-field interactions for $U(3)$ differ considerably from the $SU(3) \times U(1)$ case. All of the gauge fields associated with the Cartan subalgebra of $U(3)$ take part in both gauge and matter field interactions. So if there is somehow any vestige of a long-range charge carrier coming from $U(3)$, it will couple to both gauge and matter field mass-energy and therefore have a chance of being consistent with gravity — which ultimately was the downfall of $SU(3)_C \times U(1)_B$. Moreover, although the physical dynamics of strongly coupled gluons is difficult to intuit, one could imagine (by analogy with the photon) the $U(1) \subset U(3)$ charge-carrying gluon having a non-zero effective mass in ponderable matter on a galactic scale. (As a reminder, the $U(1)$ subgroup is *not* the electromagnetic gauge symmetry.) Evidently, the $U(1)$ charge-carrying gluon might have dark matter implications whether $U(3)$ is broken to $SU(3)$ or not. Of course this is highly speculative, and the suggestion that dark matter might be associated with strong (non-gravitational) interactions with visible matter runs contrary to orthodox opinion. Less clear and more imperative is whether unbroken $U(3)$ can somehow agree with QCD and therefore imply three generations.

4. Summary

Our analysis started with the observation that $U(n)$ gauge symmetry can be incorporated into gauge field theories via semi-direct products and not simply as direct products. As such, the generator of the $U(1)$ subgroup couples to all the other gluons producing a much richer gluon dynamics than can be achieved through $SU(n) \times U(1)$. In particular, for $U(3)$ the construction of the semi-direct product is not unique: it comes in three versions in the defining representation. We argued these three versions can be interpreted as three species of matter fields. The interpretation relies on including all three versions of the semi-direct product in the Lagrangian, the large-gauge-transformation status of certain permutation operators, and the identification $\Psi^{(r)} := P^{r-1}\Psi$.

Many phenomenological models make use of various $SU(n) \times U(1)$ gauge symmetry groups; especially string theory D-brane compactifications. We observed that model builders might profit by instead using $U(n)$. For example, again specializing to $U(3)$ we suggested an extension of the Standard Model by enlarging the QCD gauge symmetry group to $U(3)$. This will yield three species of fermions in the defining representation. Moreover, the field that carries the charge associated with the $U(1)$ subgroup will almost certainly behave in novel and perhaps unexpected ways due to the non-commutative nature of $U(3)$. Specifically, the potential energy of its interactions with the other gluons will add to the stress-energy tensor and enhance the gluon-fermion interaction dynamics. One anticipates the augmented stress-energy tensor to at least partially contribute to dark matter—ironically the 'dark' being a consequence of strong rather than weak interactions. Of course the important question is whether the gluon-fermion QCD phenomenology is realistic or not. Some initial steps to answer this question have been taken and will be reported elsewhere.

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