Imperial College of Science, Technology and Medicine
Department of Physics

Algebraic construction of Coulomb branches in $3d$ $\mathcal{N} = 4$ quiver gauge theories

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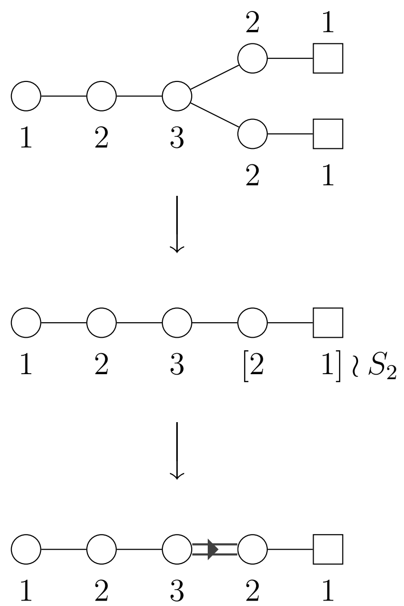

Thesis

Submitted in partial fulfilment of the requirements
for the degree of *Doctor of Philosophy in Physics of Imperial College London*
and the *Diploma of Imperial College*

Abstract

This thesis serves as a self-contained review of some recent advances in the study of three-dimensional $\mathcal{N} = 4$ quiver gauge theories and their Coulomb branch moduli spaces in particular. Our investigation leverages and develops the Hilbert series and abelianisation approaches and finds them mutually complementary. Their synthesis provides an explicit construction of the Coulomb branch with several desirable properties: for example, the global symmetry is made explicit and any complex mass deformation is easily derived. Moreover, it naturally handles two generalisations of quiver gauge theories: non-simply laced quivers and previously unknown wreathed quivers. Many concrete examples are provided to illustrate the concepts.

Dedication

This thesis is dedicated to Edward Tasker and Michael Brooks, two hilarious men with hearts of gold who cannot be replaced.

Declaration

Originality

The research presented in this thesis is the author's original work and is the result of two collaborations with Amihay Hanany and Antoine Bourget. Chapter 3 has been published in

- A. Hanany and D. Miketa, *Nilpotent orbit Coulomb branches of types AD*, *Journal of High Energy Physics* **2019** (2019) 113

Work in Chapter 4 has appeared in the preprint

- A. Bourget, A. Hanany and D. Miketa, *Quiver origami: discrete gauging and folding*, *arXiv:2005.05273 [hep-th]* (2020)

A significant portion of Chapter 2 has been adapted from introductory sections of the two publications listed above.

 Unless stated otherwise, it should be assumed that material adapted from these works has been modified from its original presentation.

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Acknowledgements

Nothing could have prepared me for the world of difference between *learning* and *researching*. Progress is much slower, results uncertain and dead ends frequent. I owe a lot to everyone who has supported me throughout the PhD and kept my spirits high.

I am grateful for my family who were always there when I needed a break.

I am indebted to my friends, who were a joy during good times and a comforting haven during the bad; special shout out to platinum-grade housemates and co-chefs Áron and Joanna, the J. P. Rolacque society, Discover Summer Academy and all my dear office mates, especially Dave, Arianna, Ed and Clement.

Knowledge is built by collaboration and I owe most of mine to my co-investigators: Amihay Hanany, who provided supervision, guidance and vision, and Antoine Bourget, who brought his deep knowledge, hard work and infectious enthusiasm for developing the few novel ideas I had to contribute. I also learnt much from other students in the "quiver subgroup": (in no particular order) Santiago Cabrera, Rudolph Kalveks, Julius Grimminger, Zhenghao Zhong, Anton Zajac and Giulia Ferlito. Finally, I would like to thank Matthew Bullimore for explaining his paper to me one afternoon – it changed everything.

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Chapter 1

Introduction

Quantum field theory is one of the greatest success stories of theoretical physics. Gravity notwithstanding, a field theory called the Standard Model can seemingly account for almost every experimental datum collected by particle experimentalists. At the same time, the theory is famously difficult and even the simplest models quickly give rise to complicated phenomena commonly known as *quantum corrections* for which one can account by mastery of Feynman diagrams, renormalisation theory, lattice simulations or other sophisticated techniques. Such things make the life of a theorist difficult. Fortunately there are upsides to being unbound by experiment. A model, though not a reflection of reality, can still be useful if it is *tractable*. This simple idea accounts for much of the interest in the study of *supersymmetry* and this thesis is no exception.

Supersymmetry allows us to say a fair amount about the possibility space of vacua, ie. field configurations devoid of perturbations colloquially known as particles. The more supersymmetry we add, the tighter it holds our hand. “Two doses” of supersymmetry, or $\mathcal{N} = 2$ in a familiar four-dimensional setting, are equivalent to four doses, or $\mathcal{N} = 4$, in three dimensions, which will be the primary arena considered in this work. Points in the space of vacua, also known as the *moduli space*, can be distinguished by different outcomes of thought experiments or, more technically, vacuum expectation values of operators. A particularly well-behaved class of such operators form the so-called *chiral ring*, which itself admits a rich structure. Morally (and imprecisely) speaking, it splits into two parts: the Higgs and Coulomb branch chiral rings. In the spirit of specialisation exhibited so far, we will be focussing almost solely on the Coulomb branch chiral ring.

Most of the Coulomb branch is inaccessible from the classical limit of a quantum field theory. New operators arise in the quantum regime and the chiral ring is endowed with the structure of an algebra. The full space of vacua parametrised by the Coulomb branch chiral ring – known as the Coulomb branch – typically carries a symmetry group. While it has been studied for some time, the evidence has been either somewhat indirect or demonstrated only on very simple examples. In other words, Coulomb branches were rarely constructed as algebraic varieties on which one could demonstrate the alleged symmetry. The work presented in this thesis is one small step towards plugging this explanatory hole.

Due to a large amount of supersymmetry the Coulomb branches which arise in this way possess a particular property of being *hyper-Kähler*. Roughly speaking, they are to quaternions what Kähler spaces are to complex numbers. Every hyper-Kähler space is also *symplectic*, and, since such spaces are typically not smooth everywhere, we will sometimes call them *symplectic singularities*. The simplest families of examples can be constructed from minimal building blocks called *nilpotent orbits* and *Słodowy slices* and we will find that many Coulomb branches considered in this thesis can be so decomposed. And in fact, this relation can be reversed: Coulomb branches are increasingly being recognised as a potent way to generate and classify symplectic singularities.

In Chapter 2 we introduce the essential background: Lie algebras and their canonical matrix realisations, quiver gauge theories and their Coulomb branches, the monopole formula and abelianisation. In Chapter 3 we combine the knowledge gained by application of the monopole formula with concreteness inherent in abelianisation to explicitly describe Coulomb branches of basic simply laced quivers as algebraic varieties. Chapter 4 explains how to generalise these methods to non-simply laced and novel wreathed quivers which are two related but inequivalent quiver analogues of non-simply laced Dynkin diagrams.

Chapter 2

Essential background

2.1 Lie algebras

One question recurs throughout this work: *what is the symmetry of this space?* The natural tools for answering it come from the theory of Lie groups and Lie algebras, which are fortunately widely known among theoretical physicists. The algebra describes the symmetry while its *representations* are objects which are acted upon by it. We will see physical observables characterising the Coulomb branch assemble into representations of a particular symmetry, by which we mean that they form one or more tensors of the symmetry algebra and that all operator relations also form tensors of the very same algebra. Once the Coulomb branch is described as a space parametrised by values of particular tensors subject to tensorial relations, the action of the symmetry group becomes manifest and the recurring question is answered; that is the overarching strategy explored in this thesis.

Following a flash review of the basics, we develop a concrete matrix realisation of type AD algebras and define a folding operation which automatically generates analogous constructions for types BCG. This review is by no means exhaustive or even particularly self-contained as this material is already well covered by many excellent sources, for example the excellent [3].

2.1.1 Lie groups and Lie algebras

We define a *group* G as a set with an associative and invertible operation $\diamond$ and a special element $e \in G$ such that for all $g \in G$, $e \diamond g = g \diamond e = g$. Invertibility means that for every g there exists a $g^{-1} \in G$ such that $g \diamond g^{-1} = g^{-1} \diamond g = e$.

A group G which is furthermore a manifold is called a *Lie group*. For example, translations and rotations form Lie groups. Synergy between differentiability and group axioms leads to a very fortunate property. The tangent bundle of a Lie group is trivial and the tangent space at the identity element e can be pushed anywhere using invertible actions of the group called the *left translations* L_g. In this way a vector at the tangent space T_eG defines a left-invariant vector field. The set of such fields, along with the standard Lie derivative, forms a *Lie algebra* $\mathfrak{g}$ which contains almost all information about the group apart from the set of connected components $\pi_0(G)$ and the fundamental group $\pi_1(G)$. In particular the groups $O(n)$ and $SO(n)$ of orthogonal matrices in n dimensions with determinant ± 1, resp 1, share the same Lie algebra, even though $O(n)$ is strictly larger than $SO(n)$. Thanks to the correspondence between left-invariant vector fields and vectors in T_eG we may think of the Lie algebra $\mathfrak{g}$ as a description of the group "around the identity"; these would correspond e.g. to small translations or rotations.

Formally a Lie algebra $\mathfrak{g}$ is a vector space with an additional bilinear and anti-symmetric operation called the *Lie bracket*:

$$[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g} \tag{2.1}$$

with the *Jacobi identity* for all $X, Y, Z \in \mathfrak{g}$:

$$[[X,Y],Z] + [[Z,X],Y] + [[Y,Z],X] = 0. \tag{2.2}$$

If $\mathfrak{g}$ is a vector space of matrices, or some subset thereof, the matrix commutator acts as a Lie bracket.

Given the fact that our need to study Lie algebras came from wanting to understand Lie groups, which are real manifolds, we might expect $\mathfrak{g}$ to be a vector space *over the field of real numbers.* It turns out that extending the defining field to the complex numbers, or *complexifying* the Lie algebra, greatly simplifies matters, as is so often the case; a single complexified Lie algebra can have several real forms, i.e. Lie algebras over real numbers which complexify to it.

We will assume that $\mathfrak{g}$, viewed as a vector space, has a finite dimension. Let $\{X_i\}$ form a basis of $\mathfrak{g}$. Then $\mathfrak{g}$ is fully specified by this data along with the *structure constants* c_{ij}^k defined by

$$[X_i, X_j] = \sum_k c_{ij}^k X_k. \tag{2.3}$$

We will call the X_i *generators* of $\mathfrak{g}$.

2.1.2 Simple Lie algebras

The algebra $\mathfrak{g}$ may contain *abelian* elements X which commute with everything in the algebra, i.e. $[X, \mathfrak{g}] = 0$. The algebra then splits as

$$\mathfrak{g} = \mathfrak{g}_{\text{abel}} \oplus \mathfrak{g}_{\text{ss}} \tag{2.4}$$

where $\mathfrak{g}_{\text{abel}}$ are the abelian elements and $\mathfrak{g}_{\text{ss}}$ is the remainder which we call a *semisimple* Lie algebra.

If there exists a basis of $\mathfrak{g}_{\text{ss}}$ such that the generators split as $\{X_i\} = \{Y_i\} \cup \{Y_i'\}$ with $[Y_i, Y_j'] = 0$ then we say $\mathfrak{g}_{\text{ss}}$ is *decomposable* and we can split the algebra into parts generated by Y_i, resp. Y_i'; each is potentially decomposable in turn. This process terminates and we get the following decomposition:

$$\mathfrak{g} = \mathfrak{g}_{\text{abel}} \oplus \bigoplus_i \mathfrak{g}_i \tag{2.5}$$

with $\mathfrak{g}_i$ non-decomposable factors called *simple* Lie algebras. From now on, unless stated otherwise, assume Lie algebras are simple by default.

2.1.3 Chevalley-Serre basis and Dynkin diagrams

Let $\mathfrak{h} \subset \mathfrak{g}$ be the maximal commutative Lie subalgebra of a simple Lie algebra $\mathfrak{g}$ and call it the *Cartan subalgebra*. Its dimension is uniquely given and referred to as the *rank* of $\mathfrak{g}$, denoted below as rank($\mathfrak{g}$) or n.

Consider a set of linear operators $[H, \cdot]$ defined for all $H \in \mathfrak{h}$ under which all $H' \in \mathfrak{h}$ have zero eigenvalues:

$$[H, H'] = 0. \tag{2.6}$$

The complement of $\mathfrak{h}$ in $\mathfrak{g}$ is spanned by vectors $\{E_\alpha;\ \alpha \in \Phi\}$ which are also eigenvectors under the aforementioned operators, but this time with non-vanishing eigenvalues:

$$[H, E_\alpha] = \alpha(H) E_\alpha. \tag{2.7}$$

α is called a *root* of $\mathfrak{g}$ and E^α is the associated *root vector*, while Φ is the root space. (Note that while $\alpha \in \Phi$ generate $\mathfrak{h}^*$, Φ is a discrete space embedded in $\mathfrak{h}$.) The root space Φ is non-degenerate, meaning that if $\alpha, \beta \in \Phi$ then $\forall H \in \mathfrak{h} : \alpha(H) = \beta(H)$ implies $E_\alpha \propto E_\beta$. Consequently there is a one-to-one correspondence between roots α and one-dimensional vector spaces spanned by E_α.

Given that $\mathfrak{h}$ is a vector space, we can see that roots α are themselves vectors in the *dual vector space* $\mathfrak{h}^*$ of the Cartan subalgebra. Endowing $\mathfrak{h}$ with a basis $\{H_i\}$ then also defines a basis for the roots:

$$[H_i, E_\alpha] = \alpha(H_i) E_\alpha = \alpha_i E_\alpha \tag{2.8}$$

With everything in place, a Lie algebra can now be decomposed into a *Cartan-Weyl basis* as

$$\mathfrak{g} = \mathfrak{h} + \{E_\alpha;\ \alpha \in \Phi\}. \tag{2.9}$$

Finite-dimensional simple Lie algebras admit a classification. The first step is to express them in a canonical *Chevalley-Serre basis* (which is a more specialised Cartan-Weyl basis) defined by the (integer-valued) Cartan matrix κ_{ij} with $1 \leq i, j \leq \operatorname{rank} \mathfrak{g}$. The Chevalley-Serre basis is generated (as a Lie algebra) by $\operatorname{rank} \mathfrak{g}$ positive simple root vectors E_{+i}, $\operatorname{rank} \mathfrak{g}$ negative simple root vectors E_{-i} and $\operatorname{rank} \mathfrak{g}$ generators H_i of the commutative Cartan subalgebra $\mathfrak{h}$ together with a Lie bracket $[\cdot, \cdot]$ subject to relations

$$[H_i, H_j] = 0 \tag{2.10}$$

$$[H_i, E_{\pm j}] = \pm\kappa_{ji} E_{\pm j} \tag{2.11}$$

$$[E_{+i}, E_{-j}] = H_i \delta_{ij} \tag{2.12}$$

$$[E_{\pm i}, \cdot]^{1-\kappa_{ij}} E_{\pm j} = 0. \tag{2.13}$$

The final relation is called the *Serre relation*. Note that while roots α are generally associated with one-dimensional vector spaces $\operatorname{span}_{\mathbb{C}}(E_\alpha)$, (2.12) partly fixes the simple root vector representatives, up to $E_{\pm i} \to c_i^{\pm 1} E_{\pm i}$.

The remaining elements of the Lie algebra $\mathfrak{g}$ are generated by repeated action of $[E_{\pm i}, \cdot]$. Note that this prescription only specifies a Lie algebra up to isomorphism.

The Cartan matrix κ_{ij} obeys several constraints following from Lie theory:

- $\kappa_{ii} = 2$
- $\kappa_{ij} = 0 \implies \kappa_{ji} = 0$
- $\kappa_{ij} \in \mathbb{Z}_{\leq 0}$ for $i \neq j$
- $\det \kappa > 0$
- The rows and columns of κ_{ij} cannot be rearranged to make the matrix block-diagonal.

This leaves only a small number of matrices which turn out to be in one-to-one correspondence with *Dynkin diagrams*, collected in Figure 2.1, at least up to isomorphism. Conversely, given a Dynkin diagram one can reconstruct a Lie algebra isomorphism class.

The correspondence between simple Lie algebras and Dynkin diagrams works in the following (invertible) way:

- The subscript n on the algebra denotes both the number of nodes in the diagram and the number of positive simple roots in the algebra.
- The Cartan matrix κ_{ij} is then an $n \times n$ matrix such that:

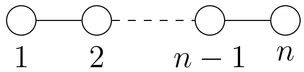
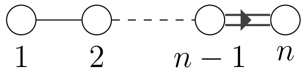
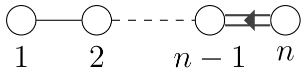
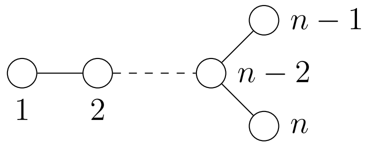
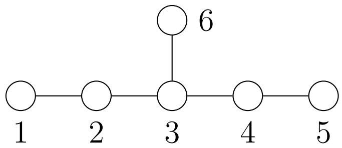
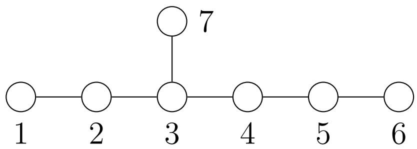
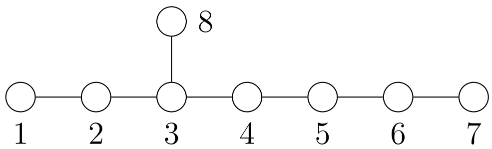
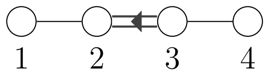
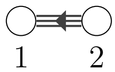
Algebra	Dynkin diagram
$A_n \simeq \mathfrak{sl}(n+1, \mathbb{C})$	
$B_n \simeq \mathfrak{so}(2n+1, \mathbb{C})$	
$C_n \simeq \mathfrak{usp}(2n, \mathbb{C})$	
$D_n \simeq \mathfrak{so}(2n, \mathbb{C})$	
E_6	
E_7	
E_8	
F_4	
G_2	

Figure 2.1: Simple Lie algebras and their corresponding Dynkin diagrams with labelled nodes

- $\kappa_{ii} = 2$
- κ_{ij} is related to the number of directed edges from node i to node j. If the nodes are connected by a single edge then $\kappa_{ij} = \kappa_{ji} = -1$. In the case of multiple edges, depicted by m links with an arrow pointing from node i to node j, the relevant entries read $\kappa_{ij} = -m$ and $\kappa_{ji} = -1$.

The set of positive root vectors is the set of root vectors which can be generated by (finitely many) Lie brackets of positive simple root vectors; its elements are labelled $E_{+\alpha}$. The set of negative root vectors is its analogue, but generated by negative simple root vectors, with elements $E_{-\alpha}$. We can also omit the sign and denote the generic root vector as E_α.

Recall from (2.8) that a root vector α is labelled by its eigenvalues α_i under the action of $[H_i, \cdot]$. The eigenvalues of simple roots α_j can be represented by Cartan matrix row vectors as $(\alpha_j)_{\pm i} = \pm(\kappa_i)_j = \pm\kappa_{ij}$. The basis in which the integers $(\alpha_j)_{\pm i}$ are evaluated is called the *basis of fundamental weights*[1]. Although important in the theory of Lie algebras and their representations, it is less suitable for our purposes than the *simple root basis* which expands a root (or equivalently a root vector's eigenvalues) in terms of simple roots (or eigenvalues of positive simple root vectors):

$$\pm\alpha = \sum_{i=1}^{n} c^i_{\pm\alpha}\alpha_i = \langle c^1_{\pm\alpha}, \ldots, c^n_{\pm\alpha}\rangle, \tag{2.14}$$

where the final expression should be understood as shorthand notation for the immediately preceding expansion in the simple root basis.

The Jacobi identity implies that

$$[H_i, [E_{\pm\alpha}, E_{\pm\beta}]] = (\pm\alpha \pm \beta)_i [E_{\pm\alpha}, E_{\pm\beta}] \tag{2.15}$$

It follows that, since the Lie algebra is generated by brackets of simple root vectors, all $c^i_{\pm\alpha}$ are integers.

Moreover, since any positive (negative) root vector E_α is constructed by finitely many bracket operations between positive (negative) simple root vectors, all its c^i_α are positive (negative).

One can easily convert the components $(\alpha)_i$ of α from the basis of fundamental weights to the simple root basis by multiplying with κ^{-1} from the right:

$$[(\alpha)_1, \ldots, (\alpha)_n](\kappa^{-1}) = \langle c^1_\alpha, \ldots, c^n_\alpha\rangle \tag{2.16}$$

[1]Weights are a generalisation of roots for other representations of the Lie algebra.

For a concrete example, consider the roots of A_3:

$$\begin{aligned}\Phi_{A_3} = \{&[1,0,1],[-1,1,1],[1,1,-1],[-1,2,-1],[2,-1,0],[0,-1,2],\\ &[0,1,-2],[-2,1,0],[1,-2,1],[-1,-1,1],[1,-1,-1],[-1,0,-1]\}\end{aligned} \tag{2.17}$$

The numbers in square brackets state components of the roots in the basis of fundamental weights. Multiplying on the right by the inverse of the Cartan matrix κ^{-1} amounts to expressing a root in terms of the simple root basis (for which we use angled brackets). For example,

$$\begin{aligned}
[1,0,1](\kappa^{-1}) &= \langle 1,1,1\rangle\\
[1,1,-1](\kappa^{-1}) &= \langle 1,1,0\rangle\\
[2,-1,0](\kappa^{-1}) &= \langle 1,0,0\rangle\\
[0,-1,2](\kappa^{-1}) &= \langle 0,0,1\rangle\\
[-1,-1,1](\kappa^{-1}) &= \langle -1,-1,0\rangle
\end{aligned}$$

All roots of A_n are given by unbroken strings of 1 or -1. We will see later that possible entries exactly correspond to a set of particularly interesting physical operators, the monopole generators of type A quivers. For now notice how the Lie bracket acts:

$$\left[E_{\langle 1,0,0\rangle}, E_{\langle 0,1,0\rangle}\right] \propto E_{\langle 1,1,0\rangle} \tag{2.18}$$

$$\left[E_{\langle 1,1,0\rangle}, E_{\langle 0,0,1\rangle}\right] \propto E_{\langle 1,1,1\rangle} \tag{2.19}$$

The precise coefficients, i.e. *structure constants*, are in this case ± 1. While many relations between structure constants can be found, the constants are not necessarily uniquely fixed. Every choice produces a different (but isomorphic) algebra, so it makes more sense to speak of Chevalley-Serre *bases*, each of which satisfies relations (2.10)–(2.13). We will select one which leaves monopole operators in their simplest form.

Chevalley-Serre basis: Matrix realisation I

This thesis describes a method of assembling generators of the Coulomb branch chiral ring into irreducible representations of the Coulomb branch symmetry. There will always be a set of generators transforming in the *coadjoint* representation of the symmetry and the remainder of this section will build towards constructing it, as a set of matrices, for the classical algebras $ABCD$ and the exceptional G_2. In order to derive the matrix realisation of the coadjoint representation we will first look for its dual, the corresponding *adjoint* representation, which is equivalent to finding a matrix realisation of one particularly nice basis of the Lie algebra itself: the aforementioned Chevalley-Serre basis.

An alternative matrix realisation of Chevalley Serre bases (but given for *all* simple Lie algebras) can be found in [4].

Type A

We start with the simplest example: the algebras $A_n \simeq \mathfrak{sl}(n+1,\mathbb{C})$ specified by a linear Dynkin diagram. To facilitate the construction of the Chevalley-Serre basis in the form of concrete matrices (with the Lie bracket implemented through commutators), we introduce one final basis for roots: the *orthonormal basis* given by $e_i - e_j$ where e_i are the orthonormal basis vectors of $\mathbb{C}^{n+1}$. Simple roots are represented as

$$\alpha_{\pm i} = \pm e_i \mp e_{i+1} \tag{2.20}$$

and brackets act by adding the orthonormal representatives up to an overall factor, e.g.

$$[E_{+1}, E_{+2}] = [E_{e_1-e_2}, E_{e_2-e_3}] \propto E_{e_1-e_3} \tag{2.21}$$

which is really a restatement of $[E_\alpha, E_\beta] \propto E_{\alpha+\beta}$, but with one notational advantage. Since any root can be expressed in the simple root basis as an unbroken string of ± 1, the $e_i - e_j$ cover and exhaust all roots. Each root is therefore labelled by two numbers, i and j, with $i < j$ for positive roots and $j < i$ for negative roots. The orthonormal representation then provides a more compact labelling scheme for simple root vectors:

$$E_{+(i:j)} \leftrightarrow e_i - e_j \ (i < j) \tag{2.22}$$
$$E_{-(i:j)} \leftrightarrow e_i - e_j \ (i > j) \tag{2.23}$$

so in particular $E_{+i} = E_{+(i:i+1)}$. In words $E_{\pm(i:j)}$ is the root whose weight vector (in the simple root basis) consists of a string of ± 1 starting at i and terminating at $j-1$.

It is now easy to guess that the matrix representative of $E_{\pm(i:j)}$ is precisely the zero matrix with the i,j or j,i component changed to ± 1. We choose the convention in which the single non-zero component of any simple root $E_{\pm(i:i+1)}$ is $+1$ and fix the remaining signs by demanding $[E_{\pm(i:j)}, E_{\pm(j,k)}] = -E_{\pm(i,k)}$. (This choice allows a neat correspondence between monopoles and roots in Sections 3.1.3 and 3.1.4.) Representatives of the Cartan subalgebra can be found by applying (2.12). The resulting Chevalley-Serre basis is

$$(\mathcal{E}_{i,j})_{ab} = \delta_{ia}\delta_{jb} \tag{2.24}$$

$$E_{+(i:j)} = (-1)^{i-j+1}\mathcal{E}_{i,j+1} \tag{2.25}$$

$$E_{-(i:j)} = (-1)^{i-j+1}\mathcal{E}_{j+1,i} \tag{2.26}$$

$$H_i = E_{i,i} - E_{i+1,i+1} \tag{2.27}$$

Remember from the discussion under 2.13 that there is some freedom in specifying the correspondence between a simple root and its root vector, namely a $\mathbb{C}^*$ factor shared between a pair of positive and negative simple roots. Here we make an ultimately arbitrary choice: we could have, for example, omitted the signs in (2.25) and (2.26). But this way we get that $[E_{+(1:2)}, E_{+(2:3)}] = -[E_{+(1:3)}]$ (for n in A_n at least 2), which will mimic an exactly parallel structure studied in Section 2.4.2 and onwards: the Poisson bracket between monopole operators. It will hopefully become apparent that this similarity is worth it, and at any rate we needed to choose *some* convention, and this one works at least as well as any other.

The resulting structure of alternating signs can already be seen in the following example of $\mathfrak{sl}(4,\mathbb{C}) = A_3$, where the coefficients c range over $\mathbb{C}$:

$$\begin{aligned}
\mathrm{ad}(\mathfrak{sl}(4,\mathbb{C})) &= \left\{ \sum_{\langle i,j,k\rangle \in \Phi} c_{\langle i,j,k\rangle} E_{\langle i,j,k\rangle} + \sum_{i=1}^{3} c_i H_i \right\} \\
&= \left\{ \begin{pmatrix} c_1 & c_{\langle 1,0,0\rangle} & -c_{\langle 1,1,0\rangle} & c_{\langle 1,1,1\rangle} \\ c_{\langle -1,0,0\rangle} & -c_1 + c_2 & c_{\langle 0,1,0\rangle} & -c_{\langle 0,1,1\rangle} \\ -c_{\langle -1,-1,0\rangle} & c_{\langle 0,-1,0\rangle} & -c_2 + c_3 & c_{\langle 0,0,1\rangle} \\ c_{\langle -1,-1,-1\rangle} & -c_{\langle 0,-1,-1\rangle} & c_{\langle 0,0,-1\rangle} & -c_3 \end{pmatrix} \right\} \\
&= \left\{ \sum_{\substack{1\le i<j\le n \\ s\in\{+,-\}}} c_{s(i:j)} E_{s(i:j)} + \sum_{i=1}^{3} c_i H_i \right\} \\
&= \left\{ \begin{pmatrix} c_1 & c^+_{1:2} & -c^+_{1:3} & c^+_{1:4} \\ c^-_{1:2} & -c_1 + c_2 & c^+_{2:3} & -c^+_{2:4} \\ -c^-_{1:3} & c^-_{2:3} & -c_2 + c_3 & c^+_{3:4} \\ c^-_{1:4} & -c^-_{2:4} & c^-_{3:4} & -c_3 \end{pmatrix} \right\}
\end{aligned} \tag{2.28}$$

The final step is to identify the corresponding coadjoint basis which is dual to the adjoint basis with respect to the scalar product

$$\langle X, Y\rangle = \mathrm{tr}(XY). \tag{2.29}$$

Labelling elements of the Chevalley-Serre basis X_m with the index m ranging from 1 to $\dim\mathfrak{g}$, we compute the matrix C

$$C_{mn} = \mathrm{tr}(X_m X_n). \tag{2.30}$$

Up to an overall multiplicative constant (the second order Dynkin index [3]), C_{mn} is precisely the Killing form $K(X_m, X_n)$. It is well known that the Killing form is non-degenerate and so C can be inverted. We use it to define matrices

$$X_m^* = \sum_p (C^{-1})_{mp} X_p \tag{2.31}$$

satisfying the property

$$\langle X_m^*, X_n \rangle = \mathrm{tr}(X_m^* X_n) = \sum_p (C^{-1})_{mp} \mathrm{tr}(X_p X_n) = (C^{-1}C)_{mn} = \delta_{mn}. \tag{2.32}$$

X_m^* constitute the desired basis for the coadjoint representation of $\mathfrak{g}$ and dualisation $* : \mathfrak{g} \to \mathfrak{g}^*$ can be defined through linear extension of (2.31).

For the Chevalley-Serre basis of type A one gets $E^*_{\pm(i:j)} = E_{\mp(i:j)}$. On the other hand the Cartan subalgebra mixes in a non-trivial way and while the dual H_i^* is still in the Cartan subalgebra, it is a linear combination of H_i. $K(H, E^\alpha) = 0$ for any $H \in \mathfrak{h}$ and $\alpha \in \Phi$. As a result the matrix C becomes block-diagonal in the Chevalley-Serre basis with $C|_\mathfrak{h}$, the restriction of this linear transformation to the Cartan subalgebra, one of the blocks. Since C is non-degenerate its block $C|_\mathfrak{h}$ must also be invertible and (2.31) specialises to

$$H_a^* = \sum_j (C|_\mathfrak{h})^{-1}_{ij} H_j. \tag{2.33}$$

Type D

The orthonormal basis for D_n is exhausted by roots of the form $\pm e_i \mp e_j$ or $\pm e_i \pm e_j$ $(1 \leq i, j \leq n)$ and the simple roots are in particular given by

$$\begin{aligned} \alpha_{\pm i} &\leftrightarrow \pm e_i \mp e_{i+1},\ 1 \leq i \leq n-1 \\ \alpha_{\pm n} &\leftrightarrow \pm e_{n-1} \pm e_n. \end{aligned} \tag{2.34}$$

The remaining root vectors (and hence their corresponding roots) are obtained through bracket products of simple root vectors. For example (using angled brackets to signify expansion in the simple root basis),

$$[E_{\langle 1,1,0,0\rangle}, E_{\langle 0,0,1,0\rangle}] \propto E_{\langle 1,1,1,0\rangle} \tag{2.35}$$

$$[E_{\langle 1,1,1,1\rangle}, E_{\langle 0,1,0,0\rangle}] \propto E_{\langle 1,2,1,1\rangle}. \tag{2.36}$$

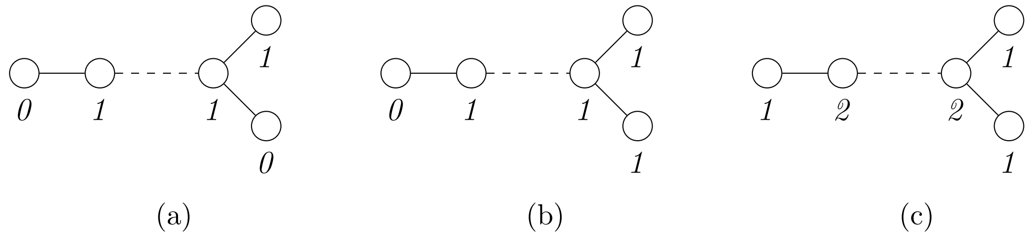


Figure 2.2: Numbers represent root components (in the simple root basis) at each node.

This corresponds to addition in the orthonormal basis:

$$\langle 1,1,1,0\rangle = e_1 - e_4 = (e_1 - e_2) + (e_2 - e_3) + (e_3 - e_4) \tag{2.37}$$

$$\langle 1,2,1,1\rangle = e_1 + e_2 = (e_1 - e_2) + 2(e_2 - e_3) + (e_3 - e_4) + (e_3 + e_4). \tag{2.38}$$

Whereas positive (negative) roots of A_n corresponded to strings of 1 (-1) in the simple root basis, the situation is marginally more complicated for D_n. The roots can be categorised as:

1. Unbroken strings of ± 1 anywhere on the Dynkin diagram (e.g. Fig. 2.2a). They are the $\pm e_i \mp e_j$ and $\pm e_i \pm e_n$ in the orthogonal basis.

2. ± 1 on both spinor ($(n-1)$-th and n-th) nodes and an arbitrarily long string of ± 1 towards the vector (first) node (e.g. Fig. 2.2b). They are the $\pm e_i \pm e_{n-1}$ in the orthogonal basis.

3. ± 1 on both spinor nodes, a string of ± 2 starting at the $(n-2)$-th node and terminating before the first node, continued by a string (of length at least 1) of ± 1 toward the first node (e.g. Fig. 2.2c). They form the rest of the $\pm e_i \pm e_j$ in the orthogonal basis.

We can therefore find two integers i, j associated to each root, just as in the case of A algebras. The complex Lie algebra of D_n, $\mathfrak{so}_{\mathbb{C}}(2n)$, acts linearly on the vector space $\mathbb{C}^{2n}$ and the adjoint representation therefore admits realisation as a $2n \times 2n$ antisymmetric matrix, which naturally breaks into 2×2 blocks indexed precisely by $i, j = 1, \ldots, n$. Antisymmetry of matrices in $\mathfrak{so}_{\mathbb{C}}(2n)$ also relates the two off-diagonal 2×2 blocks indexed by i, j and j, i (where $i \neq j$). This is schematically represented by the following matrix, which has zeroes everywhere apart from two 2×2 blocks $\mathbf{D}$ sitting in the $(2i-1)$-th and $2i$-th row, $(2j-1)$-th and $2j$-th column and vice versa, modified by an overall constant dependent on the position of the $\mathbf{D}$

block within the larger matrix:

$$\mathscr{D}^{(ij)} = \begin{array}{c} \begin{array}{ccccc} \dots & 2i-1 \;\&\; 2i & \dots & 2j-1 \;\&\; 2j & \dots \\ & \downarrow & & \downarrow & \end{array} \\ \left(\begin{array}{ccccc} & & & & \\ & & & \mathrm{i}^{i-j+1}\mathbf{D}^{(ij)} & \\ & & & & \\ & -(\mathrm{i}^{i-j+1})(\mathbf{D}^{(ij)})^T & & & \\ & & & & \end{array} \right) \end{array} \begin{array}{l} \vdots \\ \leftarrow\; 2i-1 \;\&\; 2i \\ \vdots \\ \leftarrow\; 2j-1 \;\&\; 2j \\ \vdots \end{array} \tag{2.39}$$

Since the same indices i and j also label roots through the orthonormal basis, we should expect a correspondence between the two and indeed, each pair of off-diagonal blocks $\mathbf{D}$ contains precisely 4 complex degrees of freedom: just enough to represent all of $e_i - e_j$, $e_i + e_j$, $-e_i + e_j$ and $-e_i - e_j$ for $1 \leq i < j \leq n$. Each root is represented by a slightly different $\mathbf{D}$ block, which we will denote $\mathbf{D}_{+-}$ for roots of the form $e_i - e_j$ $(i < j)$, $\mathbf{D}_{++}$ for $e_i + e_j$ and $\mathbf{D}_{-+}$, $\mathbf{D}_{--}$ for their respective counterparts among negative roots. They are given by:

$$\mathbf{D}_{+-} = \frac{\mathrm{i}}{2}\begin{pmatrix} 1 & \mathrm{i} \\ -\mathrm{i} & 1 \end{pmatrix}$$
$$\mathbf{D}_{++} = \frac{\mathrm{i}}{2}\begin{pmatrix} 1 & -\mathrm{i} \\ -\mathrm{i} & -1 \end{pmatrix}$$
$$\mathbf{D}_{-+} = \frac{\mathrm{i}}{2}\begin{pmatrix} 1 & -\mathrm{i} \\ \mathrm{i} & 1 \end{pmatrix}$$
$$\mathbf{D}_{--} = \frac{\mathrm{i}}{2}\begin{pmatrix} 1 & \mathrm{i} \\ \mathrm{i} & -1 \end{pmatrix}$$

The full block $\mathbf{D}$ is then a linear combination of the four matrices above,

$$\mathbf{D}^{(ij)} = c_{+-}^{(ij)}\mathbf{D}_{+-} + c_{++}^{(ij)}\mathbf{D}_{++} + c_{-+}^{(ij)}\mathbf{D}_{-+} + c_{--}^{(ij)}\mathbf{D}_{--}. \tag{2.40}$$

Therefore the matrix realisation represents roots as

$$e_i - e_j \leftrightarrow \mathscr{D}^{(ij)}|_{c_{+-}^{(ij)}=1} = E_{(+i,-j)} \tag{2.41}$$
$$e_i + e_j \leftrightarrow \mathscr{D}^{(ij)}|_{c_{++}^{(ij)}=1} = E_{(+i,+j)} \tag{2.42}$$
$$-e_i + e_j \leftrightarrow \mathscr{D}^{(ij)}|_{c_{-+}^{(ij)}=1} = E_{(-i,+j)} \tag{2.43}$$
$$-e_i - e_j \leftrightarrow \mathscr{D}^{(ij)}|_{c_{--}^{(ij)}=1} = E_{(-i,-j)} \tag{2.44}$$

where $1 \leq i < j \leq n$ and all other coefficients c vanish.

All that remains is to define appropriate generators of the Cartan subalgebra, but that is easily achieved by invoking (2.12). A Cartan subalgebra generator is given by

$$H_i = \begin{array}{c} \begin{array}{cccc} \dots & 2i-1 \,\&\, 2i & 2i+1 \,\&\, 2i+2 & \dots \\ & \downarrow & \downarrow & \end{array} \\ \begin{pmatrix} \mathbf{0} & & & & \\ & \ddots & & & \\ & & \mathbf{H} & \mathbf{0} & \\ & & \mathbf{0} & -\mathbf{H} & \\ & & & & \ddots \\ & & & & & \mathbf{0} \end{pmatrix} \begin{array}{l} \\ \vdots \\ \leftarrow \quad 2i-1 \,\&\, 2i \\ \leftarrow \quad 2i+1 \,\&\, 2i+2 \\ \vdots \\ \\ \end{array} \end{array} \tag{2.45}$$

for $a = 1, \dots, n-1$, where

$$\mathbf{H} = \begin{pmatrix} 0 & \mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix} \tag{2.46}$$

and the remaining entries of H_i are zero. The final Cartan generator differs only very slightly from H_{n-1}, as one might expect:

$$H_n = \begin{array}{c} \begin{array}{cccc} \dots & 2n-3 \,\&\, 2n-2 & 2n-1 \,\&\, 2n & \\ & \downarrow & \downarrow & \end{array} \\ \begin{pmatrix} \mathbf{0} & & & \\ & \ddots & & \\ & & \mathbf{H} & \mathbf{0} \\ & & \mathbf{0} & \mathbf{H} \end{pmatrix} \begin{array}{l} \vdots \\ \\ \leftarrow \quad 2n-1 \,\&\, 2n \\ \leftarrow \quad 2n+1 \,\&\, 2n+2 \end{array} \end{array} . \tag{2.47}$$

The full adjoint representation is then realised as

$$\mathrm{adj}(\mathfrak{so}(2n, \mathbb{C})) = \left\{ \sum_{\substack{1 \leq i < j \leq n \\ r,s\ \in \{+,-\}}} c_{rs}^{(ij)} E_{(ri,sj)} + \sum_{1 \leq i \leq n} c_i H_i \right\} \tag{2.48}$$

where coefficients c range over $\mathbb{C}$.

As was the case with type A Chevalley-Serre bases, we finish this section by identifying the basis of the coadjoint representation. The generalisation is completely straightforward. We define the dual of a root vector $X_m^* \equiv \sum_n (C^{-1})_{mn} X_n$ through the inverse of the matrix

$$C_{mn} = \mathrm{tr}(X_m X_n), \tag{2.49}$$

which is again proportional to the non-degenerate Killing form. As was the case with type A, positive root vectors are swapped with their negative counterparts, although now an overall rescaling factor is involved:

$$E^*_{(ri,sj)} = \frac{1}{2}E_{(-ri,-sj)}, \quad r,s \in \{+,-\} \tag{2.50}$$

There is no additional subtlety in the dualisation of the Cartan subalgebra, which again mixes non-trivially through the the restriction of the Killing form to $\mathfrak{h}$:

$$H^*_i = \sum_j (C|_{\mathfrak{h}})^{-1}_{ij} H_j. \tag{2.51}$$

2.1.4 Folding of Lie algebras

Some pairs of simple Lie algebras can be related by an operation called *folding* [5], which acts on an algebra's Dynkin diagram and its internal structure. In a prototypical example, the D_4 algebra folds into B_3; in other words, rotations in eight dimensions are restricted to seven. Moreover, we show folding maps the D_4 Chevalley-Serre basis to its B_3 counterpart.

Dynkin diagrams can be folded if there is a graph automorphism such that there is no edge connecting a node to its own image. (A node and its image may be connected through an intermediary node.) In particular, the diagrams for A_{2n-1}, D_n or E_6 satisfy this constraint as they possess S_2 graph automorphisms, while the special case D_4 is invariant under S_3. In a unique case, B_3 folds to G_2 despite lacking an obvious graph automorphism (see Figure 30.14 in [5]). The associated algebra $\mathfrak{g}$ is then folded to $\tilde{\mathfrak{g}}$ by the following recipe, which simultaneously recovers the folded Chevalley-Serre basis.

Chevalley-Serre basis: Matrix realisation II

First let us denote the set of automorphisms by Γ, which is in practice either S_2 or S_3, and its elements by $\pi \in \Gamma$. We write

$$\pi(i) = j \tag{2.52}$$

to express that under the automorphism π the i-th node is mapped to the j-th node. The fact that π is a Dynkin diagram automorphism translates into the following invariance of the Cartan matrix under the action of π: $\kappa_{\pi(i)\pi(j)} = \kappa_{ij}$.

The folding function f takes as input nodes of the unfolded Dynkin diagram and

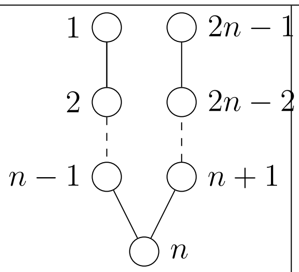
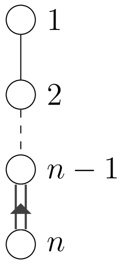
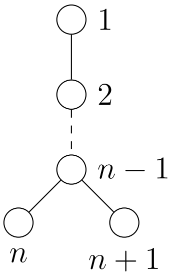
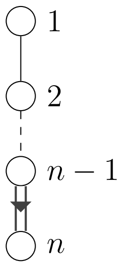
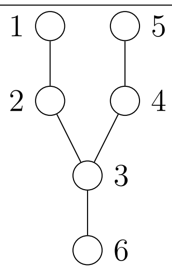
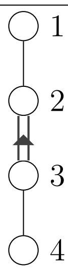
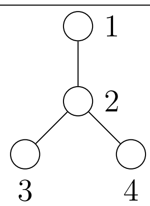
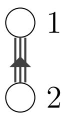
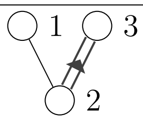
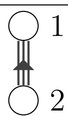
$\mathfrak{g}$	$\tilde{\mathfrak{g}}$	Dynkin diagram of $\mathfrak{g}$	Dynkin diagram of $\tilde{\mathfrak{g}}$	projection
A_{2n-1}	C_n			$S_2 : \begin{cases} 1, 2n-1 \mapsto 1 \\ \vdots \\ n \pm 1 \mapsto n-1 \\ n \mapsto n \end{cases}$
D_{n+1}	B_n			$S_2 : \begin{cases} 1 \mapsto 1 \\ \vdots \\ n-1 \mapsto n-1 \\ n, n+1 \mapsto n \end{cases}$
E_6	F_4			$S_2 : \begin{cases} 1, 5 \mapsto 1 \\ 2, 4 \mapsto 2 \\ 3 \mapsto 3 \\ 6 \mapsto 4 \end{cases}$
D_4	G_2			$S_3 : \begin{cases} 1, 3, 4 \mapsto 1 \\ 2 \mapsto 2 \end{cases}$
B_3	G_2			$S_2 : \begin{cases} 1, 3 \mapsto 1 \\ 2 \mapsto 2 \end{cases}$

Figure 2.3: Foldable simple Lie algebras. Note that numbers label nodes and do not indicate gauge groups as these are not quiver theories. The S_2 in the last row is a special case treated in several places in the main text.

maps them to appropriate nodes in the folded diagram. Consequently, $f \circ \pi = f$. As an example, take A_{2n-1} which folds to C_n and think of f as acting on indices i of the original linear diagram. f acts as $f(1) = f(2n-1) = 1$, $f(2) = f(2n-2) = 2$ and so on, but $f(n) = n$.

The folding procedure is now easily stated:

$$\tilde{H}_i = \sum_{j:f(j)=i} H_j \tag{2.53}$$

$$\tilde{E}_{\pm i} = \sum_{j:f(j)=i} E_{\pm j} \tag{2.54}$$

This defines the Chevalley-Serre basis for the folded algebra $\tilde{\mathfrak{g}}$. In the case of A_{2n-1}, the folded algebra is indeed C_n.

Special care must be taken when folding non-simple root vectors. Sometimes a sign change is required to preserve the algebra homomorphism $\mathfrak{g} \to f(\mathfrak{g})$. Consider the case of $A_3 \to C_2$. A_3 includes two elements $E_{12} = -[E_1, E_2]$ and $E_{23} = -[E_2, E_3]$. According to the definition of folding just given, $\tilde{E}_1 = E_1 + E_3$ and $\tilde{E}_2 = E_2$. Then it follows that

$$\tilde{E}_{12} = -[\tilde{E}_1, \tilde{E}_2] = -[E_1 + E_3, E_2] = -\left([E_1, E_2] + [E_3, E_2]\right) = E_{12} - E_{23}.$$

In this specific case it is clear that the sign flips because the third node, which comes after the second, is mapped to the first, which comes before the second. Likewise it is clear that such a scenario will never occur in the case $D_{n+1} \to B_n$ and only comes into play for $A_{2n-1} \to C_n$, $B_3 \to G_2$, $D_4 \to G_2$ and $E_6 \to F_4$.

The interested reader can easily check (2.10)-(2.12) and (2.13) with a bit more effort. To illustrate the typical calculation, we will confirm (2.11) for A_5 folding to C_3. The Cartan matrix of C_3 is

$$\kappa = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 2 \end{pmatrix} \tag{2.55}$$

and

$$[\tilde{H}_2, \tilde{E}_3] = [H_2 + H_4, E_3] = -E_3 - E_3 = -2\tilde{E}_3 = \kappa_{32}\tilde{E}_3 \tag{2.56}$$

$$[\tilde{H}_3, \tilde{E}_2] = [H_3, E_2 + E_4] = -E_2 - E_4 = -\tilde{E}_2 = \kappa_{23}\tilde{E}_2 \tag{2.57}$$

Folded Lie algebras sometimes preserve additional tensors. In the case of C_n

there exists a tensor J such that for every X in C_n

$$X^T J + JX = 0. \tag{2.58}$$

We can also reverse this statement: every X in A_{2n-1} which satisfies (2.58) is in C_n.

In our convention J assumes the following form:

$$J = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & -1 & 0 \\ 0 & 0 & 0 & \dots & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & -1 & \dots & 0 & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ -1 & 0 & 0 & \dots & 0 & 0 & 0 \end{pmatrix} \tag{2.59}$$

The other case of this type is G_2, which is the subalgebra of $SO(7)$ preserving the following rank 3 antisymmetric tensor ϕ:

$$\sum_{a'} \phi_{a'bc} X_{a'a} + \sum_{b'} \phi_{ab'c} X_{b'b} + \sum_{c'} \phi_{abc'} X_{c'c} = 0$$

for all $X \in G_2$. Given our choice of Chevalley-Serre basis the tensor can be defined as

$$\phi_{127} = -\phi_{136} = -\phi_{145} = \phi_{235} = -\phi_{246} = -\phi_{347} = -\phi_{567} = 1$$

with the remaining values either fixed by antisymmetry or equal to 0.

The dual Chevalley-Serre basis of linear forms $\{X_i^*\}$ is defined to obey $X_i^*(X_j) = \delta_{ij}$ for all X_i in the Chevalley-Serre basis. In practice we realise X_i^* as square matrices of the same dimension as X_i and represent the evaluation as the linear extension of

$$X_i^*(X_j) = \langle X_i^*, X_j \rangle = \langle X_j, X_i^* \rangle = \operatorname{tr}\left(X_i^* X_j\right). \tag{2.60}$$

The dual Chevalley-Serre bases of "parent" and folded algebras are related:

$$\tilde{H}_i^* = \frac{1}{\#_i} \sum_{j:f(j)=i} H_j^* \tag{2.61}$$

$$\tilde{E}_{\pm i}^* = \frac{1}{\#_i} \sum_{j:f(j)=i} E_{\pm j}^* \tag{2.62}$$

where $\#_i$ denotes the *multiplicity* of node i defined as

$$\#_i = \left|f^{-1}(i)\right| = |\{j : f(j) = i\}| \tag{2.63}$$

For example:

$$\langle \tilde{H}_i, \tilde{H}_j^* \rangle = \frac{1}{\#_j} \sum_{\substack{k:f(k)=i \\ l:f(l)=j}} \langle H_k, H_l^* \rangle = \frac{1}{\#_j} \sum_{\substack{k:f(k)=i \\ l:f(l)=j}} \delta_{kl} = \frac{1}{\#_j} \#_j \delta_{ij} = \delta_{ij} \tag{2.64}$$

where the second-to-last equality follows from the fact that $k = l$ can only occur if both fold to the same node, i.e. $i = j$, and that this can happen for $\#_j$ joint choices of (identical) k and l.

We close this section with a brief discussion of the aforementioned case of B_3 folding to G_2 despite a lack of graph automorphisms. This is easily elucidated with a quick detour through D_4:

$$\tilde{H}_3^B = H_3^D + H_4^D \tag{2.65}$$

$$\tilde{H}_2^G = H_1^D + H_3^D + H_4^D = \tilde{H}_1^B + \tilde{H}_3^B \tag{2.66}$$

where we decorate each Cartan generator with a superscript denoting its algebra. As illustrated – and the pattern holds up for remaining G_2 basis elements – G_2 can be expressed as a folding of B_3 in the same way that B_3 is a folding of D_4.

2.2 Supersymmetric quantum field theories

We assume that the reader is familiar with quantum field theory (QFT) at least at the level of any number of excellent textbooks [6, 7] and supersymmetry at the level of [8]. A quantum field theory is, as the name suggests, a theory of fields, or objects distributed across all of space-time and transforming in a representation of the global symmetry of the space-time, e.g. the Poincaré group associated to four-dimensional space-time. Unlike classical fields, quantum fields at rest can only be perturbed in discrete chunks, and the minimal such chunk is called a *particle*. Correspondingly, every fundamental (perturbative) particle arises as a perturbation of an underlying field. It follows that understanding "quantum fields at rest" is a crucial prerequisite for studying particle physics.

"Resting" configurations of quantum fields are known as *vacua* and form the focus of this thesis. While generic QFTs can have a unique vacuum, theories imbued with *supersymmetry* often possess uncountably many – each with its own characteristic particle content. Our work here is to essentially catalogue (a class of) them.

Since the world around us is – or in any case seems to be – four-dimensional, QFT is typically introduced to students in the context of four dimensions. However QFTs can exhibit radically different phenomena in different dimensions. For example, some particle-like dynamics can arise not as minimal perturbations of a field but as a topological phenomenon analogous to a twist on the Möbius strip: the twist can travel the length of the strip but such a configuration is not exactly 'close' to an untwisted cylindrical strip. The presence of such features, which are themselves heavily influenced by dimensionality, ought to have an effect on small perturbations and by extension particle content, so methods for studying vacua may vary across different dimensions – as is indeed the case. This thesis is largely concerned with three-dimensional physics but its applicability is wider: higher-dimensional supersymmetric theories can often be related to a $3d$ case, opening the possibility for a breakthrough with intrinsically three-dimensional techniques; see eg [9–11] for applications in four dimensions and [12–17] for a recent series of novel results in five- and six-dimensional physics achieved with $3d$ methods.

2.2.1 $4d$ $\mathcal{N}=1$

In keeping with the aforementioned pedagogical bias, we start with a discussion of supersymmetry in four dimensions, largely following the conventions of [8], particularly where indices and contractions are concerned. A $4d$ $\mathcal{N}=1$ theory is defined on a space-time whose symmetry is described by the four-dimensional Poincaré algebra, which is in turn extended by two *supercharges* $Q_\alpha, \bar{Q}_{\dot\alpha}$ transforming as spinors of $\mathfrak{so}(4)$; this extension is known as the *Poincaré superalgebra*, with (anti-)commutation relations

$$\{Q_\alpha, \bar{Q}_{\dot\alpha}\} = 2(\sigma^\mu)_{\alpha\dot\beta} P_\mu \tag{2.67}$$

$$[Q_\alpha, P^\mu] = [\bar{Q}_{\dot\alpha}, P^\mu] = 0 \tag{2.68}$$

$$[Q_\alpha, M^{\mu\nu}] = (\sigma^{\mu\nu})_\alpha^\beta Q_\beta \tag{2.69}$$

The four-vector σ^μ is matrix-valued:

$$\sigma^0 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$\sigma^2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.70}$$

and

$$\bar\sigma^\mu = (\sigma^0, -\sigma^i) \tag{2.71}$$

$$\sigma^{\mu\nu} = \frac{i}{4}(\sigma^\mu\bar\sigma^\nu - \sigma^\nu\bar\sigma^\mu) \tag{2.72}$$

Supercharges generate transformations which turn bosonic fields into fermions and vice versa and so a non-trivial representation of the superalgebra must involve both bosonic and fermionic components.

It's tempting to ask: why bother? What does supersymmetry bring us? Adding symmetry to a problem can drastically reduce its complexity and simplify calculations. Supersymmetry is valuable because it makes computations tractable, and the more supersymmetry there is, the easier the math becomes. Some of the truly exceptional simplifications relevant for our purposes require $\mathcal{N}=2$ supersymmetry or more, but since they are special cases of $\mathcal{N}=1$ (as one can always just "forget" about some of the symmetry), it makes sense to start with the less supersymmetric $\mathcal{N}=1$ theories.

Supersymmetric field theories are theories of fields. By complete analogy with standard non-supersymmetric theory, fields can have components: for example the vector field A_μ has four components A_0, A_1, A_2, A_3. However, *on-shell* particle states of those fields may be constrained by classical equations of motion to fewer components, e.g. the photon's two polarisations. We will now describe a few important superfields and indicate their particle components, assuming the particles are massless.

A *scalar superfield* is a function defined on an extension of space-time called *superspace*, which adds two grassmannian variables $\theta_\alpha, \bar{\theta}_{\dot{\alpha}}$:

$$\begin{aligned} S(x,\theta,\bar{\theta}) = \phi(x) + \theta\psi(x) + \bar{\theta}\bar{\chi}(x) + \theta\theta M(x) + \bar{\theta}\bar{\theta}N(x) + (\theta\sigma^\mu\bar{\theta})V_\mu(x) \\ +(\theta\theta)\bar{\theta}\bar{\lambda}(x) + (\bar{\theta}\bar{\theta})\theta\rho(x) + (\theta\theta\bar{\theta}\bar{\theta})D(x) \end{aligned} \tag{2.73}$$

S can be decomposed into *component fields* $\phi, \psi_\alpha, \bar{\chi}_{\dot{\alpha}}, M, N, V_\mu, \bar{\lambda}_{\dot{\alpha}}, \rho_\alpha$ and D which form irreducible representations of the Lorentz group. ϕ, M, N and D are scalar fields, V_μ is a vector field and both ψ_α, ρ_α and $\bar{\chi}_{\dot{\alpha}}, \bar{\lambda}_{\dot{\alpha}}$ are Weyl spinor fields transforming in conjugate spinorial representations.

More careful analysis shows that S is a reducible representation of the Poincaré superalgebra. Expressing the supercharges in terms of differential operators acting on fields,

$$\begin{aligned} Q_\alpha &= -\mathrm{i}\frac{\partial}{\partial\theta^\alpha} - (\sigma^\mu)_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}\frac{\partial}{\partial x^\mu} \\ \bar{Q}_{\dot{\alpha}} &= +\mathrm{i}\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} + \theta^\beta(\sigma^\mu)_{\beta\dot{\alpha}}\frac{\partial}{\partial x^\mu} \end{aligned} \tag{2.74}$$

we can find two other differential operators which commute with them and therefore preserve supersymmetry:

$$\begin{aligned} \mathcal{D}_\alpha &= \frac{\partial}{\partial\theta^\alpha} + \mathrm{i}(\sigma^\mu)_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}\frac{\partial}{\partial x^\mu} \\ \bar{\mathcal{D}}_{\dot{\alpha}} &= \frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} + \mathrm{i}\theta^\beta(\sigma^\mu)_{\beta\dot{\alpha}}\frac{\partial}{\partial x^\mu}. \end{aligned} \tag{2.75}$$

Since both $\mathcal{D}S$ and $\bar{\mathcal{D}}S$ are superfields, we can impose e.g. $\mathcal{D}S = 0$ as a consistent supersymmetric condition on S to restrict to a smaller representation of the Poincaré superalgebra.

A *chiral* superfield Φ satisfies $\bar{\mathcal{D}}\Phi = 0$. It can be expanded as

$$\begin{aligned}\Phi(x,\theta,\bar{\theta}) = \phi(x) + \sqrt{2}\theta\psi(x) + \theta\theta F(x) + \mathrm{i}\theta\sigma^\mu\bar{\theta}\partial_\mu\phi(x) \\ -\frac{\mathrm{i}}{\sqrt{2}}(\theta\theta)\partial_\mu\psi(x)\sigma^\mu\bar{\theta} - \frac{1}{4}(\theta\theta)(\bar{\theta}\bar{\theta})\partial_\mu\partial^\mu\phi(x)\end{aligned} \tag{2.76}$$

with four bosonic components bundled into two complex scalar fields ϕ, F and four fermionic components in ψ_α. It realises matter fields in supersymmetric theories. The auxiliary field F can be directly solved for using the equations of motion. The (on-shell) particle states form a complex scalar field and one Weyl spinor.

The superfield $\bar{\Phi}$ subject to $\mathcal{D}\bar{\Phi} = 0$ is called *antichiral*, admits an entirely analogous expansion in component fields and represents matter in the representation conjugate to that of chiral superfields. Its on-shell states are described by one complex scalar and a Weyl spinor (of the chirality opposite to that of chiral superfield Weyl spinors).

The last case of relevance for us is the *vector* superfield A constrained by $A^\dagger = A$. It expands into

$$\begin{aligned}A(x,\theta,\bar{\theta}) = C(x) + \mathrm{i}\theta\chi(x) - \mathrm{i}\bar{\theta}\bar{\chi}(x) + \frac{\mathrm{i}}{2}\theta\theta\left(M(x) + iN(x)\right) - \frac{\mathrm{i}}{2}\bar{\theta}\bar{\theta}\left(M(x) - \mathrm{i}N(x)\right) \\ +\theta\sigma^\mu\bar{\theta}A_\mu(x) + \mathrm{i}\theta\theta\bar{\theta}\left(-\mathrm{i}\bar{\lambda}(x) + \frac{\mathrm{i}}{2}\bar{\sigma}^\mu\partial_\mu\chi(x)\right) - \mathrm{i}\bar{\theta}\bar{\theta}\theta\left(\lambda(x) - \frac{\mathrm{i}}{2}\sigma^\mu\partial_\mu\bar{\chi}(x)\right) \\ +\frac{1}{2}(\theta\theta)(\bar{\theta}\bar{\theta})\left(D(x) - \frac{1}{2}\partial_\mu\partial^\mu C(x)\right)\end{aligned} \tag{2.77}$$

with C, M, N, D as real scalar fields and A_μ a vector field for a total of 8 bosonic degrees of freedom and fermionic fields χ, λ with 4 degrees of freedom each. Unsurprisingly it can act as the supersymmetric generalisation of the gauge field. Once again one of the scalar fields, D, is auxiliary, meaning that writing out the equations of motion allows one to directly solve for it. The superfield's on-shell particle states form a constrained vector field and a spinor called the *gaugino*.

Now we have enough to define $\mathcal{N} = 1$ supersymmetric electrodynamics (SQED) by specifying its Lagrangian. To that end we introduce the chiral supersymmetric abelian field strength

$$\begin{aligned}\mathcal{W}_\alpha = -\frac{1}{4}\bar{\mathcal{D}}\bar{\mathcal{D}}\mathcal{D}_\alpha A \\ \bar{\mathcal{W}}_{\dot{\alpha}} = -\frac{1}{4}\mathcal{D}\mathcal{D}\bar{\mathcal{D}}_{\dot{\alpha}} A\end{aligned} \tag{2.78}$$

It can be shown that $\mathcal{W}_\alpha$ contains the usual field strength $F_{\mu\nu}$ as a component field.

The superfields transform under the $U(1)$ gauge group in the following way:

$$\Phi_i \to e^{iq\Lambda}\Phi_i \tag{2.79}$$

$$\Phi_i^\dagger \to \Phi_i^\dagger e^{-iq\Lambda} \tag{2.80}$$

$$A \to A - \frac{i}{2}(\Lambda - \Lambda^\dagger) \tag{2.81}$$

$$\mathcal{W} \to e^{iq\Lambda}\mathcal{W}e^{-iq\Lambda} \tag{2.82}$$

where Λ is a chiral superfield.

The kinetic term for the gauge field is written as

$$\mathcal{L}_A = \frac{1}{4g^2}(\mathcal{W}\mathcal{W}\big|_{\theta\theta} + \bar{\mathcal{W}}\bar{\mathcal{W}}\big|_{\bar{\theta}\bar{\theta}}). \tag{2.83}$$

where $X\big|_{f(\theta,\bar{\theta})}$ denotes the component field of X appearing at order $f(\theta,\bar{\theta})$.

The vector superfield can also appear in the Lagrangian in another way, as long as it plays the role of a $U(1)$ gauge field, in the form of a *Fayet-Iliopoulos* term

$$\mathcal{L}_{\text{FI}} = \xi A^{U(1)}\big|_{\theta\theta\bar{\theta}\bar{\theta}} \tag{2.84}$$

where ξ is a real parameter of the theory. While Fayet-Iliopoulos parameters play an interesting role in the study of Higgs branches, in what follows we will set them to 0.

The (anti-)chiral fields Φ_i ($1 \le i \le n$) enter the Lagrangian as

$$\mathcal{L}_\Phi = \left(\Phi_i^\dagger e^{2qA}\Phi_i\right)\Big|_{\theta\theta\bar{\theta}\bar{\theta}} + \left(W(\Phi_i)\big|_{\theta\theta} + h.c.\right) \tag{2.85}$$

with $W(\Phi_i)$ a holomorphic, gauge-invariant combination of fields $W(\Phi_i)$; in the case of a single Φ we necessarily have $W(\Phi) = 0$.

Putting everything together, the Lagrangian of SQED reads

$$\mathcal{L}_{\text{SQED}} = \mathcal{L}_\Phi + \mathcal{L}_A + \mathcal{L}_{\text{FI}}. \tag{2.86}$$

This formalism can be extended to other gauge groups and supersymmetric Yang-Mills theories in the following way. Let $\mathcal{G}$ be a non-abelian gauge group. An associated gauge transformation Λ is an element of Lie$(\mathcal{G})$ whose components are chiral superfields. (E.g. if $\mathcal{G} = SU(n)$ then Λ is an $n \times n$ traceless matrix with chiral superfield components.) Its effects on chiral Φ in the fundamental representation, chiral $\tilde{\Phi}$ in the antifundamental representation and (superfield and gauge) vector A

are:

$$\Phi_i \to e^{\Lambda}\Phi_i \tag{2.87}$$
$$\tilde{\Phi}_i \to \tilde{\Phi}_i e^{-\Lambda} \tag{2.88}$$
$$e^{2A} \to e^{\Lambda^\dagger} e^{2A} e^{-\Lambda}. \tag{2.89}$$

The non-abelian field strength $\mathcal{W}$ must be modified to

$$\mathcal{W}_\alpha = -\frac{1}{8}\bar{\mathcal{D}}\bar{\mathcal{D}}\left(e^{-2A}\mathcal{D}_\alpha e^{2A}\right)$$

transforming as

$$\mathcal{W} \to e^{-\Lambda}\mathcal{W}e^{\Lambda}.$$

The kinetic term for vector superfields is amended to

$$\mathcal{L}_A = \frac{1}{2g^2}(\mathrm{tr}\mathcal{W}\mathcal{W}\big|_{\theta\theta} + \mathrm{tr}\bar{\mathcal{W}}\bar{\mathcal{W}}\big|_{\bar{\theta}\bar{\theta}}) \tag{2.90}$$

and the (fundamental) chiral field Lagrangian becomes

$$\mathcal{L}_\Phi = \left(\Phi_i^\dagger e^{2A}\Phi_i\right)\Big|_{\theta\theta\bar{\theta}\bar{\theta}} + \left(W(\Phi_i)\big|_{\theta\theta} + h.c.\right) \tag{2.91}$$

with $W(\Phi_i)$ again a holomorphic, gauge-invariant combination of Φ_i. To generalise to non-fundamental chiral fields, one need only cast A into the appropriate representation of the gauge algebra.

2.2.2 $4d$ $\mathcal{N} = 2$

Just as the Poincaré algebra was extended by two supercharges Q and $\bar{Q}$ into the $\mathcal{N} = 1$ Poincaré superalgebra, so can it in turn be extended by adding another distinct pair of $Q, \bar{Q}$ with suitable (anti-)commutative properties:

$$\{Q^A_\alpha, \bar{Q}^B_{\dot{\beta}}\} = 2(\sigma^\mu)_{\alpha\dot{\beta}}P_\mu\delta^{AB} \tag{2.92}$$
$$\{Q^A_\alpha, Q^B_\beta\} = \epsilon_{\alpha\beta}Z^{AB} \tag{2.93}$$

where the second equation introduces antisymmetric *central charges* Z^{AB}. Central charges necessarily vanish for massless representations and will be set to 0 in the following discussion.

A theory with two pairs of supercharges is appropriately denoted $\mathcal{N} = 2$. For some theories swapping the two sets of charges is also a symmetry of the theory and in some cases they may even be continuously rotated into one another (though see [18] for a counter-example). All transformations of this kind form the *R-symmetry* of the theory[2]. In this case the R-symmetry is $\mathfrak{su}(2)_R \times \mathfrak{u}(1)_r$ but the latter abelian factor is unimportant for this discussion.

$\mathcal{N}=2$ supersymmetric representations are larger than their $\mathcal{N}=1$ counterparts and can be decomposed into them. Recall that the action of a single supercharge turns a bosonic component field into a fermionic compotent field of the same supercharge (or vice versa). With two supercharges to choose from, bosonic component fields will form representations of the R-symmetry, as will their fermionic counterparts. To simplify matters, the following discussion of $\mathcal{N}=2$ fields will only concern particle states and neglect the remaining off-shell components of their respective superfields.

The $\mathcal{N}=2$ *vector* supermultiplet V consists of two $\mathcal{N}=1$ multiplets: the vector multiplet A and the chiral multiplet Φ, both transforming in some group's adjoint representation. In terms of particle states, it contains a vector field $A_\mu(x)$ and a complex scalar field $\varphi(x)$ with two real degrees of freedom each, and two Weyl spinors $\psi(x)$ and $\lambda(x)$ for a total of four fermionic degrees of freedom. While the vector and scalar are singlets under $\mathfrak{su}(2)_R$, the spinors form a doublet.

The $\mathcal{N}=2$ version of matter fields comes from *hypermultiplets* composed of one $\mathcal{N}=1$ chiral multiplet Q and one antichiral multiplet $\tilde{Q}^\dagger$. The two multiplets necessarily transform in conjugate representations of the gauge group $\mathcal{G}$, which are typically specified by reference to the representation of Q, e.g. the fundamental representation of $\mathcal{G}$. The component supermultiplets are often referred to as the *quark* and *anti-quark*, respectively. The hypermultiplet's particle states consist of two complex scalars $q(x)$ and $\tilde{q}^\dagger(x)$ in an $\mathfrak{su}(2)_R$ doublet and two Weyl spinors and $\mathfrak{su}(2)_R$ scalars ψ and $\tilde{\psi}^\dagger$.

$\mathcal{N}=2$ invariance seriously constrains the class of admissible Lagrangians. The vector contribution now takes the form

$$\mathcal{L}_V^{\mathcal{N}=2} = \frac{1}{4\pi}\mathrm{Im}[\tau \text{ tr } \Phi^\dagger e^{2[A,\cdot]}\Phi\big|_{\theta\theta\bar{\theta}\bar{\theta}} + \frac{1}{2}\tau \text{ tr } W_\alpha W^\alpha\big|_{\theta\theta}] \tag{2.94}$$

where $\tau = \frac{4\pi \mathrm{i}}{g^2} + \frac{\theta}{2\pi}$ is the complexified gauge coupling.

Massless hypermultiplets contribute to the Lagrangian as

$$\mathcal{L}_H^{\mathcal{N}=2} = \frac{1}{4\pi}\mathrm{Im}[\tau \text{ tr } \left(Q^{\dagger i} e^A Q_i + \tilde{Q}^i e^{-A}\tilde{Q}_i^\dagger\right)\Big|_{\theta\theta\bar{\theta}\bar{\theta}} + \tau\mathcal{W}\big|_{\theta\theta}] \tag{2.95}$$

where the superpotential takes the form

$$\mathcal{W} = \sqrt{2} \text{ tr } \tilde{Q}\Phi Q. \tag{2.96}$$

[2]Formally an R-symmetry is a symmetry which does not commute with the supersymmetry generators.

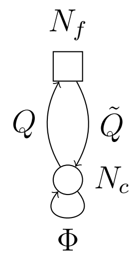
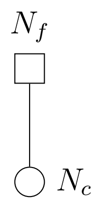
This notation is general across choices of gauge group and matter representations. For example, N_f quarks Q coupled to an $SU(N_c)$ gauge field would be represented by a $N_c \times N_f$ matrix, $\tilde{Q}$ would be an $N_f \times N_c$ matrix with Φ an $N_c \times N_c$ matrix. The trace is then understood to be over the fundamental representation, i.e. ranging over the N_f flavors.

The full Lagrangian is then written as

$$\mathcal{L}^{\mathcal{N}=2} = \mathcal{L}_V^{\mathcal{N}=2} + \mathcal{L}_H^{\mathcal{N}=2}. \tag{2.97}$$

2.2.3 Quiver gauge theories

We have seen that supersymmetry places strict constraints on Lagrangians. As a pleasant consequence, the full structure of many theories can be succinctly specified by a *quiver diagram*, or *quiver* for short. The main features are already visible in the quiver depiction of $\mathcal{N} = 2$ SQCD:

$\mathcal{N} = 1$ notation	$\mathcal{N} = 2$ notation
	

On the $\mathcal{N} = 1$ side, we see several features

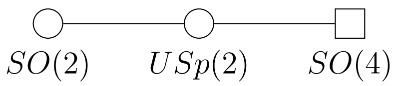
- **Circle node** N_c: gauge group $U(N_c)$ with a $\mathcal{N} = 1$ vector multiplet; also called the *gauge node*
- **Square node** N_f: flavor group $SU(N_f)$; also called the *flavor node*
- **Arrow** Q: chiral multiplet transforming in the fundamental representation of $U(N_c)$ and antifundamental representation of $SU(N_f)$
- **Arrow** $\tilde{Q}^\dagger$: antichiral multiplet transforming in the antifundamental representation of $U(N_c)$ and fundamental representation of $SU(N_f)$
- **Loop** Φ: vector multiplet transforming in the adjoint representation of $U(N_c)$

Note in particular that the loop is consistent with our notation for arrows as the adjoint representation it denotes carries one each of fundamental and antifundamental indices. (Matter can also appear in more general representations of the gauge group, but for the purposes of this thesis we will be content with the options presented above.)

Restricting to $\mathcal{N} = 2$ allows more condensed quiver notation. In the case of unitary gauge groups, $\mathcal{N} = 2$ matter always comes packaged in a hypermultiplet. $\mathcal{N} = 1$ language encodes it as two opposite arrows but since they always come as a pair, we can represent the $\mathcal{N} = 2$ hypermultiplet as a single unoriented line. And because $\mathcal{N} = 2$ gauge groups always bring an extra chiral multiplet "loop", the $\mathcal{N} = 2$ quiver can simply absorb the loop into the definition of a gauge node.

This recipe for $\mathcal{N} = 2$ quivers can only generate undirected quivers. However, directed edges have appeared in the literature since their first appearance in [11]. Such quivers are described as *non-simply laced* on account of their resemblance to non-simply laced Dynkin diagrams (i.e. of types $BCFG$). (We stress that connecting two nodes with multiple undirected edges does not make a quiver non-simply laced under to this convention.) They feature prominently in Chapter 4.

Quivers need not have (special) unitary gauge and flavor groups. Families of *orthosymplectic quivers* have been studied in [16, 19–23]. Such quivers exhibit an alternating pattern of (special) orthogonal and symplectic nodes. A link between an orthogonal and symplectic node represents matter in the *half-hypermultiplet* representation: essentially a hypermultiplet with a reality condition [19]. The quiver



(2.98)

is one such example. More general examples, mixing unitary nodes alongside orthosymplectic ones, have also been studied. In this thesis we restrict to purely unitary theories.

2.2.4 $3d$ $\mathcal{N} = 4$

Although the formalism described in the previous sections was introduced in the context of four-dimensional theories, it is easily adapted for work in three dimensions, where most of the discussion presented in this thesis takes place. For example, the quiver description can be adopted without modification. For an overview of general $3d$ $\mathcal{N} = 4$ features see [24].

A popular way to reduce the dimension of a theory is to *compactify* it. In the simplest case some of the infinite directions of space-time are replaced by circles – the theory is said to be put on a torus – and their radii are shrunk to zero. For concreteness, say we take a four-dimensional theory, whose space-time symmetry (neglecting translations) is the Lorentz group $SO(3, 1)$. It includes $SO(2, 1)$ as its subgroup and so every $SO(3, 1)$ representation must decompose into irreducible representations of $SO(2,1)$. This type of reduction turns four-dimensional $\mathcal{N}=2$ theories into their three-dimensional $\mathcal{N}=4$ counterparts. The increase in $\mathcal{N}$ signifies that the two independent sets of supercharges in four dimensions decompose further into 4 independent sets of three-dimensional supercharges. The total number of supercharge components, however, remains constant. Since there are eight of them, we say that both $4d$ $\mathcal{N}=2$ and $3d$ $\mathcal{N}=4$ are *theories with eight supercharges*[3].

The R-symmetry of $3d$ $\mathcal{N}=4$ theories is expanded to $SO(4)$, which rotates the four supercharges among each other, and decomposes as $\mathfrak{su}(2)_{\mathcal{H}} \times \mathfrak{su}(2)_{\mathcal{C}}$; this fact will be relevant when we discuss moduli spaces.

We still refer to $3d$ $\mathcal{N}=4$ representations by their $4d$ $\mathcal{N}=2$ names, and most of the internal effects of dimensional reduction are immaterial for our purposes. The hypermultiplet, for example, still contains two complex scalars.

The vector multiplet, on the other hand, gains a second complex scalar from its vector component field A_μ: without loss of generality, assume we compactify along the third dimension so $A_3 = \sigma$ forms the trivial representation under $SO(2,1)$. Moreover the remaining vector $A_i = (A_0, A_1, A_2)$ defines a two-form field strength $F = \mathrm{d}A$ which is Hodge-dual to the one-form $\star F$. A one-form can be (locally) interpreted as the exterior derivative of a scalar field γ, i.e. $\star F = \mathrm{d}\gamma$. We call γ the *dual photon*. Together with σ and the complex scalar φ contained within the chiral multiplet Φ, it forms one of four real scalar degrees of freedom in $3d$ $\mathcal{N}=4$ multiplets. φ and A_3 together transform as $(\mathbf{1},\mathbf{3})$ under $\mathfrak{su}(2)_{\mathcal{H}} \times \mathfrak{su}(2)_{\mathcal{C}}$, so we can bundle them together into a new representation ϕ with three real components. The dual photon is invariant under both factors. The vector multiplet also includes a Dirac spinor λ.

Note that both $3d$ $\mathcal{N}=4$ hypermultiplets and vector multiplets contain precisely four real scalar degrees of freedom. This is the first sign of a very special property of three-dimensional theories with eight supercharges: loosely speaking, one can exchange gauge and matter fields to get another consistent $3d$ $\mathcal{N}=4$ theory with closely related properties. We call this property *three-dimensional mirror symmetry*, as it relates "mirrored" pairs of theories and will briefly return to it in Section 2.3.7.

Three-dimensional $\mathcal{N}=4$ actions admit a supersymmetric Chern-Simons term [25]

$$\frac{k}{4\pi}\int \mathrm{tr}\left(A \wedge \mathrm{d}A - \frac{2i}{3} A \wedge A \wedge A + i\bar{\lambda}\lambda + 2D\sigma\right) \tag{2.99}$$

but $k \neq 0$ has the effect of removing massless degrees of freedom in the IR (and

[3]This is a frustrating convention as "supercharge" typically refers to a spinor representation formed of supercharge components. However, in this instance, it refers to the components themselves. So a more accurate description would be *theories with eight supercharge components*. In fact, it's now clear why $4d$ $\mathcal{N}=2$ turns to $3d$ $\mathcal{N}=4$ but remains a "theory with eight supercharges": the eight components remain but are repackaged into a larger number of smaller spinors.

furthermore reduces supersymmetry to $\mathcal{N}=3$). Since our interest in these theories stems from the complex structure of their IR degrees of freedom, we want to preserve as many as possible and uniformly set k to 0.

2.3 Moduli space

Among the simplest aspects of a quiver theory one can study is its *moduli space*, or the set of all admissible vacua.

Let $|\Omega\rangle$ be a vacuum state, defined by the property that the energy $\langle\Omega|\, H\, |\Omega\rangle$ is minimised. The vacuum preserves the space-time symmetry of the theory, so two distinct vacua can only differ in the *vacuum expectation values* (VEVs) $\langle\Omega|\, \mathcal{O}\, |\Omega\rangle$ where $\mathcal{O}$ is a Lorentz-invariant operator. Each VEV is therefore a (Lorentz) scalar. The space of all vacua is called the *moduli space*, and it is parametrised by admissible values of VEVs. Note that this is the site of a crucial conceptual switch: we transition from talking about *theories* to talking about *geometrical spaces*.

Quantum field theories can behave very differently at various energy scales and some theoretical approaches are only suitable in specific energy regimes. For example, quivers are typically specified using UV data. However, vacua, being by definition a low-energy concept, are naturally studied in IR and renormalisation can have significant effects on a theory's moduli space. Neglecting these so-called *quantum corrections* for now, one can schematically expand the scalar part of the Lagrangian in (2.97) as

$$\mathcal{L}^{\mathcal{N}=4}_{\text{scalar}} = \frac{1}{g^2}\text{tr } F(\gamma)^2 + \frac{1}{g^2}\text{tr } \partial_i\phi^j\partial^i\phi_j + \text{tr } \partial_i q\partial^i q^\dagger + \text{tr } \partial_i\tilde{q}\partial^i\tilde{q}^\dagger - V(\gamma,\phi,q,q^\dagger,\tilde{q},\tilde{q}^\dagger). \tag{2.100}$$

where we stressed that F is just the repackaged dual photon γ. A choice of $(\gamma,\phi,q,q^\dagger,\tilde{q},\tilde{q}^\dagger)$ such that $V(\gamma,\phi,q,q^\dagger,\tilde{q},\tilde{q}^\dagger) = 0$ corresponds to a choice of vacuum. While generic non-supersymmetric theories will typically have unique vacua, or perhaps a discrete set, supersymmetric theories tend to have "flat directions" in the potential V. In other words the vacua form continuous spaces, which may (and often do) contain singularities.

The F and D fields in the Lagrangian are non-dynamical and so can be easily substituted by solutions to their Euler-Lagrange equations. The potential then simplifies to the schematic form

$$V(\gamma,\phi,q,q^\dagger,\tilde{q},\tilde{q}^\dagger) = \frac{1}{2g^2}D^A D_A + \sum_{\{\varphi\}} |F_\varphi|^2 \tag{2.101}$$

where

$$D^A = \sum_{\{\varphi\}} \mathrm{tr}\ \varphi^\dagger T^A \varphi \tag{2.102}$$

$$F_\varphi = \frac{\partial \mathcal{W}}{\partial \varphi} \tag{2.103}$$

with $\{\varphi\} = \{\gamma, \phi, q, q^\dagger, \tilde{q}, \tilde{q}^\dagger\}$ the set of scalar fields, A an adjoint index, T^A the generator of the gauge group representation under which φ transforms and tr the corresponding trace. Note that F is here the auxiliary scalar field and that both D and F appear as squares so must vanish to globally minimise the potential. (Non-local minima tend to break supersymmetry whereas the global minimum maximally preserves it.)

There are three classes of vacua for $3d$ $\mathcal{N} = 4$ theories encoded by the potential V:

- **Higgs branch** $\mathcal{H}$: All scalar VEVs in vector multiplets vanish and hypermultiplet scalars take values constrained by $V(0, 0, q, q^\dagger, \tilde{q}, \tilde{q}^\dagger) = 0$. The forced vanishing of F and D terms defines a set of algebraic equations for hypermultiplet scalars. Field configurations differing only in the choice of gauge are identified. $\mathcal{H}$ is an affine algebraic variety. Only the $\mathfrak{su}(2)_\mathcal{H}$ part of R-symmetry acts non-trivially on this space.

- **(Classical) Coulomb branch** $\mathcal{C}_{\mathrm{cl}}$: All scalar VEVs in hypermultiplets vanish and scalars in vector multiplets take values in the zero locus of $V(\gamma, \phi, 0, 0, 0, 0)$. The D-terms force a set of commutation relations and $\mathcal{C}_{\mathrm{cl}}$ is therefore parametrised by scalars in the Cartan subalgebra of the gauge group $\mathcal{G}$. Only the $\mathfrak{su}(2)_\mathcal{C}$ part of R-symmetry acts non-trivially on this space.

- **Mixed branches** $\mathcal{M}_i$: Some hypermultiplet and some vector multiplet VEVs are non-zero. This case is usually omitted as it can be treated by tools developed for Higgs and Coulomb branches. Both factors of the R-symmetry act non-trivially.

The classical Coulomb branch then undergoes important quantum corrections which "enhance" it to the full (and strictly larger) Coulomb branch $\mathcal{C}$.

The overall moduli space $\mathcal{M}$ is the union of the Higgs branch, Coulomb branch, and typically several mixed branches:

$$\mathcal{M} = \mathcal{H} \cup \mathcal{C} \cup \bigcup_i \mathcal{M}_i \tag{2.104}$$

Notably, each branch carries *hyper-Kähler* structure, implying that its real dimension is divisible by 4 and that it carries three complex structures satisfying quaternion-like relations. More information on hyper-Kähler spaces is provided in Section 2.3.2

The various branches of the moduli space intersect at the *origin* where all scalar fields vanish. As the title of the thesis suggests, we will focus almost exclusively on the Coulomb branch, with very infrequent mentions of the Higgs branch and virtually no appearance of a mixed branch.

We will look at the Higgs and Coulomb branches in more detail but first we need to introduce a concept central to their study: the chiral ring.

2.3.1 Chiral ring

We restrict our attention to *chiral* operators which break one half of $\mathcal{N} = 4$ supersymmetry, i.e. $[Q_b, \mathcal{O}] \neq 0$ for Q_b drawn from a two-dimensional subspace of supercharges[4]. Operators with this property are also called *half BPS*.

Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be two such operators; moreover, since we are interested in moduli spaces, the operators will be taken as bosonic and Lorentz-invariant. Then, for the two-dimensional space of preserved supercharges Q_p:

$$[Q_p, \mathcal{O}_1\mathcal{O}_2] = [Q_p, \mathcal{O}_1]\mathcal{O}_2 + \mathcal{O}_1[Q_p, \mathcal{O}_2] = 0 \qquad (2.105)$$

and so $\mathcal{O}_1\mathcal{O}_2$ is also chiral. Since linear combinations of chiral operators are also (trivially) chiral, we say that such operators form a (generically non-commutative) *chiral ring.*

Let us investigate some properties of vacuum expectation values of chiral ring operators. A fundamental relation between supercharges states that $\{Q_p, \tilde{Q}_p\} \propto P$, the momentum operator, which acts by translating operators. So the space-time variation of $\mathcal{O}$ can be quickly computed through the supersymmetric generalisation of the Jacobi identity:

$$[P, \mathcal{O}] \propto [\{Q_p, \tilde{Q}_p\}, \mathcal{O}] = -\{[\tilde{Q}_p, \mathcal{O}], Q_p\} - \{[\mathcal{O}, Q_p], \tilde{Q}_p\} = \{Q_p, [\tilde{Q}_p, \mathcal{O}]\} \qquad (2.106)$$

However, any VEV of the form $\langle\{Q_p, \mathcal{O}'\}\rangle$ vanishes and consequently the VEV $\langle\mathcal{O}(x)\rangle$, which we may *a priori* expect to vary over space-time, is necessarily constant.

Finally we consider the space-time variation of the product operator $\mathcal{O}_1(x)\mathcal{O}_2(0)$.

$$[P, \mathcal{O}_1(x)\mathcal{O}_2(0)] \propto [\{Q_p, \tilde{Q}_p\}, \mathcal{O}_1(x)\mathcal{O}_2(0)] = \{Q_p, [\tilde{Q}_p, \mathcal{O}_1(x)\mathcal{O}_2(0)]\} \qquad (2.107)$$

[4]Note that this subspace has *four* supercharge components, since each supercharge is a two-dimensional spinor.

following the same manipulations as above. It follows that $\partial\langle\mathcal{O}_1(x)\mathcal{O}_2(0)\rangle = 0$, i.e. the product is independent of the operators' separation. We can therefore move $\mathcal{O}_1(x)$ to infinity and use the cluster decomposition principle to separate the VEV:

$$\begin{aligned}\langle\mathcal{O}_1(x)\mathcal{O}_2(0)\rangle &= \lim_{x\to\infty}\langle\mathcal{O}_1(x)\mathcal{O}_2(0)\rangle = \lim_{x\to\infty}\langle\mathcal{O}_1(x)\rangle\langle\mathcal{O}_2(0)\rangle \\ &=\langle\mathcal{O}_1(0)\rangle\langle\mathcal{O}_2(0)\rangle = \langle\mathcal{O}_1\rangle\langle\mathcal{O}_2\rangle\end{aligned} \tag{2.108}$$

where we used the fact that $\langle\mathcal{O}_i(x)\rangle$ are constant and in the last step suppressed space-time dependence.

To recapitulate, the VEV of a chiral operator is a single complex number, products are also chiral and the VEV of a product is simply the product of the VEVs. Moreover, sums of operators are chiral and the identity is also a chiral operator. The set of chiral ring VEVs therefore satisfies the necessary properties of a *commutative ring*. It is common (though imprecise) to refer to "the ring of chiral operator VEVs" as "the chiral ring", and we will follow this convention throughout the rest of this work.

Now pick a branch of the moduli space, say the Coulomb branch, and fix a $\mathcal{N} = 2$ subalgebra, which is equivalent to choosing a complex structure. Coulomb branch operators[5] form $\mathfrak{su}(2)_{\mathcal{C}}$ representations, whose action rotates the choice of complex structure (or $\mathcal{N} = 2$ subalgebra). The operator can be represented by its highest weight component under $\mathfrak{su}(2)_{\mathcal{C}}$, which turns out to be chiral for a judicious choice of supercharges. So elements of a certain chiral ring are in one-to-one correspondence to $\mathfrak{su}(2)_{\mathcal{C}}$ multiplets of Coulomb branch operators, because they form their highest weight components. We call this particular ring the *Coulomb branch chiral ring* and denote it $\mathbb{C}[\mathcal{C}]$[6]. (Put aside any worries about the fact that $\mathbb{C}[\mathcal{C}]$ would traditionally stand for the ring of holomorphic functions on $\mathcal{C}$. We will get to this point very soon.)

The chiral rings studied in this work can be presented in the following form of a freely generated ring quotiented by an ideal:

$$R = \mathbb{C}[\mathcal{O}_1, \mathcal{O}_2, \dots]/\mathcal{I} \tag{2.109}$$

We will refer to the $\mathcal{O}_i$ – which stand in for VEVs of gauge-invariant chiral operators – as *generators*. Elements of $\mathcal{I}$ are called *relations* and we usually find that the ideal is non-trivial but finitely generated.

[5]By which we mean their VEVs!
[6]A detailed recent account of this construction can be found in [26], albeit for the more complicated (but relevant) example of a theory on a sphere.

2.3.2 Interlude: Hyper-Kähler spaces

The presence of eight supercharges implies that the Higgs and Coulomb branches each carry three independent complex structures, imbuing them with overall hyper-Kähler structures, which are roughly the quaternionic equivalent of a complex manifold; as a consequence their real dimension is divisible by four. Hyper-Kähler spaces were first named in [27] and appeared in the physical literature in [28], describing a Higgs branch. Other than the three complex structures, such spaces also carry a triplet of (real) symplectic two-forms ω. We arbitrarily select one complex structure; an $SU(2)$ symmetry rotates between the possible choices. Given a fixed complex structure, we can also combine two of the real symplectic two-forms into a complex symplectic two-form; from here on out, when we mention the symplectic form, we will be referring to this complex two-form.

The $SU(2)$ symmetry also acts on holomorphic and anti-holomorphic coordinates of the hyper-Kähler space. For example, the space $\mathbb{H} = \mathbb{C}^2$ carries two holomorphic coordinates z_1, z_2 and two anti-holomorphic coordinates $\bar{z}_1, \bar{z}_2$ [29]. They form $SU(2)$ doublets $\gamma_1 = (z_1, \bar{z}_2)$ and $\gamma_2 = (z_2, \bar{z}_1)$ and z_1, resp. z_2, are their highest weight representatives. Indeed any monomial in z_1 and z_2, i.e. a homogeneous holomorphic function, is the highest weight of a suitable $SU(2)$ representation. Similarities to the discussion of chiral rings just a few paragraphs above are not accidental; in fact they are the central prerequisite for the line of inquiry advocated in this thesis.

It is generally believed that, at least for three-dimensional $\mathcal{N} = 4$ Higgs and Coulomb branches, *Higgs (Coulomb) branch chiral rings are isomorphic to the ring of holomorphic functions on the Higgs (Coulomb) branch moduli space.*

This statement ties together supersymmetric gauge theory and algebraic geometry through the medium of Hilbert's *Nullstellensatz*. Let J be an ideal of a ring R, $V(R)$ the set of points in the ambient space on which it vanishes, $I(X)$ the ideal of polynomials vanishing on a set of points X and $\sqrt{J}$ the radical of J. Then the *Nullstellensatz* states that

$$I(V(J)) = \sqrt{J}. \tag{2.110}$$

So finding the set of vanishing polynomials on $V(J)$ is *almost* a left inverse of V, and an honest left inverse if J is a radical ideal (i.e. equal to its own radical). It is worth flagging here that switching between geometry and algebra carries with it a risk and sometimes our tools may only find the radical $\sqrt{J}$ rather than J (the true ring of chiral ring VEVs). See [30] for a striking example in $5d$ Higgs branches at infinite coupling: the chiral rings in question include nilpotent elements which are absent from a reconstruction using three-dimensional Coulomb branches, since this technique is only sensitive to the radical of ring relations $\mathcal{I}$ in (2.109).

However, there are no nilpotent elements in $3d$ $\mathcal{N} = 4$ Coulomb branches and we can safely study it (as a geometric space) by analysing the ring of (physical) chiral operators. If we find the chiral ring is isomorphic to the coordinate ring of a particular geometric space, we have good grounds to claim that the Coulomb branch is isomorphic to that very space. That is the general motivation of this thesis.

Let G be a Lie group with an action on a hyper-Kähler space $\mathcal{M}$, i.e. G is a continuous symmetry[7]. Then there exists a *moment map* $\mu : \mathcal{M} \to \mathrm{Lie}(G)^*$ which encodes flows along the manifold. In particular, if ξ^X is the vector field preserving the symplectic (and complex) structure generated from $X \in \mathrm{Lie}(G)$, we have that

$$\mathrm{d}\left(\mu\left(X\right)\right)\left(\xi^Y\right) = \omega\left(\xi^X, \xi^Y\right) = \xi^{[X,Y]} \tag{2.111}$$

for any $X, Y \in \mathrm{Lie}(G)$.

The symplectic form also implies the existence of a Poisson bracket between holomorphic functions on $\mathcal{M}$ satisfying the following property:

$$\{\mu(X), \mu(Y)\} = \mu\left([X, Y]\right). \tag{2.112}$$

Finally we point out that μ is a function from the space $\mathcal{M}$ to the dual of the symmetry's Lie algebra – in other words it is a coadjoint-valued function. At the same time μ is holomorphic thanks to the hyper-Kähler structure. Taken together, the dim G components of μ must correspond to (some) chiral ring elements. Consequently, the chiral ring includes at least enough independent operators to fill out a coadjoint representation of the global symmetry of a given branch of the moduli space. Much of this thesis is concerned with constructing the Coulomb branch moment map for various families of theories.

2.3.3 Higgs branch

We provide a very terse description of the Higgs branch. The interested reader is advised to look at any one of several historical or contemporary treatments [20, 23, 29, 31–34] for more details and concrete examples.

The Higgs branch is the space of vacua in which all scalar VEVs associated to vector multiplets vanish and only VEVs associated to gauge-invariant combinations of scalars in the theory's hypermultiplets are allowed to take non-zero values. If we think of $\{V = 0\}_{\mathcal{H}}$ as the set of Higgs branch constraints, i.e. the set of equations coming from D and F terms after vector multiplet scalars are set to zero, we can write

$$\mathcal{H} = (\mathbb{C}^{2\times\text{num. of hypermultiplets}} / \{V = 0\}_{\mathcal{H}})/\mathcal{G} \tag{2.113}$$

[7]Note that $\mathcal{G}$ is the theory's gauge symmetry and G is the symmetry of its space of vacua. The two are not independent but their relation is complex and involves the matter representation $\mathcal{R}$. The curious reader should keep reading on: this connection lies at the core of this thesis.

where we also indicated that field configurations related by the action of the gauge group $\mathcal{G}$ are to be identified.

Standard non-renormalisation arguments [35] show that Higgs branches do not receive quantum corrections, so classical analysis is entirely sufficient. Moreover, they are largely robust under dimensional reduction, although a few quantum-mechanical effects appear in other dimensions. As studied in [36–43] and more recently in [14, 17], Higgs branches of supersymmetric theories in five and six dimensions (and eight supercharges) "enhance" in the UV as gauge coupling becomes infinite and instanton operators, resp. tensionless strings, turn massless. Four dimensional physics offers an analogue in *Argyres-Douglas* points [10, 44, 45]. In the present setting of three dimensions such matters need not concern us and we shall be satisfied with classical computations of the type described above.

As an example take $U(2)$ SQCD with 4 flavors and let q_p^a be a scalar field with gauge index a and flavor index p and $\tilde{q}_a^p$ the conjugate field. The F terms in (2.103) imply

$$q_p^a \tilde{q}_a^q = 0 \tag{2.114}$$

$$q_a^{\dagger p} q_p^b - \tilde{q}_p^{\dagger a} \tilde{q}_b^p = 0 \tag{2.115}$$

Define the "meson matrix" of gauge-invariant operators

$$M_p^q = q_p^a \tilde{q}_a^q \tag{2.116}$$

and notice that

$$M_p^q M_q^r = q_p^a \tilde{q}_a^q q_q^b \tilde{q}_b^r = 0 \tag{2.117}$$

The flavor group $SU(4)$ also comes with ϵ_{pqrs} and ϵ^{pqrs} tensors which allow us to check the rank of M:

$$\epsilon^{pqrs} \epsilon_{p'q'r's'} M_p^{p'} M_q^{q'} M_r^{r'} = (\epsilon^{pqrs} q_p^a q_q^b q_r^c)(\epsilon_{p'q'r's'} \tilde{q}_a^{p'} \tilde{q}_b^{q'} \tilde{q}_c^{r'}) = 0 \tag{2.118}$$

since for every term at least two of a, b, c must be the same and (wlog assuming $a = b$) $q_p^a q_q^b$ is symmetric in $p \leftrightarrow q$ but contracted to a tensor antisymmetric in the same indices. Therefore the rank of M is at most 2.

The attentive reader might notice we never made use of the D-term and indeed they are secretly made redundant by the imposition of gauge invariance, or rather invariance under the complexified gauge group; for details see [46].

The Higgs branch described above is our first example of a *nilpotent orbit closure*, a space parametrised by a single (co)adjoint matrix with a nilpotency condition (here $M^2 = 0$) and potentially other relations (here rank $M \leq 2$). In fact, M is secretly the Higgs branch moment map. This particular space is the next-to-minimal nilpotent orbit closure of $\mathfrak{su}(4)$, also known as $\bar{\mathcal{O}}_{(2^2)}$ [20, 47, 48], and consists of traceless matrices conjugate to

$$X_{(2^2)} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{2.119}$$

2.3.4 Interlude: Nilpotent orbits

Nilpotent orbits [49] form an important class of hyper-Kähler spaces, largely due to their simplicity: while any hyper-Kähler space must have at least one coadjoint representation's worth of operators to form the moment map, the coordinate ring of a nilpotent orbit of an algebra $\mathfrak{g}$ is generated by a *single* generator in the coadjoint representation of $\mathfrak{g}$ [50]. The moment map is then precisely the set of the coordinate ring's generators. Relations between ring elements can be read off from constraints on the moment map. As a result, identifying the space as a nilpotent orbit greatly simplifies matters, as we no longer have to talk about dim $\mathcal{M}$ operators subject to a (usually fairly large) number of relations. Instead, we have one moment map μ with only a handful of constraints specified by its contractions with other copies of itself or available invariant tensors.

Nilpotent orbits of $\mathfrak{g}$ are defined as coadjoint[8] orbits of nilpotent matrices in $\mathfrak{g}$. To illustrate the concept we provide a full characterisation of nilpotent orbits of $\mathfrak{sl}(n,\mathbb{C})$. Precise definitions for the remaining classical groups can be found e.g. in [49].

The first step is to take a nilpotent element $X \in \mathfrak{sl}(n,\mathbb{C})$, i.e. $X^k = 0$ for some k. It can be transformed to its Jordan-normal form

$$X_{(\lambda_1,\ldots,\lambda_N)} = \begin{pmatrix} J_{\lambda_1} & & & \\ & J_{\lambda_2} & & \\ & & \ddots & \\ & & & J_{\lambda_N} \end{pmatrix} \tag{2.120}$$

[8]Nilpotent orbits can also be defined as *adjoint* orbits of nilpotent matrices, but as long as $\mathfrak{g}$ is semisimple there is a one-to-one correspondence between coadjoint and adjoint nilpotent orbits. Since our moment maps are coadjoint, we stick with the appropriate, coadjoint definition.

where J_{λ_i} is the $\lambda_i \times \lambda_i$ block matrix

$$J_{\lambda_i} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \tag{2.121}$$

and λ_i appear in descending order, i.e. $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$. Note that $\sum_{i=1}^{N} \lambda_i = n$, so the set of Jordan-normal matrices of the form (2.120) is in one-to-one correspondence to partitions of λ. We denote a particular partition $\vec{\lambda} = (\lambda_1, \ldots, \lambda_N)$ and use it to label nilpotent orbits. Other classical groups have a similar correspondence between nilpotent orbits and a precisely defined subset of partitions, see [49].

The nilpotent orbit $\mathcal{O}_{\vec{\lambda}}$ is itself defined as

$$\mathcal{O}_{\vec{\lambda}} = \{A^{-1} X_{\vec{\lambda}} A \mid A \in SL(n, \mathbb{C})\}. \tag{2.122}$$

Note that every $X \in \mathcal{O}_{\vec{\lambda}}$ shares the nilpotency condition $X^k = 0$ with the "prototype" element $X_{\vec{\lambda}}$.

Let us consider nilpotent orbits of $\mathfrak{sl}(2, \mathbb{C})$ for the sake of concreteness. There are two partitions $\vec{\lambda}$ of $n = 2$: $(1, 1) = (1^2)$ or (2). The first partition corresponds to the Jordan-normal matrix

$$X_{(1^2)} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{2.123}$$

whose orbit is a single point:

$$\mathcal{O}_{(1^2)} = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}. \tag{2.124}$$

The case of $\vec{\lambda} = (2)$ is more interesting:

$$X_{(2)} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{2.125}$$

and the orbit is now larger:

$$\mathcal{O}_{(2)} = \left\{ A^{-1} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} A \mid A \in SL(n, \mathbb{C}) \right\} \tag{2.126}$$

where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, ad - bc = 1. \tag{2.127}$$

The elements of $\mathcal{O}_{(2)}$ therefore have the form

$$X = \begin{pmatrix} cd & d^2 \\ -c^2 & -cd \end{pmatrix} \tag{2.128}$$

Note that although $X \notin \mathcal{O}_{(1^2)}$, there is a sequence $X_i \in \mathcal{O}_{(2)}$ which can get arbitrarily close to $\mathcal{O}_{(1^2)}$, i.e. $\mathcal{O}_{(1^2)}$ is included in the *closure* $\bar{\mathcal{O}}_{(2)}$ of $\mathcal{O}_{(2)}$. In fact, the closure is just the union of the two orbits:

$$\bar{\mathcal{O}}_{(2)} = \mathcal{O}_{(2)} \cup \mathcal{O}_{(1^2)}. \tag{2.129}$$

Closures of nilpotent orbits are of particular importance to the study of supersymmetric vacua. Many of the theories we investigate will have Coulomb or Higgs branches of precisely this type. Luckily closures are as easily categorised as nilpotent orbits. Take any two partitions $\vec{\lambda}$ and $\vec{\lambda}'$ of λ. The set of partitions is partially ordered by the *domination* relation

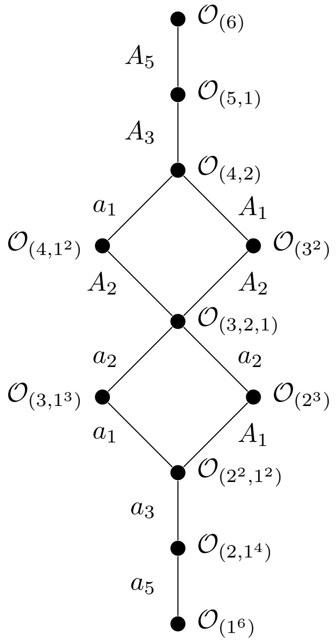
$$\vec{\lambda} \geq \vec{\lambda}' \Longleftrightarrow \forall j : \sum_{i=1}^{j} \lambda_i \geq \sum_{i=1}^{j} \lambda_i'. \tag{2.130}$$

The closure of $\mathcal{O}_{\vec{\lambda}}$ is then defined as the union of all nilpotent orbits $\mathcal{O}_{\vec{\lambda}'}$ such that $\vec{\lambda} \geq \vec{\lambda}'$. Note that this immediately implies the inclusion relation $\bar{\mathcal{O}}_{\vec{\lambda}} \supset \bar{\mathcal{O}}_{\vec{\lambda}'}$.

This structure can be represented with a *Hasse diagram*, which is best introduced on an example. Let $\mathfrak{g} = \mathfrak{sl}(6, \mathbb{C})$. The partitions of 6, and therefore the nilpotent orbits of $\mathfrak{sl}(6, \mathbb{C})$, are exhausted by

$$\{(6), (5,1), (4,2), (4,1^2), (3^2), (3,2,1), (3,1^3), (2^3), (2^2,1^2), (2,1^4), (1^6)\} \tag{2.131}$$

The Hasse diagram reads [29, 49]



(2.132)

The diagram contains a wealth of information about nilpotent orbits of $\mathfrak{sl}(6, \mathbb{C})$. First we need to understand the notation. Each node is a nilpotent orbit labelled by the corresponding partition of 6. A link connecting a higher-placed orbit to a lower-placed one implies that the higher partition dominates the lower and hence the closure of the higher orbit includes the closure of the lower orbit. In fact one can think of the closure of an orbit as "its node and everything connected to it from below".

Finally, each link is labelled by a geometric *transverse slice* between two adjacent orbits. As we have seen, a nilpotent orbit corresponds not only to a partition, but also to a nilpotent element $X \in \mathfrak{g}$. The Jacobson-Morozov theorem allows one to complete this element to a standard $\mathfrak{sl}(2, \mathbb{C})$ triple $\{X, Y, H\}$ and define the affine space [21, 51]

$$S_X = X + \ker([Y, \cdot]) \tag{2.133}$$

which we can call the *transverse (or Słodowy) slice in* $\mathfrak{g}$. Of particular interest is the intersection of this slice with another nilpotent orbit. Let us consider two nilpotent orbits $\mathcal{O}_{1,2}$ such that $\bar{\mathcal{O}}_1 \supset \mathcal{O}_2$. Select any $X \in \mathcal{O}_2$ and define $S_{\mathcal{O}_1, X} = S_X \cap \mathcal{O}_1$. Then $\bar{\mathcal{O}}_1$ and $S_{\mathcal{O}_1, X} \times \mathcal{O}_2$ are isomorphic in a neighbourhood of X. And this is precisely what the links' labels imply in the Hasse diagram: the transverse slice of a smaller orbit inside the closure of the larger orbit is (isomorphic to) a known geometric space, and the closure of the larger orbit is locally isomorphic to a product of the smaller orbit and the labelled space: for example $\mathcal{O}_{(6)}$ is locally isomorphic to $\mathcal{O}_{(5,1)} \times A_5$. The A_i denote Kleinian singularities $\mathbb{C}^2/\mathbb{Z}_{i+1}$ while the a_i are *minimal* *nilpotent orbits* of the corresponding A_i algebra (ie of $\mathfrak{sl}(i+1,\mathbb{C})$)[9]. The minimal nilpotent orbit is the smallest non-trivial orbit, here $\mathcal{O}_{(2,1^4)}$; it is always unique. The maximal orbit, here $\mathcal{O}_{(6)}$, and the subregular (next-to-maximal), here $\mathcal{O}_{(5,1)}$, are also always uniquely specified.

Closures of nilpotent orbits are generally not manifolds. Instead they form a rich family of *symplectic singularities*, i.e. singular spaces with symplectic structure (plus a few extra conditions). For example, $\mathcal{O}_{(5,1)}$ is a singular subspace of $\bar{\mathcal{O}}_{(6)}$, by which we mean the symplectic structure partially degenerates on it. The Hasse diagram clearly shows that $\bar{\mathcal{O}}_{(5,1)}$ is itself singular in turn. We say that the subspace with the same "degree of singularity" (same degeneration of the symplectic form) is a symplectic leaf of the larger space, e.g. $\mathcal{O}_{(5,1)}$ is a symplectic leaf of $\bar{\mathcal{O}}_{(6)}$, but $\bar{\mathcal{O}}_{(5,1)}$ is not on account of including even more singular subspaces (eg $\mathcal{O}_{(4,2)}$). A symplectic singularity necessarily has a unique top symplectic leaf, although this need not be true of an arbitrary symplectic space.

We will often specify closures of nilpotent orbits in terms of the degree of nilpotency as well as information about the rank or trace of (possibly powers of) each element. But (2.122) makes it clear that it is sufficient to enforce rank or trace constraints on the "prototype" $X_{\vec{\lambda}}$ (or powers thereof) to enforce constraints on every $X \in \mathcal{O}_{\vec{\lambda}}$. Notably, the same constraints also hold for every nilpotent orbit $\mathcal{O}_{\vec{\lambda}}$ contained in the closure $\bar{\mathcal{O}}_{\vec{\lambda}}$. In fact, the full set of rank and trace conditions forms algebraic relations for the closure $\bar{\mathcal{O}}_{\vec{\lambda}}$. Since our study of moduli spaces yields this type of relations, the results must be closures of nilpotent orbits rather than nilpotent orbits *simpliciter*.

From now on, in the interest of brevity, when we identify a moduli space as a nilpotent orbit, we will implicitly understand it as "the closure of this nilpotent orbit". Almost all Coulomb and Higgs branches analysed in this thesis will be closures of nilpotent orbits although a handful will have a more complicated structure. They will, in any case, be examples of symplectic singularities with an associated Hasse diagram.

Many theories whose Higgs or Coulomb branches are nilpotent orbits or other hyper-Kähler varieties were tabulated in [20, 22, 32, 52]. Hasse diagrams were used to study the geometry of nilpotent orbits in [53, 54] and served as a valuable tool in the study of moduli spaces of quiver gauge theoris in [15, 16, 23, 29, 52, 55, 56].

2.3.5 Classical Coulomb branch

The following discussion follows [57, 58].

[9] A_i and a_i do not exhaust the possibilities of transverse slices. There are also the other Kleinian singularities, D_i and E_i, and nilpotent orbits b_i, c_i, d_i, e_i, f_4, g_2, and other much less frequently encountered spaces.

The Coulomb branch is characterised by vanishing hypermultiplet VEVs. The D-term (2.102) reduces to

$$[\phi_A, \phi_B] = 0 \tag{2.134}$$

for A, B ranging over adjoint indices. This implies that the adjoint-valued ϕ_A lie in a maximal commuting subalgebra of Lie($\mathcal{G}$), which is also known as its *Cartan subalgebra.* A generic choice of ϕ_A breaks $\mathcal{G}$ to $U(1)^{\text{rank } \mathcal{G}}$. Each ϕ_A contributes three real scalar degrees of freedom while each $U(1)$ factor brings one extra massless photon. Assuming no Fayet-Iliopoulos or Chern-Simons terms, all 4(rank $\mathcal{G}$) scalars remain massless in quantum theory, i.e. no mass terms are generated by renormalisation.

Since the ϕ_A commute, they can be simultaneously diagonalised (at least for the gauge groups we consider, i.e. unitary, symplectic or orthogonal). For example, for $SU(2)$, the solitary non-vanishing VEV can be gauge-transformed to a canonical form

$$\phi = \begin{pmatrix} s & 0 \\ 0 & -s \end{pmatrix} \tag{2.135}$$

which is unique up to $s \to -s$, the Weyl group of $SU(2)$. The fourth scalar, the dual photon transforms as $\gamma \to -\gamma$ under it (as the Weyl transformation acts by charge conjugation). The classical Coulomb branch is then $((\mathbb{R}^3 - \Delta_{12}) \times S^1)/S_2$.

In general, the classical Coulomb branch takes roughly[10] the form

$$\mathcal{C}_{\text{cl}} = (\mathbb{R}^3 \times S^1)^{\text{rank } \mathcal{G}} / \mathcal{W}_{\mathcal{G}} \tag{2.136}$$

where $\mathcal{W}_{\mathcal{G}}$ is the Weyl group of the overall gauge group. Note that the Weyl group of $U(1)$ is the trivial group.

2.3.6 Quantum Coulomb branch

The Coulomb branch is greatly influenced by quantum effects. The quiver specifies the theory in the UV but the moduli space must be analysed in the IR. Historically the Coulomb branch was first understood for theories with the gauge group $\mathcal{G} = U(1)$ [57, 59]. As described above, the photon could be dualised into a scalar and massive hypermultiplets and W-bosons could be explicitly integrated out. The theory's low energy description is perturbatively renormalised, but the process mercifully terminates after one-loop corrections to provide a renormalised Lagrangian encoding the hyper-Kähler metric on the Coulomb branch. There are several issues with this approach which necessitate an overall change of strategy. The first is that dualisation of non-abelian vector multiplets remains an open problem. One can try to sidestep

[10] Here we gloss over issues arising from possibly enhanced gauge symmetry, see [58]; the classical Coulomb branch is strictly speaking smaller than what we indicate here.

this issue by considering a theory whose gauge group $\mathcal{G}$ is broken to the maximal torus (Cartan subgroup) $U(1)^{\operatorname{rank}\mathcal{G}}$ by the choice of vacuum, but such an approach is not reliable everywhere on the Coulomb branch. In particular, some choices of VEVs leave certain W-bosons massless. But even abelian theories with $\mathcal{G} = U(1)^n$ are problematic as the complexity of calculations quickly pulls results out of reach.

A more fruitful approach[11] considers the emergence of a certain type of local disorder operator, the 't Hooft *monopole operator* [62] in the IR. It has a straightforward UV description, which is then allowed to flow into the infrared. We first take the vector of real scalars $\vec{\phi}_i$, $1 \leq i \leq \operatorname{rank}\mathcal{G}$ along with the dual photon γ^i (normalised to periodicity 2π) and define

$$e^{\chi_i} = e^{\frac{2\pi}{g^2}\phi_i^3 + \mathrm{i}\gamma_i} \tag{2.137}$$

$$\varphi_i = \phi_i^1 + \mathrm{i}\phi_i^2. \tag{2.138}$$

Crucially, both types of operators are chiral and their VEVs can therefore parametrise supersymmetric vacua. (2.137) is the classical expression for the monopole while (2.138) is a complex scalar operator which we sideline for now. For now it is important that the monopole "eats up" two scalar degrees of freedom.

Roughly speaking, a monopole operator's semi-classical contribution to the (Euclidean) path integral is $e^{-S} = e^{-\chi\cdot\beta_i}$, with β_i belonging to a set of simple roots of the GNO-dual algebra, selected so that $\mathrm{Re}(\chi)\cdot\beta_i = \phi^3\cdot\beta_i \geq 0$ [24, 63, 64]. As long as ϕ^3 is sufficiently large, the classical expression (2.137) describes the magnetic monopole well. But it is clearly not suitable at small ϕ^3 and since $g \to \infty$ as we flow to the IR in three dimensions the domain of validity becomes ever more restrictive. We must therefore look for an alternative description and indeed find one in an explicit SCFT construction [65, 66] of the monopole as a singularity in the fundamental gauge and scalar fields in the Euclidean path integral.

Let us start with the $U(1)$ case. A *bare monopole operator* $V_m(x)$, labelled by m (to which we return shortly) inserts a Dirac monopole singularity at x and modifies the gauge field around the insertion point. In standard spherical coordinates (r, θ, φ) centred around x:[12]

$$A_\pm \simeq \frac{m}{2}(\pm 1 - \cos\theta)\mathrm{d}\varphi \tag{2.141}$$

[11] We follow the discussions of [26, 48, 60, 61]

[12] Monopoles should still preserve the Coulomb branch $\mathcal{N} = 2$ subalgebra, which translates to the BPS condition

$$(\mathrm{d} - iA)\phi^3 = - \star F \tag{2.139}$$

where F is the usual field strength associated to A. To satisfy the equation, an analogous singularity must be inserted in the $\phi^3(x)$ field near x:

$$\phi^3 \simeq \frac{m}{2r}. \tag{2.140}$$

The gauge connections $A_\pm$ are specified in two patches, which can be thought of as the northern and southern hemispheres of an S^2 surrounding x. The two fields must be related by a gauge transformation, constraining m by the Dirac quantisation condition:

$$\exp(2\pi \mathrm{i} m) = 1. \tag{2.142}$$

Abelian monopoles are therefore labelled by integers called *magnetic charges* – and that's almost everything we need to know. This thesis is not at all concerned with, say, profiles of gauge fields. We are interested in vacuum expectation values of operators and how they vary across the moduli space. For now, knowing how to label and relate monopoles in abelian theories is all that matters since *they* provide the VEVs.

The non-abelian case is significantly more involved but crucial to our work. A Dirac monopole is an abelian singularity, so the set of (bare) non-abelian monopoles should correspond to all the ways of inserting $U(1)$ into the gauge group $\mathcal{G}$ up to choice of gauge; let us label such operators v_m. But two insertions related by choice of gauge should be identified. The honest, gauge-invariant, physical monopole operator must then be defined as the *sum* over v_m insertions where m are related by a gauge transformation; we denote this gauge-invariant insertion V_m. We see then that the gauge-invariant expressions are labelled by $m \in \mathrm{Hom}(U(1), \mathcal{G})/\mathcal{G} \simeq \mathrm{Hom}(U(1), \mathbb{T})/\mathcal{W}_\mathcal{G}$, the set of cocharacters of $\mathcal{G}$. Another way to see this is to write down the generalised Dirac quantisation condition [67]

$$\exp(2\pi \mathrm{i} m) = \mathbb{1}_\mathcal{G}. \tag{2.143}$$

There is an isomorphism between cocharacters and Weyl orbits in the coweight lattice $\Lambda_w^\vee$, or equivalently the weight lattice of the GNO (i.e. Langlands) dual group ${}^L\mathcal{G}$ [68]. Each Weyl orbit comes with one dominant weight lying in the principal Weyl chamber. Let m^D be such a dominant weight. We now have

$$m^D \in \Gamma_{{}^L\mathcal{G}}/\mathcal{W}_{{}^L\mathcal{G}}. \tag{2.144}$$

As long as our gauge group is fully unitary, i.e. $\mathcal{G} = \prod_i U(r_i)$, then, since ${}^L U(r) = U(r)$, we have

$$m^D \in \prod_i \mathbb{Z}^{r_i}/S_{r_i}. \tag{2.145}$$

where the quotient by $S_{r_i} \simeq \mathcal{W}_{U(r_i)}$ can be taken to enforce the order $m^D_{i,1} \geq m^D_{i,2} \geq \cdots \geq m^D_{i,r_i}$. So we can now label our monopole operators V_{m^D} without fear of leaving any of them out.

Now let us range over $m \in \mathcal{W}_{\mathcal{G}} m^D$, the Weyl orbit of m^D. We have

$$V_{m^D} = \sum_{m \in \mathcal{W}_{\mathcal{G}} m^D} v_m. \tag{2.146}$$

To give a pair of concrete examples, let $\mathcal{G} = U(2)$ (for both cases). Then $m^D = (m_1^D, m_2^D)$ where $m_1^D \geq m_2^D$ and we can insert the gauge invariant monopoles

$$V_{(1,0)} = v_{(1,0)} + v_{(0,1)}, \tag{2.147}$$

$$V_{(1,1)} = v_{(1,1)}. \tag{2.148}$$

Yet again the structure of labels is of singular and overwhelming importance – at least so far. We will refine this point of view shortly. But for now we have learnt enough about bare monopole operators.

Note that the discussion above becomes considerably more involved for quivers with gauge factors other than $U(r)$ such as $SU(n)$, $SO(n)$ or $USp(n)$: in such theories a gauge-invariant monopole insertion may involve sums over abelian monopole insertions labelled by $m \notin \mathcal{W}_{\mathcal{G}} m^D$. This phenomenon, called *monopole bubbling* [69], has been studied in [70–75] and also recently in [60, 76] (with a secret appearance in (2.3) of [77]) using methods closely related to abelianisation (see Section 2.4.2).

Not all monopole operators are bare, however. Monopoles can also be *dressed* by complex scalars (2.138) in vector multiplets. Let us start by dressing the "empty" monopole: the unit operator.

The Harish-Chandra isomorphism states that elements of the centre of the universal enveloping algebra of $\mathcal{G}$ – i.e. gauge-invariant operators – are in one-to-one correspondence to polynomials in φ_i, $1 \leq i \leq \operatorname{rank} \mathcal{G}$, invariant under the Weyl group $\mathcal{W}_{\mathcal{G}}$ of $\mathcal{G}$. If $\mathcal{G} = U(n)$ then the space of such operators is generated by $\operatorname{tr}(\varphi^k) = \sum_{i=1}^n \varphi_i^k$, the Casimir operators of $U(n)$.

Now pick a monopole V_{m^D}. The choice of magnetic charge m^D typically breaks the gauge group to a subgroup. For example, if $\mathcal{G} = U(3)$, we can have $m^D = (1,1,0)$, breaking the gauge group into $\mathcal{G}(m^D) = U(2) \times U(1)$, or the subgroup which leaves m^D invariant. Its Weyl group is reduced to a S_2 subgroup. Now express the monopole as a sum over abelian insertions

$$V_{(1,1,0)} = \sum_{m \in \mathcal{W}_{\mathcal{G}}(1,1,0)} v_m = v_{(1,1,0)} + \cdots \tag{2.149}$$

We dress a polynomial by multiplying the dominant contribution with a suitable polynomial $P(\vec{\varphi}) = P(\varphi_1, \varphi_2, \varphi_3)$ and the sub-dominant contributions $v_{w \cdot m^D}$ by

$P(w \cdot \vec{\varphi}) = P(w(\varphi_1), w(\varphi_2), w(\varphi_3))$, which can be denoted

$$\left[P(\vec{\varphi})V_{(1,1,0)}\right] = \sum_{m \in \mathcal{W}_{\mathcal{G}}(1,1,0)} P(w \cdot \vec{\varphi}) v_m = P(\varphi_1, \varphi_2, \varphi_3) v_{(1,1,0)} + \cdots \tag{2.150}$$

To ensure that the newly defined operator is still gauge-invariant, it must be invariant under the action of the Weyl group. Since a subgroup $S_2 \subset S_3$ is unbroken by the bare monopole's magnetic charge, the sum over $w \in \mathcal{W}_{\mathcal{G}}$ does not permute the first two indices. Consequently dressing the monopole in a polynomial which is *not* invariant under the same S_2 would spoil its Weyl- and hence gauge-invariance. So the rule is that the dressing polynomial's contribution to the dominant insertion must be invariant under the gauge subgroup which leaves the dominant magnetic charge invariant.

Note that the trivial insertion $m = (0, \ldots, 0)$ can still be dressed by polynomials which are symmetric under the full Weyl group of the theory. We will call these scalar gauge invariants the *Casimir operators*, on account of their one-to-one correspondence with the gauge group's Casimir invariants.

This will suffice as an intro to the zoo of Coulomb branch chiral ring elements. However, just specifying – really barely listing – them is not enough. For example, a product of two operators may be equal to a third one; this would constitute a ring relation. And to understand a chiral ring as the coordinate ring of a moduli space, we need to catalogue these relations as well as list elements which they relate. Fortunately several recent works have developed tools addressing precisely these concerns. We turn to them in the section on Coulomb branch methods.

More precise descriptions of monopole operators as operators in the SQFT can be found across several modern treatments [26, 60, 78, 79] from which we draw inspiration for methods described in some of the following sections. Another recent approach models monopole operators using brane insertions in string backgrounds [61, 80]. A mathematical treatment of Coulomb branches and their coordinate rings was provided in [81–83] and recently expanded in [84] for the case of non-simply laced quivers. [85] proved using the mathematical description that Coulomb branches of simply laced quiver gauge theories without loops are symplectic singularities.

Finally, before moving on, we make the first inroad into understanding a principal feature of the Coulomb branch: its symmetry. Take a unitary quiver, i.e. one whose gauge nodes are unitary groups $U(r_i)$. Each such node includes a $U(1)$ factor. The Hodge dual of its field strength $J = \star F^{U(1)}$ is a conserved current on account of the Bianchi identity $\mathrm{d}F^{U(1)}$=0 and independently of equations of motion:

$$\star \mathrm{d} \star J = - \star \mathrm{d}F^{U(1)} = 0. \tag{2.151}$$

The conserved current J is called *topological* due to its relation to twists of the gauge group's principal bundle. Any conserved current indicates the presence of a continuous symmetry by Noether's theorem.

Monopoles can be charged under this symmetry. Their topological charge is given by:

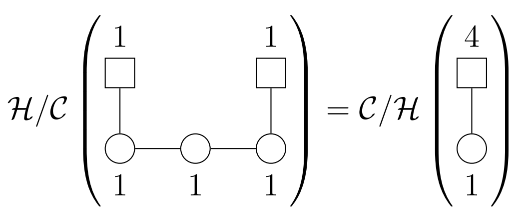
$$q_i(m^D) = \sum_{a=1}^{r_i} m^D_{i,a} \in \mathbb{Z}. \tag{2.152}$$

Note that q_i is invariant under the action of the Weyl group. Each monopole operator can have any combination of integral topological charges (even 0 at every node) while scalar operators are always topologically uncharged.

Assuming that the quiver is good[13] and its gauge group is $\mathcal{G} = \prod_{i=1}^{n} U(r_i)$, we expect the Coulomb branch to carry a $U(1)^n$ symmetry, or equivalently admit a faithful $U(1)^n$ action. The true symmetry of the space might be much larger however, and it generally is in the cases we cover in this thesis.

2.3.7 Mirror symmetry

It was noticed in [59] that $3d$ $\mathcal{N} = 4$ theories come in pairs such that the Higgs branch of one is the Coulomb branch of the other and vice versa. This mysterious property, known as $3d$ *mirror symmetry*, was given a string-theoretic explanation in [86]. Many $3d$ $\mathcal{N} = 4$ theories can be constructed as effective theories for fields living on *branes* in Type IIB string theory; see e.g. [19, 80] for later refinements. The Type IIB S-duality acts e.g. by exchanging $D5$ and $NS5$ or $D1$ and $F1$ branes. Higgs and Coulomb branches can be read off brane theories, provided one rearranges the branes in a suitable way and keeps track of so-called *Hanany-Witten transitions.* It can be shown that the effect of S-duality on the low-energy brane theory is to swap its Higgs and Coulomb branches. Since S-duality maps one brane system to another, the low energy theory of the first system is mirror-dual to the that of the second. For example,



(2.153)

In the years preceding developments covered in the following section, techniques for computing Higgs branches were much more advanced than for Coulomb branches. One would therefore often learn about a theory's Coulomb branch by taking its mirror dual and analysing the mirror's Higgs branch. We will at times use this trick

[13] Which quivers are good is explained under (2.171).

for an independent check of results achieved by standalone methods.

Mirror symmetry continues to attract active interest among researchers. [87] investigated its effect on two-dimensional supersymmetric boundary conditions. [88] recently studied more general classes of $3d$ $\mathcal{N} = 4$ operators which included Higgs and Coulomb branch chiral rings while [89] studied two classes of twist-translated operators in non-abelian theories which are related to, but more refined than the Higgs and Coulomb branch chiral rings, effectively tying together the work in [26, 33, 34, 60, 90]. A recent series of papers [77, 80, 91] revisited the string-theoretic setting of mirror symmetry. A large number of mirror pairs can be found in [20, 32, 52].

2.4 Coulomb branch methods

With essential background out of the way, we can focus on computational methods used to study Coulomb branches. A variety of approaches is available, some of which have already been mentioned. Early attempts focused on computing its metric [59], but such calculations soon became intractable. Later works [65, 66] explicitly constructed monopole operators in the QFT, but this too quickly becomes prohibitive as theories increase in complexity beyond $SU(2)$ SQCD with N_f flavors. [19] (correctly) conjectured an extension of these results and computed quantum numbers of monopole operators for a large class of theories, stating several strong results about Coulomb branches. The conjecture was later verified in [92, 93]. A major breakthrough was reported in [11, 48] where the quantum numbers were combined with the plethystic programme of [94, 95] to create the *monopole formula* which could calculate the Hilbert series of the Coulomb branch chiral ring. In [78, 79] the authors succeed at expressing monopole generators using a novel approach called *abelianisation*, which was further developed by [26, 60, 61, 77, 80, 96, 97].

This thesis documents a novel method synthesising the monopole formula and abelianisation approaches. Some familiarity with these techniques is therefore a prerequisite. The next two sections should provide a relatively self-contained account to bring the reader up to speed.

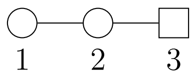
It is helpful to keep in focus why we care about the chiral ring in the first place: it corresponds to the coordinate ring of an algebraic variety, and the variety is exactly the moduli space. But this correspondence is a ring isomorphism and it is not always straightforward to look at two rings and judge whether or not they are isomorphic. You can either express both in some kind of canonical basis, or try to bypass the need for a common basis altogether. Operator counting takes the latter route while the central method of this thesis, developed over subsequent chapters, takes a stab at the former.

Gauge and matter representations

It will prove useful to adopt a particular convention for weights of matter and gauge field representations under the gauge group. As is often the case with conventions, this one was selected for its compatibility with pre-existing tools: the monopole formula (see Sec. 2.4.1) and techniques developed in [60]. There is no particular physical insight in this choice and we include it merely to improve reproducibility of our calculations.

The matter fields transform in a (usually reducible) representation $\mathcal{R}$ while gauge vectors transform under the representation $\mathcal{V}$.

Recall that (in unitary theories) the matter representation consists of two hypermultiplets in mutually conjugate representations. Consider the unitary quiver below, along with its full matter and gauge representations:



(2.154)

$$\mathcal{R} = \{(\pm 1; \mp 1, 0; 0, 0, 0), (\pm 1; 0, \mp 1; 0, 0, 0), (0; \pm 1, 0; \mp 1, 0, 0), (0; \pm 1, 0; 0, \mp 1, 0), (0; \pm 1, 0; 0, 0, \mp 1), (0; 0, \pm 1; \mp 1, 0, 0), (0; 0, \pm 1; 0, \mp 1, 0), (0; 0, \pm 1; 0, 0, \mp 1)\} \tag{2.155}$$

$$\mathcal{V} = \{(0; 0, 0; 0, 0, 0), (0; \pm 1, \mp 1; 0, 0, 0), (0; 0, 0; 0, 0, 0), (0; 0, 0; 0, 0, 0)\} \tag{2.156}$$

Each vector contains charges under $U(1)$, $U(2)$ and $SU(3)$, in this order. Entries of $\mathcal{R}$ clearly describe bifundamental fields charged under adjacent nodes. Gauge representation weights do not "cross" nodes and, in the case of unitary quivers, either carry no charge (and belong to the $U(1)$ factor or Cartan subalgebra of $SU(n)$) or carry one unit each of charge 1 and -1 in two components of the same gauge node; there is one weight for each choice of components. In this particular case, the first vector corresponds to the $U(1)$ photon and the remaining terms are the $U(2)$ gauge bosons.

Both methods driving this thesis, the monopole formula and abelianisation, require that we only use *half* the weights in $\mathcal{R}$ for the formulas and disregard the rest or, equivalently, that we iterate over *pairs of weights* related by a sign flip (i.e. charge conjugation). The choice of representative will never affect the monopole formula, but may result in an overall sign flip in abelianised calculations. However, the sign can always be reabsorbed into the definition of an abelianised variable.

We have found the following convention clean and useful as it produces uniform signs. First, let us label weight components using two indices i and a, where i ranges over gauge nodes and a over the dimensions of its Cartan torus. Then split $\mathcal{R}$ into two disjoint and equally large sets:

$$\mathcal{R} = \mathcal{R}^+_{i,a} \sqcup \mathcal{R}^-_{i,a}, \qquad \forall w \in \mathcal{R} : (w_{i,a} > 0 \Rightarrow w \in \mathcal{R}^+_{i,a}) \wedge (w \in \mathcal{R}^+_{i,a} \Leftrightarrow -w \in \mathcal{R}^-_{i,a}). \tag{2.157}$$

The convention is agnostic about whether a weight with $w_{i,a} = 0$ belongs to $\mathcal{R}^+_{i,a}$ or $\mathcal{R}^-_{i,a}$, but no calculation hinges on that fact, so it may be left indeterminate. The monopole formula is even less picky about signs and we define $\mathcal{R} = \mathcal{R}^+ \sqcup \mathcal{R}^-$ to be *any* partition such that $w \in \mathcal{R}^+ \Leftrightarrow -w \in \mathcal{R}^-$.

2.4.1 Operator counting

The operator counting, or Hilbert series approach to Coulomb branch chiral rings was pioneered in [11]. The two main insights behind this method are that we can often easily identify a set of “basic” symmetries of the Coulomb branch and that we in principle know exactly how many operators carry any particular combination of charges under them. This information is preserved by ring isomorphisms, so it has to be the same for any description of the physical chiral ring (which we can specify) and the coordinate ring of a putative geometric description of the Coulomb branch (which we would like to find) and constitutes a highly non-trivial test which is sometimes sufficient to fully specify the chiral ring presentation in (2.109). Operator counting is a state-of-the-art method for analysing Coulomb branches. It has given rise to vast swathes of novel results, helped identify novel mirror pairs and previously unknown relations between theories [11, 14–16, 20, 22, 32, 52, 98–105] and served as the inspiration behind mathematical work which finally gave the Coulomb branch a rigorous definition [81–83]. See [106] for a longer review of this approach.

To help formalise the following discussion, let us first discuss the concept of a graded ring and its Hilbert series. Consider a ring R which can be decomposed into a (potentially infinite) direct sum of vector spaces R_s for non-negative s

$$R = \bigoplus_{s \in \mathbb{Z}_{\geq 0}} R_s \tag{2.158}$$

such that the grading plays well with multiplication:

$$x \in R_r \wedge y \in R_s \Rightarrow xy \in R_{r+s}. \tag{2.159}$$

We say that R is a *graded ring*. If R is a polynomial ring then the elements of R_s are called *homogeneous polynomials*.

Its Hilbert series is defined as

$$\mathrm{HS}_R(t) = \sum_{s\in\mathbb{Z}_{\geq 0}} (\dim R_s) t^s \tag{2.160}$$

where the dummy variable t is called the *fugacity*. The series can often be explicitly summed up into a rational function.

As an example consider the coordinate ring $R = \mathbb{C}[x,y]/\langle x^2 - y\rangle$ of a parabola. If we assign grade 1 to x and grade 2 to y, we can list all independent monomials forming bases of vector subspaces R_i:

$$\begin{array}{ll} R_0 : & 1 \\ R_1 : & x \\ R_2 : & x^2 = y \\ R_3 : & x^3 = xy \\ R_4 : & x^4 = x^2 y = y^2 \\ \vdots & \vdots \end{array} \tag{2.161}$$

and so on; it is clear that each R_s is one-dimensional. The Hilbert series is therefore

$$\mathrm{HS}_R(t) = 1 + t + t^2 + t^3 + t^4 + \cdots = \frac{1}{1-t}. \tag{2.162}$$

One can also consider multigraded rings which split into vector spaces labelled by $\mathbb{Z}_{\geq 0} \times \mathbb{Z}^n$:

$$R = \bigoplus_{(s,i_1,\ldots,i_n)\in\mathbb{Z}_{\geq 0}\times\mathbb{Z}^n} R_{(s,i_1,\ldots,i_n)} \tag{2.163}$$

such that

$$x \in R_{(r,i_1,\ldots,i_n)} \wedge y \in R_{(s,j_1,\ldots,j_n)} \Rightarrow xy \in R_{(r+s,i_1+j_1,\ldots,i_n+j_n)}. \tag{2.164}$$

The Hilbert series can now be *refined* by introducing fugacities for each component of the grading:

$$\mathrm{HS}_R(t) = \sum_{(s,i_1,\ldots,i_n)\in\mathbb{Z}_{\geq 0}\times\mathbb{Z}^n} (\dim R_{(s,i_1,\ldots,i_n)}) t^s \prod_{k=1}^{n} z^{i_k}. \tag{2.165}$$

For a more sophisticated example than above, and one which uses multigrading, consider the affine variety in $\mathbb{C}^3$ defined by $xy = z^2$. Its coordinate ring is $R =$

$\mathbb{C}[x,y,z]/\langle xy - w^2\rangle$ with grading

$$\begin{aligned} x: &\quad (2,1)\\ y: &\quad (2,-1)\\ w: &\quad (2,0)\end{aligned}$$

Note that the relation $xy - w^2$ is homogeneous in both components of the grading.

We can once again provide a partial list of independent monomials:

$$\begin{array}{rc} R_0: & 1\\ R_2: & x,y,w\\ R_4: & x^2,y^2,xy=w^2\\ R_6: & x^3,x^2y=xw^2,xy^2=yw^2,y^3,w^3\\ \vdots & \vdots \end{array} \tag{2.166}$$

with trivial R_k for k odd. The associated Hilbert series calculation is slightly trickier, but also more insightful. We know that the final result looks like the Hilbert series of the freely generated ring $R_{\text{free}} = \mathbb{C}[x,y,w]$, except we should not double count the ideals $\langle xy\rangle$ and $\langle w^2\rangle$, i.e. we should subtract the contribution from the ideal $(w^2)R$:

$$\mathrm{HS}_R(t,z) = \mathrm{HS}_{R_{\text{free}}}(t,z) - t^4\mathrm{HS}_{R_{\text{free}}}(t,z) = \frac{1-t^4}{(1-t^2z)(1-t^2z^{-1})(1-t^2)}. \tag{2.167}$$

This is, in fact, the Hilbert series of one of the simplest hyper-Kähler varieties, $\mathbb{C}^2/\mathbb{Z}_2$, which happens to be isomorphic to the affine variety defined by $xy = w^2$. It expands as

$$\begin{aligned}\mathrm{HS}_R(t,z) &= 1 + (z+1+z^{-1})t^2 + (z^2+z+1+z^{-1}+z^{-2})t^4 + \cdots = \\ &= \sum_{s=0}^{\infty}\left(\sum_{j=-s}^{s} z^j\right)t^{2s} = \sum_{s=0}^{\infty}\chi_{[2s]}(z)t^{2s}\end{aligned} \tag{2.168}$$

where $\chi_{[2s]}(z) = \sum_{j=-s}^{s} z^j$ is the character of the $\mathfrak{sl}(2,\mathbb{C})$ irreducible representation (*irrep*) $[2s]$. That might not be immediately obvious since the weight content of an irrep is often stated in the basis of fundamental weights, e.g. $[2]$ is the $\mathfrak{sl}(2,\mathbb{C})$ adjoint irrep. But the exponents of z in $\chi_{[2s]}$ correspond to components of weights in the *simple root* basis. This is an unfortunate but necessary awkwardness: highest weights of irreps are most naturally expressed in the fundamental weight basis[14] while unitary quivers provide us with grading in the simple root basis, and we will

[14] All components of the highest weight, expressed in the basis of fundamental weights, must be non-negative integers, and conversely any vector of this form is a valid highest weight uniquely specifying an irrep.

use this convention throughout the thesis.

Coming back to the actual content of the series, we see that the coordinate ring breaks into representations of $\mathfrak{sl}(2,\mathbb{C})$; this is a strong indicator of the space's symmetry under the action of this algebra (and the corresponding Lie group). And indeed we can make it explicit by arranging the three generators into an $\mathfrak{sl}(2,\mathbb{C})$ adjoint matrix:

$$M = \begin{pmatrix} \mathrm{i}w & x \\ y & -\mathrm{i}w \end{pmatrix}. \tag{2.169}$$

We have that $\mathrm{tr}(M) = 0$ and $\det(M) = w^2 - xy = 0$. There is a natural action $A \in SU(2) : M \to AMA^{-1}$ mapping M to another adjoint (traceless) matrix while leaving the determinant relation invariant, i.e. $\det(AMA^{-1}) = \det(M) = 0$. So we can think of the variety as the set of $\mathfrak{sl}(2,\mathbb{C})$ matrices with vanishing determinant, which also happens to be the (closure of the) minimal nilpotent orbit of $\mathfrak{sl}(2,\mathbb{C})$. And although we just explicitly demonstrated that the space is symmetric under $SU(2)$, resp. $\mathfrak{sl}(2,\mathbb{C})$, we could have guessed that it might be – based only on the decomposition of the Hilbert series into $\mathfrak{sl}(2,\mathbb{C})$ characters.

The example demonstrates a very important point. Assuming we can endow our chiral ring with a grading and compute its Hilbert series, we have gained a powerful tool, as the Hilbert series is invariant under ring isomorphisms, at least if they play well with the grading – but hold that thought for now. Assuming we chose our grading well, and we found the chiral ring's Hilbert series, we need only take the result and compare it against a catalogue of Hilbert series of hyper-Kähler spaces to find candidates for the moduli space. But while two isomorphic hyper-Kähler spaces certainly share the exact same Hilbert series – again modulo worries about the grading – two different hyper-Kähler spaces can *share* a Hilbert series.

It is easy to come up with an almost trivial example. Take the $\mathbb{C}^2/\mathbb{Z}_2$ variety we just saw but slightly modify the defining relation to $xy = w(w + c)$, where c is a dimensionful parameter (and *not* a new ring element). Whereas the original space has a singularity at the origin (where the $\mathbb{Z}_2$ action degenerates), this deformed space removes it. However, it is easy to see that the deformation makes no difference to the Hilbert series.

This case provides us with a very important lesson: Hilbert series do not uniquely identify spaces. But they certainly rule some out. And if we can remove dimensionful parameters from the chiral ring description, we can rule out yet more.

There is only one candidate for a dimensionful parameter in a $3d\,\mathcal{N} = 4$ theory[15]: the hypermultiplet masses. Hypermultiplets transforming under the fundamental representation of a gauge group factor (rather than a bifundamental representa-

[15]Recall that we explicitly turned off all Chern-Simons and Fayet-Iloupoulos terms and the gauge coupling is infinite in the IR.

tion under *two* gauge factors) also carry a flavor symmetry, which shows up as a square node in the theory's quiver. Just as circular nodes imply the presence of associated vector multiplets, the square node encodes *background* vector multiplets, whose scalar VEVs are the background magnetic flux and complex mass of a hypermultiplet. We will put background fluxes to the side; see [98] for a practical use of fluxes in Coulomb branch computations and [103] for an investigation of this parameter's effects on the moduli space. The complex mass, on the other hand, will play a role, and at times be explicitly set to 0 but otherwise left arbitrary. Methods which rely solely on operator counting typically turn complex masses off; it is one of the main advantages of the method presented in this thesis that it can handle non-zero masses.

With all this preamble out of the way, we will now proceed to enumerate linearly independent operators in the Coulomb branch chiral ring and reparametrise the Coulomb branch Hilbert series using physical charges.

Recall from Section 2.3.6 that monopole operators can come in bare or dressed forms, where the latter is obtained by multiplying the dominant monopole insertion (i.e. the one associated to the dominant weight of the cocharacter labelling the bare monopole) with a polynomial and Weyl-symmetrising the resulting gauge-dependent operator. This dressing polynomial must in turn be invariant under whichever part of the gauge group remains unbroken by the monopole. We will count operators by considering all operators with the same monopole charge together, with a unique bare monopole and a tower of polynomially dressed descendants.

Enumerating bare monopoles is easy and we already did the bulk of the work in Section 2.3.6. They are labelled by cocharacters of the gauge group $\mathcal{G}$, or equivalently by elements from the lattice $\Gamma_{^L\mathcal{G}}/\mathcal{W}_{^L\mathcal{G}}$. We have mentioned that $U(r)$ gauge factors come with a $U(1)$ topological symmetry of the Coulomb branch. The charge under this (diagonal) $U(1)$ is the sum of magnetic weights $q_i = \sum_{i=1}^{r} m_i^D \in \mathbb{Z}$. If $\mathcal{G} = \prod_{i=1}^{n} U(r_i)$, we can assign n integers to a given monopole, corresponding to charges under the n distinct topological $U(1)$ symmetries.

There is one final charge to consider. Recall that the R-symmetry of $3d$ $\mathcal{N} = 4$ theories is $SO(4) \simeq SU(2)_{\mathcal{C}} \times SU(2)_{\mathcal{H}}$. The factor $SU(2)_{\mathcal{C}}$ acts on Coulomb branch operators while $SU(2)_{\mathcal{H}}$ acts on operators in the Higgs branch. $SU(2)_{\mathcal{C}}$ rotates the three Coulomb branch complex structures and corresponding $\mathcal{N} = 2$ subalgebras. Chiral operators are highest weight elements of $SU(2)_{\mathcal{C}}$ representations.

Irreducible representations of $SU(2)$ are characterised by a single weight or, in physics parlance, spin; we normalise this quantity so that the lowest non-trivial spin is $1/2$, as is common in physics literature. A tensor product of two $SU(2)$

representations decomposes into irreps as

$$[n] \otimes [m] = [n+m] \oplus \cdots \tag{2.170}$$

where the omitted irreps have strictly lower spin. It follows that the product of highest weight elements of two $SU(2)$ irreps is again the highest weight element for an $SU(2)$ irrep – and its spin obeys the pattern of (2.159). Moreover $SU(2)$ irreps are labelled by non-negative half-integers, so (after rescaling by a factor of 2) we see that spin under $SU(2)_{\mathcal{C}}$ could provide the (first component of a multi-) grading for the Coulomb branch chiral ring, which must be a non-negative integer. Understanding spin under $SU(2)_{\mathcal{C}}$ is now top priority.

(As a remark, since all Coulomb branch operators transform trivially under the Higgs half of the overall R-symmetry, there will be no risk of confusion in referring to $SU(2)_{\mathcal{C}}$ and R-symmetry interchangeably in the sequel.)

We now define an important property of unitary $3d$ $\mathcal{N}=4$ quiver theories with profound influence on the Coulomb branch. First, let Δ denote the IR R-symmetry spin of a Coulomb branch operator. We follow [19] and call a theory *good* if $\Delta > \frac{1}{2}$ for all such operators, *ugly* if $\Delta \geq \frac{1}{2}$ and some operators saturate the *unitarity bound* $\Delta = \frac{1}{2}$ or *bad* if the Coulomb branch includes operators with $\Delta < \frac{1}{2}$.

[19] provide a formula for the spin of monopole operators by relating it to their conformal dimension in the free UV theory. The bad news is that the correspondence is generically only guaranteed to work in the UV: whereas the conformal dimension in the UV SCFT is necessarily equal to the R-symmetry spin in the same UV theory, the UV and IR R-symmetries need not be the same. However, there is also good news: they show that in the particular case of good and ugly theories, the two R-symmetries are identified, R-symmetry spin becomes a protected quantity under the RG flow and the conformal dimension is preserved as well. And since we only concern ourselves with good theories, we are good to go. We will also henceforth refer to R-symmetry spin and conformal dimension interchangeably.

But how can we tell if a theory is good, bad or ugly? This question is readily answered in the special case of unitary quiver theories. Define the *excess* e_i of a node $U(r_i)$ as [19]

$$e_i = \#\ \text{flavors}_i - 2r_i, \tag{2.171}$$

where the first term effectively sums the ranks of nodes attached to $U(r_i)$. If a node has zero excess, we say that it is *balanced*. A unitary quiver theory is good if $e_i \geq 0$ for all nodes indexed by i. A unitary quiver with $e_i \geq -1$ for all i (and the inequality saturated by at least one node) may be ugly or bad, but it is always ugly if the inequality is only saturated by a single node. An ugly theory's chiral ring will include an even number of operators of spin $1/2$, say $2n$ of them; the Coulomb branch will then factorise as $\mathcal{C} = \mathcal{C}_{\text{red}} \times \mathbb{H}^n$. Unitary quivers with $e_i < -1$ are always bad and their RG flows are less well behaved. In particular operator counting, as presently understood, fails miserably. We focus exclusively on good quivers; bad theories have been studied in e.g. [96, 107, 108].

We will temporarily adopt a more flexible indexing and allow the result to take a more general form. Instead of assigning an integer to gauge groups, we will label them by the vertex to which they correspond. The notation will later revert to the practice of labelling gauge factors by integers, typically the usual labels for corresponding Dynkin diagrams.

Consider first a unitary simply laced quiver. The underlying graph is formed by a set of vertices V and a set of (unoriented) edges $E \subset S^2(V)$. To each vertex $v \in V$ is associated a gauge group $U(r_v)$, and to each edge $e \in E$ is associated a hypermultiplet in the bifundamental representation of $U(r_v) \times U(r_{v'})$ where $e = (v, v')$. Finally, we have a set of flavor vertices $F \neq \emptyset$ with global symmetries $SU(n_f)$ for $f \in F$, and a set of edges $E' \subset V \times F$. An edge $e' = (v, f)$ encodes n_f hypermultiplets in the fundamental representation of $U(r_v)$. The total gauge group is

$$\mathcal{G} = \prod_{v \in V} U(r_v) \tag{2.172}$$

and it has rank

$$r = \sum_{v \in V} r_v \,. \tag{2.173}$$

The Weyl group is

$$\mathcal{W} = \prod_{v \in V} S_{r_v} \,. \tag{2.174}$$

A magnetic charge is an element $m \in \mathbb{Z}^r$. The conformal dimension $\Delta(m)$ of a bare monopole (with magnetic charge m) is defined by

$$2\Delta(m) = \sum_{(v,v') \in E} \sum_{i=1}^{r_v} \sum_{i'=1}^{r_{v'}} |m_{v,i} - m_{v',i'}| + \sum_{(v,f) \in E'} \sum_{i=1}^{r_v} n_f |m_{v,i}| - \sum_{v \in V} \sum_{i=1}^{r_v} \sum_{j=1}^{r_v} |m_{v,i} - m_{v,j}| \,. \tag{2.175}$$

Incidentally, the above equation can be significantly streamlined and generalised to non-unitary simply laced quivers as

$$2\Delta(m) = \sum_{w \in \mathcal{R}^+} |w \cdot m| - \sum_{\alpha \in \mathcal{V}} |\alpha \cdot m|. \tag{2.176}$$

where we follow the conventions set out in Sec. 2.4. However, the form in (2.175) is more practical for calculations in this thesis.

We can now assemble the Hilbert series counting all gauge-invariant (and linearly independent) Coulomb branch chiral ring elements. The conformal dimension formula implies that, for every (linearly independent) bare monopole insertion of charge m, the Hilbert series will contain a contribution of $t^{2\Delta(m)}$. Note that the coefficient of t counts *double* the spin, which is always necessarily integral – and, for good and ugly theories, necessarily non-negative, with identity as the unique t^0 operator.

A bare monopole can be dressed by scalar operators as in (2.150). Each elementary operator φ has conformal dimension (and hence spin) 1. We need to count how many there are for each choice of m. Fortunately this translates to a straightforward combinatoric question. Take a single gauge factor $U(r_v)$ for now. m contains a component, or possibly a number of components, corresponding to magnetic charge under this factor – call it m_v. Then $U(r_v)$ is broken by m_v into a subgroup $\prod_i U(r_{v,i})$, $\sum_i r_{v,i} = r_v$, with the Weyl group $\prod_i S_{r_{v,i}}$. The admissible polynomial dressings are all the polynomials which are symmetric under this product Weyl group. If we can get a generating function for them of the form $g_{m,v,i}(t) = 1 + a^{(1)}_{m,v,i} t^2 + a^{(2)}_{m,v,i} t^4 + \cdots$, then we can assemble a Hilbert series contribution

$$t^{2\Delta(m)} \prod_i g_{m,v,i}(t) \tag{2.177}$$

counting *every* monopole insertion of charge m, bare or not. Note that the term for the trivial insertion, $m = (0, \ldots, 0)$, would count all gauge-invariant polynomials in φ, i.e. the Casimir operators.

The solution is well known, but before we state it, let us put this discussion in more formal terms, which will incidentally become quite useful as we later discuss wreathed quivers. For Γ a subgroup of S_r and m a magnetic charge, define the stabiliser as

$$\Gamma(m) = \{g \in \Gamma | g \cdot m = m\} \, . \tag{2.178}$$

We certainly have that $\mathcal{W} \subset S_r$. In the previous paragraph's notation, $\mathcal{W}(m) = \prod_{v,i} S_{r_{v,i}}$.

The generating function is then given by the Molien series [109]:

$$\frac{1}{|\mathcal{W}|} \sum_{\gamma \in \mathcal{W}(m)} \frac{1}{\det\left(1 - t^2 \gamma\right)}. \tag{2.179}$$

where γ can be represented e.g. as a permutation matrix acting on $\mathbb{R}^r$. In the case of $\mathcal{W}(m) = \prod_{v,i} S_{r_{v,i}}$ we get

$$\frac{1}{|\mathcal{W}|} \sum_{\gamma \in \mathcal{W}(m)} \frac{1}{\det\left(1 - t^2 \gamma\right)} = \prod_{v,i} \prod_{d=1}^{r_{v,i}} \frac{1}{\left(1 - t^{2d}\right)}. \tag{2.180}$$

Finally the (unrefined) Hilbert series for the Coulomb branch of the quiver is given by the *monopole formula* [11], which can be written as

$$\mathrm{HS}(t) = \mathrm{HS}_{\mathcal{W}}(t) = \frac{1}{|\mathcal{W}|} \sum_{m \in \mathbb{Z}^r} \sum_{\gamma \in \mathcal{W}(m)} \frac{t^{2\Delta(m)}}{\det\left(1 - t^2 \gamma\right)} \,. \tag{2.181}$$

This formula can be further refined by labelling each monopole insertion with its charge under the topological symmetry q_v as defined in (2.152). We only need introduce $|V|$ extra fugacities z_v:

$$\mathrm{HS}_{\mathrm{ref}}(t, z_v) = \mathrm{HS}_{\mathrm{ref},\mathcal{W}}(t, z_v) = \frac{1}{|\mathcal{W}|} \sum_{m \in \mathbb{Z}^r} \sum_{\gamma \in \mathcal{W}(m)} \frac{\left(\prod_v z_v^{q_v(m)}\right) t^{2\Delta(m)}}{\det\left(1 - t^2 \gamma\right)} \,. \tag{2.182}$$

Defining the formula is only half the battle: it must be also be computable in reasonable time to be of much use. We refer to [110] for an interesting algebraic look at the difficulties surrounding computations of the monopole formula; here we merely gloss over them.

Assume that we have the form (2.182) expanded as an infinite series:

$$\mathrm{HS}_{\mathrm{ref}}(t, z_v) = \sum_{s=0}^{\infty} p_s(z_v) t^s \tag{2.183}$$

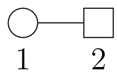
Now comes the crucial part: the polynomial $p_s(z_v)$ multiplying t^s is – trivially – a character of the topological symmetry $\prod_i U(1)$. But it may also be a character for a *larger* group. One might think that could happen by chance for a particular order in t, but it would be much less likely that *all* coefficients of t^s, for all s, are characters of the same larger group – and there are general results constraining the coefficients even further.

If there is in fact a larger symmetry group acting on the Coulomb branch (in which case we'd say that its symmetry is *enhanced*), then there would have to exist an associated set of conserved currents. And if these currents could be shown to leave a trace in the chiral ring, the Hilbert series might store a strong hint about the symmetry.

As shown in [111], a unitary $3d$ SCFT's conserved primary currents exhibit R-symmetry spin 1. [24] places conserved currents inside $\mathcal{N} = 2$ linear multiplets. (Note that the dual photon is the lowest component of a linear multiplet and that linear multiplets can be dualised to chiral multiplets.) Weaving the two together, [19] note that in $3d$ $\mathcal{N} = 4$ SCFTs conserved currents appear in multiplets whose lowest component is a scalar in a $\mathcal{N} = 2$ chiral superfield with R-symmetry spin 1. Note that this is fully consistent with the idea that the topological symmetry (with its conserved current $\star F$) is a direct result of the existence of $U(1)$ superfields which contain complex scalars of R-symmetry spin 1. Consequently the order t^2 contribution to the Hilbert series is counting at least the generators of the global symmetry.

Finally, currents associated to a global symmetry form its adjoint representation. Absent more information or physical arguments the Hilbert series is only counting the dimension of the algebra, which is of course not enough to determine its structure constants, but the presence of a term readily interpreted as an adjoint irrep's character is a strong hint that the global symmetry of the Coulomb branch is larger than $U(1)^{\sum_v r_v}$[16].

For example the Coulomb branch of the quiver



(2.184)

has topological symmetry $U(1)$ coming from its single gauge node, but the coefficient of t at every order in its Hilbert series

$$\mathrm{HS}(t, z) = \sum_{s=0}^{\infty} \chi_{[2s]}^{\mathfrak{sl}(2,\mathbb{C})}(z) t^{2s} \tag{2.185}$$

is an $\mathfrak{sl}(2, \mathbb{C})$ character, including the adjoint representation $\chi_{[2]}^{\mathfrak{sl}(2,\mathbb{C})}$ at order t^2. The Coulomb branch symmetry algebra is then likely enhanced to overall $\mathfrak{sl}(2, \mathbb{C})$. New directions on the Coulomb branch correspond to VEVs of monopole operators; we will shortly see this example worked out in explicit detail.

Note that the Hilbert series is preserved under complex mass deformation. If we read off the isometry of the SCFT Coulomb branch from the Hilbert series, and the series remains untouched upon turning on complex mass parameters, it is natural to conjecture that the isometry will also remain intact. We will be able to confirm it for worked examples.

So Hilbert series suggests the isometry; it also gives us quite a bit more than that. The coefficient at the lowest non-trivial order in t^2 must correspond to (at least some of) the generators. The Casimir operators must be linear if they are present at that order at all and the monopole operators must be bare. In fact most of the rather special quivers in this thesis have Coulomb branch chiral rings generated by operators at order t^2, i.e. by linear Casimirs and (specific) bare monopole operators.

[16] That said, it is relatively straightforward to find physical arguments which *do* bridge the divide and associate exponents of z_v with structure constants, at least in the case of unitary quivers [19] – it gets more complicated when the rank of the UV topological symmetry is less than the rank of the enhanced symmetry.

They assemble into the coadjoint representation[17] of the isometry – and the isometry is precisely the simple Lie algebra represented by the quiver reinterpreted as a Dynkin diagram. Note that generic good quivers may have chiral rings generated by operators beyond lowest order in t.

We should mention an important subtlety regarding the monopole formula's implicit grading, which we alluded to earlier. Recall that we are interested in finding an isomorphism between the Coulomb branch and a hyper-Kähler algebraic variety, and that it induces an isomorphism between the chiral ring and the coordinate ring. It should in particular map the R-symmetry (on the chiral ring side) to the $SU(2)$ which rotates the three complex structures (on the hyper-Kähler space). This is always assured: since $SU(2)$ has rank 1, the map is unique (i.e. without mixing), possibly up to a trivial scalar factor. But we would like the isomorphism to also reveal the Coulomb branch symmetry by mapping it to the symmetry on the variety, in a way that makes the symmetry obvious. This part is trickier. As we saw in the example earlier in this section, we can guess that a space is symmetric under $\mathfrak{g}$ if its Hilbert series decomposes as $\sum_s \chi^{\mathfrak{g}}(z_v)t^s$ into a sum over characters of (not necessarily irreducible) representations of $\mathfrak{g}$, further multiplied by t^s. But it is not always straightforward to identify a polynomial as a character. It would be easy if the character were put into a canonical form – perhaps the exponents of z_v correspond to components of $\mathfrak{g}$ weights in the simple root basis. The cases in this thesis happen to have this extremely useful property, and specifically for the simple root basis, but it is not guaranteed in the general case.

Operator counting can do one more thing for us: it can pin down the relations between generators. This is largely thanks to its sensitivity to the symmetry: if generators form tensors of the symmetry group then so must relations, since otherwise they would break the symmetry. Close analysis of a Hilbert series expansion will typically reveal that there are fewer operators at higher orders in t than would be expected from free (symmetric) products of generating tensors; they must be "removed" by a set of relations which transform in irreducible representations of the symmetry[18]. To be clear, this type of analysis can only ever say "representation R

[17] The Hilbert series does not distinguish between adjoint and coadjoint representations. Earlier works, which relied heavily on the monopole formula, often claimed that the order t^2 operators form an adjoint representation. We will see on many concrete examples that the natural object that comes out, the moment map, is in the coadjoint representation.

To see why, take for granted that most of the Coulomb branches studied herein are (closures of) nilpotent orbits, ie. orbits of a nilpotent element under the action of the adjoint representation, and let $p \in \mathcal{C}$ be a point on the Coulomb branch. Then the previous sentence merely says that there is an adjoint action Ad on p. Now let $x(\cdot) : \mathcal{C} \to \mathbb{C}$ be an element of the coordinate ring $\mathbb{C}[\mathcal{C}]$. The expression $x(p)$ should be invariant under the action of the symmetry group, as we simultaneously shift the point p and the coordinate function $x(\cdot)$. But that is just to say that, if $p \to \mathrm{Ad}_g p$, x transforms under the correct representation R as $x(\cdot) \to R_g x(\cdot) = x(\mathrm{Ad}_{g^{-1}} \cdot)$: precisely the defining equation of the coadjoint representation Ad^*.

[18] This claim can be recast in more technical terms of plethystic logarithms and syzygies [94, 95].

is missing at order t^s" – but if there is only one candidate tensor transforming in R, we know it to be our relation. Assume we found the recurring example (2.168) by computing the monopole formula of a quiver. We could tell that the Coulomb branch is generated by one adjoint matrix at order t^2. Then, because the second symmetric product of the adjoint representation contains the trivial representation, which is not represented at order t^4 of the Hilbert series, we see there must be a relation transforming in that same (i.e. trivial) representation. Assuming all complex mass parameters are set to 0, this relation must be $\mathrm{tr}(M^2) = 0$ or, equivalently, $\det(M) = 0$. If mass parameters M_i are allowed, the relation can be modified to $\mathrm{tr}(M^2) = q(M_i)$, where q is a quadratic polynomial (since masses always have conformal dimension 1 and hence count at order t^2). Still, this is a win: operator counting reduces the complexity of the original task to finding coefficients of q, which can be attempted by another method.

The monopole formula has seen some improvements over the years. [11] modified it with the addition of non-simply laced quivers to the world of quiver gauge theories. While they were not explicitly constructed (say, as Lagrangian theories), it was relatively straightforward to modify the monopole formula such that, when computed for non-simply laced quivers, the results made sense and followed the pattern of their simply-laced cousins. In particular, it is well known that balanced quivers' Coulomb branch symmetry enhances according to the Dynkin diagram which the quiver resembles. For example, balanced linear quivers exhibit A_n symmetry. Non-simply laced balanced quivers were found to have B_n, C_n, F_4 or G_2 symmetry.

The only difference introduced by non-simply laced quivers to the monopole formula is a modification of (2.175) to

$$\begin{aligned} 2\Delta(m) = &\sum_{(v,v')\in E} \sum_{i=1}^{n_v} \sum_{i'=1}^{n_{v'}} |\kappa_{v,v'} m_{v,i} - \kappa_{v',v} m_{v',i'}| + \sum_{(v,f)\in E'} \sum_{i=1}^{n_v} n_f |m_{v,i}| \\ &- \sum_{v\in V} \sum_{i=1}^{n_v} \sum_{j=1}^{n_v} |m_{v,i} - m_{v,j}| \end{aligned} \tag{2.186}$$

where κ is defined as follows:

- $\kappa_{vv} = 2$
- $\kappa_{vv'} = \kappa_{v'v} = -n$ if v and v' are connected by n undirected edges
- $\kappa_{vv'} = -n, \kappa_{v'v} = -1$ if v and v' are connected by an n-valent directed edge from v to v'

The similarity to Cartan matrices is, of course, not coincidental and reappears in the abelianised formalism.

We also touch upon it towards the end of this section.

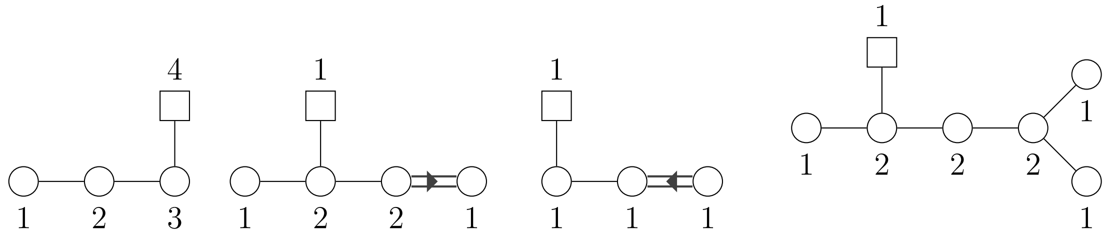


Figure 2.4: Examples of balanced quivers of type A, B, C, resp. D

Note on Coulomb branch symmetries

It was already apparent in [59] that Coulomb branches exhibit interesting patterns of symmetry enhancement. This insight was sharpened in [19] who noticed that enhancement comes from balanced nodes, i.e. those with zero excess (2.171). The extent of *visible enhancement* can be summarised by the following statement:

> Let Q be a quiver with Q_b and Q_u the subquivers consisting of balanced, resp. unbalanced gauge nodes and assume that Q has at least one flavor node. To read off the Coulomb branch symmetry of Q, reinterpret the shape of Q_b as a set of Dynkin diagrams; their corresponding simple algebras give the non-abelian part of the symmetry. The abelian part is $U(1)^{|Q_u|}$.

All quivers considered in this thesis are fully balanced like those in Fig. 2.4, although our methods should generalise well to more generic cases. Since fully balanced quivers necessarily have the same shape as a Dynkin diagram of some simple algebra $\mathfrak{g}$, we say that the quiver is of *type* $\mathfrak{g}$, eg type A_n. We can also choose to omit the index and talk about quivers of type B, for example, if the claims made apply to the whole family of type B_n quivers.

Highest weight generating functions

As noted above, Hilbert series expand into the form (2.183), in which the t^s are multiplied by a character of the Coulomb branch symmetry algebra $\mathfrak{g}$. A character can be a very large polynomial, numbering as many monomial terms as the dimension of its associated representation. But it carries no more information than the highest weights of its components' irreducible representations.

For example, take the variety $\mathbb{C}^2/\mathbb{Z}_2$ and its Hilbert series (2.168)

$$\mathrm{HS}(t,z) = \sum_{s=0}^{\infty} \chi_{[2s]}(z) t^{2s} = 1 + \chi_{[2]} t^2 + \chi_{[4]} t^4 + \cdots \tag{2.187}$$

The *highest weight generating function* [112] expresses just as much information (assuming an implicit choice of algebra $\mathfrak{g}$):

$$\mathrm{HWG}(t,\mu) = \sum_{s=0}^{\infty} \mu^{2s} t^{2s} = 1 + \mu^2 t^2 + \mu^4 t^4 + \cdots \tag{2.188}$$

A HWG is a generating function for the highest weights of irreducible representations at a particular order counted by t. In the above example, $\mu^2 t^2$ tells us that there are operators at order t^2 and that they assemble into the irrep labelled by the exponent of the fugacity μ, i.e. [2]. For rank $\mathfrak{g} > 1$ we introduce rank $\mathfrak{g}$ fugacities μ_i whose individual exponents correspond to the i-th component of the highest weight, e.g. $\mu_1\mu_3$ denotes the adjoint irrep [101] of $\mathfrak{sl}(4,\mathbb{C})$.

HWGs are not just a tidy way of expressing characters. For example, $\chi_{[2]} \cdot \chi_{[2]} = \chi_{[4]} + \chi_{[0]}$ but $\mu^2 \cdot \mu^2 = \mu^4$, so the algebra of characters and fugacities is a bit different. This can lead to neater expressions. For example, the HWG of $\mathbb{C}^2/\mathbb{Z}_2$ is just

$$\mathrm{HWG}(t,\mu) = \sum_{s=0}^{\infty} \mu^{2s} t^{2s} = \frac{1}{1-\mu^2 t^2}, \tag{2.189}$$

a simpler expression than (2.168). The relative reduction in complexity between Hilbert series and HWGs is a generic feature for nilpotent orbits, whose structure is heavily geometrically and algebraically constrained. We will use both types of generating functions throughout this thesis.

Ungauging of $U(1)$

The equation (2.175) is invariant under a simultaneous shift of all magnetic charges by the same integer, i.e. $m \to m + c$ for some fixed c, assuming the quiver is free of flavor nodes. In such theories any non-trivial monopole generates an infinite family of c-shifted monopoles with the same conformal dimension and the Hilbert series is undefined. Fortunately, we can select a particular (but arbitrary) magnetic charge m_I and use the shift by c to set this charge to 0, reflecting the fact that an overall $U(1)$ decouples from the rest of the theory. The Hilbert series of the resulting Coulomb branch $\mathcal{C}(m_I = 0)$ may be well-defined. The operation corresponds to the factorisation

$$\mathcal{C} = \mathcal{C}(m_I = 0) \times (\mathbb{R}^3 \times S^1) \tag{2.190}$$

Where the latter factor is the Coulomb branch of a free $U(1)$ theory. In string-theoretic descriptions the operation typically removes the degree of freedom corresponding to the branes' "centre of mass".

If m_I is the sole magnetic charge of a $U(1)$ node then the node is effectively converted to a flavor node. This action is often called "ungauging the $U(1)$" in the literature. If the quiver is simply laced then all possible choices lead to equivalent Coulomb branches. If no $U(1)$ node is present then one of the other magnetic charges must be set to 0; this case is much less studied and we will not make any general claims.

Note that the situation is much more complicated in the case of non-simply laced quivers, see [105], where one choice of ungauged $U(1)$ node produces a cover, resp. orbifold of the Coulomb branch associated to a different choice.

Reading relations off the Hilbert series

We briefly describe the method by which we extract chiral ring relations from the Coulomb branch Hilbert series. Assume that the Hilbert series is refined with fugacities z_i counting charge under a Cartan subalgebra of the Coulomb branch symmetry algebra $\mathfrak{g}$. The Hilbert series expands as

$$\mathrm{HS}(t, z_i) = \sum_{s \in \mathbb{Z}_{\geq 0}} p_s(z_i) t^s \tag{2.191}$$

where $p_s(z_i)$ are characters of $\mathfrak{g}$.

We first state the general strategy for a nilpotent orbit, whose coordinate ring is generated by a single (co)adjoint representation with spin 1. The quaternionic dimension of each Coulomb branch is easily calculated by summing up gauge ranks, which is unaffected by discrete gauging. Knowing the dimension and global symmetry, we can look up the space in [32][19].

We could then expand the highest weight generating function, comparing (polynomial) coefficients of t^{2n} to the character representation of the n-th symmetric product $\mathrm{Sym}^n \mathrm{adj}(\mathfrak{g})$ and find missing representations suggesting the existence of relations. Or we can perform the same computation in a more elegant fashion using the plethystic logarithm:

$$\mathrm{PL}(\mathrm{HS}(t, z_i)) = \sum_{k=1}^{\infty} \frac{\mu(k)}{k} \log\left(\mathrm{HS}\left(t^k, z_i^k\right)\right) = \sum_{s=1}^{\infty} g_s(z_i) t^s - \sum_{s=1}^{\infty} r_s(z_i) t^s \tag{2.192}$$

where $\mu(k)$ is the Möbius function and the polynomials $g_s(z_i)$ and $r_s(z_i)$ are characters of $\mathfrak{g}$ corresponding respectively to the generators and relations of the Coulomb branch. If the space is a complete intersection, the list of g_s and r_s is finite. The minimal set of relations is typically present in the first few orders of t. For example, the (closure of the) minimal nilpotent orbit of any simple algebra $\mathfrak{g}$ (whose coordinate ring is generated by one coadjoint generator [50]) is described by a set

[19] The paper differs from this thesis in the simple root convention for G_2: for this thesis the (co)adjoint representation goes by [01] whereas in [32] the two labels are swapped.

of Joseph relations [113, 114] of its coordinate ring. They are always necessarily quadratic in the coadjoint generator and remove every tensor in $\text{Sym}^2(\text{adj } \mathfrak{g})$ other than the highest weight component[20]. In more general cases we go to slightly higher order, t^6 or t^8. Then, where feasible, one can verify that that the full set of relations are identified: it suffices to calculate the Hilbert series of a ring defined by dim $\mathfrak{g}$ generators subject to the relations in question and compare it to tabulated expressions.

This procedure is only slightly modified in the few isolated cases in this thesis where the Coulomb branch is not a nilpotent orbit. The chiral ring is then generated by more generators, which are in these particular cases also coadjoint. Their contribution will be visible in the PL.

2.4.2 Abelianisation

There are drawbacks to operator counting. The monopole formula is an inherently indirect method of understanding what the Coulomb branch chiral ring operators *are*. As physicists we like to see things that plug into path integrals or between bras and kets and study relations between them. Operator counting can only do what it says on the tin: count those operators and perhaps say a thing or two about charges under various symmetries. In this section we introduce the fully explicit method of *abelianisation*, which acts as a counterbalance of sorts to operator counting.

The procedure was first introduced in [78] and given more conceptual background in [79]. The monopole operator appears as one endpoint of a vortex worldline, i.e. it can generate or annihilate a topological vortex. One can then study the physics of this vortex in the language of one-dimensional quantum mmechanics on its worldline. In this setting many calculations become much easier and the authors were able to prove several technical results which are reported below. In [26, 60] a slightly different approach was chosen: the spacetime is now a three-sphere and the worldlines are great circles. This work was able to fully generalise abelianisation to non-unitary gauge groups while making more inroads on the related project of *quantisation deformation*[21]. [80] recast the $U(1)$ results of [78] in the language of brane systems, finishing the job for non-abelian groups in [61]. [97] used abelianisation to study the class of star-shaped quivers while [96] confronted the issue of bad theories and [77] trained their eyes on $USp(2N)$ SQCD.

Chapter 3 contains very explicit and pedagogical examples of abelianised constructions and Chapter 4 builds on them by abelianising non-simply laced quivers.

[20] e.g. the $\mathfrak{sl}(n, \mathbb{C})$ case is generated by the adjoint representation $\mathfrak{sl}(n, \mathbb{C}) = [1, 0, \ldots, 0, 1]$ with $\text{Sym}^2[1, 0, \ldots, 0, 1] = [2, 0, \ldots, 0, 2] + [0, 1, 0, \ldots, 0, 1, 0] + [0, \ldots, 0]$. The space has $[0, 1, 0, \ldots, 0, 1, 0]$ and $[0, \ldots, 0]$ relations.

[21] Very briefly: every symplectic manifold admits a natural QM-like non-commutative deformation. Since Coulomb branches are symplectic, studying their deformations is interesting.

Accordingly, we intend this section to serve as a reference rather than a gentle introduction to the method. The reader is advised to skim this section on a first reading and proceed to the next chapter.

As we noted in Section 2.3.6, Coulomb branch operators can be constructed as gauge-invariant combinations of abelian insertions, where the gauge invariance is achieved by averaging over the action of the Weyl group. Abelianisation merely extends this approach.

One first defines the abelianised chiral ring, which is then reduced by the action of the gauge symmetry's Weyl group. Let i index the vertices and hence gauge group factors of a quiver gauge theory. Each gauge node $\mathcal{G}_i$ contributes several basic variables to the ring: $u^+_{i,a}$, $u^-_{i,a}$ and $\varphi_{i,a}$, where $1 \leq a \leq \text{rank}\ \mathcal{G}_i$. We will sometimes blur the distinction between the three types of variables by dropping all identifying information except for the node and gauge indices, leaving only $x_{i,a}$. The variables satisfy *abelianised relations*

$$u^+_{i,a} u^-_{i,a} = -\frac{\prod_{w \in \mathcal{R}^+_{i,a}} \langle w, \vec{\varphi} \rangle^{|w_{i,a}|}}{\prod_{\alpha \in \mathcal{V}} \langle \alpha, \vec{\varphi} \rangle^{|\alpha_{i,a}|}} \tag{2.193}$$

where $\vec{\varphi} = \left(\varphi_{1,1}, \ldots, \varphi_{n,\text{rank}\ \mathcal{G}_n}, M_{1,1}, \ldots M_{n,N^n_f} \right)$, N^i_f is the number of fundamental flavors on the i-th node, and both the roots α and weights w are expressed as weights in the weight basis of the theory's gauge group $\mathcal{G} = \prod_{i=1}^n \mathcal{G}_i$, i.e. the one introduced in Sec 2.4.

For example $U(2)$ with 4 fundamental flavors comes with the following matter and gauge representations:

$$\mathcal{R} = \{(1,0;-1,0,0,0), \ldots (1,0;0,0,0,-1), (0,1;-1,0,0,0), \ldots (0,1;0,0,0,-1)\} \tag{2.194}$$

with the first two charges belonging to $U(2)$ and the last four belonging to $SU(4)$ and

$$\mathcal{V} = \{(1,-1;0,0,0,0), (-1,1;0,0,0,0)\} \tag{2.195}$$

where the vanishing weights associated to the commuting part of $U(2)$ have been omitted.

The Coulomb branch is a symplectic space so its chiral ring carries a Poisson bracket, which descends from a bracket defined on the abelianised ring:

$$\{\varphi_{i,a}, u^\pm_{i,a}\} = \pm u^\pm_{i,a} \tag{2.196}$$

$$\{u^+_{i,a}, u^-_{i,a}\} = \frac{\partial}{\partial \varphi_{i,a}} \frac{\prod_{w \in \mathcal{R}} \langle w, \vec{\varphi} \rangle^{|w_{i,a}|}}{\prod_{\alpha \in \mathcal{V}} \langle \alpha, \vec{\varphi} \rangle^{|\alpha_{i,a}|}} \tag{2.197}$$

$$\{u^{\pm}_{i,a}, u^{\pm}_{j,b}\} = \begin{cases} \pm\kappa_{ij} \frac{u^{\pm}_{i,a} u^{\pm}_{j,b}}{\varphi_{i,a} - \varphi_{j,b}} & \text{if } (i,a) \neq (j,b) \\ 0 & \text{if } (i,a) = (j,b) \end{cases} \tag{2.198}$$

where κ_{ij} is defined as in Section 2.4.1 and the remaining Poisson brackets vanish.

These Poisson brackets, along with the abelianised relations (2.193), generate every element of the "abelianised" chiral ring $\mathbb{C}[\mathcal{C}_{\text{abel}}]$. The true Coulomb branch chiral ring $\mathbb{C}[\mathcal{C}]$ is a particular sub-ring of the Weyl-symmetrised ring $\mathbb{C}[\mathcal{C}_{\text{abel}}]^{W_G}$. But *which* subring? That is one of the central mysteries of each of the following sections – and many paths may lead to the correct answer. Ours will follow a robust and well-motivated strategy: we will leverage information from the Hilbert series to identify the correct subring, picking Weyl-invariant abelianised operators with the correct representation-theoretic properties as dictated by the physics. Appendix A offers a comparison with an alternative approach but the reader is advised to read it only after Chapter 3.

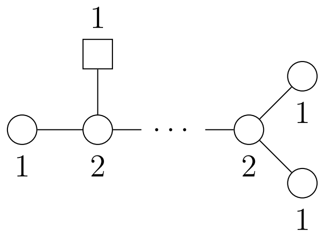
Chapter 3

Simply laced unitary quivers

We have introduced two approaches to studying the Coulomb branch: operator counting and abelianisation. Unsurprisingly, each comes with its strengths and drawbacks. Operator counting is very general, straightforwardly algorithmic and naturally captures the isometry of the Coulomb branch and representation-theoretic content of chiral ring relations, reducing the problem of finding the moduli space to identifying coefficients for finitely many linear combinations of finitely many operators. The representation-theoretic data is also often sufficient to solve this latter problem. However, turning on complex mass deformations compromises the computational utility of this method. Operator counting also rarely aids physical interpretation of particular chiral ring operators. On the other hand the recent abelianised construction leverages operators' physical properties, naturally handles complex mass deformations and in principle fully specifies the moduli space for arbitrary quivers. However, the way in which it is defined does not draw out the Coulomb branch symmetry, corresponding representation-theoretic data and as a result physical relations between gauge-invariant chiral operators are difficult to extract.

This chapter demonstrates that operator counting and abelianisation can be synthesised into a new approach which combines their strengths, removes many of their drawbacks and provides a new and powerful way to derive relations between gauge-invariant operators in $3d$ $\mathcal{N}=4$ theories. We refer to it as the *synthetic method*. To aid exposition, examples are restricted to very simple quiver theories.

First, a quick definition. *Quiver (Panyushev) height* [115] leverages the similarity between Dynkin diagrams of simple Lie algebras and subgraphs of quivers formed by all gauge nodes and can be calculated by taking the dot product between the vector of Coxeter labels and the vector of flavor ranks. For example, the family of type D quivers



(3.1)

whose Coxeter labels are $(1, 2, \ldots, 2, 1, 1)$, has height

$$(0, 1, \ldots, 0, 0, 0) \cdot (1, 2, \ldots, 2, 1, 1) = 2. \tag{3.2}$$

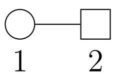
This chapter's examples are drawn from families of balanced quiver gauge theories of type A and D (ie. shaped like their namesakes among Dynkin diagrams). These specific type A quivers feature at least one $U(1)$ gauge node while type D quivers have Panyushev height 2; these examples have been studied in [20] using only operator counting. Chapter 4 expands this approach to types B, C and G, also of height 2. All of these quivers have at least one flavor node.

Although it may seem that we have narrowed the class of quivers almost out of existence, we have merely restricted to cases covered in [20], whose Coulomb branches are closures of nilpotent orbits[1]. They are the simplest exemplars of their kind and hence a suitable arena for development of a new technique. We expect that once our method is established for these basic cases most – if not all – of the imposed restrictions can be lifted and the description will generalise to varieties beyond nilpotent orbits.

3.1 Type A: generalities

3.1.1 min A_1: A simple example

The main results of this chapter are best introduced as generalisations of two concrete results, both of which originally appeared in [78] in some form. The simpler of the two concerns SQED with two electrons:



(3.3)

We will initially set both electrons' masses to 0. The Hilbert series of the theory, calculated using the monopole formula from Section 2.4.1, is

[1]We will continue to trade accuracy for brevity and refer to closures of nilpotent orbits as, simply, "nilpotent orbits" in what follows.

$$\begin{aligned}\mathrm{HS}(t) &= 1 + (z + 1 + \frac{1}{z})t^2 + (z^2 + z + 1 + \frac{1}{z} + \frac{1}{z^2})t^4 + O(t^6) \\ &= 1 + (w^2 + 1 + w^{-2})t^2 + (w^4 + w^2 + 1 + w^{-2} + w^{-4})t^4 + O(t^6) \\ &= 1 + [2]t^2 + [4]t^4 + O(t^6) \quad (3.4)\end{aligned}$$

where $z \mapsto w^2$ cast it into a manifest sum of $\mathfrak{sl}(2,\mathbb{C})$ characters $[n] = w^n + w^{n-2} + \cdots + \frac{1}{w^n}$.

The series identifies a generator – call it N – transforming in the (co)adjoint representation [2]. If the ring were freely generated then we would see a singlet [0] and a tensor transforming in [4] at quadratic order, but the singlet is absent. Hence there must be a quadratic singlet relation, which can only take the form $A \det N + B\ \mathrm{tr}(N^2) = 0$ for some A, B; a quick calculation shows that every generic choice of A, B is equivalent[2]. The relation can also be written as

$$t^4[0]:\ N^2 = 0, \tag{3.5}$$

which identifies the space of N, ie. the Coulomb branch of this theory, as a nilpotent orbit of $\mathfrak{sl}(2) \simeq A_1$. We have already seen this example several times. For future reference, note how we choose to report relations: the exponent of t is twice the R-symmetry spin, followed by the highest weight of the relation and then the explicit tensorial relation itself.

This is a good result but some information is lost. There are three operators in N, but what *are* they physically? How do they assemble into the matrix realisation of N? How should we physically interpret the relation $N^2 = 0$? If we set electrons' (complex) masses to M, would the relation change to $\mathrm{Tr}(N^2) = M^2$? Hilbert series can help with some of these questions but they are not the most suitable tools.

Let's explore this problem using the algebraic construction of the chiral ring pioneered in [78]. This approach has several virtues: it is directly connected to physics and very cleanly handles complex mass deformations of the theory. However the Coulomb branch isometry remains hidden.

The ring is generated by two monopole operators $u^\pm$ and one scalar operator φ subject to the relation

$$u^+u^- = -(\varphi - M_1)(\varphi - M_2) \tag{3.6}$$

where the M_i are complex masses of electrons. It is important to note that this relation comes "for free" from the definition of the chiral ring provided by [78]. This is a particularly simple example. There are no generators beyond $u^\pm$ and φ and no relations beyond (3.6). In other words, this is our chiral ring, but it is not

[2]Exceptions such as $A = B = 0$ would reduce the relation to $0 = 0$ and we can disregard them because the Hilbert series indicates there *is* a non-trivial scalar relation.

immediately obvious that it describes (a deformation of) a nilpotent orbit of $\mathfrak{sl}(2,\mathbb{C})$.

We want to develop a synthetic approach which adapts an important result of [78]: the Coulomb branch, being hyper-Kähler, has a moment map transforming in the coadjoint representation of $\mathfrak{sl}(2,\mathbb{C})$ and specifically given by

$$\mu = \begin{pmatrix} \varphi - \frac{M_1}{2} - \frac{M_2}{2} & u^- \\ u^+ & -\varphi + \frac{M_1}{2} + \frac{M_2}{2} \end{pmatrix} \tag{3.7}$$

Recall that the adjoint and coadjoint representations of $\mathfrak{sl}(2,\mathbb{C})$ are isomorphic and the Hilbert series has no way of distinguishing between them, so N may in fact be a coadjoint generator. We will see that it is most naturally expanded in the coadjoint representation's basis as defined in section 2.1.3.

μ also obeys the same relation as N of (3.5):

$$\mu^2 = \begin{pmatrix} \frac{(M_1+M_2-2\varphi)^2}{4} + u^+u^- & 0 \\ 0 & \frac{(M_1+M_2-2\varphi)^2}{4} + u^+u^- \end{pmatrix} = \frac{(M_1 - M_2)^2}{4}\mathbb{1}_{2\times 2} \tag{3.8}$$

where we used (3.6) to simplify some quadratic expressions. Note that when the masses are taken to 0 – that is, precisely in the case considered using Hilbert series – the equation reduces to $\mu^2 = 0$.

Several features of this result are noteworthy:

- The matrix μ is traceless and hence belongs to $\mathfrak{sl}(2)$ (or $\mathfrak{sl}(2)^*$) – but is valued in the chiral ring R rather than $\mathbb{C}$. The operator counting approach implied the existence of a coadjoint matrix N whose complex coefficients are constrained by relations. The synthetic approach defines μ as a ring-valued matrix and matrix relations are reinterpreted as consequences of chiral ring relations which can be fully specified prior to embedding into a matrix.

- $\mathfrak{sl}(2,\mathbb{C})$ has a natural (co)adjoint action on μ and components of μ generate the chiral ring – so $\mu = N$.

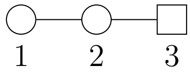
- The fact that there are no independent higher-order relations is assured by Hilbert series.

- However, the Coulomb branch Hilbert series provides no way of fixing the coefficient on the complex-mass-deformed relation.

All of the above generalises to all examples considered in this section and helps illustrate some of the utility of our synthetic method.

3.1.2 max A_2: A slightly more complicated example

For the second example we pick the theory



(3.9)

Its gauge group is $U(1) \times U(2)$. Both gauge nodes are balanced so its Coulomb branch has an $\mathfrak{sl}(3,\mathbb{C}) \simeq A_2$ symmetry. We present its Hilbert series in terms of topological fugacities z_1, z_2 and w_1, w_2 related by

$$w_i = \prod_j z_j^{\kappa_{ij}^{-1}} \tag{3.10}$$

where κ_{ij} is the Cartan matrix

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \tag{3.11}$$

and we use the notation $[\mathbf{p_1}, \mathbf{p_2}]$ as shorthand for the $\mathfrak{sl}(3,\mathbb{C})$ character with highest weight $[p_1, p_2]$, eg.

$$[\mathbf{1},\mathbf{1}] = w_1 w_2 + \frac{w_1^2}{w_2} + \frac{w_2^2}{w_1} + 2 + \frac{w_1}{w_2^2} + \frac{w_2}{w_1^2} + \frac{1}{w_1 w_2} \tag{3.12}$$

We call w_i, resp. z_j *fundamental weight*, resp. *simple root fugacities* for reasons which will shortly become apparent.

This notation significantly simplifies the Hilbert series and manifests its nature as a class function:

$$\begin{aligned} \mathrm{HS}(t) &= 1 + (z_1 z_2 + z_1 + z_2 + 2 + \frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_1 z_2})t^2 + O(t^4) \\ &= 1 + [\mathbf{1},\mathbf{1}]t^2 + ([\mathbf{2},\mathbf{2}] + [\mathbf{1},\mathbf{1}])t^4 + ([\mathbf{3},\mathbf{3}] + [\mathbf{2},\mathbf{2}] + [\mathbf{3},\mathbf{0}] + [\mathbf{0},\mathbf{3}])t^6 + O(t^8) \end{aligned} \tag{3.13}$$

A closer look at the Hilbert series (to all orders) shows that the (massless) chiral ring is generated by a single $\mathfrak{sl}(3,\mathbb{C})$ (co)adjoint tensor – whose character appears in (3.12) – subject to

$$t^4[0,0] : \mathrm{tr}(N^2) = 0 \tag{3.14}$$

$$t^6[0,0] : \mathrm{tr}(N^3) = 0 \tag{3.15}$$

which amounts to setting all eigenvalues to 0 and describes the maximal nilpotent orbit of $\mathfrak{sl}(3,\mathbb{C})$.

The Hilbert series predicts 8 generators in total, two of which are linear Casimirs. Expressing $w_1^{p_1} w_2^{p_2} = [p_1, p_2]$ and $z_1^{n_1} z_2^{n_2} = \langle n_1, n_2 \rangle$, we observe the following correspondence to bare monopoles with magnetic charges $m = (m_1; m_{2,1}, m_{2,2})$:

$$
\begin{aligned}
[2,-1] &\leftrightarrow \langle 1,0\rangle \leftrightarrow m = (1;0,0) \\
[-1,2] &\leftrightarrow \langle 0,1\rangle \leftrightarrow m = (0;1,0) \\
[1,1] &\leftrightarrow \langle 1,1\rangle \leftrightarrow m = (1;1,0) \\
[-2,1] &\leftrightarrow \langle -1,0\rangle \leftrightarrow m = (-1;0,0) \\
[-1,2] &\leftrightarrow \langle 0,-1\rangle \leftrightarrow m = (0;-1,0) \\
[-1,-1] &\leftrightarrow \langle -1,-1\rangle \leftrightarrow m = (-1;-1,0)
\end{aligned}
$$

It turns out that although the basis of fundamental weights is useful for pinning down the symmetry and representation content, going back to z_i, or the basis of simple roots, is more physically transparent so we will keep working in that basis.

We can now construct explicit generators and will label them as follows: generating monopole operators are indexed by corresponding roots, ie $V_{\langle n_1, n_2 \rangle}$, and linear Casimirs Φ carry the index of their gauge node, ie. Φ_i. [78] provides a recipe in terms of auxiliary gauge-dependent *abelianised* fields $u_1^{\pm}, \varphi_1, u_{2,1}^{\pm}, u_{2,2}^{\pm}, \varphi_{2,1}$ and $\varphi_{2,2}$:

$$
\begin{aligned}
V_{\langle 1,0\rangle} &= u_1^+ \\
V_{\langle 0,1\rangle} &= u_{2,1}^+ + u_{2,2}^+ \\
V_{\langle 1,1\rangle} &= \frac{u_1^+ u_{2,1}^+}{\varphi_1 - \varphi_{2,1}} + \frac{u_1^+ u_{2,2}^+}{\varphi_1 - \varphi_{2,2}} \\
V_{\langle -1,0\rangle} &= u_1^- \\
V_{\langle 0,-1\rangle} &= u_{2,1}^- + u_{2,2}^- \\
V_{\langle -1,-1\rangle} &= \frac{u_1^- u_{2,1}^-}{\varphi_1 - \varphi_{2,1}} + \frac{u_1^- u_{2,2}^-}{\varphi_1 - \varphi_{2,2}} \\
\Phi_1 &= \varphi_1 \\
\Phi_2 &= \varphi_{2,1} + \varphi_{2,2}
\end{aligned}
$$

The algebraic construction also posits a set of relations:

$$
\begin{aligned}
u_1^+ u_1^- &= -(\varphi_1 - \varphi_{2,1})(\varphi_1 - \varphi_{2,2}) \\
u_{2,1}^+ u_{2,1}^- &= -\frac{(\varphi_{2,1} - \varphi_1)(\varphi_{2,1} - M_{2,1})(\varphi_{2,1} - M_{2,2})(\varphi_{2,1} - M_{2,3})}{(\varphi_{2,1} - \varphi_{2,2})^2}
\end{aligned}
$$

$$u^+_{2,2}u^-_{2,2} = -\frac{(\varphi_{2,2}-\varphi_1)(\varphi_{2,2}-M_{2,1})(\varphi_{2,2}-M_{2,2})(\varphi_{2,2}-M_{2,3})}{(\varphi_{2,1}-\varphi_{2,2})^2}$$

There are several structural features to point out. Firstly, operators such as $\varphi_{2,1}$ and $\varphi_{2,2}$ are gauge-dependent quantities; in fact, the Weyl group of $U(2)$ transforms one into the other. Their sum $\Phi_2 = \varphi_{2,1} + \varphi_{2,2}$, however, is gauge-invariant, as would be eg. $\varphi_{2,1}\varphi_{2,2}$ (recall the discussion in Section 2.3.6). We will always reserve φ, resp. Φ, for gauge-dependent, resp. gauge-independent manifestations of the scalar superpartners of gauge bosons and $\varphi_{i,a}$ will refer to the a-th gauge-dependent (*abelianised*) scalar superpartner of the gauge bosons associated to the i-th node.

Secondly, complex mass parameters $M_{i,p}$, again labelled as being the p-th mass on the i-th node, enter relations in a similar way to complex scalars φ. This is explained by the fact that complex masses can be interpreted as scalar VEVs of background vector supermultiplets with analogous coupling rules.

Thirdly, monopole operators $V_{\langle \pm 1,\pm 1\rangle}$ have a curious structure of rational functions (and also the property of gauge-invariance-by-averaging which was just mentioned). The nature of such operators is, in our experience, a common source of confusion. One could think of e.g. $u^+_1 u^+_{2,1}/(\varphi_1 - \varphi_{2,1})$ as a new abstract ring element[3] along with the relation

$$\frac{u^+_1 u^+_{2,1}}{(\varphi_1-\varphi_{2,1})}(\varphi_1-\varphi_{2,1}) = u^+_1 u^+_{2,1}. \tag{3.16}$$

The chiral ring is still specifically a *ring* and division is not in general defined as a valid operation.

Fourthly, the theory's chiral ring includes the quadratic Casimir operator $\varphi_{2,1}\varphi_{2,2}$ – in fact it's already present in the UV description. It is easy to check that

$$\varphi_{2,1}\varphi_{2,2} = -\Phi_1(\Phi_1-\Phi_2) - V_{\langle 1,0\rangle}V_{\langle -1,0\rangle} \tag{3.17}$$

Our method does not provide an algorithmic recipe for deriving this relation but its existence is ensured.

Finally, relations are given in terms of the abelianised and hence gauge-dependent fields. But the Coulomb branch only has directions corresponding to gauge-independent operators. So we would like to find gauge-independent relations to complement them; indeed, they should be exactly the relations predicted by Hilbert series. Our synthetic method can determine them.

The prescription for the coadjoint moment map (and the chiral ring generator)

[3]Some practitioners (e.g. [77, 97]) like to think of it as a separate abelianised variable $u^+_{(1;1,0)}$ with the relation (3.16), but we have found it helpful for computational purposes to reduce these variables to rational functions of the more basic $u^\pm_{i,a}$ and $\varphi_{i,a}$ and our presentation follows that convention.

is

$$N = \begin{pmatrix} \Phi_1 - \frac{M_{2,1}+M_{2,2}+M_{2,3}}{3} & V_{\langle -1,0\rangle} & -V_{\langle -1,-1\rangle} \\ V_{\langle 1,0\rangle} & -\Phi_1 + \Phi_2 - \frac{M_{2,1}+M_{2,2}+M_{2,3}}{3} & V_{\langle 0,-1\rangle} \\ -V_{\langle 1,1\rangle} & V_{\langle 0,1\rangle} & 2\frac{M_{2,1}+M_{2,2}+M_{2,3}}{3} - \Phi_2 \end{pmatrix} \tag{3.18}$$

and indeed, one easily confirms that

$$t^4[0,0]: \ \mathrm{tr}(N^2) = \frac{2}{3}(M_{2,1}^2 + M_{2,2}^2 + M_{2,3}^2 - M_{2,1}M_{2,2} - M_{2,1}M_{2,3} - M_{2,2}M_{2,3}) \tag{3.19}$$

$$\begin{aligned} t^6[0,0]: \ \mathrm{tr}(N^3) =& \frac{1}{9}(2M_{2,1} - M_{2,2} - M_{2,3})(2M_{2,2} - M_{2,1} - M_{2,3}) \\ &\cdot (2M_{2,3} - M_{2,1} - M_{2,2}) \end{aligned} \tag{3.20}$$

both of which vanish in the massless limit. The last three expressions can be drastically simplified under the simultaneous reparametrisation $M_{2,1} = M - n_1, M_{2,2} = M + n_1 - n_2, M_{2,3} = M + n_2$ and shift in scalar variables $\varphi_{i,a} \mapsto \varphi_{i,a} + M$:

$$N = \begin{pmatrix} \Phi_1 & V_{\langle -1,0\rangle} & -V_{\langle -1,-1\rangle} \\ V_{\langle 1,0\rangle} & -\Phi_1 + \Phi_2 & V_{\langle 0,-1\rangle} \\ -V_{\langle 1,1\rangle} & V_{\langle 0,1\rangle} & -\Phi_2 \end{pmatrix} \tag{3.21}$$

$$t^4[0,0]: \ \mathrm{tr}(N^2) = 2(n_1^2 + n_2^2 - n_1 n_2) \tag{3.22}$$

$$t^6[0,0]: \ \mathrm{tr}(N^3) = 3n_1 n_2(n_2 - n_1) \tag{3.23}$$

We have simultaneously derived gauge-invariant relations in the chiral ring and generalised them for the case of massive quarks, demonstrating the advantages of the synthetic method over pure operator counting or algebraic construction.

3.1.3 Construction of generators and gauge-dependent relations

All balanced quivers of type A_n (of type A with n gauge nodes) and at least one gauge node of rank 1 share the same pattern of generators [20]. They always have R-symmetry spin 1 and include n linear Casimirs originating from gauge scalars at the n gauge nodes. The remaining generators are bare monopole operators labelled by their topological charges $\vec{q} = \langle q_1, \ldots, q_n \rangle$[4] uniquely without any degeneracies.

[4] Topological charge vectors are written with angled brackets in anticipation of a thorough correspondence between their associated generating monopole operators and roots in the isometry algebra.

Every monopole generator exhibits the following pattern of charges:

$$\vec{q} = \langle 0, \ldots, 0, \pm 1, \ldots, \pm 1, 0, \ldots, 0 \rangle, \tag{3.24}$$

or an uninterrupted string of ± 1 padded by zeroes. The string of ones can stretch to each end so, for example, $\langle 1, 1, 1 \rangle$ is a valid charge vector of a monopole generator in an A_3 quiver. The choice of $+1$ or -1 must be made consistently in a given charge vector so no A_3 monopole generator carries the charge vector $\langle 1, -1, 0 \rangle$ or other similarly "mixed" charges. Such monopole operators still exist within the chiral ring but we do not count them among a canonical set of generators.

Overall we get $n^2 + n$ monopole operators and n linear Casimirs which together generate the chiral ring. [78] provides a general prescription for these generators in terms of gauge-dependent quantities, or *abelianised variables* as they are described in the original paper. The prescription was tested on several linear quivers in the original paper and succeeded when compared against known results. Principles behind the proposal have received further support in [79, 80] which exploit quantum mechanics of vortices and string theory respectively. The chiral ring can be specified algorithmically:

- Label each gauge node with an index $i \in \{1, \ldots, n\}$ starting from the leftmost node. Let r_i be the rank of the unitary group $U(r_i)$ at the gauge node i.

- Define the abelianised ring R_{abel}. We merely adapt the more general procedure of Section 2.4.2 to linear quivers.

  1. Any node with gauge group $U(r_i)$ and index i gives rise to $3r_i$ abelianised variables: $u^+_{i,a}$, $u^-_{i,a}$ and $\varphi_{i,a}$, where a runs from 1 to r_i. They physically correspond to directions in the moduli space of the fully broken gauge group $U(1)^{r_i}$. As an abelian theory it gives rise to r_i different monopoles of charge $+1$ under the various $U(1)$ factors – those would be the $u^+_{i,a}$ – their counterparts with charges -1 – the $u^-_{i,a}$ – and complex scalars in the vector supermultiplet – the $\varphi_{i,a}$. They are essentially eigenvalues of the adjoint-valued scalar superpartner of gauge bosons.

  2. We identify all topologically charged generators of the abelianised ring. Some of these operators carry no topological charge except ± 1 at a single node i; we call such operators *minimally charged* and they are already represented by r_i operators $u^\pm_{i,a}$. The remaining monopole generators are topologically charged under several adjacent nodes and have to be constructed from the abelianised variables. They can be built in different (but equivalent) ways.

– [78] defines the Poisson bracket $\{\cdot,\cdot\}$ acting on the abelianised chiral ring; we reproduce it in (3.38). An abelianised monopole charged under adjacent nodes i and $i+1$ is given by

$$\{u^{\pm}_{i,a}, u^{\pm}_{i+1,b}\} \propto \frac{u^{\pm}_{i,a}u^{\pm}_{i+1,b}}{\varphi_{i,a}-\varphi_{i+1,b}} \tag{3.25}$$

with coefficient ± 1. This can be extended by the action of an adjacent node, eg. $u^{\pm}_{i+2,c}$:

$$\{\frac{u^{\pm}_{i,a}u^{\pm}_{i+1,b}}{\varphi_{i,a}-\varphi_{i+1,b}}, u^{\pm}_{i+2,c}\} \propto \frac{u^{\pm}_{i,a}u^{\pm}_{i+1,b}u^{\pm}_{i+2,c}}{(\varphi_{i,a}-\varphi_{i+1,b})(\varphi_{i+1,b}-\varphi_{i+2,c})} \tag{3.26}$$

This operator can again be extended by the action of an adjacent node; the maximal operator "stretches" between the leftmost and the rightmost nodes.

– Alternatively one can just give a general prescription for the non-minimally charged monopole generator. We will adopt this method and define a monopole charged ± 1 under nodes $i, i+1, \ldots, j-2, j-1$ as

$$u^{\pm}_{i:j,(a_i,\ldots,a_{j-1})} = \frac{u^{\pm}_{i,a_i}\cdots u^{\pm}_{j-1,a_{j-1}}}{(\varphi_{i,a_i}-\varphi_{i+1,a_{i+1}})\cdots(\varphi_{j-2,a_{j-2}}-\varphi_{j-1,a_{j-1}})} \tag{3.27}$$

In particular, $u^{\pm}_{i,a} = u^{\pm}_{i:i+1,(a)}$. Note that we selected the sign to be positive for all monopoles.

3. A flavor node of rank s_i connected to the gauge node i contributes complex mass parameters $M_{i,p}$, where p runs from 1 to s_i.

4. Define $A(i)$ as the set of gauge nodes (resp. their indices) adjacent to node i; for most nodes $A(i) = \{i-1, i+1\}$ but $A(1) = \{2\}$ and $A(n) = \{n-1\}$.

5. For each gauge node define two auxiliary polynomials:

$$P_i(z) = \prod_{1\le p\le s_i}(z - M_{i,p}) \tag{3.28}$$

$$Q_i(z) = \prod_{1\le a\le r_i}(z - \varphi_{i,a}) \tag{3.29}$$

6. Abelianised variables are subject to relations[5]

$$u_{i,a}^{+}u_{i,a}^{-} = -\frac{P_i(\varphi_{i,a})\prod_{j\in A(i)}Q_j(\varphi_{i,a})}{\prod_{b\neq a}(\varphi_{i,a}-\varphi_{i,b})^2} \tag{3.30}$$

which can be repackaged as generators of the ideal

$$I = \left\langle u_{i,a}^{+}u_{i,a}^{-} + \frac{P_i(\varphi_{i,a})\prod_{j\in A(i)}Q_j(\varphi_{i,a})}{\prod_{b\neq a}(\varphi_{i,a}-\varphi_{i,b})^2}\right\rangle \tag{3.31}$$

7. The abelianised ring R_{abel} is then a quotient of a polynomial ring freely generated by scalars and monopole generators:

$$R_{\text{abel}} = \mathbb{C}[u^{\pm}_{i:j,(a_i,\ldots,a_{j-1})}, \varphi_{i,a}]/I \tag{3.32}$$

with $1 \leq i < j \leq n+1$.

- The overall gauge group of the quiver is $\mathcal{G} = \prod_i U(r_i)$. Its Weyl group is then $\mathcal{W}(\mathcal{G}) = \prod_i S_{r_i}$. $\mathcal{W}(\mathcal{G})$ has a natural action on the $u^{\pm}_{i,a}$ and $\varphi_{i,a}$: each S_{r_i} permutes indices a for a fixed i. The true, physical chiral ring R can only include gauge-invariant operators and so must be a subset of the restriction of R_{abel} to $\mathcal{W}(\mathcal{G})$-invariant polynomials:

$$R \subset R_{\text{abel}}^{\mathcal{W}(\mathcal{G})} = \mathbb{C}[u^{\pm}_{i:j,(a_i,\ldots,a_{j-1})}, \varphi_{i,a}]^{\mathcal{W}(\mathcal{G})}/I \tag{3.33}$$

where $u^{\pm}_{i:j,(a_i,\ldots,a_{j-1})}$ are interpreted using (3.27) and indices are implicitly ranged over.

Several elements of $R_{\text{abel}}^{\mathcal{W}(\mathcal{G})}$ are significant enough to deserve a name:

$$V^{\pm}_{i:j} = \sum_{a,\ldots,d} u^{\pm}_{i:j,(a,\ldots,d)} = \sum_{a,\ldots,d} \frac{u^{\pm}_{i,a}\cdots u^{\pm}_{j-1,d}}{(\varphi_{i,a}-\varphi_{i+1,b})\cdots(\varphi_{j-2,c}-\varphi_{j-1,d})} \tag{3.34}$$

$$\Phi_i = \sum_a \varphi_{i,a} \tag{3.35}$$

Hilbert series computations for balanced type A quivers show that such operators form (at least some of) the generating set for R. It will also be helpful to repackage mass parameters into symmetric polynomials:

$$M_i = \sum_{p=1}^{s} M_{i,p} \tag{3.36}$$

[5]Note that these relations fix R-symmetry spin of bare abelianised monopoles $u^{\pm}_{i,a}$ since topological charge conjugation should commute with R-symmetry and φ have spin 1.

$$\vec{M} = (M_1, \ldots, M_n) \tag{3.37}$$

All that remains is to pin down *which* $\mathcal{W}_\mathcal{G}$-invariant subring of R_{abel} is the Coulomb branch chiral ring. In this work we advocate explicitly constructing tensors whose components generate the chiral ring, and for simple enough quivers there is only one: the moment map.

3.1.4 Moment map

The moment map of a symplectic space is a coadjoint-valued map, so we should be able to expand it in the basis (2.31). The coefficients will be precisely the VEVs of Coulomb branch operators of Section 3.1.3; in fact both the monopole generators and dual root vectors are labelled by unbroken strings of ± 1 padded by zeroes and there are as many linear Casimirs as there are generators of the Cartan subalgebra, although here the correspondence is marginally more involved.

The symplectic structure of the Coulomb branch gives rise to the Poisson bracket on operators (3.38), which is closely related to the moment map and described by its action on the abelianised variables in [78]:

$$\begin{aligned}
\{\varphi_{i,a}, u^\pm_{i,a}\} &= \pm u^\pm_{i,a} \\
\{u^+_{i,a}, u^-_{i,a}\} &= \frac{\partial}{\partial \varphi_{i,a}} \left[\frac{P_i(\varphi_{i,a}) \prod_{j \in A_i} Q_j(\varphi_{i,a})}{\prod_{b \neq a} (\varphi_{i,a} - \varphi_{i,b})^2} \right] \\
\{u^\pm_{i,a}, u^\pm_{j,b}\} &= \pm \kappa_{ij} \frac{u^\pm_{i,a} u^\pm_{j,b}}{\varphi_{i,a} - \varphi_{j,b}}
\end{aligned} \tag{3.38}$$

The remaining undetermined brackets vanish.

In fact, one can think of the moment map N as a homomorphism from the Lie algebra of the Coulomb branch symmetry to the Poisson algebra of operators. More explicitly, for all $X, Y \in \mathfrak{g}$

$$\operatorname{tr}(N[X, Y]) = \{\operatorname{tr}(NX), \operatorname{tr}(NY)\}. \tag{3.39}$$

So choosing $X, Y = E_\alpha$ or H_i, we can see that the operators $e_\alpha = \operatorname{tr}(NE_\alpha)$ and $h_i = \operatorname{tr}(NH_i)$ form the Lie algebra $\mathfrak{g}$ under the Poisson bracket. We can take this fact and work backwards: the moment map is assembled as

$$N = \sum_{\alpha \in \Phi} e_\alpha E^*_\alpha + \sum_{a=1}^{\operatorname{rank} \mathfrak{g}} h_i H^*_i \tag{3.40}$$

so if we can find operators e_α and h_i which reproduce the Lie algebra of $\mathfrak{g}$, we can explicitly construct the moment map.

Identifying e_α is easy. Recall from Section 2.1.3 that root vectors can also be labelled $E_{\pm(i:j)}$; these are the same labels as we have on $V^{\pm}_{i:j}$, and there is a neat correspondence:

$$e_{\pm(i:j)} = \mathrm{tr}(NE_{\pm(i:j)}) = V^{\pm}_{i:j} \tag{3.41}$$

with

$$e_{\pm i} = V^{\pm}_{(i:i+1)} = \sum_{a=1}^{\mathrm{rank}\,\mathcal{G}_i} u^{\pm}_{i,a} \tag{3.42}$$

the simple root vectors; it is completely determined by the monopole operators' topological charges [19].

We are still missing the operator analogues of Cartan elements H_i in (2.11), but only momentarily. One can easily check that

$$\{\sum_k \kappa_{ik}\Phi_k - M_i, e_{\pm j}\} = \pm\sum_k \kappa_{ik}\delta_{jk}e_{\pm j} = \pm\kappa_{ij}e_{\pm j} \tag{3.43}$$

and less easily, but straightforwardly on concrete cases, that

$$\{e_{+i}, e_{-i}\} = \sum_k \kappa_{ik}\Phi_k - M_i. \tag{3.44}$$

We can then define $h_i = \sum_k \kappa_{ik}\Phi_k - M_i$[6] and construct the coadjoint-valued moment map:

$$\begin{aligned} N(\vec{M}) &= \sum_{\substack{1\le i<j\le n\\ s\in\{+,-\}}} e_{s(i:j)}E^*_{s(i:j)} + \sum_{i=1}^{n} h_i H^*_i \\ &= \begin{pmatrix} \bar\Phi_1(\vec{M}) & V^-_{1:2} & -V^-_{1:3} & \cdots & (-1)^{n+1}V^-_{1:n+1} \\ V^+_{1:2} & -\bar\Phi_1(\vec{M})+\bar\Phi_2(\vec{M}) & V^-_{2:3} & \cdots & (-1)^{n}V^-_{2:n+1} \\ -V^+_{1:3} & V^+_{2:3} & -\bar\Phi_2(\vec{M})+\bar\Phi_3(\vec{M}) & \cdots & (-1)^{n-1}V^-_{3:n+1} \\ \cdots & \cdots & \cdots & \ddots & \vdots \\ (-1)^{n+1}V^+_{1:n+1} & (-1)^{n}V^+_{2:n+1} & (-1)^{n-1}V^+_{3:n+1} & \cdots & -\bar\Phi_n(\vec{M}) \end{pmatrix} \end{aligned} \tag{3.45}$$

where $\bar\Phi_i(\vec{M}) = (C^{-1}\kappa\Phi)_i - (C^{-1}\vec{M})_i$[7]. The homomorphism (3.39) follows from the definition of N and (2.32).

Hilbert series then predict that components of $N(\vec{0})$[8] will generate the Coulomb

[6] Note that M_i can be viewed as a scalar component of a background vector supermultiplet associated to the flavor node adjacent to i and that the definition of h_i treats it on the same footing as scalar components of vector supermultiplets of *gauge* nodes j adjacent to i, for which $\kappa_{ij} = -1$.

[7] $C^{-1}\kappa = \mathbb{1}$ for type A and $\frac{1}{2}\mathbb{1}$ for type D, respectively, given our choices of bases.

[8] We treat the complex masses $\vec{M}$ as *parameters* of the theory rather than new moduli. Then

branch chiral ring R:

$$R = \mathbb{C}[N_{ij}(\vec{0})]/I \tag{3.46}$$

where I is the ideal of gauge-dependent relations as defined in (3.31).

This claim is already non-trivial (and was made in [78] for cases of type A). To see this note that as a gauge-invariant operator, the Casimir invariant $\sum_{1<a<b<r_i} \varphi_{i,a}\varphi_{i,b}$ can be found in the chiral ring. It should be possible to express it in terms of ring generators $N_{ij}(\vec{0})$ but that clearly cannot be done without invoking some relations in I and we would like a guarantee that those relations are sufficient for this purpose.

However, one should expect such a guarantee on theoretical grounds. On the one hand, the abelianisation approach manifestly includes all Casimir invariants of $\varphi_{i,a}$. On the other hand, Casimir invariants of degree d exhibit R-symmetry spin d and all chiral rings considered in this section are generated by operators of spin 1, as computed using Hilbert series methods. Therefore any Casimir invariants of degree greater than 1 must be equal to some combination of spin 1 operators.

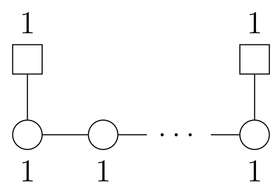
We are not aware of a generic formula for relations between Casimir invariants and moment map components but they can always be derived with a sensible ansatz: just try all linear combinations of generators with vanishing topological charges with the correct overall R-symmetry spin.

3.2 Type A: further examples

Previous sections identify gauge-invariant generators of the chiral ring and lay the groundwork for generalisation to more general quivers. The current section concludes our investigation of quivers of type A by expressing (3.46) as a ring quotiented by an ideal of gauge-invariant relations.

3.2.1 $\min A_n$

The Coulomb branch of the quiver



(3.47)

$C^{-1}\vec{M}$ is just a vector of complex numbers and components of $N(\vec{0})$ are straightforwardly generated as shifts of components of $N(\vec{M})$ by constant numbers and vice versa, so the two generating sets are equivalent.

has the highest weight generating function [20]

$$\mathrm{HWG}(t,\mu_i) = \frac{1}{1-\mu_1\mu_n t^2} \tag{3.48}$$

which identifies a single (co)adjoint generator N subject to several relations transforming in particular representations. We can compare this HWG against known tables of nilpotent orbits [20][9] and find that it is the minimal nilpotent orbit. This variety is fully described by a set of quadratic Joseph relations:

$$t^4[0,1,0,\ldots,0,1,0] : \mathrm{rank}\ N(\vec{0}) \leq 1 \tag{3.49}$$

$$t^4[0,\ldots,0] : \mathrm{tr}(N(\vec{0})^2) = 0 \tag{3.50}$$

One can now construct the chiral ring and the moment map (3.45) to explicitly check that, in fact,

$$t^4[0,1,0,\ldots,0,1,0] : (N^a_i - \delta^a_i \frac{M_1-M_n}{n+1})(N^b_j - \delta^b_j \frac{M_1-M_n}{n+1}) - (a \leftrightarrow b) = 0 \tag{3.51}$$

$$t^4[0,\ldots,0] : \mathrm{tr}(N^k) - \frac{n(M_1-M_n)^k + (-n)^k(M_1-M_n)^k}{(n+1)^k} = 0 \tag{3.52}$$

where $N = N(\vec{M})$ and we redefined $M_i =: M_{i,1}$ to reduce clutter.

This calculation is particularly tractable owing to the quiver's abelian gauge nodes and was partially performed in [78]. Note that when complex mass parameters are set equal the equations reproduce predictions from Hilbert series. Moreover, the left hand sides of (3.51) and (3.52) generate an ideal $J(\vec{M})$ of gauge-invariant operators. The chiral ring is then given by

$$R = \mathbb{C}[N_{ij}(\vec{M})]/I(\vec{M}) = \mathbb{C}[N_{ij}(\vec{M})]/J(\vec{M}) \tag{3.53}$$

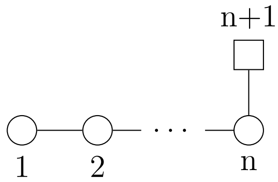
N_{ij} and $J(\vec{M})$ are both specified in terms of gauge-invariant operators, making good on our promise to define the chiral ring purely in terms of physical moduli.

The space can be identified with $\mathrm{T}^*\mathbb{P}^n$ which is known to have a single deformation parameter, here the difference of masses.

[9]This particular family of spaces had been studied earlier e.g. in [11, 48, 59, 86, 112, 116–120], appearing for example as the reduced moduli space of one $SU(n+1)$ instanton.

3.2.2 max A_n

Coulomb branches of quivers



(3.54)

are isomorphic to maximal nilpotent orbits of $\mathfrak{sl}(n+1,\mathbb{C})$ [20]. Its structure is most easily seen by calculating the unrefined Hilbert series [20]:

$$\mathrm{HS}(t) = \frac{\prod_{k=2}^{n+1}(1-t^{2k})}{(1-t^2)^{(n+1)^2-1}} = 1 + \left((n+1)^2 - 1\right)t^2 + \dots \tag{3.55}$$

We see that their chiral rings are again generated by the (co)adjoint generator N defined by (3.45). The (massless) relations can be read off from the Hilbert series' numerator:

$$t^{2k}[0,\dots,0]: \ \mathrm{tr}(N(\vec{0})^k) = 0 \tag{3.56}$$

for $2 \le k \le n+1$.

Calculating complex-mass-deformed relations for general n proves much more challenging than for minimal nilpotent orbits but numerical calculations at low enough n are viable. It suffices to replace $N(\vec{0}) \mapsto N(\vec{M})$ and straightforwardly evaluate[10]:

- $n = 1$:

$$t^4[0]: \ \mathrm{tr}(N(\vec{M})^2) = 2n_1^2 \tag{3.57}$$

- $n = 2$:

$$t^4[0,0]: \ \mathrm{tr}(N(\vec{M})^2) = 2(n_1^2 + n_2^2 - n_1 n_2) \tag{3.58}$$

$$t^6[0,0]: \ \mathrm{tr}(N(\vec{M})^3) = 3n_1 n_2(n_2 - n_1) \tag{3.59}$$

- $n = 3$:

$$t^4[0,0]: \ \mathrm{tr}(N(\vec{M})^2) = 2(n_1^2 + n_2^2 + n_3^2 - n_1 n_2 - n_2 n_3) \tag{3.60}$$

$$t^6[0,0]: \ \mathrm{tr}(N(\vec{M})^3) = 3n_2(n_3 - n_1)(n_1 - n_2 + n_3) \tag{3.61}$$

[10]Complex masses were reparametrised $M_{n,i} = M - n_i + n_{i+1}$ (with $n_{n+1} = 0$) for cleaner presentation; the parameter M automatically drops out.

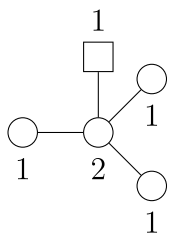
$$t^8[0,0]:\ \mathrm{tr}(N(\vec{M})^4) = 2(\sum_i n_i^4 + 3n_2^2(n_1^2+n_3^2) - 2n_2(n_1^3 + n_1 n_2^2 + n_2^2 n_3 + n_3^3)) \tag{3.62}$$

These relations are necessary and sufficient, as can be seen from their theories' Hilbert series.

3.3 Type D: generalities

3.3.1 $\mathfrak{so}(8,\mathbb{C})$: An example

The synthetic method extends to balanced quivers of type D and height 2 which we demonstrate on one of the simplest examples. The quiver



(3.63)

is shaped as the Dynkin diagram of D_4, suggesting $\mathfrak{so}(8,\mathbb{C})$ symmetry of the Coulomb branch. Its HWG shows that the chiral ring is generated by 28 generators assembled into the (co)adjoint representation N of $\mathfrak{so}(8,\mathbb{C})$ [20]:

$$\mathrm{HWG}(t) = \frac{1}{1-\mu_2 t^2} \tag{3.64}$$

The (massless) relations can also be identified through operator counting or, since the space is the minimal nilpotent orbit, simply by reading off the quadratic Joseph relations:

$$t^4([2,0,0,0]+[0,0,0,0]) : N(\vec{0})^2 = 0 \tag{3.65}$$

$$t^4([0,0,2,0]+[0,0,0,2]) : N(\vec{0})_{[ij}N(\vec{0})_{kl]} = 0 \tag{3.66}$$

The operators in N correspond to 4 generators of the Cartan subalgebra, 12 positive roots and their 12 negative root counterparts. As expressed in the simple root basis, the positive roots are:

$$\begin{aligned}\Phi^+ = \{&\langle 1,0,0,0\rangle, \langle 0,1,0,0\rangle, \langle 0,0,1,0\rangle, \langle 0,0,0,1\rangle, \langle 1,1,0,0\rangle, \langle 0,1,1,0\rangle,\\ &\langle 0,1,0,1\rangle, \langle 1,1,1,0\rangle, \langle 1,1,0,1\rangle, \langle 0,1,1,1\rangle, \langle 1,1,1,1\rangle, \langle 1,2,1,1\rangle\}\end{aligned} \tag{3.67}$$

Roots label monopole operators by specifying charges at appropriate nodes: the first integer gives the topological charge under the leftmost node, followed by topological charges at the central, top right and finally bottom right node. Each node also contributes a topologically uncharged linear Casimir corresponding to the generator of the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{so}(8,\mathbb{C})$ carrying the same label. The fully assembled coadjoint generator – again playing the role of the moment map to the theory's Coulomb branch – is

$$N(\vec{M}) = \begin{pmatrix} \mathbf{H}\bar{\Phi}_1 & \bar{\mathbf{D}}^{\langle 1,0,0,0\rangle}_{\langle 1,2,1,1\rangle} & \bar{\mathbf{D}}^{\langle 1,1,0,0\rangle}_{\langle 1,1,1,1\rangle} & \bar{\mathbf{D}}^{\langle 1,1,1,0\rangle}_{\langle 1,1,0,1\rangle} \\ -(\bar{\mathbf{D}}^{\langle 1,0,0,0\rangle}_{\langle 1,2,1,1\rangle})^T & \mathbf{H}(-\bar{\Phi}_1+\bar{\Phi}_2) & \bar{\mathbf{D}}^{\langle 0,1,0,0\rangle}_{\langle 0,1,1,1\rangle} & \bar{\mathbf{D}}^{\langle 0,1,1,0\rangle}_{\langle 0,1,0,1\rangle} \\ -(\bar{\mathbf{D}}^{\langle 1,1,0,0\rangle}_{\langle 1,1,1,1\rangle})^T & -(\bar{\mathbf{D}}^{\langle 0,1,0,0\rangle}_{\langle 0,1,1,1\rangle})^T & \mathbf{H}(-\bar{\Phi}_2+\bar{\Phi}_3+\bar{\Phi}_4) & \bar{\mathbf{D}}^{\langle 0,0,1,0\rangle}_{\langle 0,0,0,1\rangle} \\ -(\bar{\mathbf{D}}^{\langle 1,1,1,0\rangle}_{\langle 1,1,0,1\rangle})^T & -(\bar{\mathbf{D}}^{\langle 0,1,1,0\rangle}_{\langle 0,1,0,1\rangle})^T & -(\bar{\mathbf{D}}^{\langle 0,0,1,0\rangle}_{\langle 0,0,0,1\rangle})^T & \mathbf{H}(-\bar{\Phi}_3+\bar{\Phi}_4) \end{pmatrix} \tag{3.68}$$

where

$$\mathbf{D}^{\alpha}_{\beta} = \frac{1}{4}\begin{pmatrix} \mathrm{i}(V_\alpha+V_{-\alpha}+V_\beta+V_{-\beta}) & V_\alpha-V_{-\alpha}-V_\beta+V_{-\beta} \\ -V_\alpha+V_{-\alpha}-V_\beta+V_{-\beta} & \mathrm{i}(V_\alpha+V_{-\alpha}-V_\beta-V_{-\beta}) \end{pmatrix} \tag{3.69}$$

$$\mathbf{H} = \begin{pmatrix} 0 & \mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix} \tag{3.70}$$

$$\bar{\Phi}_i = \frac{1}{2}\Phi_i - (C^{-1}\vec{M})_i \tag{3.71}$$

The V_α and Φ_i are gauge-invariant objects which can be expressed in terms of gauge-dependent abelianised variables; those are in turn defined just as in Section 3.1.3. The explicit expressions are:

$$\Phi_1 = \varphi_1 \tag{3.72}$$

$$\Phi_2 = \varphi_{2,1} + \varphi_{2,2} \tag{3.73}$$

$$\Phi_3 = \varphi_3 \tag{3.74}$$

$$\Phi_4 = \varphi_4 \tag{3.75}$$

$$V_{\langle\pm1,0,0,0\rangle} = u_1^\pm \tag{3.76}$$

$$V_{\langle 0,\pm1,0,0\rangle} = u_{2,1}^\pm + u_{2,2}^\pm \tag{3.77}$$

$$V_{\langle 0,0,\pm1,0\rangle} = u_3^\pm \tag{3.78}$$

$$V_{\langle 0,0,0,\pm1\rangle} = u_4^\pm \tag{3.79}$$

$$V_{\langle\pm1,\pm1,0,0\rangle} = \frac{u_1^\pm u_{2,1}^\pm}{\varphi_1-\varphi_{2,1}} + \frac{u_1^\pm u_{2,2}^\pm}{\varphi_1-\varphi_{2,2}} \tag{3.80}$$

$$V_{\langle 0,\pm1,\pm1,0\rangle} = \frac{u_{2,1}^\pm u_3^\pm}{\varphi_{2,1}-\varphi_3} + \frac{u_{2,2}^\pm u_3^\pm}{\varphi_{2,2}-\varphi_3} \tag{3.81}$$

$$V_{\langle 0,\pm1,0,\pm1\rangle} = \frac{u^\pm_{2,1}u^\pm_4}{\varphi_{2,1}-\varphi_4} + \frac{u^\pm_{2,2}u^\pm_4}{\varphi_{2,2}-\varphi_4} \tag{3.82}$$

$$V_{\langle \pm1,\pm1,\pm1,0\rangle} = \frac{u^\pm_1 u^\pm_{2,1}u^\pm_3}{(\varphi_1-\varphi_{2,1})(\varphi_{2,1}-\varphi_3)} + \frac{u^\pm_1 u^\pm_{2,2}u^\pm_3}{(\varphi_1-\varphi_{2,2})(\varphi_{2,2}-\varphi_3)} \tag{3.83}$$

$$V_{\langle \pm1,\pm1,0,\pm1\rangle} = \frac{u^\pm_1 u^\pm_{2,1}u^\pm_4}{(\varphi_1-\varphi_{2,1})(\varphi_{2,1}-\varphi_4)} + \frac{u^\pm_1 u^\pm_{2,2}u^\pm_4}{(\varphi_1-\varphi_{2,2})(\varphi_{2,2}-\varphi_4)} \tag{3.84}$$

$$V_{\langle 0,\pm1,\pm1,\pm1\rangle} = \frac{u^\pm_{2,1}u^\pm_3 u^\pm_4}{(\varphi_{2,1}-\varphi_3)(\varphi_{2,1}-\varphi_4)} + \frac{u^\pm_{2,2}u^\pm_3 u^\pm_4}{(\varphi_{2,2}-\varphi_3)(\varphi_{2,1}-\varphi_4)} \tag{3.85}$$

$$\begin{aligned} V_{\langle \pm1,\pm1,\pm1,\pm1\rangle} &= \frac{u^\pm_1 u^\pm_{2,1}u^\pm_3 u^\pm_4}{(\varphi_1-\varphi_{2,1})(\varphi_{2,1}-\varphi_3)(\varphi_{2,1}-\varphi_4)} \\ &+ \frac{u^\pm_1 u^\pm_{2,2}u^\pm_3 u^\pm_4}{(\varphi_1-\varphi_{2,2})(\varphi_{2,2}-\varphi_3)(\varphi_{2,2}-\varphi_4)} \end{aligned} \tag{3.86}$$

$$V_{\langle \pm1,\pm2,\pm1,\pm1\rangle} = \frac{(\varphi_{2,1}-\varphi_{2,2})^2 u^\pm_1 u^\pm_{2,1}u^\pm_{2,2}u^\pm_3 u^\pm_4}{(\varphi_1-\varphi_{2,1})(\varphi_1-\varphi_{2,2})(\varphi_{2,1}-\varphi_3)(\varphi_{2,2}-\varphi_3)(\varphi_{2,1}-\varphi_4)(\varphi_{2,2}-\varphi_4)} \tag{3.87}$$

with (3.31) acting on abelianised variables as the ideal of relations. A simple exercise in computer-assisted algebra is sufficient to check that (3.65) and (3.66) are satisfied by $N(\vec{0})$ and further that the gauge-invariant Joseph relations still hold without modification for $N(\vec{M})$:

$$t^4([2,0,0,0]+[0,0,0,0]) : N(\vec{M})^2 = 0 \tag{3.88}$$

$$t^4([0,0,2,0]+[0,0,0,2]) : N(\vec{M})_{[ij}N(\vec{M})_{kl]} = 0 \tag{3.89}$$

This is not to say that complex mass parameters have no effect at all on the Coulomb branch: they modify the generator $N(\vec{M})$ itself by shifting scalar operators. However, this effect can be fully removed by redefining scalar fields with the opposite shift. The algebraic structure of relations (3.88) and (3.89) is also preserved in this particular case. Consequently, complex mass *physically reparametrises* rather than *deforms* this Coulomb branch.

Note that for (3.89) this is the only result consistent with preservation of Coulomb branch symmetry under mass deformation since there are no $\mathfrak{so}(8,\mathbb{C})$-invariant tensors which could stand on the right hand side of that particular relation. (3.88) could have been deformed by $(\vec{M}\cdot\vec{M})\mathbb{1}_{2n}$, judging solely on representational grounds. However the mass can be explicitly defined away in the abelianised treatment by $\varphi_{i,a} \mapsto \varphi_{i,a} + M_{2,1}$ without affecting the structure of the chiral ring and hence the relations.

3.3.2 Charges of chiral ring generators

If the D-type quiver is of height 2 the chiral ring is generated by spin 1 operators assembled into the adjoint representation of $\mathfrak{so}(2n, \mathbb{C})$. The generators again split into linear Casimirs, of which there is one per node, and bare monopole operators labelled by topological charges. In this section we gather our knowledge about the latter.

Extensive sets of Hilbert series calculations [20] applied to these theories show that all monopole operators at R-symmetry spin 1 belong to one of two categories. The following classification identifies a monopole generator with a labelled quiver diagram whose flavour nodes and gauge rank information have been removed:

- Unbroken (and linear) strings of either only $+1$ or only -1 stretching anywhere across the quiver – see Figure 2.2a for an example stretching all the way to the spinor node.
- Unbroken strings of ± 1 (with uniform choice of sign) with charges ± 1 on both rightmost (spinor) nodes – see Figure 2.2b. If both spinor nodes are turned on then a string of ± 2 (with the same choice of sign as ± 1) can be extended from the trivalent node arbitrarily far to the left, terminating with a string of ± 1 which must have length at least 1 – see Figure 2.2c.

It will prove convenient to arrange topological charges into linear vectors and we pick the usual convention, ie. the first $n-2$ entries describe charges on the linear segment from the first node to the trivalent node and the $n-1$-th, resp. n-th entries belong to the top right, resp. top bottom nodes.

3.3.3 Construction of the chiral ring

Construction of the chiral ring is closely analogous to that of Section 3.1.3 with differences arising only with respect to monopoles whose topological charges stretch across multiple nodes.

The simplest and cleanest way to identify monopole operators is to utilise the symplectic structure defined in [78] and captured in the Poisson brackets of operators (3.38).

Minimally charged (gauge-invariant) monopoles at node i are defined as

$$U_i^{\pm} = \sum_a u_{i,a}^{\pm} \tag{3.90}$$

and we can use the action they induce along with the Poisson bracket, $\{U_i^{\pm}, \cdot\}$, to generate the entire set of bare monopole operators. The procedure is inductive on the sum of topological charges of a monopole, $q = \sum q_i$, where we treat positive and negative monopoles separately:

- Restrict to positively charged monopole operators and take the first non-trivial case of $q = 1$. These are the minimally charged monopoles and their description is given above.

- To get the expression for a positive monopole operator V with topological charges $\vec{q}$ whose sum is $\sum_i q_i = q = r + 1$ one can start by assuming the inductive hypothesis, that is, expressions are known for all bare monopole operators up to and including overall topological charge $r > 1$. The classification of monopoles given in the previous section is enough to establish that there exists a monopole operator V' with topological charges $\vec{r}$ such that $\sum_i r_i = r$ and $\vec{q} - \vec{r}$ is the usual unit vector $\vec{e}_i$. Then the monopole V is obtained as follows:
$$V = \pm\{U_i^+, V'\} \tag{3.91}$$
and the sign is chosen so that, when scalar fields in denominators are ordered "lowest indices to the left, highest indices to the right" – eg. in combinations $(\varphi_1 - \varphi_3)$ but not $(\varphi_4 - \varphi_2)$ – the expressions are monic. This generates all positive monopoles.

- To generate negative monopoles merely replace positive abelianised monopole variables with their negative counterparts: $u_{i,a}^+ \mapsto u_{i,a}^-$.

In the $\mathfrak{so}(8, \mathbb{C})$ example the monopole operator with highest overall topological charge was obtained by
$$V_{\langle 1,2,1,1\rangle} \propto \{U_2^+, V_{\langle 1,1,1,1\rangle}\} \tag{3.92}$$
and it is worth taking a look at the structure of (3.87) to see how this monopole operator arrives at overall R-symmetry spin 1.

3.3.4 Moment map

All that remains to define the Coulomb branch moment map is to associate generators of the coadjoint basis with monopole and linear Casimir operators.

- For monopole operators use the correspondence between roots on the one hand and pairs of integers and signs on the other described in (2.41-2.44) to translate labels in the simple root basis into the orthonormal basis:
$$V_\alpha \leftrightarrow V_{rs}^{(ij)} \tag{3.93}$$

where $r, s \in \{+, -\}$ and $1 \le i < j \le n$ and pair them with the corresponding dual root vectors:

$$E^*_{(ri,sj)} \leftrightarrow e_{(ri,sj)} = V^{(ij)}_{rs} \tag{3.94}$$

- Linear Casimirs need to be suitably combined to reproduce Poisson brackets analogously to the case of type A; a mass shift is also allowed by the abelianised Poisson brackets:

$$H^*_i \leftrightarrow h_i = \sum_j \kappa_{ij}\Phi_j - M_i \tag{3.95}$$

Putting everything together the moment map comes out as

$$N = \sum_{\substack{1\le i<j\le n \\ r,s\ \in\{+,-\}}} e_{(ri,sj)} E^*_{(ri,sj)} + \sum_{1\le i\le n} h_i H^*_i \tag{3.96}$$

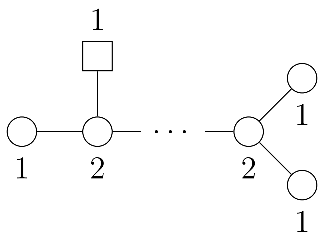
This prescription tends to lead to matrices which struggle to fit on a page so we refer to the case of $\mathfrak{so}(8,\mathbb{C})$ in (3.68) as an exemplar.

The moment map still generates the Lie algebra homomorphism (3.39), albeit for a D_n algebra.

3.4 Type D: further examples

3.4.1 $\min D_n$

The D_n analogue of quivers investigated in Section 3.2.1 is



(3.97)

Their Coulomb branches are the closures of minimal nilpotent orbits of D_n with HWG [20]

$$\mathrm{HWG}(t) = \frac{1}{1-\mu_2 t^2}. \tag{3.98}$$

The Joseph relations on such an orbit are

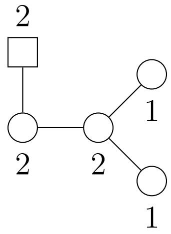
$$t^4([2,0,\ldots,0] + [0,\ldots,0]) : N(\vec{0})^2 = 0 \tag{3.99}$$

$$t^4([0,0,0,1,0,\ldots,0]) : \text{rank } N(\vec{0}) \le 1 \tag{3.100}$$

and have been numerically verified for low values of n. The lack of a complex mass deformation in the minimal nilpotent orbit of $\mathfrak{so}(8,\mathbb{C})$ generalises to minimal nilpotent orbits of $\mathfrak{so}(2n,\mathbb{C})$ with $n > 4$.

3.4.2 n. min D_4

We provide one final example of D_n nilpotent orbits, the next-to-minimal nilpotent orbit quivers



(3.101)

The relations can be deduced from [20]

$$\mathrm{HWG}(t,\mu_i) = \frac{1-\mu_3^2\mu_4^2 t^8}{(1-\mu_2 t^2)(1-\mu_3^2 t^4)(1-\mu_4^2 t^4)} \tag{3.102}$$

and are given by the tensor relations

$$t^4([0,0,0,0]) : \mathrm{tr}(N(\vec{0})^2) = 0 \tag{3.103}$$

$$t^4([0,0,2,0]+[0,0,0,2]) : N(\vec{0})_{[ij}N(\vec{0})_{kl]} = 0 \tag{3.104}$$

and have been verified by our methods. Turning on masses leads to the related set of equations

$$t^4([0,0,0,0]) : \mathrm{tr}(N(\vec{M})^2) = \frac{1}{2}(M_{1,1}-M_{1,2})^2 \tag{3.105}$$

$$t^4([0,0,2,0]+[0,0,0,2]) : N(\vec{M})_{[ij}N(\vec{M})_{kl]} = 0 \tag{3.106}$$

The trace equation shows that this Coulomb branch has a complex mass deformation.

3.5 Synthetic method: a summary

Computations in the previous chapter demonstrated that operator counting and abelianisation can be fruitfully combined in a new approach which we refer to as the *synthetic method*. We chose to adapt the method to the structural peculiarities of type AD quivers to aid exposition, but here we provide a clean, quiver-agnostic and general prescription which will serve us well in analysing quivers of BCG type in the following chapter.

Abelianised variables $\varphi_{i,a}$ have weight 2 (spin 1) under the R-symmetry[11], while the Poisson bracket scales with weight -2. Weights of $u^{\pm}_{i,a}$ can be read off from (2.193). The Coulomb branch chiral ring of any good or ugly theory is graded by R-symmetry weights as $\mathbb{C}[\mathcal{C}] = \sum_{i\in\mathbb{Z}_{\geq 0}} \mathbb{C}[\mathcal{C}]_i$ where $\mathbb{C}[\mathcal{C}]_i$ is the vector space of all Coulomb branch chiral ring operators with R-symmetry weight i.

Any Coulomb branch operator $\mathcal{O}$ with well-defined R-symmetry weight j defines a map $\{\mathcal{O},\cdot\} : \mathbb{C}[\mathcal{C}]_i \to \mathbb{C}[\mathcal{C}]_{i+j-2}$ and therefore operators in $\mathbb{C}[\mathcal{C}]_2$ form a closed Poisson algebra. This algebra is precisely the symmetry algebra $\mathfrak{g}$ of the Coulomb branch and all operators in $\mathbb{C}[\mathcal{C}]_i$ necessarily assemble into tensors of the $\mathbb{C}[\mathcal{C}]_2$ algebra $\mathfrak{g}$. In this thesis we focus almost exclusively on good (in fact, balanced) theories whose Coulomb branch chiral rings are generated by operators in $\mathbb{C}[\mathcal{C}]_2$ and whose symmetry algebra $\mathfrak{g}$ is simple. Consequently, $\mathbb{C}[\mathcal{C}]_2$ operators assemble into a single (coadjoint) representation of $\mathfrak{g}$ – the moment map of the symmetry – which has a matrix realisation for all cases in this work. We may also consider cases whose ring is generated by $\mathbb{C}[\mathcal{C}]_2$ operators transforming in the coadjoint representation along with another set of $\mathbb{C}[\mathcal{C}]_4$ operators, also in the coadjoint representation, in which case the following discussion straightforwardly generalises.

The synthetic method itself can be summarised as follows. Let $X_k \in \mathfrak{g}$ form a basis of $\mathfrak{g}$ satisfying $[X_k, X_l] = \sum_m c_{klm} X_m$. There is a basis of $\mathbb{C}[\mathcal{C}]_2$ formed by $\mathcal{O}_k$ such that $\{\mathcal{O}_k, \mathcal{O}_l\} = \sum_m c_{klm}\mathcal{O}_m$. If X^*_k are dual to X_k, ie. $\langle X^*_k, X_l\rangle = \delta_{kl}$, the moment map N[12] is explicitly constructed as

$$N = \sum_k \mathcal{O}_k X^*_k. \tag{3.107}$$

This definition guarantees that $\langle N,\cdot\rangle$ acts as a Lie algebra homomorphism:

$$\{\langle N, X_k\rangle, \langle N, X_l\rangle\} = \langle N, [X_k, X_l]\rangle \tag{3.108}$$

The choice of $\mathcal{O}_k$ is heavily constrained – enough, in fact, to allow us to select an almost unique[13] set of operators from $\mathbb{C}[\mathcal{C}_{\text{abel}}]^{W_G}$ to form components of N. And since N generates the Coulomb branch chiral ring, we have found a set of generators for $\mathbb{C}[\mathcal{C}]$.

The moment map N satisfies certain matrix relations, which can be inferred from the Hilbert series of the Coulomb branch. Let us denote the ideal they form as I_M,

[11] The R-symmetry is assumed to be the $SU(2)$ factor acting non-trivially on the Coulomb branch. An operator's weight is twice its conformal dimension.

[12] We reserve the usual symbol for moment maps, μ, for highest weight fugacities.

[13] It is enough to declare $\sum_a u^+_{i,a}$ a positive simple root operator to fix the remaining choices, at least in the quivers we consider.

and let I_A stand for the ideal of abelian relations (2.193). Then we claim that

$$\mathbb{C}[\mathcal{C}] = \mathbb{C}[N(u^{\pm}_{i,a}, \varphi_{i,a})]/I_A = \mathbb{C}[N]/I_M. \tag{3.109}$$

Note that the third object contains no abelian expressions (and hence no potentially troubling factors of $1/(\varphi_i - \varphi_j)$). In this last step abelianised relations and components of N, both expressed in terms of abelianised variables $u^{\pm}$ and φ, are replaced by symbolic components of N (ie. N_{ij}) with matrix relations also expressed in terms of components of N.

It is in effect a change of variables to a set which is well-defined even on the non-abelian locus of the moduli space. On the other hand, the less-than-perfect abelianised representation brings with it a major advantage: it is very explicit and rigid. We can use it to construct the generator N, independently check its relations and also calculate *exact* coefficients in these relations (including dependence on mass parameters) – something which is forever off-limits to any method relying solely on Hilbert series, all while keeping representation theory front and centre.

Chapter 4

Wreathed and non-simply laced unitary quivers

4.1 Introduction

In this chapter we clarify the relation between several concepts relating to $3d\ \mathcal{N}=4$ Coulomb branches of $BCFG$ type. It has been known since [11] that the Coulomb branch monopole formula [48] can be extended to quivers in the form of non-simply laced framed Dynkin diagrams. However, while all the ingredients of a simply laced Dynkin diagram – gauge and flavor nodes, hypermultiplet links – are readily interpretable, it was unknown at the time what to make of the novel multiple link. Recently [84] argued that their Coulomb branches result from a discrete folding operation on Coulomb branches of simply laced quivers. We independently derive and illustrate the same claim through the method of abelianisation [78]. We also develop a second, related but distinct discrete operation which was previously studied in [100, 101]. Both aspects expand on our previous work in [1].

The main concepts, presented in Fig. 4.1, can be summarised as follows.

Quivers with an automorphism possess a discrete symmetry relating gauge groups. By analogy with continuous gauge groups, it, or any of its subgroups, can be gauged[1], and we demonstrate that this results in a theory whose Coulomb branch is a discrete quotient of the original, where the action by which we quotient is directly induced by the quiver automorphism (or subgroup thereof). This operation, which we call *discrete gauging*, produces *wreathed quivers*. Previous work [100, 101] generated similar results on the Coulomb branch by replacing n $U(1)$ nodes by a $U(n)$ node with adjoint matter.

In contrast, *quiver folding* relates Coulomb branches of pairs of simply laced and non-simply laced quiver gauge theories. To be clear, we show the action on the

[1]Discretely gauging string backgrounds is of course an old idea which has generated a lot of discussion, for example in [121–125], and the findings of this section may be viewed as a new entry.

Coulomb branch and conjecture that one can view it as one effect of an action on the *theory*[2]. However, we have been unsuccessful in our attempts to write down the path integral or compute the Higgs branch of folded theories. Coulomb branches of balanced A_{2n-1}–type quivers, ie. framed linear quivers satisfying the balance condition[3] and exhibiting $\mathfrak{sl}(2n, \mathbb{C})$ symmetry on the Coulomb branch, can be "folded" into Coulomb branches of balanced C_n–type quivers with $\mathfrak{usp}(2n, \mathbb{C})$ symmetry. Balanced D_n–type quiver Coulomb branches, ie. Coulomb branches of balanced framed quivers shaped like D_n Dynkin diagrams, can be "folded" into Coulomb branches of balanced B_{n-1}–type quivers. G_2–type quiver Coulomb branches can be similarly obtained from D_4–type quivers while F_4–type quivers are folded E_6–type quivers. The folded spaces are fixed points under the group action induced by the quiver automorphism and we show that they are symplectic leaves of spaces obtained by discretely gauging their respective original Coulomb branches. In some cases distinct subgroups of the quiver automorphism can give identical sets of fixed points (eg. S_3 and $\mathbb{Z}_3$ of the D_4 affine quiver) and their folded spaces coincide; as a result, there are "fewer" folded than wreathed quivers.

Actions of both discrete gauging and folding on the Coulomb branch are readily interpreted through a geometric lens, see Figure 4.2. We claim that, since discrete gauging is implemented by restricting the chiral ring to invariants of a symmetry group action Γ, the resulting space is an orbifold of the initial Coulomb branch under Γ – and since the Poisson structure respects this group action, the orbifold inherits a natural symplectic structure. If the original space is a nilpotent orbit of some algebra then the orbifold is sometimes, but not always, a nilpotent orbit of the relevant folded algebra, but it is in any case symmetric under the folded algebra's action.

Folding, on the other hand, reduces the Coulomb branch to the fixed subspace under the same group action Γ. We show that it has a Poisson structure and, since the fixed subspace is (a singular) part of the corresponding orbifold, the Hasse diagram [17] of the folded space is a subdiagram of the orbifold's Hasse diagram. In all known cases a nilpotent orbit folds to another nilpotent orbit (of the folded algebra). This situation is reminiscent of a general phenomenon identified in [126], in which orbits in the small affine Grassmannian for an algebraic group G (the subvariety of the affine Grassmannian corresponding to the so-called small coweights of G; see [127] for a friendly introduction addressed to physicists) possess a $\mathbb{Z}_2$ global

[2] In the rest of this text we will elide the distinction between folding a quiver theory and folding its Coulomb branch, but wish to be clear that we present solid evidence only for the latter and at best circumstantial evidence for the former.

[3] A node is balanced when the contributions of gauge and matter to the RG flow of the gauge coupling exactly cancel out *assuming the quiver is understood as a 4d theory.* Assuming simply laced unitary quivers without loops, this amounts to the condition that twice the node's rank equals the sum of all surrounding (gauge or matter) nodes' ranks.

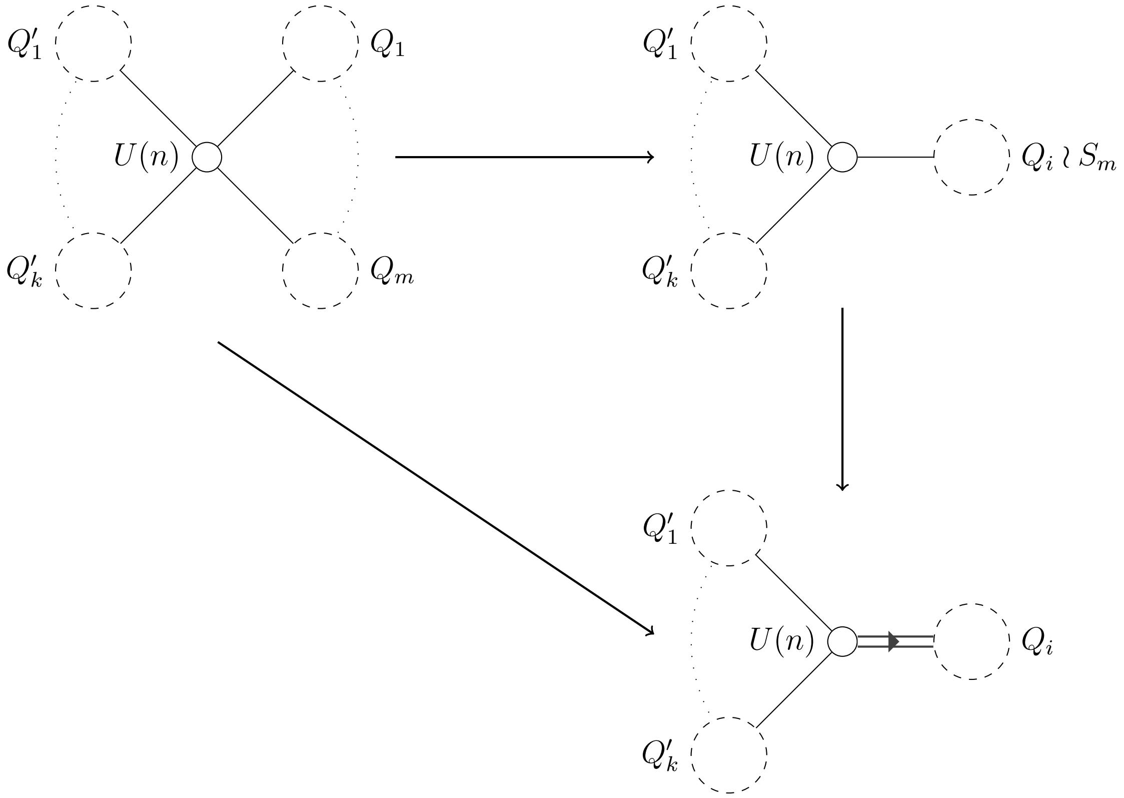


Figure 4.1: (Top left) k generic subquivers Q'_1 through Q'_k and m identical subquivers Q_1 through Q_m are connected to a common central $U(n)$ node. (Top right) Wreathed quiver. (Bottom right) Non-simply laced quiver. The multiple link has valence m, here depicted for $m = 2$.

involution – here these orbits would be depicted as the left portion of Figure 4.2. Some of these orbits can be mapped to so-called *Reeder pieces* which are the union of two nilpotent orbits of G, one which can be identified with a $\mathbb{Z}_2$ quotient of the affine Grassmannian slice, and the other as the $\mathbb{Z}_2$ fixed points – respectively the middle and right parts of Figure 4.2. Coulomb branches of framed unitary ADE quivers were identified with slices in the associated affine Grassmannian [83], following the construction [81, 82], and the $\mathbb{Z}_2$ involution of [126] is realized on the quiver as leg permutations like in Figure 4.1. As a consequence, several of the examples discussed below follow from the geometric point of view from these previous works; the present chapter sheds a new light on this topic by providing quivers for each of the three spaces, and giving formulas to compute the Hilbert series and HWGs of their closures.

Figure 4.3 features a third discrete action called *crossing*. Flavorless simply laced quiver theories possess a certain freedom of reparametrisation: the gauge group $\mathcal{G}$ factorises as $\mathcal{G}/U(1) \times U(1)$, with the decoupled $U(1)$ factor contributing a (geometrically uninteresting) factor of $\mathbb{R}^3 \times S^1$ to the Coulomb branch, which is discarded by convention. Crucially, while the choice of $U(1)$ is somewhat constrained, the allowed options are in practice equivalent and one can in particular choose to ungauge any

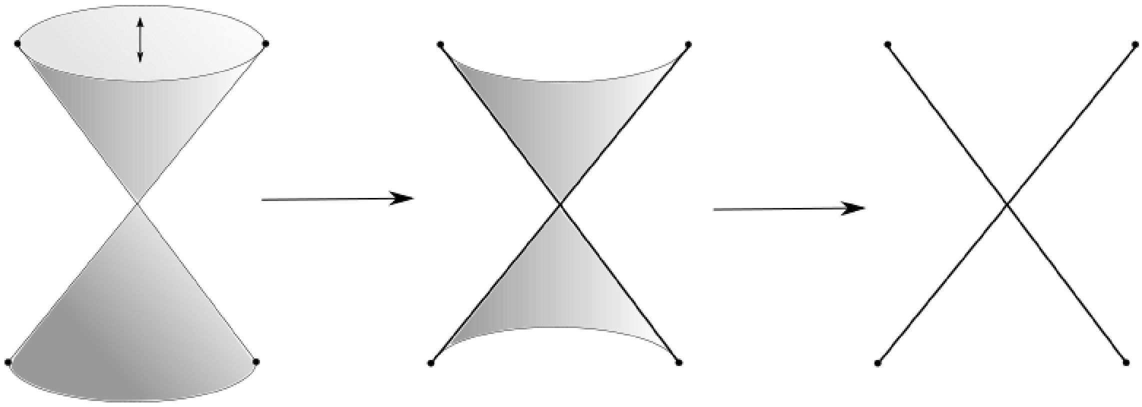


Figure 4.2: (left) Initial Coulomb branch with highlighted $\mathbb{Z}_2$ symmetry. (middle) Coulomb branch of the discretely gauged quiver depicted as an orbifold of the original space. Note that bold edges form a singular subspace under the $\mathbb{Z}_2$ symmetry. (right) Coulomb branch of the folded quiver, the subspace fixed under the $\mathbb{Z}_2$ symmetry.

given $U(1)$ node without affecting the Coulomb branch. The situation is modified for non-simply laced quivers, where ungaugings on opposite sides of the directed multiple link give rise to pairs of Coulomb branches where one is the discrete quotient of the other. We list this case for the sake of completeness, but do not study it further in this thesis. The reader could instead consult the recent treatment in [105].

Kostant-Brylinski reductions

In [128] the authors identified that discrete quotients of certain minimal nilpotent orbits were equivalent to (generically non-minimal) nilpotent orbits of other algebras; their results are summarised in Figure 4.4[4]. The same pattern is observed in discrete gauging and we claim that our construction is a physical realisation of their cases 1,2,3,4 and 9. We empirically confirmed this conjecture using both Hilbert series and abelianisation methods as in [1] up to low but non-trivial rank. The lines painted in green (cases 2, 3, 4 and 9) correspond to wreathed simply laced quivers. Case 1, painted in red, stands apart because of the non-simply laced initial quiver; although the moduli space can be described algebraically using abelianised variables, the explicit implementation of the monopole formula for non-simply laced wreathed quivers is postponed for future investigations.

A recent work [105] showed that cases 5, 6 and 7 (yellow in Figure 4.4) occur in Coulomb branches of non-simply laced quivers. The $\mathbb{Z}_n$ quotient corresponds to gauging a $U(1)$ node on the "long" end of an edge of multiplicity n and ungauging another $U(1)$ node on the "short" end. The quiver realisations of all eight known cases are collected in Figure 4.4.

[4][129] provide more examples of discrete and non-discrete quotients in nilpotent orbits.

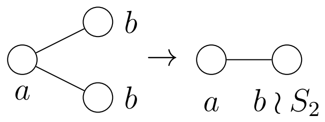
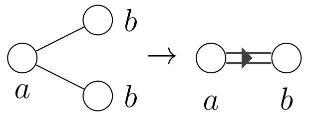
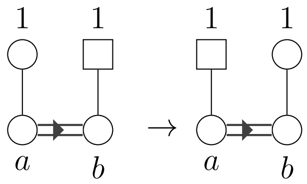
Operation on gauge theory	Quiver description	Resulting quiver	String background action	Operation on Coulomb branch
Discrete gauging		Wreathed quiver	Orbifold	Discrete quotient (dimension preserving)
Folding		Non-simply laced quiver	Orientifold	Fixed points (not dimension preserving)
Crossing		Non-simply laced quiver	?	Discrete quotient (dimension preserving)

Figure 4.3: Discrete actions on the quiver

Case number 8 still presents a challenge, and we are not aware of any quiver realisation of the corresponding $\mathbb{Z}_2^2$ quotient. However the HWGs are under control, and are discussed briefly at the end of Section 4.2.5.

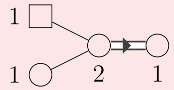
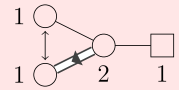
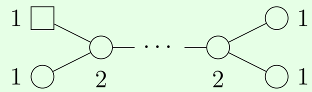
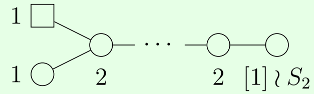
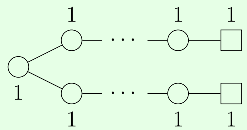
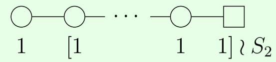
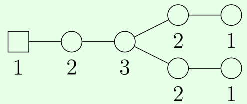
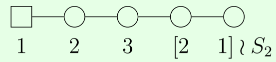
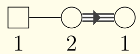
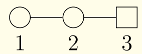
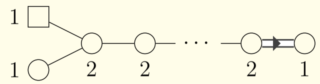
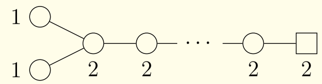
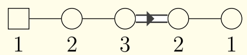
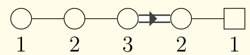
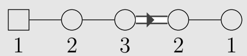
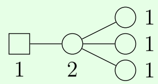
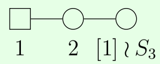
	$\mathfrak{g}$	$\tilde{\mathfrak{g}}$	V	M	$\dim_{\mathbb{H}} M$	Quiver for $\mathfrak{g}$	Quiver for $\tilde{\mathfrak{g}}$
1	B_3	G_2	$\mathbb{C}^7$	$\mathcal{O}^{G_2}_{[3,2^2]}$	4		
2	D_{n+1}	B_n	$\mathbb{C}^{2n+1}$	$\mathbb{Z}_2 \cdot \mathcal{O}^{B_n}_{[3,1^{2n-2}]}$	$2n-1$		
3	A_{2n-1}	C_n	$(\Lambda^2\mathbb{C}^{2n})/\mathbb{C}$	$\mathbb{Z}_2 \cdot \mathcal{O}^{C_n}_{[2^2,1^{2n-4}]}$	$2n-1$		
4	E_6	F_4	$\mathbb{C}^{26}$	$\mathbb{Z}_2 \cdot \mathcal{O}^{F_4}_{[3,2^8,1^7]}$	11		
5	G_2	A_2	$\mathbb{C}^3 \oplus \Lambda^2\mathbb{C}^3$	$\mathbb{Z}_3 \cdot \mathcal{O}^{A_2}_{[3]}$	3		
6	B_n	D_n	$\mathbb{C}^{2n}$	$\mathbb{Z}_2 \cdot \mathcal{O}^{D_n}_{[3,1^{2n-3}]}$	$2n-2$		
7	F_4	B_4	$\mathbb{C}^{16}$	$\mathbb{Z}_2 \cdot \mathcal{O}^{B_4}_{[2^4,1]}$	8		
8	F_4	D_4	$\mathbb{C}^8 \oplus \mathbb{C}^8 \oplus \mathbb{C}^8$	$\mathbb{Z}_2^2 \cdot \mathcal{O}^{D_4}_{[3,2^2,1]}$	8		
9	D_4	G_2	$\mathbb{C}^7 \oplus \mathbb{C}^7$	$S_3 \cdot \mathcal{O}^{G_2}_{[3^2,1]}$	5		

Figure 4.4: Augmented Table 1 of [128]. $\mathfrak{g}$ ($\tilde{\mathfrak{g}}$) is the symmetry algebra of the original (reduced) space. $V \simeq \mathfrak{g}/\tilde{\mathfrak{g}}$. M is the reduced space with discrete group prefactor Γ if the original is its Γ-cover. The initial (reduced) space is the Coulomb branch of the first (second) quiver, respectively. See Section 4.4.2 for more details on the first row and an explanation of the arrow connecting two $U(1)$ nodes.

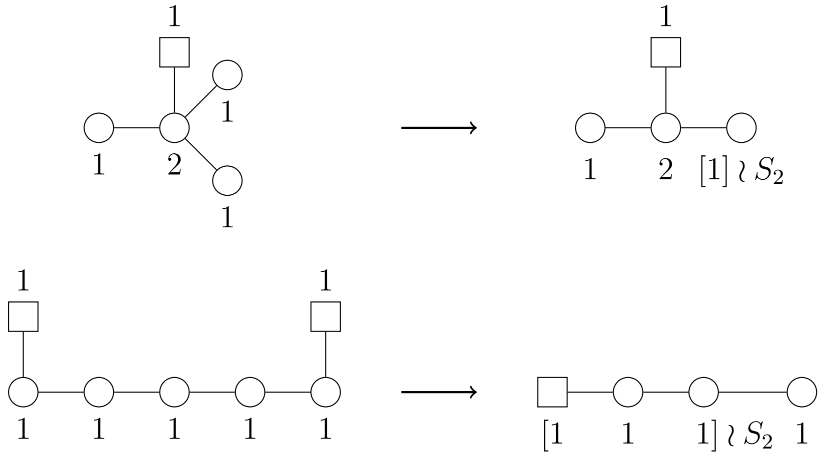


Figure 4.5: Quivers on the left wreathe into quivers on the right.

4.2 Discrete gauging

Our first example of a discrete quiver operation, discrete gauging, orbifolds the Coulomb branch by a subgroup of the quiver's automorphisms. Another operation, which also acts on the Coulomb branch as an orbifold, was previously studied in [100, 101][5]. Ours differs in several respects: it preserves the dimension of the Higgs branch as well as the Coulomb branch, allows for consistent and successive discrete gauging of nodes into "larger" nodes and generalises beyond acting on a collection of n $U(1)$ nodes (which form a $U(n)$ node with adjoint matter in [100, 101]) to acting on n copies of arbitrary gauge groups or "legs" of the quiver.

It is possible to discretely gauge any quiver of the type depicted in the top left corner of Fig. 4.1, ie. one with m identical legs[6] Q_i (and potentially other legs Q'_j) connected to a single common node which we call the *pivot*[7]. One discretely gauges the m identical legs by extending the overall gauge group with the symmetric group S_m, or a subgroup thereof, which permutes the gauge factors associated with each leg. We say that we have *gauged the quiver's automorphism*. For example, three legs composed solely of $U(1)$ nodes will arrange into a $U(1) \wr S_3$ node, while two legs with $U(2) \times U(1)$ gauge nodes will combine to give $(U(2) \times U(1)) \wr S_2$, with S_2 simultaneously exchanging $U(2)$ and $U(1)$ factors.

Our strategy in this section consists of the following steps. We first demonstrate the existence of a well-defined orbifolding operation on the Coulomb branch, giving results consistent with existing literature. Then we suggest that the operation acts on the quiver as a whole in a way that can be deduced from the Coulomb branch

[5] In contrast to our treatment, these works did not claim to discretely gauge the theory, but restricted their claims only to effects on the Coulomb branch.

[6] A leg can have arbitrary shape and in particular need not be linear.

[7] It may be possible to discretely gauge quivers without a pivot node but we do not have a successful case to present.

action and that the results of this operation should be viewed as quivers in their own right, even if they often cannot be written down using existing notation; we introduce the concept of *wreathed quivers* to get over this difficulty; see Figure 4.5 for two examples. We support this claim by generalising the monopole formula to this family of quivers and computing an example, as well as calculating a few wreathed quiver Higgs branches. We also conjecture that a well known Higgs branch operation is the 3d mirror to this operation on the Coulomb branch.

4.2.1 Wreath product

We pause for a moment to introduce the notion of the *wreath product* $G \wr \Gamma$ of a group G by a permutation group $\Gamma \subseteq S_n$ (the integer n is understood in the notation $G \wr \Gamma$, which we could denote $G \wr_n \Gamma$ if there is a risk of confusion) [130]. As a set, we define

$$G \wr_n \Gamma \equiv G \wr \Gamma = \left(\prod_{i=1}^{n} G_i \right) \times \Gamma \,, \tag{4.1}$$

where the $\times$ denotes the Cartesian product of sets, *not* the direct product of groups. There are n copies G_1, ..., G_n of the group G. An element of $(g, \sigma) \in G \wr \Gamma$ is an ordered list of n elements g_i of G together with a permutation $\sigma \in \Gamma$. The group multiplication law is given, for $(g, \sigma) \in G \wr \Gamma$ and $(g', \sigma') \in G \wr \Gamma$, by

$$(g, \sigma) \cdot (g', \sigma') = (g\sigma(g'), \sigma\sigma') \,, \text{ with } (g\sigma(g'))_i = g_i g'_{\sigma^{-1}(i)} \,. \tag{4.2}$$

Intuitively, $G \wr \Gamma$ is the direct product of n copies of G, which can in addition be permuted by Γ.

In this section we consider wreath products where G is a unitary group $U(r)$, or more generally a direct product of finitely many unitary groups $U(r_1) \times \cdots \times U(r_k)$. In this case, in particular in the quivers, we extend the usual shorthand notation in which $U(r)$ is replaced by the rank r, and we write $r \wr \Gamma$ for $U(r) \wr \Gamma$, and more generally $[r_1 \quad \cdots \quad r_k] \wr \Gamma$ for $(U(r_1) \times \cdots \times U(r_k)) \wr \Gamma$.

4.2.2 Action on the Coulomb branch

We will first study this procedure through the lens of Coulomb branch abelianisation. The goal is to show that the Coulomb branch can be reduced to an orbifold by an automorphism of the quiver.

Since each node contributes several variables to the abelianised chiral ring, there is an induced S_m action permuting them. For any $\pi \in S_m$, we have

$$\pi(x_{i,a}) := x_{\pi(i),a} \tag{4.3}$$

Action on more complicated (polynomial or rational) functions of these variables is defined by action on indices of the full expression. For example

$$\pi(u^+_{i,a}u^+_{j,b}) = u^+_{\pi(i),a}u^+_{\pi(j),b}. \tag{4.4}$$

Note that mass parameters should be treated as *numbers* (parameters) rather than ring elements (VEVs); therefore π does *not* act on them, ie.

$$\pi(M_{i,a}) = M_{i,a}. \tag{4.5}$$

In fact, this constraint forces

$$\pi(M_{i,a}) = M_{i,a} = M_{\pi(i),a}. \tag{4.6}$$

To see this consider the A_5 theory which gauges to the bottom right quiver in Fig. 4.5:

$$\pi(u^+_1u^-_1) = \pi\left(-(\varphi_1-\varphi_2)(\varphi_1-M_1)\right) = -(\varphi_5-\varphi_4)(\varphi_5-M_1) \tag{4.7}$$
$$u^+_5u^-_5 = -(\varphi_5-\varphi_4)(\varphi_5-M_5) \tag{4.8}$$

Since $\pi(u^+_1u^-_1) = u^+_5u^-_5$, the two mass parameters must be equal to preserve symmetry under π. This is a sensible constraint: if $M_1 \neq M_5$ then the mass deformation breaks the quiver's S_2 symmetry.

We should check that the form of the Poisson brackets (2.196)-(2.198) is compatible with this action in the sense that $\{\pi(x),\pi(y)\} = \pi(\{x,y\})$.

$$\{\pi(\varphi_{i,a}),\pi(u^\pm_{i,a}\}) = \{\varphi_{\pi(i),a},u^\pm_{\pi(i),a}\} = \pm u^\pm_{\pi(i),a} = \pi\left(\{\varphi_{i,a},u^\pm_{i,a}\}\right) \tag{4.9}$$
$$\{\pi(u^+_{i,a}),\pi(u^-_{i,a})\} = \{u^+_{\pi(i),a},u^-_{\pi(i),a}\} = \frac{\partial}{\partial\varphi_{\pi(i),a}}\frac{\prod_{w\in\mathcal{R}}\langle w,\pi(\vec{\varphi})\rangle^{|w_{i,a}|}}{\prod_{\alpha\in\Phi}\langle\alpha,\pi(\vec{\varphi})\rangle^{2|\alpha_{i,a}|}} = \pi\left(\{u^+_{i,a},u^-_{i,a}\}\right) \tag{4.10}$$
$$\{\pi(u^\pm_{i,a}),\pi(u^\pm_{j,b})\} = \{u^\pm_{\pi(i),a},u^\pm_{\pi(j),b}\} = \pm\kappa_{ij}\frac{u^\pm_{\pi(i),a}u^\pm_{\pi(j),b}}{\varphi_{\pi(i),a}-\varphi_{\pi(j),b}} = \pi\left(\{u^\pm_{i,a},u^\pm_{j,b}\}\right) \tag{4.11}$$

The first line is clearly compatible with the action. The second line also succeeds with a simple relabelling: $w_{\pi(i)} \leftrightarrow w_i$ and $\alpha_{\pi(i)} \leftrightarrow \alpha_i$. The third line is similarly preserved because $\kappa_{\pi(i)\pi(j)} = \kappa_{ij}$ is a consequence of the automorphism. In fact, it is noteworthy that the third line forces the action of π to preserve connectedness while the second line enforces identical gauge and matter content on each leg Q_i.

To implement the quotient on the Coulomb branch chiral ring, it is enough to declare that only S_m–invariant operators are physical. This is easily done through the use of a projector:

$$P(\cdot) = \frac{1}{m!} \sum_{\pi \in S_m} \pi(\cdot). \tag{4.12}$$

Every operator of the form $P(\mathcal{O})$ is physical.

The effect on the Coulomb branch is then transparent. If $\mathcal{C}$ and $\tilde{\mathcal{C}}$ are Coulomb branches of, respectively, the original quiver and discretely gauged quivers, the two spaces are related by

$$\tilde{\mathcal{C}} = \mathcal{C}/S_m \tag{4.13}$$

ie. the discretely gauged Coulomb branch is a S_m orbifold of the original space. This construction leads to new Coulomb branches which were previously unknown, provided that they are orbifolds of known Coulomb branches.

Note that nothing prevents generalisation from S_m to arbitrary subgroups Γ of S_m, for instance the alternating group A_m or cyclic group $\mathbb{Z}_m$. We investigate one such example in Section 4.4.2.

The projector acts remarkably simply on moment maps of type AD quivers as studied in [1]. In fact, if $N_{A_{2n-1}}$ is the moment map for a type A_{2n-1} quiver then

$$P(N_{A_{2n-1}}) = N_{C_n} \tag{4.14}$$

is a C_n moment map and P acts component-wise. Similarly,

$$P(N_{D_{n+1}}) = N_{B_n}. \tag{4.15}$$

To see this action on an example, and to illustrate why its action on moment maps is so simple, consider the top left quiver in Fig. 4.14. Select the Chevalley-Serre basis of D_4 and its operator counterpart, ie. the basis of operators in $\mathbb{C}[\mathcal{C}]_2$ which replicates (2.10)-(2.13) with Poisson brackets. We will denote the algebra elements and their duals with capital letters, reserving lower case letters for operators with appropriate commutation relations. In this notation, the moment map is

$$N_{D_4} = \sum_{1 \leq i \leq 4} h_i H_i^* + \sum_{\alpha \in \Phi^+} \left(e_\alpha E_\alpha^* + e_{-\alpha} E_{-\alpha}^* \right) . \tag{4.16}$$

In the interest of concreteness we perform the projection by P in full detail. Start with the moment map of D_4:

$$\begin{aligned}
N_{D_4} =& h_1 H_1^* + h_2 H_2^* + h_3 H_3^* + h_4 H_4^* \\
&+ e_{\langle \pm 1,0,0,0\rangle} E^*_{\langle \pm 1,0,0,0\rangle} + e_{\langle 0,\pm 1,0,0\rangle} E^*_{\langle 0,\pm 1,0,0\rangle} \\
&+ e_{\langle 0,0,\pm 1,0\rangle} E^*_{\langle 0,0,\pm 1,0\rangle} + e_{\langle 0,0,0,\pm 1\rangle} E^*_{\langle 0,0,0,\pm 1\rangle} \\
&+ e_{\langle \pm 1,\pm 1,0,0\rangle} E^*_{\langle \pm 1,\pm 1,0,0\rangle} + e_{\langle 0,\pm 1,\pm 1,0\rangle} E^*_{\langle 0,\pm 1,\pm 1,0\rangle} + e_{\langle 0,\pm 1,0,\pm 1\rangle} E^*_{\langle 0,\pm 1,0,\pm 1\rangle} \\
&+ e_{\langle \pm 1,\pm 1,\pm 1,0\rangle} E^*_{\langle \pm 1,\pm 1,\pm 1,0\rangle} + e_{\langle \pm 1,\pm 1,0,\pm 1\rangle} E^*_{\langle \pm 1,\pm 1,0,\pm 1\rangle} + e_{\langle 0,\pm 1,\pm 1,\pm 1\rangle} E^*_{\langle 0,\pm 1,\pm 1,\pm 1\rangle} \\
&+ e_{\langle \pm 1,\pm 1,\pm 1,\pm 1\rangle} E^*_{\langle \pm 1,\pm 1,\pm 1,\pm 1\rangle} + e_{\langle \pm 1,\pm 2,\pm 1,\pm 1\rangle} E^*_{\langle \pm 1,\pm 2,\pm 1,\pm 1\rangle}
\end{aligned} \tag{4.17}$$

The projector acts on operators:

$$\begin{aligned}
P(N_{D_4}) =& \sum_{1\le i\le 4} P(h_i) H_i^* + \sum_{\alpha\in\Phi^+} \big(P(e_\alpha) E^*_\alpha + P(e_{-\alpha}) E^*_{-\alpha}\big) \\
=& P(h_1) H_1^* + P(h_2) H_2^* + P(h_3) H_3^* + P(h_4) H_4^* \\
&+ P(e_{\langle \pm 1,0,0,0\rangle}) E^*_{\langle \pm\pm 1,0,0,0\rangle} + P(e_{\langle 0,\pm 1,0,0\rangle}) E^*_{\langle 0,\pm 1,0,0\rangle} \\
&+ P(e_{\langle 0,0,\pm 1,0\rangle}) E^*_{\langle 0,0,\pm 1,0\rangle} + P(e_{\langle 0,0,0,\pm 1\rangle}) E^*_{\langle 0,0,0,\pm 1\rangle} \\
&+ P(e_{\langle \pm 1,\pm 1,0,0\rangle}) E^*_{\langle \pm 1,\pm 1,0,0\rangle} + P(e_{\langle 0,\pm 1,\pm 1,0\rangle}) E^*_{\langle 0,\pm 1,\pm 1,0\rangle} \\
&+ P(e_{\langle 0,\pm 1,0,\pm 1\rangle}) E^*_{\langle 0,\pm 1,0,\pm 1\rangle} + P(e_{\langle \pm 1,\pm 1,\pm 1,0\rangle}) E^*_{\langle \pm 1,\pm 1,\pm 1,0\rangle} \\
&+ P(e_{\langle \pm 1,\pm 1,0,\pm 1\rangle}) E^*_{\langle \pm 1,\pm 1,0,\pm 1\rangle} + P(e_{\langle 0,\pm 1,\pm 1,\pm 1\rangle}) E^*_{\langle 0,\pm 1,\pm 1,\pm 1\rangle} \\
&+ P(e_{\langle \pm 1,\pm 1,\pm 1,\pm 1\rangle}) E^*_{\langle \pm 1,\pm 1,\pm 1,\pm 1\rangle} + P(e_{\langle \pm 1,\pm 2,\pm 1,\pm 1\rangle}) E^*_{\langle \pm 1,\pm 2,\pm 1,\pm 1\rangle} \\
=& h_1 H_1^* + h_2 H_2^* + \frac{h_3 + h_4}{2} H_3^* + \frac{(h_3 + h_4)}{2} H_4^* \\
&+ e_{\langle \pm 1,0,0,0\rangle} E^*_{\langle \pm 1,0,0,0\rangle} + e_{\langle 0,\pm 1,0,0\rangle} E^*_{\langle 0,\pm 1,0,0\rangle} \\
&+ \frac{e_{\langle 0,0,\pm 1,0\rangle} + e_{\langle 0,0,0,\pm 1\rangle}}{2} E^*_{\langle 0,0,\pm 1,0\rangle} + \frac{e_{\langle 0,0,\pm 1,0\rangle} + e_{\langle 0,0,0,\pm 1\rangle}}{2} E^*_{\langle 0,0,0,\pm 1\rangle} \\
&+ e_{\langle \pm 1,\pm 1,0,0\rangle} E^*_{\langle \pm 1,\pm 1,0,0\rangle} + \frac{e_{\langle 0,\pm 1,\pm 1,0\rangle} + e_{\langle 0,\pm 1,0,\pm 1\rangle}}{2} E^*_{\langle 0,\pm 1,\pm 1,0\rangle} \\
&+ \frac{e_{\langle 0,\pm 1,\pm 1,0\rangle} + e_{\langle 0,\pm 1,0,\pm 1\rangle}}{2} E^*_{\langle 0,\pm 1,0,\pm 1\rangle} + \frac{e_{\langle \pm 1,\pm 1,\pm 1,0\rangle} + e_{\langle \pm 1,\pm 1,0,\pm 1\rangle}}{2} E^*_{\langle \pm 1,\pm 1,\pm 1,0\rangle} \\
&+ \frac{e_{\langle \pm 1,\pm 1,\pm 1,0\rangle} + e_{\langle \pm 1,\pm 1,0,\pm 1\rangle}}{2} E^*_{\langle \pm 1,\pm 1,0,\pm 1\rangle} + e_{\langle 0,\pm 1,\pm 1,\pm 1\rangle} E^*_{\langle 0,\pm 1,\pm 1,\pm 1\rangle} \\
&+ e_{\langle \pm 1,\pm 1,\pm 1,\pm 1\rangle} E^*_{\langle \pm 1,\pm 1,\pm 1,\pm 1\rangle} + e_{\langle \pm 1,\pm 2,\pm 1,\pm 1\rangle} E^*_{\langle \pm 1,\pm 2,\pm 1,\pm 1\rangle} \\
=& h_{\pm 1} H^*_{\pm 1} + h_2 H_2^* + (h_3 + h_4) \frac{H_3^* + H_4^*}{2} \\
&+ e_{\langle \pm 1,0,0,0\rangle} E^*_{\langle \pm 1,0,0,0\rangle} + e_{\langle 0,\pm 1,0,0\rangle} E^*_{\langle 0,\pm 1,0,0\rangle} \\
&+ (e_{\langle 0,0,\pm 1,0\rangle} + e_{\langle 0,0,0,\pm 1\rangle}) \frac{E^*_{\langle 0,0,\pm 1,0\rangle} + E^*_{\langle 0,0,0,\pm 1\rangle}}{2} + e_{\langle \pm 1,\pm 1,0,0\rangle} E^*_{\langle \pm 1,\pm 1,0,0\rangle} \\
&+ (e_{\langle 0,\pm 1,\pm 1,0\rangle} + e_{\langle 0,\pm 1,0,\pm 1\rangle}) \frac{E^*_{\langle 0,\pm 1,\pm 1,0\rangle} + E^*_{\langle 0,\pm 1,0,\pm 1\rangle}}{2}
\end{aligned}$$

$$
\begin{aligned}
&+ (e_{\langle\pm1,\pm1,\pm1,0\rangle} + e_{\langle\pm1,\pm1,0,\pm1\rangle})\frac{E^*_{\langle\pm1,\pm1,\pm1,0\rangle} + E^*_{\langle\pm1,\pm1,0,\pm1\rangle}}{2} \\
&+ e_{\langle0,\pm1,\pm1,\pm1\rangle}E^*_{\langle0,\pm1,\pm1,\pm1\rangle} \\
&+ e_{\langle\pm1,\pm1,\pm1,\pm1\rangle}E^*_{\langle\pm1,\pm1,\pm1,\pm1\rangle} + e_{\langle\pm1,\pm2,\pm1,\pm1\rangle}E^*_{\langle\pm1,\pm2,\pm1,\pm1\rangle} \\
=&\tilde{h}_1\tilde{H}^*_1 + \tilde{h}_2\tilde{H}^*_2 + \tilde{h}_3\tilde{H}^*_3 \\
&+ \tilde{e}_{\langle\pm1,0,0\rangle}\tilde{E}^*_{\langle\pm1,0,0\rangle} + \tilde{e}_{\langle0,\pm1,0\rangle}\tilde{E}^*_{\langle0,\pm1,0\rangle} + \tilde{e}_{\langle0,0,\pm1\rangle}\tilde{E}^*_{\langle0,0,\pm1\rangle} \\
&+ \tilde{e}_{\langle\pm1,\pm1,0\rangle}\tilde{E}^*_{\langle\pm1,\pm1,0\rangle} + \tilde{e}_{\langle0,\pm1,\pm1\rangle}\tilde{E}^*_{\langle0,\pm1,\pm1\rangle} \\
&+ \tilde{e}_{\langle\pm1,\pm1,\pm1\rangle}\tilde{E}^*_{\langle\pm1,\pm1,\pm1\rangle} + \tilde{e}_{\langle0,\pm1,\pm2\rangle}\tilde{E}^*_{\langle0,\pm1,\pm2\rangle} \\
&+ \tilde{e}_{\langle\pm1,\pm1,\pm2\rangle}\tilde{E}^*_{\langle\pm1,\pm1,\pm2\rangle} + \tilde{e}_{\langle\pm1,\pm2,\pm2\rangle}\tilde{E}^*_{\langle\pm1,\pm2,\pm2\rangle} \\
=&N_{B_3}
\end{aligned}
\tag{4.18}
$$

where we defined

$$
\tilde{h}_3 = h_3 + h_4 = 2(\varphi_3 + \varphi_4) - 2(\varphi_{2,1} + \varphi_{2,2}) \tag{4.19}
$$

$$
\tilde{e}_{\langle0,0,1\rangle} = e_{\langle0,0,1,0\rangle} + e_{\langle0,0,0,1\rangle} = u^+_3 + u^+_4 \tag{4.20}
$$

$$
\tilde{e}_{\langle0,1,2\rangle} = e_{\langle0,1,1,1\rangle} = \sum_{a=1,2}\frac{u^+_{2,a}u^+_3u^+_4}{(\varphi_{2,a}-\varphi_3)(\varphi_{2,a}-\varphi_4)} \tag{4.21}
$$

and the remaining operators follow the same pattern $\tilde{e}_{\langle a,b,2c\rangle} = e_{\langle a,b,c,c\rangle}$ or $\tilde{e}_{\langle a,b,c\rangle} = e_{\langle a,b,c,0\rangle} + e_{\langle a,b,0,c\rangle}$ if $c \neq 0$ and $\tilde{e}_{\langle a,b,0\rangle} = e_{\langle a,b,0,0\rangle}$ otherwise. Notice a feature common to components of the moment map on which P acts non-trivially: the prefactor from the operator becomes the inverse multiplicity required in the definition of the new dual basis, e.g.

$$
\begin{aligned}
P(h_3)H^*_3 + P(h_4)H^*_4 =&\frac{h_3+h_4}{\#_3}H^*_3 + \frac{h_3+h_4}{\#_3}H^*_4 \\
=&(h_3+h_4)\frac{H^*_3+H^*_4}{\#_3} \\
=&\tilde{h}_3\frac{\#_3\tilde{H}^*_3}{\#_3} \\
=&\tilde{h}_3\tilde{H}^*_3.
\end{aligned}
\tag{4.22}
$$

Just as N_{D_4} satisfies certain matrix relations which identify the space it parametrises as the (closure of the) minimal nilpotent orbit of D_4, so does N_{B_3} obey several relations appropriate for a B_3 nilpotent orbit. The space should be an orbifold of the minimal orbit of D_4, so it should in particular have the same quaternionic dimension, namely 5. That is precisely the dimension of the next-to-minimal orbit of B_3

with the HWG [20]

$$\mathrm{HWG}(t,\mu_i) = \frac{1}{(1-\mu_2 t^2)(1-\mu_1^2 t^4)} \tag{4.23}$$

This space is parametrised by a matrix M satisfying the relations (computed using standard plethystic techniques)[8]

$$\begin{aligned} t^4[000] &: \mathrm{tr}N^2 = 0 \\ t^4[002] &: N \wedge N = 0 \\ t^6[010] &: N^3 = 0 \end{aligned} \tag{4.24}$$

We describe a relation by its R-symmetry weight appearing in the exponent of t and the global symmetry representation in which it transforms. This often, but not always, specifies the tensorial form of the relation, which we provide on the other side of the colon. The notation $N \wedge N$ should be understood as the contraction $\sum_{lmno} \epsilon_{ijklmno} N_{lm} N_{no}$ with the rank 7 antisymmetric invariant tensor of $\mathfrak{so}(7)$.

One can check that the moment map N_{B_3} satisfies the identities in (4.24) modulo abelianised relations. To show that there exist no *other* independent relations, or generators for that matter, one can calculate the Hilbert series of the ring as described below. This computation shows that indeed (4.24) form the complete set of relations for the next-to-minimal orbit of $\mathfrak{so}(7)$. Note that this is an instance of Case 2 of the Kostant-Brylinski Figure 4.4.

4.2.3 Wreathed quivers

The previous section establishes that some Coulomb branches can be orbifolded by a quiver automorphism. We will now argue that the orbifold can also be recovered as the Coulomb branch of the original quiver after gauging its automorphism. It is natural to ask if the resulting theory is also a quiver theory which could be studied without reference to the original, ungauged theory. This is indeed possible, albeit at the cost of generalising the notion of a quiver theory to *wreathed quivers*.

Traditionally a quiver theory is described by a quiver diagram in which nodes represent gauge or flavor groups and links represent appropriately charged matter. Wreathed quiver theories add wreathed legs denoted by $(\cdot) \wr S_n$ with an associated *wreathing group* S_n. See Fig. 4.5 for two prototypical examples. The top right quiver has a single wreathed node while the bottom right quiver is an example of a

[8] A similar set of relations appears in [131], albeit for the next-to-minimal orbit of D_4. The methods employed therein can be extended to the present case: given a general nilpotent orbit, one can construct the quiver for which it is the Higgs branch and look for matrix relations implied by the F-terms.

quiver with a longer wreathed leg.

Abelianisation of wreathed quivers

The Coulomb branch of a wreathed quiver can be studied through abelianisation with relatively minor changes, but it is cumbersome to write them down in full generality. We find much greater clarity in (entirely equivalent) abelianised calculations performed on discretely gauged non-wreathed quivers. In practice, this amounts to keeping the indices, Poisson and abelianised chiral structure from the non-wreathed quiver while imposing invariance under the projector 4.12. For illustrative purposes, and to draw a link to [100, 101], we present two particularly simple examples depicted in Fig. 4.5.

There are very few new elements in the wreathed quiver theory depicted in the top right quiver of Fig. 4.5. The third node brings six variables $u^{\pm}_{3,a}$ and $\varphi_{3,a}$, $a \in \{1,2\}$, much like a $U(2)$ node would. The wreathing group acts similarly to a Weyl group in that it permutes the index a and all physical operators are invariant under it.

Abelianised relations on the middle node read

$$u^+_{2,a}u^-_{2,a} = -\frac{(\varphi_{2,a}-M_2)(\varphi_{2,a}-\varphi_1)(\varphi_{2,a}-\varphi_{3,1})(\varphi_{2,a}-\varphi_{3,2})}{(\varphi_{2,1}-\varphi_{2,2})^2} \tag{4.25}$$

and the relations on the third node are essentially unchanged:

$$u^+_{3,a}u^-_{3,a} = -(\varphi_{3,a}-\varphi_{2,1})(\varphi_{3,a}-\varphi_{2,2}). \tag{4.26}$$

Interestingly, the latter can be read in two ways: either as the relation of a $U(1) \wr S_2$ node, or as

$$u^+_{3,a}u^-_{3,a} = -\frac{(\varphi_{3,a}-\varphi_{2,1})(\varphi_{3,a}-\varphi_{2,2})}{(\varphi_{3,1}-\varphi_{3,2})^2}(\varphi_{3,1}-\varphi_{3,2})^2, \tag{4.27}$$

which is appropriate for a $U(2)$ node with adjoint matter. This explains why in [100, 101] a "bouquet" of n $U(1)$ nodes combined into $U(n)$ with adjoint matter: at the level of the Coulomb branch, there is no difference between $U(1) \wr S_n$ and $U(n)$ with adjoint matter.

The case of the bottom right quiver in Fig. 4.5 is slightly more subtle. The first and second gauge nodes, which are inside the scope of a two-fold wreathing, each come with six variables $u^{\pm}_{i,a}$ and $\varphi_{i,a}$, $a \in \{1,2\}$. However, the pattern of abelianised relations, which can be determined by consistency with discrete gauging of the bottom left quiver in Fig. 4.5, is as follows:

$$u^+_{1,a}u^-_{1,a} = -(\varphi_{1,a}-M_1)(\varphi_{1,a}-\varphi_{2,a}) \tag{4.28}$$

$$u^+_{2,a}u^-_{2,a} = -(\varphi_{2,a} - \varphi_{1,a})(\varphi_{2,a} - \varphi_3) \tag{4.29}$$

$$u^+_3 u^-_3 = -(\varphi_3 - \varphi_{2,1})(\varphi_3 - \varphi_{2,2}) \tag{4.30}$$

Note in particular that the index a "stretches" across several nodes (but not the mass variable, which is shared by all legs). The wreathing group S_2 again acts on this index, and invariance under it is a necessary prerequisite for operator physicality. The Coulomb branch has C_3 symmetry and the moment map parametrises the next-to-minimal nilpotent orbit of this algebra. Its components include:

$$e_{\langle \pm1,0,0\rangle} = u^\pm_{1,1} + u^\pm_{1,2} \tag{4.31}$$

$$e_{\langle 0,\pm1,0\rangle} = u^\pm_{2,1} + u^\pm_{2,2} \tag{4.32}$$

$$e_{\langle 0,0,\pm1\rangle} = u^\pm_3 \tag{4.33}$$

$$e_{\langle \pm1,\pm1,0\rangle} = \frac{u^\pm_{1,1}u^\pm_{2,1}}{\varphi_{1,1} - \varphi_{2,1}} + \frac{u^\pm_{1,2}u^\pm_{2,2}}{\varphi_{1,2} - \varphi_{2,2}} \tag{4.34}$$

$$e_{\langle 0,\pm1,\pm1\rangle} = \frac{u^\pm_{2,1}u^\pm_3}{\varphi_{2,1} - \varphi_3} + \frac{u^\pm_{2,2}u^\pm_3}{\varphi_{2,2} - \varphi_3} \tag{4.35}$$

$$e_{\langle \pm1,\pm1,\pm1\rangle} = \frac{u^\pm_{1,1}u^\pm_{2,1}u^\pm_{3,1}}{(\varphi_{1,1} - \varphi_{2,1})(\varphi_{2,1} - \varphi_3)} + \frac{u^\pm_{1,2}u^\pm_{2,2}u^\pm_{3,1}}{(\varphi_{1,2} - \varphi_{2,2})(\varphi_{2,2} - \varphi_3)} \tag{4.36}$$

$$e_{\langle 0,\pm2,\pm1\rangle} = \frac{u^\pm_{2,1}u^\pm_{2,2}u^\pm_3}{(\varphi_{2,1} - \varphi_3)(\varphi_{2,2} - \varphi_3)} \tag{4.37}$$

$$e_{\langle \pm1,\pm2,\pm1\rangle} = \frac{u^\pm_{1,1}u^\pm_{2,1}u^\pm_{2,2}u^\pm_{3,1}}{(\varphi_{1,1} - \varphi_{2,1})(\varphi_{2,1} - \varphi_3)(\varphi_{2,2} - \varphi_3)} + \frac{u^\pm_{1,2}u^\pm_{2,1}u^\pm_{2,2}u^\pm_{3,1}}{(\varphi_{1,2} - \varphi_{2,1})(\varphi_{2,1} - \varphi_3)(\varphi_{2,2} - \varphi_3)} \tag{4.38}$$

$$e_{\langle \pm2,\pm2,\pm1\rangle} = \frac{u^\pm_{1,1}u^\pm_{1,2}u^\pm_{2,1}u^\pm_{2,2}u^\pm_{3,1}}{(\varphi_{1,1} - \varphi_{2,1})(\varphi_{1,2} - \varphi_{2,2})(\varphi_{2,1} - \varphi_3)(\varphi_{2,2} - \varphi_3)} \tag{4.39}$$

$$h_1 = 2(\varphi_{1,1} + \varphi_{1,2}) - (\varphi_{2,1} + \varphi_{2,2}) \tag{4.40}$$

$$h_2 = -(\varphi_{1,1} + \varphi_{3,1}) + 2(\varphi_{2,1} + \varphi_{2,2}) - 2\varphi_3 \tag{4.41}$$

$$h_3 = -(\varphi_{2,1} + \varphi_{2,2}) + 2\varphi_3 \tag{4.42}$$

4.2.4 Monopole formula for wreathed quivers

Consider now a wreathed quiver. To compute the monopole formula, we need to replace the Weyl group in (2.181) with an appropriate discrete group, necessarily a subgroup of S_r which contains the Weyl group $\mathcal{W}$ of the gauge group $\mathcal{G}$ and leaves $\Delta(m)$ invariant. Generically, several such discrete groups exist. Our choice, which we dub $\mathcal{W}_\Gamma$, is the wreath product of the Weyl group $\mathcal{W}$ by the wreathing group Γ:

$$\mathcal{W}_\Gamma = \mathcal{W} \wr \Gamma \,, \qquad \mathcal{W} \subseteq \mathcal{W}_\Gamma \subseteq S_r \,. \tag{4.43}$$

Formula (2.181) generalises readily for such a group, setting

$$\boxed{\mathrm{HS}_\Gamma(t) = \frac{1}{|\mathcal{W}_\Gamma|} \sum_{m \in \mathbb{Z}^r} \sum_{\gamma \in \mathcal{W}_\Gamma(m)} \frac{t^{2\Delta(m)}}{\det\left(1 - t^2 \gamma\right)}} \,. \tag{4.44}$$

This is the monopole formula for the wreathed quiver.

A comment on computational complexity

The monopole formula in the form (4.44) is very time-consuming to evaluate numerically in a series expansion in t. For such a task, it is preferable to preprocess it somewhat, using the high level of symmetry that it presents. In particular, if the group Γ can be written as a product of two groups $\mathcal{W}_\Gamma = \mathcal{W}_1 \times \mathcal{W}_2$, then one can split the summation into two sums.

This procedure involves finding a subset of $\mathbb{Z}^r$ which contains exactly one element of each orbit of $\mathcal{W}_\Gamma$. In the context of Weyl groups, or more generally Coxeter groups, this is called a fundamental chamber. For instance, if $\mathcal{W}_\Gamma = W$ as in (2.174), then this group can be used to order the magnetic charges in increasing order for each node. Then one uses the identity

$$P_{U(n)}(t^2; m_1, \dots, m_N) := P_{S_n}(t; m_1, \dots, m_n) = \frac{1}{n!} \sum_{\gamma \in S_n(m)} \frac{1}{\det\left(1 - t^2 \gamma\right)} \tag{4.45}$$

for the Casimir factors as defined in the appendix of [48]. This is done in the usual way of presenting the monopole formula.

For wreathed quivers, $\mathcal{W}_\Gamma$ does not decompose in general as a direct product of symmetry groups. One can introduce symmetry factors exactly as in (4.45), via

$$P_{\mathcal{W}_\Gamma}(t^2; m) = \frac{1}{|\mathcal{W}_\Gamma|} \sum_{\gamma \in \mathcal{W}_\Gamma(m)} \frac{1}{\det\left(1 - t^2 \gamma\right)} \,. \tag{4.46}$$

The formula (4.44) can then be rewritten

$$\mathrm{HS}_\Gamma(t) = \sum_{m \in \mathrm{Weyl}(\mathcal{G} \wr \Gamma)} P_{\mathcal{W}_\Gamma}(t; m) t^{2\Delta(m)} \,, \tag{4.47}$$

where $\mathrm{Weyl}(\mathcal{G} \wr \Gamma)$ is a principal Weyl chamber for the group $\mathcal{G} \wr \Gamma$. We now illustrate the procedure on three examples and most explicitly on the third.

Example 1 : subgroups of S_3

Consider the quiver corresponding to the affine D_4 Dynkin diagram (see the first column of Figure 4.18). One of the four rank-one nodes is a flavor node, so we can define the graph by the vertices

$$V = \{a, b, c, d\} \qquad F = \{e\} \tag{4.48}$$

where a denotes the central node, and

$$E = \{(a, b), (a, c), (a, d)\} \qquad E' = \{(a, e)\}\,. \tag{4.49}$$

The corresponding ranks are $n_a = 2$, $n_b = n_c = n_d = n_e = 1$. The total gauge group is $\mathcal{G} = U(2) \times U(1)^3$ with rank $r = 5$. The Weyl group is $W = S_2$. The magnetic charges are elements $m = (m_{a,1}, m_{a,2}, m_b, m_c, m_d) \in \mathbb{Z}^5$ and the conformal dimension is given by

$$2\Delta(m) = \sum_{i=1}^{2}(|m_{a,i}-m_b|+|m_{a,i}-m_c|+|m_{a,i}-m_d|+|m_{a,i}|)-2|m_{a,1}-m_{a,2}|\,. \tag{4.50}$$

The group S_5 includes 156 subgroups which can be gathered into 19 conjugacy classes. These 19 classes are partially ordered and form a Hasse diagram. For Δ to be invariant, we have to select those groups Γ which are subgroups of $S_2 \times S_3$ (this is also known as the dihedral group D_{12}), and moreover to satisfy (4.43) the groups Γ have to contain S_2 as a subgroup. Out of the 19 classes of subgroups, 9 are subgroups of D_{12}, and out of these 9, 6 contain a S_2 as a subgroup. However there are two equivalent but non-conjugate S_2 subgroups of D_{12}, and we have to pick one of them. We are then left with 4 classes of subgroups, which can be identified with the 4 classes of subgroups of S_3 (S_3, $\mathbb{Z}_3$, $\mathbb{Z}_2$ and 1). Clearly, in this simple example this analysis is slightly superfluous and the result could have been guessed. We end up with 4 inequivalent groups Γ, and we can readily evaluate the expression (4.44):

$$\mathrm{HS}_{\mathbb{Z}_2} = \frac{(1+t^2)(1+17t^2+48t^4+17t^6+t^8)}{(1-t^2)^{10}} \tag{4.51}$$

$$\mathrm{HS}_{\mathbb{Z}_2\times\mathbb{Z}_2} = \frac{(1+t^2)(1+10t^2+20t^4+10t^6+t^8)}{(1-t^2)^{10}} \tag{4.52}$$

$$\mathrm{HS}_{\mathbb{Z}_2\times\mathbb{Z}_3} = \frac{(1+t^2)(1+3t^2+20t^4+3t^6+t^8)}{(1-t^2)^{10}} \tag{4.53}$$

$$\mathrm{HS}_{\mathbb{Z}_2\times S_3} = \frac{(1+t^2)(1+3t^2+6t^4+3t^6+t^8)}{(1-t^2)^{10}} \tag{4.54}$$

Name	Generators	Cardinality
Trivial	-	1
S_2	(12)	2
Double transposition	(12)(34)	2
$\mathbb{Z}_4$	(1234)	4
Normal Klein	(12)(34), (13)(24)	4
Non-normal Klein	(12), (34)	4
Dih_4	(1234) ,(13)	8
$\mathbb{Z}_3$	(123),	3
S_3	(12) , (13)	6
A_4	(123) , (124)	12
S_4	(12) , (13) , (14)	24

Figure 4.6: Subgroups of S_4

which identify the spaces as the (closure of the) minimal nilpotent orbit of $SO(8)$, next to minimal of $SO(7)$, double cover of the subregular orbit of G_2 [126], and the subregular orbit of G_2.

Example 2 : subgroups of S_4

We now consider the same quiver as in the previous example, namely the affine D_4 quiver, but we use the fact that the gauge group of the theory is really

$$\frac{U(1)^4 \times U(2)}{U(1)} \tag{4.55}$$

where the $U(1)$ acts diagonally. This form makes the S_4 symmetry of the quiver explicit, and this S_4 contains as a subgroup the S_3 which is studied in the previous example. Following the approach of this section, one can define a wreathed quiver for each conjugacy class of subgroups of S_4. Part of the results presented here already appear in unpublished summer work by Siyul Lee [132], where the cycle index technique was used. The group S_4 admits 30 subgroups that can be organized into 11 conjugacy classes, as listed in Figure 4.6, where we give a name to each class of subgroups.

For each subgroup, one can compute the Hilbert series (4.44). The results are gathered in Figure 4.8, where they are arranged in the shape of the Hasse diagram of the subgroups of S_4. We give some details about the computation in appendix C. We also give the first orders of the series expansions of these Hilbert series, along with their plethystic logarithms, in Figure 4.7. The coefficient of the t^2 term gives the dimension of the isometry group of the Coulomb branch.

Subgroup	Perturbative Hilbert series	PLog
Trivial	$1 + 28t^2 + 300t^4 + 1925t^6 + ...$	$28t^2 - 106t^4 + 833t^6 + ...$
S_2	$1 + 21t^2 + 195t^4 + 1155t^6 + ...$	$21t^2 - 36t^4 + 140t^6 + ...$
Double transposition	$1 + 16t^2 + 160t^4 + 985t^6 + ...$	$16t^2 + 24t^4 - 215t^6 + ...$
$\mathbb{Z}_4$	$1 + 9t^2 + 83t^4 + 497t^6 + ...$	$9t^2 + 38t^4 - 10t^6 + ...$
Normal Klein	$1 + 10t^2 + 90t^4 + 515t^6 + ...$	$10t^2 + 35t^4 - 55t^6 + ...$
Non-normal Klein	$1 + 15t^2 + 125t^4 + 685t^6 + ...$	$15t^2 + 5t^4 - 70t^6 + ...$
Dih_4	$1 + 9t^2 + 69t^4 + 356t^6 + ...$	$9t^2 + 24t^4 - 25t^6 + ...$
$\mathbb{Z}_3$	$1 + 14t^2 + 118t^4 + 693t^6 + ...$	$14t^2 + 13t^4 - 49t^6 + ...$
S_3	$1 + 14t^2 + 104t^4 + 539t^6 + ...$	$14t^2 - t^4 - 7t^6 + ...$
A_4	$1 + 8t^2 + 48t^4 + 223t^6 + ...$	$8t^2 + 12t^4 + 7t^6 + ...$
S_4	$1 + 8t^2 + 48t^4 + 210t^6 + ...$	$8t^2 + 12t^4 - 6t^6 + ...$

Figure 4.7: Wreathed quivers obtained from the affine D_4 quiver by acting on the legs by all subgroups of S_4.

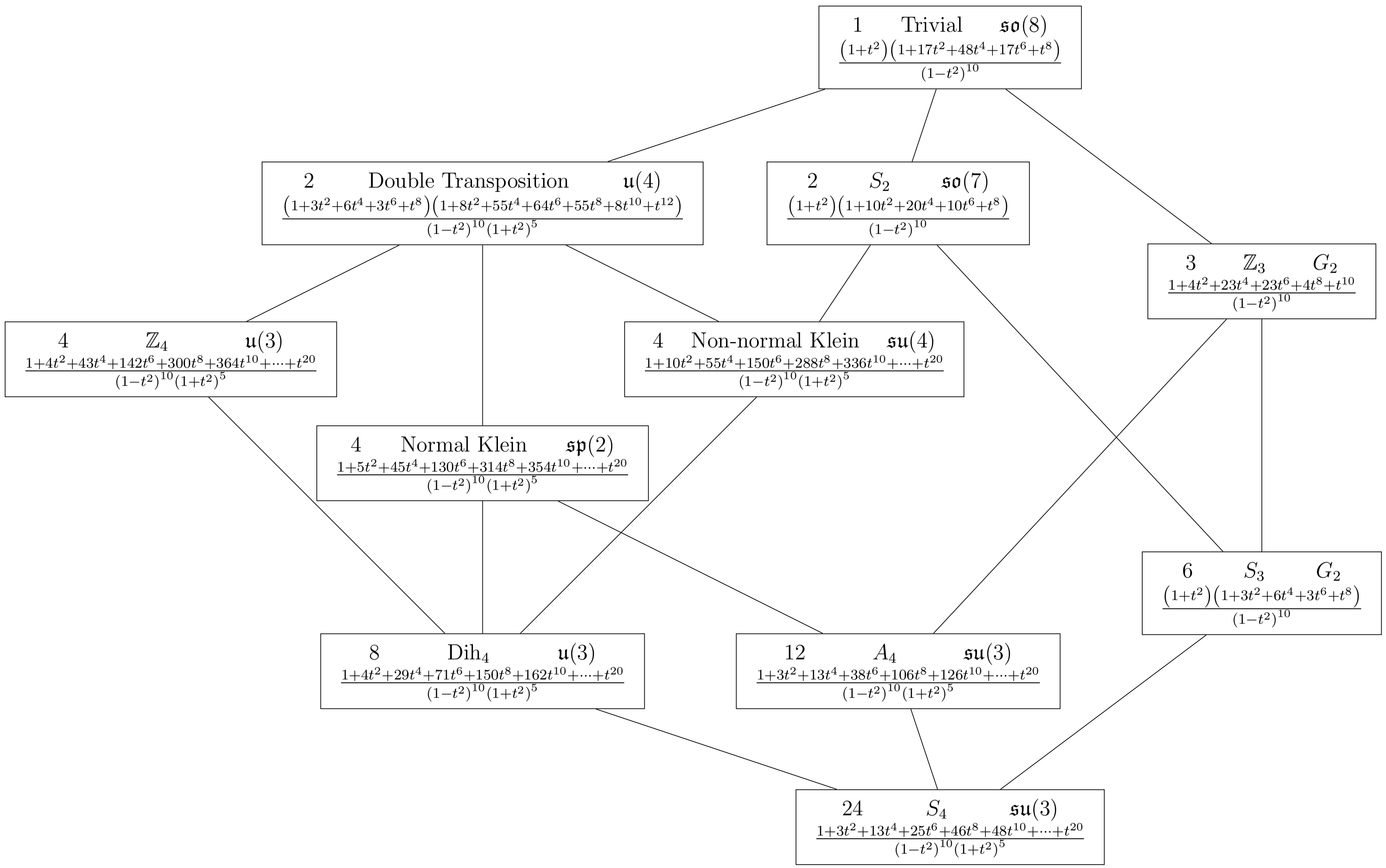
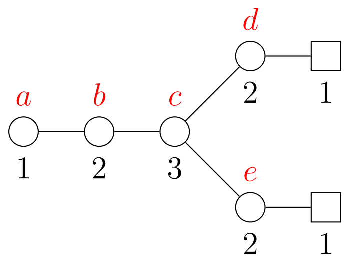


Figure 4.8: Hasse diagram of the 11 conjugacy classes of subgroups of S_4 with the Hilbert series for the Coulomb branch of the corresponding wreathed affine D_4 quiver. Dots in the numerators can be filled in using the fact that each polynomial is palindromic. In each box, the number on the left is the order of the group Γ, and the algebra on the right is the global symmetry of the Coulomb branch. The whole diagram possesses a symmetry exchanging $S_4 \leftrightarrow$ Trivial, $S_3 \leftrightarrow \mathbb{Z}_4$, Dih_4 and Double Transposition, Normal and Non-normal Klein, and fixes $\mathbb{Z}_4$. This is not obvious in the depiction because of planarity constraints.

Example 3 : wreath product of non Abelian groups

We now consider the quiver



(4.56)

whose Coulomb branch is the closure of the nilpotent orbit of $\mathfrak{so}(10)$ associated with the partition $[2^4, 1^2]$. The letters in red give our assignment of magnetic charge for the various gauge groups. The rank is $r = 10$ and the Weyl group is $W = S_1 \times S_2 \times S_3 \times S_2 \times S_2$. In order to preserve Δ, we need a symmetry of the quiver, which is given by permutation of the two legs containing the nodes d and e. So there are only two allowed groups $\mathcal{W}_\Gamma$, namely $\mathcal{W}_\Gamma = \mathcal{W}$ and an extension $\mathcal{W}_\Gamma$ of $\mathcal{W}$ of index 2. Let's focus on this second group.

The factors $S_1 \times S_2 \times S_3$ in $\mathcal{W}$ are unaffected by the permutation, so we omit them in the matrix discussion that follows. The commutant of this part in S_{10} is S_4, which acts by permuting the four magnetic fugacities (d_1, d_2, e_1, e_2). The group $\mathcal{W}_\Gamma$ is then the product $\mathcal{W}_\Gamma = S_1 \times S_2 \times S_3 \times \Gamma'$ where $S_2 \times S_2 \subset \Gamma' \subset S_4$. We can describe Γ' explicitly as generated by the following two permutation matrices:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \tag{4.57}$$

This group is isomorphic to the dihedral group of order 8, that we denote by Dih_4. With this explicit description, it is now possible to evaluate (4.44), and one finds the Hilbert series for the Coulomb branch of the wreathed quiver,

$$\mathrm{HS}_{S_1 \times S_2 \times S_3 \times \mathrm{Dih}_4} = \frac{1 + 14t^2 + 106t^4 + 454t^6 + 788t^8 + 454t^{10} + 106t^{12} + 14t^{14} + t^{16}}{(1-t^2)^{20}(1+t^2)^{-2}}. \tag{4.58}$$

The corresponding HWG and other data concerning this space are gathered in the middle column of Figure 4.21.

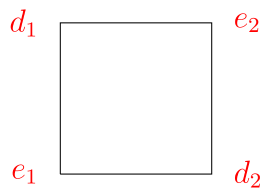
However, the sum involved in the computation is difficult to evaluate in practice, and it is useful to use the symmetries to avoid unnecessary repetitions, as explained above. In the present case, the sum in (4.44) for $\mathcal{W}_\Gamma = S_1 \times S_2 \times S_3 \times \mathrm{Dih}_4$ becomes

(4.47) where the sum over the Weyl chamber is given by:

$$\frac{1}{|\mathcal{W}_\Gamma|}\sum_{m\in\mathbb{Z}^{10}} \longrightarrow \sum_{a}\sum_{b_1\leq b_2}\sum_{c_1\leq c_2\leq c_3}\left[\sum_{\begin{smallmatrix} d_1 & \leq & d_2 \\ & & \wedge \\ e_1 & \leq & e_2 \end{smallmatrix}} + \sum_{\begin{smallmatrix} d_1 & \leq & d_2 \\ \mid\wedge & & \parallel \\ e_1 & \leq & e_2 \end{smallmatrix}}\right] \tag{4.59}$$

The notation here should be clear, with the charges $m = (a, b_1, b_2, c_1, c_2, c_3, d_1, d_2, e_1, e_2) \in \mathbb{Z}^{10}$ denoted with the letters as in (4.56). The first three sums in the right hand side of (4.59) exploit in the standard way the symmetric groups, which allow to order the charges. The same principle is used to get the summation range over indices (d_1, d_2, e_1, e_2). Inside the sum, one of course introduces symmetry factors (4.46). Let's now explain the summation range for the last four indices in (4.59).

Γ' is the dihedral group Dih_4, of order 8, or the group of symmetries of the square



(4.60)

The elements of Γ' are listed in Figure 4.9, with some of their properties. Without entering into the details of the theory of Coxeter groups, let us note that the Weyl chambers in $\mathbb{R}^4$ are delimited by subspaces fixed by the order 2 elements in the group. Formally, the Weyl group of a simple Lie algebra, the principal Weyl chamber is defined as the set of charges m which satisfy the inequalities

$$\alpha \cdot m \geq 0 \tag{4.61}$$

for every simple root α. However, in the present case the order 2 elements don't necessarily fix a hyperplane in $\mathbb{R}^4$ (the -1 eigenspace can have dimension > 1). The condition (4.61) then has to be replaced by a more general condition, which we now explain on our example. We leave the study of the general case, and the connection with Coxeter group theory, for future work.

The elements of order 2 in $\mathcal{W}_\Gamma$ can be read from Figure 4.9. For every element α of order 2 in $\mathcal{W}_\Gamma$, seen as a group of endomorphisms of its representation space, we pick a basis (δ_i^α) of the kernel of this endomorphism in a consistent way (with $i = 1, \dots, \dim\ker(\alpha)$). This is done in the third column of Figure 4.9. The principal Weyl chamber is then defined by

$$\delta^\alpha \cdot m \geq 0\,, \tag{4.62}$$

Permutation	Order	-1 eigenspace	Inequality
Identity	1		
$d_1 \leftrightarrow d_2$	2	$\begin{pmatrix} -1 & 1 & 0 & 0 \end{pmatrix}$	$d_1 \leq d_2$
$e_1 \leftrightarrow e_2$	2	$\begin{pmatrix} 0 & 0 & -1 & 1 \end{pmatrix}$	$e_1 \leq e_2$
$d_1 \leftrightarrow d_2$, $e_1 \leftrightarrow e_2$	2	$\begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}$	$e_1 < e_2$ or $e_1 = e_2$ and $d_1 \leq d_2$
$d_1 \leftrightarrow e_1$, $d_2 \leftrightarrow e_2$	2	$\begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}$	$d_2 < e_2$ or $d_2 = e_2$ and $d_1 \leq e_1$
$d_1 \leftrightarrow e_2$, $d_2 \leftrightarrow e_1$	2	$\begin{pmatrix} 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$	$d_1 < e_2$ or $d_1 = e_2$ and $d_2 \leq e_1$
$d_1 \to e_1 \to d_2 \to e_2 \to d_1$	4		
$d_1 \to e_2 \to d_2 \to e_1 \to d_1$	4		

Figure 4.9: Elements of the group Dih_4. In the first column, they are presented as permutations, acting on (d_1, d_2, e_1, e_2). The second column is the order of the element, the third gives a basis of the -1 eigenspace in the (d_1, d_2, e_1, e_2) representation. The last column displays the condition imposed by (4.62).

which generalizes (4.61). The subtlety in (4.62) comes from the cases where $\dim \ker(\alpha) > 1$. When this is the case, $\delta^\alpha \cdot m$ is an element of $\mathbb{R}^{\dim \ker(\alpha)}$ and we need to say what we mean by $\geq$. A simple choice, which we adopt here, is to pick the lexicographic order

$$(x, y) \leq (x', y') \Leftrightarrow y < y' \text{ or } (y = y' \text{ and } x \leq x') \,. \tag{4.63}$$

Doing this for every order 2 element in Figure 4.9, we get the conditions listed in the last column of that figure. Combining all these conditions together, we obtain the summation range in (4.59).

4.2.5 HWG for wreathed quivers

We now explain how to perform the orbifold at the level of the HWG. The starting point is the HWG of the Coulomb branch $\mathcal{C}$ of a quiver, which can be wreathed by a finite permutation group Γ. The goal is to compute the HWG for $\mathcal{C}/\Gamma$.

In the following, we give the general prescription, and at the same time we illustrate with three examples $\Gamma = \mathbb{Z}_2$, $\mathbb{Z}_3$ and S_3 to keep the discussion concrete. We first recall that the group Γ has a well defined character table, which is a square matrix whose columns are labelled by conjugacy classes of elements of Γ, and whose rows are labelled by irreducible representations of Γ. For our three examples, these character tables are

$\Gamma = \mathbb{Z}_2$	1	-1
Cardinality	1	1
1	1	1
ϵ	1	-1

(4.64)

$\Gamma = \mathbb{Z}_3$	1	ω	ω^2
Cardinality	1	1	1
$\mathbf{1}$	1	1	1
$\mathbf{f}$	1	ω	ω^2
$\bar{\mathbf{f}}$	1	ω^2	ω

$\Gamma = S_3$	Id	3-cycles	2-cycles
Cardinality	1	2	3
$\mathbf{1}$	1	1	1
ε	1	1	-1
$\mathbf{2}$	2	-1	0

(4.65)

These character tables contain in each entry the trace of the matrices of the conjugacy class in the corresponding representation. One way to refine this information is to give, instead of the trace, the list (unordered, and with repetitions allowed) of the eigenvalues of these matrices. We will need these eigenvalues in equation (4.70). On our three examples, we get

$\Gamma = \mathbb{Z}_2$	1	-1
Cardinality	1	1
$\mathbf{1}$	$\{1\}$	$\{1\}$
ϵ	$\{1\}$	$\{-1\}$

(4.66)

$\Gamma = \mathbb{Z}_3$	1	ω	ω^2
Cardinality	1	1	1
$\mathbf{1}$	$\{1\}$	$\{1\}$	$\{1\}$
$\mathbf{f}$	$\{1\}$	$\{\omega\}$	$\{\omega^2\}$
$\bar{\mathbf{f}}$	$\{1\}$	$\{\omega^2\}$	$\{\omega\}$

$\Gamma = S_3$	Id	3-cycles	2-cycles
Cardinality	1	2	3
$\mathbf{1}$	$\{1\}$	$\{1\}$	$\{1\}$
ε	$\{1\}$	$\{1\}$	$\{-1\}$
$\mathbf{2}$	$\{1,1\}$	$\{\omega,\omega^2\}$	$\{1,-1\}$

(4.67)

Of course in each case the sum of the eigenvalues listed in (4.66), (4.67) gives the characters (4.64), (4.65). Let us call C_j the conjugacy classes ($j = 1,\dots,n$, with n the number of conjugacy classes, and C_1 is the class of the identity element), $c_j = |C_j|$ their cardinalities, ρ_i the irreducible representations ($i = 1,\dots,n$, and ρ_1 is the trivial representation), and d_i their dimensions. Finally we denote by $\Lambda_{i,j}$ the list of eigenvalues for C_j in ρ_i. For a representation R which is not irreducible, we similarly denote by $\Lambda_{R,j}$ the list of eigenvalues of the class C_j in the representation R. The elements of $\Lambda_{R,j}$ are denoted $\lambda^k_{R,j}$ for $k = 1,\dots,\dim R$. This list is easily obtained from the decomposition of R in irreducible representations. Note that $\lambda^k_{R,1} = 1$ for all k.

We now show how to compute the HWG for an orbifold Coulomb branch based on an initial Coulomb branch that admits a *finite* HWG. We say that HWG($\mathcal{C}$) is finite is there exist two lists of monomials, that we denote $(M_1,\dots,M_K)$ and $(M'_1,\dots,M'_{K'})$, in the highest weight fugacities (μ_l) and the variable t, so that the

HWG is

$$\mathrm{HWG}(\mathcal{C}) = \frac{\prod_{k'=1}^{K'} (1 - M'_{k'})}{\prod_{k=1}^{K} (1 - M_k)} . \tag{4.68}$$

We assume that HWG($\mathcal{C}$) can be written in that way; this is a non-trivial assumption, as it is known that many Coulomb branches don't satisfy it.

The fact that Γ is a symmetry group translates into the fact that to the above expression are associated two representations R and R' of Γ, not necessarily irreducible, of respective dimensions K and K', such that the numerator and the denominator of the above expression transform according to these representations. Then HWG($\mathcal{C}$) can be written

$$\mathrm{HWG}(\mathcal{C}) = \frac{\prod_{k'=1}^{K'} (1 - \lambda^{k'}_{R',1} M'_{k'})}{\prod_{k=1}^{K} (1 - \lambda^{k}_{R,1} M_k)} . \tag{4.69}$$

From this expression it is then straightforward to write the conjectured HWG for the orbifold

$$\mathrm{HWG}(\mathcal{C}/\Gamma) = \frac{1}{|\Gamma|} \sum_{j=1}^{n} c_j \times \frac{\prod_{k'=1}^{K'} (1 - \lambda^{k'}_{R',j} M'_{k'})}{\prod_{k=1}^{K} (1 - \lambda^{k}_{R,j} M_k)} . \tag{4.70}$$

We illustrate how this formula works in practice on the example of the D_4 affine quiver. All HWGs and quivers are gathered in Figure 4.18. Consider for instance the HWGs written in terms of G_2 fugacities. The closure of the minimal nilpotent orbit of D_4 has HWG equal to $\mathrm{PE}\left[2\mu_1 t^2 + \mu_2 t^2 + \mu_2 t^4\right]$. The identification of the irreducible representations is as follows:

$$\mathbb{Z}_2 \quad : \quad \mathrm{HWG}(\mathcal{C}) = \mathrm{PE}\left[\mathbf{1}\mu_1 t^2 + \varepsilon\mu_1 t^2 + \mathbf{1}\mu_2 t^2 + \varepsilon\mu_2 t^4\right] \tag{4.71}$$

$$\mathbb{Z}_3 \quad : \quad \mathrm{HWG}(\mathcal{C}) = \mathrm{PE}\left[\mathbf{f}\mu_1 t^2 + \bar{\mathbf{f}}\mu_1 t^2 + \mathbf{1}\mu_2 t^2 + \mathbf{1}\mu_2 t^4\right] \tag{4.72}$$

$$S_3 \quad : \quad \mathrm{HWG}(\mathcal{C}) = \mathrm{PE}\left[\mathbf{2}\mu_1 t^2 + \mathbf{1}\mu_2 t^2 + \varepsilon\mu_2 t^4\right] \tag{4.73}$$

We then use equation (4.70) to obtain

$$\begin{aligned} \mathrm{HWG}(\mathcal{C}/\mathbb{Z}_2) &= \frac{1}{2}\left(\frac{1}{(1-\mu_1 t^2)^2(1-\mu_2 t^2)(1-\mu_2 t^4)} \right. \\ &\quad \left. + \frac{1}{(1-\mu_1 t^2)(1+\mu_1 t^2)(1-\mu_2 t^2)(1+\mu_2 t^4)}\right) \\ &= \frac{1-\mu_1^2\mu_2^2 t^{12}}{(1-\mu_1 t^2)(1-\mu_2 t^2)(1-\mu_1^2 t^4)(1-\mu_1\mu_2 t^6)(1-\mu_2^2 t^8)} . \end{aligned} \tag{4.74}$$

$$\begin{aligned} \mathrm{HWG}(\mathcal{C}/\mathbb{Z}_3) &= \frac{1}{3}\sum_{i=0}^{2} \frac{1}{(1-\omega^i\mu_1 t^2)(1-\omega^{-i}\mu_1 t^2)(1-\mu_2 t^2)(1-\mu_2 t^4)} \\ &= \frac{(1-\mu_1^6 t^{12})}{(1-\mu_2 t^2)(1-\mu_2 t^4)(1-\mu_1^2 t^4)(1-\mu_1^3 t^6)^2}\,. \end{aligned} \tag{4.75}$$

$$\begin{aligned} \mathrm{HWG}(\mathcal{C}/S_3) &= \frac{1}{6}\Bigg(\frac{1}{(1-\mu_1 t^2)^2(1-\mu_2 t^2)(1-\mu_2 t^4)} \\ &\quad + \frac{2}{(1-\omega\mu_1 t^2)(1-\omega^2\mu_1 t^2)(1-\mu_2 t^2)(1-\mu_2 t^4)} \\ &\quad + \frac{3}{(1-\mu_1 t^2)(1+\mu_1 t^2)(1-\mu_2 t^2)(1+\mu_2 t^4)}\Bigg) \\ &= \frac{1-\mu_1^6\mu_2^2 t^{20}}{(1-\mu_2 t^2)(1-\mu_1^2 t^4)(1-\mu_1^3 t^6)(1-\mu_2^2 t^8)(1-\mu_1^3\mu_2 t^{10})}\,. \end{aligned} \tag{4.76}$$

This reproduces the results in [105].

Eighth case of Figure 4.4 We can apply similar methods to the eighth line of Figure 4.4. The HWG for the minimal nilpotent orbit of F_4, written in terms of D_4 fugacities, is $\mathrm{PE}[(\mu_1+\mu_2+\mu_3+\mu_4)t^2]$. The weights μ_1, μ_3 and μ_4 correspond to the external nodes of the Dynkin diagram. In order to perform the $\mathbb{Z}_2^2$ quotient, we charge them under the three distinct $\mathbb{Z}_2$ subgroups and apply formula (4.70). This way one gets the HWG

$$\frac{1}{4}\sum_{\epsilon_1=\pm 1}\sum_{\epsilon_2=\pm 1} \frac{1}{(1-\epsilon_1\epsilon_2\mu_1 t^2)(1-\mu_2 t^2)(1-\epsilon_2\mu_3 t^2)(1-\epsilon_1\mu_4 t^2)} \tag{4.77}$$

which evaluates to $\mathrm{PE}[\mu_2 t^2 + (\mu_1^2+\mu_3^2+\mu_4^2)t^4 + \mu_1\mu_3\mu_4 t^6 - \mu_1^2\mu_3^2\mu_4^2 t^{12}]$. One can check that this is indeed the HWG for the closure of the $[3,2^2,1]$ orbit of $\mathfrak{so}(8)$. An alternative way of seeing the same computation relies on the fact that $\mathbb{C}^3/\mathbb{Z}_2^2$ is a weighted hypersurface in $\mathbb{C}^4$.

4.2.6 Higgs branch of wreathed quivers

In this subsection, we turn to the Higgs branch of wreathed quivers. This is in contrast with the rest of the section, which focuses on the Coulomb branch of the $3d$ $\mathcal{N}=4$ theories, but it serves several purposes. First, it demonstrates that wreathed quivers do indeed provide a well-defined hyper-Kähler quotient, which can be associated with a gauge theory whose gauge group is disconnected. Secondly, we explain how to compute the Hilbert series of such quivers, using an averaging procedure. Finally, it allows the study of the geometric action of wreathing on the Higgs branch and contrasts it with the parallel action on the Coulomb branch.

We focus on a simple but rich example, the affine D_4 quiver, and compute the Higgs branch of all the wreathed quivers that appear in Figure 4.8. Let Γ be a subgroup of S_4. We consider the wreathed quiver defined by this group acting on the four U(1) nodes. This produces (when Γ is non-trivial) a disconnected gauge group, as follows directly from the definition (4.1). Disconnected gauge groups have been considered in the context of the plethystic program in [133], where groups were extended by outer automorphisms, following a formula of Wendt [134]. Here the context is different but the techniques spelled out in [133] apply. In fact, the case considered here is particularly easy to handle because the groups which are being wreathed are all U(1) groups, therefore the Haar measure is not modified. We pick fugacities z_i ($i = 1, 2, 3, 4$) for the U(1) factors and fugacity y for the U(2) factor (after ungauging a diagonal U(1)). It follows that the Higgs branch Hilbert series is obtained via a Molien-Weyl integral which is written explicitly as

$$\mathrm{HS}_{\Gamma}^{\mathcal{H}}(t) = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \int_{z_i, y} \mathrm{d}\mu(z_i, y) \mathcal{F}^{\flat}(z_i, y, t, \gamma)\,, \tag{4.78}$$

where the measure is

$$\mathrm{d}\mu(z_i, y) = \frac{\mathrm{d}z_1}{2\pi i z_1} \frac{\mathrm{d}z_2}{2\pi i z_2} \frac{\mathrm{d}z_3}{2\pi i z_3} \frac{\mathrm{d}z_4}{2\pi i z_4} \frac{(1-y^2)\mathrm{d}y}{2\pi i y} \tag{4.79}$$

and

$$\mathcal{F}^{\flat}(z_i, y, t, \gamma) = \frac{\det\left(\mathbf{1}_4 - \gamma t^2\right)(1-t^2)(1-t^2y^2)(1-t^2y^{-2})}{\det\left(\mathbf{1}_4 - \gamma t y D\right)\det\left(\mathbf{1}_4 - \gamma t y^{-1} D\right)\det\left(\mathbf{1}_4 - \gamma t y D^{-1}\right)\det\left(\mathbf{1}_4 - \gamma t y^{-1} D^{-1}\right)} \tag{4.80}$$

with D the diagonal matrix $\mathrm{Diag}(z_1, z_2, z_3, z_4)$. The integral over the z_i and y fugacities are performed over the contours $|z_i| = |y| = 1$. Note that (4.78) makes it manifest that $\gamma \in \Gamma$ can be considered as a discrete fugacity for the disconnected gauge group $\mathrm{U}(1) \wr \Gamma$. The integrals (4.78) are readily evaluated for each of the 11 subgroups of S_4, and the resulting Hilbert series are presented in Figure 4.10.

We make a few comments on the results. First, the Hilbert series coincide with those of Du Val singularities $\mathbb{C}^2/J$, with J a finite subgroup of SU(2), of ADE type. Specifically, four instances occur, namely $J = D_4, D_6, E_6, E_7$, that can be identified using the degrees of invariants of the corresponding groups. In particular, this shows that the quaternionic dimension of the Higgs branches of all these quivers is 1.

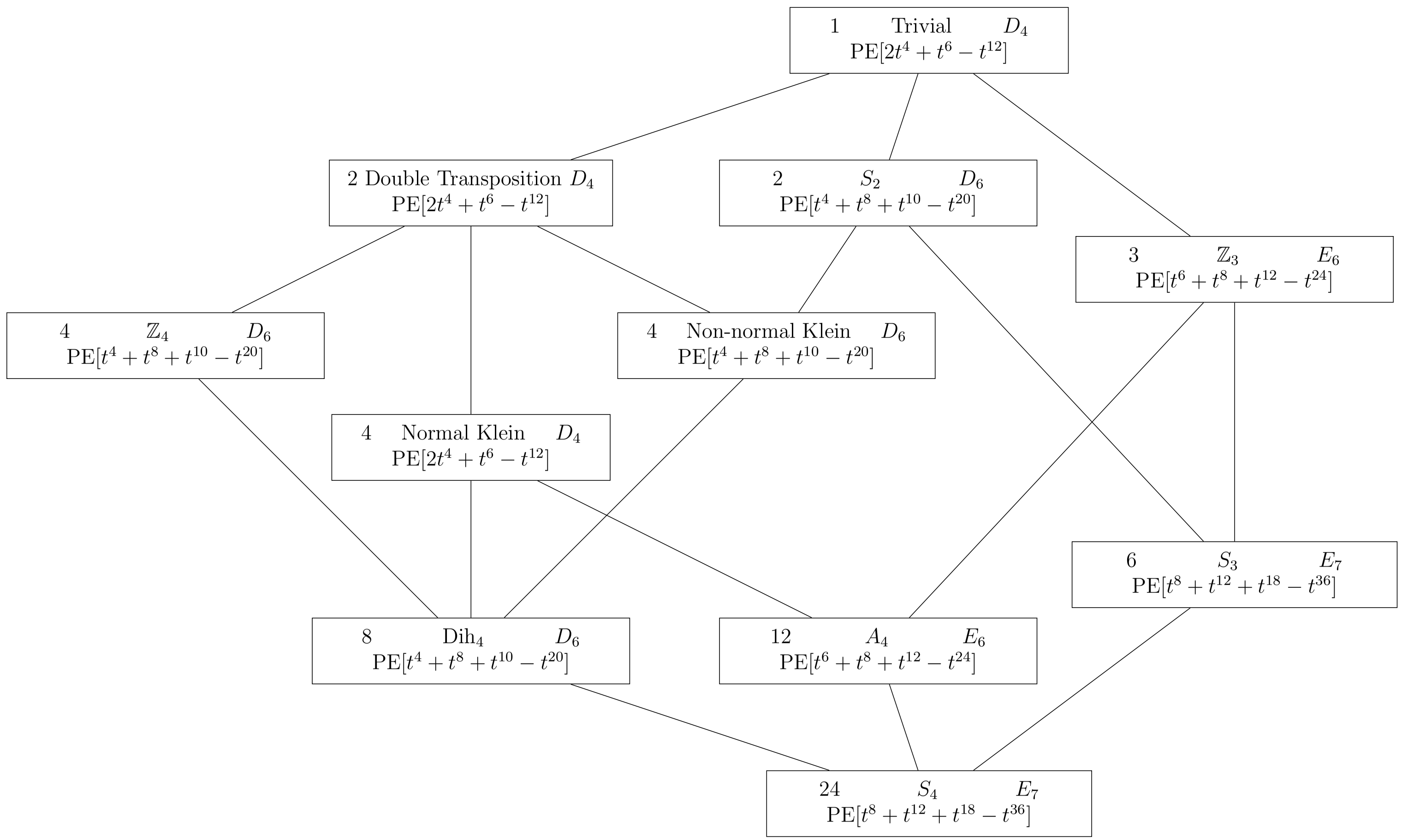


Figure 4.10: Hasse diagram of the 11 conjugacy classes of subgroups of S_4 with the Hilbert series for the Higgs branch of the corresponding wreathed affine D_4 quiver. In each box, the number on the left is the order of the group Γ, and the group on the right denotes the ADE type of the finite subgroup J of SU(2) such that the Higgs branch is $\mathbb{C}^2/J$.

Gauge invariant operators As a check of the computations presented in Figure 4.10, we briefly show how the same results can be obtained from a counting of invariant operators. We call A_i and B_i the scalars in the chiral multiplets transforming as bifundamentals of U(2) and $\mathrm{U}(1)_i$, for $i = 1, 2, 3, 4$, A_i being a column vector and B_i being a row vector[9]. For simplicity, we ungauge one of the U(1) groups, say $\mathrm{U}(1)_4$, and study the action of the wreath product by a subgroup Γ of S_3 permuting the three remaining U(1) gauge groups.

The F-term equations on $\mathrm{U}(1)_i$ are

$$\text{For } i = 1, 2, 3, \qquad B_i A_i = 0\,. \tag{4.81}$$

The F-term equations on the U(2) group are

$$\sum_{i=1}^{4} A_i B_i = 0\,. \tag{4.82}$$

Taking the trace of (4.82) and combining with (4.81) we obtain

$$B_4 A_4 = 0\,. \tag{4.83}$$

Gauge invariants are paths in the quiver of the form $B_4 \alpha_{i_1} \cdots \alpha_{i_r} A_4$ subject to the relations above, using the shorthand notation $\alpha_i = A_i B_i$. An irreducible gauge invariant is one that can not be written as a product of other non-trivial gauge invariants, so it can be written $B_4 \alpha_{i_1} \cdots \alpha_{i_r} A_4$ where the indices can not take the value 4. The F-term relations imply that

$$\alpha_i \alpha_i = 0 \text{ and } \sum_{i=1}^{4} \alpha_i = 0\,. \tag{4.84}$$

In particular an irreducible gauge invariant can not contain three α_i's or more.[10] So generators of the Higgs branch coordinate ring contain either one or two α_i's. The generators containing one α_i are $X_i = B_4 \alpha_i A_4$ ($i = 1, 2, 3$) subjected to $X_1 + X_2 + X_3 = 0$, and transform in the irreducible two-dimensional representations of S_3. The generators with two α_i's are built from $Y_{ij} = B_4 \alpha_i \alpha_j A_4$. Note that $Y_{ij} = -Y_{ji}$ and that $Y_{12} = Y_{23} = Y_{31}$, which shows that the Y_{ij} transform in the ε representation of S_3. Finally, there is a relation between the two families, for instance in the form

$$X_1 X_2 X_3 = B_4 \alpha_1 \alpha_4 \alpha_2 \alpha_4 \alpha_3 A_4 = B_4 \alpha_1 \alpha_3 \alpha_2 \alpha_1 \alpha_3 A_4 = -Y_{12}^2\,. \tag{4.85}$$

[9] Their components are the q and $\tilde{q}$ of Sec. 2.3.3.

[10] Consider for instance $B_4 \alpha_i \alpha_j \alpha_k A_4$ with $i \neq j$, $j \neq k$ and $i, j, k \neq 4$. If $i \neq k$ then one finds $B_4 \alpha_i \alpha_j \alpha_k A_4 = -B_4 \alpha_i (\alpha_i + \alpha_k + \alpha_4) \alpha_k A_4 = B_4 \alpha_i \alpha_4 \alpha_k A_4 = (B_4 \alpha_i A_4)(B_4 \alpha_k A_4)$. If $i = k$ then $B_4 \alpha_i \alpha_j \alpha_i A_4 = -B_4 \alpha_i \alpha_j \alpha_l A_4$ with $l \neq i, j, 4$ and we're back in the previous case.

Γ	Generators	Relation	Space
S_1	$t^4: \quad x = X_1$ $t^4: \quad y = X_2$ $t^6: \quad z = Y_{12}$	$xy(x+y) = z^2$	$\mathbb{C}^2/D_4$
S_2	$t^4: \quad x = X_1 + X_2$ $t^8: \quad y = X_1 X_2$ $t^{10}: \quad z = Y_{12}(X_1 - X_2)$	$xy(x^2 - 4y) = z^2$	$\mathbb{C}^2/D_6$
$\mathbb{Z}_3$	$t^6: \quad x = Y_{12}$ $t^8: \quad y = X_1^2 + X_1X_2 + X_2^2$ $t^{12}: \quad z = (X_1 - X_2)(2X_1 + X_2)(X_1 + 2X_2)$	$-27x^4 + 4y^3 = z^2$	$\mathbb{C}^2/E_6$
S_3	$t^8: \quad x = X_1^2 + X_1X_2 + X_2^2$ $t^{12}: \quad y = Y_{12}^2$ $t^{18}: \quad z = Y_{12}(X_1 - X_2)(2X_1 + X_2)(X_1 + 2X_2)$	$4x^3y - 27y^3 = z^2$	$\mathbb{C}^2/E_7$

Figure 4.11: Generators and relations for operators on the Higgs branch of the affine D_4 quiver wreathed by subgroups of S_3.

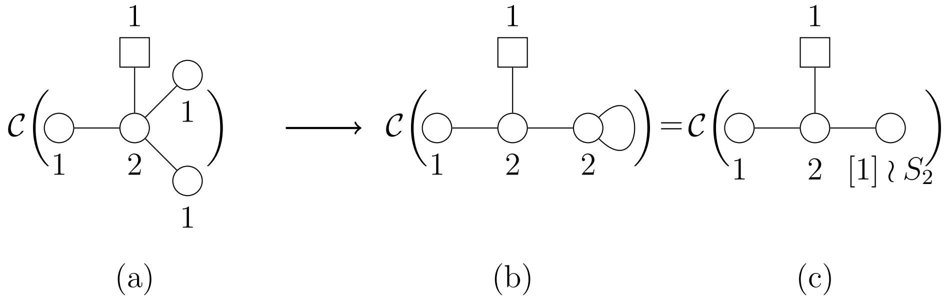


Figure 4.12: The Coulomb branch of the D_4 quiver (left) is orbifolded by an S_2 action into a Coulomb branch shared by two distinct quivers.

Putting all this together, we obtain the Hilbert series $\mathrm{PE}[2t^4 + t^6 - t^{12}]$ for the affine D_4 quiver (the X_i have weight 4 while the Y_{ij} have weight 6). To deal with the wreathed quivers, we have to impose the additional gauge invariance under the discrete factor Γ. The spectrum of operators on the Higgs branch is a subset of the one determined above for trivial Γ. The results are gathered in Figure 4.11.

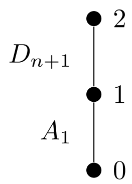
Comparison with adjoint matter Consider the case depicted in Fig. 4.12. In [100, 101] it was pointed out that the Coulomb branch of the quiver (b) is an orbifold of the Coulomb branch of the quiver (a). We have argued that the Coulomb branch of the wreathed quiver (c) is also that very same orbifold. Let's look at the Higgs branch of quiver (b).

The (quaternionic) dimension of the Higgs branch, when there is complete Higgsing, which is the case here, is equal to the number of matter multiplets minus the number of gauge multiplets. The D_4 quiver therefore has dim $\mathcal{H}_{D_4} =$

$(4 \cdot 2 \cdot 1) - (3 \cdot 1 + 2^2) = 1$, as do all the wreathed quivers. The quiver (b) has a Higgs branch of quaternionic dimension dim $\mathcal{H}_{\text{loop}}^{U(2)} = (2\cdot 2\cdot 1 + 2\cdot 2 + 2^2) - (1 + 2\cdot 2^2) = 3$, of which one dimension is a free factor $\mathbb{H}$ from the trivial factor in the adjoint loop. We can be more precise and compute the Hilbert series using the hyper-Kähler quotient, finding

$$\mathrm{PE}[2t]\mathrm{PE}[3t^2 + 2t^5 - t^{12}]\,. \tag{4.86}$$

The first term comes from a free contribution $\mathbb{H}$ which can be discarded. The second term can be identified as the Hilbert series for an intersection of a Słodowy slice and the nilpotent cone in the C_3 algebra, namely the transverse slice between the maximal orbit (of dimension 9) and the $\mathcal{O}_{[4,1^2]}$ orbit of dimension 7, see Table 12 in [21] (labelled [210] therein). The global symmetry on this space is $Sp(1)$ under which the generators of the chiral ring transform in the [2] and the [1] representations, respectively. This space makes a rare occurrence of a symplectic singularity which is also a hypersurface in $\mathbb{C}^5$. In fact it has been suggested that all hypersurface symplectic singularities of dimension 2 are intersections of Słodowy slices of the nilpotent orbit $\mathcal{O}_{[2n-2,1^2]}$ and the nilpotent cone in C_n [135]. This family appears in the context of trivertex theories where the rank of C_n is interpreted as the genus of a Riemann surface (A_1 class S theory on a Riemann surface of genus n and one puncture). See section 7.2 of [136] and equation (7.12) for the hypersurface equation. The same family also appears as a Coulomb branch of the mirror quiver in the work of [137] where the identification as a transverse slice is made, as well as an explicit form of the hypersurface equation. The Hasse diagram is



(4.87)

In summary, the two quivers on the right of Figure 4.12 share the same Coulomb branch, but only the wreathed quiver's Higgs branch shares the original quiver's Higgs branch dimension, as one would expect from discrete gauging.

4.2.7 Mirror symmetry and discrete gauging

Many $3d$ $\mathcal{N} = 4$ quiver theories admit a dual description as a theory whose Higgs branch is the original's Coulomb branch and vice versa; this property is known as $3d$ *mirror symmetry* [19, 59, 138] and is a consequence of S-duality for theories with brane realisations. One should therefore expect to be able to find the mirror dual of discrete gauging. As it turns out, it is already known.

Let us consider the paradigmatic case of quivers in Figure 4.13. The Coulomb

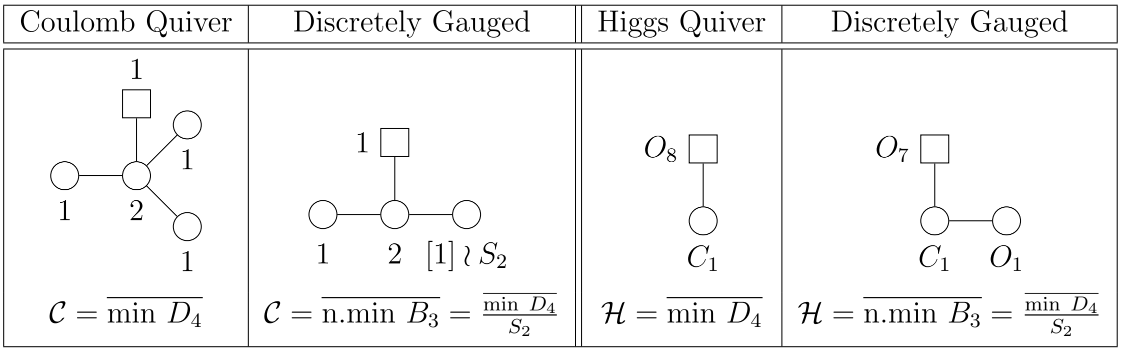


Figure 4.13: Illustration of the relation between i) discrete gauging's effects on the Coulomb branch and ii) discrete gauging's effects on the Higgs branch of a corresponding electric quiver.

branch of the quiver in the first column is the minimal nilpotent orbit of D_4. Its dual is depicted in the third column of the same figure; the symmetry of its Higgs branch is the same as the symmetry on the flavor node. Each matter hypermultiplet is coupled to a mass which can be viewed as a background vector multiplet. This vector can in turn be gauged, turning the quiver into the one depicted in the fourth column; such an operation was first reported as "the case $O(1)$" in [129]. The new gauge node $O(1) \cong \mathbb{Z}_2 \cong S_2$ represents the discrete symmetry of the gauged vector. In this case the gauge group is *enlarged.* We claim this is the mirror dual of the process covered in the previous section. Somewhat confusingly, both procedures are called discrete gauging[11] but they act differently. On the left quiver an automorphism is gauged; on the right we gauge a background vector.

If the enhancement of the mirror is discrete, so must be the original's. Moreover, since discrete gauging of background vectors is a genuine action on quiver theories, so is its mirror dual.

4.3 Quiver folding

The next discrete operation allows for a natural interpretation of non–simply laced quivers, which were identified in [11] through the use of the monopole formula. It was already well established [48] that many ADE nilpotent orbits could be recovered as Coulomb branches of unitary quiver theories and that there is a robust connection between choice of quiver and the resulting nilpotent orbit. In particular, the quiver should be balanced and shaped like the desired symmetry algebra's Dynkin diagram. Consequently one might assume that quivers whose Coulomb branches reproduce $BCFG$ nilpotent orbits would resemble the non-simply laced $BCFG$ Dynkin diagrams. [11] conjectured a minimal modification to the monopole formula

[11]We are not aware of a physics reference for discrete gauging on the Higgs branch but believe it to be fairly well known among physicists interested in Higgs branches of quiver gauge theories.

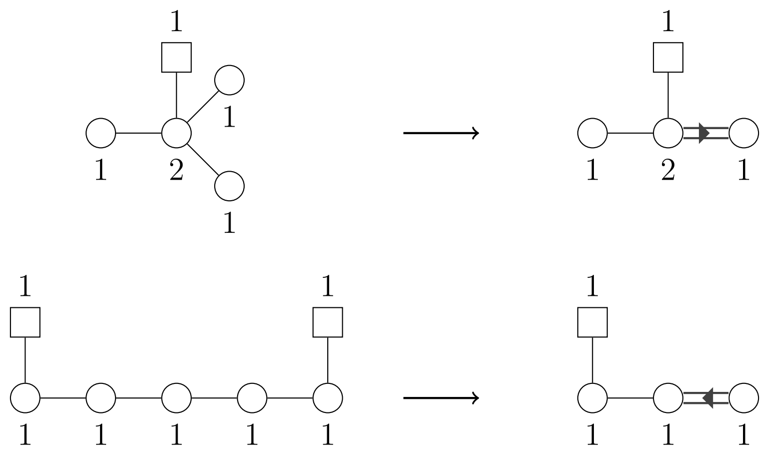


Figure 4.14: Quivers on the left fold into quivers on the right.

which reflected the enigmatic multiple link, checking against earlier tentative results of [139] on F_4 and G_2 spaces. Although the conjecture was highly successful in its goal, giving support to the existence of non-simply laced quivers and allowing further study [140], precise details of multiple links remained elusive[12]. They have made an appearance in the study of little string theory [141] or gauge-vortex duality [142] and W-algebras associated to them were studied in [143]. A mathematical treatment of folding and Coulomb branches of non-simply laced quivers was recently provided in [84]. Some of the phenomena in [144, 145] can be reinterpreted as folding the five-dimensional theories' magnetic quivers [15].

In this section we show (using an alternative approach to [84]) that the multiple link can be interpreted as the result of *quiver folding*; see Fig. 4.14 for examples. We first utilise abelianisation to show that Coulomb branches of A_{2n-1} (D_{n+1}) quivers fold into spaces with C_n (B_n) symmetry and derive the effects of folding on the monopole formula. We then reinterpret folding as an action on the quiver itself, showing that it produces non-simply laced quivers; in particular, our analysis of the monopole formula on examples reproduces the form in [11, 84].

Note that the examples below focus on nilpotent orbit quivers only because they are most easily studied using tools we have developed. We expect folding to be a completely general operation. For example, the quiver of Section 4.1.2 in [102] folds into the quiver in (7.1) of [105] as can be guessed by mapping $\mu_{2N-i} \mapsto \mu_i$ for $i < N$ in the former's HWG and comparing to the HWG of the latter quiver.

4.3.1 Action on the Coulomb branch

Although one can fold a quiver directly, the operation can also be performed on a discretely gauged Coulomb branch. The prerequisites for folding and discrete

[12]According to [11], Jan Troost suggested that quivers of this type might be understood as folded simply laced quivers, an idea that ultimately finds validation in [84] and our results.

gauging are identical: a quiver with an automorphism. We start yet again with the example of a D_4 quiver in the bottom left of Fig. 4.14. Recall that in (4.20) the discretely gauged quiver's operator $\tilde{e}_{\langle 0,0,1\rangle}$ is defined as $e_{\langle 0,0,1,0\rangle} + e_{\langle 0,0,0,1\rangle} = u_3^+ + u_4^+$. A space is folded by restricting it to the subspace fixed under the action of the symmetry, which in this case generates the constraints $u_3^+ = u_4^+$ as well as $\varphi_3 = \varphi_4$ and so on; we denote this space $\hat{\mathcal{C}}$ and in general use hats to denote variables on the folded space. Note that mass parameters must be set to identical values across folded legs; sometimes this removes all independent mass parameters but one and, as a result, even though the original space is mass-deformable, the folded space is not.

As long as we stay on $\hat{\mathcal{C}}$ there is no more need to track each individual wreathed variable. To reduce to a minimal necessary set we introduce the *folding map*

$$F(x_{i,a}) = \frac{\hat{x}_{I,a}}{\#_i} \tag{4.88}$$

$$F(x+y) = F(x) + F(y) \tag{4.89}$$

$$F(cx^m y^n) = cF(x)^m F(y)^n \tag{4.90}$$

where the *multiplicity* $\#_i$ denotes the number of nodes that fold onto the same node as node i, x and y are arbitrary operators, c is a complex number and $I = \min_j\{j : \pi(j) = \pi(i)\}$. In particular, $F(u_3^+) = F(u_4^+) = \frac{\hat{u}_3^+}{2}$. As a result, $F(\tilde{e}_{\langle 0,0,1\rangle}) = \hat{u}_3^+ = \hat{e}_{\langle 0,0,1\rangle}$.

The folding map has a simple interpretation. Abelianised variables of the initial, unfolded quiver, partition into orbits of the automorphism. The folding map merely sets every single variable in that orbit to the same value; for convenience, basic abelianised variables are normalised by node multiplicity. In other words, the folded Coulomb branch is a restricted subspace of the discretely gauged quiver's Coulomb branch.

While abelianised variables fold in a completely trivial manner, composite operators are more interesting. For example, let's fold the operator in (4.20):

$$\begin{aligned} \hat{e}_{\langle 0,1,2\rangle} &= F(\tilde{e}_{\langle 0,1,2\rangle}) = \\ &= F(e_{\langle 0,1,1,1\rangle}) = \\ &= \sum_{a=1,2} \frac{F(u_{2,a}^+ u_3^+ u_4^+)}{F(\varphi_{2,a}-\varphi_3)F(\varphi_{2,a}-\varphi_4)} = \\ &\sum_{a=1,2} \frac{\hat{u}_{2,a}^+ \hat{u}_3^{+2}/4}{(\varphi_{2,a}-\hat{\varphi}_3/2)^2} = \\ &\sum_{a=1,2} \frac{\hat{u}_{2,a}^+ \hat{u}_3^{+2}}{(2\varphi_{2,a}-\hat{\varphi}_3)^2} \end{aligned} \tag{4.91}$$

If the folded space is to retain the original's hyper-Kähler property, the symplectic property in particular must be preserved and the Poisson brackets on the folded space must close. In other words for any $\hat{f}, \hat{g} \in \mathbb{C}[\hat{\mathcal{C}}]$ we require $\{\hat{f}, \hat{g}\} \in \mathbb{C}[\hat{\mathcal{C}}]$, ie. $\{\hat{f}, \hat{g}\} = \pi\left(\{\hat{f}, \hat{g}\}\right)$. It is enough to show that generators $\hat{x}_{i,a}$ of the Poisson algebra satisfy this property:

$$\{x_{i,a}, x_{j,b}\} = F(x_{k,c}) = F(x_{\pi(k),c}) = \pi\left(F(x_{k,c})\right) = \pi\left(\{x_{i,a}, x_{j,b}\}\right) \tag{4.92}$$

where we restrict to the folding locus

$$x_{i,a} = x_{\pi(i),a}, \forall \pi \in \Gamma \subset \text{Aut } Q. \tag{4.93}$$

where Γ is the subgroup by whose action we fold.

So we have in our hands two pieces: a "folded" subspace (with its coordinate ring) and a Poisson bracket on this space. If we assume that the complex structures also properly restrict to the subspace, we have a new hyper-Kähler space to study. What is it? What is its symmetry?

Now we re-establish contact with discrete gauging. For $\tilde{\mathcal{O}}_i \in \mathbb{C}[\tilde{\mathcal{C}}]_2$ and $\hat{\mathcal{O}}_i \in \mathbb{C}[\hat{\mathcal{C}}]_2$, we have

$$\{\tilde{\mathcal{O}}_i, \tilde{\mathcal{O}}_j\} = \sum_k c_{ij}{}^k \tilde{\mathcal{O}}_k \tag{4.94}$$

and therefore the relations in particular hold on the automorphism's fixed point, which is the folded subspace:

$$\{\hat{\mathcal{O}}_i, \hat{\mathcal{O}}_j\} = \sum_k c_{ij}{}^k \hat{\mathcal{O}}_k. \tag{4.95}$$

Therefore, unless some folded $\hat{\mathcal{O}}_k$ identically vanish, the two algebras have identical structure constants and are in fact isomorphic as Lie algebras. A simple proof in appendix B shows that $\hat{\mathcal{O}}_k$ is not 0 everywhere on the folded space so we conclude that folded spaces have the same continuous symmetries as their discretely gauged counterparts.

In particular, a A_{2n-1} (D_{n+1}) quiver's Coulomb branch folds into a C_n (B_n)-symmetric space of strictly lower dimension and the minimal nilpotent orbit of D_4 folds into the minimal nilpotent orbit of B_3. Of course this space is just the Coulomb branch of a non-simply laced quiver, and we claim this is no coincidence: although we have so far only explored folding as an action on the Coulomb branch, we conjecture it is in fact merely one facet of an action on the *quiver theory* and that all non-simply laced quivers can be understood as folded simply laced quivers.

As was hinted in Section 2.1.4, in some special cases a B_3 non-simply laced quiver, eg. the bottom right quiver in Fig. 4.14, can fold into G_2 despite the lack of an obvious symmetry. There is one major difference however: multiplicities are assigned in a more involved manner. As a prerequisite, the "short root" (i.e. third) gauge node must have the same rank and number of flavors as the "vector root" (i.e. first node). We can unfold the B_3 quiver into a D_4 shape by simply reversing the folding procedure. Let us denote the variables of that quiver's Coulomb branch e.g. $\varphi_i^{D_4}$, with $\varphi_i^{B_3}$ and $\varphi_i^{G_2}$ the partially and fully folded counterparts. Then at the $D_4 \to G_2$ folding locus the following holds:

$$\varphi_1^{D_4} = \varphi_3^{D_4} = \varphi_4^{D_4} = \varphi_1^{B_3} = \frac{\varphi_3^{B_3}}{2} = \frac{\varphi_1^{G_2}}{3} \tag{4.96}$$

So the B_3 quiver can fold to G_2 as if $\mu_1 = 3$ and $\mu_3 = \frac{3}{2}$.

4.3.2 Monopole formula: examples

To show that folded quivers become non-simply laced, we compute two explicit examples and conjecture that the pattern generalises.

min $A_3 \to$ min C_2

The first check will be done on quivers in Figure 4.16 by folding two $U(1)$ nodes.

Let HS^A and HS^C be the Hilbert series of the initial and folded quivers, respectively:

$$\mathrm{HS}^A(t,x,y,z) = \frac{1}{(1-t^2)^3} \sum_{q_1,q_2,q_3\in\mathbb{Z}} t^{|q_1|+|q_1-q_2|+|q_2-q_3|+|q_3|}(xy)^{q_1}\left(\frac{x}{y}\right)^{q_3} z^{q_2} \tag{4.97}$$

$$\mathrm{HS}^C(t,x,z) = \frac{1}{(1-t^2)^2} \sum_{r_1,r_2\in\mathbb{Z}} t^{|r_1|+|r_1-2r_2|} x^{r_1} z^{r_2}\,. \tag{4.98}$$

The unrefined Hilbert series are:

$$\mathrm{HS}^A(t,1,1,1) = \frac{(1+t^2)\,(1+8t^2+t^4)}{(1-t^2)^6} \tag{4.99}$$

$$\mathrm{HS}^C(t,1,1) = \frac{1+6t^2+t^4}{(1-t^2)^4}\,. \tag{4.100}$$

Note the unusual fugacity y in HS^A which is crucial in the following calculations. By comparison with known Hilbert series, we find that the two Coulomb branches are the (closures of the) minimal nilpotent orbits of A_3 and C_2.

We will now derive the action $\mathrm{HS}^A \to \mathrm{HS}^C$ in two steps.

At the level of bare monopole operators, many become duplicate. For example, $(u_1^+)^2$, $u_1^+u_3^+$ and $(u_3^+)^2$ all fold to $\frac{(\hat{u}_1^+)^2}{4}$. More generally, a bare monopole monomial in the A_3 theory can be expressed (not necessarily uniquely) as a product of generators

$$\mathcal{O}_{q_1,q_2,q_3} = \prod_i e_{\langle q_1^i, q_2^i, q_3^i \rangle} \tag{4.101}$$

where $q_j = \sum_i q_j^i$. Note that $|q_1^i - q_3^i| \in \{0, 1\}$. The S_2 symmetry exchanges $e_{\langle q_1^i, q_2^i, q_3^i \rangle} \leftrightarrow e_{\langle q_3^i, q_2^i, q_1^i \rangle}$ and acting with it on *any number* of operators in the product produces a monopole in the A_3 theory which folds to the exact same monopole in the C_2 theory. "Flipping" a single operator in this way leaves q_1 and q_3 unchanged or changes both by ± 1 with opposite signs so that $q_1 + q_3$ is preserved. *Sequential* action on all the monopoles in the product produces $\mathcal{O}_{q_3,q_2,q_1}$. It follows that in this chain of flips there is an operator $\mathcal{O}_{\frac{q_1+q_3}{2},q_2,\frac{q_1+q_3}{2}}$ or $\mathcal{O}_{\frac{q_1+q_3+1}{2},q_2,\frac{q_1+q_3-1}{2}}$, depending on the parity of $q_1 + q_3$. Since all operators in the chain fold to the same operator, the C_2 monopole formula better count *precisely one of them*. We will pick either $\mathcal{O}_{\frac{q_1+q_3}{2},q_2,\frac{q_1+q_3}{2}}$ or $\mathcal{O}_{\frac{q_1+q_3+1}{2},q_2,\frac{q_1+q_3-1}{2}}$ (of which precisely one exists), which translates to selecting only monopoles with $q_1 = q_3$ or $q_1 = q_3 + 1$, respectively.

To accomplish this we must extract only the terms constant and linear in y, as can be seen from (4.97): terms constant in y come from the charge sublattice $q_1 = q_3$ while linear terms all satisfy $q_1 = q_3 + 1$. To set up later generalisation we further slightly modify the prescription to an equivalent form: we will extract every operator at order y^0 and *average over* operators at order y and y^{-1}.

The second step corrects for scalar dressing: one extraneous scalar field must be removed since $\varphi_1 = \varphi_3 = \frac{\hat{\varphi}_1}{2}$. We need only multiply the entire expression with $1 - t^2$ to remove the newly duplicate $U(1)$ dressing factor $\frac{1}{1-t^2}$.

We conjecture that these two modifications are sufficient to represent the action of folding on the Hilbert series.

To implement them, we multiply the (unsummed) monopole formula by the kernel $\frac{1}{2\pi i y}\left(1 + \frac{1}{2}\left(y + \frac{1}{y}\right)\right)$ and and integrate around $y = 0$, picking up the desired contributions by the residue theorem. Finally we multiply by the scalar factor $(1 - t^2)$:

$$\mathrm{HS}^C(t, x, z) = (1 - t^2) \oint \frac{\mathrm{d}y}{2\pi i y} \left(1 + \frac{1}{2}\left(y + \frac{1}{y}\right)\right) \mathrm{HS}^A(t, x, y, z) \tag{4.102}$$

And indeed:

$$\begin{aligned} RHS &= \frac{1}{2}(1 - t^2)^{-2} \oint \frac{\mathrm{d}y}{2\pi i y} \left(y + \frac{1}{y} + 2\right) \sum_{q_1,q_2,q_3 \in \mathbb{Z}} t^{|q_1|+|q_1-q_2|+|q_2-q_3|+|q_3|} (xy)^{q_1} \left(\frac{x}{y}\right)^{q_3} z^{q_2} \\ &= \frac{1}{2}(1 - t^2)^{-2} \oint \frac{\mathrm{d}y}{2\pi i y} \sum_{q_1,q_2,q_3 \in \mathbb{Z}} t^{|q_1|+|q_1-q_2|+|q_2-q_3|+|q_3|} x^{q_1+q_3} y^{q_1-q_3-1} z^{q_2} + \end{aligned}$$

$$
\begin{aligned}
& \frac{1}{2}(1-t^2)^{-2}\oint \frac{\mathrm{d}y}{2\pi i y}\sum_{q_1,q_2,q_3\in\mathbb{Z}} t^{|q_1|+|q_1-q_2|+|q_2-q_3|+|q_3|}x^{q_1+q_3}y^{q_1-q_3+1}z^{q_2}+\\
& (1-t^2)^{-2}\oint \frac{\mathrm{d}y}{2\pi i y}\sum_{q_1,q_2,q_3\in\mathbb{Z}} t^{|q_1|+|q_1-q_2|+|q_2-q_3|+|q_3|}x^{q_1+q_3}y^{q_1-q_3}z^{q_2}\\
= & \frac{1}{2}(1-t^2)^{-2}\sum_{r_2\in\mathbb{Z},r_1\in(2\mathbb{Z}+1)} t^{|(r_1+1)/2|+|(r_1+1)/2-r_2|+|r_2-(r_1-1)/2|+|(r_1-1)/2|}x^{r_1}z^{r_2}+\\
& \frac{1}{2}(1-t^2)^{-2}\sum_{r_2\in\mathbb{Z},r_1\in(2\mathbb{Z}+1)} t^{|(r_1-1)/2|+|(r_1-1)/2-r_2|+|r_2-(r_1+1)/2|+|(r_1+1)/2|}x^{r_1}z^{r_2}+\\
& (1-t^2)^{-2}\sum_{r_2\in\mathbb{Z},r_1\in(2\mathbb{Z})} t^{|r_1/2|+|r_1/2-r_2|+|r_2-r_1/2|+|r_1/2|}x^{r_1}z^{r_2}\\
= & (1-t^2)^{-2}\sum_{r_2\in\mathbb{Z},r_1\in(2\mathbb{Z}+1)} t^{|(r_1+1)/2|+|(r_1+1)/2-r_2|+|r_2-(r_1-1)/2|+|(r_1-1)/2|}x^{r_1}z^{r_2}+\\
& (1-t^2)^{-2}\sum_{r_2\in\mathbb{Z},r_1\in(2\mathbb{Z})} t^{|r_1|+|r_1-2r_2|}x^{r_1}z^{r_2}\\
= & (1-t^2)^{-2}\sum_{r_1,r_2\in\mathbb{Z}} t^{|r_1|+|r_1-2r_2|}x^{r_1}z^{r_2}
\end{aligned}
$$

In particular note the appearance of 2 in $|r_1-2r_2|$, the novel feature in non-simply laced quivers' monopole formulas.

$\min D_4 \to \min G_2$

We now look at the folding of three $U(1)$ gauge nodes of the D_4 minimal nilpotent orbit quiver. We again assign fugacities to the nodes: call z the fugacity for the $U(2)$ node, and xy_1, $x\frac{y_2}{y_1}$, $x\frac{1}{y_2}$ the fugacities for the three $U(1)$ nodes. This parametrisation is chosen so that folding corresponds to an integration over the y_i, which have an A_2 symmetry. Note that this prescription generalises the previous example, where the "folding fugacity" appeared as y and y^{-1}, which are related by an A_1 symmetry.

The folding equation becomes

$$
\mathrm{HS}^{G_2}(t,x,z) = (1-t^2)^2\oint \frac{\mathrm{d}y_1}{2\pi i y_1}\frac{\mathrm{d}y_2}{2\pi i y_2}\, f(y_1,y_2)H^{D_4}(t,x,y_1,y_2,z)
$$

with

$$
f(y_1,y_2) = 1+\frac{1}{3}\left(y_1+\frac{1}{y_1}+y_2+\frac{1}{y_2}+\frac{y_2}{y_1}+\frac{y_1}{y_2}\right).
$$

Note that this kernel is a natural generalization of the previous case $f(y) = 1+\frac{1}{2}(y+y^{-1})$. We conjecture that the monopole formula of a quiver with n $U(1)$ legs folds by integration over the kernel

$$
f(y_1,\dots y_{n-1}) = 1+\frac{1}{n}\chi_f^{A_{n-1}}(y_1,\dots,y_{n-1}) \tag{4.103}
$$

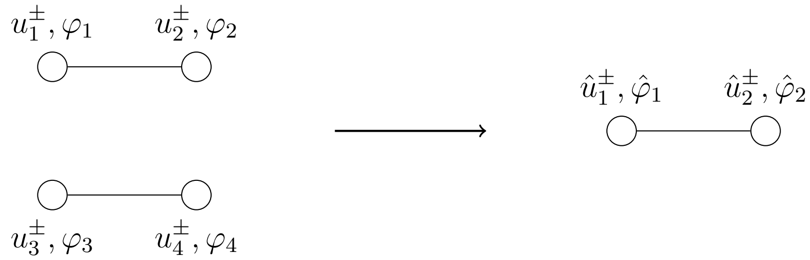


Figure 4.15: Example of folding two "parallel" links which do not originate from the same node. Note that folding does not introduce a multiple link in this case.

where $\chi_f^{A_{n-1}}$ is the character of the A_{n-1} fundamental representation.

The steps outlined above can be generalised to longer legs, larger gauge groups and, presumably, to completely arbitrary legs. However, rather than undertaking this task ourselves, we refer to [84] for a systematic look at the link between folding and the modified monopole formula of [11].

4.3.3 Non-simply laced quivers

It is possible to generalise abelianisation, including the Poisson structure, directly to non-simply laced framed quivers; the generalisation of the monopole formula was already achieved in [11]. The input data are a list of gauge nodes with optional fundamental matter and a connectivity matrix κ defined precisely like the Cartan matrix of a Dynkin diagram. One can always unfold the quiver $\hat{Q}$ into a simply laced quiver Q. Keeping with the term's use in previous sections, the number of nodes of Q which fold onto the i-th node of $\hat{Q}$ is called the *multiplicity* $\#_i$ *of node* i.

Each node still contributes three abelianised variables $\hat{u}^\pm_{i,a}$ and $\hat{\varphi}_{i,a}$ but the relations are slightly modified. They can be derived by demanding consistency with folding; recall that $x_{i,a} = \hat{x}_{i,a}/\#_i$ on the subspace preserved by discrete action. For simplicity we present them in the case of quivers with one multiple edge:

$$\hat{u}^+_{i,a}\hat{u}^-_{i,a} = -\#_i^2 \frac{\prod_{w\in\mathcal{R}^+_{i,a}} \langle w, \vec{\hat{\varphi}}/\vec{\#}\rangle^{g_i(w)|w_{i,a}|}}{\prod_{\alpha\in\Phi}\langle \alpha, \vec{\hat{\varphi}}/\vec{\#}\rangle^{|\alpha_{i,a}|}} \tag{4.104}$$

where $\mathcal{R}$ is defined as if the quiver were simply laced (ie. the multiple link were replaced with one simple link), $\vec{\hat{\varphi}}/\vec{\#}$ denotes a vector of $\hat{\varphi}_{i,a}/\#_i$ and $g_i(w)$ is an auxiliary function defined as

$$g_i(w) = \begin{cases} |\kappa_{ji}| & \text{if } w \text{ connects the node } i \text{ to node } j, \\ 1 & \text{otherwise} \end{cases} \tag{4.105}$$

and κ is the Cartan matrix of the non-simply laced quiver.

The derivation of Poisson brackets is slightly more subtle. As a concrete example, consider a quiver with nodes 1 to 4 (plus possibly others) such that 1 and 2 , resp. 3 and 4 are connected, and 3 and 4 fold onto 1 and 2, respectively (see Fig 4.15). Then

$$\begin{aligned}\{\hat{\varphi}_1, \hat{u}_1^+\} = \{\varphi_1 + \varphi_3, u_1^+ + u_3^+\}\big|_{x_1=x_3} = (\{\varphi_1, u_1^+\} + \{\varphi_3, u_3^+\})\big|_{x_1=x_3} = \\ = \varphi_1 + \varphi_3\big|_{\substack{x_1=x_3\\x_2=x_4}} = \hat{\varphi}_1 \quad (4.106)\end{aligned}$$

Similarly, and keeping to the same quiver for this example,

$$\{\hat{u}_1^+, \hat{u}_2^+\} = \{u_1^+ + u_3^+, u_2^+ + u_4^+\}\big|_{\substack{x_1=x_3\\x_2=x_4}} = 2\{u_1^+, u_2^+\} = 2\kappa_{12}\frac{u_1^+ u_2^+}{\varphi_1 - \varphi_2} = 2\kappa_{12}\frac{\hat{u}_1^+/2\ \hat{u}_2^+/2}{\hat{\varphi}_1/2 - \hat{\varphi}_2/2} \quad (4.107)$$

Note that the factor of 2 comes from the two links which fold onto each other.

Now that the procedure is clear it readily generalises:

$$\{\hat{\varphi}_{i,a}, \hat{u}_{i,a}^\pm\} = \pm\hat{u}_{i,a}^\pm \quad (4.108)$$

$$\{\hat{u}_{i,a}^+, \hat{u}_{i,a}^-\} = \#_i^2 \frac{\partial}{\partial\hat{\varphi}_{i,a}} \frac{\prod_{w\in\mathcal{R}}\langle w, \vec{\hat{\varphi}}/\vec{\#}\rangle^{g_i(w)|w_{i,a}|}}{\prod_{\alpha\in\Phi}\langle \alpha, \vec{\hat{\varphi}}/\vec{\#}\rangle^{|\alpha_{i,a}|}} \quad (4.109)$$

$$\{\hat{u}_{i,a}^\pm, \hat{u}_{j,b}^\pm\} = \pm\kappa_{ij}^S \frac{\#_{ij}}{\#_i\#_j} \frac{\hat{u}_{i,a}^\pm \hat{u}_{j,b}^\pm}{\hat{\varphi}_{i,a}/\#_i - \hat{\varphi}_{j,b}/\#_j} \quad (4.110)$$

where κ^S is a "simply laced" Cartan matrix defined as $\kappa_{ij}^S = \max(\kappa_{ij}, \kappa_{ji})$ (essentially throwing away information about multiplicity of edges) and $\#_{ij}$ is the *link multiplicity* of the edge between nodes i and j defined as the number of its pre-images in the unfolded quiver. Remember that just as in the case of abelianised relations this form is appropriate for quivers with one multiple edge.

4.4 Examples

In this section we study several cases of nilpotent orbit quivers, ie. quiver theories whose Coulomb branches are nilpotent orbits. Their chiral rings are generated by moment maps, which we explicitly construct; recall that such moment maps transform in the coadjoint representation $\mathrm{coadj}(\mathfrak{g}) \simeq \mathrm{adj}(\mathfrak{g})$ of the Coulomb branch symmetry. The chiral ring data is completed by providing a set of relations which also form representations of the Coulomb branch symmetry. We discretely gauge and fold such quivers and examine the resulting Coulomb branches in turn.

Most spaces encountered in this section are nilpotent orbits; their coordinate rings are generated by a single coadjoint representation. But there are a few cases which are not nilpotent orbits: their Coulomb branches are generated not only by

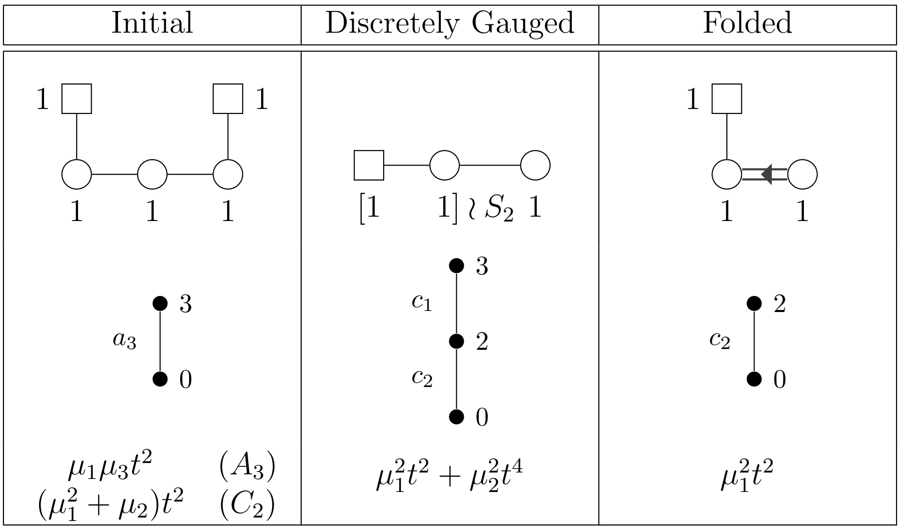


Figure 4.16: A_3 minimal nilpotent orbit and its discrete reductions.

coadj($\mathfrak{g}$) but also by chiral ring elements in other representations of $\mathfrak{g}$. If the quiver is balanced, for the examples studied in this section, we find that the remaining generators *also* assemble into coadjoint (or sometimes trivial) representations and the bulk of our techniques still applies. One such case appears in Sec. 4.4.2. The resulting spaces are not as comprehensively tabulated as nilpotent orbits and we generally have to turn to more varied sources to find their Hilbert series or highest weight generating functions.

4.4.1 min $A_3 \to$ (n.)min C_2

A-type quivers tend to have very simple moment maps which can be presented in reasonably compact form, allowing us to present the action of discrete gauging and folding.

The quivers we choose, as exhibited in Figure 4.16, also exhibit an interesting pattern of complex mass deformation. As a general rule, all $\varphi_{i,a}$ abelian moduli and $M_{i,a}$ parameters only appear in the abelian algebra as differences and as a result the moduli space is invariant under reparametrisations $\varphi_{i,a} \to \varphi_{i,a} + c$, $M_{i,a} \to M_{i,a} + c$. Since there are precisely two mass parameters, the moduli space relations can be modified by terms proportional to $M_1 - M_3$, ie. a complex mass deformation. However both discrete gauging and folding remove one half of mass parameters by forcing $M_1 = M_3$, which can in turn be set to 0 by the reparametrisation above. As a result only the original space can be deformed by one triplet of mass parameters.

Initial quiver

To remind the reader we reproduce abelianised relations restricting $u_i^{\pm}, \varphi_i$ for $i = 1, 2, 3$:

$$u_1^+ u_1^- = -(\varphi_1 - M_1)(\varphi_1 - \varphi_2) \tag{4.111}$$
$$u_2^+ u_2^- = -(\varphi_2 - \varphi_1)(\varphi_2 - \varphi_3) \tag{4.112}$$
$$u_3^+ u_3^- = -(\varphi_3 - \varphi_2)(\varphi_3 - M_3) \tag{4.113}$$

The Coulomb branch is generated by

$$N_{A_3} = \begin{pmatrix} \varphi_1 - \frac{3M_1+M_3}{4} & u_1^- & -\frac{u_1^- u_2^-}{\varphi_1-\varphi_2} & \frac{u_1^- u_2^- u_3^-}{(\varphi_1-\varphi_2)(\varphi_2-\varphi_3)} \\ u_1^+ & -\varphi_1+\varphi_2+\frac{M_1-M_3}{4} & u_2^- & -\frac{u_2^- u_3^-}{(\varphi_2-\varphi_3)} \\ -\frac{u_1^+ u_2^+}{\varphi_1-\varphi_2} & u_2^+ & -\varphi_2+\varphi_3+\frac{M_1-M_3}{4} & u_3^- \\ \frac{u_1^+ u_2^+ u_3^+}{(\varphi_1-\varphi_2)(\varphi_2-\varphi_3)} & -\frac{u_2^+ u_3^+}{(\varphi_2-\varphi_3)} & u_3^+ & -\varphi_3+\frac{M_1+3M_3}{4} \end{pmatrix} \tag{4.114}$$

and one can read its relations either from the HWG [20]

$$\mathrm{HWG}(t, \mu_i) = \frac{1}{1 - \mu_1 \mu_3 t^2} \tag{4.115}$$

or simply from the Joseph relations, which are obeyed by any minimal nilpotent orbit:

$$t^4\left([101] + [000]\right) : N^2 = -\frac{1}{2}(M_1 - M_3)N + \frac{3}{16}(M_1 - M_3)^2 \mathbf{1} \tag{4.116}$$
$$t^4[020] : \sum_{a',b'} \varepsilon_{a'b'[cd} N_a^{a'} N_{b]}^{b'} = -\frac{1}{16}(M_1 - M_3)^2 \varepsilon_{abcd} \tag{4.117}$$

Discrete Gauging

The A_3 moment map discretely gauges to the following expression:

$$N_{C_2} = \begin{pmatrix} \frac{1}{2}(\tilde\varphi_1 + \tilde\varphi_3) & \frac{1}{2}(\tilde u_1^- + \tilde u_3^-) & \frac{1}{2}\left(-\frac{\tilde u_1^- \tilde u_2^-}{\tilde\varphi_1-\tilde\varphi_2} + \frac{\tilde u_2^- \tilde u_3^-}{\tilde\varphi_2-\tilde\varphi_3}\right) & \frac{\tilde u_1^- \tilde u_2^- \tilde u_3^-}{(\tilde\varphi_1-\tilde\varphi_2)(\tilde\varphi_2-\tilde\varphi_3)} \\ \frac{1}{2}(\tilde u_1^+ + \tilde u_2^+) & -\frac{1}{2}(\tilde\varphi_1+\tilde\varphi_3)+\tilde\varphi_2 & \tilde u_2^- & \frac{1}{2}\left(\frac{\tilde u_1^- \tilde u_2^-}{\tilde\varphi_1-\tilde\varphi_2} - \frac{\tilde u_2^- \tilde u_3^-}{\tilde\varphi_2-\tilde\varphi_3}\right) \\ \frac{1}{2}\left(-\frac{\tilde u_1^+ \tilde u_2^+}{\tilde\varphi_1-\tilde\varphi_2} + \frac{\tilde u_2^+ \tilde u_3^+}{\tilde\varphi_2-\tilde\varphi_3}\right) & \tilde u_2^+ & -\tilde\varphi_2 + \frac{1}{2}(\tilde\varphi_1+\tilde\varphi_3) & \frac{1}{2}(\tilde u_1^- + \tilde u_3^-) \\ \frac{\tilde u_1^+ \tilde u_2^+ \tilde u_3^+}{(\tilde\varphi_1-\tilde\varphi_2)(\tilde\varphi_2-\tilde\varphi_3)} & \frac{1}{2}\left(\frac{\tilde u_1^+ \tilde u_2^+}{\tilde\varphi_1-\tilde\varphi_2} - \frac{\tilde u_2^+ \tilde u_3^+}{\tilde\varphi_2-\tilde\varphi_3}\right) & \frac{1}{2}(\tilde u_1^+ + \tilde u_2^+) & \frac{1}{2}(-\tilde\varphi_1 - \tilde\varphi_3) \end{pmatrix} \tag{4.118}$$

and the resulting space is expected to have quaternionic dimension 3 and exhibit C_2 symmetry. The next-to-minimal nilpotent orbit of C_2 is a suitable candidate. Its HWG reads [20]

$$\mathrm{HWG}(t, \mu_i) = \frac{1}{(1 - \mu_1^2 t^2)(1 - \mu_2^2 t^4)} \tag{4.119}$$

suggesting several relations:

$$t^4\left([00]+[01]\right) : N^2 = 0 \tag{4.120}$$

$$t^6[20] : \text{rank}(N) \leq 2 \tag{4.121}$$

(Note that in our convention we multiply C_n matrices as ordinary matrices, ie. without insertion of an ϵ tensor.) The second of these relations can be written equivalently as

$$\sum_{b,c,d,b',c',d'} \varepsilon_{a'b'c'd'}\varepsilon^{abcd} N_b^{b'} N_c^{c'} N_d^{d'} = 0.$$

In other words, an explicit algebraic description of the Coulomb branch of the discretely gauged quiver is

$$\left\{N \in \mathfrak{gl}(4,\mathbb{C}) | N^2 = 0, \quad \text{rank}(N) \leq 2, \quad N^T J - JN = 0\right\}. \tag{4.122}$$

Folding

The folded moment map is similar:

$$N_{C_2} = \begin{pmatrix} \frac{1}{2}\hat{\varphi}_1 & \frac{1}{2}\hat{u}_1^- & -\frac{\hat{u}_1^-\hat{u}_2^-}{\hat{\varphi}_1-2\hat{\varphi}_2} & -\frac{(\hat{u}_1^-)^2\hat{u}_2^-}{(\hat{\varphi}_1-2\hat{\varphi}_2)^2} \\ \frac{1}{2}\hat{u}_1^+ & -\frac{1}{2}\hat{\varphi}_1+\hat{\varphi}_2 & \hat{u}_2^- & \frac{\hat{u}_2^-\hat{u}_1^-}{(\hat{\varphi}_1-2\hat{\varphi}_2)} \\ -\frac{\hat{u}_1^+\hat{u}_2^+}{\hat{\varphi}_1-2\hat{\varphi}_2} & \hat{u}_2^+ & -\hat{\varphi}_2+\frac{1}{2}\hat{\varphi}_1 & \frac{1}{2}\hat{u}_1^- \\ -\frac{(\hat{u}_1^+)^2\hat{u}_2^+}{(\hat{\varphi}_1-2\hat{\varphi}_2)^2} & \frac{\hat{u}_2^+\hat{u}_1^+}{(\hat{\varphi}_1-2\hat{\varphi}_2)} & \frac{1}{2}\hat{u}_1^+ & -\frac{1}{2}\hat{\varphi}_1 \end{pmatrix} \tag{4.123}$$

The Coulomb branch has dimension 2 and C_2 symmetry, which agrees with the minimal nilpotent orbit with HWG [20]

$$\text{HWG}(t,\mu_i) = \frac{1}{1-\mu_1^2 t^2} \tag{4.124}$$

This space satisfies slightly more stringent (Joseph) relations:

$$t^4\left([00]+[01]\right) : N^2 = 0 \tag{4.125}$$

$$t^4[02] : \text{rank}(N) \leq 1 \tag{4.126}$$

The second of these relations can be written equivalently as $N_{[a}^{a'} N_{b]}^{b'} = 0$. In other words, an explicit algebraic description of the Coulomb branch of the folded quiver is

$$\left\{N \in \mathfrak{gl}(4,\mathbb{C}) | N^2 = 0, \quad \text{rank}(N) \leq 1, \quad N^T J - JN = 0\right\}. \tag{4.127}$$

4.4.2 min $D_4 \to G_2$

G_2 is small yet non-trivial enough to serve as an excellent illustration of the techniques studied in this section. Since it is only fourteen-dimensional, we provide the complete folding prescription from both D_4 and B_3:

$$G_2 = \mathrm{span}_{\mathbb{C}}\left(E^{G_2}_{\pm 1^3 2^2}, E^{G_2}_{\pm 1^3 2}, E^{G_2}_{\pm 1^2 2}, E^{G_2}_{\pm 12}, E^{G_2}_{\pm 1}, E^{G_2}_{\pm 2}, H^{G_2}_{1}, H^{G_2}_{2}\right)$$

$$\begin{aligned}
E^{G_2}_{\pm 1^3 2^2} &= E^{D_4}_{\pm 12^2 34} = E^{B_3}_{\pm 12^2 3^2} \\
E^{G_2}_{\pm 1^3 2} &= E^{D_4}_{\pm 1234} = E^{B_3}_{\pm 123^2} \\
E^{G_2}_{\pm 1^2 2} &= -E^{D_4}_{\pm 123} - E^{D_4}_{\pm 124} + E^{D_4}_{\pm 234} = -E^{B_3}_{\pm 123} + E^{B_3}_{\pm 23^2} \\
E^{G_2}_{\pm 12} &= E^{D_4}_{\pm 12} - E^{D_4}_{\pm 23} - E^{D_4}_{\pm 24} = E^{B_3}_{\pm 12} - E^{B_3}_{\pm 23} \\
E^{G_2}_{\pm 1} &= E^{D_4}_{\pm 1} + E^{D_4}_{\pm 3} + E^{D_4}_{\pm 4} = E^{B_3}_{\pm 1} + E^{B_3}_{\pm 3} \\
E^{G_2}_{\pm 2} &= E^{D_4}_{\pm 2} = E^{B_3}_{\pm 2} \\
H^{G_2}_{\pm 1} &= H^{D_4}_{\pm 1} + H^{D_4}_{\pm 3} + H^{D_4}_{\pm 4} = H^{B_3}_{\pm 1} + H^{B_3}_{\pm 3} \\
H^{G_2}_{\pm 2} &= H^{D_4}_{\pm 2} = H^{B_3}_{\pm 2}
\end{aligned}$$

Recall that G_2 is characterised as the subalgebra of B_3 which preserves a particular rank 3 antisymmetric tensor ϕ; for more details see Section 2.1.4.

The goal of this subsection is to identify quivers whose Coulomb branches are generated by operators in one G_2 coadjoint representation [01]; such spaces are necessarily nilpotent orbits. We also study one related space whose coordinate ring is generated by coadjoint generators but is not a nilpotent orbit. The following sections should be read alongside Figures 4.17 and 4.18.

Note that because the quiver has only flavor node of rank 1, the G_2 spaces studied below cannot be deformed by a complex mass.

We provide the first few symmetric products of the (co)adjoint representation for reference:

$$\mathrm{Sym}^2[01] = [20] + [00] + [02] \tag{4.128}$$

$$\mathrm{Sym}^3[01] = [30] + [21] + [01] + [10] + [03] \tag{4.129}$$

$$\mathrm{Sym}^4[01] = [40] + [31] + [22] + [11] + 2[20] + [00] + 2[02] + [04] \tag{4.130}$$

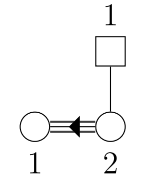
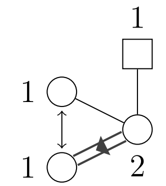
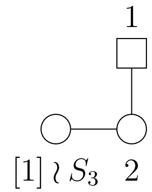
Label	Dimension	Relations	PL(HWG)	Hilbert Series	Quiver
[00]	0	$N=0$	0	1	
[01]	3	$N^2=0$	$\mu_2 t^2$	$\frac{(1+t^2)(1+7t^2+t^4)}{(1-t^2)^6}$	
[10]	4	$\mathrm{tr}(N^2)=0$ $N\wedge N\wedge N=0$ $\mathrm{rank}(N^2)\leq 1$	$\mu_2 t^2+\mu_1^2 t^4+\mu_1^3 t^6-\mu_1^6 t^{12}$	$\frac{1+6t^2+20t^4+43t^6-7t^8-7t^{10}}{(1-t^2)^8}$	
[02]	5	$\mathrm{tr}(N^2)=0$ $N\wedge N\wedge N=0$	$\mu_2 t^2+\mu_1^2 t^4+\mu_1^3 t^6+\mu_2^2 t^8-\mu_1^3\mu_2 t^{10}$	$\frac{(1+t^2)(1+3t^2+6t^4+3t^6+t^8)}{(1-t^2)^{10}}$	
[22]	6	$\mathrm{tr}(N^2)=\mathrm{tr}(N^6)=0$	too long to display	$\frac{(1+t^2)^2(1+t^4+t^8)}{(1-t^2)^{12}}$	

Figure 4.17: G_2 nilpotent orbits. We use the convention that the root α_1 is short and the root α_2 is long, so that μ_1 is the fundamental and μ_2 is the adjoint. N is a matrix in the fundamental 7-dimensional representation of the $\mathfrak{g}_2$, $N \in \mathfrak{so}(7,\mathbb{C})$. The relation $N\wedge N\wedge N=0$ stands for $N_{ab}N_{cd}N_{ef}\varepsilon^{abcdefgh}=0$. The Hilbert series in the last column are computed using Macaulay2. They agree with the computation of [32]. The Hasse diagram for nilpotent orbits of G_2 was computed in [146]. The [10] quiver is constructed with an unusual combination of folding and discrete gauging as describe in Sec. 4.4.2.

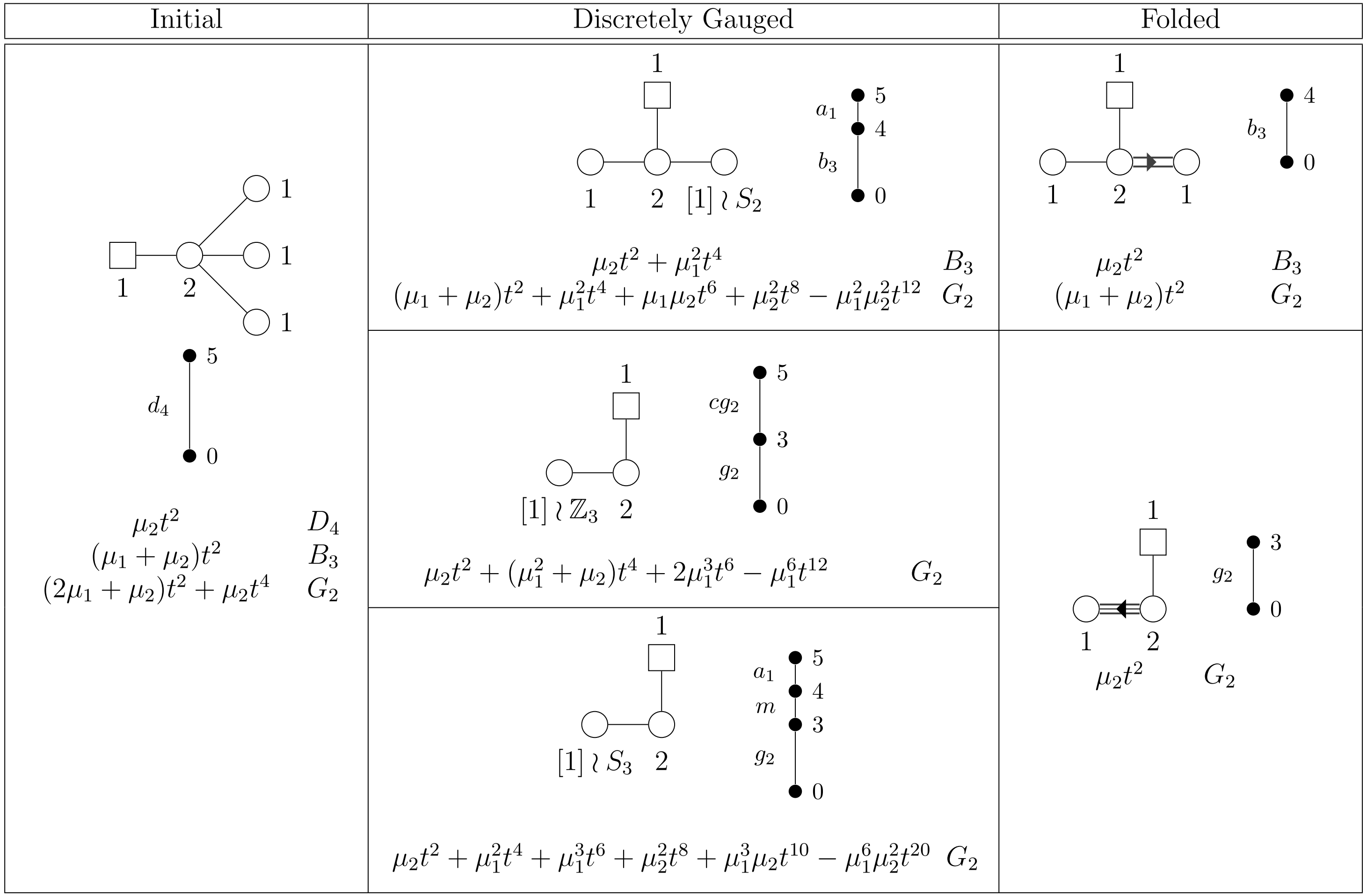


Figure 4.18: Overview of discrete actions on the D_4 minimal nilpotent orbit quiver by groups S_2, S_3 and $\mathbb{Z}_3$, previously studied in [105]. The non-normal transverse slice m was introduced in [146] and the slice cg_2 was described in [147]. Its Hilbert series is expressed in characters of $SU(2)$: $\sum_{n\neq 1}[n]t^n$. Highest weight generating functions of the corresponding moduli space are written in terms of fugacities for the algebras D_4, B_3, G_2, as specified next to them.

Initial quiver

The next few examples share the quiver on the left of Figure 4.18 as the common starting point. Its Coulomb branch is the minimal nilpotent orbit of D_4 which is parametrised by a coadjoint (antisymmetric) matrix M subject to the Joseph relations

$$([2000] + [0000])\, t^4 : N^2 = 0 \tag{4.131}$$

$$([0020] + [0002])\, t^4 : N \wedge N = 0 \tag{4.132}$$

We refer the reader to its treatment in [1] for more details.

Folding

The minimal nilpotent orbit of D_4 folds into the minimal nilpotent orbit of G_2 whose quiver is depicted in Figure 4.17 under the label [01]. To verify this claim we can look at the highest weight generating function of the minimal nilpotent orbit of G_2 [20]

$$\mathrm{HWG}(t) = \frac{1}{1 - \mu_2 t^2} = 1 + \mu_2 t^2 + \mu_2^2 t^4 + \dots \tag{4.133}$$

or recall that the Joseph relations tell us that the coadjoint generator is constrained by the quadratic relation

$$([20] + [00])\, t^4 : \; N^2 = 0. \tag{4.134}$$

Direct computation shows that the relation is satisfied by N defined either by folding the moment map of the D_4 minimal nilpotent orbit quiver or directly using the non-simply laced prescription.

S_3 discrete gauging

The five-dimensional subregular orbit of G_2 is known to be the S_3 quotient of the minimal nilpotent orbit of D_4 [100] so it should be the Coulomb branch of the appropriate D_4 quiver after discrete gauging, see row [02] of Figure 4.17. One can either symmetrise the D_4 moment map using the projector defined in (4.12) or, given the G_2 Chevalley Serre basis $\{X_i\}$, form the G_2 moment map N_{G_2} from its D_4 counterpart N_{D_4} as

$$N_{G_2} = \sum_i X_i^* \; \mathrm{tr}\left(N_{D_4} X_i\right).$$

The highest weight generating function is

$$\mathrm{HWG}(t) = \frac{1+\mu_1^3\mu_2 t^{10}}{(1-\mu_2 t^2)(1-\mu_1^2 t^4)(1-\mu_1^3 t^6)(1-\mu_2^2 t^8)} =$$
$$1+\mu_2 t^2+\left(\mu_1^2+\mu_2^2\right)t^4+\left(\mu_1^3+\mu_2\mu_1^2+\mu_2^3\right)t^6+\left(\mu_1^4+\mu_2\mu_1^3+\mu_2^2\mu_1^2+\mu_2^4+\mu_2^2\right)t^8+\ldots \tag{4.135}$$

Two relations are needed this time:

$$[00]t^4 : \mathrm{tr}N^2 = 0 \tag{4.136}$$
$$[10]t^6 : N \wedge N \wedge N = 0 \tag{4.137}$$

and both are satisfied by the coadjoint N_{G_2}.

Mixed folding and S_2 gauging

Midway between the two previous examples lies a nilpotent orbit of dimension 4. It is known [32] to be non-normal[13] and hence not expected to be the Coulomb branch of any quiver since both simply and non-simply laced quivers are necessarily normal [82, 84]. However we conjecture that it *can* be recovered by using a specific and non-generic discrete operation on the minimal nilpotent orbit quiver of B_3, which is itself four-dimensional. This would make our construction the first non-normal Coulomb branch in the literature.

We first construct the moment map N_{B_3} of the underlying B_3 quiver. The quiver has no obvious automorphism so rather than using the projector form in (4.12) we define the Chevalley-Serre basis $\{X_i\}$ of G_2 and project using the trick from the previous quiver calculation:

$$N_{G_2} = \sum_i X_i^* \ \mathrm{tr}\left(N_{B_3} X_i\right) \tag{4.138}$$

We depict the conjectured quiver theory in Figure 4.17 on row 10.

[13] An irreducible affine variety is *normal* if its coordinate ring is an integrally closed domain [148].

The HWG of this orbit is given by[20][14]

$$\begin{aligned}\mathrm{HWG}(t) &= \frac{1-\mu_1^6 t^{12}}{(1-\mu_2 t^2)(1-\mu_1^2 t^4)(1-\mu_1^3 t^6)} \\ &= 1+\mu_2 t^2 + \left(\mu_1^2+\mu_2^2\right) t^4 + \left(\mu_1^3+\mu_1^2\mu_2+\mu_2^3\right) t^6 + \left(\mu_1^4+\mu_1^3\mu_2+\mu_1^2\mu_2^2+\mu_2^4\right) t^8 + \ldots\end{aligned} \tag{4.141}$$

Compared to the subregular nilpotent orbit we find an extra relation at t^8 in the [02] representation. The condition that N^2 is of rank at most 1 is of this type.

In total the moment map is expected to satisfy three relations:

$$[00]t^4 : \mathrm{tr}(N^2) = 0 \tag{4.142}$$

$$[10]t^6 : N \wedge N \wedge N = 0 \tag{4.143}$$

$$[02]t^8 : \ \mathrm{rank}(N^2) \leq 1 \tag{4.144}$$

and indeed all are met by our coadjoint N_{G_2}. The last relation (4.144) can be written as $\sum_{m,n} (N_{am}N_{mb}N_{cn}N_{nd} - N_{am}N_{md}N_{cn}N_{nb}) = 0$. We have checked analytically that the three relations above form a complete set of relations.

$\mathbb{Z}_3$ discrete gauging

Although elsewhere in the section we discretely gauge or fold S_m quiver automorphisms, discrete gauging by a subset of S_m is perfectly well defined. Here we consider the $\mathbb{Z}_3$ discrete gauging of the D_4 quiver studied in this section. Its Coulomb branch was previously investigated in [105] under the name $\mathcal{C}_{D_4^{(3)}}$. The plethystic logarithm of its highest weight generating function was reported as[15]

$$PL(t) = [01]t^2 + ([01]-[00])t^4 - ([01]+[10]+[20]+[00])t^6 - ([01]+[10]-[02])t^8 + O(t^{10}). \tag{4.145}$$

This space is not a nilpotent orbit. It is generated by two coadjoint matrices at quadratic and quartic order in t respectively. The lower coadjoint matrix N is also

[14] We can compare this expression with the HWG for the minimal B_3 orbit, written in terms of G_2 fugacities, which reads [20]

$$\begin{aligned}\mathrm{HWG}(t) = \frac{1}{(1-\mu_1 t^2)(1-\mu_2 t^2)} &= 1 + (\mu_1+\mu_2) t^2 + \left(\mu_1^2+\mu_1\mu_2+\mu_2^2\right) t^4 \\ &+ \left(\mu_1^3+\mu_1^2\mu_2+\mu_1\mu_2^2+\mu_2^3\right) t^6 + \left(\mu_1^4+\mu_1^3\mu_2+\mu_1^2\mu_2^2+\mu_1\mu_2^3+\mu_2^4\right) t^8 + \ldots\end{aligned} \tag{4.139}$$

The difference between the two expressions is

$$\frac{\mu_1 t^2}{1-\mu_2 t^2} = \mu_1 t^2 + \mu_1\mu_2 t^4 + \mu_1\mu_2^2 t^6 + \mu_1\mu_2^3 t^8 + \ldots \tag{4.140}$$

[15] Paper [105] also follows the opposite root convention to the present discussion.

the moment map and looks precisely like the one obtained by S_3 symmetric gauging. Since $\mathbb{Z}_3 \subset S_3$, there are operators in this theory which are removed if the remaining $S_2 \subset S_3$ symmetry is imposed. One of the simplest operators is

$$\tilde{e}^4_{\langle 10 \rangle} = u_1^+(\varphi_4 - \varphi_3) + u_3^+(\varphi_1 - \varphi_4) + u_4^+(\varphi_3 - \varphi_1). \tag{4.146}$$

As its label suggests, $\tilde{e}^4_{\langle 10 \rangle}$ is a t^4 operator which acts as the first simple root under action of the moment map's components. And just as one can "rotate" a simple root into any other root by repeated action of the Lie bracket, it is possible to repeatedly act with the Poisson bracket on $\tilde{e}^4_{\langle 10 \rangle}$ to generate an entire t^4 adjoint representation's worth of operators which can be bundled together to form the second coadjoint matrix R. For example:

$$\tilde{e}^4_{\langle 01 \rangle} = -\{\tilde{e}_{\langle -10 \rangle}, \{\tilde{e}_{\langle 01 \rangle}, \tilde{e}^4_{\langle 10 \rangle}\}\} \tag{4.147}$$

The plethystic logarithm suggests several relations between N and R but we find it is not too helpful in this case. For example, its syzygies obscure several relations at order t^8. Accordingly, we opt for a different approach to identify the relations. [105] identifies a non-simply laced quiver with the same Coulomb branch, which is itself a folded version of the quiver in Figure 8 of [131]; the latter paper reports matrix relations. In general folded relations follow the form of the original quiver's; indeed they must as they are merely the original relations restricted to the folded subspace. Accounting for several coincidences in G_2 (eg $N^3 \propto (\mathrm{tr}N^2)N, \{N, R\} \propto N \wedge R$) and a different numerical factor in the last relation, we are left with the following relations:

$$[00]t^4 : \mathrm{tr}N^2 = 0 \tag{4.148}$$

$$[10]t^6 : N \wedge N \wedge N = 0 \tag{4.149}$$

$$[01]t^6 : [N, R] = 0 \tag{4.150}$$

$$([20] + [00])t^6 : \{N, R\} = 0 \tag{4.151}$$

$$([20] + [00])t^8 : R^2 = 0 \tag{4.152}$$

$$[02]t^8 : (N^2)_{[a}{}^{[b}(N^2)_{c]}{}^{d]} = \frac{1}{54} R_a{}^b R_c{}^d \tag{4.153}$$

We are able to verify all of them symbolically, but cannot guarantee that they form a minimal set of relations as our current techniques run against a computational limit.

4.4.3 $D_5 \to B_4$

We close off by studying discrete gauging and folding on a family of quivers. Figures 4.19, 4.20 and 4.21 present results of discrete gauging and folding on three D_5 nilpotent orbit quivers. The Hilbert series, HWGs and quivers were originally reported in [20, 22].

Figures 4.19-4.21 follow the same pattern. The first line shows the unitary magnetic quivers. The second line shows the equivalent orthosymplectic magnetic quivers (ie. with the same Coulomb branch); our discrete gauging appears to be the unitary analogue of gauging an $O(1)$ group in these quivers as studied in [22]. The third line shows an electric quiver, by which we mean a classical quiver theory whose Higgs branch is the Coulomb branch under study. Several quivers may share this property; in particular the ones chosen here need not be the $3d$ mirrors. Note in those electric quivers the appearance of an $O_1 = \mathbb{Z}_2$ gauge group in the middle column. The last lines show the Hasse diagrams, HWG and relations. The HWG use B_4 fugacities except in the first column where D_5 fugacities are also used.

We draw the reader's attention to several interesting properties.

Firstly, a D–type moment map in the Chevalley-Serre basis is too long to print but both discrete gauging and folding have clear and discernible effects on it. The original, unfolded moment map transforms in the coadjoint (antisymmetric) matrix representation of $\mathfrak{so}(10, \mathbb{C})$. Upon either discrete operation, all components along the last row and column vanish and the originally 10×10 matrix effectively becomes a 9×9 antisymmetric matrix padded by zeroes – and hence transforms in the coadjoint representation of $\mathfrak{so}(9, \mathbb{C})$.

Secondly, in the case of the next-to-next-to-minimal nilpotent orbit we wreathe a $U(2)$ node rather than the simple and well understood case of $U(1)$, demonstrating that discrete gauging generalises to gauge ranks higher than 1. Finally, in the same example, each wreathed $U(2)$ node comes with one flavor so the triplet of spaces exhibits interesting complex mass deformation behaviour analogous to that of Section 4.4.1: only the initial space can be deformed by complex mass, and turning on two inequivalent mass parameters spoils the S_2 symmetry required for both discrete gauging and folding.

Note that notation of the form $N \wedge \cdots \wedge N$ denotes antisymmetrisation over all indices, or equivalently contraction with the appropriate Levi-Civita tensor.

In Figures 4.19-4.21, we have colored the terms of the HWG which are charged under the $\mathbb{Z}_2$ action in violet.

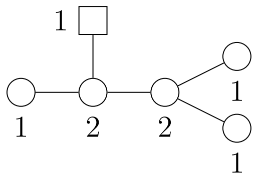
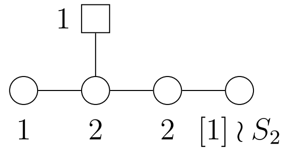
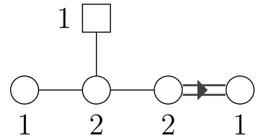
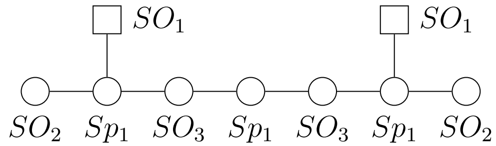
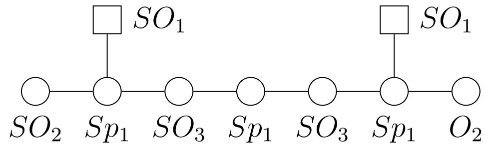
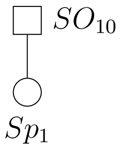
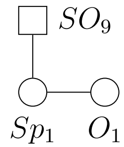
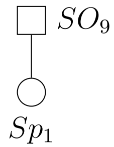
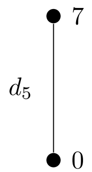
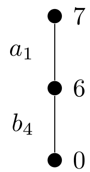
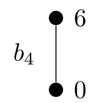
Initial	Discretely Gauged	Folded
		
		Non-special
		
 $\bar{\mathcal{O}}^{\mathfrak{so}(10)}_{[2^2,1^6]}$	 $\bar{\mathcal{O}}^{\mathfrak{so}(9)}_{[3,1^6]}$	 $\bar{\mathcal{O}}^{\mathfrak{so}(9)}_{[2^2,1^5]}$
$\mu_2 t^2$ D_5 $(\mu_2+\mu_1)t^2$ B_4	$\mu_2 t^2 + \mu_1^2 t^4$ B_4	$\mu_2 t^2$ B_4
$t^4\left([20000]+[00000]\right): \quad N^2=0$ $t^4[00011]: \quad N\wedge N=0$	$t^4[0000]: \quad \mathrm{tr}N^2=0$ $t^4[0002]: \quad N\wedge N=0$ $t^6[0100]: \quad N^3=0$	$t^4\left([2000]+[00000]\right): \quad N^2=0$ $t^4[0002]: \quad N\wedge N=0$

Figure 4.19: D_5 minimal nilpotent orbit quiver and its discrete reductions. The folded space is non-special [22] and therefore without an orthosymplectic realisation.

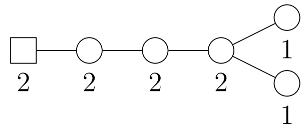
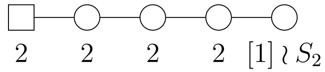
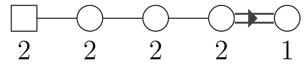
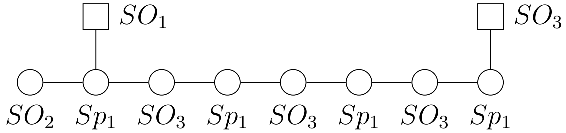
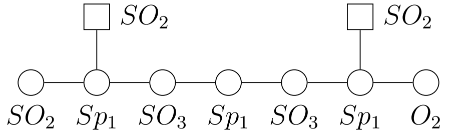
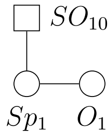
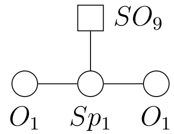
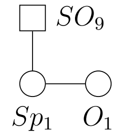
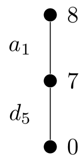
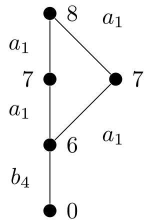
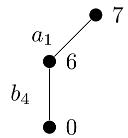
Initial	Discretely Gauged	Folded
		
	? (not a nilpotent orbit)	
		
 $\bar{\mathcal{O}}^{\mathfrak{so}(10)}_{[3,1^7]}$		 $\bar{\mathcal{O}}^{\mathfrak{so}(9)}_{[3,1^6]}$
$\mu_2 t^2 + \mu_1^2 t^4$ D_5 $(\mu_1 + \mu_2)t^2 + (1 + \mu_1^2 + \mu_1)t^4 - \mu_1^2 t^8$ B_4	$\mu_2 t^2 + (1 + \mu_1^2 + \mu_1^2)t^4 + \mu_1^2 t^6 - \mu_1^4 t^{12}$ B_4	$\mu_2 t^2 + \mu_1^2 t^4$ B_4
$t^4[00000]$: $\mathrm{tr}N^2 = \frac{1}{2}(M_{1,1} - M_{1,2})^2$ $t^4[00011]$: $N \wedge N = 0$ $t^6[01000]$: $N^3 = \frac{1}{4}(M_{1,1} - M_{1,2})^2 N$	$t^4[0002]$: $N \wedge N = 0$? ?	$t^4[0000]$: $\mathrm{tr}N^2 = \frac{1}{2}(M_{1,1} - M_{1,2})^2$ $t^4[0002]$: $N \wedge N = 0$ $t^6[0100]$: $N^3 = \frac{1}{4}(M_{1,1} - M_{1,2})^2 N$

Figure 4.20: D_5 next-to-minimal nilpotent orbit quiver and its discrete reductions. $M_{1,1}$ and $M_{1,2}$ are mass parameters associated to the flavor node of the top quiver. The discretely gauged space is generated by a coadjoint matrix N and another generator R transforming as [2000]. The methods in this thesis need to be generalised to non-coadjoint representations to verify relations between N and R and such work is beyond the current scope. Its Hasse diagram should also be regarded as a conjecture.

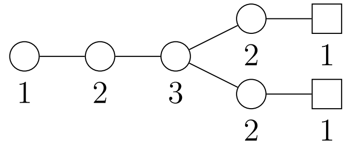
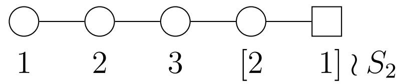
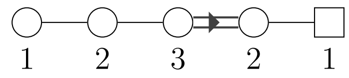
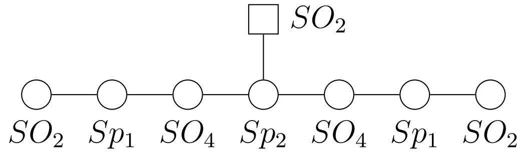
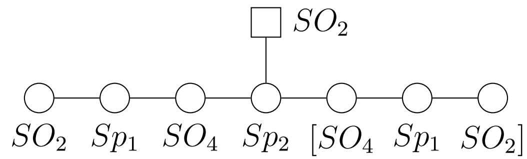
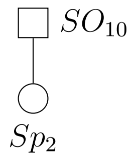
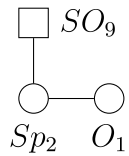
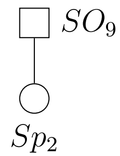
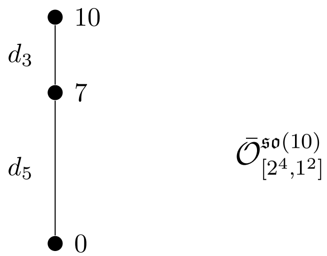
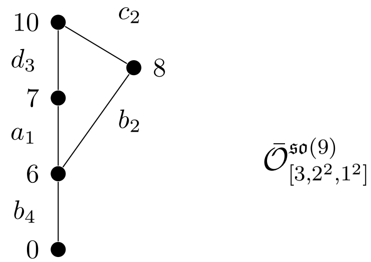
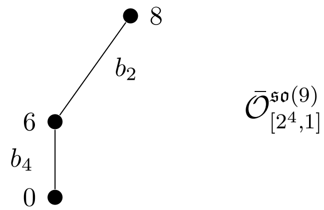
Initial	Discretely Gauged	Folded
		
		Non-special
		
		
$\mu_2 t^2 + \mu_4\mu_5 t^4$ D_5 $(\mu_2 + \mu_1)t^2 + (\mu_4^2 + \mu_3)t^4$ B_4	$\mu_2 t^2 + (\mu_1^2 + \mu_4^2)t^4 + \mu_1\mu_3 t^6 + \mu_3^2 t^8 - \mu_1^2\mu_3^2 t^{12}$	$\mu_2 t^2 + \mu_4^2 t^4$
$t^4[20000]: \quad N^2 = \frac{1}{16}(M_4 - M_5)^2\mathbf{1}$ $t^6[00011]: \quad N \wedge N \wedge N = \frac{3\mathrm{i}(M_4-M_5)}{2\cdot 6!} \star (N \wedge N)$	$t^4[0000]: \quad \mathrm{tr}(N^2) = 0$ $t^6[0100]: \quad N^3 = 0$ $t^6[0010]: \quad N \wedge N \wedge N = 0$ $t^8[0200]: \quad \mathrm{rank}(N^2) \leq 1$	$t^4[0100]: \quad N^2 = 0$ $t^6[0010]: \quad N \wedge N \wedge N = 0$

Figure 4.21: D_5 next-to-next-to-minimal nilpotent orbit quiver and its discrete reductions. M_4 and M_5 are mass parameters associated to the flavor nodes of the top quiver. The folded space is non-special [22] and therefore without an orthosymplectic realisation.

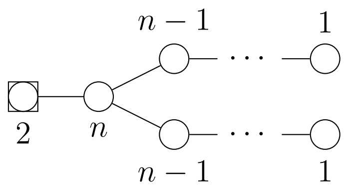
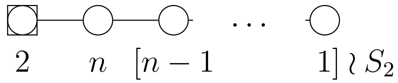
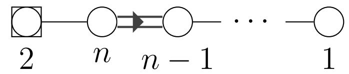
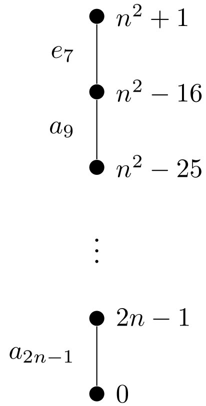
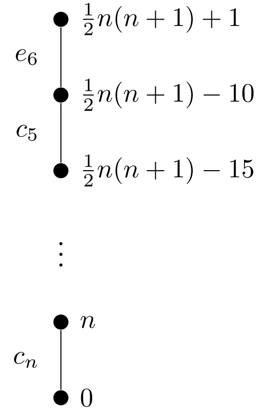
Initial	Discretely Gauged	Folded
		
	?	
$\left(\sum_{i=1}^{n-1} \mu_i \mu_{2n-i} t^{2i}\right) + t^4 + \mu_n (t^{n-2} + t^n)$	?	$\left(\sum_{i=1}^{n-1} \mu_i^2 t^{2i}\right) + t^4 + \mu_n (t^{n-2} + t^n)$

Figure 4.22: C_n

4.4.4 The C_n family

We end this section with a family of quivers labelled by an integer $n \geq 4$ whose Coulomb branch global symmetry is E_7 for $n = 4$ and $SU(2n)$ for $n > 4$; see Figure 4.22. The S_2 symmetry exchanging the two legs can be discretely gauged and folded. After folding, the global symmetry is $Sp(n)$. The resulting family of quivers, appearing on the rightmost column in Figure 4.22, is called the C_n family. As can be seen from the Hasse diagrams, both Coulomb branches are stratified into $n-2$ symplectic leaves. The quivers can then be interpreted as magnetic quivers for the Higgs branch of rank $n-3$ SCFTs in four dimensions. For $n = 4$ one recovers the E_7 theory studied in [149, 150], and for $n = 5$ one of the rank-2 theories identified in [151, 152].

Chapter 5

Conclusion

The author hopes that this work can serve as an approachable introduction to several recent methods in the study of $3d\,\mathcal{N}=4$ Coulomb branches. It covered the necessary background and introduced a certain kind of workflow for a typical calculation:

1. Calculate the Hilbert series and identify representations of generators and relations under the Coulomb branch isometry.
2. Explicitly construct gauge-invariant monopole operators and scalar operators out of abelianised variables and attempt to assemble them into the aforementioned generator representations.
3. Test gauge-invariant relations at the SCFT point and, if successful, turn on complex mass parameters to identify SUSY-preserving deformations of the Coulomb branch.

We built on several results of [78], particularly the explicit and physically interpretable construction of the Coulomb branch moment map for many balanced unitary quivers of type A. We were able to extend our understanding to a subclass of type D quivers. We also found two natural extensions to non-simply laced quivers: *quiver folding* and *wreathing*.

While our examples only cover a narrow slice of available quiver theories we believe the general workflow fully generalises to many (all?) $3d\ \mathcal{N}=4$ theories and that future practitioners will carry forward the methods used and in some cases developed by the author. Several interesting and rich research problems are within reach.

For example, increase in quiver height adds several new generators to the chiral ring of type D quivers. It would be interesting to express them in terms of abelianised variables and construct their gauge-invariant relations. A similar phenomenon appears upon generalisation to quivers without a $U(1)$ node and our methods could provide a novel window into quiver subtractions of [52].

We may also sacrifice balance. Quivers with one overbalanced node (excess greater than 0) were recently identified as relevant to the vacuum structure of five-dimensional supersymmetric theories. Such quivers' chiral rings are generated by a tensor in the coadjoint representation and additional tensors in another representation of the overall symmetry.

Finally, it should be possible to extend our methods to orthosymplectic quivers but such a move would require a generalisation of the analysis in [78] along the lines of [77]. We have carried out preliminary investigations and found that the moment map is encoded differently than in the unitary case, as if adapted to a different choice of basis of the symmetry algebra.

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Appendix A

$U(2)$ with 4 flavors and abelianised rings

The main text describes a two-stage method to explicitly construct the chiral ring: first generate a set of abelianised operators and then, guided by representation-theoretic data from the Hilbert series, select certain Weyl-invariant combinations to form tensor generators of the ring and verify that they satisfy the correct relations. The method does not require a deep dive into abelianisation and the precise relation between $\mathbb{C}[\mathcal{C}_{\text{abel}}]$ and $\mathbb{C}[\mathcal{C}]$, so we leave such discussion out of the main body and address some of the potential concerns in this appendix.

First, we set the stage. Starting with abelianised variables, the Poisson bracket (2.198) generates new elements of the abelianised ring which cannot be expressed by adding and multiplying basic variables. The full abelianised ring[1] $\mathbb{C}[\mathcal{C}_{\text{abel}}]$ is constructed as the ring underlying the Poisson algebra generated by abelianised variables $u^{\pm}_{i,a}$ and $\varphi_{i,a}$ subject to relations (2.193) and (2.196-2.198).

[1] We stress that this notion of the abelianised ring is a departure from that of [78], where it is introduced in (4.9) (with minor notational differences) as

$$\mathbb{C}[\mathcal{M}^{\text{abel}}_{\mathcal{C}}] = \left(\mathbb{C}[\{u^{\pm}_A\}, \{\varphi_a\}, \{(M^W_j)^{-1}\}_{j\in\text{roots}}]/(\text{abelianised relations})\right)^{\mathcal{W}_G}. \tag{A.1}$$

In essence, this "abelianised" ring is the Weyl-invariant part of a ring generated by all abelianised monopoles, scalar operators and all inverse masses (i.e. $1/(\varphi_a - \varphi_b)$) modulo relations between abelianised monopoles. The authors follow with clarification that this is emphatically *not* the Coulomb branch chiral ring, since a) the inverse masses are not defined everywhere on it (i.e. when $\varphi_a = \varphi_b$), and even if one restricts to the "discriminant locus" of the Coulomb branch (that is, the points where $\varphi_a \neq \varphi_b$), b) this ring contains *extra* elements.

To see that the two "abelianised" rings $\mathbb{C}[\mathcal{M}^{\text{abel}}_{\mathcal{C}}]$ and $\mathbb{C}[\mathcal{C}_{\text{abel}}]$ are inequivalent, take the theory $U(2)$ with 4 flavors (as in the remainder of this appendix). Then the element $1/(\varphi_1 - \varphi_2)^2$ is included in $\mathbb{C}[\mathcal{M}^{\text{abel}}_{\mathcal{C}}]$ but absent from $\mathbb{C}[\mathcal{C}_{\text{abel}}]$. (A comparison between $\mathbb{C}[\mathcal{M}^{\text{abel}}_{\mathcal{C}}]$ and $\mathbb{C}[\mathcal{C}_{\text{abel}}]^{W_G}$ may appear to be fairer, since both objects are Weyl-symmetric, but it makes no difference since $\mathbb{C}[\mathcal{C}_{\text{abel}}]^{W_G} \leq \mathbb{C}[\mathcal{C}_{\text{abel}}]$ implies that the element in question does not belong to the Weyl-invariant ring $\mathbb{C}[\mathcal{C}_{\text{abel}}]^{W_G}$ either.)

We use $\mathbb{C}[\mathcal{C}_{\text{abel}}]$ as an active computational precursor to the Coulomb branch chiral ring and therefore reserve it the name *abelianised ring* despite any confusion it may cause when compared to the similarly denoted ring in [78].

In the interest of concreteness consider the case of a $U(2)$ theory with 4 flavors. Elements such as $1/(\varphi_1-\varphi_2)$ (inverse masses of W-bosons) never appear in $\mathbb{C}[\mathcal{C}_{\text{abel}}]$ on their own. On the other hand, the abelianised relation

$$u_1^+u_1^- = -\frac{\prod_{1\le i\le 4}(\varphi_1-M_i)}{(\varphi_1-\varphi_2)^2} \tag{A.2}$$

does contain an inverse mass as part of an expression. This is worrying because, as $\varphi_2\to\varphi_1$, this function diverges and so indicates that either the locus $\varphi_1=\varphi_2$ is not in the Coulomb branch (a solution which should be rejected on physical grounds), or else the function is not in the Coulomb branch chiral ring. The latter is correct: every element of $\mathbb{C}[\mathcal{C}]$ must be Weyl-invariant, and this one clearly is not. But that only raises another worry. Consider the Weyl-invariant polynomial

$$u_1^+u_1^- + u_2^+u_2^- = -\frac{\prod_i(\varphi_1-M_i)}{(\varphi_1-\varphi_2)^2} - \frac{\prod_i(\varphi_2-M_i)}{(\varphi_1-\varphi_2)^2}. \tag{A.3}$$

It also diverges as $\varphi_2\to\varphi_1$. Any workable prescription for reducing $\mathbb{C}[\mathcal{C}_{\text{abel}}]$ to $\mathbb{C}[\mathcal{C}]$ must exclude this polynomial from the ring[2].

The author is aware of two candidate prescriptions. [78] suggest in their Section 6.3 that $\mathbb{C}[\mathcal{C}]$ can be generated as a Poisson algebra by a certain small set of Weyl-invariant operators, namely all single-node bare monopoles $\sum_a u_{i,a}^\pm$ and single-node elementary symmetric polynomials in $\varphi_{i,a}$, i.e. $\sum_a \varphi_{i,a}$, $\sum_{a<b}\varphi_{i,a}\varphi_{i,b}$ etc. This is borne out in all cases which we have studied in depth and which do not involve monopole bubbling. In many cases an alternative method is available and we are able to define the Coulomb branch chiral ring by directly specifying its generators[3] in the abelianised formalism, with relations following directly from (2.193) as summarised in Section 3.5. The two approaches have given identical results on every unitary quiver which the author has studied.

First approach Consider first the construction in [78] which only uses $u_1^\pm + u_2^\pm$, $\varphi_1+\varphi_2$ and $\varphi_1\varphi_2$ to generate the entire ring as a Poisson algebra. We wish to construct $u_1^+u_1^- + u_2^+u_2^-$. There are only a few things one could try. Since the Poisson bracket $\{\cdot,\cdot\}$ has dimension -1, the action of $\{\varphi_1\varphi_2,\cdot\}$ increases dimension by 1 while $\{\varphi_1+\varphi_2,\cdot\}$ and $\{u_1^\pm+u_2^\pm,\cdot\}$ preserve it. At the same time the space of $\Delta=2$ operators is finite so successive use of $\{\varphi_1+\varphi_2,\cdot\}$ will only bring finitely many new elements to consider. And we are only interested in operators of vanishing topological charge.

Let's start with $(u_1^+ + u_2^+)(u_1^- + u_2^-)$ which includes an extra contribution of

[2] The simple, but incorrect prescription $\mathbb{C}[\mathcal{C}]=\mathbb{C}[\mathcal{C}_{\text{abel}}]^{\mathcal{W}_G}$ would fail it. We thank an anonymous reviewer for bringing this to our attention.

[3] That is, generators of a *ring*, as opposed to a *Poisson algebra*.

$u_1^+u_2^- + u_2^+u_1^-$; it must be removed to create (A.3). There is nothing else left to do without using the dimension-raising Poisson bracket:

$$\{\varphi_1\varphi_2, u_1^+ + u_2^+\} = u_1^+\varphi_2 + u_2^+\varphi_1 \tag{A.4}$$

$$\{u_1^+\varphi_2 + u_2^+\varphi_1, u_1^- + u_2^-\} = -\varphi_1^2 - \varphi_2^2 - (u_1^+ + u_2^+)(u_1^- + u_2^-) \tag{A.5}$$

This ring element does not help and it does not take much work to convince oneself that we have exhausted all non-trivial options, which implies the construction in [78] is free of the problematic element (A.3).

Second approach We can also explicitly construct ring generators in representations of the moduli space symmetry. The Coulomb branch Hilbert series of this theory reads

$$\mathrm{HS}(t) = 1 + t^2[2] + t^4([4] + [2] + [0]) + \mathcal{O}(t^3)$$

with plethystic logarithm

$$PL(\mathrm{HS})(t) = t^2[2] + t^4[2] - t^6 - t^8$$

implying that its chiral ring is generated by six generators forming two adjoint representations [2] which we label N and R. This representation forces constraints on any putative abelianised construction. For any $X, Y \in \mathfrak{su}(2)$, expressed as square 2×2 matrices,

$$\{\mathrm{tr}(NX), \mathrm{tr}(NY)\} = \mathrm{tr}(N[X,Y]) \tag{A.6}$$

and

$$\{\mathrm{tr}(NX), \mathrm{tr}(RY)\} = \mathrm{tr}(R[X,Y]). \tag{A.7}$$

One can then use these constraints to find abelianised expressions of N and R. In the massless case they are

$$N = \begin{pmatrix} \varphi_1 + \varphi_2 & u_1^- + u_2^- \\ u_1^+ + u_2^+ & -\varphi_1 - \varphi_2 \end{pmatrix}$$

(as in [1]) and

$$R = \begin{pmatrix} -\frac{1}{2}\left(\varphi_1^2 + \varphi_2^2 + (u_1^+ + u_2^+)(u_1^- + u_2^-)\right) & u_1^-\varphi_2 + u_2^-\varphi_1 \\ u_1^+\varphi_2 + u_2^+\varphi_1 & \frac{1}{2}\left(\varphi_1^2 + \varphi_2^2 + (u_1^+ + u_2^+)(u_1^- + u_2^-)\right) \end{pmatrix}.$$

One can check that the Lie algebra homomorphisms (A.6) and (A.7) are satisfied, and further that this choice of N, consistent with previous work, fixes the choice of R up to a scalar factor (assuming the positively charged monopole is of the form

$u^+\varphi$). One can also check that these two generators satisfy precisely the relations predicted by the Coulomb branch Hilbert series.

Agreement with the Hilbert series licences us to claim that components of N and R (two of which are redundant, for a total of 6 independent entries) generate $\mathbb{C}[\mathcal{C}]$. Observe that there is no way to generate eg. $(u_1^+)^2 + (u_2^+)^2$ from the generators presented here. In fact the $\Delta = 2$ subspace of $\mathbb{C}[\mathcal{C}]$ is generated by i) quadratic combinations of components of N and ii) components of R. Notably, it is impossible to construct the counter-example (A.3) with these building blocks.

Comparison between the two approaches It is interesting to compare the two approaches to reducing the abelianised ring to $\mathbb{C}[\mathcal{C}]$: the new off-diagonal terms in R appear as $\{\varphi_1\varphi_2, u_1^\pm + u_2^\pm\}$ and diagonal terms are straightforwardly constructed from $\varphi_1\varphi_2$ and components of N even without the use of a Poisson bracket.

In summary neither approach generates the divergent function (A.3). One can therefore interpret the appearance of inverse masses in (A.2) as a sign that we were dealing with the wrong variables and relations. The correct variables are i) elements of a certain Poisson algebra or ii) components of two coadjoint tensors, which themselves satisfy several tensorial relations. In the latter case, abelianised relations are crucial in deriving and testing the form of these coadjoint tensors, but ultimately serve as a ladder to be thrown away once the tensors are obtained. As mentioned above the two approaches agree on every unitary quiver of which the author is aware.

Appendix B

Folded Lie algebras are the same as discretely gauged Lie algebras

As mentioned in the main text, the Lie algebra of the discretely gauged space is given by

$$\{\tilde{\mathcal{O}}_i, \tilde{\mathcal{O}}_j\} = \sum_k c_{ij}{}^k \tilde{\mathcal{O}}_k. \tag{B.1}$$

for $\tilde{\mathcal{O}}_i$ which form a basis of $\mathbb{C}[\tilde{\mathcal{C}}]_2$. In particular these operators vary across the moduli space. Restricting to the folded subspace, we find

$$\{\hat{\mathcal{O}}_i, \hat{\mathcal{O}}_j\} = \sum_k c_{ij}{}^k \hat{\mathcal{O}}_k. \tag{B.2}$$

This does not necessarily mean that the two Lie algebras are isomorphic as some of the RHS terms could vanish if $\hat{\mathcal{O}}_k$ vanishes identically. We will now prove that this does not happen.

$\tilde{\mathcal{O}}_k$ is a non-constant symmetric function in variables attached to wreathed legs; call them $\vec{x}_i$ where i labels the leg. So we can rewrite the operator as $f(\vec{x}_1, \ldots \vec{x}_n)$ for some n. At the fixed point $\vec{x}_i = \vec{x}$, so the operator becomes $f(\vec{x}, \ldots, \vec{x})$. Assume it vanishes everywhere. Then

$$\nabla_{\vec{x}} f(\vec{x}, \ldots, \vec{x}) = \sum_i \nabla_{\vec{x}_i} f(\vec{x}_1, \ldots \vec{x}_n)\Big|_{\vec{x}_j = x}. \tag{B.3}$$

However all the summands are identical under the restriction:

$$\begin{aligned}
(\nabla_{\vec{x}_i} f(\vec{x}_1, \ldots \vec{x}_n))_j &= \lim_{\varepsilon \to 0} \frac{f(\vec{x}_1, \ldots, \vec{x}_i + \varepsilon e_i, \ldots, \vec{x}_n) - f(\vec{x}_1, \ldots, \vec{x}_i, \ldots, \vec{x}_n)}{\varepsilon} \\
&= \lim_{\varepsilon \to 0} \frac{f(\vec{x}_i + \varepsilon e_i, \ldots, \vec{x}_1, \ldots, \vec{x}_n) - f(\vec{x}_i, \ldots, \vec{x}_1, \ldots \vec{x}_n)}{\varepsilon} \\
&= \left(\nabla_{\vec{x}'_1} f(\vec{x}'_1, \ldots \vec{x}'_n)\right)_j \Big|_{\substack{\vec{x}'_1 = \vec{x}_i \\ \vec{x}'_i = \vec{x}_1 \\ \vec{x}'_{j \neq 1, i} = \vec{x}_j}}
\end{aligned} \tag{B.4}$$

so

$$(\nabla_{\vec{x}_i} f(\vec{x}_1, \ldots \vec{x}_n))_j \Big|_{\vec{x}_j = \vec{x}} = (\nabla_{\vec{x}_1} f(\vec{x}_1, \ldots \vec{x}_n))_j \Big|_{\vec{x}_j = \vec{x}}. \tag{B.5}$$

Then

$$\nabla_{\vec{x}} f(\vec{x}, \ldots, \vec{x}) = n \nabla_{\vec{x}_1} f(\vec{x}_1, \ldots \vec{x}_n)\big|_{\vec{x}_j = x} \neq 0 \tag{B.6}$$

unless $\nabla_{\vec{x}_i} f(\vec{x}_1, \ldots \vec{x}_n)\big|_{\vec{x}_j = \vec{x}}$ vanishes, i.e. $f(\vec{x}_1, \ldots \vec{x}_n)$ is a constant, which contradicts the assumption that $\tilde{\mathcal{O}}_k$ is non-constant. It follows that both the discretely gauged and folded spaces have isomorphic Lie algebras and hence share the same continuous symmetry.

Appendix C

Computation of Hilbert series with S_4 wreathing

The computation of the exact Hilbert series presented in Figure 4.8 can be done in principle using (4.44). However it is often useful to massage this formula until a more manageable form can be used in practice. In this appendix, we give the result of such manipulations in the case of the quiver at hand. Derivations use simple algebra and are not detailed here.

Using the notations of (4.48), but using the gauge group (4.55), one can set $m_{a,2} = 0$ and $m_{a,1} \equiv m_a$ and the conformal dimension can be expressed in terms of

$$m = (m_a, m_b, m_c, m_d, m_e) \tag{C.1}$$

as

$$2\Delta(m) = |m_a - m_b| + |m_a - m_c| + |m_a - m_d| + |m_a - m_e| + |m_b| + |m_c| + |m_d| + |m_e| - 2|m_a| \,. \tag{C.2}$$

One then computes the auxiliary sums

$$\Sigma_i = \sum_{m_a=0}^{\infty} \sum_{(m_b,m_c,m_d,m_e)\in \mathrm{Range}_i} t^{2\Delta(m)} \tag{C.3}$$

where Range_i is defined in Figure C.1. The exact value of the sums Σ_i is straightforward to compute (note the absence of Casimir factors!) and is given in Figure C.1 as well.

Let's now pick a subgroup Γ of S_4. For $\mu \in \mathbb{Z}^4$ we call $O^{S_4}(\mu)$ the orbit of μ under the action of S_4. This orbit can be written as a disjoint union of $n(\mu)$ orbits under Γ,

$$O^{S_4}(\mu) = \coprod_{j=1}^{n(\mu)} O^{\Gamma}(\mu_j) \tag{C.4}$$

i	Range_i	Σ_i
1	$m_b < m_c < m_d < m_e$	$\frac{t^6\left(4+16t^2+18t^4+13t^6+4t^8+t^{10}\right)}{(1-t^2)^6(1+t^2)^3}$
2	$m_b = m_c < m_d < m_e$ $m_b < m_c < m_d = m_e$	$\frac{t^4\left(2+9t^2+10t^4+9t^6+3t^8+t^{10}\right)}{(1-t^2)^5(1+t^2)^3}$
3	$m_b < m_c = m_d < m_e$	$\frac{t^4\left(4+7t^2+7t^4+3t^6+t^8\right)}{(1-t^2)^5(1+t^2)^2}$
4	$m_b = m_c = m_d < m_e$ $m_b < m_c = m_d = m_e$	$\frac{t^2\left(2+3t^2+4t^4+2t^6+t^8\right)}{(1-t^2)^4(1+t^2)^2}$
5	$m_b = m_c < m_d = m_e$	$\frac{t^2+5t^4+5t^6+6t^8+2t^{10}+t^{12}}{(1-t^2)^4(1+t^2)^3}$
6	$m_b = m_c = m_d = m_e$	$\frac{\left(1-t+t^2\right)\left(1+t+t^2\right)\left(1+t^4\right)}{(1-t^2)^3(1+t^2)^2}$

Figure C.1: Definitions of the ranges involved in the sums (C.3), and exact values of these sums. When there are two possible ranges, this means that the two choices lead to the same sums.

where the $\mu_j \in \mathbb{Z}^4$ are representatives of these orbits, (not uniquely!) determined by the above equation. Using the notation (4.46), that we recall here,

$$P_\Gamma(t^2;\mu) = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma(\mu)} \frac{1}{\det(1 - t^2\gamma)} , \tag{C.5}$$

one can define the modified Casimir factor

$$\tilde{P}_\Gamma(t^2;\mu) = \sum_{j=1}^{n(\mu)} P_\Gamma(t^2;\mu_j) . \tag{C.6}$$

The rationale behind this definition is that we have evaluated the sums (C.3) which are adapted to the full group S_4, and the Casimir factors for the group Γ have to be collected accordingly. This being done, the Hilbert series for the Coulomb branch of the wreathed quivers are simply

$$\mathrm{HS}_\Gamma(t) = \sum_{i=1}^{6} \tilde{P}_\Gamma(t^2;\mu_i)\Sigma_i , \tag{C.7}$$

where $\mu_i \in \mathbb{Z}^4$ is any element satisfying the condition Range_i. Using this formula, all the Hilbert series of Figure 4.8 are evaluated in a fraction of a second on a standard computer.