Quantum Toroidal Algebras and their Representation Theory

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Les Algèbres Toroïdales Quantiques et leur Théorie des Représentations

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Titre: Les Algèbres Toroïdales Quantiques et leurs Théorie des Représentations

Mots clés: Algèbres Quantiques, Théorie des Représentations, Groupe de Tresses

Résumé:A toute algèbre de Lie sur le corps des complexes, nous pouvons lui associer le groupe quantique considéré comme généralisation de l'algèbre. C'est la deformation de l'algèbre enveloppante universelle U(g). En prenant la limite q tend vers 1, nous retrouvons l'algèbre enveloppante universelle. L'algèbre de Lie possède une généralisation naturelle en dimension infinie qui est l'algèbre de Lie affine. La déformation de l'algèbre enveloppante d'une algèbre de Lie affine non-tordue nous permet de définir les algebres affines quantiques. Due à V.G. Drinfel'd les algèbres affines quantiques possèdent une deuxième réalisation en terme de générateurs de Drinfel'd. Cet isomorphisme est prouvé Par I. Damiani et J. Beck. Ceci nous permet de dire qu'on peut effectuer l'affinisation avant ou bien après la quantification. On a un diagramme commutative. En plus, on peut definir la quantification affine qui nous permet d'associer à toute algèbre de Lie de type finie une algèbre quantique affine dans la réalisation de Drinfel'd. Le procédé de quantification affine peut être effectué sur une algèbre affine non tordue. Ceci est la definition des algèbres toroidales quantiques. Le résultat est une algèbre qui est doublement affine.

Dans cette thèse nous étudions les algèbres toroidales quantiques et leurs représentations. La première partie est consacrée à l'étude de l'algèbre toroidale quantique de type A1. Par action du groupe des tresses, nous construisons une nouvelle presentation de l'algèbre qui nous donne une nouvelle décomposition triangulaire. Dans la seconde partie, nous utilisons ce résultat pour définir et classifier les représentations simples de plus hauts t-poids. Finalement, nous généralisons les résultats de la première partie pour obtenir une action du groupe des tresses sur tout autres systems de racines.

Title: Quantum Toroidal Algebras and their Representation Theory

Keywords: Quantum Algebras, Representation Theory, Braid Group.

Abstract: With every irreducible finitenite root system, one can associate the corresponding Drinfel'd-Jimbo quantum group.This is a Hopf algebra, which can be thought of as a deformation of the universal enveloping algebra of the Lie algebra of the same Cartan type. It naturally comes equipped with a universal R-matrix, thus providing solutions of the Yang-Baxter equation which plays a definitional role in the theory of quantum integrable systems and underlies the algebraic Bethe ansatz. In case the initial root system is affine instead of finite, the resulting Drinfel'd- Jimbo quantum groups are known as quantum affine algebras. Drinfel'd proposed an alternative presentation of these algebras though, closer in spirit to their classic current or loop presentation as Lie algebras. It is now widely referred to as the Drinfel'd presentation and was rigorously established by Damiani and Beck, making crucial use of Lusztig's affine braid group symmetries; a quantum analogue of the classical Weyl group symmetries of simple Lie algebras. As one expects in view of the classical current Lie algebra case, Drinfel'd's presentation only depends on the underlying finite root system, i.e. the one with the extra affine simple root removed. Now it turns out that this inherently affine presentation still makes sense if, instead of a finite root system, one takes an affine root system. In that case, the doubly affine algebra one obtains is known as a quantum toroidal algebra. Although the latter are believed to be relevant in various areas of theoretical physics, ranging from quantum integrable systems to CFT, not much is presently known about their representation theory. From a more mathematical perspective, the interest in these algebras essentially stems form the fact that, in type A, they are known to be Frobenius-Schur duals of the widely studied doubly affine Hecke algebras or DAHA originally introduced by Cherednik in order to prove MacDonald's conjectures.

In this thesis we study quantum toroidal algebras and their representation theory. In the first section, we construct a new presentation of the algebra using the braid group action on the generators and show the existence of an isomorphism between both presentations. This allows us to define a new triangular decomposition. Using these results, we define and classify highest-weight representations. Finally, we generalize the action of the braid group to any root system.

Contents

0.1 Synthèse en Français 10
  0.1.1 Double Affinisation Quantique 10
  0.1.2 Représentations des Algèbres Toroidales Quantique de Type $\mathfrak{a}_1$ 13
  0.1.3 Action du Groupe de Tresses: Le Cas Général 14

1 A Brief History **17**
1.1 Quantized Enveloping Algebra 17
  1.1.1 Representation Theory 18
  1.1.2 Representation Theory of $\mathrm{U}_q(\mathfrak{g})$ 19
1.2 Quantum Affine Algebras 22
  1.2.1 Damiani-Beck Isomorphism 23
  1.2.2 Automorphisms and Braid Group Action on $\mathrm{U}_q(\dot{\mathfrak{g}})$ 24
  1.2.3 Representation Theory of Quantum Affine Algebras 25
  1.2.4 q-Characters 30
1.3 Quantum Toroidal Algebras 31
  1.3.1 Schur-Weyl Duality 31
  1.3.2 A word on Quiver Quantum Toroidal Algebras 34

2 On Double Quantum Affinization: Type $\mathfrak{a}_1$ **35**
2.1 Introduction 35
2.2 The quantum toroidal algebra of type $\mathfrak{a}_1$ and its completion 38
  2.2.1 Definition 38
  2.2.2 Automorphisms of $\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)$ 40
  2.2.3 The completions $\widehat{\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)}$ and $\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)^{\widehat{\otimes} m\geq 2}$ 41
  2.2.4 Continuous Lusztig automorphisms 43
  2.2.5 Topological Hopf algebra structure on $\widehat{\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)}$ 45
  2.2.6 Non-degenerate Hopf algebra pairing 45
2.3 Double quantum affinization in type $\mathfrak{a}_1$ 47
  2.3.1 Definition of $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)$ 47
  2.3.2 The subalgebra $\ddot{\mathrm{U}}_q^0(\mathfrak{a}_1)$ and the elliptic Hall algebra 51
  2.3.3 $\dot{\mathrm{U}}_q(\mathfrak{a}_1)$ subalgebras of $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)$ 53
  2.3.4 Automorphisms of $\widehat{\ddot{\mathrm{U}}'_q(\mathfrak{a}_1)}$ 54
  2.3.5 Triangular decomposition of $\widehat{\ddot{\mathrm{U}}_q(\mathfrak{a}_1)}$ 54
  2.3.6 Weight-finite highest t-weight modules 55

2.3.7 Topological Hopf algebra structure on $\widehat{\ddot{\mathrm{U}}_q'(\mathfrak{a}_1)}$ 56
2.4 Doubly Affine Damiani-Beck isomorphism 57
2.4.1 Double loop generators 57
2.4.2 Exchange relations 63
2.4.3 Weight grading relations 67
2.4.4 The central elements $\mathbf{\Gamma}^{\pm}_{m>2}(z)$ 68
2.4.5 Relations in $\Psi^{-1}(\ddot{\mathrm{U}}^0_q(\mathfrak{a}_1))$ 72
2.4.6 The Serre relations of the elliptic Hall algebra 73

3 Weight-finite modules over the quantum affine and double quantum affine algebras of type $\mathfrak{a}_1$ 77
3.1 Introduction 77
3.2 Weight-finite modules over the quantum affine algebra $\dot{\mathrm{U}}_q(\mathfrak{a}_1)$ 80
3.2.1 The quantum affine algebra $\dot{\mathrm{U}}_q(\mathfrak{a}_1)$ 80
3.2.2 Finite dimensional $\dot{\mathrm{U}}_q'(\mathfrak{a}_1)$-modules 81
3.2.3 Weight-finite simple $\mathrm{U}_q(\mathrm{L}\mathfrak{a}_1)$-modules 83
3.3 Double quantum affinization of type $\mathfrak{a}_1$ 87
3.3.1 Definition of $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)$ 87
3.3.2 $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)$ as a topological algebra 89
3.3.3 The double quantum loop algebra 90
3.3.4 Triangular decomposition of $\widehat{\ddot{\mathrm{U}}_q'(\mathfrak{a}_1)}$ 91
3.3.5 The closed subalgebra $\widehat{\ddot{\mathrm{U}}^0_q(\mathfrak{a}_1)}$ as a topological Hopf algebra 91
3.3.6 The closed subalgebra $\widehat{\ddot{\mathrm{U}}^0_q(\mathfrak{a}_1)}$ and the elliptic Hall algebra 93
3.3.7 The quantum toroidal algebra $\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)$ 95
3.3.8 $\dot{\mathrm{U}}_q(\mathfrak{a}_1)$ subalgebras of $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)$ 97
3.3.9 (Anti-)Automorphisms of $\widehat{\ddot{\mathrm{U}}_q'(\mathfrak{a}_1)}$ 98
3.3.10 Topological Hopf algebra structure on $\widehat{\ddot{\mathrm{U}}_q'(\mathfrak{a}_1)}$ 98
3.4 t-weight $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)$-modules 101
3.4.1 ℓ-weight modules over $\ddot{\mathrm{U}}^0_q(\mathfrak{a}_1)$ 101
3.4.2 t-weight $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)$-modules 109
3.5 An evaluation homomorphism and evaluation modules 117
3.5.1 The quantum Heisenberg algebras $\mathcal{H}_t^+$ and $\mathcal{H}_t^-$ 117
3.5.2 A PBW basis for $\mathcal{H}_t^{\pm}$ 119
3.5.3 The dressing factors $\mathbf{L}^{\pm}_m(z)$ and $\mathbf{R}^{\pm}_m(z)$ 120
3.5.4 The algebra $\mathcal{B}_t$ 121
3.5.5 The completion $\widehat{\mathcal{B}}_t$ of $\mathcal{B}_t$ 124
3.5.6 The shift factors 124
3.5.7 The evaluation algebra $\widehat{\mathcal{A}}_t$ 126
3.5.8 The evaluation homomorphism 127
3.5.9 Evaluation modules 128

4 Topological Braid Group Action on $\dot{\mathrm{U}}_q(\dot{\mathfrak{g}})$ 131

4.1 Introduction . . . 131

4.2 Definition of $\dot{\mathrm{U}}_q(\dot{\mathfrak{g}})$ . . . 131

4.3 Automorphisms of $\dot{\mathrm{U}}_q(\dot{\mathfrak{g}})$ . . . 132

4.4 Braid group action . . . 133

4.5 Proof of the main theorem . . . 138

4.5.1 Proof of the Serre relations . . . 147

5 Further Directions 154

5.1 Formal distributions . . . 156

5.1.1 Definitions and main properties . . . 156

5.1.2 Laurent expansion and the Dirac formal distribution . . . 158

5.1.3 The structure power series $G_{ij}^{\pm}(z)$ . . . 160

A well fitting motto: **"This was scary... Let's do more of it!"**

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0.1 Synthèse en Français

A toute algèbre de Lie de type fini $\mathfrak{g}$, on peut lui associer l'algèbre de Kac-Moody affine non-tordue $\dot{\mathfrak{g}}$. Les algèbres de Kac-Moody affines sont obtenues en remplaçant les données d'un système de racine fini par les données d'un système de racines affine. C'est le procédé d'affinization classique des algèbres de Lie. Un autre procédé est celui de la quantification de ces algèbres. Dans ce cas, on obtient d'une part les groupes quantiques $U_q(\mathfrak{g})$ qui ont étaient introduits par V.G. Drifel'd et M. Jimbo. Les groupes quantiques sont une déformation de l'algèbre enveloppente d'une algèbre de Lie finie. D'autre part, quand on applique le procéssus de quantification sur une algèbre de Kac-Moody affine non-tordue, on obtient les algèbres quantiques affines. Les algèbres quantiques affines possèdent deux présentations en termes de générateurs et relations d'algèbre. La première, celle de Dinfel'd et Jimbo, $\mathrm{U}_q(\dot{\mathfrak{g}})$, est obtenue en remplaçant les données du système de racine fini par celles d'un système de racines affine. La deuxième, $\dot{\mathrm{U}}_q(\mathfrak{g})$ due à Drinfel'd est en terme de générateurs de Drinfel'd et depend uniquement d'un système de racines fini. L'isomorphisme entre ces deux présentations a été démontré par I. Damiani et J. Beck. Ceci nous permet d'écrire le diagramme commutatif suivant:

$$\begin{tikzcd}
\mathfrak{g} \arrow[r, "\text{Affinisation Classique}"] \arrow[d, "\text{Affinisation Quantique}"'] & \dot{\mathfrak{g}} \arrow[d, "\text{Quantification}"] \\
\dot{\mathrm{U}}_q(\mathfrak{g}) \arrow[r, "\sim", "\text{isom. Damiani-Beck}"'] & \mathrm{U}_q(\dot{\mathfrak{g}})
\end{tikzcd}$$

Le fait que la présentation des algèbres affine quantiques $\dot{\mathrm{U}}_q(\mathfrak{g})$ dépend d'un système de racines fini nous permet de definir une troisiéme algèbre, $\dot{\mathrm{U}}_q(\dot{\mathfrak{g}})$ doublement affine, en remplaçant encore une fois le système de racine fini par un système de racine affine. C'est la définition algèbrique des algèbres toroidales quantiques. Les algèbres quantiques toroidales, introduientt par Ginzburg-Kapranov-Vasserot, apparaissent naturellement dans certaines constructions géométriques des groupes quantiques reliées aux théories de gauge en physique, plus précisement les théories de jauge-carquois. Le sujet de cette thèse est de dèfinir une présentation des algèbres toroidales quantiques à la Drinfel'd en terme de générateurs à lacets doubles. La thèse est divisée en trois parties:

- Dans la première nous présentons les demarches nécessaires pour obtenir une presentation à la Drinfel'd de l'algèbre toroidale quantique de type A_1.
- Dans la deuxième nous utilisons cette nouvelle présentation pour définir une nouvelle catégorie de module et nous les classifions.
- Dans la troisiéme nous donnons l'outils crucial pour généraliser ces résultats pour couvrir tous système de racine.

0.1.1 Double Affinisation Quantique

Dans la section "On Double Quantum Affinization" nous suivons les mêmes étapes suivies par I. Damiani et J. Beck pour definir un isomorphisme dans le cas de algèbres toroidales quantiques. Voici les ingrédients principaux pour construire la nouvelle présentation:

- Automorphismes de l'algèbre $\dot{\mathrm{U}}_q(\ddot{\mathfrak{g}})$
- Extension du groupe de tresses toroidale
- Définition des générateurs lacets doubles

Pour les automorphismes d'algèbres:

Proposition 0.1.1. *i. Pour tout automorphisme de diagramme de Dynkin: $\pi : \dot{I} \xrightarrow{\sim} \dot{I}$, il existe un unique automorphisme de $\mathbb{F}$-algèbre $T_\pi \in \mathrm{Aut}(\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1))$ tel que:*

$$T_\pi(\mathbf{x}_i^\pm(z)) = \mathbf{x}_{\pi(i)}^\pm(z)\,, \qquad T_\pi(\mathbf{k}_i^\pm(z)) = \mathbf{k}_{\pi(i)}^\pm(z)\,, \qquad T_\pi(C^{1/2}) = C^{1/2}\,, \qquad T_\pi(D) = D\,. \tag{0.1.1}$$

ii. Pour tout $i \in \dot{I}$, il existe un unique automorphism de $\mathbb{F}$-algèbre $T_{\omega_i^\vee} \in \mathrm{Aut}(\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1))$ tel que

$$T_{\omega_i^\vee}(\mathbf{x}_j^\pm(z)) = z^{\pm\delta_{ij}}\mathbf{x}_j^\pm(z) \qquad T_{\omega_i^\vee}(\mathbf{k}_j^\pm(z)) = C^{\mp\delta_{ij}}\mathbf{k}_j^\pm(z) \qquad T_{\omega_i^\vee}(C^{1/2}) = C^{1/2} \qquad T_{\omega_i^\vee}(D) = D \tag{0.1.2}$$

iii. Il existe un unique anti-homomorphisme involutive de $\mathbb{F}$-algèbre $\eta \in \mathrm{Aut}(\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1))$ tel que:

$$\eta(\mathbf{x}_i^\pm(z)) = \mathbf{x}_i^\pm(1/z) \qquad \eta(\mathbf{k}_i^\pm(z)) = \mathbf{k}_i^\mp(1/z) \qquad \eta(C^{1/2}) = C^{1/2} \qquad \eta(D) = D \tag{0.1.3}$$

iv. Il existe un unique anti-homomorphisme involutive de $\mathbb{F}$-algèbre $\varphi \in \mathrm{Aut}(\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1))$ tel que:

$$\varphi(\mathbf{x}_i^\pm(z)) = \mathbf{x}_i^\mp(1/z) \qquad \varphi(\mathbf{k}_i^\pm(z)) = \mathbf{k}_i^\mp(1/z) \qquad \varphi(C^{1/2}) = C^{-1/2} \qquad \varphi(D) = D^{-1} \qquad \varphi(q) = q^{-1} \tag{0.1.4}$$

Proposition 0.1.2. *Il existe un unique automorphisme d'algèbre $T \in \mathrm{Aut}(\widehat{\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)})$ tel que:*

$$T(C^{1/2}) = C^{1/2} \quad T(D) = D \quad T(\mathbf{k}_0^\pm(z)) = \mathbf{k}_0^\pm(zq^2)\mathbf{k}_1^\pm(z)\mathbf{k}_1^\pm(zq^2) \quad T(\mathbf{k}_1^\pm(z)) = \mathbf{k}_1^\pm(z)^{-1} \tag{0.1.5}$$

$$T(\mathbf{x}_0^+(z)) = \frac{1}{[2]_q}\operatorname*{res}_{z_1,z_2} z_1^{-1}z_2^{-1}\left[\mathbf{x}_1^+(z_1), \left[\mathbf{x}_1^+(z_2), \mathbf{x}_0^+(zq^2)\right]_{G_{10}^-(z_2/zq^2)}\right]_{G_{11}^-(z_1/z_2)G_{10}^-(z_1/zq^2)} \tag{0.1.6}$$

$$T(\mathbf{x}_0^-(z)) = \frac{1}{[2]_q}\operatorname*{res}_{z_1,z_2} z_1^{-1}z_2^{-1}\left[\left[\mathbf{x}_0^-(zq^2), \mathbf{x}_1^-(z_1)\right]_{G_{10}^+(zq^2/z_1)}, \mathbf{x}_1^-(z_2)\right]_{G_{11}^+(z_1/z_2)G_{10}^+(zq^2/z_2)} \tag{0.1.7}$$

$$T(\mathbf{x}_1^+(z)) = -\mathbf{x}_1^-(C^{-1}z)\mathbf{k}_1^+(C^{-1/2}z)^{-1} \tag{0.1.8}$$

$$T(\mathbf{x}_1^-(z)) = -\mathbf{k}_1^-(C^{-1/2}z)^{-1}\mathbf{x}_1^+(C^{-1}z) \tag{0.1.9}$$

Ces automorphismes seront associés aux générateurs du groupe de tresse toroidale pour construire les générateurs de la nouvelle présentation.

Definition 0.1.3. Soit $\dot{\mathfrak{B}}$ le groupe de tresses affine de type $\dot{\mathfrak{a}}_1$. Soit $\ddot{\mathfrak{B}} := \dot{\mathfrak{B}} \ltimes P^\vee$, i.e. $\ddot{\mathfrak{B}}$ est isomorphe au groupe de dont les générateurs sont t, y and $(x_\lambda)_{\lambda\in P^\vee}$ tels que:

$$ty^{-1}t = y\,, \qquad tx_\lambda t^{-1} = x_{s_{\alpha_1}(\lambda)}\,, \qquad x_\lambda y = yx_\lambda\,, \tag{0.1.10}$$

Pour tout $\lambda \in P^\vee$.

Ceci definit l'extension du groupe de tresses toroidale de type $\dot{\mathfrak{a}}_1$. Le théorème suivant est la base de la construction des générateurs double-Drinfel'd:

Theorem 0.1.4.

$$t \mapsto T \qquad y \mapsto Y := T_\pi \circ T \qquad x_{\omega_i^\vee} \mapsto T_{\omega_i^\vee} \tag{0.1.11}$$

s'étend à un homomorphisme de groupe $\ddot{\mathfrak{B}} \to \mathrm{Aut}(\widehat{\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)})$.

En appliquant l'automorphisme Y, nous construisons $\boldsymbol{\psi}^+_{1,m}(z)$:

$$\left[Y^m\left(\mathbf{k}_1^-(z)^{-1}\mathbf{x}_1^-(C^{1/2}z)\right), \mathbf{x}_1^+(v)\right]_{G^-_{01}\left(z/C^{1/2}v\right)} = -\delta\left(\frac{q^{2m}z}{C^{1/2}v}\right)\boldsymbol{\psi}^+_{1,m}(v) \tag{0.1.12}$$

où

$$G^\pm_{ij}(z_1/z_2) = \left(\frac{z_1 q^{\mp c_{ij}} - z_2}{z_1 - q^{\mp c_{ij}} z_2}\right)_{|z_2| \gg |z_1|}. \tag{0.1.13}$$

les génerateurs de al sous algèbre de Cartan sont definis tels que:

$$\mathbf{K}^+_{1,m}(v) := (q - q^{-1})\mathbf{k}_1^-(C^{1/2}vq^{-2m})\boldsymbol{\psi}^+_{1,m}(v),$$

où $m \in \mathbb{N}^\times$ et

$$\mathbf{X}^\pm_{1,m}(z) := Y^{\mp m}(\mathbf{x}_1^\pm(z)).$$

Pour obtenir les relations dans cette nouvelle presentation il faut:

- établir les relations entre $\boldsymbol{\psi}^+_{1,m}(z)$ et les générateurs de la presentation de Drinfel'd,
- établir les relations dans $\ddot{\mathrm{U}}_q^+(\mathfrak{a}_1)$, $\ddot{\mathrm{U}}_q^0(\mathfrak{a}_1)$ et $\ddot{\mathrm{U}}_q^-(\mathfrak{a}_1)$,
- établir les relations entre les trois sous-algèbres: $\ddot{\mathrm{U}}_q^+(\mathfrak{a}_1)$, $\ddot{\mathrm{U}}_q^0(\mathfrak{a}_1)$ et $\ddot{\mathrm{U}}_q^-(\mathfrak{a}_1)$.

Nous pouvons maintenant fournir la nouvelle presentation:

Definition 0.1.5. L'algèbre de double affinization quantique $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)$ de type $\mathfrak{a}_1$ est la $\mathbb{F}$-algèbre dont les générateurs sont

$$\{\mathsf{D}_1, \mathsf{D}_1^{-1}, \mathsf{D}_2, \mathsf{D}_2^{-1}, \mathsf{C}^{1/2}, \mathsf{C}^{-1/2}, \mathsf{c}^+_m, \mathsf{c}^-_{-m}, \mathsf{K}^+_{1,0,m}, \mathsf{K}^-_{1,0,-m}, \mathsf{K}^+_{1,n,r}, \mathsf{K}^-_{1,-n,r}, \mathsf{X}^+_{1,r,s}, \mathsf{X}^-_{1,r,s} : m \in \mathbb{N}, n \in \mathbb{N}^\times, r, s \in \mathbb{Z}\}$$

tels que:

$$\mathsf{C}^{\pm 1/2} \text{ et } \mathbf{c}^\pm(z) \text{ sont les charges centrales} \tag{0.1.14}$$

$$\operatorname*{res}_{v,w} \frac{1}{vw}\mathbf{c}^\pm(v)\mathbf{c}^\mp(w) = 1\,, \tag{0.1.15}$$

$$\mathsf{D}_1^{\pm 1}\mathsf{D}_1^{\mp 1} = 1 \qquad \mathsf{D}_2^{\pm 1}\mathsf{D}_2^{\mp 1} = 1 \qquad \mathsf{D}_1\mathsf{D}_2 = \mathsf{D}_2\mathsf{D}_1 \tag{0.1.16}$$

$$\mathsf{D}_1\mathbf{K}^\pm_{1,\pm m}(z)\mathsf{D}_1^{-1} = q^{\pm m}\mathbf{K}^\pm_{1,\pm m}(z) \qquad \mathsf{D}_1\mathbf{X}^\pm_{1,r}(z)\mathsf{D}_1^{-1} = q^r\mathbf{X}^\pm_{1,r}(z)\,, \tag{0.1.17}$$

$$\mathsf{D}_2\mathbf{K}^\pm_{1,\pm m}(z)\mathsf{D}_2^{-1} = \mathbf{K}^\pm_{1,\pm m}(zq^{-1}) \qquad \mathsf{D}_2\mathbf{X}^\pm_{1,r}(z)\mathsf{D}_2^{-1} = \mathbf{X}^\pm_{1,r}(zq^{-1})\,, \tag{0.1.18}$$

$$\operatorname*{res}_{v,w} \frac{1}{vw}\mathbf{K}^\pm_{1,0}(v)\mathbf{K}^\mp_{1,0}(w) = 1\,, \tag{0.1.19}$$

$$(v-q^{\pm 2}z)(v-q^{2(m-n\mp 1)}z)\mathsf{K}^{\pm}_{1,\pm m}(v)\mathsf{K}^{\pm}_{1,\pm n}(z)=(vq^{\pm 2}-z)(vq^{\mp 2}-q^{2(m-n)}z)\mathsf{K}^{\pm}_{1,\pm n}(z)\mathsf{K}^{\pm}_{1,\pm m}(v)\,, \tag{0.1.20}$$

$$(\mathsf{C}q^{2(1-m)}v-w)(q^{2(n-1)}v-\mathsf{C}w)\mathsf{K}^{+}_{1,m}(v)\mathsf{K}^{-}_{1,-n}(w)=(\mathsf{C}q^{-2m}v-q^{2}w)(q^{2n}v-\mathsf{C}q^{-2}w)\mathsf{K}^{-}_{1,-n}(w)\mathsf{K}^{+}_{1,m}(v)\,, \tag{0.1.21}$$

$$(v-q^{\pm 2}z)\mathsf{K}^{\pm}_{1,\pm m}(v)\mathsf{X}^{\pm}_{1,r}(z)=(q^{\pm 2}v-z)\mathsf{X}^{\pm}_{1,r}(z)\mathsf{K}^{\pm}_{1,\pm m}(v)\,, \tag{0.1.22}$$

$$(\mathsf{C}v-q^{2(m\mp 1)}z)\mathsf{K}^{\pm}_{1,\pm m}(v)\mathsf{X}^{\mp}_{1,r}(z)=(\mathsf{C}q^{\mp 2}v-q^{2m}z)\mathsf{X}^{\mp}_{1,r}(z)\mathsf{K}^{\pm}_{1,\pm m}(v)\,, \tag{0.1.23}$$

$$(v-q^{\pm 2}w)\mathsf{X}^{\pm}_{1,r}(v)\mathsf{X}^{\pm}_{1,s}(w)=(vq^{\pm 2}-w)\mathsf{X}^{\pm}_{1,s}(w)\mathsf{X}^{\pm}_{1,r}(v)\,, \tag{0.1.24}$$

$$\begin{aligned}[\mathsf{X}^{+}_{1,r}(v),\mathsf{X}^{-}_{1,s}(z)] \quad = \quad & \frac{1}{q-q^{-1}}\left\{\delta\left(\frac{\mathsf{C}v}{q^{2(r+s)}z}\right)\prod_{p=1}^{|s|}\mathsf{c}^{-}\left(\mathsf{C}^{-1/2}q^{(2p-1)\mathrm{sign}(s)-1}z\right)^{-\mathrm{sign}(s)}\mathsf{K}^{+}_{1,r+s}(v)\right.\\ & \left.-\delta\left(\frac{\mathsf{C}^{-1}v}{q^{2(r+s)}z}\right)\prod_{p=1}^{|r|}\mathsf{c}^{+}\left(\mathsf{C}^{-1/2}q^{(1-2p)\mathrm{sign}(r)-1}v\right)^{\mathrm{sign}(r)}\mathsf{K}^{-}_{1,r+s}(z)\right\}\,,\end{aligned} \tag{0.1.25}$$

avec $m,n\in\mathbb{N}$, $r,s\in\mathbb{Z}$ et

$$\mathsf{c}^{\pm}(z)=\sum_{m\in\mathbb{N}}\mathsf{c}^{\pm}_{\pm m}z^{\mp m}\,, \tag{0.1.26}$$

$$\mathsf{K}^{\pm}_{1,0}(z)=\sum_{m\in\mathbb{N}}\mathsf{K}^{\pm}_{1,0,\pm m}z^{\pm m}\,, \tag{0.1.27}$$

et pour tout $m\in\mathbb{N}^{\times}$ et $r\in\mathbb{Z}$,

$$\mathsf{K}^{\pm}_{1,\pm m}(z)=\sum_{s\in\mathbb{Z}}\mathsf{K}^{\pm}_{1,\pm m,s}z^{-s}\,, \tag{0.1.28}$$

$$\mathsf{X}^{\pm}_{1,r}(z)=\sum_{s\in\mathbb{Z}}\mathsf{X}^{\pm}_{1,r,s}z^{-s}\,. \tag{0.1.29}$$

In (5.0.6), we further assume that $\mathsf{K}^{\pm}_{1,\mp m}(z)=0$ pour tout $m\in\mathbb{N}^{\times}$.

0.1.2 Représentations des Algèbres Toroidales Quantique de Type $\mathfrak{a}_1$

La nouvelle définition de l'algèbre toroidale quatique nous donne une nouvelle decomposition triangulaire pour etudier et classifier les modules de cette algèbre.

Definition 0.1.6. Soit M un module de $\ddot{\mathrm{U}}_q(\mathfrak{g})$. On dit que M est un t-poids module s'il existe un ensemble denombrable $\{M_\alpha : \alpha\in A\}$ indecomposables appelés espaces de t-poids tel que:

$$M\cong\bigoplus_{\alpha\in A}M_\alpha \tag{0.1.30}$$

où chaque M_α est un module de $\ddot{\mathrm{U}}^0_q(\mathfrak{g})$.

Nous pouvons ensuite classifier les modules de plus hauts t-poids.

Theorem 0.1.7. *L'unique module simple de plus haut t-poids possède un nombre de poids classique fini ssi sont espace de plus haut t-poids est un module de $\ddot{\mathrm{U}}^0_q(\mathfrak{g})$ simple t-dominant.*

0.1.3 Action du Groupe de Tresses: Le Cas Général

Dans ce qui suit nous donnons le théorème principale pour généraliser les résultats de cette thèse à toute autre système de racines.

Theorem 0.1.8. *$\forall i \neq j \in \dot{I}$, nous définissons T_i tel que:*

$$\begin{aligned}
&T_i(C) = C, \qquad T_i(D) = D \\
&T_i(\mathbf{x}_i^+(z)) = -\mathbf{x}_i^-(zC^{-1})\mathbf{k}_i^+(zC^{-1/2})^{-1}, \qquad T_i(\mathbf{x}_i^-(z)) = -\mathbf{k}_i^-(zC^{-1/2})^{-1}\mathbf{x}_i^+(zC^{-1}) \\
&T_i(\mathbf{k}_i^\pm(z)) = \mathbf{k}_i^\pm(z)^{-1}, \qquad T_i(\mathbf{k}_j^\pm(z)) = \prod_{p=1}^{|-a_{ij}|} \mathbf{k}_i^\pm(zq_i^{2-2p-a_{ij}})\mathbf{k}_j^\pm(zq_i^2) \\
&T_i(\mathbf{x}_j^+(z)) = \frac{\mathbf{x}^+_{i^{-a_{ij}}j}(zq_i^{-a_{ij}})}{[-a_{ij}]_{q_i}}, \quad T_i(\mathbf{x}_j^-(z)) = \frac{\mathbf{x}^-_{ji^{-a_{ij}}}(zq_i^{-a_{ij}})}{[-a_{ij}]_{q_i}}.
\end{aligned}$$

en plus, on donne

$$\begin{aligned}
&T_i^{-1}(C) = C, \qquad T_i^{-1}(D) = D \\
&T_i^{-1}(\mathbf{x}_i^+(z)) = -\mathbf{k}_i^+(zC^{1/2})^{-1}\mathbf{x}_i^-(zC), \qquad T_i^{-1}(\mathbf{x}_i^-(z)) = -\mathbf{x}_i^+(zC)\mathbf{k}_i^-(zC^{1/2})^{-1} \\
&T_i^{-1}(\mathbf{k}_i^\pm(z)) = \mathbf{k}_i^\pm(z)^{-1}, \qquad T_i^{-1}(\mathbf{k}_j^\pm(z)) = \prod_{p=1}^{|-a_{ij}|} \mathbf{k}_i^\pm(zq_i^{2-2p-a_{ij}})\mathbf{k}_j^\pm(zq_i^{-2}) \\
&T_i^{-1}(\mathbf{x}_j^+(z)) = \frac{\mathbf{x}^+_{ji^{-a_{ij}}}(zq^{a_{ij}})}{[-a_{ij}]_{q_i}}, \quad T_i^{-1}(\mathbf{x}_j^-(z)) = \frac{\mathbf{x}^-_{i^{-a_{ij}}j}(zq^{a_{ij}})}{[-a_{ij}]_{q_i}}.
\end{aligned}$$

Ceci nous permet d'avoir une action par automorphismes T_i du groupe de tresse $\ddot{\mathfrak{B}}$ sur l'algèbre $\widehat{\dot{\mathrm{U}}_q(\dot{\mathfrak{g}})}$.

Introduction

Quantum Groups

Quantum groups have been a subject of interest for the past 35 years and interestingly enough, to this day, do not have an agreed upon satisfactory definition. The name was popularized by V.G. Drinfel'd and defined by M. Jimbo and V.G. Drinfel'd [Drin85] [Jim86] as the q-deformation of the universal envoloping algebra $U(\mathfrak{g})$.

Quantum groups quickly gained popularity in physics too. Specifically, they were first used to address statistical mechanics problems such as inverse scattering methods. Ever since, the field exploded in multiple research directions in both mathematics and physics. On the mathematical side, one can look at multiple results ranging from geometry and knot theory to representation theory. When the deformation parameter q is not a root of unity, the representation theory of quantum groups is quite similar to that of complex simple Lie algebras. However, a remarkable result appears in the case q is a root of unity. In this case, if q is the p-th root of unity the representation theory becomes closely linked to that of the Lie algebra over a finite field with p elements.

Whereas for physics [PS90], we can see applications to integrable systems, conformal field theories, and lattice models relevant to statistical mechanics problems. Quantum groups in the case of finite integrable systems play a similar role as the Virasoro algebra in the case of conformal theories. In the case of the XXX open spin chain, there is an $\mathfrak{su}_2$ symmetry. Its counterpart is a $U_q(\mathfrak{su}_2)$ symmetry for an open XXZ spin chain with an appropriate boundary condition. We can think of this as deforming the spin chain in one direction so that the interaction along one of the directions is different. Moreover, $U_q(\mathfrak{sl}_2)$ is the Schur-Weyl dual of the Temperley-Lieb algebra for 2D lattice systems,such as loop models, or equivalently spin chains that are built out of the Temperley-Leib algebra.

Quantum Affine Algebras

For every Lie algebra $\mathfrak{g}$ we denote by $\dot{\mathfrak{g}}$ the untwisted affine Kac-Moody algebra associated to $\mathfrak{g}$. The quantum affine algebra $\mathrm{U}_q(\dot{\mathfrak{g}})$ is obtained through a Drinfel'd-Jimbo quantisation of $\dot{\mathfrak{g}}$. These algebras are still a field of very active research in both physics and mathematics too. Although it is practically impossible to list all of them, especially on the mathematical side, we will list a few and focus on what was most relevant during the development of this work. For physicists these algebras are especially interesting because of the trigonometric R-matrix, solution to the Yang-Baxter equation. Quantum affine algebras also arise in the context of integrable quantum field theories. However we will start shifting our interest to the mathematical side here. The quantum affine algebra admits two presentations. One in terms of the Drinfel'd-Jimbo generators and the second, denoted here as $\dot{\mathrm{U}}_q(\mathfrak{g})$ in terms of Drinfel'd's current generators. The isomorphism between the two presentations is due to I. Damiani and J. Beck. [Dam93],[Bec94] and the latter depends on the finite root system data only. In fact, Drinfel'd's presentation was the key that unlocked the study of the representation theory of quantum affine algberas, an idea that we will come back to in the toroidal setting. Contrary to classical Lie algebras, highest weight representations of quantum affine algebras are infinite dimensional. However, V. Chari and A. Pressley [CP95] established that a representation of $\dot{\mathrm{U}}_q(\mathfrak{g})$ is finite dimensional if and only if there exists a polynomial called Drinfel'd polynomial with constant coefficient 1 called ℓ-weight.

Quantum Toroidal Algebras

Quantum toroidal algebras were first introduced by V. Ginzburg, M. Kapranov, and E. Vasserot for type A in [GKV95] and then generally by H. Nakajima and N. Jing in [Nak01][Jin98]. Then, M. Varagnolo and E. Vasserot established a Schur-Weyl duality between quantum toroidal algebras and the double affine Hecke algebra (DAHA) [VV96]. Most recently, G. Noshita, and A. Watanabe in [NW21] introduced quiver quantum toroidal algebras as a q-deformation of the quiver Yangian as the quantum toroidal $\mathfrak{gl}_1$. One can also define quantum toroidal algebras in the same fashion as quantum affine algebras. By which we mean, if we use Drinfel'd's quantum affinization on an untwisted affine Kac-Moody algebra, then we obtain quantum toroidal algebras. The representation theory of quantum toroidal algebras is far from being understood although one finds several results such as plane partition representations of quantum toroidal $\mathfrak{gl}_1$ by B. Feigin, M. Jimbo, T. Miwa and E. Mukhin [FJMM12] as well as vertex representations by Y. Saito [S98].

The thesis is organized as follows. In chapter 1, we will give a small review on the results that are most relevant for this work. This review can also be seen as a road map because the results we are presenting in this thesis are an affinized version of the review. In chapter 2, we give a double Drinfel'd current presentation for quantum toroidal $\mathfrak{sl}_2$ through an affinized Damiani-Beck isomorphism. In chapter 3, we use this new presentation to discuss some representation theoretic consequences by defining a new notion of finiteness as well as providing an evaluation homomorphism. As we know, one does not expect to have finite dimensional representations for quantum toroidal algebras. Therefore, we introduce the idea of classical weight-finiteness and prove that it generalizes the classification result of V. Chari and A. Pressley for $\dot{\mathrm{U}}_q(\mathfrak{sl}_2)$. Then, in chapter 4 we give the action of the braid group on the quantum toroidal algebras of any rank. This chapter is the stepping stone for generalizing chapter 2 to higher ranks. Finally, we give possible future directions and conjecture how the double Drinfel'd presentation $\ddot{\mathrm{U}}_q(\mathfrak{g})$ should look like for any simple finite dimensional Lie algebra $\mathfrak{g}$.

Chapter 1

A Brief History

1.1 Quantized Enveloping Algebra

When it comes to quantized enveloping algebras, a lot of its structure carries over from Lie algebras all thanks to the fact that semisimple complex Lie algebras are at the heart of the quantized enveloping algebra denoted $\mathrm{U}_q(\mathfrak{g})$.

We start by a reminder that for a root system with basis Π, a Cartan matrix A, and a_{ij} the Cartan matrix entries, the Lie algebra $\mathfrak{g}$ in the Chevalley-Serre presentation is generated by the generators e_i^+, e_i^-, and h_i satisfying the following relations:

$$[h_i, h_j] = 0, \quad [h_i, e_i^+] = a_{ij} e_i^+, \tag{1.1.1}$$

$$[h_i, e_i^+] = -a_{ij} e_i^-, \quad [e_i^+, e_j^-] = \delta_{ij} h_i, \tag{1.1.2}$$

$$(\mathrm{ad}\, e_i^+)^{1-a_{ij}} e_j^+ = 0 \tag{1.1.3}$$

$$(\mathrm{ad}\, e_i^-)^{1-a_{ij}} e_j^- = 0 \tag{1.1.4}$$

Then, the enveloping algebra is defined as the associative algebra with the same generators and relations where we quotient by $[x, y] = xy - yx$. This means that we can rewrite the last relations as:

$$\sum_{i=0}^{1-a_{ij}} (-1)^i \begin{bmatrix} 1-a_{ij} \\ i \end{bmatrix} (e_i^+)^{1-a_{ij}-i} e_j^+ (e_i^+)^i \tag{1.1.5}$$

$$\sum_{i=0}^{1-a_{ij}} (-1)^i \begin{bmatrix} 1-a_{ij} \\ i \end{bmatrix} (e_i^-)^{1-a_{ij}-i} e_j^- (e_i^-)^i \tag{1.1.6}$$

Now let $q \in \mathbb{C}$ such that $q \neq 0$ and q is not a root of unity. Let $q_i = q^{(\alpha_i, \alpha_i)/2}$ and define the q-numbers by:

$$[n]_{q_i} = \frac{q_i^n - q_i^{-n}}{q_i - q_i^{-1}} \tag{1.1.7}$$

Then, the quantized enveloping algebra $\mathrm{U}_q(\mathfrak{g})$ is the algebra with generators $E_i^{\pm}$ and K_i, and K_i^{-1} satisfying:

$$K_i K_j = K_j K_i \tag{1.1.8}$$

$$K_iK_i^{-1} = K_i^{-1}K_i = 1 \tag{1.1.9}$$

$$K_iE_j^+K_i^{-1} = q_i^{(\alpha_i,\alpha_j)}E_j^+ \tag{1.1.10}$$

$$K_iE_j^-K_i^{-1} = q_i^{-(\alpha_i,\alpha_j)}E_j^- \tag{1.1.11}$$

$$\sum_{i=0}^{1-a_{ij}}(-1)^i \begin{bmatrix}1-a_{ij}\\ i\end{bmatrix}_{q_i} (E_i^+)^{1-a_{ij}-i}E_j^+(E_i^+)^i \tag{1.1.12}$$

$$\sum_{i=0}^{1-a_{ij}}(-1)^i \begin{bmatrix}1-a_{ij}\\ i\end{bmatrix}_{q_i} (E_i^-)^{1-a_{ij}-i}E_j^-(E_i^-)^i \tag{1.1.13}$$

We can clearly see all the parallels between the two presentations of a Lie algebra and the quantized enveloping algebra. As it was the case for Lie algebras, we denote by $\mathrm{U}_q(\mathfrak{g})^{\pm}$, and $\mathrm{U}_q(\mathfrak{g})^0$ the subalgebras respectively generated by $E_i^{\pm}$, and K_i together with K_i^{-1}. We also have the following triangular decomposition:

$$\mathrm{U}_q(\mathfrak{g}) \cong \mathrm{U}_q(\mathfrak{g})^+ \otimes \mathrm{U}_q(\mathfrak{g})^0 \otimes \mathrm{U}_q(\mathfrak{g})^- \tag{1.1.14}$$

which will once again be relevant for the representation theory part of course.

Furthermore, by setting:

$$\Delta(E_i^+) = E_i \otimes 1 + K_i \otimes E_i^+, \quad \epsilon(E_i^+) = 0 \tag{1.1.15}$$

$$\Delta(E_i^+) = E_i \otimes K_i^{-1} + 1 \otimes E_i^-, \quad \epsilon(E_i^-) = 0 \tag{1.1.16}$$

$$\Delta(K_i) = K_i \otimes K_i, \quad \epsilon(K_i) = 1 \tag{1.1.17}$$

$$S(E_i^+) = -K_i^{-1}E_i^+, \quad S^{-1}(E_i^+) = -E_i^+K_i^{-1} \tag{1.1.18}$$

$$S(E_i^-) = -E_i^-K_i, \quad S^{-1}(E_i^-) = -E_i^-K_i \tag{1.1.19}$$

$$S(K_i) = K_i^{-1}, \quad S^{-1}(K_i) = K_i^{-1} \tag{1.1.20}$$

for the comultiplication Δ, the counit ϵ, and the antipode S, we make the quantized enveloping algebra into a Hopf algebra.

We can now give the two unique automorphisms on $\mathrm{U}_q(\mathfrak{g})$ denoted ω, and τ given by:

$$\omega(E_i^+) = E_i^-, \quad \omega(E_i^-) = E_i^+, \quad \omega(K_i) = K_i^{-1} \tag{1.1.21}$$

$$\tau(E_i^+) = E_i^+, \quad \tau(E_i^-) = E_i^-, \quad \tau(K_i) = K_i^{-1} \tag{1.1.22}$$

The proof is straightforward and all we have to do is check this on the algebra relations.

1.1.1 Representation Theory

As it is the case for Lie algebras, the representation theory of $\mathrm{U}_q(\mathfrak{g})$ stems mostly from the representation theory of $U_q(\mathfrak{sl}_2)$. We will start by giving some of the important results of that in the case where q is not a root of unity and the field $\mathbb{F}$ is of characteristic zero.

The presentation of $U_q(\mathfrak{sl}_2)$ is the same as the one presented in the previous section where we get rid of the subscript i because it can only take one value since we only have one simple root.

If M is a $U_q(\mathfrak{sl}_2)$-module then set for all $\lambda \in \mathbb{F}^\times$:

$$M_\lambda = \{m \in M | K.m = \lambda.m\} \tag{1.1.23}$$

This means that M_λ is the eigenspace of K with eigenvalue λ. As we have done for Lie algebras, the λ's will be called the weights. Taking the algebra relations into account, it is clear that we have:

$$E^+.M_\lambda \subset M_{q^2\lambda}, \quad E^-.M_\lambda \subset M_{q^{-2}\lambda} \tag{1.1.24}$$

More precisely, we have that the direct sum of any weight spaces of the form $M_{q^{2n}\lambda}$ is a submodule. From this we can conclude that if M is simple, we have $M = \bigoplus_n M_{q^{2n}\lambda}$. We now have the following proposition:

Proposition 1.1.1. *Suppose M is a finite-dimensional $U_q(\mathfrak{sl}_2)$-module, then M is a direct sum of its weight spaces with weights of the form $\pm q^a$ for $a \in \mathbb{Z}$.*

For each $\lambda \in \mathbb{F}$, there is an infinite dimensional $U_q(\mathfrak{sl}_2)$-module with basis $m_0, m_1, ...$ where the algebra generators act as:

$$K.m_i = \lambda q^{-2i} m_i, \quad F.m_i = m_{i+1}, \tag{1.1.25}$$

$$E.m_i = 0,\ if\ i = 0, \tag{1.1.26}$$

$$E.m_i = [i]_q \frac{\lambda q^{1-i} - \lambda^{-1} q^{i-1}}{q - q^{-1}}, \text{ otherwise} \tag{1.1.27}$$

We now finish the part about the representation theory of $U_q(\mathfrak{sl}_2)$ with the following two theorems:

Theorem 1.1.2. *Let M be a finite dimensional $U_q(\mathfrak{sl}_2)$-module that is a direct sum of its weight spaces, then M is a semisimple module.*

Analogously to $\mathfrak{sl}_2$,

Theorem 1.1.3. *For each $n \geq 0$ there are two simple $U_q(\mathfrak{sl}_2)$-modules denoted respectively $L(N, +)$, and $L(N, -)$ with basis $m_0, m_1, ..., m_n$, and $m'_0, m'_1, ..., m'_n$ such that:*

$$K.m_i^{(')} = q^{n-2i} m_i^{(')} \tag{1.1.28}$$

$$E^+.m_i^{(')} = 0 \quad if\ i = 0 \tag{1.1.29}$$

$$E^+.m_i^{(')} = [i]_q[n+1-i]_q m_{i-1}^{(')} \quad \text{otherwise} \tag{1.1.30}$$

$$E^-.m_i^{(')} = 0 \quad if\ i = n \tag{1.1.31}$$

$$E^-.m_i^{(')} = m_{i+1}^{(')} \quad \text{otherwise} \tag{1.1.32}$$

1.1.2 Representation Theory of $\mathrm{U}_q(\mathfrak{g})$

Most of the representation theory results of $\mathrm{U}_q(\mathfrak{g})$ arise from what we have seen in the case of $U_q(\mathfrak{sl}_2)$. Moreover, there are a lot of similarities between the representation theory of $\mathrm{U}_q(\mathfrak{g})$ and that of $\mathfrak{g}$. In this section, we will introduce the category of finite dimensional $\mathrm{U}_q(\mathfrak{g})$-modules and give the classification theorem of its objects.

Let λ be a weight and μ an element of the root lattice $\mathbb{Z}\Phi$. For any $\mathrm{U}_q(\mathfrak{g})$-module M, let for all λ and all $\sigma : \mathbb{Z}\Phi \mapsto \{\pm 1\}$ $M_{\lambda,\sigma}$ be the subspace of M given by:

$$M_{\lambda,\sigma} = \{m \in M | K_\mu m = \sigma(\mu) q^{(\lambda,\mu)} m\}. \tag{1.1.33}$$

$M_{\lambda,\sigma}$ are the weight spaces of M.

In case M is a finite dimensional module, then

$$M = \bigoplus_{\sigma,\lambda} M_{\lambda,\sigma} \tag{1.1.34}$$

since all the K_i's are simultaneously digonalizable. Moreover, we have for all λ, and σ:

$$E_i^+ M_{\lambda,\sigma} \subset M_{\lambda+\alpha_i,\sigma}, \quad \text{and} \quad E_i^- M_{\lambda,\sigma} \subset M_{\lambda-\alpha_i,\sigma} \tag{1.1.35}$$

for all simple roots α_i. This is clear from the algebra relations.

Furthermore, the generators $E_i^{\pm}$ act nilpotently. This is because the module M holds $U_q(\mathfrak{sl}_2)$-submodules for each α_i.

For a module M given by the direct sum as above, we will say that it is of type σ if $M = M_{\lambda,\sigma}$ and of type 1 if in addition to that we have $\sigma(\alpha) = 1$ for all α.

The previous results can be summarized as follows: the category of finite dimensional $\mathrm{U}_q(\mathfrak{g})$-modules is the direct sum of the categories of all finite dimensional modules of type σ.

However, we can define the involutary automorphism $\tilde{\sigma}$ by:

$$\tilde{\sigma}(E_i^{\pm}) = \sigma(\alpha_i)^{\frac{1\pm 1}{2}} E_i^{\pm}, \quad \tilde{\sigma}(K_i) = \sigma(\alpha_i) K_i \tag{1.1.36}$$

that allows us to twist any module of type σ into a module of type 1. Clearly, this is a functor that allows us to say that we have an equivalence between the category of finite dimensional modules of type 1 and that of type σ.

Now, since $\mathrm{U}_q(\mathfrak{g})$ has the structure of a Hopf algebra, most of the results of the section on $U_q(\mathfrak{sl}_2)$ generalize to $\mathrm{U}_q(\mathfrak{g})$. The coproduct allows us to define the tensor product of modules, and since we have: $\Delta(K_i) = K_i \otimes K_i$ then we have:

$$M_{\lambda,\sigma} \otimes M'_{\lambda',\sigma'} \subset (M \otimes M')_{\lambda+\lambda',\sigma\sigma'} \tag{1.1.37}$$

The fact that the tensor product of two modules of type 1 is of type 1 follows immediately. Once more, we have a trivial one dimensional module given by the ϵ and defining for each $\sigma : \mathbb{Z}\Phi \to \{\pm 1\}$ such that:

$$\epsilon_\sigma(E_i^{\pm}) = 0, \quad \text{and} \quad \epsilon_\sigma(K_i) = \sigma(\alpha_i) \tag{1.1.38}$$

gives us a one dimensional module denoted $L(0, \sigma)$.

From now on, since we can twist any module, we will stick to type 1 modules.

Since the set of weights is always finite, there exists for every module M a weight λ such that $M_\lambda \neq 0$ but $M'_\lambda = 0$ for any $\lambda' > \lambda$. This holds true in particular if λ' is a simple root. This means that if M is a finite dimensional module, there exists $v \in M_\lambda$, $v \neq 0$, but $E_i^+ v = 0$ for all i. In this case, λ is called a dominant weight. Moreover, we have: $(E_i^-)^{a+1} v = 0$ with $a = 2\frac{(\lambda,\alpha_i)}{(\alpha_i,\alpha_i)}$. This holds because otherwise, the

$U_q(\mathfrak{sl}_2)$-submodule corresponding to the simple root α_i would be infinite dimensional.

We will now construct the universal highest weight module (Verma module) of highest weight λ.

Any weight λ defines a one dimensional $\mathrm{U}_q(\mathfrak{g})^{\geqslant 0}$-module where we have:

$$K_i m = q^{(\lambda,\alpha_i)}m, E_i^+ m = 0 \tag{1.1.39}$$

for all $m \in M$. The kernel of this representation is an ideal $I_\lambda^{\geqslant 0}$ given by:

$$I_\lambda^{\geqslant 0} = \sum_{\alpha_i \in \Pi} \mathrm{U}_q(\mathfrak{g})^{\geqslant 0} E_i^+ + \sum_{\alpha_i \in \Pi} \mathrm{U}_q(\mathfrak{g})^{\geqslant 0}(K_i - q^{(\lambda,\alpha_i)}). \tag{1.1.40}$$

Clearly, we have:

$$\mathrm{U}_q(\mathfrak{g}) = \mathrm{U}_q(\mathfrak{g})^- \oplus I_\lambda \tag{1.1.41}$$

where,

$$I_\lambda = \sum_{\alpha_i \in \Pi} \mathrm{U}_q(\mathfrak{g}) E_i^+ + \sum_{\alpha_i \in \Pi} \mathrm{U}_q(\mathfrak{g})(K_i - q^{(\lambda,\alpha_i)}). \tag{1.1.42}$$

due to I_λ being a left ideal and the triangular decomposition of $\mathrm{U}_q(\mathfrak{g})$. Taking the quotient: $M(\lambda) = \mathrm{U}_q(\mathfrak{g})/I_\lambda$, where it is clear that we have:

$$E_i^+ v_\lambda, \quad K_i v_\lambda = q^{(\lambda,\alpha_i)} v_\lambda \tag{1.1.43}$$

makes $M(\lambda)$ into a universal highest weight module. The existence of such module means that we have a bijection between elements in $\mathrm{U}_q(\mathfrak{g})_\mu$ and $M(\lambda)_{\lambda-\mu}$ where each $\mathrm{U}_q(\mathfrak{g})_\mu$ is finite dimensional where we get:

$$M(\lambda)_\lambda = \mathbb{F} v_\lambda, \quad M(\lambda)_{\lambda - n\alpha_i} = \mathbb{F}(E_i^-)^n v_\lambda. \tag{1.1.44}$$

for all integers $n \geq 0$. We obtain the unique up to isomorphisms submodule of $L(\lambda)$ by taking the quotient: $L(\lambda) = M(\lambda)/N(\lambda)$ where N is the unique maximal submodule. This λ is unique therefore it is the largest weight of the module meaning that it is a dominant weight. In order to complete the classification of simple finite dimensional modules we need to show that if λ is dominant, then the module is finite dimensional.

We can now define a homomorphism of $\mathrm{U}_q(\mathfrak{g})$-modules ϕ:

$$\phi : v_{\lambda-(n+1)\alpha_i} \mapsto (E_i^-)^{n+1} v_\lambda. \tag{1.1.45}$$

This can be easily proven by using the relation between E_i^+ and E_j^- on $(E_i^-)^{n+1}v_\lambda$ to verify the universal property of the module.

Another interesting result that will be useful for proving that the modules are finite dimensional is the fact $E_i^\pm$ act nilpotently on the on the module $\mathrm{U}_q(\mathfrak{g})/I$ where I is the ideal generated by $(E_i^+)^{m(\alpha_i)}$ and $(E_i^-)^{n(\alpha_i)}$ where $m(\alpha_i)$, and $n(\alpha_i)$ are both positive integers.

Now, for an important result for the classification theorem, let $\phi_{\alpha_i} : M(\lambda - (n(\alpha_i)+1)\alpha_i) \to M(\lambda)$. The $\mathrm{U}_q(\mathfrak{g})$-module $\tilde{L}(\lambda) \simeq M(\lambda)/\sum_{\alpha_i} im(\phi_{\alpha_i})$ is finite dimensional.

It is clear that upon identifying $M(\lambda)$ with $\mathrm{U}_q(\mathfrak{g})^-$, we identify the image of ϕ_{α_i} with $\mathrm{U}_q(\mathfrak{g})^-(E_i^-)^{n(\alpha_i+1)}$. Then,

$$\tilde{L}(\lambda) \simeq \mathrm{U}_q(\mathfrak{g})/(\sum_{\alpha_i \in \Pi} \mathrm{U}_q(\mathfrak{g}) E_i^+ + \sum_{\alpha_i \in \Pi} \mathrm{U}_q(\mathfrak{g})(E_i^-)^{n(\alpha_i)+1} + \sum_{\alpha_i \in \Pi} \mathrm{U}_q(\mathfrak{g})(K_i - q^{(\lambda,\alpha_i)}).) \tag{1.1.46}$$

which means that $E_i^{\pm}$ act nilpotently on $\tilde{L}(\lambda)$. The approach will be strictly weight oriented. We will show that the weights are stable under the Weyl group action and the weight space contains finitely many dominant weights in the orbits under this action we get a finite dimensional module. It suffices to show that for a weight β the image after a simple reflection is still a weight of $\tilde{L}(\lambda)$. This relies on the fact that we have $U_q(\mathfrak{sl}_2)$-modules for each simple root and that since $E_i^{\pm}$ act nilpotently then that the $U_q(\mathfrak{sl}_2)$-modules $V = \bigoplus_n V_n$ are finite dimensional and $\dim\, V_n = \dim\, V_{-n}$. In fact, consider:

$$V = \bigoplus_{n\in\mathbb{Z}} \tilde{L}(\lambda)_{\beta+n\alpha_i} \tag{1.1.47}$$

We can make the identification $V_m = \tilde{L}(\lambda)_{\beta+n\alpha_i}$ with $m = 2\frac{(\beta,\alpha_i)}{(\alpha_i,\alpha_i)} + 2n$. Then by setting $r = 2\frac{(\beta,\alpha_i)}{(\alpha_i,\alpha_i)}$, we have:

$$V_r = \tilde{L}(\lambda)_{\beta}, \quad V_{-r} = \tilde{L}(\lambda)_{s_{\alpha_i}(\beta)} \tag{1.1.48}$$

which means that $s_{\alpha_i}(\beta)$ is a weight in $\tilde{L}(\lambda)$.

Finally, the classification theorem follows:

Theorem 1.1.4. *For each dominant weight λ the module $L(\lambda)$ is finite dimensional and each finite dimensional $\mathrm{U}_q(\mathfrak{g})$-module is isomorphic to exactly one $L(\lambda)$ with λ dominant.*

1.2 Quantum Affine Algebras

After constructing the quantum group $\mathrm{U}_q(\mathfrak{g})$ for a simple finite dimensional Lie algebra, an obvious question is whether we can repeat this process but for an affine Lie algebra. Clearly, the answer is yes. Upon replacing the root system data by that of an affine Lie algebra, i.e. instead of using a finite type Cartan matrix we use a Cartan matrix of an untwisted affine Kac-Moody algebra, we obtain what is called a quantum affine Lie algebra denoted $\mathrm{U}_q(\dot{\mathfrak{g}})$.

Then, the quantum affine algebra $\mathrm{U}_q(\mathfrak{g})$ is the algebra with generators $E_i^{\pm}$ and K_i, and K_i^{-1} satisfying:

$$K_iK_j = K_jK_i \tag{1.2.1}$$

$$K_iK_i^{-1} = K_i^{-1}K_i = 1 \tag{1.2.2}$$

$$K_iE_j^{+}K_i^{-1} = q_i^{(\alpha_i,\alpha_j)}E_j^{+} \tag{1.2.3}$$

$$K_iE_j^{-}K_i^{-1} = q_i^{-(\alpha_i,\alpha_j)}E_j^{-} \tag{1.2.4}$$

$$\sum_{i=0}^{1-a_{ij}} (-1)^i \begin{bmatrix} 1-a_{ij} \\ i \end{bmatrix}_{q_i} (E_i^{+})^{1-a_{ij}-i}E_j^{+}(E_i^{+})^i \tag{1.2.5}$$

$$\sum_{i=0}^{1-a_{ij}} (-1)^i \begin{bmatrix} 1-a_{ij} \\ i \end{bmatrix}_{q_i} (E_i^{-})^{1-a_{ij}-i}E_j^{-}(E_i^{-})^i \tag{1.2.6}$$

where, $i \in \dot{I}$ with $\dot{I} = 0, 1, ..., n$ the set corresponding to the nodes of the Dynkin diagram. We remind you that each node corresponds to a simple root and the $0th$ node is the affine root. Therefore, we can simply that we have the following diagram:

$$\begin{tikzcd}
\mathfrak{g} \arrow[r, "\text{Classical Affinization}"] \arrow[d, "\text{Quantization}"'] & \dot{\mathfrak{g}} \arrow[d, "\text{Quantization}"] \\
\mathrm{U}_q(\mathfrak{g}) \arrow[r, "\text{Classical Affinization}"'] & \mathrm{U}_q(\dot{\mathfrak{g}})
\end{tikzcd}$$

As usual, our goal is to study the representation theory of these algebras. However, it turns out that this presentation of quantum affine algebras isn't the best one for understanding and classifying its representations. Drinfel'd then proposed another presentation usually also known as Drinfel'd current presentation of quantum affine algebras. The name comes from the fact that this new presentation is in fact very similar in spirit to that of the central extension of the loop algebras in the case of Kac-Moody algebras. In fact, Drinfel'd gave the presentation but the proof that the two algebras were actually isomorphic was done by I. Damiani (for the injectivity part of the isomorphism) and J. Beck (for the surjectivity part of the isomorphism). Therefore, in the next part we will review the main points that lead to constructing Drinfel'd's presentation which unlocked the representation theory of quantum affine algebras.

1.2.1 Damiani-Beck Isomorphism

We will start this section by stating the isomorphism theorem that gives us the new presentation and then work our way through defining all the relevant material that leads to it.

Theorem 1.2.1. *Let $\dot{\mathrm{U}}_q(\mathfrak{g})$ be the associative algebra generated by the generators*

$$\left\{D, D^{-1}, C^{1/2}, C^{-1/2}, k^+_{i,n}, k^-_{i,-n}, x^+_{i,m}, x^-_{i,m} : i \in I, m \in \mathbb{Z}, n \in \mathbb{N}\right\}$$

subject to the following relations

$$C^{\pm 1/2} \text{ is central} \qquad C^{\pm 1/2}C^{\mp 1/2} = 1 \qquad D^{\pm 1}D^{\mp 1} = 1 \tag{1.2.7}$$

$$D\mathbf{k}^{\pm}_i(z)D^{-1} = \mathbf{k}^{\pm}_i(zq^{-1}) \qquad D\mathbf{x}^{\pm}_i(z)D^{-1} = \mathbf{x}^{\pm}_i(zq^{-1}) \tag{1.2.8}$$

$$\operatorname*{res}_{z_1,z_2} \frac{1}{z_1 z_2}\mathbf{k}^{\pm}_i(z_1)\mathbf{k}^{\mp}_i(z_2) = 1 \tag{1.2.9}$$

$$\mathbf{k}^{\pm}_i(z_1)\mathbf{k}^{\pm}_j(z_2) = \mathbf{k}^{\pm}_j(z_2)\mathbf{k}^{\pm}_i(z_1) \tag{1.2.10}$$

$$\mathbf{k}^{-}_i(z_1)\mathbf{k}^{+}_j(z_2) = G^{-}_{ij}(C^{-1}z_1/z_2)G^{+}_{ij}(Cz_1/z_2)\mathbf{k}^{+}_j(z_2)\mathbf{k}^{-}_i(z_1) \tag{1.2.11}$$

$$G^{\mp}_{ij}(C^{\mp 1/2}z_2/z_1)\mathbf{k}^{+}_i(z_1)\mathbf{x}^{\pm}_j(z_2) = \mathbf{x}^{\pm}_j(z_2)\mathbf{k}^{+}_i(z_1) \tag{1.2.12}$$

$$\mathbf{k}^{-}_i(z_1)\mathbf{x}^{\pm}_j(z_2) = G^{\mp}_{ij}(C^{\mp 1/2}z_1/z_2)\mathbf{x}^{\pm}_j(z_2)\mathbf{k}^{-}_i(z_1) \tag{1.2.13}$$

$$(z_1 - q^{\pm c_{ij}}z_2)\mathbf{x}^{\pm}_i(z_1)\mathbf{x}^{\pm}_j(z_2) = (z_1 q^{\pm c_{ij}} - z_2)\mathbf{x}^{\pm}_j(z_2)\mathbf{x}^{\pm}_i(z_1) \tag{1.2.14}$$

$$[\mathbf{x}^{+}_i(z_1), \mathbf{x}^{-}_j(z_2)] = \frac{\delta_{ij}}{q_i - q_i^{-1}}\left[\delta\left(\frac{z_1}{Cz_2}\right)\mathbf{k}^{+}_i(z_1C^{-1/2}) - \delta\left(\frac{z_1 C}{z_2}\right)\mathbf{k}^{-}_i(z_2C^{-1/2})\right] \tag{1.2.15}$$

$$\sum_{\sigma \in S_{1-a_{ij}}} \sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k}_{q_i} \mathbf{x}^{\pm}_i(z_{\sigma(1)})\cdots\mathbf{x}^{\pm}_i(z_{\sigma(k)})\mathbf{x}^{\pm}_j(z)\mathbf{x}^{\pm}_i(z_{\sigma(k+1)})\cdots\mathbf{x}^{\pm}_i(z_{\sigma(1-a_{ij})}) = 0 \tag{1.2.16}$$

where, for every $i \in I$, we define the following $\dot{\mathrm{U}}_q(\mathfrak{g})[[z, z^{-1}]]$-valued formal distributions

$$\mathbf{x}_i^{\pm}(z) := \sum_{m \in \mathbb{Z}} x_{i,m}^{\pm} z^{-m} \in \dot{\mathrm{U}}_q(\mathfrak{g})[[z, z^{-1}]]\,; \tag{1.2.17}$$

$$\mathbf{k}_i^{\pm}(z) := \sum_{n \in \mathbb{N}} k_{i,\pm n}^{\pm} z^{\mp n} \in \dot{\mathrm{U}}_q(\mathfrak{g})[[z^{\mp 1}]]\,, \tag{1.2.18}$$

for every $i, j \in \dot{I}$, we define the following $\mathbb{F}$-valued formal power series

$$G_{ij}^{\pm}(z) := q_i^{\pm a_{ij}} + (q_i - q_i^{-1})[\pm a_{ij}]_{q_i} \sum_{m \in \mathbb{N}^{\times}} q_i^{\pm m a_{ij}} z^m \in \mathbb{F}[[z]] \tag{1.2.19}$$

and

$$\delta(z) := \sum_{m \in \mathbb{Z}} z^m \in \mathbb{F}[[z, z^{-1}]] \tag{1.2.20}$$

is an $\mathbb{F}$-valued formal distribution.

Note that $G_{ij}^{\pm}(z)$ is invertible in $\mathbb{F}[[z]]$ with inverse $G_{ij}^{\mp}(z)$, i.e.

$$G_{ij}^{\pm}(z) G_{ij}^{\mp}(z) = 1\,, \tag{1.2.21}$$

and that it can be viewed as the power series expansion of a rational function of $(z_1, z_2) \in \mathbb{C}^2$ as $|z_2| \gg |z_1|$, which we shall denote as follows

$$G_{ij}^{\pm}(z_1/z_2) = \left(\frac{z_1 q_i^{\mp a_{ij}} - z_2}{z_1 - q_i^{\mp a_{ij}} z_2} \right)_{|z_2| \gg |z_1|}. \tag{1.2.22}$$

We will now see how the generators:

$$\left\{ k_{i,n}^{+}, k_{i,-n}^{-}, x_{i,m}^{+}, x_{i,m}^{-} : i \in I, m \in \mathbb{Z}, n \in \mathbb{N} \right\}$$

are defined from the generators of the Drinfel'd-Jimbo presentation. In order to do this, we need to make a small but necessary detour and talk about the braid group action on $\mathrm{U}_q(\dot{\mathfrak{g}})$.

1.2.2 Automorphisms and Braid Group Action on $\mathrm{U}_q(\dot{\mathfrak{g}})$

We start by introducing the $\mathbb{C}$-algebra automorphisms ϕ, and Ω which are given by:

$$\phi(E_i^{\pm}) = E_i^{\mp}, \quad \phi(K_i) = K_i, \quad \phi(q) = q^{-1} \tag{1.2.23}$$

$$\Omega(E_i^{\pm}) = E_i^{\mp}, \quad \Omega(K_i) = K_i^{-1}, \quad \Omega(q) = q^{-1} \tag{1.2.24}$$

The braid group associated to the Weyl group W with generators T_i acts on the generators of the algebra as follows:

$$T_i(E_i^{+}) = -E_i^{-} K_i, \quad T_i(E_i^{-}) = -K_i^{-1} E_i^{-} \tag{1.2.25}$$

$$T_i(E_j^{+}) = \sum_{n=0}^{-a_{ij}} (-1)^{n - a_{ij}} q_i^{-n} (E_i^{+})^{(-n - a_{ij})} E_j^{+} (E_i^{+})^{(n)} \tag{1.2.26}$$

$$T_i(E_j^-) = \sum_{n=0}^{-a_{ij}} (-1)^{n-a_{ij}} q_i^{-n} (E_i^-)^{(-n-a_{ij})} E_j^- (E_i^-)^{(n)} \tag{1.2.27}$$

$$T_i(K_\alpha) = K_{s_i(\alpha)} \tag{1.2.28}$$

Extend that group with T_τ where τ is Dynkin diagram automorphism and denote $\mathcal{P}$ the group of automorphisms generated by $T_{\omega_i^\vee}$ and their inverses. Note that for $w \in W$ such that $w = \tau s_{i_1} ... s_{i_n}$ we have $T_w = T_\tau T_{i_1} ... T_{i_n}$.

We can now define the current generators of $\dot{\mathrm{U}}_q(\mathfrak{g})$ by using the braid group action as follows:

$$k_{i,+n}^+ = (q_i - q_i^{-1}) K_i \psi_{i,+n} \tag{1.2.29}$$

where $n \in \mathbb{N}^\times$, and

$$\psi_{i,r} = C^{-k/2} (q_i^{-2} E_i^+ T_{\omega_i}^k (K_i^{-1} E_i^-) - T_{\omega_i}^k (K_i^{-1} E_i^-) E_i^+) \tag{1.2.30}$$

The definition of $k_{i,+n}^+$ follows from using the automorphisms above and the fact that we have $\Omega T_i = T_i \Omega$.

We now move to defining the generators $x_{i,k}^\pm$:

$$x_{i,k}^\pm = \tilde{T}_{\omega_i}^{\mp k}(E_i^\pm) \tag{1.2.31}$$

for $i \in I$, $k \in \mathbb{Z}$.

The surjectivity part is checked directly on the relations of the algebra, whereas for the injectivity, I. Damiani shows it by restricting to the case $q = 1$ and showing that on one side one obtains the central extension of the loop algebra and on the other side the affine Lie algebra.

1.2.3 Representation Theory of Quantum Affine Algebras

As in the case of Lie algebras and quantum groups, the representation theory of quantum affine algebras is heavily based on that of $\dot{\mathrm{U}}_q(\mathfrak{sl}_2)$. This section is a review of the classification papers by V. Chari and A. Pressley. [CP91] Set $q \in \mathbb{C}^\times$ not a root of unity.

By looking at the presentation of $U_q(\mathfrak{sl}_2)$ and $\dot{\mathrm{U}}_q(\mathfrak{sl}_2)$, it is clear that $\dot{\mathrm{U}}_q(\mathfrak{sl}_2)$ has subalgebras which are isomorphic to $U_q(\mathfrak{sl}_2)$ given by the map:

$$E^+ \mapsto x_k^+, \quad E^- \mapsto C^- k x_{-k}^-, \quad K \mapsto K C^k \tag{1.2.32}$$

for all $k \in \mathbb{Z}$. The image of this map is the diagram subalgebras U^i, for $i = 0, 1$ corresponding to the nodes of the Dynkin diagram of $\dot{\mathrm{U}}_q(\mathfrak{sl}_2)$.

Since it will be useful down the line we remind you that the two irreducible represention $V_{n,\epsilon}$ of $U_q(\mathfrak{sl}_2)$ with basis $\{v_0, ... v_n\}$ are given by:

$$K.v_i = \epsilon q^{n-2i} v_i, \quad E^+.v_i = \epsilon [n-i+1]_q v_{i-1}, \quad E^-.v_i = [i+1]_q v_{i+1} \tag{1.2.33}$$

with $\epsilon = \pm 1$. We can also twist one choice of ϵ into the other. Therefore, one can stick to one value of ϵ. Moreover, when it comes to finite-dimensional irreducible representations of $\dot{\mathrm{U}}_q(\mathfrak{sl}_2)$, the central charge acts as 1. We now define the subalgebras $\dot{\mathrm{U}}_q(\mathfrak{sl}_2)^\pm$, and $\dot{\mathrm{U}}_q(\mathfrak{sl}_2)^0$ generated by $x_k^\pm$ and $\{k_{\pm n}, C\}$ respectively. It follows that

$$\dot{\mathrm{U}}_q(\mathfrak{sl}_2) = \dot{\mathrm{U}}_q(\mathfrak{sl}_2)^- \dot{\mathrm{U}}_q(\mathfrak{sl}_2)^0 \dot{\mathrm{U}}_q(\mathfrak{sl}_2)^+ \tag{1.2.34}$$

which allows us to say that a vector v in a representation of $\dot{\mathrm{U}}_q(\mathfrak{sl}_2)$ is a highest weight vector if v is annihilated by x_k^+ for all $k \in \mathbb{Z}$.

We can now present the first result in the following proposition

Proposition 1.2.2. *Every finite-dimensional irreducible representation of* $\dot{\mathrm{U}}_q(\mathfrak{sl}_2)$ *is highest weight.*

This is easily proven by contradiction. By letting V be as above, assume there are no non-zero vectors annihilated by any of the x_k^+. This means that for an eigenvector v of k_0^+, there exists an infinite sequence of vectors given by

$$v,\ x_{k_1}^+.v,\ x_{k_2}^+.v,\ x_{k_3}^+.v, ... \tag{1.2.35}$$

all non-zero and eigenvectors of k_0^+ with distinct eigenvalues. This makes them linearly independent. Clearly this contradicts the finite-dimensional aspect of V.

Furthermore, since C acts as 1 on V and we can always obtain any representation by twisting a type 1 representation, it suffices to consider representations of $\mathrm{U}_q(L(\mathfrak{sl}_2))$ where we remind you that $L(\mathfrak{sl}_2)$ is the loop algebra of $\mathfrak{sl}_2$.

A representation of $uqls$ is highest weight if it is generated by a vector v which is annihilated by x_k^+ for all k and such that:

$$k_{+n}^+.v = d_n^+ v, \quad k_{-n}^-.v = d_{-n}^+ v \tag{1.2.36}$$

for $n \in \mathbb{N}$, and $d_{\pm n}^{\pm} \in \mathbb{C}$. The collection $\underline{d} = \{d_n\}$ is the highest weight.

As it was the case for Lie algebras, Kac-Moody algebras, and quantum groups, we construct the universal highest weight module $M(\underline{d})$ by taking the quotient of $\mathrm{U}_q(L(\mathfrak{sl}_2))$ by the left ideal generated by

$$\{x_k^+, k_{+n}^+ - d_n^+.1, k_{-n}^- - d_{-n}^-.1, k \in \mathbb{Z}, n \in \mathbb{N}\} \tag{1.2.37}$$

and once more any representation of highest weight $\underline{d}$ is a quotientof $M(\underline{d})$ and there exists a unique irreducible quotient $L(\underline{d})$. Now for the main theorem:

Theorem 1.2.3. *The irreducible highest weight representation* $L(\underline{d})$ *is finite dimensional if and only if there exists a polynomial* P *with non-zero constant term such that*

$$\sum_{n=0}^{\infty} d_n^+ z^n = q^{deg(P)} \left(\frac{P(zq^{-2})}{P(z)} \right)_{|z| \ll 1}, \tag{1.2.38}$$

$$\sum_{n=0}^{\infty} d_{-n}^- z^{-n} = q^{deg(P)} \left(\frac{P(zq^{-2})}{P(z)} \right)_{|z^{-1}| \ll 1}, \tag{1.2.39}$$

The polynomial P, is unique once its constant coefficient is normalized to 1 and called Drinfel'd's polynomial. The proof of this theorem depends on two inputs. The "only if" part is proven by using some elements in the subalgebra $\dot{\mathrm{U}}_q(\mathfrak{sl}_2)^0$, and the "if" part makes use of the evaluation homomorphism defined by Jimbo as well as some properties of the tensor product of representations. Starting with the only if part, there exists $P_r, Q_r \in \dot{\mathrm{U}}_q(\mathfrak{sl}_2)^0$ given by:

$$P_r \equiv (-1)^r q^{r^2} (x_0^+)^{(r)} (x_1^-)^{-(r)} (k_0)^{-r} \mod \dot{\mathrm{U}}_q(\mathfrak{sl}_2) \dot{\mathrm{U}}_q(\mathfrak{sl}_2)^+ \tag{1.2.40}$$

$$Q_r \equiv (-1)^r q^{-r^2} (x_{-1}^+)^{(r)} (x_0^-)^{-(r)} (k_0)^r \mod \dot{\mathrm{U}}_q(\mathfrak{sl}_2)\dot{\mathrm{U}}_q(\mathfrak{sl}_2)^+ \tag{1.2.41}$$

$$P_r = \frac{q^{-r}}{(q-q^{-1})[r]_q} \sum_{j=0}^{r-1} k_{j+1}^+ P_{r-j-1} K^{-1}, \tag{1.2.42}$$

$$Q_r = \frac{-q^{r}}{(q-q^{-1})[r]_q} \sum_{j=0}^{r-1} k_{-j-1}^- Q_{r-j-1} K^{-1}, \tag{1.2.43}$$

$$(-1)^r q^{r(r-1)} (x_0^+)^{(}r-1)(x_1^-)^{(}r) \equiv -\sum_{j=0}^{r-1} x_{j+1}^- P_{r-j-1} K^{r-1} \mod \dot{\mathrm{U}}_q(\mathfrak{sl}_2)\dot{\mathrm{U}}_q(\mathfrak{sl}_2)^+ \tag{1.2.44}$$

$$(-1)^r q^{-r(r-1)} (x_{-1}^+)^{(}r-1)(x_0^-)^{-(r)} \equiv -\sum_{j=0}^{r-1} x_{-j}^- Q_{r-j-1} K^{-r+1} \mod \dot{\mathrm{U}}_q(\mathfrak{sl}_2)\dot{\mathrm{U}}_q(\mathfrak{sl}_2)^+). \tag{1.2.45}$$

where,

$$X^{(r)} = \frac{X^r}{[r]_q} \tag{1.2.46}$$

Then, by definig

$$\mathcal{P}(z) = \sum_{r=0}^{\infty} P_r z^r, \quad \mathcal{Q}(z) = \sum_{r=0}^{\infty} Q_r z^{-r} \tag{1.2.47}$$

we have

$$k^+(z) = K \frac{\mathcal{P}(zq^{-2})}{\mathcal{P}(z)}, \quad k^-(z) = K^{-1} \frac{\mathcal{Q}(zq^2)}{\mathcal{Q}(z)} \tag{1.2.48}$$

Assume now that $\dim \ L(\underline{d}) < \infty$ with highest weight $r \in \mathbb{Z}_+$ for the action of the $U_q(\mathfrak{sl}_2)$-subalgebra of $\dot{\mathrm{U}}_q(\mathfrak{sl}_2)$ associated to the 0th-node of the Dynkin diagram. From our previous results we know that for the highest weight vector v, we have :

$$K.v = q^r v \tag{1.2.49}$$

and it follows that the subrepresentation generated by this highest weight vector is an $(r+1)$ dimensional irreducible representation and in particular we have :

$$(x_0^-)^{r+1}.v = 0 \tag{1.2.50}$$

which gives

$$\mathcal{P}(z).v = P(z)v \tag{1.2.51}$$

as an immediate consequence. Then, due to the expression of $k^{\pm}(z)$ in terms of $\mathcal{P}(z)$ the only if part of the theorem follows.

For the remaining part of the theorem, we have to introduce the evaluation homomorphism ev_a.

For any $a \in \mathbb{C}^{\times}$, there is a homomorphism of algebras $\dot{\mathrm{U}}_q(\mathfrak{sl}_2) \to U_q(\mathfrak{sl}_2)$ such that:

$$ev_a(x_k^+) = q^{-k} a^k K^k E^+ \tag{1.2.52}$$

$$ev_a(x_k^-) = q^{-k} a^k E^- K^k \tag{1.2.53}$$

which allows us to deduce how the homomorphism maps the rest of the generators by using the algebra relations. Therefore, we can construct representations by pulling back representations of $U_q(\mathfrak{sl}_2)$ by the evaluation homomorphism. Then, we deduce the action of $x_k^{\pm}$ on $L_n(a)$ which is given by:

$$x_k^+.v = a^k q^{k(n-2i+1)}[n-i+1]_q v_{i-1} \tag{1.2.54}$$

$$x_k^+.v = a^k q^{k(n-2i-1)}[i+1]_q v_{i+1} \tag{1.2.55}$$

This is a highest weight representation with v_0 its highest weight vector and polynomial P given by:

$$P(z) = (1-q^{n-1}az)(1-q^{n-3}az)...(1-q^{-n+1}az) \tag{1.2.56}$$

Moreover, we clearly have that:

$$q^n \frac{P(q^{-2}z)}{P(z)} = q^n \frac{(1-q^{-n-1}az)}{(1-aq^{n-1}z)}. \tag{1.2.57}$$

Finally, as we promised, the final part to complete this proof is the properties of the weights corresponding tensor product of irreducible finite dimensional representations. For an irreducible tensor product $V \otimes W$ of two irreducible repressentation of $\mathrm{U}_q(L(\mathfrak{sl}_2))$, we have $P_{V\otimes W} = P_V P_W$. This is due to the group-like structure of the comultiplication of $k^+(z)$ and the fact that $V \otimes W$ is isomorphic to $W \otimes V$ when the tensor product is irreducible. The group-like structure of the coproduct means that a tensor product of highest weight vectors is a highest weight vector. Now take the tensor product

$$V = V_1(a_1) \otimes V_1(a_2) \otimes ... \otimes V_1(a_r) \tag{1.2.58}$$

where the a_i^{-1} are the roots of the Drinfel'd polynomial. Clearly, V contains a vector with weight q^r. This is the tensor product of all the highest weight vectors in each factor. This vector generates a subrepreentation V' which contains a maximal subrepresentation N. Then, the finite dimensional representation V'/N is irreducible and has $P(z) = \prod_{i=1}^r (1-a_i z)$ as the associated polynomial. This concludes the proof of the theorem.

Clearly, there exists a generalized version if this theorem for $\dot{\mathrm{U}}_q(\mathfrak{sl}_2)$.

Theorem 1.2.4. *Let* $(d_{i,r})$ *be a pair of* $I \times \mathbb{Z}$*-tuples of complex numbers. Then, the irreducible representation* $V(\underline{d})$ *of* $\dot{\mathrm{U}}_q(\mathfrak{sl}_2)$ *is finite dimensional if and only if there exists* $\underline{P} = (P_i)_{i\in I}$ *such that:*

$$\sum_{r=0}^{\infty} d_{i,r}^+ z^r = q^{deg\,(P_i)} \frac{P_i(q^{-2}z)}{P_i(z)} = \sum_{r=0}^{\infty} d_{i,-r}^- z^{-r} \tag{1.2.59}$$

Proof. Starting with the "if" part of the proof, we clearly have $k_{i,0}^{\pm} = K_i^{\pm 1}$. This means that we can set $\lambda(i) = deg\,(P_i)$ with $\lambda(i)$ the classical weight of the highest weight vector v_P of the highest weight representation $V(\underline{P})$. We also have that $V(\underline{P}) = \mathbb{C}v_P$ and

$$V(\underline{P}) = \bigoplus_{\alpha\in Q^+} V(\underline{P})_{\lambda-\alpha}. \tag{1.2.60}$$

This part of the proof boils down to the following two statements:

i. $V(\underline{P})_{\lambda-\alpha} \neq 0$ for finitely many $\alpha \in Q$.

ii. $V(\underline{P})_{\lambda-\alpha}$ is finite dimensional for all $\alpha \in Q$.

However, if we have that for a vector $v \in V(\underline{P})_\mu$, $V_i = U_i.v$, where U_i is the diagram subalgebra $U_{q_i}(\mathfrak{sl}_2)$,

i. would be a consequence of having the weights stable under the action of the Weyl group of the finite dimensional Lie algebra $\mathfrak{g}$ and that for any $\mu \in P$ such that $V(\underline{P})_\mu \neq 0$, $w(\mu) \in W.\{\alpha \in P^+ | \alpha \leq \lambda\}$. Clearly this also follows from the fact that there exists an $N > 0$ such that for $r > N$, $V(\underline{P})_{\lambda - r\alpha} = V(\underline{P})_{\lambda + r\alpha} = 0$ for $r > 3h + \lambda(i)$ where h is the ehight of the $\lambda - \mu$.

Going one step further, the last statement follows from the fact that $V(\underline{P})_{\lambda - r\alpha}$ is spanned by:

$$X_1^- x_{i_1,k_1}^- X_2^- x_{i_2,k_2}^- ... X_h^- x_{i_h,k_h}^- X_{h+1}^- v_P \tag{1.2.61}$$

for

$$\lambda - \mu = \alpha_{i_1} + \alpha_{i_2} + ... + \alpha_{i_h},$$

$$X_p^- = x_{i,l_{1,p}}^- x_{i,l_{2,p}}^- ... x_{i,l_{r_p,p}}^-,$$

and

$$r_1 + r_2 + ... + r_{h+1} = r.$$

This is actually straightforward because of the weak PBW theorem giving us $\dot{\mathrm{U}}_q(\mathfrak{g}) = \dot{\mathrm{U}}_q(\mathfrak{g})^+ . \dot{\mathrm{U}}_q(\mathfrak{g})^0 . \dot{\mathrm{U}}_q(\mathfrak{g})^-$ and by making use of the algebra relations between $x_{i,k}^-$ and $x_{i',k'}^-$.

When it comes to ii. by induction on on the height h of α, we have nothing to prove in case $h = 0$. For $h = 1$, $x_{i,k}^-.v_P$ is in the finite dimensional space $\tilde{U_i}.v_P$, where $\tilde{U_i}$ is the $\dot{\mathrm{U}}_q(\mathfrak{g})$-subalgebra generated by $\{x_{i,k}^\pm, k_{i,\pm n}^\pm\}$ which is finite dimensional due to the results of $\mathrm{U}_q(L(\mathfrak{sl}_2))$. Assume we have proven all the cases up to but not the one of height h. The weight space $V(\underline{P})_{\lambda - \alpha}$ for $\alpha = \alpha_{i_1} + \alpha_{i_2} + ... + \alpha_{i_h}$ is spanned by vectors of the form:

$$x_{i_1,k_1} x_{i_2,k_2} ... x_{i_h,k_h}.v_P \tag{1.2.62}$$

We can fix a set $\{i_1, ..., i_h\}$ and prove that the vectors above span a finite dimensional space. Now, by the induction hypothesis, there exists an $M \in \mathbb{N}$ such that for all $i \in \{i_1, ..., i_h\}$, $V(\underline{P})_{\lambda - \alpha + \alpha_i}$ is panned by vectors

$$x_{j_2,l_2} x_{j_2,l_2} ... x_{j_h,l_h}.v_P \tag{1.2.63}$$

with $|l_1|, |l_2|, ..., |l_h| \leq M$. Thus it suffices to prove that the space for $\alpha = \alpha_{i_1} + \alpha_{i_2} + ... + \alpha_{i_h}$ is contained in:

$$V = \sum_{k_2=-M}^{M} x_{i_2,k_2}^-.V(\underline{P})_{\lambda - \alpha + \alpha_{i_2}} + x_{i_1,0}^-.V(\underline{P})_{\lambda - \alpha + \alpha_{i_1}} \tag{1.2.64}$$

Clearly, any vector of the form $x_{i_1,k_1} x_{i_2,k_2} ... x_{i_h,k_h}.v_P$ can be written as a linear combination of

$$x_{i_2,k_2} x_{i_1,k_1} ... x_{i_h,k_h}.v_P \tag{1.2.65}$$

$$x_{i_2,k_2+1} x_{i_1,k_1-1} ... x_{i_h,k_h}.v_P \tag{1.2.66}$$

$$x_{i_1,k_1-1} x_{i_2,k_2+1} ... x_{i_h,k_h}.v_P \tag{1.2.67}$$

by using the algebra relations. The first two are clearly in V and the last one can be shown to be in V by an induction on k_1. This completes this part of the proof.

For the "only if" part of the proof, it is very similar to that of $\mathrm{U}_q(L(\mathfrak{sl}_2))$ case. In fact, all you have to do is define $P_{i,r}$, and $Q_{i,r}$ for each node i. The elements are in $\dot{\mathrm{U}}_q(\mathfrak{g})^0$ and are defined by: Starting with the only if part, there exists $P_r, Q_r \in \dot{\mathrm{U}}_q(\mathfrak{sl}_2)^0$ given by:

$$P_{i,r} \equiv (-1)^r q_i^{r^2} (x_{i,0}^+)^{(r)} (x_{i,1}^-)^{-(r)} \tag{1.2.68}$$

$$Q_{i,r} \equiv (-1)^r q_i^{-r^2} (x_{i,-1}^+)^{(r)} (x_{i,0}^-)^{-(r)} \tag{1.2.69}$$

$$P_r = \frac{q_i^{-r}}{(q_i - q_i^{-1})[r]_{q_i}} \sum_{j=0}^{r-1} k_{i,j+1}^+ P_{i,r-j-1} K_i^{-1}, \tag{1.2.70}$$

$$Q_r = \frac{-q_i^r}{(q_i - q_i^{-1})[r]_{q_i}} \sum_{j=0}^{r-1} k_{i,-j-1}^- Q_{i,r-j-1} K_i, \tag{1.2.71}$$

$$(-1)^r q^{r(r-1)_i} (x_{i,0}^+)^{(r-1)} (x_{i,1}^-)^{(r)} \equiv - \sum_{j=0}^{r-1} x_{i,j+1}^- P_{i,r-j-1} K_i^{r-1} \tag{1.2.72}$$

$$(-1)^r q^{-r(r-1)_i} (x_{i,-1}^+)^{(r-1)} (x_{i,0}^-)^{-(r)} \equiv - \sum_{j=0}^{r-1} x_{i,-j}^- Q_{i,r-j-1} K_i^{-r+1}. \tag{1.2.73}$$

The rest of the proof follows the same steps as of that of $\mathfrak{sl}_2$. □

1.2.4 q-Characters

When it comes to characters, the classical notion of characters does not offer much insight in the case of the quantum affine algebras. This prevents us from trying to understand the Grothendieck ring structure of finite dimensional representation. However, E. Frenkel and N. Reshetikhin in [FR98] introduced the idea of q-characters. Moreover, q-characters were a very useful tool for D. Hernandez and B. Leclerc, to show that there exists a cluster algebra structure on that Grothendieck ring.

Definition 1.2.5. Let $\mathcal{R}$ be the universal R-matrix satisfying the Yang-Baxter equation and let (V, π_V) be a finite-dimensional representation of $\dot{\mathrm{U}}_q(\mathfrak{g})$. Then, define the the following operator:

$$L_V = L_V(z) = (\pi_{V(z)}\mathrm{id})(\mathcal{R}). \tag{1.2.74}$$

This allows us to define the transfer matrix t_V as:

$$t_V = t_V(z) = Tr_V q^{2\rho} L_V(z) \tag{1.2.75}$$

where,

$$q^{2\rho} = \tilde{k_1^2} ... \tilde{k_n^2}. \tag{1.2.76}$$

The following proposition is crucial for defining χ_q and showing its properties as a character.

Proposition 1.2.6. *The linear map ν_q sending V in*$\mathrm{Rep}(\dot{\mathrm{U}}_q(\mathfrak{g}))$ *to $t_V(z) \in \dot{\mathrm{U}}_q(\mathfrak{b}_-)[[z]]$ is a $\mathbb{C}^\times$-equivariant ring homomorphism from* $\mathrm{Rep}(\dot{\mathrm{U}}_q(\mathfrak{g}))$ *to* $\dot{\mathrm{U}}_q(\mathfrak{b}_-)[[z]]$.

Now, we define the second map that will play an equally important role in giving us the q-character.

Definition 1.2.7. Let $\tilde{U}_q(\mathfrak{g})$ be the subalgebra of $\dot{\mathrm{U}}_q(\mathfrak{g})$ generated by $x_{i,n}^\pm, k_i, h_{i,n}$ for $i \in I$, $n \leq 0$. Now denote by $\mathbf{h_q}$ the restriction to $U_q(\mathfrak{b}_-)$ of the projection from $\tilde{U}_q(\mathfrak{g})$ to $\tilde{U}_q(\mathfrak{h})$.

Theorem 1.2.8. *The map* $\chi_q : \mathrm{Rep}(\dot{\mathrm{U}}_q(\mathfrak{g})) \to \tilde{U}_q(\mathfrak{h})[[z]]$ *given by the composition of* ν_q *and* $\mathbf{h_q}$ *is an injective ring homomorphism such that* $\chi_q : \mathrm{Rep}(\dot{\mathrm{U}}_q(\mathfrak{g})) \to \mathbb{Z}[Y_{i,a_i}]_{i \in I, a_i \in \mathbb{C}^\times} \subset \tilde{U}_q(\mathfrak{h})$.

1.3 Quantum Toroidal Algebras

1.3.1 Schur-Weyl Duality

In this section we review the main results that led to the Schur-Weyl duality theorem by M. Varagnolo and E. Vasserot in [VV96].

Definition 1.3.1. The toroidal Hecke algebra $\ddot{\mathbf{H}}'$ of type $\mathfrak{gl}_n$ is the unital associative algebra over $\mathbb{A} = \mathbb{C}[\mathbf{x}^{\pm 1}, \mathbf{y}^{\pm 1}, \mathbf{q}^{\pm 1}]$ with generators:

$$T_i^{\pm 1}, X_j^{\pm 1}, Y_j^{\pm 1}, \quad i \in [\![n-1]\!]^\times, \quad j \in [\![n]\!]^\times \tag{1.3.1}$$

subject to the following relations:

$$(T_i - q^2)(T_i + 1) = 0 \tag{1.3.2}$$

$$T_i T_i^{-1} = T_i^{-1} T_i = 0 \tag{1.3.3}$$

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} = 1 \tag{1.3.4}$$

$$T_i T_j = T_j T_i = 1 \quad |i - j| > 1 \tag{1.3.5}$$

$$X_0 Y_1 = \mathbf{x} Y_1 X_0, \qquad X_i X_j = X_i X_j, \qquad Y_i Y_j = Y_i Y_j, \tag{1.3.6}$$

$$X_j T_i = T_i X_j, \qquad Y_j T_i = T_i Y_j, \quad if j \neq i, i+1 \tag{1.3.7}$$

$$T_i X_i T_i = \mathbf{q}^2 X_{i+1}, \qquad T_i^{-1} Y_i T_i^{-1} = \mathbf{q}^{-2} Y_{i+1} \tag{1.3.8}$$

$$X_2 Y_1^{-1} X_2^{-1} Y_1 = \mathbf{q}^{-2} \mathbf{y} T_1^2. \tag{1.3.9}$$

where, $X_0 = X_1 X_2 ... X_n$.

Taking $\mathbf{x} = 1$ gives the double affine Hecke algebra.

Let $\dot{\mathbf{H}}'^{(1)}, \dot{\mathbf{H}}'^{(2)} \subset \ddot{\mathbf{H}}'$ be the subalgebras generated respectively by $T_i^{\pm 1}$, $Y_j^{\pm 1}$, and $T_i^{\pm 1}$, $X_j^{\pm 1}$ $(i \in [\![n-1]\!]^\times, \quad j \in [\![n]\!]^\times)$. These two subalgebras are siomorphic to the affine Hecke algebra.

Let $\mathbf{H}' \subset \ddot{\mathbf{H}}'$ be the subalgebra generated by $T_i^{\pm 1}$, $(i \in [\![n-1]\!]^\times)$. This is the subalgebra isomorphic to the Hecke algebra of type $\mathfrak{gl}_n$.

Now we define:

$$\ddot{\mathbf{H}} = \ddot{\mathbf{H}}' \otimes_{\mathbb{A}} \mathbb{C}_{x,y,q}, \qquad \ddot{\mathbf{H}} = \mathbf{H}' \otimes_{\mathbb{A}} \mathbb{C}_{x,y,q} \tag{1.3.10}$$

$$\dot{\mathbf{H}}'^{(1)} = \dot{\mathbf{H}}' \otimes_{\mathbb{A}} \mathbb{C}_{x,y,q}, \qquad \dot{\mathbf{H}}'^{(1)} = \mathbf{H}' \otimes_{\mathbb{A}} \mathbb{C}_{x,y,q} \tag{1.3.11}$$

Definition 1.3.2. Let $\dot{\mathrm{U}}_q(\dot{\mathfrak{sl}}_{n+1})$ be the associative $\mathbb{F} = \mathbb{C}[\mathbf{c}^{\pm 1}, \mathbf{d}^{\pm 1}, \mathbf{q}^{\pm 1}]$-algebra generated by the generators

$$\left\{ D, D^{-1}, C^{1/2}, C^{-1/2}, k_{i,n}^{+}, k_{i,-n}^{-}, x_{i,m}^{+}, x_{i,m}^{-} : i \in \dot{I}, m \in Z, n \in \mathbb{N} \right\}$$

subject to the following relations

$$C^{\pm 1/2} \text{ is central} \qquad C^{\pm 1/2}C^{\mp 1/2} = 1 \qquad D^{\pm 1}D^{\mp 1} = 1 \tag{1.3.12}$$

$$D\mathbf{k}_i^{\pm}(z)D^{-1} = \mathbf{k}_i^{\pm}(zq^{-1}) \qquad D\mathbf{x}_i^{\pm}(z)D^{-1} = \mathbf{x}_i^{\pm}(zq^{-1}) \tag{1.3.13}$$

$$\operatorname*{res}_{z_1,z_2} \frac{1}{z_1 z_2}\mathbf{k}_i^{\pm}(z_1)\mathbf{k}_i^{\mp}(z_2) = 1 \tag{1.3.14}$$

$$\mathbf{k}_i^{\pm}(z_1)\mathbf{k}_j^{\pm}(z_2) = \mathbf{k}_j^{\pm}(z_2)\mathbf{k}_i^{\pm}(z_1) \tag{1.3.15}$$

$$\mathbf{k}_i^{-}(z_1)\mathbf{k}_j^{+}(z_2) = G_{ij}^{-}(C^{-1}z_1/z_2)G_{ij}^{+}(Cz_1/z_2)\mathbf{k}_j^{+}(z_2)\mathbf{k}_i^{-}(z_1) \tag{1.3.16}$$

$$G_{ij}^{\mp}(C^{\mp 1/2}z_2/z_1)\mathbf{k}_i^{+}(z_1)\mathbf{x}_j^{\pm}(z_2) = \mathbf{x}_j^{\pm}(z_2)\mathbf{k}_i^{+}(z_1) \tag{1.3.17}$$

$$\mathbf{k}_i^{-}(z_1)\mathbf{x}_j^{\pm}(z_2) = G_{ij}^{\mp}(C^{\mp 1/2}z_1/z_2)\mathbf{x}_j^{\pm}(z_2)\mathbf{k}_i^{-}(z_1) \tag{1.3.18}$$

$$(z_1 - q^{\pm c_{ij}}z_2)\mathbf{x}_i^{\pm}(z_1)\mathbf{x}_j^{\pm}(z_2) = (z_1 q^{\pm c_{ij}} - z_2)\mathbf{x}_j^{\pm}(z_2)\mathbf{x}_i^{\pm}(z_1) \tag{1.3.19}$$

$$[\mathbf{x}_i^{+}(z_1), \mathbf{x}_j^{-}(z_2)] = \frac{\delta_{ij}}{q_i - q_i^{-1}}\left[\delta\left(\frac{z_1}{Cz_2}\right)\mathbf{k}_i^{+}(z_1 C^{-1/2}) - \delta\left(\frac{z_1 C}{z_2}\right)\mathbf{k}_i^{-}(z_2 C^{-1/2})\right] \tag{1.3.20}$$

$$\sum_{\sigma \in S_{1-a_{ij}}} \sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k}_{q_i} \mathbf{x}_i^{\pm}(z_{\sigma(1)}) \cdots \mathbf{x}_i^{\pm}(z_{\sigma(k)})\mathbf{x}_j^{\pm}(z)\mathbf{x}_i^{\pm}(z_{\sigma(k+1)}) \cdots \mathbf{x}_i^{\pm}(z_{\sigma(1-a_{ij})}) = 0 \tag{1.3.21}$$

where, for every $i \in \dot{I}$, we define the following $\dot{\mathrm{U}}_q(\dot{\mathfrak{sl}}_{n+1})[[z, z^{-1}]]$-valued formal distributions

$$\mathbf{x}_i^{\pm}(z) := \sum_{m \in \mathbb{Z}} x_{i,m}^{\pm} z^{-m} \in \dot{\mathrm{U}}_q(\dot{\mathfrak{sl}}_{n+1})[[z, z^{-1}]]\,; \tag{1.3.22}$$

$$\mathbf{k}_i^{\pm}(z) := \sum_{n \in \mathbb{N}} k_{i,\pm n}^{\pm} z^{\mp n} \in \dot{\mathrm{U}}_q(\dot{\mathfrak{sl}}_{n+1})[[z^{\mp 1}]]\,, \tag{1.3.23}$$

for every $i, j \in \dot{I}$, we define the following $\mathbb{F}$-valued formal power series

$$G_{ij}^{\pm}(z) := q_i^{\pm a_{ij}} + (q_i - q_i^{-1})[\pm a_{ij}]_{q_i} \sum_{m \in \mathbb{N}^{\times}} q_i^{\pm m a_{ij}} z^m \in \mathbb{F}[[z]] \tag{1.3.24}$$

and

$$\delta(z) := \sum_{m \in \mathbb{Z}} z^m \in \mathbb{F}[[z, z^{-1}]] \tag{1.3.25}$$

is an $\mathbb{F}$-valued formal distribution.

Similarly to the toroidal Hecke algebra, we define:

i) $\dot{\mathrm{U}}_q^{\prime(1)}, \dot{\mathrm{U}}_q^{\prime(2)} \subset \dot{\mathrm{U}}_q(\dot{\mathfrak{sl}}_{n+1})$ as the subalgebras of $\dot{\mathrm{U}}_q(\dot{\mathfrak{sl}}_{n+1})$ respectively generated by: $k_{i,n}^{+}, k_{i,-n}^{-}, x_{i,m}^{+}, x_{i,m}^{-}$, and $k_{i,0}^{+}, k_{i,0}^{-}, x_{i,0}^{+}, x_{i,0}^{-}$, $i \in \dot{I}$. These subalgebras are isomorphic to the quantum affine algebra of $\mathfrak{sl}_{n+1}$, one in Drinfel'd's current presentation and the other in the Drinfel'd Jimbo presentation.

ii) $U_q' \subset \dot{\mathrm{U}}_q(\dot{\mathfrak{sl}}_{n+1})$ be the subalgebra generated by $k_i^{+} = k_{i,0}^{+}, k_i^{-} = k_{i,0}^{-}, x_i^{+} = x_{i,0}^{+}, x_i^{-} = x_{i,0}^{-}$, $i \in I$. This subalgebra is ismorphic to the quantum group of $\mathfrak{sl}_{n+1}$.

Let $\mathbb{C}_{c,d,q} = \mathbb{F}/\mathcal{N}_{c,d,q}$ where $\mathcal{N}_{c,d,q}$ is the maximal ideal generated by $\mathbf{d}-d, \mathbf{c}-c, \mathbf{q}-q$. Then,

$$\ddot{U} = \dot{\mathrm{U}}_q(\dot{\mathfrak{sl}}_{n+1}) \otimes_{\mathbb{F}} \mathbb{C}_{c,d,q}, \qquad U = U'_q \otimes_{\mathbb{F}} \mathbb{C}_{c,d,q} \tag{1.3.26}$$

$$\dot{U}^{(1)} = \dot{U}_q^{\prime(1)} \otimes_{\mathbb{F}} \mathbb{C}_{c,d,q}, \qquad \dot{U}^{(2)} = \dot{U}_q^{\prime(2)} \otimes_{\mathbb{F}} \mathbb{C}_{c,d,q} \tag{1.3.27}$$

Definition 1.3.3. A module M is integrable if

$$M = \bigoplus_{\lambda \in \mathbb{Z}^n} M_\lambda, \qquad M_{\{\lambda_0,\lambda_1,\ldots\lambda_n\}} = \{m \in M | k_{i,0}^+ m = q^{\lambda_i} m\} \tag{1.3.28}$$

and $x_{i,0}^+, x_{i,0}^-$ are locally nilpotent on M.

Let $\mathbf{V}$ be the fundamental representation of U and $\mathbf{V}^{\otimes n}$ the left U-module induced by the coproduct as defined in chapter 1. This action of U on the module commutes with a left $\mathbf{H}$-action given by: $T_i = 1^{\otimes i-1} \otimes T \otimes 1^{\otimes n-i-1}$, where $T \in End(V^{\otimes 2})$ satisfies the following relations:

$$T(v_r \otimes v_s) = q^2 v_r \otimes v_s \qquad if r = s \tag{1.3.29}$$

$$T(v_r \otimes v_s) = q v_s \otimes v_r \qquad if r < s \tag{1.3.30}$$

$$T(v_r \otimes v_s) = q v_s \otimes v_r + (q^2-1) v_r \otimes v_s \qquad if r > s \tag{1.3.31}$$

Definition 1.3.4. We can define T'_i, $i \in [\![n+1^\times]\!]$ as the automorphism of $\dot{U}$ by setting:

$$T'_i(x_i^+) = -x_i^- k_i, \qquad T'_i(x_j^+) = \sum_{s=0}^{-a_{ij}} (-1)^{s-a_{ij}} q^{-s} (x_i^+)^{(-a_{ij}-s)} x_j^+ (x_i^+)^{(}s), \quad i \neq j \tag{1.3.32}$$

$$T'_i(x_i^-) = -k_i^{-1} x_i^+, \qquad T'_i(x_j^+) = \sum_{s=0}^{-a_{ij}} (-1)^{s-a_{ij}} q^{s} (x_i^-)^{(s)} x_j^+ (x_i^+)^{(-a_{ij}-s)}, \quad i \neq j \tag{1.3.33}$$

$$T'_i(k_j) = k_{s_i(j)} \tag{1.3.34}$$

where s_i is the transposition $(i\, i+1)$. Moreover, we can define the braid action on an integrable $\dot{U}$-module M' by setting:

$$T''_i(m') = \sum_{r-s+t=0} (-1)^{s+k} q^{s-rt} (x_i^+)^{(r)} (x_i^+)^{(s)} (x_i^+)^{(t)} m'. \tag{1.3.35}$$

and for all $m' \in M'$, and all $u \in \dot{U}$,

$$T''_i(um') = T'_i(u) T''_i(m') \tag{1.3.36}$$

Let M be a right $\dot{\mathbf{H}}$-module. By the previous statements, M is also a right $\mathbf{H}$- module. We consider the dual left U-module $M \otimes_{\mathbf{H}} V^{\otimes n}$. This module has the structure of a left $\dot{U}$-module given by:

$$x_{n+1}^+(m \otimes \mathbf{v}) = \sum_{j=1}^{n} m Y_j^{-1} \otimes x_\theta^-(\mathbf{v}), \qquad x_{n+1}^-(m \otimes \mathbf{v}) = \sum_{j=1}^{n} m Y_j \otimes x_\theta^+(\mathbf{v}) \tag{1.3.37}$$

$$k_{n+1}(m \otimes \mathbf{v}) = m \otimes (k_\theta^{-1})^{\otimes n}(\mathbf{v}). \tag{1.3.38}$$

where $x_\theta^{\pm}$, and $k_\theta \in End_{\mathbb{C}}(\mathbf{V})$. Similarly, let M be a right $\ddot{\mathbf{H}}$-module therefore, $M \otimes_{\mathbf{H}} V^{\otimes n}$ has the structure of a $\dot{U}$-module. Moreover, by introducing the map $\psi : M \otimes_{\mathbf{H}} V^{\otimes n} \to M \otimes_{\mathbf{H}} V^{\otimes n}$ given by:

$$\psi(m \otimes \mathbf{v_j}) = mX_1^{-\delta_{n+1,j_1}} ... X_1^{-\delta_{n+1,j_n}} v_{1+j_1} \otimes v_{1+j_2} \otimes ... \otimes v_{1+j_n} \tag{1.3.39}$$

with the condition: $v_{n+2} = v_1$, we can show that:

Proposition 1.3.5. *For $i \in \llbracket n+1 \rrbracket^{\times}$, we have the following:*

$$\begin{aligned} \psi^{-1}\mathbf{x}_i^+(z)\psi &= \mathbf{x}_{i-1}^+(q^{-1}dz), & \psi^{-2}\mathbf{x}_1^+(xz)\psi^2 &= \mathbf{x}_n^+(q^{n-1}d^{1-n}z), \\ \psi^{-1}\mathbf{x}_i^-(z)\psi &= \mathbf{x}_{i-1}^-(q^{-1}dz), & \psi^{-2}\mathbf{x}_1^-(xz)\psi^2 &= \mathbf{x}_n^-(q^{n-1}d^{1-n}z), \\ \psi^{-1}\mathbf{k}_i(z)\psi &= \mathbf{k}_{i-1}(q^{-1}dz), & \psi^{-2}\mathbf{k}_1(xz)\psi^2 &= \mathbf{k}_n(q^{n-1}d^{1-n}z). \end{aligned}$$

We can now state the theorems that lead to the duality theorem.

Theorem 1.3.6. *Suppose that $x = d^{-n-1}q^{n+1}$, and $c = y = 1$. Then for any right $\ddot{\mathbf{H}}$-module, the following formulas:*

$$\mathbf{x}_0^+(m \otimes \mathbf{v}) = \sum_{j=1}^{n} mX_j \otimes \mathbf{x}_{\theta,j}^-(\mathbf{v}), \qquad \mathbf{x}_0^-(m \otimes \mathbf{v}) = \sum_{j=1}^{n} mX_j^{-1} \otimes \mathbf{x}_{\theta,j}^+(\mathbf{v}) \tag{1.3.40}$$

$$\mathbf{k}_0(m \otimes \mathbf{v}) = m \otimes ({\mathbf{k}_\theta}^{-1})^{\otimes n}(\mathbf{v}) \tag{1.3.41}$$

give a left integrable $\ddot{U}$-module.

The proof of this theorem relies on the previous proposition.

Theorem 1.3.7. *Let M' be an integrable left $\ddot{U}$-module with trivial central charge and level n. There exists a $\ddot{\mathbf{H}}$-module M, such that $M' \cong M \otimes_H V^{\otimes n}$ as $\ddot{U}$-module.*

Finally, we give the duality theorem which stems from the previous two.

Theorem 1.3.8. *The functor $M \to M \times_H \mathbf{V}^{\otimes n}$ is an equivalence between the category of right $\ddot{\mathbf{H}}$-modules and the category of left integrable $\ddot{U}$-modules with trivial central charge and level n.*

1.3.2 A word on Quiver Quantum Toroidal Algebras

In their recent paper, G. Noshita and A. Watanabe [NW21] provide a presentation of the quantum toroidal algebra associated to a quiver Q. However, they do not provide a set of Serre relations as it remains an open question for that topic. When it comes to the other relations, their presentation is obtained by replacing the Cartan data by a set of rules on the quiver Q. Specifically, it is rules on the set of arrows and loops of the quiver that would help identify the data replacing the Cartan matrix elements. Moreover, they proceed to show that there exists a Hopf algebra structure on the quiver quantum toroidal algebra.

Chapter 2

On Double Quantum Affinization: Type $\mathfrak{a}_1$

2.1 Introduction

Let $\mathfrak{g}$ be a simple Lie algebra and denote by $\dot{\mathfrak{g}}$ the corresponding untwisted affine Kač-Moody algebra. Starting from $\mathfrak{g}$ and $\dot{\mathfrak{g}}$ or from their respective root systems, one can construct two a priori different algebras: on one hand, the quantum affine algebra $\mathrm{U}_q(\dot{\mathfrak{g}})$ is the standard Drinfel'd-Jimbo algebra associated with $\dot{\mathfrak{g}}$; whereas on the other hand, the quantum affinization $\dot{\mathrm{U}}_q(\mathfrak{g})$ of $\mathfrak{g}$, which we define as $\mathrm{U}_q(\dot{\mathfrak{g}})$ in its Drinfel'd current presentation, is associated with the simple finite root system of $\mathfrak{g}$. Now $\dot{\mathrm{U}}_q(\mathfrak{g})$ and $\mathrm{U}_q(\dot{\mathfrak{g}})$ are isomorphic by virtue of a theorem established by Damiani and Beck, [Dam93, Bec94], which can be regarded as a quantum version of the classic result that each affine Lie algebra is isomorphic to the corresponding untwisted affine Kač-Moody Lie algebra. The situation can be summarized by the following diagram

$$\begin{tikzcd}
\mathfrak{g} \arrow[r, "\text{Classical Affinization}"] \arrow[d, "\text{Quantum Affinization}"'] & \dot{\mathfrak{g}} \arrow[d, "\text{Quantization}"] \\
\dot{\mathrm{U}}_q(\mathfrak{g}) \arrow[r, "\sim", "\text{Damiani-Beck isom.}"'] & \mathrm{U}_q(\dot{\mathfrak{g}})
\end{tikzcd}$$

It turns out that quantum affinization still makes sense for the already affine Lie algebra $\dot{\mathfrak{g}}$, thus yielding a doubly affine quantum algebra known as the *quantum toroidal algebra* $\dot{\mathrm{U}}_q(\dot{\mathfrak{g}})$. These originally appeared in type $\mathfrak{a}_n$ in the work of Ginzburg, Kapranov and Vasserot, [GKV95]. Quantum toroidal algebras have received a lot of attention in the past – see [Her09] for a review – and are presently the subject of a renewed interest due to their relevance for integrable systems – see e.g. [FFJ+11, FJMM12, FJMM15] – and for 5 dimensional supersymmetric Yang-Mills theory and related AGT like correspondence – see [AKM+17]. From a more mathematical perspective, it is well known – see [VV96] – that they are the Frobenius-Schur duals of Cherednik's Doubly Affine Hecke Algebras (DAHA) – see [Che05, Mac03] for classic references on the latter.

The purpose of the present work is to reconsider quantum toroidal algebras as topological Hopf algebras. On the one hand, this is only natural since the existence of an algebraic comultiplication for quantum toroidal algebras is still essentially open to this date – although see [GNW17] for recent results on algebraic comultiplications for affine Yangians that may suggest the existence of similar results for quantum toroidal algebras – and only a topological coalgebra structure is provided by the so-called Drinfel'd current coproduct. On the other hand, the existence of a braid group action by bicontinuous algebra automorphisms, generalizing those in [DK00], provides us with a topological version of the Lusztig symmetries that prove pivotal in both Damiani's and Beck's proofs of Drinfel'd's current presentation. We may therefore expect, in that context, the existence of an alternative presentation for quantum toroidal algebras, in terms of double current generators. In the same spirit as Drinfel'd's current presentation, such a presentation could be regarded as defining the *double quantum affinization* $\ddot{\mathrm{U}}_q(\mathfrak{g})$ of $\mathfrak{g}$ and (a subalgebra $\ddot{\mathrm{U}}'_q(\mathfrak{a}_1)$ of) $\ddot{\mathrm{U}}_q(\mathfrak{g})$ should be isomorphic to (the completion of) $\dot{\mathrm{U}}_q(\dot{\mathfrak{g}})$ – see section 2.3. We therefore expect a diagram of the form

$$\begin{tikzcd}
\mathfrak{g} \arrow[rr, "\text{Classical Affinization}"] \arrow[d, "\text{Double Quantum Affinization}"'] & & \dot{\mathfrak{g}} \arrow[d, "\text{Quantum Affinization}"] \\
\ddot{\mathrm{U}}'_q(\mathfrak{g}) \arrow[d, "\text{Completion}"'] & & \dot{\mathrm{U}}_q(\dot{\mathfrak{g}}) \arrow[d, "\text{Completion}"] \\
\widehat{\ddot{\mathrm{U}}'_q(\mathfrak{g})} \arrow[rr, "\sim", "\text{Affine Damiani-Beck isom.}"'] & & \widehat{\dot{\mathrm{U}}_q(\dot{\mathfrak{g}})}
\end{tikzcd}$$

In the present paper we prove such results in the particular case where $\mathfrak{g}$ is of type $\mathfrak{a}_1$. It is fairly natural to conjecture that similar results hold for higher rank root systems, thus yielding

Conjecture 2.1.1. Every simple Lie algebra $\mathfrak{g}$ admits a (unique up to isomorphisms) double quantum affinization $\ddot{\mathrm{U}}_q(\mathfrak{g})$.

and

Conjecture 2.1.2. Every untwisted affine Kač-Moody Lie algebra $\dot{\mathfrak{g}}$ admits a (unique up to isomorphisms) double quantum affinization $\ddot{\mathrm{U}}_q(\dot{\mathfrak{g}})$.

Note that the latter would naturally provide a definition for the so far elusive triply affine quantum algebras. The latter are believed to play an important role in mathematical physics, as the conformal block side of an AGT type correspondence with 6-dimensional super Yang-Mills theories, [AKM+17].

In any case, $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)$ – and presumably other double quantum affinizations if any – admits a triangular decomposition $(\ddot{\mathrm{U}}_q^-(\mathfrak{a}_1), \ddot{\mathrm{U}}_q^0(\mathfrak{a}_1), \ddot{\mathrm{U}}_q^+(\mathfrak{a}_1))$. The latter naturally leads to an alternative notion of weight and highest weight modules that we shall refer to as t-weight and highest t-weight modules. Natural analogues of the finite dimensional modules over quantum affine algebras also appear, that we refer to as *weight-finite* modules – see section 2.3 for definitions. We actually expect that it will be possible to classify simple weight-finite modules over $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)$, by essentially classifying those simple $\ddot{\mathrm{U}}_q^0(\mathfrak{a}_1)$-modules that appear as their highest t-weight spaces – see section 2.3 for the corresponding discussion. This is the subject of ongoing work.

Quite remarkably, there exists an algebra homomorphism $f : \mathcal{E}_{q^{-4},q^2,q^2} \to \ddot{\mathrm{U}}_q^{0^+}(\mathfrak{a}_1)$, where $\ddot{\mathrm{U}}_q^{0^+}(\mathfrak{a}_1)$ is a closed subalgebra of $\ddot{\mathrm{U}}_q^0(\mathfrak{a}_1)$ and, for every q_1, q_2, q_3 such that $q_1 q_2 q_3 = 1$, $\mathcal{E}_{q_1,q_2,q_3}$ is the corresponding elliptic Hall algebra – see section 2.3. The latter was first defined by Miki in [Mik07] as a (q,γ)-analogue of the $W_{1+\infty}$ algebra. It reappeared later in [FFJ+11], as the quantum continuous $\mathfrak{gl}_\infty$ algebra. Schiffmann then identified it with the Hall algebra of the category of coherent sheaves on some elliptic curve whose Weil numbers are related to q_1, q_2, q_3, [Sch12]. More recently, it also appeared in [FJMM12] and in subsequent works by Feigin et al. as the quantum toroidal algebra associated with $\mathfrak{gl}_1$. As we shall see, it appears natural to make the following

Conjecture 2.1.3. $\ddot{\mathrm{U}}_q^{0^+}(\mathfrak{a}_1)$ is isomorphic to the completion of $\mathcal{E}_{q^{-4},q^2,q^2}$.

If it held true, the above conjecture would have many interesting implications. On one hand, in view of Schiffmann's results, it seems reasonable to expect that the double quantum affinization $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)$ admits a K-theoretic realization, in the spirit of Nakajima's quiver varieties realization of quantum affine algebras [Nak01], wherein the generators outside of the elliptic Hall algebras would be realized as correspondences. At the level of representation theory on the other hand, conjecture 2.1.3 would imply that the classification of the simple $\ddot{\mathrm{U}}_q^0(\mathfrak{a}_1)$-modules that appear as highest t-weight spaces of simple weight-finite $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)$-modules would almost entirely reduce to a classification of the corresponding subclass of simple modules over the elliptic Hall algebra. Again, we leave these questions for future work.

The paper is organized as follows. In section 2.2, we briefly review some well known facts about quantum toroidal algebras, including their definition and natural gradings. We endow them with a topology and construct the corresponding completion. On the latter, we construct a set of automorphisms, including affinized versions of Lusztig's symmetry. Analogues of these for simply laced untwisted affine $\dot{\mathfrak{a}}_{n\geq 2}$-types appeared in the work of Ding and Khoroshkin [DK00]. The $\dot{\mathfrak{a}}_1$ version we give here plays a crucial role in section 2.4 where we prove the main result of this paper. In section 2.3, we define the double quantum affinization of type $\mathfrak{a}_1$, $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)$. We prove that there exists an algebra homomorphism from the elliptic Hall algebra $\mathcal{E}_{q_1,q_2,q_3}$ to its subalgebra $\ddot{\mathrm{U}}_q^0(\mathfrak{a}_1)$. We also ellaborate on the consequences at the level of representation theory and introduce the notions of (highest) t-weights and of weight-finiteness. Finally, in section 2.4, we prove the affinized version of the Damiani-Beck isomorphism. The appendix contains a short review of formal distributions as relevant to the present work. This is already covered in the literature – see e.g. [Kac98] –, however, since our conventions slightly differ from the standard ones, we included it for the sake of clarity.

Notations and conventions

We let $\mathbb{N} = \{0, 1, \dots\}$ be the set of natural integers including 0. We denote by $\mathbb{N}^\times$ the set $\mathbb{N} - \{0\}$. For every $m \leq n \in \mathbb{N}$, we denote by $[\![m, n]\!] = \{m, m+1, \dots, n\}$. We also let $[\![n]\!] = [\![1, n]\!]$ for every $n \in \mathbb{N}$. For every finite subset $\Sigma \subset \mathbb{N}$ with $\mathrm{card}\, \Sigma = N$, any $n \leq N$ and any $m_1, \dots, m_n \in \mathbb{N}$ such that $m_1 + \dots + m_n = N$, we let $\mathsf{P}_\Sigma^{(m_1,\dots,m_n)}$ denote the set of ordered $(m_1, \dots, m_n)$ set n-partitions, i.e. any $\mathbf{A} = (A^{(1)}, \dots, A^{(n)}) \in \mathsf{P}_\Sigma^{(m_1,\dots,m_n)}$ is such that

i. for every $p \in [\![n]\!]$, $\mathrm{card}\, A^{(p)} = m_p$;

ii. for every $p \in [\![n]\!]$, $A^{(p)} = \{A_1^{(p)}, \dots, A_{m_p}^{(p)}\} \subset \Sigma$, with $A_1^{(p)} < \dots < A_{m_p}^{(p)}$;

iii. $A^{(1)} \sqcup \dots \sqcup A^{(n)} = \Sigma$.

We let $\mathrm{sign} : \mathbb{Z} \to \{-1, 0, 1\}$ be defined by setting, for any $n \in \mathbb{Z}$,

$$\mathrm{sign}(n) = \begin{cases} -1 & \text{if } n < 0; \\ 0 & \text{if } n = 0; \\ 1 & \text{if } n > 0. \end{cases}$$

We assume throughout that $\mathbb{K}$ is a field of characteristic 0 and we let $\mathbb{F} := \mathbb{K}(q)$ denote the field of rational functions over $\mathbb{K}$ in the formal variable q. As usual, we let $\mathbb{K}^\times = \mathbb{K} - \{0\}$ and $\mathbb{F}^\times = \mathbb{F} - \{0\}$. We make $\mathbb{F}$ a topological field by endowing it with the discrete topology.

For every $m, n \in \mathbb{N}$, we define the following elements of $\mathbb{F}$

$$[n]_q := \frac{q^n - q^{-n}}{q - q^{-1}}\,, \qquad [n]_q^! := \begin{cases} [n]_q [n-1]_q \cdots [1]_q & \text{if } n \in \mathbb{N}^\times; \\ 1 & \text{if } n = 0; \end{cases} \qquad \binom{n}{m}_q := \frac{[n]_q^!}{[m]_q^! [n-m]_q^!}\,. \tag{2.1.1}$$

We shall let

$$ {}_a[A,B]_b = aAB - bBA\,, $$

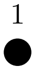
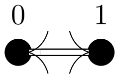
for any symbols a, b, A and B provided the r.h.s of the above equations makes sense. At some point we may need the following obvious identities

$$ [[A,B]_a, C]_b = [[A,C]_b, B]_a + [A,[B,C]]_{ab}\,, \tag{2.1.2} $$

$$ [_a[A,B], C]_b = {}_a[[A,C]_b, B] + {}_a[A,[B,C]]_b\,. \tag{2.1.3} $$

We refer to the Appendix for conventions and more details on formal distributions.

The Dynkin diagrams and correponding Cartan matrices in type $\mathfrak{a}_1$ and $\dot{\mathfrak{a}}_1$ are reminded in the following table.

Type	Dynkin diagram	Simple roots	Cartan matrix
$\mathfrak{a}_1$		$\Phi = \{\alpha_1\}$	(2)
$\dot{\mathfrak{a}}_1$		$\dot{\Phi} = \{\alpha_0, \alpha_1\}$	$\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$

2.2 The quantum toroidal algebra of type $\mathfrak{a}_1$ and its completion

2.2.1 Definition

Let $\dot{I} = \{0,1\}$ be the above labeling of the nodes of the Dynkin diagram of type $\dot{\mathfrak{a}}_1$ and let $\dot{\Phi} = \{\alpha_0, \alpha_1\}$ be a choice of simple roots for the corresponding root system. We denote by $(c_{ij})_{i,j=0,1}$ the entries of the associated Cartan matrix. Let $\dot{Q}^\pm = \mathbb{Z}^\pm\alpha_0 \oplus \mathbb{Z}^\pm\alpha_1$ and let $\dot{Q} = \mathbb{Z}\alpha_0 \oplus \mathbb{Z}\alpha_1$ be the type $\dot{\mathfrak{a}}_1$ root lattice.

Definition 2.2.1. The *quantum toroidal algebra* $\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)$ is the associative $\mathbb{F}$-algebra generated by the generators

$$ \left\{D, D^{-1}, C^{1/2}, C^{-1/2}, k^+_{i,n}, k^-_{i,-n}, x^+_{i,m}, x^-_{i,m} : i \in \dot{I}, m \in \mathbb{Z}, n \in \mathbb{N}\right\} $$

subject to the following relations

$$ C^{\pm 1/2} \text{ is central} \qquad C^{\pm 1/2}C^{\mp 1/2} = 1 \qquad D^{\pm 1}D^{\mp 1} = 1 \tag{2.2.1} $$

$$ D\mathbf{k}^\pm_i(z)D^{-1} = \mathbf{k}^\pm_i(zq^{-1}) \qquad D\mathbf{x}^\pm_i(z)D^{-1} = \mathbf{x}^\pm_i(zq^{-1}) \tag{2.2.2} $$

$$ \operatorname*{res}_{z_1,z_2} \frac{1}{z_1 z_2}\mathbf{k}^\pm_i(z_1)\mathbf{k}^\mp_i(z_2) = 1 \tag{2.2.3} $$

$$ \mathbf{k}^\pm_i(z_1)\mathbf{k}^\pm_j(z_2) = \mathbf{k}^\pm_j(z_2)\mathbf{k}^\pm_i(z_1) \tag{2.2.4} $$

$$ \mathbf{k}^-_i(z_1)\mathbf{k}^+_j(z_2) = G^-_{ij}(C^{-1}z_1/z_2)G^+_{ij}(Cz_1/z_2)\mathbf{k}^+_j(z_2)\mathbf{k}^-_i(z_1) \tag{2.2.5} $$

$$G_{ij}^{\mp}(C^{\mp 1/2}z_2/z_1)\mathbf{k}_i^{+}(z_1)\mathbf{x}_j^{\pm}(z_2) = \mathbf{x}_j^{\pm}(z_2)\mathbf{k}_i^{+}(z_1) \tag{2.2.6}$$

$$\mathbf{k}_i^{-}(z_1)\mathbf{x}_j^{\pm}(z_2) = G_{ij}^{\mp}(C^{\mp 1/2}z_1/z_2)\mathbf{x}_j^{\pm}(z_2)\mathbf{k}_i^{-}(z_1) \tag{2.2.7}$$

$$(z_1 - q^{\pm c_{ij}}z_2)\mathbf{x}_i^{\pm}(z_1)\mathbf{x}_j^{\pm}(z_2) = (z_1 q^{\pm c_{ij}} - z_2)\mathbf{x}_j^{\pm}(z_2)\mathbf{x}_i^{\pm}(z_1) \tag{2.2.8}$$

$$[\mathbf{x}_i^{+}(z_1), \mathbf{x}_j^{-}(z_2)] = \frac{\delta_{ij}}{q-q^{-1}}\left[\delta\left(\frac{z_1}{Cz_2}\right)\mathbf{k}_i^{+}(z_1 C^{-1/2}) - \delta\left(\frac{z_1 C}{z_2}\right)\mathbf{k}_i^{-}(z_2 C^{-1/2})\right] \tag{2.2.9}$$

$$\sum_{\sigma\in S_{1-c_{ij}}}\sum_{k=0}^{1-c_{ij}}(-1)^k\binom{1-c_{ij}}{k}_q \mathbf{x}_i^{\pm}(z_{\sigma(1)})\cdots\mathbf{x}_i^{\pm}(z_{\sigma(k)})\mathbf{x}_j^{\pm}(z)\mathbf{x}_i^{\pm}(z_{\sigma(k+1)})\cdots\mathbf{x}_i^{\pm}(z_{\sigma(1-c_{ij})}) = 0 \tag{2.2.10}$$

where, for every $i \in \dot{I}$, we define the following $\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)$-valued formal distributions

$$\mathbf{x}_i^{\pm}(z) := \sum_{m\in\mathbb{Z}} x_{i,m}^{\pm} z^{-m} \in \dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)[[z, z^{-1}]]\,; \tag{2.2.11}$$

$$\mathbf{k}_i^{\pm}(z) := \sum_{n\in\mathbb{N}} k_{i,\pm n}^{\pm} z^{\mp n} \in \dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)[[z^{\mp 1}]]\,, \tag{2.2.12}$$

for every $i, j \in \dot{I}$, we define the following $\mathbb{F}$-valued formal power series

$$G_{ij}^{\pm}(z) := q^{\pm c_{ij}} + (q - q^{-1})[\pm c_{ij}]_q \sum_{m\in\mathbb{N}^{\times}} q^{\pm m c_{ij}} z^m \in \mathbb{F}[[z]] \tag{2.2.13}$$

and

$$\delta(z) := \sum_{m\in\mathbb{Z}} z^m \in \mathbb{F}[[z, z^{-1}]] \tag{2.2.14}$$

is an $\mathbb{F}$-valued formal distribution.

Note that $G_{ij}^{\pm}(z)$ is invertible in $\mathbb{F}[[z]]$ with inverse $G_{ij}^{\mp}(z)$, i.e.

$$G_{ij}^{\pm}(z)G_{ij}^{\mp}(z) = 1\,, \tag{2.2.15}$$

and that it can be viewed as the power series expansion of a rational function of $(z_1, z_2) \in \mathbb{C}^2$ as $|z_2| \gg |z_1|$, which we shall denote as follows

$$G_{ij}^{\pm}(z_1/z_2) = \left(\frac{z_1 q^{\mp c_{ij}} - z_2}{z_1 - q^{\mp c_{ij}} z_2}\right)_{|z_2|\gg|z_1|}\,. \tag{2.2.16}$$

Observe furthermore that we have the following useful identity in $\mathbb{F}[[z, z^{-1}]]$

$$\frac{G_{ij}^{\pm}(z) - G_{ij}^{\mp}(z^{-1})}{q - q^{-1}} = [\pm c_{ij}]_q \delta\left(z q^{\pm c_{ij}}\right)\,. \tag{2.2.17}$$

Remark 2.2.2. In type $\mathfrak{a}_1$, $\dot{I} = \{0, 1\}$, $c_{ij} = 4\delta_{ij} - 2$ and we have an additional identity, namely $G_{10}^{\pm}(z) = G_{11}^{\mp}(z)$. We refer to section 5.1.3 of the Appendix for more identities involving the formal power series $G_{ij}^{\pm}(z)$.

$\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)$ is obviously a $\mathbb{Z}$-graded algebra, i.e. we have

$$\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1) = \bigoplus_{n\in\mathbb{Z}} \dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)_n\,, \qquad \text{where for all } n\in\mathbb{Z} \qquad \dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)_n := \{x\in\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1) : DxD^{-1} = q^n x\}\,. \tag{2.2.18}$$

It was proven in [Her05] to admit a triangular decomposition $(\dot{\mathrm{U}}_q^-(\dot{\mathfrak{a}}_1), \dot{\mathrm{U}}_q^0(\dot{\mathfrak{a}}_1), \dot{\mathrm{U}}_q^+(\dot{\mathfrak{a}}_1))$, where $\dot{\mathrm{U}}_q^\pm(\dot{\mathfrak{a}}_1)$ and $\dot{\mathrm{U}}_q^0(\dot{\mathfrak{a}}_1)$ are the subalgebras of $\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)$ respectively generated by $\left\{x^\pm_{i,m} : i\in\dot{I}, m\in\mathbb{Z}\right\}$ and

$$\left\{C^{1/2}, C^{-1/2}, D, D^{-1}, k^+_{i,m}, k^-_{i,m} : i\in\dot{I}, m\in\mathbb{Z}\right\}\,.$$

Observe that $\dot{\mathrm{U}}_q^\pm(\dot{\mathfrak{a}}_1)$ admits a natural gradation over $\dot{Q}^\pm$ that we shall denote by

$$\dot{\mathrm{U}}_q^\pm(\dot{\mathfrak{a}}_1) = \bigoplus_{\alpha\in\dot{Q}^\pm} \dot{\mathrm{U}}_q^\pm(\dot{\mathfrak{a}}_1)_\alpha\,. \tag{2.2.19}$$

Of course $\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)$ is graded over the root lattice $\dot{Q}$. We finally remark that the two Dynkin diagram subalgebras $\dot{\mathrm{U}}_q(\mathfrak{a}_1)^{(0)}$ and $\dot{\mathrm{U}}_q(\mathfrak{a}_1)^{(1)}$ of $\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)$ generated by

$$\left\{D, D^{-1}, C^{1/2}, C^{-1/2}, k^+_{i,n}, k^-_{i,-n}, x^+_{i,m}, x^-_{i,m} : m\in\mathbb{Z}, n\in\mathbb{N}\right\}\,,$$

with $i=0$ and $i=1$ respectively, are both isomorphic to $\dot{\mathrm{U}}_q(\mathfrak{a}_1)$, thus yielding two injective algebra homomorphisms $\iota^{(i)} : \dot{\mathrm{U}}_q(\mathfrak{a}_1) \hookrightarrow \dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)$.

2.2.2 Automorphisms of $\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)$

Proposition 2.2.3. *i. For every Dynkin diagram automorphism $\pi : \dot{I} \xrightarrow{\sim} \dot{I}$, there exists a unique $\mathbb{F}$-algebra automorphism $T_\pi \in \mathrm{Aut}(\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1))$ such that*

$$T_\pi(\mathbf{x}^\pm_i(z)) = \mathbf{x}^\pm_{\pi(i)}(z)\,, \qquad T_\pi(\mathbf{k}^\pm_i(z)) = \mathbf{k}^\pm_{\pi(i)}(z)\,, \qquad T_\pi(C^{1/2}) = C^{1/2}\,, \qquad T_\pi(D) = D\,. \tag{2.2.20}$$

ii. For every $i\in\dot{I}$, there exists a unique $\mathbb{F}$-algebra automorphism $T_{\omega_i^\vee} \in \mathrm{Aut}(\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1))$ such that

$$T_{\omega_i^\vee}(\mathbf{x}^\pm_j(z)) = z^{\pm\delta_{ij}}\mathbf{x}^\pm_j(z) \qquad T_{\omega_i^\vee}(\mathbf{k}^\pm_j(z)) = C^{\mp\delta_{ij}}\mathbf{k}^\pm_j(z) \qquad T_{\omega_i^\vee}(C^{1/2}) = C^{1/2} \qquad T_{\omega_i^\vee}(D) = D \tag{2.2.21}$$

iii. There exists a unique involutive $\mathbb{F}$-algebra anti-homomorphism $\eta \in \mathrm{Aut}(\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1))$ such that

$$\eta(\mathbf{x}^\pm_i(z)) = \mathbf{x}^\pm_i(1/z) \qquad \eta(\mathbf{k}^\pm_i(z)) = \mathbf{k}^\mp_i(1/z) \qquad \eta(C^{1/2}) = C^{1/2} \qquad \eta(D) = D \tag{2.2.22}$$

iv. There exists a unique involutive $\mathbb{K}$-algebra anti-homomorphism φ such that

$$\varphi(\mathbf{x}^\pm_i(z)) = \mathbf{x}^\mp_i(1/z) \qquad \varphi(\mathbf{k}^\pm_i(z)) = \mathbf{k}^\mp_i(1/z) \qquad \varphi(C^{1/2}) = C^{-1/2} \qquad \varphi(D) = D^{-1} \qquad \varphi(q) = q^{-1} \tag{2.2.23}$$

Remark 2.2.4. In the present case, the Dynkin diagram being that of type $\dot{\mathfrak{a}}_1$, $\dot{I} = \{0,1\}$ and the only nontrivial diagram automorphism is defined by setting $\pi(0)=1$ and $\pi(1)=0$.

Remark 2.2.5. Note that φ restricts as a non-trivial automorphism of the field $\mathbb{F}$ and that, as such, it yields e.g.

$$\varphi(G_{ij}^{\pm}(z)) = G_{ij}^{\mp}(z)\,. \tag{2.2.24}$$

2.2.3 The completions $\widehat{\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)}$ and $\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)^{\widehat{\otimes} m\geq 2}$

Let, for every $n \in \mathbb{N}$,

$$\Omega_n := \bigoplus_{\substack{r\geq n\\ s\geq n}} \dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)\cdot \dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)_{-r}\cdot \dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)\cdot \dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)_{s}\cdot \dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)\,.$$

Proposition 2.2.6. *The following hold true:*

i. For every $n \in \mathbb{N}$, Ω_n is a two-sided ideal of $\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)$;

ii. For every $n \in \mathbb{N}$, $\Omega_n \supseteq \Omega_{n+1}$;

iii. $\Omega_0 := \bigcup_{n\in\mathbb{N}} \Omega_n = \dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)$;

iv. $\bigcap_{n\in\mathbb{N}} \Omega_n = \{0\}$;

v. For every $m, n \in \mathbb{N}$, $\Omega_m + \Omega_n \subseteq \Omega_{\min(m,n)}$;

vi. For every $m, n \in \mathbb{N}$, $\Omega_m \cdot \Omega_n \subseteq \Omega_{\max(m,n)}$.

Proof. Points *i.* and *ii.* are obvious. As sets, it is clear that $\Omega_0 \subseteq \dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)$. Now, $1 \in \dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)_0$ and for every $x \in \dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)$, we can write $x = 1 \cdot x \cdot 1$ thus proving that $x \in \Omega_0$. Point *iii.* follows. Point *v.* is an easy consequence of point *ii.*. Point *vi.* is obvious given *i.*. So let us finally prove point *iv.*. In order to do so, it suffices to prove that for every $x \in \dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1) - \{0\}$, there exists a largest integer $\nu_x \in \mathbb{N}$ such that $x \in \Omega_{\nu_x}$; for then indeed $x \notin \Omega_{\nu_x+1}$, whereas obviously $0 \in \Omega_n$, for every $n \in \mathbb{N}$. Relations ((4.2.5) – (4.2.9)) respectively imply that, for every $i, j \in \dot{I}$, every $m \in \mathbb{N}$ and every $n \in \mathbb{N}^{\times}$,

$$k_{i,m}^{+}k_{j,-n}^{-} = k_{j,-n}^{-}k_{i,m}^{+} - (q^{c_{ij}} - q^{-c_{ij}})(C - C^{-1}) \sum_{p=1}^{\min(m,n)} \frac{q^{-pc_{ij}}C^{p} - q^{pc_{ij}}C^{-p}}{q^{-c_{ij}}C - q^{c_{ij}}C^{-1}} k_{j,-n+p}^{-}k_{i,m-p}^{+}\,,$$

$$k_{i,m}^{+}x_{j,-n}^{\pm} = q^{\pm c_{ij}}x_{j,-n}^{\pm}k_{i,m}^{+} + (q^{\pm c_{ij}} - q^{\mp c_{ij}})\sum_{p=0}^{m} C^{\mp p/2}q^{\pm pc_{ij}}x_{j,-n+p}^{\pm}k_{i,m-p}^{+}\,,$$

$$x_{i,m}^{\pm}k_{j,-n}^{-} = q^{\pm c_{ij}}k_{j,-n}^{-}x_{i,m}^{\pm} + (q^{\pm c_{ij}} - q^{\mp c_{ij}})\sum_{p=0}^{n} C^{\mp p/2}q^{\pm pc_{ij}}k_{j,-n+p}^{-}x_{i,m-p}^{\pm}\,,$$

$$\begin{aligned} x_{i,m}^{\pm}x_{j,-n}^{\pm} \;&=\; q^{\pm c_{ij}}x_{j,-n}^{\pm}x_{i,m}^{\pm} + (q^{\pm c_{ij}} - q^{\mp c_{ij}}) \sum_{p=0}^{\min(m,n)-1} q^{\pm pc_{ij}}x_{j,-n+p}^{\pm}x_{i,m-p}^{\pm} \\ &\quad - q^{\pm(\min(m,n)-1)c_{ij}}x_{j,\min(m,n)-n}^{\pm}x_{i,m-\min(m,n)}^{\pm} + q^{\pm\min(m,n)c_{ij}}x_{i,m-\min(m,n)}^{\pm}x_{j,\min(m,n)-n}^{\pm}\,, \end{aligned}$$

$$x_{i,m}^{\pm}x_{j,-n}^{\mp} = x_{j,-n}^{\mp}x_{i,m}^{\pm} \pm \frac{\delta_{ij}}{q - q^{-1}} \begin{cases} C^{\pm\frac{m+n}{2}}k_{i,m-n}^{+} & \text{if } m > n; \\ -C^{\mp\frac{m+n}{2}}k_{i,n-m}^{-} & \text{if } m < n; \\ \left[C^{\pm m}k_{i,0}^{+} - C^{\mp m}k_{i,0}^{-}\right] & \text{if } m = n. \end{cases}$$

Now let

$$B = \left\{ b_{\mathbf{a},\mathbf{m}} = \overrightarrow{\prod_{p \in [\![n]\!]}} \xi_{a_p, m_p} : n \in \mathbb{N}, \quad \mathbf{a} = (a_1, \dots, a_n) \in (\dot{\Phi} \sqcup -\dot{\Phi} \sqcup \dot{I})^n, \quad \mathbf{m} = (m_1, \dots, m_n) \in \mathbb{Z}^n \right\},$$

where, for every $(a, m) \in (\dot{\Phi} \sqcup -\dot{\Phi} \sqcup \dot{I}) \times \mathbb{Z}$,

$$\xi_{a,m} = \begin{cases} x^{\pm}_{i,m} & \text{if } a = \pm\alpha_i \in \pm\dot{\Phi},\ i \in \dot{I}; \\ k^{\pm}_{i,m} & \text{if } a = i \in \dot{I} \text{ and } m \in \mathbb{Z}^{\pm}. \end{cases}$$

If we omit $C^{\pm 1/2}$ and $D^{\pm 1}$ which are clearly irrelevant for the present discussion, B is obviously a spanning set for $\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)$. Making repeated use of the above relations, one then easily shows that, for every $n \in \mathbb{N}$, every $\mathbf{a} \in (\dot{\Phi} \sqcup -\dot{\Phi} \sqcup \dot{I})^n$ and every $\mathbf{m} \in \mathbb{Z}^n$,

$$b_{\mathbf{a},\mathbf{m}} - c_{\mathbf{a},\mathbf{m}} \overrightarrow{\prod_{\substack{p \in [\![n]\!] \\ m_p < 0}}} \xi_{a_p, m_p} \overrightarrow{\prod_{\substack{p \in [\![n]\!] \\ m_p \geq 0}}} \xi_{a_p, m_p} \in \Omega_{N(\mathbf{m})-1} - \Omega_{N(\mathbf{m})},$$

where $c_{\mathbf{a},\mathbf{m}} \in \mathbb{F}^{\times}$ and

$$N(\mathbf{m}) = \min \left(- \sum_{\substack{p \in [\![n]\!] \\ m_p < 0}} m_p, \sum_{\substack{p \in [\![n]\!] \\ m_p \geq 0}} m_p \right).$$

As a consequence, $\nu_{b_{\mathbf{a},\mathbf{m}}} \leq N(\mathbf{m})$, which concludes the proof. □

Similarly, making use of the natural $\mathbb{Z}$-grading of the tensor algebras $\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)^{\otimes m}$, $m \in \mathbb{N}^{\times}$, we let, for every $n \in \mathbb{N}$,

$$\Omega^{(m)}_n := \bigoplus_{\substack{r \geq n \\ s \geq n}} \dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)^{\otimes m} \cdot \left(\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)^{\otimes m} \right)_{-r} \cdot \dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)^{\otimes m} \cdot \left(\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)^{\otimes m} \right)_{s} \cdot \dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)^{\otimes m}.$$

One easily checks that for every $m \in \mathbb{N}^{\times}$, $\{\Omega^{(m)}_n : n \in \mathbb{N}\}$ has the same properties as the ones established in proposition 3.3.6 for $\{\Omega_n = \Omega^{(1)}_n : n \in \mathbb{N}\}$.

Definition-Proposition 2.2.7. We endow $\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)$ with the topology τ whose open sets are either $\emptyset$ or nonempty subsets $\mathcal{O} \subseteq \dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)$ such that for every $x \in \mathcal{O}$, $x + \Omega_n \subseteq \mathcal{O}$ for some $n \in \mathbb{N}$. Similarly, we endow each tensor power $\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)^{\otimes m \geq 2}$ with the topology induced by $\{\Omega^{(m)}_n : n \in \mathbb{N}\}$. These turn $\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)$ into a (separated) topological algebra. We then let $\widehat{\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)}$ denote its completion and we extend by continuity to $\widehat{\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)}$ all the (anti-)automorphisms defined over $\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)$ in the previous section. We eventually denote by $\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)^{\widehat{\otimes} m \geq 2}$ the corresponding completions of $\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)^{\otimes m \geq 2}$.

Proof. The addition is automatically continuous in the above defined topology of $\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)$. The continuity of the multiplication follows from point *vi.* of proposition 3.3.6. Point *iv.*, in turn, implies that $\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)$, as a topological space, is Hausdorff. The continuity of the unit map $\eta : \mathbb{F} \to \dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)$ is easily checked – remember that $\mathbb{F}$ is given the discrete topology. □

Remark 2.2.8. It is worth noting that the above topology is actually ultrametrizable. In the notations of the previous proof, let indeed, for every $x \in \dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)$,

$$\|x\| = \begin{cases} \exp(-\nu_x) & \text{if } x \in \dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1) - \{0\}; \\ 0 & \text{if } x = 0. \end{cases}$$

Since obviously $\nu_{x+y} \geq \min(\nu_x, \nu_y)$ for every $x, y \in \dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)$, the ultrametric inequality for the metric defined by $d(x,y) = \|x - y\|$ follows immediately as a consequence of the inequality $\|x + y\| \leq \max(\|x\|, \|y\|)$.

2.2.4 Continuous Lusztig automorphisms

Following [Mac03] we make the following

Definition 2.2.9. The affine braid group $\dot{\mathfrak{B}}$ of type $\dot{\mathfrak{a}}_1$ is generated by t and y subject to the relation $ty^{-1}t = y$.

The coweight lattice $P^\vee$ of $\dot{\mathfrak{a}}_1$ is an abelian group whose generators we shall denote as x_λ for every $\lambda \in P^\vee$. In particular, we shall write

$$x_\lambda x_\mu = x_\mu x_\lambda = x_{\lambda+\mu}\,, \tag{2.2.25}$$

assuming that $x_0 = 1$. There exists a unique group homomorphism $\dot{\mathfrak{B}} \to \mathrm{Aut}(P^\vee)$ defined by letting

$$t(x_\lambda) = x_{s_{\alpha_1}(\lambda)}\,, \qquad y(x_\lambda) = x_\lambda\,, \tag{2.2.26}$$

where s_{α_1} denotes the reflection in the simple root α_1, i.e. $s_{\alpha_1}(\lambda) = \lambda - (\alpha_1^\vee, \lambda)\alpha_1$. This action allows us to make the following

Definition 2.2.10. We let $\ddot{\mathfrak{B}} := \dot{\mathfrak{B}} \ltimes P^\vee$, i.e. $\ddot{\mathfrak{B}}$ is isomorphic to the group with generators t, y and $(x_\lambda)_{\lambda \in P^\vee}$ obeying the relations

$$ty^{-1}t = y\,, \qquad t x_\lambda t^{-1} = x_{s_{\alpha_1}(\lambda)}\,, \qquad x_\lambda y = y x_\lambda\,, \tag{2.2.27}$$

for every $\lambda \in P^\vee$.

We now define an action of $\ddot{\mathfrak{B}}$ on $\widehat{\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)}$ by bicontinuous algebra automorphisms, i.e. we construct a group homomorphism $\ddot{\mathfrak{B}} \to \mathrm{Aut}(\widehat{\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)})$. In order to do so, we first describe the image of the latter, following [DK00].

Proposition 2.2.11. *There exists a unique bicontinuous algebra automorphism $T \in \mathrm{Aut}(\widehat{\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)})$ such that*

$$T(C^{1/2}) = C^{1/2} \qquad T(D) = D \qquad T(\mathbf{k}_0^\pm(z)) = \mathbf{k}_0^\pm(zq^2)\mathbf{k}_1^\pm(z)\mathbf{k}_1^\pm(zq^2) \qquad T(\mathbf{k}_1^\pm(z)) = \mathbf{k}_1^\pm(z)^{-1} \tag{2.2.28}$$

$$T(\mathbf{x}_0^+(z)) = \frac{1}{[2]_q} \operatorname*{res}_{z_1,z_2} z_1^{-1} z_2^{-1} \left[\mathbf{x}_1^+(z_1), \left[\mathbf{x}_1^+(z_2), \mathbf{x}_0^+(zq^2)\right]_{G_{10}^-(z_2/zq^2)}\right]_{G_{11}^-(z_1/z_2)G_{10}^-(z_1/zq^2)} \tag{2.2.29}$$

$$T(\mathbf{x}_0^-(z)) = \frac{1}{[2]_q} \operatorname*{res}_{z_1,z_2} z_1^{-1} z_2^{-1} \left[\left[\mathbf{x}_0^-(zq^2), \mathbf{x}_1^-(z_1)\right]_{G_{10}^+(zq^2/z_1)}, \mathbf{x}_1^-(z_2)\right]_{G_{11}^+(z_1/z_2)G_{10}^+(zq^2/z_2)} \tag{2.2.30}$$

$$T(\mathbf{x}_1^+(z)) = -\mathbf{x}_1^-(C^{-1}z)\mathbf{k}_1^+(C^{-1/2}z)^{-1} \tag{2.2.31}$$

$$T(\mathbf{x}_1^-(z)) = -\mathbf{k}_1^-(C^{-1/2}z)^{-1}\mathbf{x}_1^+(C^{-1}z) \tag{2.2.32}$$

Proof. It suffices to check all the relations, which is cumbersome but straightforward. The inverse automorphism is given by

$$T^{-1}(C^{1/2}) = C^{1/2} \qquad T^{-1}(D) = D \qquad T^{-1}(\mathbf{k}_0^\pm(z)) = \mathbf{k}_0^\pm(zq^{-2})\mathbf{k}_1^\pm(z)\mathbf{k}_1^\pm(zq^{-2}) \qquad T^{-1}(\mathbf{k}_1^\pm(z)) = \mathbf{k}_1^\pm(z)^{-1} \tag{2.2.33}$$

$$T^{-1}(\mathbf{x}_0^+(z)) = \frac{1}{[2]_q} \operatorname*{res}_{z_1,z_2} z_1^{-1}z_2^{-1}\left[\left[\mathbf{x}_0^+(zq^{-2}), \mathbf{x}_1^+(z_1)\right]_{G_{10}^-(zq^{-2}/z_1)}, \mathbf{x}_1^+(z_2)\right]_{G_{11}^-(z_1/z_2)G_{10}^-(zq^{-2}/z_2)} \tag{2.2.34}$$

$$T^{-1}(\mathbf{x}_0^-(z)) = \frac{1}{[2]_q} \operatorname*{res}_{z_1,z_2} z_1^{-1}z_2^{-1}\left[\mathbf{x}_1^-(z_1), \left[\mathbf{x}_1^-(z_2), \mathbf{x}_0^-(zq^{-2})\right]_{G_{10}^+(z_2/zq^{-2})}\right]_{G_{11}^+(z_1/z_2)G_{10}^+(z_1/zq^{-2})} \tag{2.2.35}$$

$$T^{-1}(\mathbf{x}_1^+(z)) = -\mathbf{k}_1^-(C^{1/2}z)^{-1}\mathbf{x}_1^-(Cz) \tag{2.2.36}$$

$$T^{-1}(\mathbf{x}_1^-(z)) = -\mathbf{x}_1^+(Cz)\mathbf{k}_1^+(C^{1/2}z)^{-1} \tag{2.2.37}$$

□

Remark 2.2.12. Making use of the defining relations of $\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)$, one easily shows that indeed

$$\left[\mathbf{x}_1^+(z_1), \left[\mathbf{x}_1^+(z_2), \mathbf{x}_0^+(zq^2)\right]_{G_{10}^-(z_2/zq^2)}\right]_{G_{11}^-(z_1/z_2)G_{10}^-(z_1/zq^2)} = [2]_q\, \delta\left(\frac{z_1}{q^2z_2}\right)\delta\left(\frac{z_2}{z}\right)T\left(\mathbf{x}_0^+(z)\right)\,, \tag{2.2.38}$$

$$\left[\left[\mathbf{x}_0^-(zq^2), \mathbf{x}_1^-(z_1)\right]_{G_{10}^+(zq^2/z_1)}, \mathbf{x}_1^-(z_2)\right]_{G_{11}^+(z_1/z_2)G_{10}^+(zq^2/z_2)} = [2]_q\, \delta\left(\frac{z_1q^2}{z_2}\right)\delta\left(\frac{z_1}{z}\right)T(\mathbf{x}_0^-(z))\,. \tag{2.2.39}$$

The following is straightforward but will be useful.

Proposition 2.2.13. *We have*

i. $\varphi \circ T_\pi = T_\pi \circ \varphi$;

ii. $\varphi \circ T = T \circ \varphi$;

iii. $T^{-1} = \eta \circ T \circ \eta$.

We have finally,

Theorem 2.2.14. *The assignement*

$$t \mapsto T \qquad y \mapsto Y := T_\pi \circ T \qquad x_{\omega_i^\vee} \mapsto T_{\omega_i^\vee} \tag{2.2.40}$$

extends to a group homomorphism $\ddot{\mathfrak{B}} \to \mathrm{Aut}(\widehat{\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)})$.

Proof. This is a cumbersome but straightforward exercise that we leave to the reader. □

Remark 2.2.15. In [Mik99], Miki constructed an algebraic action by automorphisms of the extended elliptic braid group on $\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)$ which should not be confused with the topological action of $\ddot{\mathfrak{B}}$ on $\widehat{\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)}$ provided by the above theorem.

2.2.5 Topological Hopf algebra structure on $\widehat{\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)}$

Definition 2.2.16. We endow the topological $\mathbb{F}$-algebra $\widehat{\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)}$ with:

i. the comultiplication $\Delta : \widehat{\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)} \to \dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)\widehat{\otimes}\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)$ defined by

$$\Delta(C^{\pm 1/2}) = C^{\pm 1/2} \otimes C^{\pm 1/2}\,, \qquad \Delta(D^{\pm 1}) = D^{\pm 1} \otimes D^{\pm 1}\,, \tag{2.2.41}$$

$$\Delta(\mathbf{k}_i^{\pm}(z)) = \mathbf{k}_i^{\pm}(zC_{(2)}^{\pm 1/2}) \otimes \mathbf{k}_i^{\pm}(zC_{(1)}^{\mp 1/2})\,, \tag{2.2.42}$$

$$\Delta(\mathbf{x}_i^{+}(z)) = \mathbf{x}_i^{+}(z) \otimes 1 + \mathbf{k}_i^{-}(zC_{(1)}^{1/2})\widehat{\otimes}\mathbf{x}_i^{+}(zC_{(1)})\,, \tag{2.2.43}$$

$$\Delta(\mathbf{x}_i^{-}(z)) = \mathbf{x}_i^{-}(zC_{(2)})\widehat{\otimes}\mathbf{k}_i^{+}(zC_{(2)}^{1/2}) + 1 \otimes \mathbf{x}_i^{-}(z)\,, \tag{2.2.44}$$

where $C_{(1)}^{\pm 1/2} = C^{\pm 1/2} \otimes 1$ and $C_{(2)}^{\pm 1/2} = 1 \otimes C^{\pm 1/2}$;

ii. the counit $\varepsilon : \widehat{\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)} \to \mathbb{F}$, defined by $\varepsilon(D^{\pm 1}) = \varepsilon(C^{\pm 1/2}) = \varepsilon(\mathbf{k}_i^{\pm}(z)) = 1$, $\varepsilon(\mathbf{x}_i^{\pm}(z)) = 0$ and;

iii. the antipode $S : \widehat{\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)} \to \widehat{\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)}$, defined by $S(D^{\pm 1}) = D^{\mp 1}$, $S(C^{\pm 1/2}) = C^{\mp 1/2}$ and

$$S(\mathbf{k}_i^{\pm}(z)) = \mathbf{k}_i^{\pm}(z)^{-1}\,, \qquad S(\mathbf{x}_i^{+}(z)) = -\mathbf{k}_i^{-}(zC^{-1/2})^{-1}\mathbf{x}_i^{+}(zC^{-1})\,, \qquad S(\mathbf{x}_i^{-}(z)) = -\mathbf{x}_i^{-}(zC^{-1})\mathbf{k}_i^{+}(zC^{-1/2})^{-1}\,.$$

With these operations so defined and the topologies defined in section 3.3.2, $\widehat{\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)}$ is a topological Hopf algebra.

2.2.6 Non-degenerate Hopf algebra pairing

Define $\dot{\mathrm{U}}_q^{\geq}(\dot{\mathfrak{a}}_1)$ (resp. $\dot{\mathrm{U}}_q^{\leq}(\dot{\mathfrak{a}}_1)$) as the subalgebra of $\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)$ generated by $\left\{k_{i,-m}^{-}, x_{i,n}^{+} : i \in I, m \in \mathbb{N}, n \in \mathbb{Z}\right\}$ (resp. $\left\{k_{i,m}^{+}, x_{i,n}^{-} : i \in I, m \in \mathbb{N}, n \in \mathbb{Z}\right\}$). In view of the triangular decompositon of $\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)$ – see [Her05] – and of its defining relations, it is clear that $\dot{\mathrm{U}}_q^{\geq}(\dot{\mathfrak{a}}_1)$ (resp. $\dot{\mathrm{U}}_q^{\leq}(\dot{\mathfrak{a}}_1)$), as an $\mathbb{F}$-vector space, is spanned by

$$\begin{aligned}\Big\{x_{i_1,r_1}^{+}\cdots x_{i_m,r_m}^{+}k_{j_1,-s_1}^{-}\cdots k_{j_n,-s_n}^{-} \quad : \quad & m,n \in \mathbb{N}, ((i_1,r_1),\dots,(i_m,r_m)) \in (\dot{I}\times\mathbb{Z})^m \\ & ((j_1,s_1),\dots,(j_n,s_n)) \in (\dot{I}\times\mathbb{N})^n\Big\}\end{aligned} \tag{2.2.45}$$

$$\begin{aligned}\Big(\text{resp.} \quad \Big\{x_{i_1,r_1}^{-}\cdots x_{i_m,r_m}^{-}k_{j_1,s_1}^{+}\cdots k_{j_n,s_n}^{+} \quad : \quad & m,n \in \mathbb{N}, ((i_1,r_1),\dots,(i_m,r_m)) \in (\dot{I}\times\mathbb{Z})^m, \\ & ((j_1,s_1),\dots,(j_n,s_n)) \in (\dot{I}\times\mathbb{N})^n\Big\}\Big)\,.\end{aligned} \tag{2.2.46}$$

Proposition 2.2.17. *There exists a unique non-degenerate Hopf algebra pairing* $\langle , \rangle : \dot{\mathrm{U}}_q^{\geq}(\dot{\mathfrak{a}}_1) \times \dot{\mathrm{U}}_q^{\leq}(\dot{\mathfrak{a}}_1) \to \mathbb{F}$, *defined by setting*

$$\left\langle \mathbf{x}_i^{+}(z), \mathbf{x}_j^{-}(v) \right\rangle = \frac{\delta_{ij}}{q-q^{-1}}\delta\left(\frac{z}{v}\right)\,, \tag{2.2.47}$$

$$\left\langle \mathbf{k}_i^{-}(z), \mathbf{k}_j^{+}(v) \right\rangle = G_{ij}^{-}\left(\frac{z}{v}\right)\,, \tag{2.2.48}$$

$$\left\langle \mathbf{k}_i^{-}(z), \mathbf{x}_j^{-}(v) \right\rangle = \left\langle \mathbf{x}_i^{+}(z), \mathbf{k}_j^{+}(v) \right\rangle = 0\,. \tag{2.2.49}$$

By definition, it is such that, for every $a, b \in \dot{\mathrm{U}}_q^{\geq}(\dot{\mathfrak{a}}_1)$ and every $x, y \in \dot{\mathrm{U}}_q^{\leq}(\dot{\mathfrak{a}}_1)$,

$$\langle a, xy \rangle = \sum \left\langle a_{(1)}, x \right\rangle \left\langle a_{(2)}, y \right\rangle ,$$

$$\langle ab, x \rangle = \sum \left\langle a, x_{(2)} \right\rangle \left\langle b, x_{(1)} \right\rangle ,$$

$$\langle a, 1 \rangle = \varepsilon_{\geq}(a) \qquad \langle 1, x \rangle = \varepsilon_{\leq}(x) ,$$

where we have set $\varepsilon_{\leq} = \varepsilon_{|\dot{\mathrm{U}}_q^{\leq}(\dot{\mathfrak{a}}_1)}$, $\varepsilon_{\geq} = \varepsilon_{|\dot{\mathrm{U}}_q^{\geq}(\dot{\mathfrak{a}}_1)}$ and we have made use of Sweedler's notation for the comultiplication

$$\Delta(x) = \sum x_{(1)} \widehat{\otimes} x_{(2)} .$$

Proof. A proof can be found in [Neg13]. □

Before we can establish the continuity of the above defined pairing, we need the following

Lemma 2.2.18. *For every $m_+, m_-, n_+, n_- \in \mathbb{N}$, $(i_1^{\pm}, \dots, i_{m_{\pm}}^{\pm}) \in \dot{I}^{m_{\pm}}$ and every $(j_1^{\pm}, \dots, j_{n_{\pm}}^{\pm}) \in \dot{I}^{n_{\pm}}$, we have*

$$\left\langle \mathbf{x}^+_{i_1^+}(u_1) \cdots \mathbf{x}^+_{i_{m_+}^+}(u_{m_+}) \mathbf{k}^-_{j_1^+}(v_1) \cdots \mathbf{k}^-_{j_{n_+}^+}(v_{n_+}), \mathbf{x}^-_{i_1^-}(w_1) \cdots \mathbf{x}^-_{i_{m_-}^-}(w_{m_-}) \mathbf{k}^+_{j_1^-}(z_1) \cdots \mathbf{k}^+_{j_{n_-}^-}(z_{n_-}) \right\rangle$$
$$= \delta_{m_+, m_-} \left(\prod_{\substack{r \in [\![n_+]\!] \\ s \in [\![n_-]\!]}} G^-_{j_r^+, j_s^-}\left(\frac{v_r}{z_s}\right) \right) \sum_{\sigma \in S_{m_+}} \left(\prod_{\substack{1 \leq r < s \leq m_+ \\ \sigma(r) > \sigma(s)}} G^-_{i_r^+, i_s^+}\left(\frac{u_r}{u_s}\right) \right) \prod_{t \in [\![m_+]\!]} \frac{\delta_{i_t^+, i_{\sigma(t)}^-}}{q - q^{-1}} \delta\left(\frac{w_{\sigma(t)}}{u_t}\right) . \tag{2.2.50}$$

Proof. One easily proves by recursion the results for $n_+ = n_- = 0$ and $m_+ = m_- = 0$, respectively. The general case then follows by a straightforward calculation. □

It follows that – remember $\mathbb{F}$ is given the discrete topology –

Corollary 2.2.19. *The Hopf algebra pairing $\langle , \rangle$ is (separately) continuous.*

Proof. It suffices to prove that for every $x \in \dot{\mathrm{U}}_q^{\geq}(\dot{\mathfrak{a}}_1)$ there exists an $m \in \mathbb{N}$ such that, for every $n \geq m$

$$\left\langle x, \Omega_n \cap \dot{\mathrm{U}}_q^{\leq}(\dot{\mathfrak{a}}_1) \right\rangle = \{0\} .$$

In order to prove the latter, it suffices to prove it over the spanning sets of (2.2.45) and (2.2.46). Now this easily follows by inspection, making use of lemma 2.2.18 and of the fact that, for any $y \in \dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1) - \{0\}$, there exists $\nu_y \in \mathbb{N}$ such that $y \notin \Omega_{\nu_y + 1}$ – see proof of proposition 3.3.6. □

We can now extend $\langle , \rangle$ from $\dot{\mathrm{U}}_q^{\geq}(\dot{\mathfrak{a}}_1) \times \dot{\mathrm{U}}_q^{\leq}(\dot{\mathfrak{a}}_1)$ to $\dot{\mathrm{U}}_q^{\geq}(\dot{\mathfrak{a}}_1) \times \widehat{\dot{\mathrm{U}}_q^{\leq}(\dot{\mathfrak{a}}_1)}$ by continuity. Importantly, we have

Proposition 2.2.20. *The extended pairing $\langle , \rangle : \dot{\mathrm{U}}_q^{\geq}(\dot{\mathfrak{a}}_1) \times \widehat{\dot{\mathrm{U}}_q^{\leq}(\dot{\mathfrak{a}}_1)} \to \mathbb{F}$ is non-degenerate in the sense that, if for every $x \in \dot{\mathrm{U}}_q^{\geq}(\dot{\mathfrak{a}}_1)$, $\langle x, y \rangle = 0$ for some $y \in \widehat{\dot{\mathrm{U}}_q^{\leq}(\dot{\mathfrak{a}}_1)}$, then $y = 0$.*

Proof. Let $\{\mathcal{O}_n : n \in \mathbb{N}\}$ be any neighbourhood basis at $0 \in \mathbb{F}$ for the discrete topology on $\mathbb{F}$. Then, let for every $n \in \mathbb{N}$,

$$A_n := \left\langle \dot{\mathrm{U}}_q^{\geq}(\dot{\mathfrak{a}}_1), - \right\rangle^{-1} (\mathcal{O}_n) = \left\{ y \in \dot{\mathrm{U}}_q^{\leq}(\dot{\mathfrak{a}}_1) : \forall x \in \dot{\mathrm{U}}_q^{\geq}(\dot{\mathfrak{a}}_1) \qquad \langle x, y \rangle \in \mathcal{O}_n \right\} .$$

We clearly have, for every $n \in \mathbb{N}$, $\{0\} \subseteq A_n \subseteq \dot{\mathrm{U}}_q^{\leq}(\dot{\mathfrak{a}}_1)$ and $A_n \supseteq A_{n+1}$. The non-degeneracy of the pairing further implies that

$$\bigcap_{n \in \mathbb{N}} A_n = \{0\} .$$

As a consequence, for every $n \in \mathbb{N}$ and every $y \in A_n - \{0\}$, there exists an $N \in \mathbb{N}$ such that for every $m \geq N$, $y \notin A_m$. Now, given $n_1 \in \mathbb{N}$, let $\mu(n_1) \in \mathbb{N}$ be the largest integer such that $A_{n_1} \subseteq \Omega_{\mu(n_1)}$. By the previous discussion, for every point $y \in A_{n_1} - \Omega_{\mu(n_1)+1}$, there exists (a smallest) $n_2 \in \mathbb{N}$ such that for every $m \geq n_2$, $y \notin A_m$. Hence, for every $m \geq n_2$, $A_m \subseteq \Omega_{\mu(n_1)+1}$ and we conclude that $\mu(n) = \mu(n_1)$ for every $n \in [\![n_1, n_2 - 1]\!]$, whereas $\mu(n_2) = \mu(n_1) + 1$. By induction, it follows that $\mu : \mathbb{N} \to \mathbb{N}$ so defined is increasing and that, as a consequence, $\lim_{n \to +\infty} \mu(n) = +\infty$. We have therefore proven that, for every $n \in \mathbb{N}$,

$$\forall x \in \dot{\mathrm{U}}_q^{\geq}(\dot{\mathfrak{a}}_1) \qquad \langle x, y \rangle \in \mathcal{O}_n \quad \Rightarrow \quad y \in \Omega_{\mu(n)} . \tag{2.2.51}$$

If we finally let $(y_n)_{n \in \mathbb{N}} \in \dot{\mathrm{U}}_q^{\leq}(\dot{\mathfrak{a}}_1)^{\mathbb{N}}$ be any Cauchy sequence that does not converge to 0, the proposition is obviously equivalent to claiming that there exists an $x \in \dot{\mathrm{U}}_q^{\geq}(\dot{\mathfrak{a}}_1)$ such that

$$\lim_{n \to +\infty} \langle x, y_n \rangle \neq 0 .$$

Indeed, since $(y_n)_{n \in \mathbb{N}}$ does not converge to 0, there exist $m \in \mathbb{N}$ such that for every $N \in \mathbb{N}$, $y_n \notin \Omega_m$ for some $n \geq N$. We can therefore extract a subsequence $(y_{n_k})_{k \in \mathbb{N}}$ such that $y_{n_k} \notin \Omega_m$ for every $k \in \mathbb{N}$. The contrapositive of (2.2.51) then implies that there exists $(x_k)_{k \in \mathbb{N}} \in \dot{\mathrm{U}}_q^{\geq}(\dot{\mathfrak{a}}_1)^{\mathbb{N}}$ such that, for every $k \in \mathbb{N}$,

$$\langle x_k, y_{n_k} \rangle \notin \mathcal{O}_{\nu(m)}$$

where $\nu(m) = \min\{n \in \mathbb{N} : \mu(n) = m\}$. But since $(y_n)_{n \in \mathbb{N}}$ is Cauchy, so is $(y_{n_k})_{k \in \mathbb{N}}$ and, upon taking $k, l \in \mathbb{N}$ large enough, we can make $\langle x_k, y_{n_l} - y_{n_k} \rangle$ arbitrary small. This eventually concludes the proof. ☐

2.3 Double quantum affinization in type $\mathfrak{a}_1$

We now define and study the main object of interest in this paper; the double quantum affinization in type $\mathfrak{a}_1$, $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)$. We let $I = \{1\}$ be the labeling of the unique node of the type $\mathfrak{a}_1$ Dynkin diagram and we let $Q^{\pm} = \mathbb{Z}^{\pm}\alpha_1$. We denote by $Q = \mathbb{Z}\alpha_1$ the type $\mathfrak{a}_1$ root lattice.

2.3.1 Definition of $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)$

Definition 2.3.1. The *double quantum affinization* $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)$ of type $\mathfrak{a}_1$ is defined as the $\mathbb{F}$-algebra generated by

$$\{\mathsf{D}_1, \mathsf{D}_1^{-1}, \mathsf{D}_2, \mathsf{D}_2^{-1}, \mathsf{C}^{1/2}, \mathsf{C}^{-1/2}, \mathsf{c}_m^+, \mathsf{c}_{-m}^-, \mathsf{K}_{1,0,m}^+, \mathsf{K}_{1,0,-m}^-, \mathsf{K}_{1,n,r}^+, \mathsf{K}_{1,-n,r}^-, \mathsf{X}_{1,r,s}^+, \mathsf{X}_{1,r,s}^- : m \in \mathbb{N}, n \in \mathbb{N}^\times, r, s \in \mathbb{Z}\}$$

subject to the relations

$$\mathsf{C}^{\pm 1/2} \text{ and } \mathbf{c}^{\pm}(z) \text{ are central} \tag{2.3.1}$$

$$\operatorname*{res}_{v,w} \frac{1}{vw} \mathbf{c}^{\pm}(v) \mathbf{c}^{\mp}(w) = 1 , \tag{2.3.2}$$

$$\mathsf{D}_1^{\pm 1}\mathsf{D}_1^{\mp 1} = 1 \qquad \mathsf{D}_2^{\pm 1}\mathsf{D}_2^{\mp 1} = 1 \qquad \mathsf{D}_1\mathsf{D}_2 = \mathsf{D}_2\mathsf{D}_1 \tag{2.3.3}$$

$$\mathsf{D}_1\boldsymbol{\mathsf{K}}^{\pm}_{1,\pm m}(z)\mathsf{D}_1^{-1} = q^{\pm m}\boldsymbol{\mathsf{K}}^{\pm}_{1,\pm m}(z) \qquad \mathsf{D}_1\boldsymbol{\mathsf{X}}^{\pm}_{1,r}(z)\mathsf{D}_1^{-1} = q^{r}\boldsymbol{\mathsf{X}}^{\pm}_{1,r}(z)\,, \tag{2.3.4}$$

$$\mathsf{D}_2\boldsymbol{\mathsf{K}}^{\pm}_{1,\pm m}(z)\mathsf{D}_2^{-1} = \boldsymbol{\mathsf{K}}^{\pm}_{1,\pm m}(zq^{-1}) \qquad \mathsf{D}_2\boldsymbol{\mathsf{X}}^{\pm}_{1,r}(z)\mathsf{D}_2^{-1} = \boldsymbol{\mathsf{X}}^{\pm}_{1,r}(zq^{-1})\,, \tag{2.3.5}$$

$$\operatorname*{res}_{v,w}\frac{1}{vw}\boldsymbol{\mathsf{K}}^{\pm}_{1,0}(v)\boldsymbol{\mathsf{K}}^{\mp}_{1,0}(w) = 1\,, \tag{2.3.6}$$

$$(v - q^{\pm 2}z)(v - q^{2(m-n\mp 1)}z)\boldsymbol{\mathsf{K}}^{\pm}_{1,\pm m}(v)\boldsymbol{\mathsf{K}}^{\pm}_{1,\pm n}(z) = (vq^{\pm 2} - z)(vq^{\mp 2} - q^{2(m-n)}z)\boldsymbol{\mathsf{K}}^{\pm}_{1,\pm n}(z)\boldsymbol{\mathsf{K}}^{\pm}_{1,\pm m}(v)\,, \tag{2.3.7}$$

$$(\mathsf{C}q^{2(1-m)}v - w)(q^{2(n-1)}v - \mathsf{C}w)\boldsymbol{\mathsf{K}}^{+}_{1,m}(v)\boldsymbol{\mathsf{K}}^{-}_{1,-n}(w) = (\mathsf{C}q^{-2m}v - q^2w)(q^{2n}v - \mathsf{C}q^{-2}w)\boldsymbol{\mathsf{K}}^{-}_{1,-n}(w)\boldsymbol{\mathsf{K}}^{+}_{1,m}(v)\,, \tag{2.3.8}$$

$$(v - q^{\pm 2}z)\boldsymbol{\mathsf{K}}^{\pm}_{1,\pm m}(v)\boldsymbol{\mathsf{X}}^{\pm}_{1,r}(z) = (q^{\pm 2}v - z)\boldsymbol{\mathsf{X}}^{\pm}_{1,r}(z)\boldsymbol{\mathsf{K}}^{\pm}_{1,\pm m}(v)\,, \tag{2.3.9}$$

$$(\mathsf{C}v - q^{2(m\mp 1)}z)\boldsymbol{\mathsf{K}}^{\pm}_{1,\pm m}(v)\boldsymbol{\mathsf{X}}^{\mp}_{1,r}(z) = (\mathsf{C}q^{\mp 2}v - q^{2m}z)\boldsymbol{\mathsf{X}}^{\mp}_{1,r}(z)\boldsymbol{\mathsf{K}}^{\pm}_{1,\pm m}(v)\,, \tag{2.3.10}$$

$$(v - q^{\pm 2}w)\boldsymbol{\mathsf{X}}^{\pm}_{1,r}(v)\boldsymbol{\mathsf{X}}^{\pm}_{1,s}(w) = (vq^{\pm 2} - w)\boldsymbol{\mathsf{X}}^{\pm}_{1,s}(w)\boldsymbol{\mathsf{X}}^{\pm}_{1,r}(v)\,, \tag{2.3.11}$$

$$\begin{aligned}
[\boldsymbol{\mathsf{X}}^{+}_{1,r}(v), \boldsymbol{\mathsf{X}}^{-}_{1,s}(z)] \quad = \quad & \frac{1}{q-q^{-1}}\left\{\delta\left(\frac{\mathsf{C}v}{q^{2(r+s)}z}\right)\prod_{p=1}^{|s|}\boldsymbol{\mathsf{c}}^{-}\left(\mathsf{C}^{-1/2}q^{(2p-1)\mathrm{sign}(s)-1}z\right)^{-\mathrm{sign}(s)}\boldsymbol{\mathsf{K}}^{+}_{1,r+s}(v)\right.\\
& \left. -\delta\left(\frac{\mathsf{C}^{-1}v}{q^{2(r+s)}z}\right)\prod_{p=1}^{|r|}\boldsymbol{\mathsf{c}}^{+}\left(\mathsf{C}^{-1/2}q^{(1-2p)\mathrm{sign}(r)-1}v\right)^{\mathrm{sign}(r)}\boldsymbol{\mathsf{K}}^{-}_{1,r+s}(z)\right\}\,,
\end{aligned} \tag{2.3.12}$$

where $m, n \in \mathbb{N}$, $r, s \in \mathbb{Z}$ and we have set

$$\boldsymbol{\mathsf{c}}^{\pm}(z) = \sum_{m\in\mathbb{N}}\mathsf{c}^{\pm}_{\pm m}z^{\mp m}\,, \tag{2.3.13}$$

$$\boldsymbol{\mathsf{K}}^{\pm}_{1,0}(z) = \sum_{m\in\mathbb{N}}\mathsf{K}^{\pm}_{1,0,\pm m}z^{\pm m}\,, \tag{2.3.14}$$

and, for every $m \in \mathbb{N}^{\times}$ and $r \in \mathbb{Z}$,

$$\boldsymbol{\mathsf{K}}^{\pm}_{1,\pm m}(z) = \sum_{s\in\mathbb{Z}}\mathsf{K}^{\pm}_{1,\pm m,s}z^{-s}\,, \tag{2.3.15}$$

$$\boldsymbol{\mathsf{X}}^{\pm}_{1,r}(z) = \sum_{s\in\mathbb{Z}}\mathsf{X}^{\pm}_{1,r,s}z^{-s}\,. \tag{2.3.16}$$

In (5.0.6), we further assume that $\boldsymbol{\mathsf{K}}^{\pm}_{1,\mp m}(z) = 0$ for every $m \in \mathbb{N}^{\times}$.

Definition 2.3.2. We define $\ddot{\mathrm{U}}^0_q(\mathfrak{a}_1)$ as the subalgebra of $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)$ generated by

$$\left\{\mathsf{C}^{1/2}, \mathsf{C}^{-1/2}, \mathsf{c}^{+}_{m}, \mathsf{c}^{-}_{-m}, \mathsf{K}^{+}_{1,0,m}, \mathsf{K}^{-}_{1,0,-m}, \mathsf{K}^{+}_{1,n,r}, \mathsf{K}^{-}_{1,-n,r} : m \in \mathbb{N}, n \in \mathbb{N}^{\times}, r \in \mathbb{Z}\right\}\,.$$

We define similarly $\ddot{\mathrm{U}}^{\pm}_q(\mathfrak{a}_1)$ as the subalgebra of $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)$ generated by $\left\{\mathsf{X}^{\pm}_{1,r,s} : r, s \in \mathbb{Z}\right\}$.

Remark 2.3.3. Obviously, $\ddot{\mathrm{U}}^{\pm}_q(\mathfrak{a}_1)$ is graded over $Q^{\pm}$ whereas $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)$ is graded over the root lattice Q of

$\mathfrak{a}_1$. $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)$ is also graded over $\mathbb{Z}^2 = \mathbb{Z}_{(1)} \times \mathbb{Z}_{(2)}$;

$$\ddot{\mathrm{U}}_q(\mathfrak{a}_1) = \bigoplus_{(n_1,n_2)\in\mathbb{Z}^2} \ddot{\mathrm{U}}_q(\mathfrak{a}_1)_{(n_1,n_2)} \, ,$$

where, for every $(n_1, n_2) \in \mathbb{Z}^2$, we let

$$\ddot{\mathrm{U}}_q(\mathfrak{a}_1)_{(n_1,n_2)} = \left\{ x \in \ddot{\mathrm{U}}_q(\mathfrak{a}_1) : \mathsf{D}_1 x \mathsf{D}_1^{-1} = q^{n_1} x, \quad \mathsf{D}_2 x \mathsf{D}_2^{-1} = q^{n_2} x \right\} .$$

Remark 2.3.4. It is worth emphasizing that, were it not for relation (5.0.6), the above definition of $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)$ would be purely algebraic. However, the r.h.s. of (5.0.6) involves two infinite series and we may equip $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)$ with a topology, along the lines of what was done in section 3.3.2 for $\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)$, making use of the $\mathbb{Z}_{(2)}$-grading in order to construct a basis $\left\{\dot{\Omega}_n : n \in \mathbb{N}\right\}$ of open neighbourhoods of 0. In that case, both series are convergent in the corresponding completion $\widehat{\ddot{\mathrm{U}}_q(\mathfrak{a}_1)}$ and we shall further require that the subalgebras $\ddot{\mathrm{U}}_q^-(\mathfrak{a}_1)$, $\ddot{\mathrm{U}}_q^0(\mathfrak{a}_1)$ and $\ddot{\mathrm{U}}_q^+(\mathfrak{a}_1)$ be defined as closed subalgebras of $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)$. We shall eventually denote with a hat their respective completions. An alternative point of view on this question, which might actually prove more useful when it comes to studying representation theory, consists in observing that $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)$ is *proalgebraic*. Indeed, for every $N \in \mathbb{N}$, let $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)^{(N)}$ be the $\mathbb{F}$-algebra generated by

$$\{\mathsf{C}^{1/2}, \mathsf{C}^{-1/2}, \mathsf{c}_n^+, \mathsf{c}_{-n}^-, \mathsf{K}_{1,0,m}^+, \mathsf{K}_{1,0,-m}^-, \mathsf{K}_{1,p,r}^+, \mathsf{K}_{1,-p,r}^-, \mathsf{X}_{1,r,s}^+, \mathsf{X}_{1,r,s}^- : m \in \mathbb{N}, n \in [\![0, N]\!], p \in \mathbb{N}^\times, r, s \in \mathbb{Z}\}$$

subject to relations ((3.3.1) – (5.0.6)), where, this time,

$$\mathbf{c}^\pm(z) = \sum_{m=0}^{N} \mathsf{c}_{\pm m}^\pm z^{\mp m} \, . \tag{2.3.17}$$

Now clearly, each $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)^{(N)}$ is algebraic since the sums on the r.h.s. of (5.0.6) are both finite – whenever $\mathbf{c}^\pm(z)^{-1}$ is involved, just multiply through by $\mathbf{c}^\pm(z)$ to get an equivalent algebraic relation. Moreover, letting $\mathcal{I}_N$ be the two-sided ideal of $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)^{(N)}$ generated by $\{\mathsf{c}_N^+, \mathsf{c}_{-N}^-\}$ (resp. $\{\mathsf{c}_0^+ - 1, \mathsf{c}_0^- - 1\}$) for every $N > 1$ (resp. for $N = 0$), we obviously have a surjective algebra homomorphism

$$\ddot{\mathrm{U}}_q(\mathfrak{a}_1)^{(N)} \longrightarrow \ddot{\mathrm{U}}_q(\mathfrak{a}_1)^{(N-1)} \cong \frac{\ddot{\mathrm{U}}_q(\mathfrak{a}_1)^{(N)}}{\mathcal{I}_N} \tag{2.3.18}$$

and we can define $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)$ as the inverse limit

$$\ddot{\mathrm{U}}_q(\mathfrak{a}_1) = \varprojlim \ddot{\mathrm{U}}_q(\mathfrak{a}_1)^{(N)}$$

of the system of algebras

$$\cdots \longrightarrow \ddot{\mathrm{U}}_q(\mathfrak{a}_1)^{(N)} \longrightarrow \ddot{\mathrm{U}}_q(\mathfrak{a}_1)^{(N-1)} \longrightarrow \cdots \longrightarrow \ddot{\mathrm{U}}_q(\mathfrak{a}_1)^{(0)} \longrightarrow \ddot{\mathrm{U}}_q(\mathfrak{a}_1)^{(-1)} \, .$$

We shall refer to the quotient of $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)^{(-1)}$ by the two-sided ideal generated by $\{\mathsf{C}^{1/2} - 1\}$ as the *double quantum loop algebra* of type $\mathfrak{a}_1$.

Definition 2.3.5. In $\widehat{\ddot{\mathrm{U}}^0_q(\mathfrak{a}_1)}$, we define

$$\boldsymbol{\mathsf{p}}^{\pm}(z) = \sum_{m\in\mathbb{N}} \mathsf{p}^{\pm}_{\pm m} z^{\mp m} = \boldsymbol{\mathsf{c}}^{\pm}(z)\boldsymbol{\mathsf{K}}^{\mp}_{1,0}(\mathsf{C}^{-1/2}z)^{-1}\boldsymbol{\mathsf{K}}^{\mp}_{1,0}(\mathsf{C}^{-1/2}zq^2)$$

and for every $m\in\mathbb{N}^\times$,

$$\boldsymbol{\mathsf{t}}^{+}_{1,m}(z) = \sum_{n\in\mathbb{N}} \mathsf{t}^{+}_{1,m,n} z^{-n} = -\frac{1}{q-q^{-1}}\boldsymbol{\mathsf{K}}^{+}_{1,0}(zq^{-2m})^{-1}\boldsymbol{\mathsf{K}}^{+}_{1,m}(z)\,,$$

$$\boldsymbol{\mathsf{t}}^{-}_{1,-m}(z) = \sum_{n\in\mathbb{N}} \mathsf{t}^{-}_{1,-m,n} z^{n} = \frac{1}{q-q^{-1}}\boldsymbol{\mathsf{K}}^{-}_{1,-m}(z)\boldsymbol{\mathsf{K}}^{-}_{1,0}(zq^{-2m})^{-1}\,.$$

Then, we let $\ddot{\mathrm{U}}^{0^+}_q(\mathfrak{a}_1)$ be the closed subalgebra of $\widehat{\ddot{\mathrm{U}}^0_q(\mathfrak{a}_1)}$ generated by

$$\{\mathsf{C}^{1/2},\mathsf{C}^{-1/2},\mathsf{p}^{+}_{m},\mathsf{p}^{-}_{-m},\mathsf{t}^{+}_{1,p,n},\mathsf{t}^{-}_{1,-p,n} : m\in\mathbb{N}, n\in\mathbb{Z}, p\in\mathbb{N}^\times\}\,.$$

Definition 2.3.6. We denote by $\ddot{\mathrm{U}}'_q(\mathfrak{a}_1)$ the subalgebra of $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)$ generated by

$$\{\mathsf{D}_2,\mathsf{D}_2^{-1},\mathsf{C}^{1/2},\mathsf{C}^{-1/2},\mathsf{c}^{+}_{m},\mathsf{c}^{-}_{-m},\mathsf{K}^{+}_{1,0,m},\mathsf{K}^{-}_{1,0,-m},\mathsf{K}^{+}_{1,n,r},\mathsf{K}^{-}_{1,-n,r},\mathsf{X}^{+}_{1,r,s},\mathsf{X}^{-}_{1,r,s} : m\in\mathbb{N}, n\in\mathbb{N}^\times, r,s\in\mathbb{Z}\}\,,$$

i.e. the subalgebra generated by all the generators of $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)$ except D_1 and D_1^{-1}. We shall denote by

$$\jmath : \ddot{\mathrm{U}}'_q(\mathfrak{a}_1) \hookrightarrow \ddot{\mathrm{U}}_q(\mathfrak{a}_1)$$

the natural injective algebra homomorphism. We extend it by continuity into

$$\widehat{\jmath} : \widehat{\ddot{\mathrm{U}}'_q(\mathfrak{a}_1)} \hookrightarrow \widehat{\ddot{\mathrm{U}}_q(\mathfrak{a}_1)}\,.$$

The main result of the present paper is the following

Theorem 2.3.7. *There exists a bicontinuous $\mathbb{F}$-algebra isomorphism $\widehat{\Psi} : \widehat{\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)} \xrightarrow{\sim} \widehat{\ddot{\mathrm{U}}'_q(\mathfrak{a}_1)}$.*

Proof. Relations ((3.3.7)-(3.3.10)) respectively imply

$$\boldsymbol{\mathsf{K}}^{\pm}_{1,0}(v)\boldsymbol{\mathsf{K}}^{\pm}_{1,0}(z) = \boldsymbol{\mathsf{K}}^{\pm}_{1,0}(z)\boldsymbol{\mathsf{K}}^{\pm}_{1,0}(v)\,, \tag{2.3.19}$$

$$\boldsymbol{\mathsf{K}}^{+}_{1,0}(v)\boldsymbol{\mathsf{K}}^{-}_{1,0}(w) = G^{+}_{11}(\mathsf{C}v/w)G^{-}_{11}(\mathsf{C}^{-1}v/w)\boldsymbol{\mathsf{K}}^{-}_{1,0}(w)\boldsymbol{\mathsf{K}}^{+}_{1,0}(v) \tag{2.3.20}$$

$$\boldsymbol{\mathsf{K}}^{\pm}_{1,0}(v)\boldsymbol{\mathsf{X}}^{\pm}_{1,r}(z) = G^{\mp}_{11}(v/z)\boldsymbol{\mathsf{X}}^{\pm}_{1,r}(z)\boldsymbol{\mathsf{K}}^{\pm}_{1,0}(v)\,, \tag{2.3.21}$$

$$\boldsymbol{\mathsf{K}}^{\pm}_{1,0}(v)\boldsymbol{\mathsf{X}}^{\mp}_{1,r}(z) = G^{\pm}_{11}(\mathsf{C}v/z)\boldsymbol{\mathsf{X}}^{\mp}_{1,r}(z)\boldsymbol{\mathsf{K}}^{\pm}_{1,0}(v)\,, \tag{2.3.22}$$

since $\boldsymbol{\mathsf{K}}^{\pm}_{1,0}(z)\in\ddot{\mathrm{U}}'_q(\mathfrak{a}_1)[[z^{\pm 1}]]$. It also easily follows from relation (5.0.5) that

$$\left[\boldsymbol{\mathsf{X}}^{+}_{1,0}(v),\boldsymbol{\mathsf{X}}^{+}_{1,-1}(w)\right]_{G^{-}_{11}(v/w)} = \delta\left(\frac{vq^{-2}}{w}\right)\Upsilon^{+}(w)\,, \tag{2.3.23}$$

$$\left[\boldsymbol{\mathsf{X}}^{-}_{1,1}(v),\boldsymbol{\mathsf{X}}^{-}_{1,0}(w)\right]_{G^{+}_{11}(v/w)} = \delta\left(\frac{vq^{2}}{w}\right)\Upsilon^{-}(w)\,, \tag{2.3.24}$$

for some $\Upsilon^{\pm}(w) \in \widehat{\ddot{\mathrm{U}}'_q(\mathfrak{a}_1)}[[w, w^{-1}]]$. Hence, the only possible obstructions to setting

$$\Psi(D^{\pm 1}) = \mathsf{D}_2^{\pm 1} \qquad \Psi(C^{\pm 1/2}) = \mathsf{C}^{\pm 1/2}\,,$$

$$\Psi(\mathbf{k}_0^{\pm}(z)) = -\boldsymbol{\mathsf{c}}^{\pm}(z)\boldsymbol{\mathsf{K}}^{\mp}_{1,0}(\mathsf{C}^{-1/2}z)^{-1} \qquad \Psi(\mathbf{k}_1^{\pm}(z)) = -\boldsymbol{\mathsf{K}}^{\mp}_{1,0}(\mathsf{C}^{-1/2}z)$$

$$\Psi(\mathbf{x}_0^{+}(z)) = -\boldsymbol{\mathsf{c}}^{-}(\mathsf{C}^{1/2}z)\boldsymbol{\mathsf{K}}^{+}_{1,0}(z)^{-1}\boldsymbol{\mathsf{X}}^{-}_{1,1}(\mathsf{C}z) \qquad \Psi(\mathbf{x}_0^{-}(z)) = -\boldsymbol{\mathsf{X}}^{+}_{1,-1}(\mathsf{C}z)\boldsymbol{\mathsf{c}}^{+}(\mathsf{C}^{1/2}z)\boldsymbol{\mathsf{K}}^{-}_{1,0}(z)^{-1}$$

$$\Psi(\mathbf{x}_1^{\pm}(z)) = \boldsymbol{\mathsf{X}}^{\pm}_{1,0}(z)\,,$$

and to extending it as an algebra homomorphism $\Psi : \dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1) \to \widehat{\ddot{\mathrm{U}}'_q(\mathfrak{a}_1)}$ are $\Upsilon^{\pm}(w)$ and the images under Ψ of the l.h.s. of the quantum Serre relations (4.2.10). We shall see in section 2.4 that both obstructions actually vanish. We also postpone until section 2.4 the construction of the continuous algebra homomorphism $\Psi^{-1} : \ddot{\mathrm{U}}'_q(\mathfrak{a}_1) \longrightarrow \widehat{\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)}$. □

2.3.2 The subalgebra $\ddot{\mathrm{U}}^0_q(\mathfrak{a}_1)$ and the elliptic Hall algebra

Another remarkable feature of $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)$ and, more particularly of its subalgebra $\ddot{\mathrm{U}}^0_q(\mathfrak{a}_1)$, is the existence of an algebra homomorphism onto it, from the elliptic Hall algebra that we now define.

Definition 2.3.8. Let q_1, q_2, q_3 be three (dependent) formal variables such that $q_1q_2q_3 = 1$. The *elliptic Hall algebra* $\mathcal{E}_{q_1,q_2,q_3}$ is the $\mathbb{Q}(q_1, q_2, q_3)$-algebra generated by $\left\{C^{1/2}, C^{-1/2}, \psi^+_m, \psi^-_{-m}, e^+_n, e^-_n : m \in \mathbb{N}, n \in \mathbb{Z}\right\}$, with $\psi^{\pm}_0$ invertible, subject to the relations

$$C^{\pm 1/2} \text{ is central}\,, \tag{2.3.25}$$

$$\boldsymbol{\psi}^{\pm}(z)\boldsymbol{\psi}^{\pm}(w) = \boldsymbol{\psi}^{\pm}(w)\boldsymbol{\psi}^{\pm}(z)\,, \tag{2.3.26}$$

$$g(Cz, w)g(Cw, z)\boldsymbol{\psi}^{+}(z)\boldsymbol{\psi}^{-}(w) = g(z, Cw)g(w, Cz)\boldsymbol{\psi}^{-}(w)\boldsymbol{\psi}^{+}(z)\,, \tag{2.3.27}$$

$$g(C^{\frac{1\pm1}{2}}z, w)\boldsymbol{\psi}^{\pm}(z)\mathbf{e}^{+}(w) = -g(w, C^{\frac{1\pm1}{2}}z)\mathbf{e}^{+}(w)\boldsymbol{\psi}^{\pm}(z)\,, \tag{2.3.28}$$

$$g(w, C^{\frac{1\mp1}{2}}z)\boldsymbol{\psi}^{\pm}(z)\mathbf{e}^{-}(w) = -g(C^{\frac{1\mp1}{2}}z, w)\mathbf{e}^{-}(w)\boldsymbol{\psi}^{\pm}(z)\,, \tag{2.3.29}$$

$$[\mathbf{e}^{+}(z), \mathbf{e}^{-}(w)] = \frac{1}{g(1,1)}\left[\delta\left(\frac{Cw}{z}\right)\boldsymbol{\psi}^{+}(w) - \delta\left(\frac{w}{Cz}\right)\boldsymbol{\psi}^{-}(z)\right]\,, \tag{2.3.30}$$

$$g(z, w)\mathbf{e}^{+}(z)\mathbf{e}^{+}(w) = -g(w, z)\mathbf{e}^{+}(w)\mathbf{e}^{+}(z)\,, \tag{2.3.31}$$

$$g(w, z)\mathbf{e}^{-}(z)\mathbf{e}^{-}(w) = -g(z, w)\mathbf{e}^{-}(w)\mathbf{e}^{-}(z)\,, \tag{2.3.32}$$

$$\operatorname*{res}_{v,w,z}(vwz)^m(v+z)(w^2 - vz)\mathbf{e}^{\pm}(v)\mathbf{e}^{\pm}(w)\mathbf{e}^{\pm}(z) = 0\,, \tag{2.3.33}$$

where $m \in \mathbb{Z}$ and we have introduced

$$g(z, w) = (z - q_1w)(z - q_2w)(z - q_3w)\,, \tag{2.3.34}$$

$$\boldsymbol{\psi}^{\pm}(z) = \sum_{m \in \mathbb{N}} \psi^{\pm}_{\pm m} z^{\mp m}\,, \tag{2.3.35}$$

$$\mathbf{e}^{\pm}(z) = \sum_{m \in \mathbb{Z}} e^{\pm}_m z^{-m}\,. \tag{2.3.36}$$

Remark 2.3.9. The elliptic Hall algebra $\mathcal{E}_{q_1,q_2,q_3}$ is $\mathbb{Z}$-graded and can be equipped with a natural topology along the lines of what we did for $\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)$ in section 3.3.2. It then becomes a topological algebra and we denote by $\widehat{\mathcal{E}_{q_1,q_2,q_3}}$ its completion.

Proposition 2.3.10. *There exists a unique continuous $\mathbb{F}$-algebra homomorphism $f : \widehat{\mathcal{E}_{q^{-4},q^2,q^2}} \to \ddot{\mathrm{U}}_q^{0+}(\mathfrak{a}_1)$ such that*

$$f(C^{1/2}) = \mathsf{C}^{1/2}\,, \tag{2.3.37}$$

$$f(\boldsymbol{\psi}^{\pm}(z)) = (q^2 - q^{-2})^2\, \boldsymbol{p}^{\pm}(\mathsf{C}^{1/2} z q^{-2})\,, \tag{2.3.38}$$

$$f(\boldsymbol{e}^{\pm}(z)) = \boldsymbol{t}^{\pm}_{1,\pm1}(z)\,. \tag{2.3.39}$$

Proof. We prove that, starting from ((3.3.50) – (2.3.39)), we can extend f as an algebra homomorphism. For that purpose, it suffices to check the relations in $\mathcal{E}_{q^{-4},q^2,q^2}$, observing that, in addition to (2.3.19) and (2.3.20), we also have

$$\mathsf{K}^{\pm}_{1,0}(v)\mathsf{K}^{\pm}_{1,\pm1}(z) = G^{\mp}_{11}(v/z)G^{\pm}_{11}(vq^2/z)\mathsf{K}^{\pm}_{1,\pm1}(z)\mathsf{K}^{\pm}_{1,0}(v)\,, \tag{2.3.40}$$

$$\mathsf{K}^{\mp}_{1,0}(v)\mathsf{K}^{\pm}_{1,\pm1}(w) = G^{\mp}_{11}(\mathsf{C}v/w)G^{\pm}_{11}(\mathsf{C}^{-1}q^2v/w)\mathsf{K}^{\pm}_{1,\pm1}(w)\mathsf{K}^{\mp}_{1,0}(v)\,, \tag{2.3.41}$$

as direct consequences of (3.3.7) and (3.3.8) respectively, since $\mathsf{K}^{\pm}_{1,0}(z) \in \ddot{\mathrm{U}}'_q(\mathfrak{a}_1)[[z^{\pm1}]]$. One then easily obtains ((3.3.33) – (3.3.36)) and ((3.3.38) – (3.3.39)). For example, we have

$$\begin{aligned} g(v,z)f(\mathbf{e}^+(v))f(\mathbf{e}^+(z)) &= \frac{1}{(q-q^{-1})^2} g(v,z)G^{+}_{11}(z/v)G^{-}_{11}(zq^{-2}/v)\mathsf{K}^{+}_{1,0}(vq^{-2})\mathsf{K}^{+}_{1,0}(q^{-2}z)\mathsf{K}^{+}_{1,1}(v)\mathsf{K}^{+}_{1,1}(z) \\ &= \frac{v-z}{(q-q^{-1})^2}(v-q^2z)(v-q^{-2}z)\mathsf{K}^{+}_{1,0}(vq^{-2})\mathsf{K}^{+}_{1,0}(q^{-2}z)\mathsf{K}^{+}_{1,1}(v)\mathsf{K}^{+}_{1,1}(z) \end{aligned} \tag{2.3.42}$$

$$\begin{aligned} &= \frac{v-z}{(q-q^{-1})^2}(vq^2-z)(vq^{-2}-z)\mathsf{K}^{+}_{1,0}(vq^{-2})\mathsf{K}^{+}_{1,0}(q^{-2}z)\mathsf{K}^{+}_{1,1}(z)\mathsf{K}^{+}_{1,1}(v) \\ &= \frac{v-z}{(q-q^{-1})^2}(vq^2-z)(vq^{-2}-z)G^{+}_{11}(vq^{-2}/z)G^{-}_{11}(v/z) \\ &\qquad \times\mathsf{K}^{+}_{1,0}(q^{-2}z)\mathsf{K}^{+}_{1,1}(z)\mathsf{K}^{+}_{1,0}(vq^{-2})\mathsf{K}^{+}_{1,1}(v) \\ &= -g(z,v)f(\mathbf{e}^+(z))f(\mathbf{e}^+(v))\,. \end{aligned}$$

Considering (3.3.37), we observe that (3.3.8) implies that there exist $\theta^{\pm}(z) \in \widehat{\ddot{\mathrm{U}}'_q(\mathfrak{a}_1)}[[z,z^{-1}]]$ such that

$$\left[\mathsf{K}^{+}_{1,1}(v), \mathsf{K}^{-}_{1,-1}(w)\right]_{G^{+}_{11}(\mathsf{C}vq^{-2}/w)G^{-}_{11}(\mathsf{C}^{-1}vq^2/w)} = \delta\left(\frac{\mathsf{C}v}{w}\right)\theta^-(v) + \delta\left(\frac{v}{\mathsf{C}w}\right)\theta^+(w)$$

and one easily sees that

$$\begin{aligned} \left[f(\mathbf{e}^+(v)), f(\mathbf{e}^-(w))\right] &= -\frac{1}{(q-q^{-1})^2}\mathsf{K}^{+}_{1,0}(vq^{-2})^{-1}\left[\mathsf{K}^{+}_{1,1}(v), \mathsf{K}^{-}_{1,-1}(w)\right]_{G^{+}_{11}(\mathsf{C}vq^{-2}/w)G^{-}_{11}(\mathsf{C}^{-1}vq^2/w)}\mathsf{K}^{-}_{1,0}(wq^{-2})^{-1} \\ &= -\frac{1}{(q-q^{-1})^2}\mathsf{K}^{+}_{1,0}(vq^{-2})^{-1}\left\{\delta\left(\frac{\mathsf{C}v}{w}\right)\theta^-(v) + \delta\left(\frac{v}{\mathsf{C}w}\right)\theta^+(w)\right\}\mathsf{K}^{-}_{1,0}(wq^{-2})^{-1}\,. \end{aligned}$$

Therefore, it suffices to prove that

$$-\frac{1}{(q-q^{-1})^2}\mathsf{K}^+_{1,0}(\mathsf{C}wq^{-2})^{-1}\theta^+(w)\mathsf{K}^-_{1,0}(wq^{-2})^{-1} = \frac{(q^2-q^{-2})^2}{g(1,1)}\mathsf{p}^+(\mathsf{C}^{1/2}q^{-2}w) \tag{2.3.43}$$

$$-\frac{1}{(q-q^{-1})^2}\mathsf{K}^+_{1,0}(vq^{-2})^{-1}\theta^-(v)\mathsf{K}^-_{1,0}(\mathsf{C}vq^{-2})^{-1} = -\frac{(q^2-q^{-2})^2}{g(1,1)}\mathsf{p}^-(\mathsf{C}^{1/2}q^{-2}v) \tag{2.3.44}$$

We postpone the proof of ((2.3.43) – (2.3.44)), as well as that of

$$\operatorname*{res}_{v,w,z}(vwz)^m(v+z)(w^2-vz)f(\mathbf{e}^\pm(v))f(\mathbf{e}^\pm(w))f(\mathbf{e}^\pm(z)) = 0\,, \tag{2.3.45}$$

until section 2.4. □

We now naturally make the following

Conjecture 2.3.11. $f:\widehat{\mathcal{E}_{q^{-4},q^2,q^2}} \to \ddot{\mathrm{U}}^{0+}_q(\mathfrak{a}_1)$ is a bicontinuous $\mathbb{F}$-algebra isomorphism.

Remark 2.3.12. It is worth mentioning that the above conjecture is supported by the fact that, in view of ((3.3.38) – (3.3.39)), there clearly exists $\mathbf{e}^\pm_{\pm 2}(z) \in \widehat{\mathcal{E}_{q_1,q_2,q_3}}[[z,z^{-1}]]$ such that

$$_{G^\mp_{01}(q^{\mp 2}v/w)G^\mp_{11}(v/w)}\left[\mathbf{e}^\pm(w),\mathbf{e}^\pm(v)\right]_{G^\mp_{01}(q^{\mp 2}w/v)G^\mp_{11}(w/v)} = \pm\,[2]_q\left\{\delta\left(\frac{q^2v}{w}\right)\mathbf{e}^\pm_{\pm 2}(w) - \delta\left(\frac{wq^2}{v}\right)\mathbf{e}^\pm_{\pm 2}(v)\right\}$$

and that we can therefore set

$$f^{-1}(\mathbf{t}^\pm_{1,\pm 2}(v)) = \mathbf{e}^\pm_{\pm 2}(v)\,.$$

In order to complete the proof, one would similarly need to construct $f^{-1}(\mathbf{t}^\pm_{1,\pm m}(v))$ for any $m>2$.

2.3.3 $\dot{\mathrm{U}}_q(\mathfrak{a}_1)$ subalgebras of $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)$

Interestingly, $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)$ admits countably many embeddings of the quantum affine algebra $\dot{\mathrm{U}}_q(\mathfrak{a}_1)$. This is the content of the following

Proposition 2.3.13. *For every $m\in\mathbb{Z}$, there exists a unique injective algebra homomorphism ι_m : $\dot{\mathrm{U}}_q(\mathfrak{a}_1) \hookrightarrow \widehat{\ddot{\mathrm{U}}_q(\mathfrak{a}_1)}$ such that*

$$\iota_m(C^{\pm 1/2}) = \mathsf{C}^{\pm 1/2} \qquad \iota_m(D^{\pm 1}) = \mathsf{D}_2^{\pm 1} \tag{2.3.46}$$

$$\iota_m(\mathbf{k}^\pm_1(z)) = -\prod_{p=1}^{|m|}\boldsymbol{\mathsf{c}}^\pm\left(q^{(1-2p)\mathrm{sign}(m)-1}z\right)^{\mathrm{sign}(m)}\boldsymbol{\mathsf{K}}^\mp_{1,0}(\mathsf{C}^{-1/2}z)\,, \tag{2.3.47}$$

$$\iota_m(\mathbf{x}^\pm_1(z)) = \boldsymbol{\mathsf{X}}^\pm_{1,\pm m}(z)\,. \tag{2.3.48}$$

Proof. Let $\iota^{(1)}:\dot{\mathrm{U}}_q(\mathfrak{a}_1) \hookrightarrow \dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)$ be the injective algebra homomorphism mapping $\dot{\mathrm{U}}_q(\mathfrak{a}_1)$ to the Dynkin diagram subalgebra of $\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)$ associated with the vertex labeled $1\in\dot{I}$ – see section 2.2.1. It naturally extends to an injective algebra homomorphism $\widehat{\iota}^{(1)}:\dot{\mathrm{U}}_q(\mathfrak{a}_1) \hookrightarrow \widehat{\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)}$. Then, let for every $m\in\mathbb{Z}$, ι_m be the composite

$$\iota_m:\dot{\mathrm{U}}_q(\mathfrak{a}_1) \underset{\widehat{\iota}^{(1)}}{\hookrightarrow} \widehat{\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)} \xrightarrow[Y^{-m}]{\sim} \widehat{\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)} \xrightarrow[\widehat{\Psi}]{\sim} \widehat{\ddot{\mathrm{U}}'_q(\mathfrak{a}_1)} \underset{\widehat{\jmath}}{\hookrightarrow} \widehat{\ddot{\mathrm{U}}_q(\mathfrak{a}_1)}\,.$$

Thus, ι_m is clearly injective. Moreover, one easily checks ((3.3.71) – (3.3.73)) – see next section. □

2.3.4 Automorphisms of $\widehat{\ddot{\mathrm{U}}'_q(\mathfrak{a}_1)}$

$\widehat{\ddot{\mathrm{U}}'_q(\mathfrak{a}_1)}$ naturally inherits, through $\widehat{\Psi}$, the automorphisms defined over $\widehat{\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)}$ in the previous section.

Proposition 2.3.14. *Conjugation by* $\widehat{\Psi}$ *clearly provides a group isomorphism* $\mathrm{Aut}(\widehat{\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)}) \cong \mathrm{Aut}(\widehat{\ddot{\mathrm{U}}'_q(\mathfrak{a}_1)})$. *In particular, for every* $f \in \mathrm{Aut}(\widehat{\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)})$, *we let* $\dot{f} = \widehat{\Psi} \circ f \circ \widehat{\Psi}^{-1} \in \mathrm{Aut}(\widehat{\ddot{\mathrm{U}}'_q(\mathfrak{a}_1)})$.

2.3.5 Triangular decomposition of $\widehat{\ddot{\mathrm{U}}_q(\mathfrak{a}_1)}$

Definition 2.3.15. Let A be a complete topological algebra with closed subalgebras $A^{\pm}$ and A^0. We shall say that (A^-, A^0, A^+) is a *triangular decomposition* of A if the multiplication induces a bicontinuous isomorphism of vector spaces $A^-\widehat{\otimes}A^0\widehat{\otimes}A^+ \xrightarrow{\sim} A$.

In order to prove the triangular decomposition of $\widehat{\ddot{\mathrm{U}}_q(\mathfrak{a}_1)}$, we shall make use of the following classic

Lemma 2.3.16. *Let A be a complete topological algebra with a triangular decomposition (A^-, A^0, A^+). Let $\mathcal{I}^{\pm}$ be a closed two-sided ideal of $A^{\pm}$ such that $\mathcal{I}^+.A \subseteq A.\mathcal{I}^+$ and $A.\mathcal{I}^- \subseteq \mathcal{I}^-.A$. Then the quotient algebra $B = A/(A.(\mathcal{I}^+ + \mathcal{I}^-).A)$ admits a triangular decomposition (B^-, A^0, B^+) where $B^{\pm}$ is the set of equivalence classes of $A^{\pm}$ in B. Moreover, there exists a bicontinuous algebra isomorphism $B^{\pm} \cong A^{\pm}/\mathcal{I}^{\pm}$.*

Proof. See e.g. [Jan96]. □

Recalling the definitions of $\ddot{\mathrm{U}}^{\pm}_q(\mathfrak{a}_1)$ and $\ddot{\mathrm{U}}^{0}_q(\mathfrak{a}_1)$ from definition 3.3.1, we have

Proposition 2.3.17. $(\ddot{\mathrm{U}}^{-}_q(\mathfrak{a}_1), \ddot{\mathrm{U}}^{0}_q(\mathfrak{a}_1), \ddot{\mathrm{U}}^{+}_q(\mathfrak{a}_1))$ *is a triangular decomposition of* $\widehat{\ddot{\mathrm{U}}_q(\mathfrak{a}_1)}$ *and* $\ddot{\mathrm{U}}^{\pm}_q(\mathfrak{a}_1)$ *is bicontinuously isomorphic to the algebra generated by* $\{\mathsf{X}^{\pm}_{1,r,s} : r, s \in \mathbb{Z}\}$ *subject to relation (5.0.5).*

Proof. Let A be the $\mathbb{F}$-algebra generated by

$$\{\mathsf{D}_1, \mathsf{D}_1^{-1}, \mathsf{D}_2, \mathsf{D}_2^{-1}, \mathsf{C}^{1/2}, \mathsf{C}^{-1/2}, \mathsf{c}^+_m, \mathsf{c}^-_{-m}, \mathsf{K}^+_{1,0,m}, \mathsf{K}^-_{1,0,-m}, \mathsf{K}^+_{1,n,r}, \mathsf{K}^-_{1,-n,r}, \mathsf{X}^+_{1,r,s}, \mathsf{X}^-_{1,r,s} : m \in \mathbb{N}, n \in \mathbb{N}^{\times}, r, s \in \mathbb{Z}\}$$

subject to the relations ((3.3.2) – (3.3.10)) and (5.0.6), i.e. all the defining relations of $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)$ but relation (5.0.5). Endow A with a topology along the lines of what was done in section 3.3.2 for $\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)$, making use of its $\mathbb{Z}_{(2)}$-grading. This yields a basis $\left\{\dot{\Omega}_n : n \in \mathbb{N}\right\}$ of open neighbourhoods of 0. Let furthermore A^0 be the closed subalgebra of A generated by

$$\left\{\mathsf{D}_1, \mathsf{D}_1^{-1}, \mathsf{D}_2, \mathsf{D}_2^{-1}, \mathsf{C}^{1/2}, \mathsf{C}^{-1/2}, \mathsf{c}^+_m, \mathsf{c}^-_{-m}, \mathsf{K}^+_{1,0,m}, \mathsf{K}^-_{1,0,-m}, \mathsf{K}^+_{1,n,r}, \mathsf{K}^-_{1,-n,r} : m \in \mathbb{N}, n \in \mathbb{N}^{\times}, r \in \mathbb{Z}\right\}$$

and $A^{\pm}$ be the closed subalgebra of A generated by $\left\{\mathsf{X}^{\pm}_{1,r,s} : r, s \in \mathbb{Z}\right\}$. An easy recursion proves that relations (5.0.1) and (3.3.10) imply that, for every $N \in \mathbb{N}$ and every $m \in \mathbb{N}$, $l, r, s \in \mathbb{Z}$,

$$\mathsf{X}^+_{1,r,s}\mathsf{K}^+_{1,m,l} - q^2\mathsf{K}^+_{1,m,l}\mathsf{X}^+_{1,r,s} - (q^2 - q^{-2})\sum_{p=1}^{N} q^{2p}\mathsf{K}^+_{1,m,l+p}\mathsf{X}^+_{1,r,s-p} + q^{2N}\mathsf{K}^+_{1,m,l+N+1}\mathsf{X}^+_{1,r,s-N-1} \in \dot{\Omega}_{\nu^+_{s,l}(N)}$$

$$\mathsf{K}^-_{1,-m,l}\mathsf{X}^-_{1,r,s} - q^{-2}\mathsf{X}^-_{1,r,s}\mathsf{K}^-_{1,-m,l} + (q^2 - q^{-2})\sum_{p=1}^{N} q^{-2p}\mathsf{X}^-_{1,r,s+p}\mathsf{K}^-_{1,-m,l-p} + q^{2N}\mathsf{X}^-_{1,r,s+N+1}\mathsf{K}^-_{1,-m,l-N-1} \in \dot{\Omega}_{\nu^-_{s,l}(N)}$$

$$\mathsf{K}^+_{1,m,l}\mathsf{X}^-_{1,r,s} - q^{-2}\mathsf{X}^-_{1,r,s}\mathsf{K}^+_{1,m,l} + (q^2-q^{-2})\sum_{p=1}^{N}\mathsf{C}^{-p}q^{2p(m-1)}\mathsf{X}^-_{1,r,s+p}\mathsf{K}^+_{1,m,l-p}$$
$$+\mathsf{C}^{-(N+1)}q^{2(N+1)(m-1)+2}\mathsf{X}^-_{1,r,s+N+1}\mathsf{K}^+_{1,m,l-N-1} \in \dot{\Omega}_{\nu^-_{s,l}(N)}$$

$$\mathsf{X}^+_{1,r,s}\mathsf{K}^-_{1,-m,l} - q^{2}\mathsf{K}^-_{1,-m,l}\mathsf{X}^+_{1,r,s} - (q^2-q^{-2})\sum_{p=1}^{N}\mathsf{C}^{p}q^{2p(1-m)}\mathsf{K}^-_{1,-m,l+p}\mathsf{X}^+_{1,r,s-p}$$
$$+\mathsf{C}^{N+1}q^{2(N+1)(1-m)}\mathsf{K}^-_{1,-m,l+N+1}\mathsf{X}^+_{1,r,s-N-1} \in \dot{\Omega}_{\nu^+_{s,l}(N)}$$

where $\nu^{\pm}_{s,l}(N) = \min(\pm l, \mp s) + N + 1$. It obviously follows that (A^-, A^0, A^+) is a triangular decomposition of A. Now let $\mathcal{I}^{\pm}$ be the closed two-sided ideal of $A^{\pm}$ generated by

$$\left\{\mathsf{X}^{\pm}_{1,r,m+1}\mathsf{X}^{\pm}_{1,s,n} - q^{\pm 2}\mathsf{X}^{\pm}_{1,r,m}\mathsf{X}^{\pm}_{1,s,n+1} - q^{\pm 2}\mathsf{X}^{\pm}_{1,s,n}\mathsf{X}^{\pm}_{1,r,m+1} + \mathsf{X}^{\pm}_{1,s,n+1}\mathsf{X}^{\pm}_{1,r,m} : r,s,m,n \in \mathbb{Z}\right\} .$$

Clearly $\ddot{\mathrm{U}}_q(\mathfrak{a}_1) \cong A/(A.(\mathcal{I}^+ + \mathcal{I}^-).A)$. In view of the above rewritings of (5.0.1) and (3.3.10), it is clear that $\mathcal{I}^+.A^0 \subseteq A^0.\mathcal{I}^+$ and $A^0.\mathcal{I}^- \subseteq \mathcal{I}^-.A^0$. Moreover, relations (5.0.1), (3.3.10) and (5.0.6) are easily shown to imply that, for every $r, s, t \in \mathbb{Z}$,

$$\left[(v - q^{\pm 2}w)\boldsymbol{\mathsf{X}}^{\pm}_{1,r}(v)\boldsymbol{\mathsf{X}}^{\pm}_{1,s}(w) - (vq^{\pm 2} - w)\boldsymbol{\mathsf{X}}^{\pm}_{1,s}(w)\boldsymbol{\mathsf{X}}^{\pm}_{1,r}(v), \boldsymbol{\mathsf{X}}^{\mp}_{1,t}(u)\right] = 0\,,$$

hence proving that $\mathcal{I}^+.A^- \subseteq A.\mathcal{I}^+$ and $A^+.\mathcal{I}^- \subseteq \mathcal{I}^-.A$. The claim eventually follows as a consequence of lemma 2.3.16 □

2.3.6 Weight-finite highest t-weight modules

Definition 2.3.18. For every $N \in \mathbb{N}^{\times}$, we shall say that a (topological) module M over $\ddot{\mathrm{U}}'_q(\mathfrak{a}_1)$ is of *type* $(1, N)$ if:

i. $\mathsf{C}^{\pm 1/2}$ acts as id on M;

ii. $\mathsf{c}^{\pm}_{\pm m}$ acts by multiplication by 0 on M, for every $m \geq N$.

We shall say that M is of *type* $(1, 0)$ if points i. and ii. above hold for every $m > 0$ and, in addition, $\mathsf{c}^{\pm}_0$ acts as id on M.

Remark 2.3.19. Let $N \in \mathbb{N}^{\times}$. Then the $\ddot{\mathrm{U}}'_q(\mathfrak{a}_1)$-modules of type $(1, N)$ are in one-to-one correspondence with the $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)^{(N-1)}/(\mathsf{C}^{1/2} - 1)$-modules – see remark 3.3.3 for a definition of $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)^{(N)}$. Similarly, $\ddot{\mathrm{U}}'_q(\mathfrak{a}_1)$-modules of type $(1, 0)$ descend to modules over the double quantum loop algebra of type $\mathfrak{a}_1$, $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)^{(-1)}/(\mathsf{C}^{1/2} - 1)$.

In view of the triangular decomposition $(\ddot{\mathrm{U}}^-_q(\mathfrak{a}_1), \ddot{\mathrm{U}}^0_q(\mathfrak{a}_1), \ddot{\mathrm{U}}^+_q(\mathfrak{a}_1))$ of $\widehat{\ddot{\mathrm{U}}'_q(\mathfrak{a}_1)}$ – see proposition 3.3.11 –, we naturally expect that a new, adapted notion of highest weight modules exists, in which $\ddot{\mathrm{U}}^0_q(\mathfrak{a}_1)$, although non-abelian, plays the role usually played by the Cartan subalgebra. Thus, we restrict our attention to modules over $\ddot{\mathrm{U}}'_q(\mathfrak{a}_1)$ which, regarded as $\ddot{\mathrm{U}}^0_q(\mathfrak{a}_1)$-modules, split as direct sums of indecomposable modules over $\ddot{\mathrm{U}}^0_q(\mathfrak{a}_1)$. We refer to those summands as *t-weight spaces*. Moreover, the injective algebra homomorphism ι_0 of proposition 2.3.13 restricts to an injective algebra homomorphism $\dot{\mathrm{U}}^0_q(\mathfrak{a}_1) \to \ddot{\mathrm{U}}^0_q(\mathfrak{a}_1)$ from the quantum Heisenberg subalgebra $\dot{\mathrm{U}}_q^0(\mathfrak{a}_1)$ of $\dot{\mathrm{U}}_q(\mathfrak{a}_1)$ to $\ddot{\mathrm{U}}_q^0(\mathfrak{a}_1)$. Therefore, considering any $\widehat{\ddot{\mathrm{U}}'_q(\mathfrak{a}_1)}$-module M of type $(1,0)$, we get an action of the infinite-dimensional abelian algebra $\dot{\mathrm{U}}_q^0(\mathfrak{a}_1)/(C^{1/2}-1)$ on all the t-weight spaces of M. Whenever the latter decompose into direct sums of generalized eigenspaces of the commuting family of linear operators $\{\mathsf{K}^+_{1,0,m}, \mathsf{K}^-_{1,0,-m} : m \in \mathbb{N}\}$, we shall say that the t-weight-spaces are ℓ-weight. In the latter case, we let $\mathrm{Sp}(M)$ denote the set of all the eigenvalues of $\mathsf{K}^+_{1,0,0}$ over M.

Definition 2.3.20. We shall say that a (topological) $\ddot{\mathrm{U}}'_q(\mathfrak{a}_1)$-module M is a *t-weight* module if there exists a countable set $\{M_\alpha : \alpha \in A\}$ of indecomposable ℓ-weight $\ddot{\mathrm{U}}_q^0(\mathfrak{a}_1)$-modules, called the *t-weight spaces* of M, such that, as $\ddot{\mathrm{U}}_q^0(\mathfrak{a}_1)$-modules,

$$M \cong \bigoplus_{\alpha \in A} M_\alpha \,. \tag{2.3.49}$$

We shall say that M is *weight-finite* if, in addition, $\mathrm{Sp}(M)$ is finite. A vector $v \in M - \{0\}$ is a *highest t-weight vector* of M if $v \in M_\alpha$ for some $\alpha \in A$ and, for every $r, s \in \mathbb{Z}$,

$$\mathsf{X}^+_{1,r,s}.v = 0 \,. \tag{2.3.50}$$

We shall say that M is *highest t-weight* if $M \cong \ddot{\mathrm{U}}'_q(\mathfrak{a}_1).v$ for some highest t-weight vector $v \in M - \{0\}$.

It is reasonably clear that, owing to the triangular decomposition $(\ddot{\mathrm{U}}_q^-(\mathfrak{a}_1), \ddot{\mathrm{U}}_q^0(\mathfrak{a}_1), \ddot{\mathrm{U}}_q^+(\mathfrak{a}_1))$ of $\widehat{\ddot{\mathrm{U}}'_q(\mathfrak{a}_1)}$, for every highest t-weight $\ddot{\mathrm{U}}'_q(\mathfrak{a}_1)$-module M and every highest t-weight vector $v \in M - \{0\}$, we have

$$M \cong \ddot{\mathrm{U}}_q^-(\mathfrak{a}_1).\ddot{\mathrm{U}}_q^0(\mathfrak{a}_1).v \,. \tag{2.3.51}$$

Remark 2.3.21. In view of (2.3.51), simple highest t-weight $\ddot{\mathrm{U}}'_q(\mathfrak{a}_1)$-modules, including simple weight-finite $\ddot{\mathrm{U}}'_q(\mathfrak{a}_1)$-modules, are entirely determined as $M \cong \ddot{\mathrm{U}}_q^-(\mathfrak{a}_1).M_0$, by the data of their unique highest t-weight space $M_0 \cong \ddot{\mathrm{U}}_q^0(\mathfrak{a}_1).v$. Classifying simple weight-finite $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)$-modules therefore amounts to classifying those simple $\ddot{\mathrm{U}}_q^0(\mathfrak{a}_1)$-modules that appear as their highest t-weight spaces. We intend to undertake that classification in a future work.

Remark 2.3.22. (2.3.51) induces a partial ordering of the t-weight spaces through the Q^--grading of $\ddot{\mathrm{U}}_q^-(\mathfrak{a}_1)$.

2.3.7 Topological Hopf algebra structure on $\widehat{\ddot{\mathrm{U}}'_q(\mathfrak{a}_1)}$

Definition-Proposition 2.3.23. We define

$$\dot{\Delta} = \left(\widehat{\Psi} \widehat{\otimes} \widehat{\Psi}\right) \circ \Delta \circ \widehat{\Psi}^{-1} \,, \tag{2.3.52}$$

$$\dot{S} = \widehat{\Psi} \circ S \circ \widehat{\Psi}^{-1} \,, \tag{2.3.53}$$

$$\dot{\varepsilon} = \varepsilon \circ \widehat{\Psi}^{-1} \,. \tag{2.3.54}$$

Equipped with the above comultiplication, antipode and counit, $\widehat{\ddot{\mathrm{U}}'_q(\mathfrak{a}_1)}$ is a topological Hopf algebra. The latter is easily extended into a topological Hopf algebraic structure on $\widehat{\ddot{\mathrm{U}}_q(\mathfrak{a}_1)}$ by setting, in addition,

$$\dot{\Delta}(\mathsf{D}_1^{\pm 1}) = \mathsf{D}_1^{\pm 1} \otimes \mathsf{D}_1^{\pm 1} \,, \qquad \dot{S}(\mathsf{D}_1^{\pm 1}) = \mathsf{D}_1^{\mp 1} \qquad \text{and} \qquad \dot{\varepsilon}(\mathsf{D}_1^{\pm 1}) = 1 \,.$$

2.4 Doubly Affine Damiani-Beck isomorphism

In this last section, we complete the proof of theorem 3.3.22 by constructing $\Psi^{-1} : \ddot{\mathrm{U}}'_q(\mathfrak{a}_1) \to \widehat{\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)}$; i.e. by constructing a realization of the generators of $\ddot{\mathrm{U}}'_q(\mathfrak{a}_1)$ in $\widehat{\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)}$.

2.4.1 Double loop generators

Definition 2.4.1. For every $m \in \mathbb{Z}$, we set $\mathbf{X}^{\pm}_{1,m}(z) := Y^{\mp m}(\mathbf{x}^{\pm}_1(z))$.

It is clear that

Proposition 2.4.2. *For every $m \in \mathbb{Z}$, we have*

$$\varphi\left(\mathbf{X}^{\pm}_{1,m}(z)\right) = \mathbf{X}^{\mp}_{1,-m}(1/z)\,. \tag{2.4.1}$$

Proposition 2.4.3. *i. There exists a unique $\boldsymbol{\psi}^{+}_{1,1}(z) \in \widehat{\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)}[[z, z^{-1}]]$ such that*

$$\left[Y\left(\mathbf{k}^{-}_1(w)^{-1}\mathbf{x}^{-}_1(C^{1/2}w)\right), \mathbf{x}^{+}_1(z)\right]_{G^{-}_{10}(C^{-1/2}w/z)} = -\delta\left(\frac{C^{-1/2}q^2w}{z}\right)\boldsymbol{\psi}^{+}_{1,1}(z)\,. \tag{2.4.2}$$

ii. Set $\boldsymbol{\psi}^{-}_{1,-1}(z) = \varphi\left(\boldsymbol{\psi}^{+}_{1,1}(1/z)\right)$. Then, we have

$$\left[\mathbf{x}^{-}_1(z), Y\left(\mathbf{x}^{+}_1(C^{1/2}w)\mathbf{k}^{+}_1(w)^{-1}\right),\right]_{G^{+}_{10}(C^{1/2}z/w)} = -\delta\left(\frac{C^{-1/2}q^2w}{z}\right)\boldsymbol{\psi}^{-}_{1,-1}(z)\,. \tag{2.4.3}$$

Proof. The proof of *i.* is immediate from the definitions. *ii.* then follows by applying φ to (2.4.2). ☐

Remark 2.4.4. It is worth noting that $\boldsymbol{\psi}^{\pm}_{1,\pm1}(z) \notin \dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)[[z, z^{-1}]]$.

Corollary 2.4.5. *For every $i \in \dot{I}$, we have*

i. $\mathbf{k}^{-}_i(v)\boldsymbol{\psi}^{\pm}_{1,\pm1}(z) = G^{\mp}_{i,0}(C^{\mp1/2}q^2v/z)G^{\mp}_{i,1}(C^{\mp1/2}v/z)\boldsymbol{\psi}^{\pm}_{1,\pm1}(z)\mathbf{k}^{-}_i(v)$;

ii. $\boldsymbol{\psi}^{\pm}_{1,\pm1}(z)\mathbf{k}^{+}_i(v) = G^{\mp}_{i,0}(C^{\mp1/2}q^{-2}z/v)G^{\mp}_{i,1}(C^{\mp1/2}z/v)\mathbf{k}^{+}_i(v)\boldsymbol{\psi}^{\pm}_{1,\pm1}(z)$;

Proof. ii. follows by applying φ to *i.* and *i.* is a direct consequence of (2.4.2) and (2.4.3) on one hand and of (4.2.6) and (4.2.7) on the other hand. ☐

Let us then define the following $\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)$-valued formal power series

$$\boldsymbol{\Gamma}^{\pm}_0(z) := \mathbf{k}^{\pm}_0(z)\mathbf{k}^{\pm}_1(z) \in \dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)[[z^{\mp1}]]\,. \tag{2.4.4}$$

Denoting by $\mathcal{Z}(\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1))$ the center of $\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)$, it is straightforward to check that indeed

Proposition 2.4.6. $\boldsymbol{\Gamma}^{\pm}_0(z) \in \mathcal{Z}(\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1))[[z^{\mp1}]]$.

Similarly, define

$$\wp^{\pm}(z) := \mathbf{k}_0^{\pm}(z)\mathbf{k}_1^{\pm}(zq^2) \in \dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)[[z^{\mp 1}]] \,. \tag{2.4.5}$$

Then we establish an important result.

Proposition 2.4.7. *We have the following fixed points of Y;*

$$Y\left(\wp^{\pm}(z)\right) = \wp^{\pm}(z)\,, \tag{2.4.6}$$

$$Y\left(\boldsymbol{\psi}^{\pm}_{1,\pm 1}(z)\right) = \boldsymbol{\psi}^{\pm}_{1,\pm 1}(z)\,. \tag{2.4.7}$$

Moreover

$$Y\left(\boldsymbol{\Gamma}_0^{\pm}(z)\right) = \boldsymbol{\Gamma}_0^{\pm}(zq^2)\,, \tag{2.4.8}$$

Proof. (2.4.6) and (2.4.8) are obvious. We prove (2.4.7) for the upper choice of signs. In order to do so, we first rewrite (2.4.2) as

$$\left[\mathbf{x}_0^+(w), \mathbf{x}_1^+(z)\right]_{G^-_{10}(w/z)} = \delta\left(\frac{q^2 w}{z}\right)\boldsymbol{\psi}^+_{1,1}(z)\,.$$

Now, (2.2.38) and the definition of Y imply that, on one hand,

$$\begin{aligned}
&\left[\left[\mathbf{x}_0^+(z_1), \left[\mathbf{x}_0^+(z_2), \mathbf{x}_1^+(wq^2)\right]_{G^-_{10}(z_2/wq^2)}\right]_{G^-_{11}(z_1/z_2)G^-_{10}(z_1/wq^2)}, \mathbf{x}_0^-(C^{-1}z)\mathbf{k}_0^+(C^{-1/2}z)^{-1}\right]_{G^-_{10}(w/z)} \\
= \;& -[2]_q\,\delta\left(\frac{z_1}{z_2q^2}\right)\delta\left(\frac{z_2}{w}\right)\left[Y\left(\mathbf{x}_0^+(w)\right), Y\left(\mathbf{x}_1^+(z)\right)\right]_{G^-_{10}(w/z)} \\
= \;& -[2]_q\,\delta\left(\frac{z_1}{z_2q^2}\right)\delta\left(\frac{z_2}{w}\right)Y\left(\left[\mathbf{x}_0^+(w), \mathbf{x}_1^+(z)\right]_{G^-_{10}(w/z)}\right) = -[2]_q\,\delta\left(\frac{z_1}{z_2q^2}\right)\delta\left(\frac{z_2}{w}\right)\delta\left(\frac{wq^2}{z}\right)Y\left(\boldsymbol{\psi}^+_{1,1}(z)\right)\,,
\end{aligned}$$

whereas, on the other hand, (2.2.38), (4.2.6), (4.2.7) and (4.2.9), as well as corollary 2.4.5, imply that

$$\begin{aligned}
&\left[\left[\mathbf{x}_0^+(z_1), \left[\mathbf{x}_0^+(z_2), \mathbf{x}_1^+(wq^2)\right]_{G^-_{10}(z_2/wq^2)}\right]_{G^-_{11}(z_1/z_2)G^-_{10}(z_1/wq^2)}, \mathbf{x}_0^-(C^{-1}z)\mathbf{k}_0^+(C^{-1/2}z)^{-1}\right]_{G^-_{10}(w/z)} \\
= \;& \left[\left[\mathbf{x}_0^+(z_1), \left[\mathbf{x}_0^+(z_2), \mathbf{x}_1^+(wq^2)\right]_{G^-_{10}(z_2/wq^2)}\right]_{G^-_{11}(z_1/z_2)G^-_{10}(z_1/wq^2)}, \mathbf{x}_0^-(C^{-1}z)\right]\mathbf{k}_0^+(C^{-1/2}z)^{-1} \\
= \;& \frac{1}{q-q^{-1}}\left\{\delta\left(\frac{z_1}{z}\right)\delta\left(\frac{z_2}{w}\right)\left[\mathbf{k}_0^+(z_1C^{-1/2}), \boldsymbol{\psi}^+_{1,1}(wq^2)\right]_{G^-_{11}(z_1/z_2)G^-_{10}(z_1/wq^2)}\right. \\
& \left.+\,\delta\left(\frac{z_2}{z}\right)\left[\mathbf{x}_0^+(z_1), \left[\mathbf{k}_0^+(z_2C^{-1/2}), \mathbf{x}_1^+(wq^2)\right]_{G^-_{10}(z_2/wq^2)}\right]_{G^-_{11}(z_1/z_2)G^-_{10}(z_1/wq^2)}\right\}\mathbf{k}_0^+(C^{-1/2}z)^{-1} \\
= \;& \delta\left(\frac{z_1}{z}\right)\delta\left(\frac{z_2}{w}\right)\frac{G^+_{00}(w/z_1)G^+_{01}(q^2w/z_1) - G^-_{11}(z_1/w)G^-_{10}(z_1/wq^2)}{q-q^{-1}}\boldsymbol{\psi}^+_{1,1}(wq^2) \\
& +\delta\left(\frac{z_2}{z}\right)\delta\left(\frac{z_1}{w}\right)\frac{G^+_{01}(q^2w/z_2) - G^-_{10}(z_2/wq^2)}{q-q^{-1}}\boldsymbol{\psi}^+_{1,1}(wq^2)\,.
\end{aligned}$$

Making use of (3.3.68) and (5.1.5) – for the latter, see Appendix –, we eventually get

$$\begin{aligned}
&\left[\left[\mathbf{x}_0^+(z_1), \left[\mathbf{x}_0^+(z_2), \mathbf{x}_1^+(wq^2)\right]_{G_{10}^-(z_2/wq^2)}\right]_{G_{11}^-(z_1/z_2)G_{10}^-(z_1/wq^2)}, \mathbf{x}_0^-(C^{-1}z)\mathbf{k}_0^+(C^{-1/2}z)^{-1}\right]_{G_{10}^-(w/z)} \\
= \;& [2]_q\,\delta\left(\frac{z_1}{z}\right)\delta\left(\frac{z_2}{w}\right)\left[\delta\left(\frac{w}{z_1}\right) - \delta\left(\frac{wq^2}{z_1}\right)\right]\boldsymbol{\psi}_{1,1}^+(wq^2) - [2]_q\,\delta\left(\frac{z_2}{z}\right)\delta\left(\frac{z_1}{w}\right)\delta\left(\frac{w}{z_2}\right)\boldsymbol{\psi}_{1,1}^+(wq^2) \\
= \;& -[2]_q\,\delta\left(\frac{z_1}{z}\right)\delta\left(\frac{z_2}{w}\right)\delta\left(\frac{wq^2}{z_1}\right)\boldsymbol{\psi}_{1,1}^+(z)\,,
\end{aligned}$$

thus proving the result. The case with lower choice of signs follows by applying φ. ☐

Proposition 2.4.8. *For every $m \in \mathbb{Z}$, we have*

i. $\left[\boldsymbol{\psi}_{1,1}^+(z), \mathbf{X}_{1,m}^-(v)\right] = -[2]_q\delta\left(\frac{Cz}{v}\right)\wp^-(C^{1/2}q^{-2}z)\mathbf{X}_{1,m+1}^-(Cq^{-2}z)$;

ii. $\left[\boldsymbol{\psi}_{1,1}^+(z), \mathbf{X}_{1,m}^+(v)\right]_{G_{10}^-(z/vq^2)G_{11}^-(z/v)} = [2]_q\delta\left(\frac{z}{vq^2}\right)\mathbf{X}_{1,m+1}^+(z)$.

iii. $\left[\boldsymbol{\psi}_{1,-1}^-(z), \mathbf{X}_{1,-m}^+(v)\right] = [2]_q\delta\left(\frac{Cz}{v}\right)\mathbf{X}_{1,-(m+1)}^+(Cq^{-2}z)\wp^+(C^{1/2}q^{-2}z)$;

iv. ${}_{G_{10}^+(vq^2/z)G_{11}^+(v/z)}\left[\boldsymbol{\psi}_{1,-1}^-(z), \mathbf{X}_{1,-m}^-(v)\right] = -[2]_q\delta\left(\frac{z}{vq^2}\right)\mathbf{X}_{1,-(m+1)}^-(z)$.

v. $\left[\boldsymbol{\psi}_{1,1}^+(z), \boldsymbol{\psi}_{1,-1}^-(v)\right] = \frac{[2]_q}{q-q^{-1}}\left[\delta\left(\frac{z}{Cv}\right)\wp^+(C^{-1/2}q^{-2}z) - \delta\left(\frac{Cz}{v}\right)\wp^-(C^{-1/2}q^{-2}v)\right]$.

Proof. *i.* and *ii.* are readily checked for $m = 0$. Then, assuming they hold for some $m \in \mathbb{Z}$ and applying $Y^{\pm 1}$, it follows from propositon 2.4.7 that they also hold for $m \pm 1$. *iii.* and *iv.* are obtained by applying φ to *i.* and *ii.* respectively. Finally *v.* is obtained by direct calculation from the definitions of $\boldsymbol{\psi}_{1,1}^+(z)$ and $\boldsymbol{\psi}_{1,-1}^-(v)$, i.e.

$$\begin{aligned}
&\delta\left(\frac{C^{-1/2}q^2w}{z}\right)\delta\left(\frac{C^{1/2}q^{-2}u}{v}\right)\left[\boldsymbol{\psi}_{1,1}^+(z), \boldsymbol{\psi}_{1,-1}^-(u)\right] \\
= \;& \left[\left[\mathbf{x}_0^+(C^{-1/2}w), \mathbf{x}_1^+(z)\right]_{G_{10}^-(C^{-1/2}w/z)}, \left[\mathbf{x}_1^-(u), \mathbf{x}_0^-(C^{-1/2}v)\right]_{G_{10}^+(C^{1/2}u/v)}\right] \\
= \;& [2]_q\delta\left(\frac{C^{-1/2}q^2v}{u}\right)\left\{\delta\left(\frac{z}{Cu}\right)\left[\mathbf{x}_0^+(C^{-1/2}w), \mathbf{x}_0^-(C^{-1/2}v)\mathbf{k}_1^+(C^{-1/2}z)\right]_{G_{10}^-(C^{-1/2}w/z)}\right. \\
&\left. -\delta\left(\frac{Cw}{v}\right)\left[\mathbf{k}_0^-(C^{-1}v)\mathbf{x}_1^-(u), \mathbf{x}_1^+(z)\right]_{G_{10}^-(C^{-1/2}w/z)}\right\} \\
= \;& \frac{[2]_q}{q-q^{-1}}\delta\left(\frac{C^{1/2}q^{-2}u}{v}\right)\delta\left(\frac{C^{-1/2}q^2w}{z}\right)\left\{\delta\left(\frac{z}{Cu}\right)\mathbf{k}_0^+(C^{-1}w)\mathbf{k}_1^+(C^{-1/2}z)\right. \\
&\left. -\delta\left(\frac{Cz}{u}\right)\mathbf{k}_0^-(C^{-1}v)\mathbf{k}_1^-(C^{-1/2}u)\right\}.
\end{aligned}$$

Compare with (2.4.5) to conclude the proof. ☐

Definition-Proposition 2.4.9. For every $m \in \mathbb{N}^\times$ there exist $\boldsymbol{\psi}_{1,m}^+(z), \boldsymbol{\Gamma}_m^+(z) \in \widehat{\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)}[[z, z^{-1}]]$, such that

$$\boldsymbol{\Gamma}_1^+(v) = 0 \tag{2.4.9}$$

and, for every $m, n \in \mathbb{N}^\times$,

$$\begin{aligned}\left[Y^m\left(\mathbf{k}_1^-(z)^{-1}\mathbf{x}_1^-(C^{1/2}z)\right), \mathbf{x}_1^+(v)\right]_{G_{01}^-\left(z/C^{1/2}v\right)} &= -\delta\left(\frac{z}{C^{1/2}v}\right)\mathbf{\Gamma}_m^+(v) \\ &\quad +(q-q^{-1})\sum_{k=1}^{m-2}\delta\left(\frac{q^{2k}z}{C^{1/2}v}\right)\boldsymbol{\psi}_{1,k}^+(v)\mathbf{\Gamma}_{m-k}^+(v) \\ &\quad -\delta\left(\frac{q^{2m}z}{C^{1/2}v}\right)\boldsymbol{\psi}_{1,m}^+(v)\,,\end{aligned} \tag{2.4.10}$$

$$Y\left(\boldsymbol{\psi}_{1,m}^+(v)\right) = \boldsymbol{\psi}_{1,m}^+(v)\,, \tag{2.4.11}$$

$$Y\left(\mathbf{\Gamma}_m^+(v)\right) = \mathbf{\Gamma}_m^+(vq^2)\,, \tag{2.4.12}$$

$$\begin{aligned}{}_{G_{01}^-(q^{-2m}v/w)G_{11}^-(q^{2(1-m)}v/w)}\left[\boldsymbol{\psi}_{1,1}^+(w), \boldsymbol{\psi}_{1,m}^+(v)\right]_{G_{01}^-(w/vq^2)G_{11}^-(w/v)} &= [2]_q\delta\left(\frac{w}{vq^2}\right)\boldsymbol{\psi}_{1,m+1}^+(q^2v) \\ &\quad -[2]_q\delta\left(\frac{q^{2m}w}{v}\right)\boldsymbol{\psi}_{1,m+1}^+(v)\,,\end{aligned} \tag{2.4.13}$$

$$[\boldsymbol{\psi}_{1,n}^+(w), \mathbf{\Gamma}_m^+(v)] = 0\,. \tag{2.4.14}$$

Proof. It suffices to prove the proposition with $n = 1$ since the general case follows by an easy recursion on n once we have (3.3.82). The proof for $n = 1$ is by recursion on m. For $m = 1$, (2.4.9) and (2.4.10) are definition-proposition 2.4.3, whereas (2.4.11) is proposition 2.4.7. (2.4.12) and (2.4.14) – with $n = 1$ – automatically follow from (2.4.9). Making use of propsition 2.4.8, it is straightforward to prove that, for every $m \in \mathbb{N}^\times$,

$$\begin{aligned} & [2]_q\delta\left(\frac{z}{uq^2}\right)Y^{-1}\left(\left[Y^{m+1}\left(\mathbf{k}_1^-(C^{-1/2}v)^{-1}\mathbf{x}_1^-(v)\right), \mathbf{x}_1^+(uq^2)\right]_{G_{01}^-(C^{-1}q^{-2}v/u)}\right) \\ - \; & [2]_q\delta\left(\frac{Cz}{v}\right)\left[Y^{m+1}\left(\mathbf{k}_1^-(C^{1/2}q^{-2}z)^{-1}\mathbf{x}_1^-(Cq^{-2}z)\right), \mathbf{x}_1^+(u)\right]_{G_{01}^-(z/uq^2)} \\ = \; & {}_{G_{10}^-(v/Cz)G_{11}^-(vq^2/Cz)}\left[\boldsymbol{\psi}_{1,1}^+(z), \left[Y^m\left(\mathbf{k}_1^-(C^{-1/2}v)^{-1}\mathbf{x}_1^-(v)\right), \mathbf{x}_1^+(u)\right]_{G_{01}^-(C^{-1}v/u)}\right]_{G_{10}^-(z/uq^2)G_{11}^-(z/u)}\,. \end{aligned} \tag{2.4.15}$$

If $m = 1$, (3.3.82) is an easy consequence of the above equation. Now assume that the proposition holds up to some $m \in \mathbb{N}^\times$. Then (2.4.15) reads, for that m,

$$\begin{aligned}
& [2]_q \delta\left(\frac{z}{uq^2}\right) Y^{-1}\left(\left[Y^{m+1}\left(\mathbf{k}_1^-(C^{-1/2}v)^{-1}\mathbf{x}_1^-(v)\right), \mathbf{x}_1^+(uq^2)\right]_{G_{01}^-(C^{-1}q^{-2}v/u)}\right) \\
- \quad & [2]_q \delta\left(\frac{Cz}{v}\right) \left[Y^{m+1}\left(\mathbf{k}_1^-(C^{1/2}q^{-2}z)^{-1}\mathbf{x}_1^-(Cq^{-2}z)\right), \mathbf{x}_1^+(u)\right]_{G_{01}^-(z/uq^2)} \\
= \quad & -\delta\left(\frac{v}{Cu}\right)_{G_{10}^-(v/Cz)G_{11}^-(vq^2/Cz)} \left[\boldsymbol{\psi}_{1,1}^+(z), \boldsymbol{\Gamma}_m^+(u)\right]_{G_{10}^-(z/uq^2)G_{11}^-(z/u)} \\
& +(q-q^{-1})\sum_{k=1}^{m-2} \delta\left(\frac{q^{2k}v}{Cu}\right)_{G_{10}^-(v/Cz)G_{11}^-(vq^2/Cz)} \left[\boldsymbol{\psi}_{1,1}^+(z), \boldsymbol{\psi}_{1,k}^+(u)\right]_{G_{10}^-(z/uq^2)G_{11}^-(z/u)} \boldsymbol{\Gamma}_{m-k}^+(u) \\
& -\delta\left(\frac{q^{2m}v}{Cu}\right)_{G_{10}^-(v/Cz)G_{11}^-(vq^2/Cz)} \left[\boldsymbol{\psi}_{1,1}^+(z), \boldsymbol{\psi}_{1,m}^+(u)\right]_{G_{10}^-(z/uq^2)G_{11}^-(z/u)} \\
= \quad & -[2]_q(q-q^{-1})\delta\left(\frac{v}{Cu}\right)\left\{\delta\left(\frac{v}{Cz}\right) - \delta\left(\frac{vq^2}{Cz}\right)\right\} \boldsymbol{\psi}_{1,1}^+(z)\boldsymbol{\Gamma}_m^+(u) \\
& +[2]_q(q-q^{-1})\sum_{k=1}^{m-2} \delta\left(\frac{q^{2k}v}{Cu}\right)\left\{\delta\left(\frac{z}{uq^2}\right)\boldsymbol{\psi}_{1,k+1}^+(uq^2) - \delta\left(\frac{zq^{2k}}{u}\right)\boldsymbol{\psi}_{k+1}^+(u)\right\}\boldsymbol{\Gamma}_{m-k}^+(u) \\
& -[2]_q\delta\left(\frac{q^{2m}v}{Cu}\right)\left\{\delta\left(\frac{z}{uq^2}\right)\boldsymbol{\psi}_{1,m+1}^+(uq^2) - \delta\left(\frac{zq^{2m}}{u}\right)\boldsymbol{\psi}_{1,m+1}^+(u)\right\}.
\end{aligned}$$

It immediately follows that (2.4.10) holds at rank $m+1$, for some $\boldsymbol{\Gamma}_{m+1}^+(z) \in \widehat{\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)}[[z, z^{-1}]]$ satisfying (2.4.12). Considering (2.4.15) at rank $m+1$, and substituting the above results, we get

$$\begin{aligned}
& [2]_q \delta\left(\frac{z}{uq^2}\right) Y^{-1}\left(\left[Y^{m+2}\left(\mathbf{k}_1^-(C^{-1/2}v)^{-1}\mathbf{x}_1^-(v)\right), \mathbf{x}_1^+(uq^2)\right]_{G_{01}^-(C^{-1}q^{-2}v/u)}\right) \\
- \quad & [2]_q \delta\left(\frac{Cz}{v}\right) \left[Y^{m+2}\left(\mathbf{k}_1^-(C^{1/2}q^{-2}z)^{-1}\mathbf{x}_1^-(Cq^{-2}z)\right), \mathbf{x}_1^+(u)\right]_{G_{01}^-(z/uq^2)} \\
= \quad & -\delta\left(\frac{v}{Cu}\right)_{G_{10}^-(v/Cz)G_{11}^-(vq^2/Cz)} \left[\boldsymbol{\psi}_{1,1}^+(z), \boldsymbol{\Gamma}_{m+1}^+(u)\right]_{G_{10}^-(z/uq^2)G_{11}^-(z/u)} \\
& +[2]_q(q-q^{-1})\sum_{k=1}^{m-1} \delta\left(\frac{q^{2k}v}{Cu}\right)\left\{\delta\left(\frac{z}{uq^2}\right)\boldsymbol{\psi}_{1,k+1}^+(uq^2) - \delta\left(\frac{zq^{2k}}{u}\right)\boldsymbol{\psi}_{k+1}^+(u)\right\}\boldsymbol{\Gamma}_{m+1-k}^+(u) \\
& -\delta\left(\frac{q^{2(m+1)}v}{Cu}\right)_{G_{10}^-(v/Cz)G_{11}^-(vq^2/Cz)} \left[\boldsymbol{\psi}_{1,1}^+(z), \boldsymbol{\psi}_{1,m+1}^+(u)\right]_{G_{10}^-(z/uq^2)G_{11}^-(z/u)}.
\end{aligned}$$

It readily follows that, on one hand, there exists some $\boldsymbol{\psi}_{1,m+2}^+(v) \in \widehat{\ddot{\mathrm{U}}_q(\mathfrak{sl}_2)}[[v, v^{-1}]]$ such that (3.3.82) holds for $m+1$ and that, on the other hand,

$$(uq^2 - z)(u - z)\left[\boldsymbol{\psi}_{1,1}^+(z), \boldsymbol{\Gamma}_{m+1}^+(u)\right] = 0\,.$$

Since $Y(\boldsymbol{\Gamma}_{m+1}^+(u)) = \boldsymbol{\Gamma}_{m+1}^+(uq^2)$, we have that

$$(uq^{2(p+1)} - z)(uq^{2p} - z)\left[\boldsymbol{\psi}_{1,1}^+(z), \boldsymbol{\Gamma}_{m+1}^+(u)\right] = 0$$

for every $p \in \mathbb{Z}$ and, as a consequence, (2.4.14) holds for $m+1$. Finally, (2.4.11) for $m+1$ follows from the corresponding case of (3.3.82), which concludes the proof. □

Remark 2.4.10. Note that since $[\boldsymbol{\psi}^+_{1,n}(z), \boldsymbol{\Gamma}^+_m(v)] = 0$ for every $m, n \in \mathbb{N}^\times$, we have that

$$\boldsymbol{\psi}^+_{1,n,k}\boldsymbol{\Gamma}^+_{m,l} = \boldsymbol{\Gamma}^+_{m,l}\boldsymbol{\psi}^+_{1,n,k} \in \Omega_{l-k} \cap \Omega_{k-l}\,, \tag{2.4.16}$$

guaranteeing the convergence in $\widehat{\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)}$ of each of the terms of the series $\boldsymbol{\psi}^+_{1,k}(z)\boldsymbol{\Gamma}^+_{m-k}(z)$ on the the r.h.s of eq. (2.4.10).

Definition 2.4.11. For every $m \in \mathbb{N}^\times$, let

$$\boldsymbol{\Gamma}^-_{-m}(z) = \varphi(\boldsymbol{\Gamma}^+_m(1/z)) \qquad \text{and} \qquad \boldsymbol{\psi}^-_{1,-m}(z) = \varphi(\boldsymbol{\psi}^+_{1,m}(1/z))\,. \tag{2.4.17}$$

Then,

Corollary 2.4.12. *We have*

$$\boldsymbol{\Gamma}^-_{-1}(v) = 0 \tag{2.4.18}$$

and, for every $m, n \in \mathbb{N}^\times$,

$$\begin{aligned}\left[\mathbf{x}^-_1(v), Y^m\left(\mathbf{x}^+_1(C^{1/2}z)\mathbf{k}^+_1(z)^{-1}\right)\right]_{G^+_{01}(C^{1/2}v/z)} &= -\delta\left(\frac{z}{C^{1/2}v}\right)\boldsymbol{\Gamma}^-_{-m}(v)\\ &\quad -(q-q^{-1})\sum_{k=1}^{m-2}\delta\left(\frac{q^{2k}z}{C^{1/2}v}\right)\boldsymbol{\Gamma}^-_{-(m-k)}(v)\boldsymbol{\psi}^-_{1,-k}(v)\\ &\quad -\delta\left(\frac{q^{2m}z}{C^{1/2}v}\right)\boldsymbol{\psi}^-_{1,-m}(v)\,,\end{aligned} \tag{2.4.19}$$

$$Y\left(\boldsymbol{\psi}^-_{1,-m}(v)\right) = \boldsymbol{\psi}^-_{1,-m}(v)\,, \tag{2.4.20}$$

$$Y\left(\boldsymbol{\Gamma}^-_{-m}(v)\right) = \boldsymbol{\Gamma}^-_{-m}(vq^2)\,, \tag{2.4.21}$$

$$\begin{aligned}{}_{G^+_{01}(q^{2m}w/v)G^+_{11}(q^{2(m-1)}w/v)}\left[\boldsymbol{\psi}^-_{1,-m}(v), \boldsymbol{\psi}^-_{1,-1}(w)\right]_{G^+_{01}(vq^2/w)G^+_{11}(v/w)} &= [2]_q\delta\left(\frac{w}{vq^2}\right)\boldsymbol{\psi}^-_{1,-(m+1)}(q^2v)\\ &\quad -[2]_q\delta\left(\frac{q^{2m}w}{v}\right)\boldsymbol{\psi}^-_{1,-(m+1)}(v)\,,\end{aligned} \tag{2.4.22}$$

$$[\boldsymbol{\psi}^-_{1,-n}(w), \boldsymbol{\Gamma}^-_{-m}(v)] = 0\,. \tag{2.4.23}$$

Proof. It suffice to apply φ to the results of the previous proposition. ☐

Proposition 2.4.13. *For every $i \in \dot{I}$ and for every $m \in \mathbb{N}^\times$, we have*

- *i.* $\mathbf{k}^-_i(v)\boldsymbol{\psi}^\pm_{1,\pm m}(z) = G^\mp_{i,0}(C^{\mp 1/2}q^{2m}v/z)G^\mp_{i,1}(C^{\mp 1/2}v/z)\boldsymbol{\psi}^\pm_{1,\pm m}(z)\mathbf{k}^-_i(v)$;
- *ii.* $\boldsymbol{\psi}^\pm_{1,\pm m}(z)\mathbf{k}^+_i(v) = G^\mp_{i,0}(C^{\mp 1/2}q^{-2m}z/v)G^\mp_{i,1}(C^{\mp 1/2}z/v)\mathbf{k}^+_i(v)\boldsymbol{\psi}^\pm_{1,\pm m}(z)$;

Proof. Clearly *ii.* follows by applying φ to *i.*. We prove *i.* by induction on $m \in \mathbb{N}^\times$. The case $m = 1$ is corollary 2.4.5*i*. Now, assuming that *i.* holds for some $m \in \mathbb{N}^\times$, we can make use of (3.3.82) and (2.4.22) to show that

$$\begin{aligned}\mathbf{k}_i^-(v)\boldsymbol{\psi}^\pm_{1,\pm(m+1)}(z) &= G^\mp_{i,0}(C^{\mp 1/2}q^{2(m+1)}v/z)G^\mp_{i,1}(C^{\mp 1/2}q^{2m}v/z) \\ &\qquad \times G^\mp_{i,0}(C^{\mp 1/2}q^{2m}v/z)G^\mp_{i,1}(C^{\mp 1/2}z/v)\boldsymbol{\psi}^\pm_{1,\pm m}(z)\mathbf{k}_i^-(v) \\ &= G^\mp_{i,0}(C^{\mp 1/2}q^{2(m+1)}v/z)G^\mp_{i,1}(C^{\mp 1/2}z/v)\boldsymbol{\psi}^\pm_{1,\pm m}(z)\mathbf{k}_i^-(v)\end{aligned}$$

whch completes the recursion. □

The above proposition has the obvious

Corollary 2.4.14. *For every $m \in \mathbb{N}^\times$, we have*

$$\begin{aligned}\wp^-(v)\boldsymbol{\psi}^\pm_{1,\pm m}(z) &= G^\mp_{00}(C^{\mp 1/2}q^{2m}v/z)G^\mp_{01}(C^{\mp 1/2}v/z) \\ &\qquad G^\mp_{01}(C^{\mp 1/2}q^{2(m+1)}v/z)G^\mp_{11}(C^{\mp 1/2}q^{2}v/z)\boldsymbol{\psi}^\pm_{1,\pm m}(z)\wp^-(v)\end{aligned} \tag{2.4.24}$$

$$\begin{aligned}\boldsymbol{\psi}^\pm_{1,\pm m}(z)\wp^+(v) &= G^\mp_{00}(C^{\mp 1/2}q^{-2m}z/v)G^\mp_{01}(C^{\mp 1/2}z/v) \\ &\qquad G^\mp_{01}(C^{\mp 1/2}q^{-2(m+1)}z/v)G^\mp_{11}(C^{\mp 1/2}q^{-2}z/v)\wp^+(v)\boldsymbol{\psi}^\pm_{1,\pm m}(z)\end{aligned} \tag{2.4.25}$$

Proposition 2.4.15. *For every $m, n \in \mathbb{N}^\times$, we have*

$$\begin{aligned}\left[\boldsymbol{\psi}^+_{1,m}(v), \boldsymbol{\psi}^-_{1,-n}(w)\right] &= [2]_q\,(q-q^{-1})\left\{\delta\left(\frac{Cq^{2(1-m)}v}{w}\right)\wp^-(C^{-1/2}q^{-2m}v)\boldsymbol{\psi}^-_{1,-(n-1)}(wq^{-2})\boldsymbol{\psi}^+_{1,m-1}(v)\right. \\ &\qquad \left. -\delta\left(\frac{q^{2(n-1)}v}{Cw}\right)\boldsymbol{\psi}^-_{1,-(n-1)}(w)\boldsymbol{\psi}^+_{1,m-1}(vq^{-2})\wp^+(C^{1/2}q^{-2}v)\right\},\end{aligned}$$

where we assume that

$$\boldsymbol{\psi}^\pm_{1,0}(z) = \frac{1}{q-q^{-1}}\,. \tag{2.4.26}$$

Proof. The case $m = n = 1$ follows immediately by proposition 2.4.8.*v*. Now, applying $a \mapsto [a, \boldsymbol{\psi}^-_{1,-n}(w)]$ and $a \mapsto [\boldsymbol{\psi}^+_{1,n}(w), a]$ to (3.3.82) and (2.4.22) respectively and making use of corollary 2.4.14, one easily completes the recursion. □

2.4.2 Exchange relations

Proposition 2.4.16. *For every $m \in \mathbb{N}$, there exists some $\xi_m(z) \in \widehat{\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)}[[z, z^{-1}]]$ such that, for every $n \in \mathbb{Z}$,*

$$[\mathbf{X}^-_{1,m+n+1}(w), \mathbf{X}^-_{1,n}(z)]_{G^-_{01}(w/z)} = -[\mathbf{X}^-_{1,n+1}(w), \mathbf{X}^-_{1,m+n}(z)]_{G^-_{01}(w/z)} = \delta\left(\frac{wq^2}{z}\right)Y^n\left(\xi_m(z)\right)\,. \tag{2.4.27}$$

Proof. Assume first that $n = 0$. The case $m = 0$ then follows immediately from the definition of $\mathbf{X}^-_{1,1}(w)$ and relations (4.2.7) and (4.2.9), leading to $\xi_0(z) = 0$, as it should. Taking the commutator of the case

$m=0$ with $\boldsymbol{\psi}^+_{1,1}(v)$, we get

$$\begin{aligned}
0 &= [[\mathbf{X}^-_{1,1}(w), \mathbf{X}^-_{1,0}(z)]_{G^-_{01}(w/z)}, \boldsymbol{\psi}^+_{1,1}(v)] \\
&= [[\mathbf{X}^-_{1,1}(w), \boldsymbol{\psi}^+_{1,1}(v)], \mathbf{X}^-_{1,0}(z)]_{G^-_{01}(w/z)} + [\mathbf{X}^-_{1,1}(w), [\mathbf{X}^-_{1,0}(z), \boldsymbol{\psi}^+_{1,1}(v)]]_{G^-_{01}(w/z)} \\
&= [2]_q \wp^-(v) \left\{ \delta\left(\frac{C^{1/2}q^2v}{w}\right) [\mathbf{X}^-_{1,2}(wq^{-2}), \mathbf{X}^-_{1,0}(z)]_{G^-_{01}(wq^{-2}/z)} \right. \\
&\qquad \left. +\delta\left(\frac{C^{1/2}q^2v}{z}\right) {}_{G^-_{01}(zq^{-2}/w)G^-_{11}(z/w)}[\mathbf{X}^-_{1,1}(w), \mathbf{X}^-_{1,1}(zq^{-2})]_{G^-_{01}(w/z)} \right\} .
\end{aligned}$$

The latter implies that

$$[\mathbf{X}^-_{1,2}(wq^{-2}), \mathbf{X}^-_{1,0}(z)]_{G^-_{01}(wq^{-2}/z)} = \delta\left(\frac{w}{z}\right) \xi_1(z)\,, \tag{2.4.28}$$

$${}_{G^-_{01}(zq^{-2}/w)G^-_{11}(z/w)}[\mathbf{X}^-_{1,1}(w), \mathbf{X}^-_{1,1}(zq^{-2})]_{G^-_{01}(w/z)} = -\delta\left(\frac{w}{z}\right) \xi_1(z)\,, \tag{2.4.29}$$

for some $\xi_1(z) \in \widehat{\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)}[[z, z^{-1}]]$. Multiplying (2.4.29) by $(zq^{-2}-w)$ and subsequently factoring $(z-q^{-2}w)$, we get that

$${}_{G^-_{01}(zq^{-2}/w)}[\mathbf{X}^-_{1,1}(w), \mathbf{X}^-_{1,1}(zq^{-2})] = \delta\left(\frac{w}{z}\right) \xi_1(z) + \delta\left(\frac{w}{zq^2}\right) \eta_0(z)\,, \tag{2.4.30}$$

for some $\eta_0(z) \in \widehat{\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)}[[z, z^{-1}]]$. Multiplying the above equation by $q^{-2}(z-w)$, we get

$$(zq^{-4}-w)\mathbf{X}^-_{1,1}(w)\mathbf{X}^-_{1,1}(zq^{-2}) - q^{-2}(z-w)\mathbf{X}^-_{1,1}(zq^{-2})\mathbf{X}^-_{1,1}(w) = z(1-q^2)\delta\left(\frac{w}{zq^2}\right) \eta_0(z)\,. \tag{2.4.31}$$

But, on the other hand,

$$\begin{aligned}
& (zq^{-4}-w)\mathbf{X}^-_{1,1}(w)\mathbf{X}^-_{1,1}(zq^{-2}) - q^{-2}(z-w)\mathbf{X}^-_{1,1}(zq^{-2})\mathbf{X}^-_{1,1}(w) \\
=\ & Y\left((zq^{-4}-w)\mathbf{x}^-_1(w)\mathbf{x}^-_1(zq^{-2}) - q^{-2}(z-w)\mathbf{x}^-_1(zq^{-2})\mathbf{x}^-_1(w)\right) = 0
\end{aligned}$$

by relation (4.2.8). Substituting back into (2.4.31) proves that $\eta_0(z) = 0$ and that (2.4.30) eventually reads

$${}_{G^-_{01}(zq^{-2}/w)}[\mathbf{X}^-_{1,1}(w), \mathbf{X}^-_{1,1}(zq^{-2})] = \delta\left(\frac{w}{z}\right) \xi_1(z)\,. \tag{2.4.32}$$

Combining (2.4.28) and (2.4.32), we get the case $m=1$. Now assume that the result holds for all nonnegative integer less than $m \in \mathbb{N}$. Taking the commutator of (2.4.27) with $\boldsymbol{\psi}^{+}_{1,1}(v)$ yields

$$\begin{aligned}
&[2]_q \boldsymbol{\wp}^{-}(v) \left\{ \delta\left(\frac{C^{1/2}q^2 v}{w}\right) [\mathbf{X}^{-}_{1,m+2}(wq^{-2}), \mathbf{X}^{-}_{1,0}(z)]_{G^{-}_{01}(wq^{-2}/z)} \right. \\
&\qquad \left. +\delta\left(\frac{C^{1/2}q^2 v}{z}\right) {}_{G^{-}_{01}(zq^{-2}/w)G^{-}_{11}(z/w)}[\mathbf{X}^{-}_{1,m+1}(w), \mathbf{X}^{-}_{1,1}(zq^{-2})]_{G^{-}_{01}(w/z)} \right\} \\
= \;& -[2]_q \boldsymbol{\wp}^{-}(v) \left\{ \delta\left(\frac{C^{1/2}q^2 v}{w}\right) [\mathbf{X}^{-}_{1,2}(wq^{-2}), \mathbf{X}^{-}_{1,m}(z)]_{G^{-}_{01}(wq^{-2}/z)} \right. \\
&\qquad \left. +\delta\left(\frac{C^{1/2}q^2 v}{z}\right) {}_{G^{-}_{01}(zq^{-2}/w)G^{-}_{11}(z/w)}[\mathbf{X}^{-}_{1,1}(w), \mathbf{X}^{-}_{1,m+1}(zq^{-2})]_{G^{-}_{01}(w/z)} \right\} \\
= \;& \delta\left(\frac{wq^2}{z}\right) [\xi_m(z), \boldsymbol{\psi}^{+}_{1,1}(v)]
\end{aligned}$$

The latter implies that

$$[\mathbf{X}^{-}_{1,m+2}(wq^{-2}), \mathbf{X}^{-}_{1,0}(z)]_{G^{-}_{01}(wq^{-2}/z)} = \delta\left(\frac{w}{z}\right) \xi_{m+1}(z) + \delta\left(\frac{wq^2}{z}\right) \eta_1(z)\,, \tag{2.4.33}$$

$${}_{G^{-}_{01}(zq^{-2}/w)G^{-}_{11}(z/w)}[\mathbf{X}^{-}_{1,m+1}(w), \mathbf{X}^{-}_{1,1}(zq^{-2})]_{G^{-}_{01}(w/z)} = -\delta\left(\frac{w}{z}\right) \xi_{m+1}(z) + \delta\left(\frac{wq^2}{z}\right) \eta_1(z)\,, \tag{2.4.34}$$

$$[\mathbf{X}^{-}_{1,2}(wq^{-2}), \mathbf{X}^{-}_{1,m}(z)]_{G^{-}_{01}(wq^{-2}/z)} = \delta\left(\frac{w}{z}\right) \eta_3(z) - \delta\left(\frac{wq^2}{z}\right) \eta_1(z)\,, \tag{2.4.35}$$

$${}_{G^{-}_{01}(zq^{-2}/w)G^{-}_{11}(z/w)}[\mathbf{X}^{-}_{1,1}(w), \mathbf{X}^{-}_{1,m+1}(zq^{-2})]_{G^{-}_{01}(w/z)} = -\delta\left(\frac{w}{z}\right) \eta_3(z) - \delta\left(\frac{wq^2}{z}\right) \eta_2(z)\,, \tag{2.4.36}$$

for some $\xi_{m+1}(z), \eta_1(z), \eta_2(z), \eta_3(z) \in \widehat{\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)}[[z, z^{-1}]]$. Multiplying (2.4.36) by $(z - wq^2)$ and subsequently factoring $(zq^2 - w)$, we get that

$$[\mathbf{X}^{-}_{1,m+1}(z), \mathbf{X}^{-}_{1,1}(w)]_{G^{-}_{01}(z/w)} = -\delta\left(\frac{w}{zq^2}\right) \eta_3(w) + \delta\left(\frac{w}{zq^4}\right) \eta_4(z)\,, \tag{2.4.37}$$

for some $\eta_4(z) \in \widehat{\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)}[[z, z^{-1}]]$. But, by the recursion hypothesis,

$$[\mathbf{X}^{-}_{1,m+1}(z), \mathbf{X}^{-}_{1,1}(w)]_{G^{-}_{01}(z/w)} = Y\left([\mathbf{X}^{-}_{1,m}(z), \mathbf{X}^{-}_{1,0}(w)]_{G^{-}_{01}(z/w)}\right) = \delta\left(\frac{w}{zq^2}\right) Y\left(\xi_{m-1}(w)\right)\,.$$

Comparing with (2.4.37), it follows that

$$\eta_3(w) = -Y\left(\xi_{m-1}(w)\right) \qquad \text{and} \qquad \eta_4(z) = 0\,.$$

By the recursion hypothesis, we also have

$$\begin{aligned}
[\mathbf{X}^{-}_{1,2}(wq^{-2}), \mathbf{X}^{-}_{1,m}(z)]_{G^{-}_{01}(wq^{-2}/z)} &= Y\left([\mathbf{X}^{-}_{1,1}(wq^{-2}), \mathbf{X}^{-}_{1,m-1}(z)]_{G^{-}_{01}(wq^{-2}/z)}\right) = -\delta\left(\frac{w}{z}\right) Y\left(\xi_{m-1}(z)\right) \\
&= \delta\left(\frac{w}{z}\right) \eta_3(z)
\end{aligned}$$

Comparing the above result with (2.4.35), we conclude that $\eta_1(z) = 0$. As a consequence, (2.4.33) now reads

$$[\mathbf{X}^-_{1,m+2}(w), \mathbf{X}^-_{1,0}(z)]_{G^-_{01}(w/z)} = \delta\left(\frac{wq^2}{z}\right)\xi_{m+1}(z)\,. \tag{2.4.38}$$

On the other hand, multiplying (2.4.34) by $(z - wq^2)$ and subsequently factoring $(zq^2 - w)$, we get that

$${}_{G^-_{01}(zq^{-2}/w)}[\mathbf{X}^-_{1,m+1}(w), \mathbf{X}^-_{1,1}(zq^{-2})] = \delta\left(\frac{w}{z}\right)\xi_{m+1}(z) + \delta\left(\frac{w}{zq^2}\right)\eta_5(z)\,, \tag{2.4.39}$$

for some $\eta_5(z) \in \widehat{\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)}[[z, z^{-1}]]$. Multiplying the above equation by $(z - w)$ yields

$$Y\left((zq^{-2} - wq^2)\mathbf{X}^-_{1,m}(w)\mathbf{X}^-_{1,0}(zq^{-2}) - (z - w)\mathbf{X}^-_{1,0}(zq^{-2})\mathbf{X}^-_{1,m}(w)\right) = z(1 - q^2)\delta\left(\frac{zq^2}{w}\right)\eta_5(z)\,. \tag{2.4.40}$$

But the recursion hypothesis

$$[\mathbf{X}^-_{1,m}(w), \mathbf{X}^-_{1,0}(zq^{-2})]_{G^-_{01}(wq^2/z)} = \delta\left(\frac{wq^4}{z}\right)\xi_{m-1}(z) \tag{2.4.41}$$

implies, upon multiplication by $(zq^{-2} - wq^2)$, that

$$(zq^{-2} - wq^2)\mathbf{X}^-_{1,m}(w)\mathbf{X}^-_{1,0}(zq^{-2}) - (z - w)\mathbf{X}^-_{1,0}(zq^{-2})\mathbf{X}^-_{1,m}(w) = 0\,. \tag{2.4.42}$$

Substituting back into (2.4.40) proves that $\eta_5(z) = 0$ and that (2.4.39) eventually reads

$${}_{G^-_{01}(w/z)}[\mathbf{X}^-_{1,m+1}(z), \mathbf{X}^-_{1,1}(w)] = \delta\left(\frac{wq^2}{z}\right)\xi_{m+1}(z)\,. \tag{2.4.43}$$

Combining (2.4.38) and (2.4.43) completes the recursion and the result holds for any $m \in \mathbb{N}$, assuming $n = 0$. The cases $n \in \mathbb{Z}^\times$ are then obtained by applying Y^n to the case $n = 0$. □

Corollary 2.4.17. *For every $m \in \mathbb{N}$ and every $n \in \mathbb{Z}$, we have*

$$[\mathbf{X}^+_{1,m+n+1}(z), \mathbf{X}^+_{1,n}(w)]_{G^+_{01}(z/w)} = -[\mathbf{X}^+_{1,n+1}(z), \mathbf{X}^+_{1,m+n}(w)]_{G^+_{01}(z/w)} = \delta\left(\frac{wq^2}{z}\right)\varphi\circ Y^{-m-n-1}\left(\xi_m(1/z)\right)\,. \tag{2.4.44}$$

Proof. It suffices to apply $\varphi \circ Y^{-m-n-1}$ to (2.4.27). □

We now return to the proof of theorem 3.3.22 and to the map $\Psi : \dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1) \to \widehat{\ddot{\mathrm{U}}'_q(\mathfrak{a}_1)}$.

Corollary 2.4.18. *We have*

i. $\Upsilon^\pm(w) = 0$*;*

ii. and for every $i \neq j$,

$$\sum_{\sigma\in S_3}\sum_{k=0}^{3}(-1)^k\binom{3}{k}_q \Psi(\mathbf{x}_i^\pm(z_{\sigma(1)}))\cdots\Psi(\mathbf{x}_i^\pm(z_{\sigma(k)}))\Psi(\mathbf{x}_j^\pm(z))\Psi(\mathbf{x}_i^\pm(z_{\sigma(k+1)}))\cdots\Psi(\mathbf{x}_i^\pm(z_{\sigma(3)})) = 0\,.$$

Proof. The proof of proposition 2.4.16 makes it clear that the relations (4.2.9) with $i \neq j$ there, both follow from the relations

$$\left[\mathbf{X}^+_{1,0}(v), \mathbf{X}^+_{1,-1}(w)\right]_{G^-_{11}(v/w)} = 0 \tag{2.4.45}$$

and

$$\left[\mathbf{X}^-_{1,1}(v), \mathbf{X}^-_{1,0}(w)\right]_{G^+_{11}(v/w)} = 0 \tag{2.4.46}$$

in the completion $\widehat{\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)}$. A tedious but straightforward calculation shows that the quantum Serre relations (4.2.10) similarly follow from

$$\left[\mathbf{X}^+_{1,-1}(v), \mathbf{X}^+_{1,-2}(w)\right]_{G^-_{11}(v/w)} = 0 \tag{2.4.47}$$

and

$$\left[\mathbf{X}^-_{1,2}(v), \mathbf{X}^-_{1,1}(w)\right]_{G^+_{11}(v/w)} = 0\,, \tag{2.4.48}$$

which in turn are a consequence of ((2.4.45) – (2.4.46)) – just apply Y there. We can therefore extend $\Psi : \dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1) \to \widehat{\ddot{\mathrm{U}}'_q(\mathfrak{a}_1)}$ by continuity [1] into $\widehat{\Psi} : \widehat{\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)} \to \widehat{\ddot{\mathrm{U}}'_q(\mathfrak{a}_1)}$ and it suffices to check point *i.* Since by construction $\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)$ is dense in $\widehat{\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)}$, there exists a sequence $(u_n(v,w))_{n\in\mathbb{N}} \in \dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)[[v, v^{-1}, w, w^{-1}]]^{\mathbb{N}}$ such that

$$\lim_{n\to+\infty} u_n(v,w) = 0\,, \tag{2.4.49}$$

whereas, on the other hand,

$$\lim_{n\to+\infty} \widehat{\Psi}(u_n(v,w)) = \delta\left(\frac{vq^{\mp 2}}{w}\right)\Upsilon^\pm(w)\,. \tag{2.4.50}$$

Take for example the partial sum of the series involved on the l.h.s. of equations ((2.4.45) – (2.4.46)) above. The result now follows by the continuity of $\widehat{\Psi}$. □

Remark 2.4.19. We have therefore completed the proof of that part of theorem 3.3.22 that claims the existence of a continuous algebra homomorphism $\widehat{\Psi} : \widehat{\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)} \to \widehat{\ddot{\mathrm{U}}'_q(\mathfrak{a}_1)}$. We still have to construct the inverse continuous algebra homomorphism $\widehat{\Psi}^{-1} : \widehat{\ddot{\mathrm{U}}'_q(\mathfrak{a}_1)} \to \widehat{\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)}$. This shall be done at the end of the present section.

2.4.3 Weight grading relations

The results of the previous subsection have the following

Corollary 2.4.20. *For every $m \in \mathbb{N}^\times$ and every $n \in \mathbb{Z}$, we have:*

i. $[\mathbf{\Gamma}^+_{m+1}(u), \mathbf{X}^-_{1,n}(z)] = 0$;

ii. $[\boldsymbol{\psi}^+_{1,m+1}(u), \mathbf{X}^-_{1,n}(z)] = -\wp^-(C^{1/2}uq^{-2(m+1)})_{G^+_{01}(Cuq^{2(1-m)}/z)}[\mathbf{X}^-_{1,n+1}(zq^{-2}), \boldsymbol{\psi}^+_{1,m}(u)]_{G^-_{01}(z/Cuq^{2(1-m)})} \propto \delta\left(\frac{Cu}{zq^2m}\right)$;

iii. $[\mathbf{\Gamma}^+_{m+1}(u), \mathbf{X}^+_{1,n}(z)] = 0$;

iv. $[\boldsymbol{\psi}^+_{1,m+1}(v), \mathbf{X}^+_{1,n}(z)]_{G^+_{01}(v/z)G^+_{11}(v/zq^{2(m+1)})} = -_{G^-_{01}(z/vq^{2m})}[\mathbf{X}^+_{1,n+1}(v), \boldsymbol{\psi}^+_{1,m}(z)]_{G^+_{01}(v/z)} \propto \delta\left(\frac{zq^2}{v}\right)$.

Proof. It suffices to prove the proposition for $n = 0$ as the general case then follows by applying Y^n for any $n \in \mathbb{Z}$. Assuming that $n = 0$ in *i.* and *ii.*, it then suffices to take the commutator of (2.4.27) – for $n = 1$ there – with $\mathbf{x}^+_1(z)$. □

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[1] Ψ is obviously $\mathbb{Z}_{(2)}$-graded, hence continuous.

Remark 2.4.21. It turns out the, for every $m \in \mathbb{N}^\times$, $\boldsymbol{\Gamma}^+_m(z) \in \mathscr{Z}(\widehat{\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)})[[z, z^{-1}]]$. Indeed, in the next section we actually establish that these central elements consistently vanish.

2.4.4 The central elements $\boldsymbol{\Gamma}^\pm_{m>2}(z)$

Before we can actually establish that these central elements vanish, we need to establish a few lemmas. In what follows, we let $\dot{\mathrm{U}}_q^<(\dot{\mathfrak{a}}_1) = \dot{\mathrm{U}}_q^\leq(\dot{\mathfrak{a}}_1) - \dot{\mathrm{U}}_q^\leq(\dot{\mathfrak{a}}_1) \cap \dot{\mathrm{U}}_q^0(\dot{\mathfrak{a}}_1)$.

Lemma 2.4.22. *For every $p \in \mathbb{N}^\times$,*

i. $\Delta(\boldsymbol{\psi}^-_{1,-p}(v)) = 1 \otimes \boldsymbol{\psi}^-_{1,-p}(v) \mod \dot{\mathrm{U}}_q^<(\dot{\mathfrak{a}}_1)\widehat{\otimes}\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)$*;*

ii. $\Delta(\mathbf{X}^+_{1,-p}(v)) = \prod_{\ell=1}^{p-1} \boldsymbol{\Gamma}^+_0(C^{-1/2}q^{2\ell}v)^{-1}\mathbf{k}^+_0(C^{-1/2}v)^{-1}\widehat{\otimes}\mathbf{X}^+_{1,-p}(v) \mod \dot{\mathrm{U}}_q^<(\dot{\mathfrak{a}}_1)\widehat{\otimes}\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)$.

Proof. First one easily checks that

$$\Delta(\boldsymbol{\psi}^-_{1,-1}(z)) = 1 \otimes \boldsymbol{\psi}^-_{1,-1}(z) + [2]_q\,(q - q^{-1})\mathbf{x}^-_1(z)\widehat{\otimes}\mathbf{x}^-_0(q^{-2}z)\mathbf{k}^+_1(z) + \boldsymbol{\psi}^-_{1,-1}(z)\widehat{\otimes}\boldsymbol{\wp}^+(q^{-2}z)\,,$$

which proves *i.* for $p = 1$. Assuming *i.* holds for some $p \in \mathbb{N}$, the result for $p + 1$ easily holds making use of (2.4.22) and of the recursion hypothesis.

Similarly, one easily checks that

$$\Delta(\mathbf{X}^+_{1,-1}(v)) = \mathbf{X}^+_{1,-1}(v) \otimes 1 + \mathbf{k}^+_0(C^{-1/2}v)^{-1}\widehat{\otimes}\mathbf{X}^+_{1,-1}(v)\,,$$

which proves *ii.* in the case $p = 1$. Assuming the result holds for some $p \in \mathbb{N}$, the result for $p + 1$ easily follows making use of proposition 2.4.8.*iii.* and of the recusrion hypothesis. $\square$

For every $N \in \mathbb{N}^\times$, we let

$$\begin{aligned} S^<_{2N-1} := \{\sigma \in S_{2N-1} \quad : \quad & \sigma(1) = 1 \\ & \forall p \in [\![N-1]\!] \quad \sigma(2p) < \sigma(2p+1) \\ & \sigma(2N-4) < \sigma(2N-1)\} \end{aligned} \tag{2.4.51}$$

Define $\varpi : \mathbb{Z} \to \dot{I} = \{0, 1\}$ by setting, for every $n \in \mathbb{Z}$,

$$\varpi(n) := \begin{cases} 0 & \text{if } n \text{ is even;} \\ 1 & \text{if } n \text{ is odd.} \end{cases} \tag{2.4.52}$$

Lemma 2.4.23. *For every $r \in \mathbb{N}$ and every $i_1, \dots, i_{2r-1} \in \dot{I}$, there exists $(\beta_{r,\sigma})_{\sigma \in S^<_{2r-1}} \in \mathbb{F}^{S^<_{2r-1}}$ such that*

$$\left\langle \mathbf{x}^+_{i_1}(z_1) \dots \mathbf{x}^+_{i_{2r-1}}(z_{2r-1}), \mathbf{X}^+_{1,-r}(v) \right\rangle = -\frac{[2]_q^{r-1}}{q - q^{-1}} \sum_{\sigma \in S^<_{2r-1}} \beta_{r,\sigma} \prod_{n=1}^{2r-1} \delta_{i_{\sigma(n)},\pi(n)}\delta\left(\frac{z_{\sigma(n)}q^{\nu_r(n)}}{v}\right) \tag{2.4.53}$$

where we have defined $\pi : \mathbb{N} \to \dot{I}$ and $\nu_r : \mathbb{N} \to \mathbb{Z}$ by setting, for every $n \in \mathbb{N}$,

$$\pi(n) = \begin{cases} 0 & \text{if } n = 1; \\ \varpi(m) & \text{if } n > 1 \end{cases} \tag{2.4.54}$$

and

$$\nu_r(n) = \begin{cases} 2(1-r) & \textit{if } n=1; \\ 2(1-r)+n-3\varpi(n) & \textit{if } n>1. \end{cases} \tag{2.4.55}$$

Proof. The case $r=0$ holds by definition of the pairing. Assume that (2.4.53) holds for some $r \in \mathbb{N}$. Then, making use of the previous lemma, one easily shows that, for every $i_1, \dots, i_{2r+1} \in \dot{I}$

$$\begin{aligned} &\left\langle \mathbf{x}^+_{i_1}(z_1) \dots \mathbf{x}^+_{i_{2r+1}}(z_{2r+1}), \left[\boldsymbol{\psi}^-_{1,-1}(z), \mathbf{X}^+_{1,-r}(v)\right] \right\rangle \\ = \quad & \frac{[2]_q}{q-q^{-1}} \sum_{\mathbf{A} \in \mathsf{P}^{(2,2r-1)}_{\llbracket 2r+1 \rrbracket}} \prod_{m \in \llbracket 2 \rrbracket} \delta_{i_{A^{(1)}_m}, 1-\varpi(m)} \delta\left(\frac{z_{A^{(1)}_m} q^{2\varpi(m)}}{z}\right) \left\langle \mathbf{x}^+_{i_{A^{(2)}_1}}(z_{A^{(2)}_1}) \dots \mathbf{x}^+_{i_{A^{(2)}_{2r-1}}}(z_{A^{(2)}_{2r-1}}), \mathbf{X}^+_{1,-r}(v) \right\rangle \\ & \times \left\{ R^<_{\mathbf{A}}(z_{\mathbf{A}}) - G^+_{i_{A^{(1)}_1},0}(C^{-1/2} z_{A^{(1)}_1}/v) G^+_{i_{A^{(1)}_2},0}(C^{-1/2} z_{A^{(1)}_2}/v) R^>_{\mathbf{A}}(z^{-1}_{\mathbf{A}}) \right\} , \end{aligned}$$

where

$$R^<_{\mathbf{A}}(z_{\mathbf{A}}) = \prod_{\substack{m \in \llbracket 2 \rrbracket \\ n \in \llbracket 2r-1 \rrbracket \\ A^{(2)}_n < A^{(1)}_m}} G^-_{i_{A^{(2)}_n}, i_{A^{(1)}_m}}(C^{-1/2} z_{A^{(2)}_n}/z_{A^{(1)}_m}) ;$$

$$R^>_{\mathbf{A}}(z^{-1}_{\mathbf{A}}) = \prod_{\substack{m \in \llbracket 2 \rrbracket \\ n \in \llbracket 2r-1 \rrbracket \\ A^{(2)}_n > A^{(1)}_m}} G^-_{i_{A^{(2)}_n}, i_{A^{(1)}_m}}(C^{1/2} z_{A^{(1)}_m}/z_{A^{(2)}_n}) .$$

Making use of proposition 2.4.8.*iii.* on the l.h.s. and of the recursion hypothesis on the r.h.s., we get

$$\begin{aligned} &[2]_q\, \delta\left(\frac{Cz}{v}\right) \left\langle \mathbf{x}^+_{i_1}(z_1) \dots \mathbf{x}^+_{i_{2r+1}}(z_{2r+1}), \mathbf{X}^+_{1,-(r+1)}(vq^{-2}) \right\rangle \\ = \quad & -\frac{[2]^r_q}{(q-q^{-1})^2} \sum_{\substack{\mathbf{A} \in \mathsf{P}^{(2,2r-1)}_{\llbracket 2r+1 \rrbracket} \\ \sigma \in S^<_{2r-1}}} \beta_{r,\sigma} \prod_{m \in \llbracket 2 \rrbracket} \delta_{i_{A^{(1)}_m}, 1-\varpi(m)} \delta\left(\frac{z_{A^{(1)}_m} q^{2\varpi(m)}}{z}\right) \prod_{n \in \llbracket 2r-1 \rrbracket} \delta_{i_{A^{(2)}_{\sigma(n)}}, \pi(n)} \delta\left(\frac{z_{A^{(2)}_{\sigma(n)}} q^{\nu_r(n)}}{v}\right) \\ & \times \left\{ Q^<_{\sigma,\mathbf{A}}(v/z) - G^+_{0,0}(C^{-1/2} z q^{-2}/v) G^+_{1,0}(C^{-1/2} z/v) Q^>_{\sigma,\mathbf{A}}(z/v) \right\} , \end{aligned} \tag{2.4.56}$$

where

$$Q^<_{\sigma,\mathbf{A}}(v/z) = \prod_{\substack{m \in \llbracket 2 \rrbracket \\ n \in \llbracket 2r-1 \rrbracket \\ A^{(2)}_{\sigma(n)} < A^{(1)}_m}} G^-_{\pi(n), 1-\varpi(m)}(C^{-1/2} v q^{\lambda_r(m,n)}/z) ;$$

and

$$Q^>_{\sigma,\mathbf{A}}(z/v) = \prod_{\substack{m \in \llbracket 2 \rrbracket \\ n \in \llbracket 2r-1 \rrbracket \\ A^{(2)}_{\sigma(n)} > A^{(1)}_m}} G^-_{\pi(n), 1-\varpi(m)}(C^{1/2} z/v q^{\lambda_r(m,n)}) ;$$

where $\lambda_r(m,n) = 2\varpi(m) - \nu_r(n)$. In view of the $\delta(Cz/v)$ factor on the l.h.s of (2.4.56), it is clear that the relevant factors in $Q^<_{\sigma,\mathbf{A}}(v/z)$ and $Q^>_{\sigma,\mathbf{A}}(z/v)$ are the ones contributing to a pole at $Cz = v$, i.e. the ones for which $\lambda_r(m,n) = c_{\pi(n),1-\varpi(m)}$ or $\lambda_r(m,n) = -c_{\pi(n),1-\varpi(m)}$ respectively. We thus let

$$L^\pm_r := \left\{ (m,n) \in \llbracket 2 \rrbracket \times \llbracket 2r-1 \rrbracket : \lambda_r(m,n) = \pm c_{\pi(n),1-\varpi(m)} \right\}$$

and determine, by inspection, that, for every $r \geq 3$,

$$L_r^+ = \{(1, 2r-2), (2, 2r-3)\}\,, \qquad \text{whereas} \qquad L_r^- = \{(2, 2r-4)\}\,.$$

Since we cannot have $A^{(2)}_{\sigma(2r-4)} > A^{(1)}_2$ while $A^{(2)}_{\sigma(2r-3)} < A^{(1)}_2$ for $\sigma \in S^<_{2r-1}$, we see that the relevant pole is necessarily a simple pole; as one might have expected, given the absence of a $\delta'(Cz/v)$ factor on the l.h.s of (2.4.56). It easily follows that

$$\left\{Q^<_{\sigma,\mathbf{A}}(v/z) - G^+_{0,0}(C^{-1/2}zq^{-2}/v)G^+_{1,0}(C^{-1/2}z/v)Q^>_{\sigma,\mathbf{A}}(z/v)\right\} = [2]_q\,(q-q^{-1})\gamma_{\sigma,\mathbf{A}}\delta\left(\frac{Cz}{v}\right)$$

for every $(\sigma, \mathbf{A}) \in S^<_{2r-1} \times \mathsf{P}^{(2,2r-1)}_{[\![2r+1]\!]}$ such that $A^{(2)}_{\sigma(2r-2)} < A^{(1)}_1$ and either:

- $A^{(2)}_{\sigma(2r-4)} > A^{(1)}_2$ (and then necessarily, $A^{(2)}_{\sigma(2r-3)} > A^{(1)}_2$); or
- $A^{(2)}_{\sigma(2r-4)} < A^{(1)}_2$ and $A^{(2)}_{\sigma(2r-3)} < A^{(1)}_2$;

and, for each such pair $(\sigma, \mathbf{A})$, $\gamma_{\sigma,\mathbf{A}} \in \mathbb{F}$. Note that the above conditions impose that $A^{(2)}_{\sigma(1)=1} < A^{(1)}_1$ and hence $A^{(2)}_1 = 1$. Now, for each pair $(\sigma, \mathbf{A})$ as above, define

$$\sigma' := \begin{pmatrix} 1 & 2 & \dots & 2r-1 & 2r & 2r+1 \\ 1 & A^{(2)}_{\sigma(2)} & \dots & A^{(2)}_{\sigma(2r-1)} & A^{(1)}_1 & A^{(1)}_2 \end{pmatrix}.$$

It is obvious that $\sigma' \in S^<_{2r+1}$. Actually, setting $(\sigma, \mathbf{A}) \mapsto \sigma'$ defines a map $S^<_{2r-1} \times \mathsf{P}^{(2,2r-1)}_{[\![2r+1]\!]} \to S^<_{2r+1}$ which is easily seen to be a bijection. Observing furthermore that $\nu_r - 2 = \nu_{r+1}$ and setting $\beta_{r+1,\sigma'} = \beta_{r,\sigma}\gamma_{\sigma,\mathbf{A}}$, we can rewrite (2.4.56) as

$$\left\langle \mathbf{x}^+_{i_1}(z_1)\dots\mathbf{x}^+_{i_{2r+1}}(z_{2r+1}), \mathbf{X}^+_{1,-(r+1)}(v)\right\rangle = -\frac{[2]_q^r}{q-q^{-1}} \sum_{\sigma' \in S^<_{2r+1}} \beta_{r+1,\sigma'} \prod_{n=1}^{2r+1} \delta_{i_{\sigma'(n)},\pi(n)}\delta\left(\frac{z_{\sigma'(n)}q^{\nu_{r+1}(n)}}{v}\right),$$

which completes the recursion. □

Proposition 2.4.24. *For every $m \in \mathbb{N}^\times$, we actually have $\mathbf{\Gamma}^+_m(v) = \mathbf{\Gamma}^-_{-m}(v) = 0$.*

Proof. It suffices to prove that, say $\mathbf{\Gamma}^-_{-m}(z) = 0$ for every $m \in \mathbb{N}^\times$ and to apply φ^{-1} to get the result for $\mathbf{\Gamma}^+_m(z)$. Considering the root space decomposition, it is obvious that having

$$\left\langle \mathbf{x}^+_{i_1}(z_1)\cdots\mathbf{x}^+_{i_{2m}}(z_{2m}), \mathbf{\Gamma}^-_{-m}(z)\right\rangle = 0\,,$$

for every $i_1, \dots, i_{2m} \in I$, is a sufficient condition. Now, making use of the previous lemma, one easily shows that

$$\begin{aligned}
&\left\langle \mathbf{x}^+_{i_1}(z_1)\cdots \mathbf{x}^+_{i_{2m}}(z_{2m}), \left[\mathbf{X}^+_{1,-m}(v), \mathbf{x}^-_1(z)\right]\right\rangle \\
&= -\frac{[2]_q}{(q-q^{-1})^2} \sum_{\substack{\mathbf{A}\in\mathsf{P}^{(1,2m-1)}_{\llbracket 2m\rrbracket}\\ \sigma\in S^<_{2m-1}}} \beta_{m,\sigma}\delta_{i_{A^{(1)}_1},1}\,\delta\left(\frac{z_{A^{(1)}_1}}{z}\right) \prod_{n\in\llbracket 2m-1\rrbracket} \delta_{i_{A^{(2)}_n},\pi(n)}\,\delta\left(\frac{z_{A^{(2)}_{\sigma(n)}}\,q^{\nu_m(n)}}{v}\right)\\
&\quad\times\left\{G^+_{i_{A^{(1)}_1},0}(C^{1/2}z_{A^{(1)}_1}/v)R^<_{\mathbf{A}}(z_{\mathbf{A}}) - R^>_{\mathbf{A}}(z^{-1}_{\mathbf{A}})\right\}\,,
\end{aligned}$$

where

$$R^<_{\mathbf{A}}(z_{\mathbf{A}}) = \prod_{\substack{n\in\llbracket 2m-1\rrbracket\\ A^{(2)}_n > A^{(1)}_1}} G^-_{i_{A^{(1)}_1}, i_{A^{(2)}_n}}(C^{-1/2}z_{A^{(1)}_1}/z_{A^{(2)}_n})\,,$$

$$R^>_{\mathbf{A}}(z^{-1}_{\mathbf{A}}) = \prod_{\substack{n\in\llbracket 2m-1\rrbracket\\ A^{(2)}_n < A^{(1)}_1}} G^-_{i_{A^{(1)}_1}, i_{A^{(2)}_n}}(C^{1/2}z_{A^{(2)}_n}/z_{A^{(1)}_1})\,.$$

Hence, upon rewriting, we get

$$\begin{aligned}
&\left\langle \mathbf{x}^+_{i_1}(z_1)\cdots \mathbf{x}^+_{i_{2m}}(z_{2m}), \left[\mathbf{X}^+_{1,-m}(v), \mathbf{x}^-_1(z)\right]\right\rangle \\
&= -\frac{[2]_q}{(q-q^{-1})^2} \sum_{\substack{\mathbf{A}\in\mathsf{P}^{(1,2m-1)}_{\llbracket 2m\rrbracket}\\ \sigma\in S^<_{2m-1}}} \beta_{m,\sigma}\delta_{i_{A^{(1)}_1},1}\,\delta\left(\frac{z_{A^{(1)}_1}}{z}\right) \prod_{n\in\llbracket 2m-1\rrbracket} \delta_{i_{A^{(2)}_n},\pi(n)}\,\delta\left(\frac{z_{A^{(2)}_{\sigma(n)}}\,q^{\nu_m(n)}}{v}\right)\\
&\quad\times\left\{G^+_{1,0}(C^{1/2}z/v)Q^<_{\sigma,\mathbf{A}}(z/v) - Q^>_{\sigma,\mathbf{A}}(v/z)\right\}\,,
\end{aligned}$$

where

$$Q^<_{\sigma,\mathbf{A}}(z/v) = \prod_{\substack{n\in\llbracket 2m-1\rrbracket\\ A^{(2)}_{\sigma(n)} > A^{(1)}_1}} G^-_{1,\pi(n)}(C^{-1/2}zq^{\nu_m(n)}/v)\,,$$

$$Q^>_{\sigma,\mathbf{A}}(v/z) = \prod_{\substack{n\in\llbracket 2m-1\rrbracket\\ A^{(2)}_{\sigma(n)} < A^{(1)}_1}} G^-_{1,\pi(n)}(C^{1/2}vq^{-\nu_m(n)}/z)\,.$$

In view of (2.4.19), the contributions to $\left\langle \mathbf{x}^+_{i_1}(z_1)\cdots \mathbf{x}^+_{i_{2m}}(z_{2m}), \mathbf{\Gamma}^-_{-m}(z)\right\rangle$ in the above expression must come from terms with a pole at $z = C^{1/2}v$. The latter happen for factors in $Q^<_{\sigma,\mathbf{A}}(z/v)$ or $Q^>_{\sigma,\mathbf{A}}(v/z)$ such that $\nu_m(n) = c_{1,\pi(n)}$ or $\nu_m(n) = -c_{1,\pi(n)}$ respectively. We thus let

$$M^\pm_m = \{n\in\llbracket 2m-1\rrbracket : \nu_m(n) = \pm c_{1,\pi(n)}\}\,.$$

Upon inspection, one easily sees that

$$M^+_m = \{2m-4\}\,, \qquad \text{whereas} \qquad M^-_m = \{2m-1\}\,.$$

Now, for $\sigma\in S^<_{2m-1}$, we have $\sigma(2m-4) < \sigma(2m-1)$ and no term has a pole at $z = C^{1/2}v$. We conclude that $\left\langle \mathbf{x}^+_{i_1}(z_1)\cdots \mathbf{x}^+_{i_{2m}}(z_{2m}), \mathbf{\Gamma}^-_{-m}(z)\right\rangle = 0$. □

2.4.5 Relations in $\Psi^{-1}(\ddot{\mathrm{U}}_q^0(\mathfrak{a}_1))$

Definition 2.4.25. We set $\mathbf{K}^+_{1,0}(v) := -\mathbf{k}^-_1(C^{1/2}v)$ and, for every $m \in \mathbb{N}^\times$,

$$\mathbf{K}^+_{1,m}(v) := (q-q^{-1})\mathbf{k}^-_1(C^{1/2}vq^{-2m})\boldsymbol{\psi}^+_{1,m}(v)\,.$$

We then let

$$\mathbf{K}^-_{1,0}(v) := \varphi\left(\mathbf{K}^+_{1,0}(1/v)\right) = -\mathbf{k}^+_1(C^{1/2}v)$$

and, for every $m \in \mathbb{N}^\times$,

$$\mathbf{K}^-_{1,-m}(v) := \varphi\left(\mathbf{K}^+_{1,m}(1/v)\right) = -(q-q^{-1})\boldsymbol{\psi}^-_{1,-m}(v)\mathbf{k}^+_1(C^{1/2}vq^{-2m})\,.$$

It is straigthforward to establish that

$$\mathbf{k}^-_1(C^{1/2}w)\boldsymbol{\psi}^+_{1,m}(v) = G^+_{11}\left(\frac{wq^{2m}}{v}\right)G^-_{11}\left(\frac{w}{v}\right)\boldsymbol{\psi}^+_{1,m}(v)\mathbf{k}^-_1(C^{1/2}w)\,. \tag{2.4.57}$$

By making repeated use of the above relation, one readily checks that, in terms of $(\mathbf{K}^+_{1,m}(v))_{m\in\mathbb{N}^\times}$, the relations (2.4.10) and (3.3.82), as well as the relations in corollary 2.4.20*ii.* and *iv.* of the previous subsections respectively read

$$[\mathbf{x}^+_1(v), \mathbf{X}^-_{1,n}(z)] \quad = \quad \frac{1}{q-q^{-1}}\delta\left(\frac{zq^{2n}}{Cv}\right)\left(\prod_{p=0}^{n-1}\boldsymbol{\Gamma}^-_0(C^{-1/2}zq^{2p})^{-1}\right)\mathbf{K}^+_{1,n}(v) \tag{2.4.58}$$

$$\begin{aligned}[\mathbf{K}^+_{1,1}(w), \mathbf{K}^+_{1,m}(v)]_{G^-_{11}(w/v)G^+_{11}(wq^{2(m-1)}/v)} \quad = \quad & [2]_q\left\{\delta\left(\frac{wq^{2m}}{v}\right)\mathbf{K}^+_{1,0}(vq^{-2m})\mathbf{K}^+_{1,m+1}(v)\right. \\ & \left. -\delta\left(\frac{w}{vq^2}\right)\mathbf{K}^+_{1,0}(v)\mathbf{K}^+_{1,m+1}(vq^2)\right\}\end{aligned} \tag{2.4.59}$$

$$\begin{aligned}[\mathbf{K}^+_{1,m+1}(v), \mathbf{X}^-_{1,n}(z)]_{G^+_{11}(Cv/zq^{2(m+1)})} \quad & = \quad -\boldsymbol{\Gamma}^-_0(C^{1/2}vq^{-2(m+1)})[\mathbf{X}^-_{1,n+1}(zq^{-2}), \mathbf{K}^+_{1,m}(v)]_{G^+_{11}(zq^{2(m-1)}/Cv)} \\ & \propto \quad \delta\left(\frac{zq^{2m}}{Cv}\right)\end{aligned} \tag{2.4.60}$$

$$[\mathbf{K}^+_{1,m+1}(v), \mathbf{X}^+_{1,n}(z)]_{G^-_{11}(v/z)} = -[\mathbf{X}^+_{1,n+1}(v), \mathbf{K}^+_{1,m}(z)]_{G^-_{11}(v/z)} \propto \delta\left(\frac{zq^2}{v}\right) \tag{2.4.61}$$

Proposition 2.4.26. *For every $m, n \in \mathbb{N}$, we have*

$$(v-q^{\pm 2}z)(v-q^{2(m-n\mp 1)}z)\mathbf{K}^\pm_{1,\pm m}(v)\mathbf{K}^\pm_{1,\pm n}(z) = (vq^{\pm 2}-z)(vq^{\mp 2}-q^{2(m-n)}z)\mathbf{K}^\pm_{1,\pm n}(z)\mathbf{K}^\pm_{1,\pm m}(v)\,, \tag{2.4.62}$$

Proof. We apply the map $a \mapsto [a, \mathbf{X}^-_{1,n}(u)]_{G^+_{11}(Cv/uq^{2(m+1)})}$ to the relation (2.4.61) with $n = 0$ there. Making use of identity (2.1.2) on the left hand side, we get

$$\underbrace{[[\mathbf{K}^+_{1,m+1}(v), \mathbf{X}^-_{1,n}(u)]_{G^+_{11}(Cq^{-2(m+1)}v/u)}}_{\propto\delta\left(\frac{C^{-1}uq^{2m}}{v}\right)}, \mathbf{x}^+_1(z)]_{G^+_{10}(v/z)}$$
$$+\left[\mathbf{K}^+_{1,m+1}(v), \left[\mathbf{x}^+_1(z), \mathbf{X}^-_{1,n}(u)\right]\right]_{G^+_{10}(v/z)G^+_{11}(Cq^{-2(m+1)}v/u)} \propto \delta\left(\frac{zq^2}{v}\right)$$

Multiplying through by $\left(C^{-1}uq^{2m} - v\right)\left(zq^2 - v\right)$ and making use of (2.4.58), it follows that

$$\begin{aligned} 0 &= \left(C^{-1}uq^{2m} - v\right)\left(zq^2 - v\right)\delta\left(\frac{uq^{2n}}{Cz}\right)\left[\mathbf{K}^+_{1,m+1}(v), \mathbf{K}^+_{1,n}(z)\right]_{G^+_{10}(v/z)G^+_{11}(Cq^{-2(m+1)}v/u)} \\ &= \left(zq^{2(m-n)} - v\right)\left(zq^2 - v\right)\delta\left(\frac{uq^{2n}}{Cz}\right)\left[\mathbf{K}^+_{1,m+1}(v), \mathbf{K}^+_{1,n}(z)\right]_{G^+_{10}(v/z)G^+_{11}(q^{2(n-m-1)}v/z)} \end{aligned}$$

Hence the result for the upper choice of signs in (2.4.62). The case with lower choice of signs follows by applying φ to the above equation. $\square$

At this point it should be clear that we have obtained Ψ^{-1}. Indeed, it suffices to let, for every $m \in \mathbb{N}$ and every $n \in \mathbb{Z}$,

$$\Psi^{-1}(\mathsf{D}_2^{\pm 1}) = D^{\pm 1} \tag{2.4.63}$$

$$\Psi^{-1}(\mathsf{C}^{\pm 1/2}) = C^{\pm 1/2} \tag{2.4.64}$$

$$\Psi^{-1}(\mathsf{c}^{\pm}(z)) = \mathbf{\Gamma}^{\pm}_0(z) \tag{2.4.65}$$

$$\Psi^{-1}(\mathsf{K}^{\pm}_{1,\pm m}(z)) = \mathbf{K}^{\pm}_{1,\pm m}(z) \tag{2.4.66}$$

$$\Psi^{-1}(\mathsf{X}^{\pm}_{1,n}(z)) = \mathbf{X}^{\pm}_{1,n}(z) \tag{2.4.67}$$

The relations in $\ddot{\mathrm{U}}'_q(\mathfrak{a}_1)$ are obviously all the relations we have derived in the present section. Ψ^{-1} therefore extends as an algebra homomorphism. This concludes the proof of theorem 3.3.22.

Returning to the proof of proposition 3.3.18, it is also clear that

$$f(\psi^{\pm}(z)) = (q^2 - q^{-2})^2 \Psi(\wp^{\pm}(C^{1/2}zq^{-2})) \tag{2.4.68}$$

$$f(\mathsf{e}^{\pm}(z)) = \Psi(\boldsymbol{\psi}^{\pm}_{1,\pm 1}(z)) \tag{2.4.69}$$

Therefore ((2.3.43) – (2.3.44)) follow from proposition 2.4.8.*v*. In order to complete the proof of proposition 3.3.18, we still have to prove the compatibility of f with the Serre relations (3.3.40) of $\mathcal{E}_{q_1,q_2,q_3}$. This is the purpose of the next section.

2.4.6 The Serre relations of the elliptic Hall algebra

By the compatibility of f with (3.3.40), we actually mean that we should have, for every $m \in \mathbb{Z}$,

$$\operatorname*{res}_{v,w,z} (vwz)^m (v+z)(w^2 - vz) f(\mathsf{e}^{\pm}(v)) f(\mathsf{e}^{\pm}(w)) f(\mathsf{e}^{\pm}(z)) = 0\,. \tag{2.4.70}$$

Now we have already identified $f(\mathsf{e}^{\pm}(z))$ with $\Psi(\boldsymbol{\psi}^{\pm}_{1,\pm1}(z))$ in (2.4.69) above. The latter means that proving (2.4.70) is equivalent to proving

Proposition 2.4.27. *For every $m \in \mathbb{Z}$, we have*

$$\operatorname*{res}_{v_1,v_2,v_3} (v_1v_2v_3)^m(v_1+v_3)(v_2^2-v_1v_3)\boldsymbol{\psi}^{\pm}_{1,\pm1}(v_1)\boldsymbol{\psi}^{\pm}_{1,\pm1}(v_2)\boldsymbol{\psi}^{\pm}_{1,\pm1}(v_3) = 0\,. \tag{2.4.71}$$

Proof. The upper choice of signs immediately follows from the lower one upon applying φ. Moreover, considering the root space decomposition, it is clear that having

$$\operatorname*{res}_{v_1,v_2,v_3} (v_1v_2v_3)^m(v_1+v_3)(v_2^2-v_1v_3)\left\langle \mathbf{x}^+_{i_1}(z_1)\dots\mathbf{x}^+_{i_6}(z_6), \boldsymbol{\psi}^-_{1,-1}(v_1)\boldsymbol{\psi}^-_{1,-1}(v_2)\boldsymbol{\psi}^-_{1,-1}(v_3)\right\rangle = 0$$

for every $i_1,\dots,i_6 \in \dot{I}$ is a sufficient condition for the result to hold. Now, making use of lemma 2.4.22, one easily obtains that

$$\begin{aligned}&\left\langle \mathbf{x}^+_{I_1}(z_1)\dots\mathbf{x}^+_{i_6}(z_6), \boldsymbol{\psi}^-_{1,-1}(v_1)\boldsymbol{\psi}^-_{1,-1}(v_2)\boldsymbol{\psi}^-_{1,-1}(v_3)\right\rangle\\ &= \left(\frac{[2]_q}{q-q^{-1}}\right)^3 \sum_{\mathbf{A}\in\mathsf{P}^{(2,2,2)}_{[\![6]\!]}} \prod_{p=1}^{3} \prod_{\substack{m\in A^{(p+1)}\sqcup\dots\sqcup A^{(3)}\\ n\in A^{(p)}\\ n>m}} \prod_{k=1}^{2} \delta_{i_{A_k^{(p)}},\varpi(k)}\,\delta\left(\frac{z_{A_k^{(p)}}q^{\varpi(k)}}{v_k}\right) G^-_{i_m,i_n}(C^{-1/2}z_m/z_n)\,.\end{aligned}$$

There is obviously an action of S_3 on $\mathsf{P}^{(2,2,2)}_{[\![6]\!]}$ given by setting $\sigma(\mathbf{A}) = (A^{(\sigma(1))}, A^{(\sigma(2))}, A^{(\sigma(3))})$ for every $\sigma \in S_3$ and every $\mathbf{A} \in \mathsf{P}^{(2,2,2)}_{[\![6]\!]}$. It is also quite clear that

$$\frac{\mathsf{P}^{(2,2,2)}_{[\![6]\!]}}{S_3} \cong \mathsf{T}^{(2,2,2)}_{[\![6]\!]}\,,$$

where

$$\mathsf{T}^{(2,2,2)}_{[\![6]\!]} := \left\{\mathbf{A} \in \mathsf{P}^{(2,2,2)}_{[\![6]\!]} : A_1^{(1)} < A_1^{(2)} < A_1^{(3)}\right\}\,.$$

For every triple $\mathbf{n} = \{n_1,n_2,n_3\} \subset [\![6]\!]$, we further let

$$\mathsf{T}^{(2,2,2)}_{[\![6]\!]}(\mathbf{n}) := \left\{\mathbf{A} \in \mathsf{T}^{(2,2,2)}_{[\![6]\!]} : \left\{A_2^{(p)} : p \in [\![3]\!]\right\} = \mathbf{n}\right\}\,.$$

With these notations in place, we can now write

$$\begin{aligned}&\operatorname*{res}_{v_1,v_2,v_3} (v_1v_2v_3)^m(v_1+v_3)(v_2^2-v_1v_3)\left\langle \mathbf{x}^+_{I_1}(z_1)\dots\mathbf{x}^+_{i_6}(z_6), \boldsymbol{\psi}^-_{1,-1}(v_1)\boldsymbol{\psi}^-_{1,-1}(v_2)\boldsymbol{\psi}^-_{1,-1}(v_3)\right\rangle\\ &= \left(\frac{[2]_q}{q-q^{-1}}\right)^3 \sum_{\substack{\mathbf{n}\subset[\![6]\!]\\ \operatorname{card}\mathbf{n}=3}} z_{\mathbf{n}}^m \delta_{i_{[\![6]\!]-\mathbf{n}},1}\delta_{i_{\mathbf{n}},0} \sum_{\mathbf{A}\in\mathsf{T}^{(2,2,2)}_{[\![6]\!]}(\mathbf{n})} \prod_{p=1}^{3} \delta\left(\frac{z_{A_1^{(p)}}q^2}{z_{A_2^{(p)}}}\right) c_{\mathbf{A}}\,,\end{aligned}$$

where, by definition,

$$z_{\mathbf{n}}^m = \prod_{i=1}^{3} z_{n_i}^m\,, \qquad \delta_{i_{\mathbf{n}},0} = \prod_{j=1}^{3} \delta_{i_{n_j},0}\,, \qquad \delta_{i_{[\![6]\!]-\mathbf{n}},1} = \prod_{m\in[\![6]\!]-\mathbf{n}} \delta_{i_m,1} \tag{2.4.72}$$

$$c_{\mathbf{A}} = \sum_{\sigma \in S_3} F(z_{A_2^{(\sigma(1))}}, z_{A_2^{(\sigma(2))}}, z_{A_2^{(\sigma(3))}}) \prod_{1 \leq p' < p \leq 3} H_{\mathbf{A},\sigma,p,p'}(z_{A_2^{(\sigma(p))}}/z_{A_2^{(\sigma(p'))}}) \tag{2.4.73}$$

$$F(x,y,z) = (x+z)(y^2 - xz) \tag{2.4.74}$$

$$H_{\mathbf{A},\sigma,p,p'}(z_{A_2^{(\sigma(p))}}/z_{A_2^{(\sigma(p'))}}) = \prod_{k,k'=1}^{2} G^{-}_{\varpi(k),\varpi(k')}(C^{-1/2}q^{2(k-k')}z_{A_2^{(\sigma(p))}}/z_{A_2^{(\sigma(p'))}})^{\epsilon(\mathbf{A},\sigma,p,p',k,k')} \tag{2.4.75}$$

$$\epsilon(\mathbf{A},\sigma,p,p',k,k') = \begin{cases} 1 & \text{if } A_k^{(\sigma(p))} < A_{k'}^{(\sigma(p'))}; \\ 0 & \text{otherwise.} \end{cases} \tag{2.4.76}$$

Denoting each $\mathbf{A} \in \mathsf{T}^{(2,2,2)}_{[\![6]\!]}$ as the tableau

$$\begin{array}{|c|c|c|}\hline A_1^{(1)} & A_1^{(2)} & A_1^{(3)} \\ \hline A_2^{(1)} & A_2^{(2)} & A_2^{(3)} \\ \hline \end{array} \; ,$$

one easily checks that, actually,

$$\mathsf{T}^{(2,2,2)}_{[\![6]\!]} = \mathsf{T}^{(2,2,2)}_{[\![6]\!]}(\{2,4,6\}) \sqcup \mathsf{T}^{(2,2,2)}_{[\![6]\!]}(\{2,5,6\}) \sqcup \mathsf{T}^{(2,2,2)}_{[\![6]\!]}(\{3,4,6\}) \sqcup \mathsf{T}^{(2,2,2)}_{[\![6]\!]}(\{3,5,6\}) \sqcup \mathsf{T}^{(2,2,2)}_{[\![6]\!]}(\{4,5,6\}) \; ,$$

with

$$\mathsf{T}^{(2,2,2)}_{[\![6]\!]}(\{2,4,6\}) = \left\{ \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 & 6 \\ \hline \end{array} \right\} ,$$

$$\mathsf{T}^{(2,2,2)}_{[\![6]\!]}(\{2,5,6\}) = \left\{ \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 6 & 5 \\ \hline \end{array} , \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 & 6 \\ \hline \end{array} \right\} ,$$

$$\mathsf{T}^{(2,2,2)}_{[\![6]\!]}(\{3,4,6\}) = \left\{ \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 4 & 3 & 6 \\ \hline \end{array} , \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 & 6 \\ \hline \end{array} \right\} ,$$

$$\mathsf{T}^{(2,2,2)}_{[\![6]\!]}(\{3,5,6\}) = \left\{ \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 6 & 3 & 5 \\ \hline \end{array} , \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 6 & 5 \\ \hline \end{array} , \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 5 & 3 & 6 \\ \hline \end{array} , \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 & 6 \\ \hline \end{array} \right\} ,$$

$$\mathsf{T}^{(2,2,2)}_{[\![6]\!]}(\{4,5,6\}) = \left\{ \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 6 & 5 & 4 \\ \hline \end{array} , \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 5 & 6 & 4 \\ \hline \end{array} , \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 6 & 4 & 5 \\ \hline \end{array} , \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 6 & 5 \\ \hline \end{array} , \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 5 & 4 & 6 \\ \hline \end{array} , \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ \hline \end{array} \right\} .$$

A tedious but straightforward calculation – see appendix for useful identities – shows that, e.g.

$$\begin{aligned} c_{\begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 6 & 3 & 5 \\ \hline \end{array}} \;=\;& (q^2 - q^{-2})^2(1+q^2)(1-q^2)z_3^3 \left[q^2 \delta\left(\frac{z_3 q^2}{z_6}\right) \delta\left(\frac{z_6}{z_5}\right) - \delta\left(\frac{z_3}{z_6}\right) \delta\left(\frac{z_6 q^2}{z_5}\right) \right] \\ & + q^{-2}(q^2 - q^{-2})(1+q^2)^2(1-q^2)^6 H_1(z_3/z_5) \left[z_5 \delta\left(\frac{z_6}{z_5}\right) - z_3 \delta\left(\frac{z_3}{z_6}\right) \right] \end{aligned}$$

$$\begin{aligned} c_{\begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3&6&5\\\hline\end{array}} &= (q^2-q^{-2})^2(1+q^2)(1-q^2)z_3^3\delta\left(\frac{z_3}{z_6}\right)\delta\left(\frac{z_6q^2}{z_5}\right)\\ &\quad +q^{-2}(q^2-q^{-2})(1+q^2)^2(1-q^2)^6H_1(z_3/z_5)z_3\delta\left(\frac{z_3}{z_6}\right)\\ &\quad +q^{-2}(q^2-q^{-2})(1+q^2)^2(1-q^2)^6H_2(z_3/z_5)z_5\delta\left(\frac{z_6}{z_5}\right)\end{aligned}$$

$$\begin{aligned} c_{\begin{array}{|c|c|c|}\hline 1&2&4\\\hline 5&3&6\\\hline\end{array}} &= -q^2(q^2-q^{-2})^2(1+q^2)(1-q^2)z_3^3\delta\left(\frac{z_5}{z_6}\right)\delta\left(\frac{z_3q^2}{z_5}\right)\\ &\quad -q^{-2}(q^2-q^{-2})(1+q^2)(1-q^2)^6H_1(z_3/z_5)z_5\delta\left(\frac{z_5}{z_6}\right)\\ &\quad -q^{-2}(q^2-q^{-2})(1+q^2)(1-q^2)^6H_2(z_3/z_6)z_3\delta\left(\frac{z_3}{z_5}\right)\end{aligned}$$

$$c_{\begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3&5&6\\\hline\end{array}} = q^{-2}(q^2-q^{-2})(1+q^2)(1-q^2)^6H_2(z_3/z_6)\left[z_3\delta\left(\frac{z_3}{z_5}\right)-z_6\delta\left(\frac{z_5}{z_6}\right)\right].$$

where we have set

$$H_1(z_3/z_5) = \left(\frac{z_3^2z_5^2(z_3+z_5)^3}{(z_3q^2-z_5)(q^4z_3-z_5)(z_3-q^2z_5)^3}\right)_{|z_5|\gg|z_3|},$$

$$H_2(z_3/z_5) = \left(\frac{z_3^2z_5^2(z_3+z_5)^3}{(z_3q^4-z_5)(z_3-q^2z_5)^4}\right)_{|z_5|\gg|z_3|}.$$

It easily follows that

$$\sum_{\mathbf{A}\in\mathsf{T}^{(2,2,2)}_{[\![6]\!]}(\{3,5,6\})}\prod_{p=1}^{3}\delta\left(\frac{z_{A_1^{(p)}}q^2}{z_{A_2^{(p)}}}\right)c_{\mathbf{A}} = 0. \tag{2.4.77}$$

Similar calculations show that, eventually, for every $\mathbf{n}\subset[\![6]\!]$ such that $\operatorname{card}\mathbf{n}=3$ and $\mathsf{T}^{(2,2,2)}_{[\![6]\!]}(\mathbf{n})\neq\emptyset$, we have

$$\sum_{\mathbf{A}\in\mathsf{T}^{(2,2,2)}_{[\![6]\!]}(\mathbf{n})}\prod_{p=1}^{3}\delta\left(\frac{z_{A_1^{(p)}}q^2}{z_{A_2^{(p)}}}\right)c_{\mathbf{A}} = 0, \tag{2.4.78}$$

thus proving the result. □

Chapter 3

Weight-finite modules over the quantum affine and double quantum affine algebras of type $\mathfrak{a}_1$

3.1 Introduction

The representation theory of quantum affine algebras is a vast and extremely rich theory which is still the subject of an intense research activity after more than three decades. The recent discovery of its relevance to the monoidal categorification of cluster algebras provides one of the latest and most striking illustrations of it – see [HL09] for a review on that subject. Probably standing as one of the most significant breakthroughs in the early days of this research area, the classification of the simple finite-dimensional modules over the quantum affine algebra of type $\mathfrak{a}_1$, $\mathrm{U}_q(\dot{\mathfrak{a}}_1)$, is due to Chari and Pressley [CP91]. It relies, on one hand, on a careful analysis of the ℓ-weight structure of those modules made possible by the existence of Drinfel'd's presentation $\dot{\mathrm{U}}_q(\mathfrak{a}_1)$ of $\mathrm{U}_q(\dot{\mathfrak{a}}_1)$ – see [Dam93] for the proof that $\dot{\mathrm{U}}_q(\mathfrak{a}_1) \cong \mathrm{U}_q(\dot{\mathfrak{a}}_1)$ – and, on the other hand, on the existence of evaluation modules, proven earlier by Jimbo, [Jim86]. This seminal work paved the way for a more systematic study of the representation theory of quantum affine algebras of all Cartan types, leading to the development of powerful tools such as q-characters, (q,t)-characters and, consequently, to a much better understanding of the categories $\mathbf{FinMod}$ of their finite-dimensional modules that recently culminated with the realization that the Grothendieck rings of certain subcategories of the categories $\mathbf{FinMod}$ actually have the structure of a cluster algebra, [HL10].

By contrast, it is fair to say that the representation theory of quantum toroidal algebras, which were initially introduced in type $\mathfrak{a}_n$ by Ginzburg, Kapranov and Vasserot [GKV95] and later generalized to higher rank types, is significantly less well understood and remains, to this date, much more mysterious – although see [Her09] for a review and references therein. In our previous work, [MZ], we constructed a new (topological) Hopf algebra $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)$, called double quantum affinization of type $\mathfrak{a}_1$, and proved that its completion (in an appropriate topology) is bicontinuously isomorphic to (a corresponding completion) of the quantum toroidal algebra $\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)$. Whereas $\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)$ is naturally graded over $\mathbb{Z} \times \dot{Q}$, where $\dot{Q}$ stands for the root lattice of the untwisted affine root system $\dot{\mathfrak{a}}_1$ of type $A_1^{(1)}$, $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)$ is naturally graded over $\mathbb{Z}^2 \times Q$, where Q stands for the root lattice of the finite root system $\mathfrak{a}_1$ of type A_1. Thus $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)$ turns out to be to $\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)$ what $\dot{\mathrm{U}}_q(\mathfrak{a}_1)$ is to $\mathrm{U}_q(\dot{\mathfrak{a}}_1)$, i.e. its Drinfel'd presentation. The latter, in the quantum affine case, has a natural triangular decomposition which allows one to define an adapted class of highest weight modules, namely highest ℓ-weight modules, in which finite-dimensional modules are singled out by the particular form of their highest ℓ-weights. Therefore, it is only natural to ask the question of whether $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)$ plays a similar role for the representation theory of $\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)$, leading, in particular, to a new notion of highest weight modules. We answer positively that question and introduce the corresponding notion of highest t-weight modules. Schematically, whereas the transition from the classical Lie theoretic weights to ℓ-weights can be regarded as trading numbers for (rational) functions, the transition from ℓ-weights to t-weights can be regarded as trading (rational) functions for entire modules over the non-commutative $\ddot{\mathrm{U}}^0_q(\mathfrak{a}_1)$-subalgebra of $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)$. That substitution can be interpreted from the perspective of a conjecture in [MZ], stating that $\ddot{\mathrm{U}}^0_q(\mathfrak{a}_1)$ is isomorphic to a split extension of the elliptic Hall algebra $\mathcal{E}_{q^{-4},q^2,q^2}$ which was initially defined by Miki, in [Mik07], as a (q,γ)-analogue of the $W_{1+\infty}$ algebra and reappeared later on in different guises; the quantum continuous $\mathfrak{gl}_\infty$ algebra in [FFJ$^+$11], the Hall algebra of the category of coherent sheaves on some elliptic curve in [Sch12], or the quantum toroidal algebra associated with $\mathfrak{gl}_1$ in [FJMM12] and in subsequent works by Feigin et al. Our conjecture is actually supported by the existence of an algebra homomorphism between $\mathcal{E}_{q^{-4},q^2,q^2}$ and $\ddot{\mathrm{U}}^0_q(\mathfrak{a}_1)$ which we promote, in the present paper, to a (continuous) homomorphism of (topological) Hopf algebras. Intuitively, the weights adapted to our new triangular decomposition can therefore be regarded as representations of a quantized algebra of functions on a non-commutative 2-torus.

On the other hand, unless the value of some scalar depending on the deformation parameter is taken to be a root of unity, the question of the existence of finite-dimensional modules over quantum toroidal algebras of type $\mathfrak{a}_{n\geq 2}$ was already answered negatively by Varagnolo and Vasserot in [VV96]. However, it is possible to push further the analogy with the quantum affine situation by defining another type of finiteness condition, namely weight-finiteness. It turns out that, in type 1, i.e. when the central charges act trivially, $\ddot{\mathrm{U}}^0_q(\mathfrak{a}_1)$ admits an infinite dimensional abelian subalgebra that, itself, admits as a subalgebra the Cartan subalgebra $\mathrm{U}^0_q(\mathfrak{a}_1)$ of the Drinfel'd-Jimbo quantum algebra $\mathrm{U}_q(\mathfrak{a}_1)$ of type $\mathfrak{a}_1$. Hence, we can assign classical Lie theoretic weights to the t-weight spaces of our modules and declare that a $\ddot{\mathrm{U}}'_q(\mathfrak{a}_1)$-module is weight-finite whenever it has only finitely many classical weights. The same notion is readily defined for modules over $\dot{\mathrm{U}}_q(\mathfrak{a}_1)$ and we then focus on $\mathbf{WFinMod}^\bullet$ (resp. $\mathbf{WFinMod}$), i.e. the full subcategory of the category $\mathbf{Mod}^\bullet$ (resp. $\mathbf{Mod}$) of $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)$-modules (resp. $\dot{\mathrm{U}}_q(\mathfrak{a}_1)$-modules) whose modules are weight-finite. Of course, the widely studied category $\mathbf{FinMod}$ of finite-dimensional $\dot{\mathrm{U}}_q(\mathfrak{a}_1)$-modules is a full subcategory of $\mathbf{WFinMod}$. The main results of the present paper consist in showing that, on one hand, the simple objects in $\mathbf{WFinMod}$ are all finite-dimensional and therefore coincide with the simple finite-dimensional $\dot{\mathrm{U}}_q(\mathfrak{a}_1)$-modules classified by Chari and Pressley, and, on the other hand, in classifying the simple objects in $\mathbf{WFinMod}^\bullet$ in terms of their highest t-weight spaces. These results clearly establish $\mathbf{WFinMod}^\bullet$ as the natural quantum toroidal analogue of $\mathbf{FinMod}$ and suggest studying further its structure and, in particular, the structure of its Grothendieck ring. Another natural development at this point would be to generalize to the quantum toroidal setting the interesting classes of $\dot{\mathrm{U}}_q(\mathfrak{a}_1)$-modules outside of $\mathbf{FinMod}$, for example by constructing a quantum toroidal analogue of category $\mathcal{O}$. We leave these questions for future work.

The present paper is organized as follows. In section 3.2, we briefly review classic results about the quantum affine algebra $\dot{\mathrm{U}}_q(\mathfrak{a}_1)$ and its finite-dimensional modules. Then, we prove that simple objects in $\mathbf{WFinMod}$ are actually finite-dimensional. In section 3.3, we review the main relevant results of [MZ] and establish a few new results, as relevant for the subsequent sections. We define highest t-weight modules in section 3.4 and, by thoroughly analyzing their structure, we establish one implication in our classification theorem, namely theorem

3.4.22. The opposite implication is established in section 3.5 by explicitly constructing a quantum toroidal analogue of the quantum affine evaluation modules. That construction is obtained after proving the existence of an evaluation homomorphism between $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)$ and an evaluation algebra built as a double semi-direct product of $\dot{\mathrm{U}}_q(\mathfrak{a}_1)$ with the completions of two Heisenberg algebras. The evaluation modules are then obtained by pulling back induced modules over the evaluation algebra along the evaluation homomorphism.

Notations and conventions

We let $\mathbb{N} = \{0, 1, \dots\}$ be the set of natural integers including 0. We denote by $\mathbb{N}^\times$ the set $\mathbb{N} - \{0\}$. For every $m \leq n \in \mathbb{N}$, we denote by $[\![m, n]\!] = \{m, m+1, \dots, n\}$. We also let $[\![n]\!] = [\![1, n]\!]$ for every $n \in \mathbb{N}$. For every $m, n \in \mathbb{N}^\times$, we let

$$C_m(n) := \left\{\lambda = (\lambda_1, \dots, \lambda_m) \in \left(\mathbb{N}^\times\right)^m : \lambda_1 + \cdots + \lambda_m = n\right\} ,$$

denote the set of m-compositions of n, i.e. of compositions of n having m summands.

We let $\mathrm{sign} : \mathbb{Z} \to \{-1, 0, 1\}$ be defined by setting, for any $n \in \mathbb{Z}$,

$$\mathrm{sign}(n) = \begin{cases} -1 & \text{if } n < 0; \\ 0 & \text{if } n = 0; \\ 1 & \text{if } n > 0. \end{cases}$$

We assume throughout that $\mathbb{K}$ is an algebraically closed field of characteristic 0 and we let $\mathbb{F} := \mathbb{K}(q)$ denote the field of rational functions over $\mathbb{K}$ in the formal variable q. As usual, we let $\mathbb{K}^\times = \mathbb{K} - \{0\}$ and $\mathbb{F}^\times = \mathbb{F} - \{0\}$. Whenever we wish to evaluate q to some element of $\mathbb{K}^\times$, we shall always do so under the restriction that $1 \notin q^{\mathbb{Z}^\times}$. For every $m, n \in \mathbb{N}$, we define the following elements of $\mathbb{F}$

$$[n]_q := \frac{q^n - q^{-n}}{q - q^{-1}} , \qquad [n]_q^! := \begin{cases} [n]_q [n-1]_q \cdots [1]_q & \text{if } n \in \mathbb{N}^\times; \\ 1 & \text{if } n = 0; \end{cases} \qquad \binom{n}{m}_q := \frac{[n]_q^!}{[m]_q^! [n-m]_q^!} . \tag{3.1.1}$$

Given an $\mathbb{F}$-algebra $(\mathcal{A}, \eta)$ with unit $\eta : \mathbb{F} \to \mathcal{A}$, we shall write the image of any $a \in \mathcal{A}$ under the canonical algebra homomorphisms $\eta^{\otimes p-1} \otimes \mathrm{id}_{\mathcal{A}} \otimes \eta^{\otimes n-p} : \mathcal{A} \to \mathcal{A}^{\otimes n}$, $1 \leq p \leq n \in \mathbb{N}^\times$, as $a_{(p)}$, always assuming that the value of n should be clear from the context. This is easily extended to $\mathcal{A}$-valued formal distributions in $\mathcal{A}[[z, z^{-1}]]$, essentially by applying the canonical algebra homomorphisms term by term to their coefficients, and the corresponding image of $a(z) \in \mathcal{A}[[z, z^{-1}]]$ can be naturally denoted by $a_{(p)}(z) \in \mathcal{A}^{\otimes n}[[z, z^{-1}]]$. As is customary though, in order to avoid the proliferation of unnecessary subscripts, we shall abuse notations and prefer e.g. to the more rigorous expression $a_{(1)}\left(zc_{(2)}\right) b_{(2)}(z') \in \mathcal{A}^{\otimes 2}[[z, z^{-1}, z', z'^{-1}]]$, with $a(z) \in \mathcal{A}[[z, z^{-1}]]$, $b(z') \in \mathcal{A}[[z', z'^{-1}]]$ and $c \in \mathcal{A}$, the somewhat less rigorous but more transparent $a\left(zc_{(2)}\right) \otimes b(z')$.

We shall say that a polynomial $P(z) \in \mathbb{F}[z]$ is *monic* if $P(0) = 1$. For every rational function $P(z)/Q(z)$, where $P(z)$ and $Q(z)$ are relatively prime polynomials, we denote by

$$\left(\frac{P(z)}{Q(z)}\right)_{|z| \ll 1} \qquad \text{(resp. } \left(\frac{P(z)}{Q(z)}\right)_{|z|^{-1} \ll 1} \text{)}$$

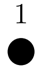
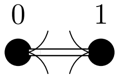
the Laurent series of $P(z)/Q(z)$ at 0 (resp. at ∞).

We shall let

$$_{a}\left[A,B\right]_{b} = aAB - bBA\,,$$

for any symbols a, b, A and B provided the r.h.s of the above equations makes sense.

The Dynkin diagrams and correponding Cartan matrices of the root systems $\mathfrak{a}_1$ and $\dot{\mathfrak{a}}_1$ are reminded in the following table.

Type	Dynkin diagram	Simple roots	Cartan matrix
$\mathfrak{a}_1$		$\Phi = \{\alpha_1\}$	(2)
$\dot{\mathfrak{a}}_1$		$\dot{\Phi} = \{\alpha_0, \alpha_1\}$	$\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$

3.2 Weight-finite modules over the quantum affine algebra $\dot{\mathrm{U}}_q(\mathfrak{a}_1)$

3.2.1 The quantum affine algebra $\dot{\mathrm{U}}_q(\mathfrak{a}_1)$

Definition 3.2.1. The *quantum affine algebra* $\dot{\mathrm{U}}_q(\mathfrak{a}_1)$ is the associative $\mathbb{K}(q)$-algebra generated by

$$\left\{D, D^{-1}, C^{1/2}, C^{-1/2}, k^+_{1,n}, k^-_{1,-n}, x^+_{1,m}, x^-_{1,m} : m \in \mathbb{Z}, n \in \mathbb{N}\right\}$$

subject to the following relations

$$C^{\pm 1/2} \text{ is central} \qquad C^{\pm 1/2}C^{\mp 1/2} = 1 \qquad D^{\pm 1}D^{\mp 1} = 1 \tag{3.2.1}$$

$$D\mathbf{k}_1^\pm(z)D^{-1} = \mathbf{k}_1^\pm(zq^{-1}) \qquad D\mathbf{x}_1^\pm(z)D^{-1} = \mathbf{x}_1^\pm(zq^{-1}) \tag{3.2.2}$$

$$\mathbf{k}_1^\pm(z_1)\mathbf{k}_1^\pm(z_2) = \mathbf{k}_1^\pm(z_2)\mathbf{k}_1^\pm(z_1) \tag{3.2.3}$$

$$\mathbf{k}_1^-(z_1)\mathbf{k}_1^+(z_2) = G^-(C^{-1}z_1/z_2)G^+(Cz_1/z_2)\mathbf{k}_1^+(z_2)\mathbf{k}_1^-(z_1) = 1 \mod z_1/z_2 \tag{3.2.4}$$

$$G^\mp(C^{\mp 1/2}z_2/z_1)\mathbf{k}_1^+(z_1)\mathbf{x}_1^\pm(z_2) = \mathbf{x}_1^\pm(z_2)\mathbf{k}_1^+(z_1) \tag{3.2.5}$$

$$\mathbf{k}_1^-(z_1)\mathbf{x}_1^\pm(z_2) = G^\mp(C^{\mp 1/2}z_1/z_2)\mathbf{x}_1^\pm(z_2)\mathbf{k}_1^-(z_1) \tag{3.2.6}$$

$$(z_1 - q^{\pm 2}z_2)\mathbf{x}_1^\pm(z_1)\mathbf{x}_1^\pm(z_2) = (z_1q^{\pm 2} - z_2)\mathbf{x}_1^\pm(z_2)\mathbf{x}_1^\pm(z_1) \tag{3.2.7}$$

$$[\mathbf{x}_1^+(z_1), \mathbf{x}_1^-(z_2)] = \frac{1}{q-q^{-1}}\left[\delta\left(\frac{z_1}{Cz_2}\right)\mathbf{k}_1^+(z_1C^{-1/2}) - \delta\left(\frac{z_1C}{z_2}\right)\mathbf{k}_1^-(z_2C^{-1/2})\right] \tag{3.2.8}$$

where we define the following $\dot{\mathrm{U}}_q(\mathfrak{a}_1)$-valued formal distributions

$$\mathbf{x}_1^\pm(z) := \sum_{m\in\mathbb{Z}} x^\pm_{1,m}z^{-m} \in \dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)[[z, z^{-1}]]\,; \tag{3.2.9}$$

$$\mathbf{k}_1^\pm(z) := \sum_{n\in\mathbb{N}} k_{1,\pm n}^\pm z^{\mp n} \in \dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)[[z^{\mp 1}]] \,, \tag{3.2.10}$$

the following $\mathbb{F}$-valued formal power series

$$G^\pm(z) := q^{\pm 2} + (q-q^{-1})[\pm 2]_q \sum_{m\in\mathbb{N}^\times} q^{\pm 2m} z^m \in \mathbb{F}[[z]] \tag{3.2.11}$$

and

$$\delta(z) := \sum_{m\in\mathbb{Z}} z^m \in \mathbb{F}[[z,z^{-1}]] \tag{3.2.12}$$

is an $\mathbb{F}$-valued formal distribution. We denote by $\dot{\mathrm{U}}'_q(\mathfrak{a}_1)$ the subalgebra of $\dot{\mathrm{U}}_q(\mathfrak{a}_1)$ generated by

$$\left\{C^{1/2}, C^{-1/2}, k_{1,n}^+, k_{1,-n}^-, x_{1,m}^+, x_{1,m}^- : m\in\mathbb{Z}, n\in\mathbb{N}\right\} .$$

We denote by $\dot{\mathrm{U}}^0_q(\mathfrak{a}_1)$ the subalgebra of $\dot{\mathrm{U}}'_q(\mathfrak{a}_1)$ generated by

$$\left\{C^{1/2}, C^{-1/2}, k_{1,n}^+, k_{1,-n}^- : n\in\mathbb{N}\right\} .$$

We let $\dot{\mathrm{U}}^{\geq}_q(\mathfrak{a}_1)$ (resp. $\dot{\mathrm{U}}^{\leq}_q(\mathfrak{a}_1)$) denote the subalgebra of $\dot{\mathrm{U}}'_q(\mathfrak{a}_1)$ generated by

$$\left\{C^{1/2}, C^{-1/2}, k_{1,n}^+, k_{1,-n}^-, x_{1,m}^+ : m\in\mathbb{Z}, n\in\mathbb{N}\right\}$$

(resp.

$$\left\{C^{1/2}, C^{-1/2}, k_{1,n}^+, k_{1,-n}^-, x_{1,m}^- : m\in\mathbb{Z}, n\in\mathbb{N}\right\}$$

). We let $\dot{\mathrm{U}}_q(\mathfrak{a}_1)^{\smile}$denote the $\mathbb{F}$-algebra generated by the same generators as $\dot{\mathrm{U}}'_q(\mathfrak{a}_1)$, subject to the relations (3.2.3 - 3.2.7) – i.e. we omit relation (3.2.8). We define the type $\mathfrak{a}_1$ *quantum loop algebra* $\mathrm{U}_q(\mathrm{L}\mathfrak{a}_1)$ as the quotient of $\dot{\mathrm{U}}'_q(\mathfrak{a}_1)$ by its two-sided ideal $(C^{1/2}-1)$ generated by $\left\{C^{1/2}-1, C^{-1/2}-1\right\}$. Similarly, we let $\mathrm{U}^{\geq}_q(\mathrm{L}\mathfrak{a}_1) = \dot{\mathrm{U}}^{\geq}_q(\mathfrak{a}_1)/(C^{1/2}-1)$ and $\mathrm{U}^{\leq}_q(\mathrm{L}\mathfrak{a}_1) = \dot{\mathrm{U}}^{\leq}_q(\mathfrak{a}_1)/(C^{1/2}-1)$. We eventually set $\breve{\mathrm{U}}_q(\mathrm{L}\mathfrak{a}_1) = \dot{\mathrm{U}}_q(\mathfrak{a}_1)^{\smile}/(C^{1/2}-1)$.

Obviously,

Proposition 3.2.2. *There exists a surjective $\mathbb{F}$-algebra homomorphism $\breve{\mathrm{U}}_q(\mathrm{L}\mathfrak{a}_1) \to \mathrm{U}_q(\mathrm{L}\mathfrak{a}_1)$.*

3.2.2 Finite dimensional $\dot{\mathrm{U}}'_q(\mathfrak{a}_1)$-modules

Let **Mod** be the category of $\dot{\mathrm{U}}'_q(\mathfrak{a}_1)$-modules. We denote by **FinMod** the full subcategory of **Mod** whose objects are finite-dimensional. Following [CP91], we make the following

Definition 3.2.3. We shall say that a $\dot{\mathrm{U}}'_q(\mathfrak{a}_1)$-module M is:

- a *weight module* if $k_{1,0}^+$ acts semisimply on M;
- of *type* 1 if it is a weight module and $C^{1/2}$ acts on M as id;
- *highest ℓ-weight* if it is of type 1 and there exists $v\in M-\{0\}$ such that

$$\mathbf{x}_1^+(z).v = 0\,, \qquad \mathbf{k}_1^\pm(z).v = \kappa^\pm(z)v$$

for some $\kappa^{\pm}(z) \in \mathbb{F}[[z^{\mp 1}]]$ and $M = \dot{\mathrm{U}}'_q(\mathfrak{a}_1).v$. We shall refer to any such v as a *highest ℓ-weight vector* and to $\boldsymbol{\kappa} = (\kappa^+(z), \kappa^-(z))$ as the corresponding highest ℓ-weight.

Clearly, type 1 $\dot{\mathrm{U}}'_q(\mathfrak{a}_1)$-modules coincide with $\mathrm{U}_q(\mathrm{L}\mathfrak{a}_1)$-modules.

Definition 3.2.4. For every $\boldsymbol{\kappa} \in \mathbb{F}[[z^{-1}]] \times \mathbb{F}[[z]]$, we construct a one-dimensional $\mathrm{U}_q^{\geq}(\mathrm{L}\mathfrak{a}_1)$-module $\mathbb{F}_{\boldsymbol{\kappa}} \cong \mathbb{F}$ by setting

$$\mathbf{x}_1^+(z).1 = 0\,, \qquad \text{and} \qquad \mathbf{k}_1^{\pm}(z).1 = \kappa^{\pm}(z)\,.$$

We then define the *universal* highest ℓ-weight $\dot{\mathrm{U}}'_q(\mathfrak{a}_1)$-module with highest ℓ-weight $\boldsymbol{\kappa}$ by setting

$$M(\boldsymbol{\kappa}) := \mathrm{U}_q(\mathrm{L}\mathfrak{a}_1) \underset{\mathrm{U}_q^{\geq}(\mathrm{L}\mathfrak{a}_1)}{\otimes} \mathbb{F}_{\boldsymbol{\kappa}}$$

as $\mathrm{U}_q(\mathrm{L}\mathfrak{a}_1)$-modules. Let $N(\boldsymbol{\kappa})$ be the maximal $\mathrm{U}_q(\mathrm{L}\mathfrak{a}_1)$-submodule of $M(\boldsymbol{\kappa})$ such that $N(\boldsymbol{\kappa}) \cap \mathbb{F}_{\boldsymbol{\kappa}} = \{0\}$ and set

$$L(\boldsymbol{\kappa}) := M(\boldsymbol{\kappa})/N(\boldsymbol{\kappa})\,.$$

By construction, $L(\boldsymbol{\kappa})$ is a simple highest ℓ-weight $\mathrm{U}_q(\mathrm{L}\mathfrak{a}_1)$-module with highest ℓ-weight $\boldsymbol{\kappa}$. It is unique up to isomorphisms.

The simple objects in **FinMod** were classified by Chari and Pressley in [CP91]. The main result is the following

Theorem 3.2.5 (Chari-Pressley)**.** *The following hold:*

i. any simple finite-dimensional $\dot{\mathrm{U}}'_q(\mathfrak{a}_1)$-module M can be obtained by twisting a simple finite-dimensional $\dot{\mathrm{U}}'_q(\mathfrak{a}_1)$-module of type 1 with an algebra automorphism of $\mathrm{Aut}(\dot{\mathrm{U}}'_q(\mathfrak{a}_1))$;

ii. every simple finite dimensional $\dot{\mathrm{U}}'_q(\mathfrak{a}_1)$-module of type 1 is highest ℓ-weight;

iii. the simple highest ℓ-weight module $L(\boldsymbol{\kappa})$ is finite-dimensional if and only if

$$\kappa^{\pm}(z) = q^{\deg(P)} \left(\frac{P(q^{-2}/z)}{P(1/z)} \right)_{|z|^{\mp 1} \ll 1}\,,$$

for some monic polynomial $P(1/z) \in \mathbb{F}[z^{-1}]$ called Drinfel'd polynomial *of $L(\boldsymbol{\kappa})$.*

Proof. The proof can be found in [CP91]. □

Up to isomorphisms, the simple objects in **FinMod** are uniquely parametrized by their Drinfel'd polynomials and we shall therefore denote by $L(P)$ the (isomorphism class of the) simple $\dot{\mathrm{U}}'_q(\mathfrak{a}_1)$-module with Drinfel'd polynomial P. Note that the roles of $\dot{\mathrm{U}}_q^{\geq}(\mathfrak{a}_1)$ and $\dot{\mathrm{U}}_q^{\leq}(\mathfrak{a}_1)$ in the above constructions are clearly symmetrical and we could have equivalently considered lowest ℓ-weight modules. In particular, point iii of the above theorem immediately translates into

Proposition 3.2.6. *The simple lowest ℓ-weight module with lowest ℓ-weight $\boldsymbol{\kappa} = (\kappa^+(z), \kappa^-(z)) \in \mathbb{F}[[z^{-1}]] \times \mathbb{F}[[z]]$ is finite-dimensional if and only if*

$$\kappa^{\pm}(z) = q^{-\deg(P)} \left(\frac{P(1/z)}{P(q^{-2}/z)} \right)_{|z|^{\mp 1} \ll 1}\,,$$

for some monic polynomial $P(1/z) \in \mathbb{F}[z^{-1}]$. In the latter case, we denote it by $\bar{L}(P)$.

3.2.3 Weight-finite simple $\mathrm{U}_q(\mathrm{L}\mathfrak{a}_1)$-modules

We now wish to consider a slightly broader family of modules over $\dot{\mathrm{U}}'_q(\mathfrak{a}_1)$. In particular, we want to allow these modules to be infinite-dimensional, while retaining some of the nice features of finite dimensional $\dot{\mathrm{U}}'_q(\mathfrak{a}_1)$-modules such as the fact that they decompose into ℓ-weight spaces. This is achieved by introducing the following notion.

Definition 3.2.7. We shall say that a (not necessarily finite-dimensional) $\dot{\mathrm{U}}'_q(\mathfrak{a}_1)$-module M is *ℓ-weight* if there exists a countable set $\{M_\alpha : \alpha \in A\}$ of indecomposable locally finite-dimensional $\dot{\mathrm{U}}^0_q(\mathfrak{a}_1)$-modules, called the *$\ell$-weight spaces* of M, such that, as $\dot{\mathrm{U}}^0_q(\mathfrak{a}_1)$-modules,

$$M \cong \bigoplus_{\alpha \in A} M_\alpha \,.$$

We shall say that M is of type 1 if $C^{1/2}$ acts on M by id.

Definition-Proposition 3.2.8. Let M be an ℓ-weight $\dot{\mathrm{U}}'_q(\mathfrak{a}_1)$-module. Then:

i. C^2 acts as id over M;

ii. for every ℓ-weight space M_α, $\alpha \in A$, of M, there exists $\kappa_{\alpha,0} \in \mathbb{F}^\times$ and $(\kappa^\pm_{\alpha,\pm m})_{m \in \mathbb{N}^\times} \in \mathbb{F}^{\mathbb{N}^\times}$ such that

$$M_\alpha \subseteq \left\{ v \in M : \exists n \in \mathbb{N}^\times , \forall m \in \mathbb{N} \quad \left(k^\pm_{1,\pm m} - \kappa^\pm_{\alpha,\pm m}\mathrm{id}\right)^n .v = 0 \right\} ,$$

where we have set $\kappa^\pm_{\alpha,0} = \kappa^{\pm 1}_{\alpha,0}$.

We let $\mathrm{Sp}(M) = \{\kappa_{\alpha,0} : \alpha \in A\}$ and refer to the formal power series

$$\kappa^\pm_\alpha(z) = \sum_{m \in \mathbb{N}} \kappa^\pm_{\alpha,\pm m} z^{\mp m}$$

as the *ℓ-weight* of the ℓ-weight space M_α.

Proof. Let M_α be an ℓ-weight space of M and let $v \in M_\alpha - \{0\}$. By definition, there exists a finite dimensional $\dot{\mathrm{U}}^0_q(\mathfrak{a}_1)$-submodule $\tilde{M}_\alpha$ of M_α such that $v \in \tilde{M}_\alpha$. Over $\tilde{M}_\alpha$, C must admit an eigenvector and, since C is central, it follows that C acts over $\tilde{M}_\alpha$ by a scalar mutliple of id. Assume for a contradiction that $C - C^{-1}$ does not act by multiplication by zero. Then, it is possible to pull back $\tilde{M}_\alpha$ into a finite-dimensional module over the Weyl algebra $\mathcal{A}_1(\mathbb{K}) = \mathbb{K}\langle x, y\rangle/(xy - yx - 1)$ by the obvious algebra homomorphism $\mathcal{A}_1(\mathbb{K}) \hookrightarrow \dot{\mathrm{U}}^0_q(\mathfrak{a}_1)$. But the Weyl algebra is known to admit no finite-dimensional modules. A contradiction. It follows that C^2 acts as id over $\tilde{M}_\alpha$. But this could be repeated for any non-zero vector in any ℓ-weight space of M. i follows. As for ii, observe that, as a consequence of i and of the defining relations (3.2.3) and (3.2.4), $\left\{k^+_{1,m}, k^-_{1,-m} : m \in \mathbb{N}\right\}$ acts by a family of commuting linear operators over M. Thus ii follows from the decomposition of locally finite-dimensional vector spaces into the generalized eigenspaces of a commuting family of linear operators; the indecomposability of M_α further imposing that it coincides with a single block in a single generalized eigenspace. □

Remark 3.2.9. It is worth emphasizing that definition 3.2.7 and definition-proposition 3.2.8 straightforwardly generalize to (topological) modules over any (topological) algebra $\mathcal{A}$ containing $\dot{\mathrm{U}}^0_q(\mathfrak{a}_1)$ as a (closed) subalgebra.

Definition 3.2.10. We shall say that an ℓ-weight $\dot{\mathrm{U}}_q'(\mathfrak{a}_1)$-module M is *weight-finite* if $Sp(M)$ is a finite set. We let **WFinMod** denote the full subcategory of the category **Mod** of $\dot{\mathrm{U}}_q'(\mathfrak{a}_1)$-modules whose objects are weight-finite.

Clearly, finite dimensional $\dot{\mathrm{U}}_q'(\mathfrak{a}_1)$-modules are objects in **WFinMod**, but not every object in **WFinMod** is in **FinMod**. However we have

Theorem 3.2.11. *The following hold:*

i. every simple ℓ-weight $\dot{\mathrm{U}}_q'(\mathfrak{a}_1)$-module can be obtained by twisting a simple ℓ-weight $\dot{\mathrm{U}}_q'(\mathfrak{a}_1)$-module of type 1 *with an algebra automorphism of* $\mathrm{Aut}(\dot{\mathrm{U}}_q(\mathfrak{a}_1))$;

ii. every weight-finite simple $\mathrm{U}_q(\mathrm{L}\mathfrak{a}_1)$*-module is highest ℓ-weight;*

iii. every weight-finite simple $\mathrm{U}_q(\mathrm{L}\mathfrak{a}_1)$*-module is finite dimensional.*

Proof. In view of definition-proposition 3.2.8, C^2 acts as id over M. Since the latter is simple and since $C^{1/2}$ is central, it is clear that C acts over M either as id or as $-$id. In the former case, there is nothing to do; whereas in the latter, upon twisting as in the finite-dimensional case – see [CP91] –, we can ensure that $C^{1/2}$ acts as id. This proves i. As for ii, the same proof as for part ii of theorem 3.2.5 can be used. So, we eventually prove iii. Let M be a weight-finite simple $\mathrm{U}_q(\mathrm{L}\mathfrak{a}_1)$-module. By ii it is highest ℓ-weight. Hence, there exists $v \in M - \{0\}$ such that $M \cong \mathrm{U}_q(\mathrm{L}\mathfrak{a}_1).v$, $\mathbf{x}_1^+(z).v = 0$ and $\mathbf{k}_1^\pm(z).v = \kappa^\pm(z)v$, for some $\kappa^\pm(z) \in \mathbb{F}[[z^{\mp1}]]$ with $\mathrm{res}_{z_1,z_2}\, z_1^{-1}z_2^{-1}\kappa^+(z_1)\kappa^-(z_2) = 1$. The triangular decomposition of $\mathrm{U}_q(\mathrm{L}\mathfrak{a}_1)$ implies that $M = \mathrm{U}_q^-(\mathrm{L}\mathfrak{a}_1).v$ and, setting for every $n \in \mathbb{N}$

$$v(z_1, \dots, z_n) = \mathbf{x}_1^-(z_1) \cdots \mathbf{x}_1^-(z_n).v\,, \tag{3.2.13}$$

it is clear that

$$\left\{ v_{m_1,\dots,m_n} = \operatorname*{res}_{z_1,\dots,z_n} z_1^{-1-m_1} \cdots z_n^{-1-m_n} v(z_1, \dots, z_n) : n \in \mathbb{N}, m_1, \dots, m_n \in \mathbb{Z} \right\} \tag{3.2.14}$$

is a spanning set of M. The defining relations (3.2.5) and (3.2.6) of $\mathrm{U}_q(\mathrm{L}\mathfrak{a}_1)$ easily imply that, for every $n \in \mathbb{N}$,

$$\mathbf{k}_1^\pm(z).v(z_1, \dots, z_n) = \kappa^\pm(z) \prod_{p=1}^{n} G^\mp\left((z_p/z)^{\pm1}\right) v(z_1, \dots, z_n) \tag{3.2.15}$$

and, in particular,

$$k_0^\pm.v(z_1, \dots, z_n) = (\kappa_0^+)^{\pm1} q^{-2n} v(z_1, \dots, z_n)\,.$$

Therefore, M being weight-finite, there must exist an $N \in \mathbb{N}$ such that

$$\mathbf{x}_1^-(z).v(z_1, \dots, z_N) = 0\,. \tag{3.2.16}$$

Making use of (3.2.8), one easily proves that, for every $n \in [\![0, N-1]\!]$,

$$\begin{aligned}\mathbf{x}_1^+(z).v(z_0,\dots,z_n) \quad = \quad & \frac{1}{q-q^{-1}}\sum_{p=0}^{n}\delta\left(\frac{z_p}{z}\right)\Bigg[\kappa^+(z)\prod_{r=p+1}^{n}G^-(z_r/z) \\ & \qquad -\kappa^-(z)\prod_{r=p+1}^{n}G^+(z/z_r)\Bigg]\, v(z_0,\dots,\widehat{z_p},\dots,z_n)\,,\end{aligned} \tag{3.2.17}$$

where a hat over a variable indicates that it should be omitted. Combining (3.2.16) and (3.2.8), we get

$$\begin{aligned}-\mathbf{x}_1^-(z_0)\mathbf{x}_1^+(z).v(z_1,\dots,z_N) \quad &= \quad [\mathbf{x}_1^+(z),\mathbf{x}_1^-(z_0)].v(z_1,\dots,z_N) \\ &= \quad \frac{1}{q-q^{-1}}\delta\left(\frac{z_0}{z}\right)\left[\kappa^+(z)\prod_{p=1}^{N}G^-(z_p/z)-\kappa^-(z)\prod_{p=1}^{N}G^+(z/z_p)\right] v(z_1,\dots,z_N)\,.\end{aligned}$$

Making use of (3.2.17) and (3.2.13), the above equation eventually yields

$$\sum_{p=0}^{N}\delta\left(\frac{z_p}{z}\right)\left[\kappa^+(z_p)\prod_{r=p+1}^{N}G^-(z_r/z_p)-\kappa^-(z_p)\prod_{r=p+1}^{N}G^+(z_p/z_r)\right] v(z_0,\dots,\widehat{z_p},\dots,z_N)=0\,.$$

Acting on the l.h.s of the above equation with $\mathbf{x}_1^+(\zeta_N)\cdots\mathbf{x}_1^+(\zeta_1)$ and making repeated use of (3.2.17), one easily shows that

$$\sum_{\sigma\in S_{N+1}}\prod_{i=0}^{N}\delta\left(\frac{z_i}{\zeta_{\sigma(i)}}\right)\left[\kappa^+(z_i)\prod_{\substack{r=i+1\\ \sigma(r)>\sigma(i)}}^{N}G^-(z_r/z_i)-\kappa^-(z_i)\prod_{\substack{r=i+1\\ \sigma(r)>\sigma(i)}}^{N}G^+(z_i/z_r)\right] v=0\,, \tag{3.2.18}$$

where we have set $\zeta_0 = z$. Since $v \neq 0$, its prefactor in the above equation must vanish. Now, it is clear that multiplication of the latter by $\prod_{j=0}^{N-1}(z_0-\zeta_j)$ annihilates all the summands with σ such that $\sigma(0)\neq N$. Similarly, multiplication by $\prod_{i=0}^{1}\prod_{j=0}^{N-i-1}(z_i-\zeta_j)$ annihilates all the summands with σ such that $\sigma(0)\neq N$ and $\sigma(1)\neq N-1$. Repeating the argument finitely many times, we arrive at the fact that multiplication by $\prod_{i=0}^{N}\prod_{j=0}^{N-i-1}(z_i-\zeta_j)$ annihilates all the summands with $\sigma \neq (N, N-1,\dots,0)$, so that, eventually,

$$0=\prod_{i=0}^{N}\delta\left(\frac{z_i}{\zeta_{N-i}}\right)\prod_{j=0}^{N-i-1}(z_i-\zeta_j)\left[\kappa^+(z_i)-\kappa^-(z_i)\right]=\prod_{i=0}^{N}\delta\left(\frac{z_i}{\zeta_{N-i}}\right)\prod_{j=0}^{N-i-1}(z_i-z_{N-j})\left[\kappa^+(z_i)-\kappa^-(z_i)\right]\,.$$

Taking the zeroth order term in ζ_j for $j=0,\dots,N$, we get

$$\begin{aligned}0 \quad &= \quad \prod_{i=0}^{N}\prod_{j=i+1}^{N}(z_i-z_j)\left[\kappa^+(z_i)-\kappa^-(z_i)\right] \\ &= \quad \begin{vmatrix} [\kappa^+(z_0)-\kappa^-(z_0)] & [\kappa^+(z_1)-\kappa^-(z_1)] & \dots & [\kappa^+(z_N)-\kappa^-(z_N)] \\ z_0\left[\kappa^+(z_0)-\kappa^-(z_0)\right] & z_1\left[\kappa^+(z_1)-\kappa^-(z_1)\right] & \dots & z_N\left[\kappa^+(z_N)-\kappa^-(z_N)\right] \\ \vdots & \vdots & \dots & \vdots \\ z_0^{N-1}\left[\kappa^+(z_0)-\kappa^-(z_0)\right] & z_1^{N-1}\left[\kappa^+(z_1)-\kappa^-(z_1)\right] & \dots & z_N^{N-1}\left[\kappa^+(z_N)-\kappa^-(z_N)\right]\end{vmatrix}\,.\end{aligned}$$

Hence, the rows of the matrix on the r.h.s. of the above equation are linearly dependent and it follows that there exists a $P(z) \in \mathbb{F}[z] - \{0\}$ of degree at most $N-1$, such that

$$P(z)\left[\kappa^{+}(z) - \kappa^{-}(z)\right] = 0\,. \tag{3.2.19}$$

As a consequence, there clearly exists $Q(z) \in \mathbb{F}[z]$ such that $\deg Q = \deg P$ and

$$\kappa^{\pm}(z) = \left(\frac{Q(z)}{P(z)}\right)_{|z|^{\mp 1} \ll 1}\,.$$

Now considering (3.2.17) with $n = 0$ and multiplying it by $P(z_0)$ obviously yields

$$\mathbf{x}_1^{+}(z).P(z_0)v(z_0) = 0\,. \tag{3.2.20}$$

Set for every $m \in \mathbb{Z}$,

$$w_m = \operatorname*{res}_{z_0} z_0^{-1-m} P(z_0)v(z_0)\,. \tag{3.2.21}$$

Then, (3.2.20), together with (3.2.15) for $n = 1$, implies that

$$\bigoplus_{m \in \mathbb{Z}} \dot{\mathrm{U}}_q(\mathfrak{a}_1).w_m$$

is a strict submodule of the simple $\mathrm{U}_q(\mathrm{L}\mathfrak{a}_1)$-module M and it follows that $w_m = 0$ for every $m \in \mathbb{Z}$. Consequently, in view of (3.2.21),

$$P(z_0)v(z_0) = 0\,. \tag{3.2.22}$$

On the other hand, all the vectors in $\{v_m : m \in \mathbb{Z}\}$ – see (3.2.14) – can be expressed as linear combinations of the vectors in, say, $\left\{v_1, \dots, v_{\deg(P)}\right\}$ and the linear span of $\{v_m : m \in \mathbb{Z}\}$ turns out to be finite dimensional.

Similarly, assume we have proven that

$$\forall k \in [\![0, n]\!]\,, \qquad \left(\prod_{p=k+1}^{n} (z_p - q^2 z_k)\right) P(z_k)v(z_0, \dots, z_n) = 0$$

for some $n \in [\![0, N-1]\!]$, as we did with (3.2.22) above for $n = 0$. It is clear, in view of (3.2.17), that, for every $k \in [\![0, n+1]\!]$,

$$\mathbf{x}_1^{+}(z).\left[\left(\prod_{p=k+1}^{n+1} (z_p - q^2 z_k)\right) P(z_k)v(z_0, \dots, z_{n+1})\right] = 0$$

and the same argument as above, making use of the simplicity of M, implies that indeed

$$\forall k \in [\![0, n+1]\!]\,, \qquad \left(\prod_{p=k+1}^{n+1} (z_p - q^2 z_k)\right) P(z_k)v(z_0, \dots, z_{n+1}) = 0\,.$$

By recursion, the above equation therefore holds for every $n \in [\![0, N-1]\!]$ and for every $k \in [\![0, n]\!]$. But this means that, for every $n \in [\![0, N-1]\!]$, the linear span of $\{v_{m_1, \dots, m_n} : m_1, \dots, m_n \in \mathbb{Z}\}$ is finite dimensional, which eventually concludes the proof. □

Corollary 3.2.12. *Let M be a weight-finite simple highest (resp. lowest) ℓ-weight $\mathrm{U}_q(\mathrm{L}\mathfrak{a}_1)$-module. Then $M \cong L(P)$ (resp. $M \cong \bar{L}(P)$), for some monic polynomial P.*

Proof. In the highest ℓ-weight case, this follows directly by the previous theorem and the classification of the simple finite dimensional $\mathrm{U}_q(\mathrm{L}\mathfrak{a}_1)$-modules, theorem 3.2.5. In the lowest ℓ-weight case, see proposition 3.2.6. □

3.3 Double quantum affinization of type $\mathfrak{a}_1$

3.3.1 Definition of $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)$

Definition 3.3.1. The *double quantum affinization* $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)$ of type $\mathfrak{a}_1$ is defined as the $\mathbb{F}$-algebra generated by

$$\{\mathsf{D}_1, \mathsf{D}_1^{-1}, \mathsf{D}_2, \mathsf{D}_2^{-1}, \mathsf{C}^{1/2}, \mathsf{C}^{-1/2}, \mathsf{c}^+_m, \mathsf{c}^-_{-m}, \mathsf{K}^+_{1,0,m}, \mathsf{K}^-_{1,0,-m}, \mathsf{K}^+_{1,n,r}, \mathsf{K}^-_{1,-n,r}, \mathsf{X}^+_{1,r,s}, \mathsf{X}^-_{1,r,s} : m \in \mathbb{N}, n \in \mathbb{N}^\times, r, s \in \mathbb{Z}\}$$

subject to the relations

$$\mathsf{C}^{\pm 1/2} \text{ and } \mathbf{\mathsf{c}}^\pm(z) \text{ are central} \tag{3.3.1}$$

$$\operatorname*{res}_{v,w} \frac{1}{vw} \mathbf{\mathsf{c}}^\pm(v) \mathbf{\mathsf{c}}^\mp(w) = 1\,, \tag{3.3.2}$$

$$\mathsf{D}_1^{\pm 1}\mathsf{D}_1^{\mp 1} = 1 \qquad \mathsf{D}_2^{\pm 1}\mathsf{D}_2^{\mp 1} = 1 \qquad \mathsf{D}_1\mathsf{D}_2 = \mathsf{D}_2\mathsf{D}_1 \tag{3.3.3}$$

$$\mathsf{D}_1 \mathbf{\mathsf{K}}^\pm_{1,\pm m}(z) \mathsf{D}_1^{-1} = q^{\pm m} \mathbf{\mathsf{K}}^\pm_{1,\pm m}(z) \qquad \mathsf{D}_1 \mathbf{\mathsf{X}}^\pm_{1,r}(z) \mathsf{D}_1^{-1} = q^r \mathbf{\mathsf{X}}^\pm_{1,r}(z)\,, \tag{3.3.4}$$

$$\mathsf{D}_2 \mathbf{\mathsf{K}}^\pm_{1,\pm m}(z) \mathsf{D}_2^{-1} = \mathbf{\mathsf{K}}^\pm_{1,\pm m}(zq^{-1}) \qquad \mathsf{D}_2 \mathbf{\mathsf{X}}^\pm_{1,r}(z) \mathsf{D}_2^{-1} = \mathbf{\mathsf{X}}^\pm_{1,r}(zq^{-1})\,, \tag{3.3.5}$$

$$\operatorname*{res}_{v,w} \frac{1}{vw} \mathbf{\mathsf{K}}^\pm_{1,0}(v) \mathbf{\mathsf{K}}^\mp_{1,0}(w) = 1\,, \tag{3.3.6}$$

$$(v - q^{\pm 2}z)(v - q^{2(m-n\mp 1)}z)\mathbf{\mathsf{K}}^\pm_{1,\pm m}(v)\mathbf{\mathsf{K}}^\pm_{1,\pm n}(z) = (vq^{\pm 2} - z)(vq^{\mp 2} - q^{2(m-n)}z)\mathbf{\mathsf{K}}^\pm_{1,\pm n}(z)\mathbf{\mathsf{K}}^\pm_{1,\pm m}(v)\,, \tag{3.3.7}$$

$$(\mathsf{C}q^{2(1-m)}v - w)(q^{2(n-1)}v - \mathsf{C}w)\mathbf{\mathsf{K}}^+_{1,m}(v)\mathbf{\mathsf{K}}^-_{1,-n}(w) = (\mathsf{C}q^{-2m}v - q^2 w)(q^{2n}v - \mathsf{C}q^{-2}w)\mathbf{\mathsf{K}}^-_{1,-n}(w)\mathbf{\mathsf{K}}^+_{1,m}(v)\,, \tag{3.3.8}$$

$$(v - q^{\pm 2}z)\mathbf{\mathsf{K}}^\pm_{1,\pm m}(v)\mathbf{\mathsf{X}}^\pm_{1,r}(z) = (q^{\pm 2}v - z)\mathbf{\mathsf{X}}^\pm_{1,r}(z)\mathbf{\mathsf{K}}^\pm_{1,\pm m}(v)\,, \tag{3.3.9}$$

$$(\mathsf{C}v - q^{2(m\mp 1)}z)\mathbf{\mathsf{K}}^\pm_{1,\pm m}(v)\mathbf{\mathsf{X}}^\mp_{1,r}(z) = (\mathsf{C}q^{\mp 2}v - q^{2m}z)\mathbf{\mathsf{X}}^\mp_{1,r}(z)\mathbf{\mathsf{K}}^\pm_{1,\pm m}(v)\,, \tag{3.3.10}$$

$$(v - q^{\pm 2}w)\mathbf{\mathsf{X}}^\pm_{1,r}(v)\mathbf{\mathsf{X}}^\pm_{1,s}(w) = (vq^{\pm 2} - w)\mathbf{\mathsf{X}}^\pm_{1,s}(w)\mathbf{\mathsf{X}}^\pm_{1,r}(v)\,, \tag{3.3.11}$$

$$\begin{aligned}
[\mathbf{\mathsf{X}}^+_{1,r}(v), \mathbf{\mathsf{X}}^-_{1,s}(z)] \quad = \quad & \frac{1}{q-q^{-1}} \left\{ \delta\left(\frac{\mathsf{C}v}{q^{2(r+s)}z}\right) \prod_{p=1}^{|s|} \mathbf{\mathsf{c}}^- \left(\mathsf{C}^{-1/2} q^{(2p-1)\mathrm{sign}(s)-1} z\right)^{-\mathrm{sign}(s)} \mathbf{\mathsf{K}}^+_{1,r+s}(v) \right. \\
& \left. - \delta\left(\frac{\mathsf{C}^{-1}v}{q^{2(r+s)}z}\right) \prod_{p=1}^{|r|} \mathbf{\mathsf{c}}^+ \left(\mathsf{C}^{-1/2} q^{(1-2p)\mathrm{sign}(r)-1} v\right)^{\mathrm{sign}(r)} \mathbf{\mathsf{K}}^-_{1,r+s}(z) \right\}\,,
\end{aligned} \tag{3.3.12}$$

where $m, n \in \mathbb{N}$, $r, s \in \mathbb{Z}$ and we have set

$$\mathsf{c}^{\pm}(z) = \sum_{m \in \mathbb{N}} \mathsf{c}^{\pm}_{\pm m} z^{\mp m}\,, \tag{3.3.13}$$

$$\mathsf{K}^{\pm}_{1,0}(z) = \sum_{m \in \mathbb{N}} \mathsf{K}^{\pm}_{1,0,\pm m} z^{\pm m}\,, \tag{3.3.14}$$

and, for every $m \in \mathbb{N}^{\times}$ and $r \in \mathbb{Z}$,

$$\mathsf{K}^{\pm}_{1,\pm m}(z) = \sum_{s \in \mathbb{Z}} \mathsf{K}^{\pm}_{1,\pm m,s} z^{-s}\,, \tag{3.3.15}$$

$$\mathsf{X}^{\pm}_{1,r}(z) = \sum_{s \in \mathbb{Z}} \mathsf{X}^{\pm}_{1,r,s} z^{-s}\,. \tag{3.3.16}$$

In (5.0.6), we further assume that $\mathsf{K}^{\pm}_{1,\mp m}(z) = 0$ for every $m \in \mathbb{N}^{\times}$.

Definition 3.3.2. We denote by $\ddot{\mathrm{U}}'_q(\mathfrak{a}_1)$ the subalgebra of $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)$ generated by

$$\{\mathsf{D}_2, \mathsf{D}_2^{-1}, \mathsf{C}^{1/2}, \mathsf{C}^{-1/2}, \mathsf{c}^{+}_{m}, \mathsf{c}^{-}_{-m}, \mathsf{K}^{+}_{1,0,m}, \mathsf{K}^{-}_{1,0,-m}, \mathsf{K}^{+}_{1,n,r}, \mathsf{K}^{-}_{1,-n,r}, \mathsf{X}^{+}_{1,r,s}, \mathsf{X}^{-}_{1,r,s} : m \in \mathbb{N}, n \in \mathbb{N}^{\times}, r, s \in \mathbb{Z}\}\,,$$

i.e. the subalgebra generated by all the generators of $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)$ except D_1 and D_1^{-1}. We shall denote by

$$\jmath : \ddot{\mathrm{U}}'_q(\mathfrak{a}_1) \hookrightarrow \ddot{\mathrm{U}}_q(\mathfrak{a}_1)$$

the natural injective algebra homomorphism.

Definition 3.3.3. We denote by $\ddot{\mathrm{U}}^0_q(\mathfrak{a}_1)$ the subalgebra of $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)$ generated by

$$\left\{\mathsf{C}^{1/2}, \mathsf{C}^{-1/2}, \mathsf{c}^{+}_{m}, \mathsf{c}^{-}_{-m}, \mathsf{K}^{+}_{1,0,m}, \mathsf{K}^{-}_{1,0,-m}, \mathsf{K}^{+}_{1,n,r}, \mathsf{K}^{-}_{1,-n,r} : m \in \mathbb{N}, n \in \mathbb{N}^{\times}, r \in \mathbb{Z}\right\}$$

and by $\ddot{\mathrm{U}}^{0,0}_q(\mathfrak{a}_1)$ the subalgebra of $\ddot{\mathrm{U}}^0_q(\mathfrak{a}_1)$ generated by

$$\left\{\mathsf{C}^{1/2}, \mathsf{C}^{-1/2}, \mathsf{c}^{+}_{m}, \mathsf{c}^{-}_{-m}, \mathsf{K}^{+}_{1,0,m}, \mathsf{K}^{-}_{1,0,-m} : m \in \mathbb{N}\right\}\,.$$

Similarly, we denote by $\ddot{\mathrm{U}}^{\pm}_q(\mathfrak{a}_1)$ the subalgebra of $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)$ generated by $\left\{\mathsf{X}^{\pm}_{1,r,s} : r, s \in \mathbb{Z}\right\}$. We eventually denote by $\ddot{\mathrm{U}}^{\geq}_q(\mathfrak{a}_1)$ (resp. $\ddot{\mathrm{U}}^{\leq}_q(\mathfrak{a}_1)$) the subalgebra of $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)$ generated by

$$\left\{\mathsf{C}^{1/2}, \mathsf{C}^{-1/2}, \mathsf{c}^{+}_{m}, \mathsf{c}^{-}_{-m}, \mathsf{K}^{+}_{1,0,m}, \mathsf{K}^{-}_{1,0,-m}, \mathsf{K}^{+}_{1,n,r}, \mathsf{K}^{-}_{1,-n,r}, \mathsf{X}^{+}_{1,r,s} : m \in \mathbb{N}, n \in \mathbb{N}^{\times}, r, s \in \mathbb{Z}\right\}$$

(resp.

$$\left\{\mathsf{C}^{1/2}, \mathsf{C}^{-1/2}, \mathsf{c}^{+}_{m}, \mathsf{c}^{-}_{-m}, \mathsf{K}^{+}_{1,0,m}, \mathsf{K}^{-}_{1,0,-m}, \mathsf{K}^{+}_{1,n,r}, \mathsf{K}^{-}_{1,-n,r}, \mathsf{X}^{-}_{1,r,s} : m \in \mathbb{N}, n \in \mathbb{N}^{\times}, r, s \in \mathbb{Z}\right\}$$

)

Remark 3.3.4. Obviously, $\ddot{\mathrm{U}}^{\pm}_q(\mathfrak{a}_1)$ is graded over $Q^{\pm}$ whereas $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)$ is graded over the root lattice Q of $\mathfrak{a}_1$. $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)$ is also graded over $\mathbb{Z}^2 = \mathbb{Z}_{(1)} \times \mathbb{Z}_{(2)}$;

$$\ddot{\mathrm{U}}_q(\mathfrak{a}_1) = \bigoplus_{(n_1,n_2) \in \mathbb{Z}^2} \ddot{\mathrm{U}}_q(\mathfrak{a}_1)_{(n_1,n_2)}\,,$$

where, for every $(n_1, n_2) \in \mathbb{Z}^2$, we let

$$\ddot{\mathrm{U}}_q(\mathfrak{a}_1)_{(n_1,n_2)} = \left\{x \in \ddot{\mathrm{U}}_q(\mathfrak{a}_1) : \mathsf{D}_1 x \mathsf{D}_1^{-1} = q^{n_1} x, \quad \mathsf{D}_2 x \mathsf{D}_2^{-1} = q^{n_2} x\right\} .$$

In the coming section, we shall also need the $\mathbb{Z}_{(2)}$-grades

$$\ddot{\mathrm{U}}_q(\mathfrak{a}_1)_n = \left\{x \in \ddot{\mathrm{U}}_q(\mathfrak{a}_1) : \mathsf{D}_2 x \mathsf{D}_2^{-1} = q^{n} x\right\} ,$$

for every $n \in \mathbb{Z}$.

Proposition 3.3.5. *The set* $\left\{C^{1/2}, C^{-1/2}, \mathsf{K}^+_{1,0,m}, \mathsf{K}^-_{1,0,-m} : m \in \mathbb{N}\right\}$ *generates a subalgebra of* $\ddot{\mathrm{U}}_q^{0,0}(\mathfrak{a}_1)$ *that is isomorphic to* $\dot{\mathrm{U}}_q^0(\mathfrak{a}_1)$*.*

Proof. This can be directly checked from the defining relations. Otherwise, it suffices to observe that the algebra isomorphism $\widehat{\Psi} : \widehat{\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)} \to \widehat{\ddot{\mathrm{U}}'_q(\mathfrak{a}_1)}$ – see theorem 3.3.22 – restricts on that set to

$$\widehat{\Psi}(C^{\pm 1/2}) = \mathsf{C}^{\pm 1/2} \qquad \text{and} \qquad \widehat{\Psi}(\mathbf{k}_1^{\pm}(z)) = -\mathsf{K}^{\mp}_{1,0}(\mathsf{C}^{-1/2} z) .$$

□

3.3.2 $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)$ as a topological algebra

Because of relation (5.0.6), the definition of $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)$ is not purely algebraic. Indeed, the r.h.s. of (5.0.6) involves two infinite series. One way to make sense of that relation is to equip $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)$ – and, for later use, its tensor powers – with a topology, such that both series be convergent in the corresponding completion $\widehat{\ddot{\mathrm{U}}_q(\mathfrak{a}_1)}$ of $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)$. Making use of the natural $\mathbb{Z}_{(2)}$-grading of the tensor algebras $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)^{\otimes m}$, $m \in \mathbb{N}^\times$, we let, for every $n \in \mathbb{N}$,

$$\dot{\Omega}_n^{(m)} := \bigoplus_{\substack{r \geq n \\ s \geq n}} \ddot{\mathrm{U}}_q(\mathfrak{a}_1)^{\otimes m} \cdot \left(\ddot{\mathrm{U}}_q(\mathfrak{a}_1)^{\otimes m}\right)_{-r} \cdot \ddot{\mathrm{U}}_q(\mathfrak{a}_1)^{\otimes m} \cdot \left(\ddot{\mathrm{U}}_q(\mathfrak{a}_1)^{\otimes m}\right)_{s} \cdot \ddot{\mathrm{U}}_q(\mathfrak{a}_1)^{\otimes m} .$$

One easily checks that

Proposition 3.3.6. *The following hold true for every* $m \in \mathbb{N}^\times$*:*

i. For every $n \in \mathbb{N}$, $\dot{\Omega}_n^{(m)}$ *is a two-sided ideal of* $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)^{\otimes m}$*;*

ii. For every $n \in \mathbb{N}$, $\dot{\Omega}_n^{(m)} \supseteq \dot{\Omega}_{n+1}^{(m)}$*;*

iii. $\dot{\Omega}_0^{(m)} = \bigcup_{n \in \mathbb{N}} \dot{\Omega}_n^{(m)} = \ddot{\mathrm{U}}_q(\mathfrak{a}_1)^{\otimes m}$*;*

iv. $\bigcap_{n \in \mathbb{N}} \dot{\Omega}_n^{(m)} = \{0\}$*;*

v. For every $n, p \in \mathbb{N}$, $\dot{\Omega}_n^{(m)} + \dot{\Omega}_p^{(m)} \subseteq \dot{\Omega}_{\min(n,p)}$*;*

vi. For every $n, p \in \mathbb{N}$, $\dot{\Omega}_n^{(m)} \cdot \dot{\Omega}_p^{(m)} \subseteq \dot{\Omega}_{\max(n,p)}$*.*

Proof. See [MZ] for a proof in the $\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)$ case that can be transposed to the present situation. □

Definition-Proposition 3.3.7. We endow $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)$ with the topology τ whose open sets are either $\emptyset$ or nonempty subsets $\mathcal{O} \subseteq \ddot{\mathrm{U}}_q(\mathfrak{a}_1)$ such that for every $x \in \mathcal{O}$, $x + \dot{\Omega}_n^{(1)} \subseteq \mathcal{O}$ for some $n \in \mathbb{N}$. Similarly, we endow each tensor power $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)^{\otimes m \geq 2}$ with the topology induced by $\{\dot{\Omega}_n^{(m)} : n \in \mathbb{N}\}$. These turn $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)$ into a (separated) topological algebra. We then let $\widehat{\ddot{\mathrm{U}}_q(\mathfrak{a}_1)}$ denote its completion and we extend by continuity to $\widehat{\ddot{\mathrm{U}}_q(\mathfrak{a}_1)}$ all the (anti)-automorphisms defined over $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)$ and its subalgebras in the previous section In particular, we extend $\jmath : \ddot{\mathrm{U}}'_q(\mathfrak{a}_1) \hookrightarrow \ddot{\mathrm{U}}_q(\mathfrak{a}_1)$ into

$$\widehat{\jmath} : \widehat{\ddot{\mathrm{U}}'_q(\mathfrak{a}_1)} \hookrightarrow \widehat{\ddot{\mathrm{U}}_q(\mathfrak{a}_1)} .$$

Similarly, we denote with a hat the completion of any subalgebra of $\widehat{\ddot{\mathrm{U}}_q(\mathfrak{a}_1)}$, like e.g. $\widehat{\ddot{\mathrm{U}}_q^-(\mathfrak{a}_1)}$, $\widehat{\ddot{\mathrm{U}}_q^0(\mathfrak{a}_1)}$ and $\widehat{\ddot{\mathrm{U}}_q^+(\mathfrak{a}_1)}$. We eventually denote by $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)^{\widehat{\otimes} m \geq 2}$ the corresponding completions of $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)^{\otimes m \geq 2}$.

Proof. This was proven in [MZ]. ☐

Remark 3.3.8. As was noted in [MZ], the above defined topology is actually ultrametrizable.

3.3.3 The double quantum loop algebra

An alternative way to make sense of relations (5.0.6) consists in observing that $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)$ is *proalgebraic*. Indeed, for every $N \in \mathbb{N}$, let $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)^{(N)}$ be the $\mathbb{F}$-algebra generated by

$$\{\mathsf{C}^{1/2}, \mathsf{C}^{-1/2}, \mathsf{c}_n^+, \mathsf{c}_{-n}^-, \mathsf{K}_{1,0,m}^+, \mathsf{K}_{1,0,-m}^-, \mathsf{K}_{1,p,r}^+, \mathsf{K}_{1,-p,r}^-, \mathsf{X}_{1,r,s}^+, \mathsf{X}_{1,r,s}^- : m \in \mathbb{N}, n \in [\![0, N]\!], p \in \mathbb{N}^\times, r, s \in \mathbb{Z}\}$$

subject to relations ((3.3.1) – (5.0.6)), where, this time,

$$\mathbf{c}^\pm(z) = \sum_{m=0}^{N} \mathsf{c}_{\pm m}^\pm z^{\mp m} . \tag{3.3.17}$$

Similarly, define $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)^{(-1)}$ as the $\mathbb{F}$-algebra generated by

$$\{\mathsf{C}^{1/2}, \mathsf{C}^{-1/2}, \mathsf{K}_{1,0,m}^+, \mathsf{K}_{1,0,-m}^-, \mathsf{K}_{1,p,r}^+, \mathsf{K}_{1,-p,r}^-, \mathsf{X}_{1,r,s}^+, \mathsf{X}_{1,r,s}^- : m \in \mathbb{N}, p \in \mathbb{N}^\times, r, s \in \mathbb{Z}\}$$

subject to relations ((3.3.1) – (5.0.6)), where $\mathbf{c}^\pm(z) = 1$.

Now clearly, each $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)^{(N)}$, $N \in \mathbb{N} \cup \{-1\}$, is algebraic since the sums on the r.h.s. of (5.0.6) are both finite – whenever $\mathbf{c}^\pm(z)^{-1}$ is involved, just multiply through by $\mathbf{c}^\pm(z)$ to get an equivalent algebraic relation. Moreover, letting $\mathcal{I}_N$ be the two-sided ideal of $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)^{(N)}$ generated by $\{\mathsf{c}_N^+, \mathsf{c}_{-N}^-\}$ (resp. $\{\mathsf{c}_0^+ - 1, \mathsf{c}_0^- - 1\}$) for every $N > 1$ (resp. for $N = 0$), we obviously have a surjective algebra homomorphism

$$\ddot{\mathrm{U}}_q(\mathfrak{a}_1)^{(N)} \longrightarrow \ddot{\mathrm{U}}_q(\mathfrak{a}_1)^{(N-1)} \cong \frac{\ddot{\mathrm{U}}_q(\mathfrak{a}_1)^{(N)}}{\mathcal{I}_N} \tag{3.3.18}$$

and we can define $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)$ as the inverse limit

$$\ddot{\mathrm{U}}_q(\mathfrak{a}_1) = \varprojlim \ddot{\mathrm{U}}_q(\mathfrak{a}_1)^{(N)}$$

of the system of algebras

$$\cdots \longrightarrow \ddot{\mathrm{U}}_q(\mathfrak{a}_1)^{(N)} \longrightarrow \ddot{\mathrm{U}}_q(\mathfrak{a}_1)^{(N-1)} \longrightarrow \cdots \longrightarrow \ddot{\mathrm{U}}_q(\mathfrak{a}_1)^{(0)} \longrightarrow \ddot{\mathrm{U}}_q(\mathfrak{a}_1)^{(-1)} .$$

Definition 3.3.9. We shall refer to the quotient of $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)^{(-1)}$ by the two-sided ideal generated by $\left\{\mathsf{C}^{1/2}-1\right\}$ as the *double quantum loop algebra* of type $\mathfrak{a}_1$ and denote it by $\ddot{\mathrm{L}}_q(\mathfrak{a}_1)$. Correspondingly, we denote by $\ddot{\mathrm{L}}_q^{\pm}(\mathfrak{a}_1)$ and $\ddot{\mathrm{L}}_q^{0}(\mathfrak{a}_1)$, the subalgebras of $\ddot{\mathrm{L}}_q(\mathfrak{a}_1)$ respectively generated by $\left\{\mathsf{X}^{\pm}_{1,r,s} : r,s\in\mathbb{Z}\right\}$ and

$$\left\{\mathsf{K}^{+}_{1,0,m}, \mathsf{K}^{-}_{1,0,-m}, \mathsf{K}^{+}_{1,n,r}, \mathsf{K}^{-}_{1,-n,r} : m\in\mathbb{N}, n\in\mathbb{N}^{\times}, r\in\mathbb{Z}\right\} .$$

We denote by $\ddot{\mathrm{L}}_q^{0,0}(\mathfrak{a}_1)$ the subalgebra of $\ddot{\mathrm{L}}_q^{0}(\mathfrak{a}_1)$ generated by

$$\left\{\mathsf{K}^{+}_{1,0,m}, \mathsf{K}^{-}_{1,0,-m} : m\in\mathbb{N}\right\} .$$

It is worth emphasizing that $\ddot{\mathrm{L}}_q^{0,0}(\mathfrak{a}_1)$ is abelian.

3.3.4 Triangular decomposition of $\widehat{\ddot{\mathrm{U}}'_q(\mathfrak{a}_1)}$

In [MZ], we proved that $\widehat{\ddot{\mathrm{U}}'_q(\mathfrak{a}_1)}$ has a triangular decomposition in the following sense.

Definition 3.3.10. Let A be a complete topological algebra with closed subalgebras $A^{\pm}$ and A^0. We shall say that (A^-, A^0, A^+) is a *triangular decomposition* of A if the multiplication induces a bicontinuous isomorphism of vector spaces $A^-\widehat{\otimes}A^0\widehat{\otimes}A^+ \xrightarrow{\sim} A$.

Recalling the definitions of $\ddot{\mathrm{U}}_q^{\pm}(\mathfrak{a}_1)$ and $\ddot{\mathrm{U}}_q^{0}(\mathfrak{a}_1)$ from definition 3.3.1, we have

Proposition 3.3.11. $(\ddot{\mathrm{U}}_q^{-}(\mathfrak{a}_1), \ddot{\mathrm{U}}_q^{0}(\mathfrak{a}_1), \ddot{\mathrm{U}}_q^{+}(\mathfrak{a}_1))$ *is a triangular decomposition of* $\widehat{\ddot{\mathrm{U}}'_q(\mathfrak{a}_1)}$ *and* $\ddot{\mathrm{U}}_q^{\pm}(\mathfrak{a}_1)$ *is bicontinuously isomorphic to the algebra generated by* $\{\mathsf{X}^{\pm}_{1,r,s} : r,s\in\mathbb{Z}\}$ *subject to relation (5.0.5).*

Proof. See [MZ]. □

3.3.5 The closed subalgebra $\widehat{\ddot{\mathrm{U}}_q^0(\mathfrak{a}_1)}$ as a topological Hopf algebra

Definition 3.3.12. In $\widehat{\ddot{\mathrm{U}}_q^0(\mathfrak{a}_1)}$, we define

$$\mathbf{p}^{\pm}(z) = \sum_{m\in\mathbb{N}} \mathsf{p}^{\pm}_{\pm m} z^{\mp m} = \mathbf{c}^{\pm}(z)\mathbf{K}^{\mp}_{1,0}(\mathsf{C}^{-1/2}z)^{-1}\mathbf{K}^{\mp}_{1,0}(\mathsf{C}^{-1/2}zq^2) \tag{3.3.19}$$

and for every $m\in\mathbb{N}^{\times}$,

$$\mathbf{t}^{+}_{1,m}(z) = \sum_{n\in\mathbb{N}} \mathsf{t}^{+}_{1,m,n} z^{-n} = -\frac{1}{q-q^{-1}}\mathbf{K}^{+}_{1,0}(zq^{-2m})^{-1}\mathbf{K}^{+}_{1,m}(z) , \tag{3.3.20}$$

$$\mathbf{t}^{-}_{1,-m}(z) = \sum_{n\in\mathbb{N}} \mathsf{t}^{-}_{1,-m,n} z^{n} = \frac{1}{q-q^{-1}}\mathbf{K}^{-}_{1,-m}(z)\mathbf{K}^{-}_{1,0}(zq^{-2m})^{-1} . \tag{3.3.21}$$

Then, we let $\ddot{\mathrm{U}}_q^{0+}(\mathfrak{a}_1)$ be the subalgebra of $\widehat{\ddot{\mathrm{U}}_q^{0}(\mathfrak{a}_1)}$ generated by

$$\{\mathsf{C}^{1/2}, \mathsf{C}^{-1/2}, \mathsf{p}^+_m, \mathsf{p}^-_{-m}, \mathsf{t}^+_{1,p,n}, \mathsf{t}^-_{1,-p,n} : m \in \mathbb{N}, n \in \mathbb{Z}, p \in \mathbb{N}^\times\}\,.$$

and we let $\widehat{\ddot{\mathrm{U}}_q^{0+}(\mathfrak{a}_1)}$ be its completion in the inherited topology.

Clearly, the closed subalgebra $\widehat{\ddot{\mathrm{U}}_q^{0}(\mathfrak{a}_1)}$ can be presented as in definition 3.3.3 or, equivalently, in terms of the generators in

$$\{\mathsf{C}^{1/2}, \mathsf{C}^{-1/2}, \mathsf{K}^+_{1,0,m}, \mathsf{K}^-_{1,0,-m}, \mathsf{p}^+_m, \mathsf{p}^-_{-m}, \mathsf{t}^+_{1,p,n}, \mathsf{t}^-_{1,-p,n} : m \in \mathbb{N}, n \in \mathbb{Z}, p \in \mathbb{N}^\times\}\,.$$

In section 3.3.10, we will endow $\widehat{\ddot{\mathrm{U}}'_q(\mathfrak{a}_1)}$ with a topological Hopf algebraic structure. It turns out that, for that structure, the closed subalgebra $\widehat{\ddot{\mathrm{U}}_q^{0}(\mathfrak{a}_1)}$ is not a closed Hopf subalgebra of $\widehat{\ddot{\mathrm{U}}_q(\mathfrak{a}_1)}$ – see lemma 4.22 in [MZ] or lemma 3.3.30 below. However, it is possible to endow $\widehat{\ddot{\mathrm{U}}_q^{0}(\mathfrak{a}_1)}$ with its own topological Hopf algebraic structure as follows.

Definition-Proposition 3.3.13. We endow $\widehat{\ddot{\mathrm{U}}_q^{0}(\mathfrak{a}_1)}$ with:

i. the comultiplication $\Delta^0 : \widehat{\ddot{\mathrm{U}}_q^{0}(\mathfrak{a}_1)} \to \ddot{\mathrm{U}}_q^{0}(\mathfrak{a}_1)\widehat{\otimes}\ddot{\mathrm{U}}_q^{0}(\mathfrak{a}_1)$ defined by

$$\Delta^0(\mathsf{C}^{\pm 1/2}) = \mathsf{C}^{\pm 1/2} \otimes \mathsf{C}^{\pm 1/2} \tag{3.3.22}$$

$$\Delta^0(\mathbf{K}^\pm_{1,0}(z)) = -\mathbf{K}^\pm_{1,0}(z\mathsf{C}_{(2)}^{\frac{1\mp1}{2}}) \otimes \mathbf{K}^\pm_{1,0}(z\mathsf{C}_{(1)}^{\frac{1\pm1}{2}})\,, \tag{3.3.23}$$

$$\Delta^0(\mathbf{p}^\pm(z)) = \mathbf{p}^\pm(z\mathsf{C}_{(2)}^{\pm1/2}) \otimes \mathbf{p}^\pm(z\mathsf{C}_{(1)}^{\mp1/2})\,, \tag{3.3.24}$$

$$\begin{aligned}\Delta^0(\mathbf{t}^+_{1,m}(z)) \;&=\; \mathbf{t}^+_{1,m}(z)\otimes 1 + \prod_{k=1}^{m}\mathbf{p}^-(zq^{-2k}\mathsf{C}_{(1)}^{1/2})\widehat{\otimes}\mathbf{t}^+_{1,m}(z\mathsf{C}_{(1)})\\ &\quad -(q-q^{-1})\sum_{k=1}^{m-1}\prod_{l=k+1}^{m}\mathbf{p}^-(zq^{-2l}\mathsf{C}_{(1)}^{1/2})\mathbf{t}^+_{1,k}(z)\widehat{\otimes}\mathbf{t}^+_{1,m-k}(zq^{-2k}\mathsf{C}_{(1)})\,,\end{aligned} \tag{3.3.25}$$

$$\begin{aligned}\Delta^0(\mathbf{t}^-_{1,-m}(z)) \;&=\; \mathbf{t}^-_{1,-m}(z\mathsf{C}_{(2)})\widehat{\otimes}\prod_{k=1}^{m}\mathbf{p}^+(zq^{-2k}\mathsf{C}_{(2)}^{1/2}) + 1\otimes\mathbf{t}^-_{1,-m}(z)\\ &\quad +(q-q^{-1})\sum_{k=1}^{m-1}\mathbf{t}^-_{1,-(m-k)}(zq^{-2k}\mathsf{C}_{(2)})\widehat{\otimes}\mathbf{t}^-_{1,-m}(z)\prod_{l=+1}^{m}\mathbf{p}^+(zq^{-2l}\mathsf{C}_{(2)}^{1/2})\,,\end{aligned} \tag{3.3.26}$$

for every $m \in \mathbb{N}$,

ii. the counit $\varepsilon(\mathsf{C}) = \varepsilon^0(\mathbf{K}^\pm_{1,0}(z)) = \varepsilon^0(\mathbf{p}^\pm(z)) = 1$, $\varepsilon^0(\mathbf{t}^\pm_{1,\pm m}(z)) = 0$, for every $m \in \mathbb{N}$,

iii. and the antipode defined by

$$S^0(\mathsf{C}^{\pm1/2}) = \mathsf{C}^{\mp1/2}\,, \tag{3.3.27}$$

$$S^0(\mathbf{K}^\pm_{1,0}(z)) = \mathbf{K}^\pm_{1,0}(z\mathsf{C}^{-1})^{-1}\,, \tag{3.3.28}$$

$$S^0(\mathsf{p}^\pm(z)) = \mathsf{p}^\pm(z)^{-1}\,, \tag{3.3.29}$$

$$S^0(\mathbf{t}^+_{1,m}(z)) = -\prod_{k=1}^{m} \mathsf{p}^-(zq^{-2k}\mathsf{C}^{-1/2})^{-1} \sum_{n=1}^{m} \sum_{\lambda\in C_n(m)} (-1)^{n-1} c_{m,\lambda}\mathbf{t}^+_{1,\lambda}(z\mathsf{C}^{-1})\,, \tag{3.3.30}$$

$$S^0(\mathbf{t}^-_{1,-m}(z)) = -\sum_{n=1}^{m} \sum_{\lambda\in C_n(m)} c_{m,\lambda}\mathbf{t}^-_{1,-\lambda}(z\mathsf{C}^{-1}) \prod_{k=1}^{m} \mathsf{p}^+(zq^{-2k})^{-1}\,, \tag{3.3.31}$$

where we have set, for every $m \in \mathbb{N}^\times$ and every $\lambda \in C_n(m)$,

$$c_{m,\lambda} = (q-q^{-1})^{n-1}\frac{[m+1]_q}{[m-1]_q}\prod_{i=1}^{n}\frac{[\lambda_i-1]_q}{[\lambda_i+1]_q}$$

and

$$\mathbf{t}^+_{1,\lambda}(z\mathsf{C}^{-1}) = \overleftarrow{\prod_{i\in[\![n]\!]}} \mathbf{t}^+_{1,\lambda_i}(zq^{-2\sum_{k=i+1}^{n}\lambda_k}\mathsf{C}^{-1})\,,$$

$$\mathbf{t}^-_{1,-\lambda}(z\mathsf{C}^{-1}) = \overrightarrow{\prod_{i\in[\![n]\!]}} \mathbf{t}^-_{1,-\lambda_i}(zq^{-2\sum_{k=i+1}^{n}\lambda_k}\mathsf{C}^{-1})\,.$$

for every $m \in \mathbb{N}$.

With these operations, $\widehat{\ddot{\mathrm{U}}^0_q(\mathfrak{a}_1)}$ is a topological Hopf algebra.

Proof. One easily checks that Δ^0 as defined by (3.3.22 – 3.3.26) is compatible with the defining relations of $\widehat{\ddot{\mathrm{U}}^0_q(\mathfrak{a}_1)}$ and that S^0 is compatible with both the multiplication and the comultiplication. ☐

In that presentation, one readily checks that

Proposition 3.3.14. *$\widehat{\ddot{\mathrm{U}}^{0+}_q(\mathfrak{a}_1)}$ is a closed Hopf subalgebra of $\widehat{\ddot{\mathrm{U}}^0_q(\mathfrak{a}_1)}$.*

Proof. $\widehat{\ddot{\mathrm{U}}^{0+}_q(\mathfrak{a}_1)}$ is a closed subalgebra of $\widehat{\ddot{\mathrm{U}}^0_q(\mathfrak{a}_1)}$ and it is clearly stable under Δ^0 and S^0. ☐

3.3.6 The closed subalgebra $\widehat{\ddot{\mathrm{U}}^0_q(\mathfrak{a}_1)}$ and the elliptic Hall algebra

As emphasized in [MZ], another remarkable feature of $\widehat{\ddot{\mathrm{U}}'_q(\mathfrak{a}_1)}$ and, more particularly of its closed subalgebra $\widehat{\ddot{\mathrm{U}}^0_q(\mathfrak{a}_1)}$, is the existence of an algebra homomorphism onto it, from the elliptic Hall algebra that we now define.

Definition 3.3.15. Let q_1, q_2, q_3 be three (dependent) formal variables such that $q_1q_2q_3 = 1$. The *elliptic Hall algebra* $\mathcal{E}_{q_1,q_2,q_3}$ is the $\mathbb{Q}(q_1,q_2,q_3)$-algebra generated by $\left\{C^{1/2}, \mathbb{C}^{-1/2}, \psi^+_m, \psi^-_{-m}, e^+_n, e^-_n : m\in\mathbb{N}, n\in\mathbb{Z}\right\}$, with $\psi^\pm_0$ invertible, subject to the relations

$$C^{\pm 1/2} \text{ is central}\,, \tag{3.3.32}$$

$$\boldsymbol{\psi}^\pm(z)\boldsymbol{\psi}^\pm(w) = \boldsymbol{\psi}^\pm(w)\boldsymbol{\psi}^\pm(z)\,, \tag{3.3.33}$$

$$g(Cz,w)g(Cw,z)\boldsymbol{\psi}^+(z)\boldsymbol{\psi}^-(w) = g(z,Cw)g(w,Cz)\boldsymbol{\psi}^-(w)\boldsymbol{\psi}^+(z)\,, \tag{3.3.34}$$

$$g(C^{\frac{1\pm1}{2}}z,w)\boldsymbol{\psi}^\pm(z)\mathbf{e}^+(w) = -g(w,C^{\frac{1\pm1}{2}}z)\mathbf{e}^+(w)\boldsymbol{\psi}^\pm(z)\,, \tag{3.3.35}$$

$$g(w,C^{\frac{1\mp1}{2}}z)\boldsymbol{\psi}^\pm(z)\mathbf{e}^-(w) = -g(C^{\frac{1\mp1}{2}}z,w)\mathbf{e}^-(w)\boldsymbol{\psi}^\pm(z)\,, \tag{3.3.36}$$

$$[\mathbf{e}^+(z), \mathbf{e}^-(w)] = \frac{1}{g(1,1)} \left[\delta\left(\frac{Cw}{z}\right) \boldsymbol{\psi}^+(w) - \delta\left(\frac{w}{Cz}\right) \boldsymbol{\psi}^-(z) \right], \tag{3.3.37}$$

$$g(z,w)\mathbf{e}^+(z)\mathbf{e}^+(w) = -g(w,z)\mathbf{e}^+(w)\mathbf{e}^+(z)\,, \tag{3.3.38}$$

$$g(w,z)\mathbf{e}^-(z)\mathbf{e}^-(w) = -g(z,w)\mathbf{e}^-(w)\mathbf{e}^-(z)\,, \tag{3.3.39}$$

$$\operatorname*{res}_{v,w,z} (vwz)^m (v+z)(w^2 - vz)\mathbf{e}^\pm(v)\mathbf{e}^\pm(w)\mathbf{e}^\pm(z) = 0\,, \tag{3.3.40}$$

where $m \in \mathbb{Z}$ and we have introduced

$$g(z,w) = (z - q_1 w)(z - q_2 w)(z - q_3 w)\,, \tag{3.3.41}$$

$$\boldsymbol{\psi}^\pm(z) = \sum_{m \in \mathbb{N}} \psi^\pm_{\pm m} z^{\mp m}\,, \tag{3.3.42}$$

$$\mathbf{e}^\pm(z) = \sum_{m \in \mathbb{Z}} e^\pm_m z^{-m}\,. \tag{3.3.43}$$

Remark 3.3.16. The elliptic Hall algebra $\mathcal{E}_{q_1,q_2,q_3}$ is $\mathbb{Z}$-graded and can be equipped with a natural topology along the lines of what we did for $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)$ in section 3.3.2. It then becomes a topological algebra and we denote by $\widehat{\mathcal{E}_{q_1,q_2,q_3}}$ its completion. Similar topologies can be constructed on its tensor powers.

Definition-Proposition 3.3.17. We endow $\widehat{\mathcal{E}_{q_1,q_2,q_3}}$ with:

i. the comultiplication $\Delta_\mathcal{E} : \widehat{\mathcal{E}_{q_1,q_2,q_3}} \to \mathcal{E}_{q_1,q_2,q_3} \widehat{\otimes} \mathcal{E}_{q_1,q_2,q_3}$ defined by

$$\Delta_\mathcal{E}(\boldsymbol{\psi}^\pm(z)) = \boldsymbol{\psi}^\pm(zC_{(2)}^{\frac{1\pm1}{2}}) \otimes \boldsymbol{\psi}^\pm(zC_{(1)}^{\frac{1\mp1}{2}})\,, \tag{3.3.44}$$

$$\Delta_\mathcal{E}(\mathbf{e}^+(z)) = \mathbf{e}^+(z) \otimes 1 + \boldsymbol{\psi}^-(z) \widehat{\otimes} \mathbf{e}^+(zC_{(1)})\,, \tag{3.3.45}$$

$$\Delta_\mathcal{E}(\mathbf{e}^-(z)) = \mathbf{e}^-(zC_{(2)}) \widehat{\otimes} \boldsymbol{\psi}^+(z) + 1 \otimes \mathbf{e}^-(z)\,, \tag{3.3.46}$$

ii. the counit $\varepsilon_\mathcal{E} : \widehat{\mathcal{E}_{q_1,q_2,q_3}} \to \mathbb{F}$ defined by $\varepsilon_\mathcal{E}(C^{\pm 1/2}) = \varepsilon_\mathcal{E}(\boldsymbol{\psi}^\pm(z)) = 1$, $\varepsilon_\mathcal{E}(\mathbf{e}^\pm(z)) = 0$,

iii. the antipode $S_\mathcal{E} : \widehat{\mathcal{E}_{q_1,q_2,q_3}} \to \widehat{\mathcal{E}_{q_1,q_2,q_3}}$ defined by

$$S_\mathcal{E}(\boldsymbol{\psi}^\pm(z)) = \boldsymbol{\psi}^\pm(zC^{-1})^{-1}\,, \tag{3.3.47}$$

$$S_\mathcal{E}(\mathbf{e}^+(z)) = -\boldsymbol{\psi}^-(zC^{-1})^{-1}\mathbf{e}^+(zC^{-1})\,, \tag{3.3.48}$$

$$S_\mathcal{E}(\mathbf{e}^-(z)) = -\mathbf{e}^-(zC^{-1})\boldsymbol{\psi}^+(zC^{-1})^{-1}\,. \tag{3.3.49}$$

With the above defined operations, $\widehat{\mathcal{E}_{q_1,q_2,q_3}}$ is a topological Hopf algebra.

Proposition 3.3.18. *There exists a unique continuous Hopf algebra homomorphism* $f : \widehat{\mathcal{E}_{q^{-4},q^2,q^2}} \to \widehat{\ddot{\mathrm{U}}^{0+}_q(\mathfrak{a}_1)}$ *such that*

$$f(C^{1/2}) = \mathcal{C}^{1/2}\,, \tag{3.3.50}$$

$$f(\boldsymbol{\psi}^\pm(z)) = \boldsymbol{p}^\pm(\mathcal{C}^{1/2} z q^{-2})\,, \tag{3.3.51}$$

$$f(\boldsymbol{e}^+(z)) = \boldsymbol{t}^+_{1,1}(z)\,, \tag{3.3.52}$$

$$f(\boldsymbol{e}^-(z)) = \frac{\boldsymbol{t}^-_{1,-1}(z)}{(q^2-q^{-2})^2}\,. \tag{3.3.53}$$

Proof. In [MZ], we proved that the assignment

$$C^{1/2} \mapsto \mathsf{C}^{1/2} \quad \boldsymbol{\psi}^\pm(z) \mapsto (q^2-q^{-2})^2\, \boldsymbol{\mathsf{p}}^\pm(\mathsf{C}^{1/2} z q^{-2})\,, \quad \mathbf{e}^\pm(z) \mapsto \boldsymbol{\mathsf{t}}^\pm_{1,\pm1}(z)$$

defined an $\mathbb{F}$-algebra homomorphism. Hence, f, which is obtained from the above assignment by rescaling the images of $\boldsymbol{\mathsf{p}}^\pm(z)$ and $\mathbf{e}^-(z)$, is obviously an $\mathbb{F}$-algebra homomorphism. Moreover, it suffices to write (3.3.24), (3.3.25) and (3.3.26) with $m=1$, to get

$$\Delta^0(\boldsymbol{\mathsf{p}}^\pm(z)) = \boldsymbol{\mathsf{p}}^\pm(z\mathsf{C}_{(2)}^{\pm1/2}) \otimes \boldsymbol{\mathsf{p}}^\pm(z\mathsf{C}_{(1)}^{\mp1/2})\,,$$

$$\Delta^0(\boldsymbol{\mathsf{t}}^+_{1,1}(z)) = \boldsymbol{\mathsf{t}}^+_{1,1}(z) \otimes 1 + \boldsymbol{\mathsf{p}}^-(zq^{-2}\mathsf{C}_{(1)}^{1/2}) \widehat{\otimes} \boldsymbol{\mathsf{t}}^+_{1,1}(z\mathsf{C}_{(1)})\,,$$

$$\Delta^0(\boldsymbol{\mathsf{t}}^-_{1,-1}(z)) = \boldsymbol{\mathsf{t}}^-_{1,-1}(z\mathsf{C}_{(2)}) \widehat{\otimes} \boldsymbol{\mathsf{p}}^+(zq^{-2}\mathsf{C}_{(2)}^{1/2}) + 1 \otimes \boldsymbol{\mathsf{t}}^-_{1,-1}(z)\,,$$

as well as (3.3.29), (3.3.30) and (3.3.31), with $m=1$, to get

$$S^0(\boldsymbol{\mathsf{p}}^\pm(z)) = \boldsymbol{\mathsf{p}}^\pm(z)^{-1}\,,$$

$$S^0(\boldsymbol{\mathsf{t}}^+_{1,1}(z)) = -\boldsymbol{\mathsf{p}}^-(zq^{-2}\mathsf{C}^{-1/2})^{-1}\boldsymbol{\mathsf{t}}^+_{1,1}(z\mathsf{C}^{-1})\,,$$

$$S^0(\boldsymbol{\mathsf{t}}^-_{1,-1}(z)) = -\boldsymbol{\mathsf{t}}^-_{1,-1}(z\mathsf{C}^{-1})\boldsymbol{\mathsf{p}}^+(zq^{-2}\mathsf{C}^{-1/2})^{-1}\,,$$

and thus to prove that $(f\widehat{\otimes}f) \circ \Delta_{\mathcal{E}} = \Delta^0 \circ f$ and $f \circ S_{\mathcal{E}} = S^0 \circ f$ as claimed. □

Remark 3.3.19. Note that we have $f(\psi_0^+)f(\psi_0^-) = f(\psi_0^-)f(\psi_0^+) = 1$, meaning that f descends to the quotient of $\mathcal{E}_{q^{-4},q^2,q^2}$ by the two-sided ideal generated by $\{\psi_0^+\psi_0^- - 1, \psi_0^-\psi_0^+ - 1\}$. That quotient is actually Miki's (q,γ)-analogue of the $W_{1+\infty}$ algebra [Mik07].

3.3.7 The quantum toroidal algebra $\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)$

Let $\dot{I} = \{0,1\}$ be a labeling of the nodes of the Dynkin diagram of type $\dot{\mathfrak{a}}_1$ and let $\dot{\Phi} = \{\alpha_0, \alpha_1\}$ be a choice of simple roots for the corresponding root system. Let $\dot{Q}^\pm = \mathbb{Z}^\pm\alpha_0 \oplus \mathbb{Z}^\pm\alpha_1$ and let $\dot{Q} = \mathbb{Z}\alpha_0 \oplus \mathbb{Z}\alpha_1$ be the type $\dot{\mathfrak{a}}_1$ root lattice.

Definition 3.3.20. The *quantum toroidal algebra* $\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)$ is the associative $\mathbb{F}$-algebra generated by the generators

$$\left\{D, D^{-1}, C^{1/2}, C^{-1/2}, k^+_{i,n}, k^-_{i,-n}, x^+_{i,m}, x^-_{i,m} : i \in \dot{I}, m \in Z, n \in \mathbb{N}\right\}$$

subject to the following relations

$$C^{\pm1/2} \text{ is central} \qquad C^{\pm1/2}C^{\mp1/2} = 1 \qquad D^{\pm1}D^{\mp1} = 1 \tag{3.3.54}$$

$$D\mathbf{k}_i^\pm(z)D^{-1} = \mathbf{k}_i^\pm(zq^{-1}) \qquad D\mathbf{x}_i^\pm(z)D^{-1} = \mathbf{x}_i^\pm(zq^{-1}) \tag{3.3.55}$$

$$\mathbf{k}_i^\pm(z_1)\mathbf{k}_j^\pm(z_2) = \mathbf{k}_j^\pm(z_2)\mathbf{k}_i^\pm(z_1) \tag{3.3.56}$$

$$\mathbf{k}_i^-(z_1)\mathbf{k}_j^+(z_2) = G_{ij}^-(C^{-1}z_1/z_2)G_{ij}^+(Cz_1/z_2)\mathbf{k}_j^+(z_2)\mathbf{k}_i^-(z_1) = 1 \mod z_1/z_2 \tag{3.3.57}$$

$$G_{ij}^{\mp}(C^{\mp 1/2}z_2/z_1)\mathbf{k}_i^{+}(z_1)\mathbf{x}_j^{\pm}(z_2) = \mathbf{x}_j^{\pm}(z_2)\mathbf{k}_i^{+}(z_1) \tag{3.3.58}$$

$$\mathbf{k}_i^{-}(z_1)\mathbf{x}_j^{\pm}(z_2) = G_{ij}^{\mp}(C^{\mp 1/2}z_1/z_2)\mathbf{x}_j^{\pm}(z_2)\mathbf{k}_i^{-}(z_1) \tag{3.3.59}$$

$$(z_1 - q^{\pm c_{ij}}z_2)\mathbf{x}_i^{\pm}(z_1)\mathbf{x}_j^{\pm}(z_2) = (z_1 q^{\pm c_{ij}} - z_2)\mathbf{x}_j^{\pm}(z_2)\mathbf{x}_i^{\pm}(z_1) \tag{3.3.60}$$

$$[\mathbf{x}_i^{+}(z_1), \mathbf{x}_j^{-}(z_2)] = \frac{\delta_{ij}}{q-q^{-1}}\left[\delta\left(\frac{z_1}{Cz_2}\right)\mathbf{k}_i^{+}(z_1C^{-1/2}) - \delta\left(\frac{z_1C}{z_2}\right)\mathbf{k}_i^{-}(z_2C^{-1/2})\right] \tag{3.3.61}$$

$$\sum_{\sigma\in S_{1-c_{ij}}}\sum_{k=0}^{1-c_{ij}}(-1)^k\binom{1-c_{ij}}{k}_q \mathbf{x}_i^{\pm}(z_{\sigma(1)})\cdots\mathbf{x}_i^{\pm}(z_{\sigma(k)})\mathbf{x}_j^{\pm}(z)\mathbf{x}_i^{\pm}(z_{\sigma(k+1)})\cdots\mathbf{x}_i^{\pm}(z_{\sigma(1-c_{ij})}) = 0 \tag{3.3.62}$$

where, for every $i \in \dot{I}$, we define the following $\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)$-valued formal distributions

$$\mathbf{x}_i^{\pm}(z) := \sum_{m\in\mathbb{Z}} x_{i,m}^{\pm} z^{-m} \in \dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)[[z, z^{-1}]]\,; \tag{3.3.63}$$

$$\mathbf{k}_i^{\pm}(z) := \sum_{n\in\mathbb{N}} k_{i,\pm n}^{\pm} z^{\mp n} \in \dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)[[z^{\mp 1}]]\,, \tag{3.3.64}$$

for every $i, j \in \dot{I}$, we define the following $\mathbb{F}$-valued formal power series

$$G_{ij}^{\pm}(z) := q^{\pm c_{ij}} + (q-q^{-1})[\pm c_{ij}]_q \sum_{m\in\mathbb{N}^{\times}} q^{\pm m c_{ij}} z^m \in \mathbb{F}[[z]] \tag{3.3.65}$$

is an $\mathbb{F}$-valued formal distribution,

Note that $G_{ij}^{\pm}(z)$ is invertible in $\mathbb{F}[[z]]$ with inverse $G_{ij}^{\mp}(z)$, i.e.

$$G_{ij}^{\pm}(z)G_{ij}^{\mp}(z) = 1\,, \tag{3.3.66}$$

and that it can be viewed as the power series expansion of a rational function of $(z_1, z_2) \in \mathbb{C}^2$ as $|z_2| \gg |z_1|$, which we shall denote as follows

$$G_{ij}^{\pm}(z_1/z_2) = \left(\frac{z_1 q^{\mp c_{ij}} - z_2}{z_1 - q^{\mp c_{ij}} z_2}\right)_{|z_2|\gg|z_1|}\,. \tag{3.3.67}$$

Observe furthermore that we have the following useful identity in $\mathbb{F}[[z, z^{-1}]]$

$$\frac{G_{ij}^{\pm}(z_1/z_2) - G_{ij}^{\mp}(z_2/z_1)}{q-q^{-1}} = [\pm c_{ij}]_q \delta\left(\frac{z_1 q^{\pm c_{ij}}}{z_2}\right)\,. \tag{3.3.68}$$

Remark 3.3.21. In type $\mathfrak{a}_1$, $\dot{I} = \{0, 1\}$, $c_{ij} = 4\delta_{ij} - 2$ and we have an additional identity, namely $G_{10}^{\pm}(z) = G_{11}^{\mp}(z)$.

$\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)$ is obviously a $\mathbb{Z}$-graded algebra, i.e. we have

$$\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1) = \bigoplus_{n\in\mathbb{Z}} \dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)_n\,, \qquad \text{where for all } n \in \mathbb{Z} \qquad \dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)_n := \{x \in \dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1) : DxD^{-1} = q^n x\}\,. \tag{3.3.69}$$

It was proven in [Her05] to admit a triangular decomposition $(\dot{\mathrm{U}}_q^{-}(\dot{\mathfrak{a}}_1), \dot{\mathrm{U}}_q^{0}(\dot{\mathfrak{a}}_1), \dot{\mathrm{U}}_q^{+}(\dot{\mathfrak{a}}_1))$, where $\dot{\mathrm{U}}_q^{\pm}(\dot{\mathfrak{a}}_1)$ and

$\dot{\mathrm{U}}_q^0(\dot{\mathfrak{a}}_1)$ are the subalgebras of $\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)$ respectively generated by $\left\{x_{i,m}^{\pm} : i \in \dot{I}, m \in \mathbb{Z}\right\}$ and

$$\left\{C^{1/2}, C^{-1/2}, D, D^{-1}, k_{i,m}^{+}, k_{i,m}^{-} : i \in \dot{I}, m \in \mathbb{Z}\right\} .$$

Observe that $\dot{\mathrm{U}}_q^{\pm}(\dot{\mathfrak{a}}_1)$ admits a natural gradation over $\dot{Q}^{\pm}$ that we shall denote by

$$\dot{\mathrm{U}}_q^{\pm}(\dot{\mathfrak{a}}_1) = \bigoplus_{\alpha \in \dot{Q}^{\pm}} \dot{\mathrm{U}}_q^{\pm}(\dot{\mathfrak{a}}_1)_{\alpha} . \tag{3.3.70}$$

Of course $\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)$ is graded over the root lattice $\dot{Q}$. We finally remark that the two Dynkin diagram subalgebras $\dot{\mathrm{U}}_q(\mathfrak{a}_1)^{(0)}$ and $\dot{\mathrm{U}}_q(\mathfrak{a}_1)^{(1)}$ of $\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)$ generated by

$$\left\{D, D^{-1}, C^{1/2}, C^{-1/2}, k_{i,n}^{+}, k_{i,-n}^{-}, x_{i,m}^{+}, x_{i,m}^{-} : m \in Z, n \in \mathbb{N}\right\} ,$$

with $i = 0$ and $i = 1$ respectively, are both isomorphic to $\dot{\mathrm{U}}_q(\mathfrak{a}_1)$, thus yielding two injective algebra homomorphisms $\iota^{(i)} : \dot{\mathrm{U}}_q(\mathfrak{a}_1) \hookrightarrow \dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)$. In [MZ], making use of their natural $\mathbb{Z}$-grading, $\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)$ and all its tensor powers were endowed with a topology along the lines of what we did in section 3.3.2 for $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)$ and its tensor powers, and subsequently completed into $\widehat{\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)}$ and $\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)^{\widehat{\otimes} r}$. The main result in [MZ] – theorem 3.7 there – is the following

Theorem 3.3.22. *There exists a unique bicontinuous $\mathbb{F}$-algebra isomorphism $\widehat{\Psi} : \widehat{\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)} \xrightarrow{\sim} \widehat{\ddot{\mathrm{U}}'_q(\mathfrak{a}_1)}$ such that*

$$\widehat{\Psi}(D^{\pm 1}) = \mathsf{D}_2^{\pm 1} \qquad \widehat{\Psi}(C^{\pm 1/2}) = \mathsf{C}^{\pm 1/2} ,$$

$$\widehat{\Psi}(\mathbf{k}_0^{\pm}(z)) = -\boldsymbol{\mathsf{c}}^{\pm}(z)\boldsymbol{\mathsf{K}}_{1,0}^{\mp}(\mathsf{C}^{-1/2}z)^{-1} \qquad \widehat{\Psi}(\mathbf{k}_1^{\pm}(z)) = -\boldsymbol{\mathsf{K}}_{1,0}^{\mp}(\mathsf{C}^{-1/2}z)$$

$$\widehat{\Psi}(\mathbf{x}_0^{+}(z)) = -\boldsymbol{\mathsf{c}}^{-}(\mathsf{C}^{1/2}z)\boldsymbol{\mathsf{K}}_{1,0}^{+}(z)^{-1}\boldsymbol{\mathsf{X}}_{1,1}^{-}(\mathsf{C}z) \qquad \widehat{\Psi}(\mathbf{x}_0^{-}(z)) = -\boldsymbol{\mathsf{X}}_{1,-1}^{+}(\mathsf{C}z)\boldsymbol{\mathsf{c}}^{+}(\mathsf{C}^{1/2}z)\boldsymbol{\mathsf{K}}_{1,0}^{-}(z)^{-1}$$

$$\widehat{\Psi}(\mathbf{x}_1^{\pm}(z)) = \boldsymbol{\mathsf{X}}_{1,0}^{\pm}(z) .$$

Proof. See [MZ] for a proof. ☐

3.3.8 $\dot{\mathrm{U}}_q(\mathfrak{a}_1)$ subalgebras of $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)$

Interestingly, $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)$ admits countably many embeddings of the quantum affine algebra $\dot{\mathrm{U}}_q(\mathfrak{a}_1)$. This is the content of the following

Proposition 3.3.23. *For every $m \in \mathbb{Z}$, there exists a unique injective algebra homomorphism $\iota_m : \dot{\mathrm{U}}_q(\mathfrak{a}_1) \hookrightarrow \widehat{\ddot{\mathrm{U}}'_q(\mathfrak{a}_1)}$ such that*

$$\iota_m(C^{\pm 1/2}) = \mathsf{C}^{\pm 1/2} \qquad \iota_m(D^{\pm 1}) = \mathsf{D}_2^{\pm 1} \tag{3.3.71}$$

$$\iota_m(\mathbf{k}_1^{\pm}(z)) = -\prod_{p=1}^{|m|} \boldsymbol{\mathsf{c}}^{\pm}\left(q^{(1-2p)\mathrm{sign}(m)-1}z\right)^{\mathrm{sign}(m)} \boldsymbol{\mathsf{K}}_{1,0}^{\mp}(\mathsf{C}^{-1/2}z) , \tag{3.3.72}$$

$$\iota_m(\mathbf{x}_1^{\pm}(z)) = \boldsymbol{\mathsf{X}}_{1,\pm m}^{\pm}(z) . \tag{3.3.73}$$

Proof. See proposition 3.13 in [MZ]. ☐

Remark 3.3.24. The injective algebra homomorphisms ι_m, $m \in \mathbb{Z}$, defined above should not be mistaken with the injective algebra homomorphisms $\iota^{(i)}$, $i \in \{0, 1\}$, from the Dynkin diagram subalgebras $\dot{\mathrm{U}}_q(\mathfrak{a}_1)^{(0)}$ and $\dot{\mathrm{U}}_q(\mathfrak{a}_1)^{(1)}$ to $\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)$ – see discussion before theorem 3.3.22 for a definition of the latter.

We also have

Proposition 3.3.25. *For every $i \in \dot{I} = \{0, 1\}$, $\widehat{\Psi} \circ \iota^{(i)}$ is an injective algebra homomorphism.*

Proof. This is obvious since $\widehat{\Psi}$ is an isomorphism and $\iota^{(i)}$ is an injective algebra homomorphism. □

3.3.9 (Anti-)Automorphisms of $\widehat{\ddot{\mathrm{U}}'_q(\mathfrak{a}_1)}$

$\widehat{\ddot{\mathrm{U}}'_q(\mathfrak{a}_1)}$ naturally inherits, through $\widehat{\Psi}$, all the continuous (anti-)automorphisms defined over $\widehat{\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)}$.

Proposition 3.3.26. *Conjugation by $\widehat{\Psi}$ clearly provides a group isomorphism $\mathrm{Aut}(\widehat{\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)}) \cong \mathrm{Aut}(\widehat{\ddot{\mathrm{U}}'_q(\mathfrak{a}_1)})$. In particular, for every $f \in \mathrm{Aut}(\widehat{\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)})$, we let $\dot{f} = \widehat{\Psi} \circ f \circ \widehat{\Psi}^{-1} \in \mathrm{Aut}(\widehat{\ddot{\mathrm{U}}'_q(\mathfrak{a}_1)})$.*

As an example, consider the Cartan anti-involution φ of $\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)$ defined in proposition 2.3.*iv.* of [MZ]. It extends by continuity into an anti-involution $\widehat{\varphi}$ over $\widehat{\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)}$ which eventually yields, upon conjugation by $\widehat{\Psi}$, an anti-involution $\dot{\varphi}$ over $\widehat{\ddot{\mathrm{U}}'_q(\mathfrak{a}_1)}$. One can easily check that,

$$\dot{\varphi}(q) = q^{-1}\,, \qquad \dot{\varphi}(\mathsf{D}_2^{\pm 1}) = \mathsf{D}_2^{\mp 1}\,, \qquad \dot{\varphi}(\mathsf{C}^{\pm 1/2}) = \mathsf{C}^{\mp 1/2}\,, \qquad \dot{\varphi}(\mathbf{c}^{\pm}(z)) = \mathbf{c}^{\mp}(1/z)\,,$$

$$\dot{\varphi}(\mathbf{K}^{\pm}_{1,\pm m}(z)) = \mathbf{K}^{\mp}_{1,\mp m}(1/z)\,, \qquad \dot{\varphi}(\mathbf{X}^{\pm}_{1,r}(z)) = \mathbf{X}^{\mp}_{1,-r}(1/z)\,,$$

for every $m \in \mathbb{N}$ and every $r \in \mathbb{Z}$.

In addition to the above, $\widehat{\ddot{\mathrm{U}}'_q(\mathfrak{a}_1)}$ also admits the following automorphisms that will prove useful in the study of its representation theory.

Proposition 3.3.27. *i. There exists a unique $\mathbb{F}$-algebra automorphism τ of $\widehat{\ddot{\mathrm{U}}'_q(\mathfrak{a}_1)}$ such that, for every $m \in \mathbb{N}$ and every $n \in \mathbb{Z}$,*

$$\tau(\mathsf{C}) = -\mathsf{C}\,, \quad \tau(\boldsymbol{c}^{\pm}(\mathsf{C}^{-1/2}z)) = \boldsymbol{c}^{\pm}(\mp \mathsf{C}^{-1/2}z)\,, \quad \tau(\boldsymbol{K}^{\pm}_{1,\pm m}(z)) = \boldsymbol{K}^{\pm}_{1,\pm m}(\mp z)\,, \quad \tau(\boldsymbol{X}^{\pm}_{1,n}(z)) = \boldsymbol{X}^{\pm}_{1,n}(\mp z)\,.$$

ii. There exists a unique $\mathbb{F}$-algebra automorphism σ of $\widehat{\ddot{\mathrm{U}}'_q(\mathfrak{a}_1)}$ such that

$$\sigma(\mathsf{C}^{1/2}) = -\mathsf{C}^{1/2}\,, \quad \sigma(\boldsymbol{c}^{\pm}(z)) = \boldsymbol{c}^{\pm}(z)\,, \quad \tau(\boldsymbol{K}^{\pm}_{1,\pm m}(z)) = \boldsymbol{K}^{\pm}_{1,\pm m}(-z)\,, \quad \tau(\boldsymbol{X}^{\pm}_{1,n}(z)) = \boldsymbol{X}^{\pm}_{1,n}(-z)\,.$$

Proof. It suffices to check the defining relations of $\widehat{\ddot{\mathrm{U}}_q(\mathfrak{a}_1)}$. □

3.3.10 Topological Hopf algebra structure on $\widehat{\ddot{\mathrm{U}}'_q(\mathfrak{a}_1)}$

Definition 3.3.28. We endow the topological $\mathbb{F}$-algebra $\widehat{\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)}$ with:

i. the comultiplication $\Delta : \widehat{\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)} \to \dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1) \widehat{\otimes} \dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)$ defined by

$$\Delta(C^{\pm 1/2}) = C^{\pm 1/2} \otimes C^{\pm 1/2}\,, \qquad \Delta(D^{\pm 1}) = D^{\pm 1} \otimes D^{\pm 1}\,, \tag{3.3.74}$$

$$\Delta(\mathbf{k}_i^{\pm}(z)) = \mathbf{k}_i^{\pm}(zC_{(2)}^{\pm 1/2}) \otimes \mathbf{k}_i^{\pm}(zC_{(1)}^{\mp 1/2}) \,, \tag{3.3.75}$$

$$\Delta(\mathbf{x}_i^{+}(z)) = \mathbf{x}_i^{+}(z) \otimes 1 + \mathbf{k}_i^{-}(zC_{(1)}^{1/2}) \widehat{\otimes} \mathbf{x}_i^{+}(zC_{(1)}) \,, \tag{3.3.76}$$

$$\Delta(\mathbf{x}_i^{-}(z)) = \mathbf{x}_i^{-}(zC_{(2)}) \widehat{\otimes} \mathbf{k}_i^{+}(zC_{(2)}^{1/2}) + 1 \otimes \mathbf{x}_i^{-}(z) \,, \tag{3.3.77}$$

ii. the counit $\varepsilon : \widehat{\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)} \to \mathbb{F}$, defined by $\varepsilon(D^{\pm 1}) = \varepsilon(C^{\pm 1/2}) = \varepsilon(\mathbf{k}_i^{\pm}(z)) = 1$, $\varepsilon(\mathbf{x}_i^{\pm}(z)) = 0$ and;

iii. the antipode $S : \widehat{\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)} \to \widehat{\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)}$, defined by $S(D^{\pm 1}) = D^{\mp 1}$, $S(C^{\pm 1/2}) = C^{\mp 1/2}$ and

$$S(\mathbf{k}_i^{\pm}(z)) = \mathbf{k}_i^{\pm}(z)^{-1} \,, \quad S(\mathbf{x}_i^{+}(z)) = -\mathbf{k}_i^{-}(zC^{-1/2})^{-1} \mathbf{x}_i^{+}(zC^{-1}) \,, \quad S(\mathbf{x}_i^{-}(z)) = -\mathbf{x}_i^{-}(zC^{-1}) \mathbf{k}_i^{+}(zC^{-1/2})^{-1} \,.$$

With the operations so defined and the topologies defined in section 3.3.7, $\widehat{\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)}$ becomes a topological Hopf algebra – see definition 2.16 in [MZ].

In view of theorem 3.3.22, it is clear that $\widehat{\ddot{\mathrm{U}}_q(\mathfrak{a}_1)}$ inherits that topological Hopf algebraic structure.

Definition-Proposition 3.3.29. We define

$$\dot{\Delta} = \left(\widehat{\Psi} \widehat{\otimes} \widehat{\Psi} \right) \circ \Delta \circ \widehat{\Psi}^{-1} \,, \tag{3.3.78}$$

$$\dot{S} = \widehat{\Psi} \circ S \circ \widehat{\Psi}^{-1} \,, \tag{3.3.79}$$

$$\dot{\varepsilon} = \varepsilon \circ \widehat{\Psi}^{-1} \,. \tag{3.3.80}$$

Equipped with the above comultiplication, antipode and counit, $\widehat{\ddot{\mathrm{U}}'_q(\mathfrak{a}_1)}$ is a topological Hopf algebra.

Before we move on to introducing t-weight $\ddot{\mathrm{U}}'_q(\mathfrak{a}_1)$-modules, we give the following

Lemma 3.3.30. *For every $m \in \mathbb{N}$ and every $r \in \mathbb{Z}$, we have*

i. $\dot{\Delta}(\boldsymbol{\mathsf{K}}^{\pm}_{1,\pm m}(z)) = \Delta^0(\boldsymbol{\mathsf{K}}^{\pm}_{1,\pm m}(z)) \mod \ddot{\mathrm{U}}_q^{<}(\mathfrak{a}_1) \widehat{\otimes} \ddot{\mathrm{U}}_q^{>}(\mathfrak{a}_1)[[z, z^{-1}]]$*;*

ii. $\dot{\Delta}(\boldsymbol{\mathsf{X}}^{+}_{1,r}(z)) \in \left(\ddot{\mathrm{U}}_q^{>}(\mathfrak{a}_1) \widehat{\otimes} \ddot{\mathrm{U}}_q^{0}(\mathfrak{a}_1) \oplus \ddot{\mathrm{U}}_q(\mathfrak{a}_1) \widehat{\otimes} \ddot{\mathrm{U}}_q^{>}(\mathfrak{a}_1) \right) [[z, z^{-1}]]$*;*

where we have set $\ddot{\mathrm{U}}_q^{>}(\mathfrak{a}_1) = \ddot{\mathrm{U}}_q^{\geq}(\mathfrak{a}_1) - \ddot{\mathrm{U}}_q^{\geq}(\mathfrak{a}_1) \cap \ddot{\mathrm{U}}_q^{0}(\mathfrak{a}_1)$ *and* $\ddot{\mathrm{U}}_q^{<}(\mathfrak{a}_1) = \ddot{\mathrm{U}}_q^{\leq}(\mathfrak{a}_1) - \ddot{\mathrm{U}}_q^{\leq}(\mathfrak{a}_1) \cap \ddot{\mathrm{U}}_q^{0}(\mathfrak{a}_1)$*.*

Proof. We first prove i for upper choices of signs. Observe that (3.3.20) equivalently reads

$$\boldsymbol{\mathsf{K}}^{+}_{1,m}(z) = -(q - q^{-1}) \boldsymbol{\mathsf{K}}^{+}_{1,0}(zq^{-2m}) \mathbf{t}^{+}_{1,m}(z) \,,$$

for every $m \in \mathbb{N}^{\times}$. For every $m \in \mathbb{N}^{\times}$, let

$$\mathbf{K}^{\pm}_{1,\pm m}(z) = \widehat{\Psi}^{-1}(\boldsymbol{\mathsf{K}}^{\pm}_{1,\pm m}(z)) \in \widehat{\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)}[[z, z^{-1}]] \,.$$

In [MZ] – see proposition-definition 4.9, definition 4.25 and eq. (4.66) –, we proved that $\mathbf{K}^{+}_{1,0}(z) = -\mathbf{k}_1^{-}(C^{1/2}z)$ and that, for every $m \in \mathbb{N}^{\times}$,

$$\mathbf{K}^{+}_{1,m}(z) = (q - q^{-1}) \mathbf{k}_1^{-}(C^{1/2} zq^{-2m}) \boldsymbol{\psi}^{+}_{1,m}(z) \,,$$

where $\boldsymbol{\psi}^+_{1,m}(z)$ can be recursively defined by setting

$$\left[\mathbf{x}_0^+(w), \mathbf{x}_1^+(z)\right]_{G^-_{10}(w/z)} = \delta\left(\frac{q^2 w}{z}\right) \boldsymbol{\psi}^+_{1,1}(z) \tag{3.3.81}$$

and

$$\begin{aligned} {}_{G^-_{01}(q^{-2m}v/w)G^-_{11}(q^{2(1-m)}v/w)} \left[\boldsymbol{\psi}^+_{1,1}(w), \boldsymbol{\psi}^+_{1,m}(v)\right]_{G^-_{01}(w/vq^2)G^-_{11}(w/v)} \quad = \quad & [2]_q \delta\left(\frac{w}{vq^2}\right) \boldsymbol{\psi}^+_{1,m+1}(q^2 v) \\ & -[2]_q \delta\left(\frac{q^{2m} w}{v}\right) \boldsymbol{\psi}^+_{1,m+1}(v) \end{aligned} \tag{3.3.82}$$

Hence, i for $m = 0$ is clear. From (3.3.81) and definition 3.3.28, and making use of relations (4.2.6) and (4.2.7) as well as of the identity (3.3.68), we deduce that

$$\Delta(\boldsymbol{\psi}^+_{1,1}(z)) = \boldsymbol{\psi}^+_{1,1}(z) \otimes 1 + \wp^-(zq^{-2}C^{1/2}_{(1)}) \widehat{\otimes} \boldsymbol{\psi}^+_{1,1}(zC_{(1)}) - [2]_q (q - q^{-1}) \mathbf{k}^-_1(zC^{1/2}_{(1)}) \mathbf{x}^+_0(zq^{-2}) \widehat{\otimes} \mathbf{x}^+_1(zC_{(1)}) \,,$$

where $\wp^-(v) = \mathbf{k}^-_0(v)\mathbf{k}^-_1(vq^2)$. Applying $\widehat{\Psi}\widehat{\otimes}\widehat{\Psi}$ to the first two terms obviously yields $\Delta^0(\mathbf{t}^+_{1,1}(z))$. Since, on the other hand, $\widehat{\Psi}(\mathbf{x}^+_0(z)) \in \ddot{\mathrm{U}}^<_q(\mathfrak{a}_1)[[z, z^{-1}]]$ – see theorem 3.3.22 –, applying $\widehat{\Psi}\widehat{\otimes}\widehat{\Psi}$ to the third term yields an element of $\ddot{\mathrm{U}}^<_q(\mathfrak{a}_1)\widehat{\otimes}\ddot{\mathrm{U}}^>_q(\mathfrak{a}_1)[[z, z^{-1}]]$ and it follows that i holds for $m = 1$ and for upper choices of signs. Suppose it holds for upper choices of signs and for some $m \in \mathbb{N}^\times$. Then, making use of (3.3.82), one easily checks that i holds for $m + 1$ and for upper choices of signs, which completes the proof of i for upper choices of signs. Observe that $\left(\varphi\widehat{\otimes}\varphi\right) \circ \Delta^{\mathrm{cop}} = \Delta \circ \varphi$ and that, as a consequence,

$$\dot{\Delta} \circ \dot{\varphi} = \left(\dot{\varphi}\widehat{\otimes}\dot{\varphi}\right) \circ \dot{\Delta}^{\mathrm{cop}} \,.$$

Similarly, one easily checks that

$$\Delta^0 \circ \dot{\varphi}_{|\ddot{\mathrm{U}}^0_q(\mathfrak{a}_1)} = \left(\dot{\varphi}\widehat{\otimes}\dot{\varphi}\right)_{|\ddot{\mathrm{U}}^0_q(\mathfrak{a}_1)} \circ \Delta^{0,\mathrm{cop}} \,.$$

Thus, i for lower choices of signs follows after applying $\dot{\varphi}$ to i with upper choices of signs.

As for ii, we let, for every $r \in \mathbb{Z}$,

$$\mathbf{X}^+_{1,r}(z) = \widehat{\Psi}^{-1}(\mathbf{X}^+_{1,r}(z)) \,.$$

In [MZ] – see definition 4.1 and proposition 4.8 –, we proved that $\mathbf{X}^+_{1,r}(z)$ could be defined recursively by setting $\mathbf{X}^+_{1,0}(z) = \mathbf{x}^+_1(z)$ and letting, for every $r \in \mathbb{N}$,

$$\left[\boldsymbol{\psi}^+_{1,1}(z), \mathbf{X}^+_{1,r}(v)\right]_{G^-_{10}(z/vq^2)G^-_{11}(z/v)} = [2]_q \delta\left(\frac{z}{vq^2}\right) \mathbf{X}^+_{1,r+1}(z) \tag{3.3.83}$$

and

$$\left[\boldsymbol{\psi}^-_{1,-1}(z), \mathbf{X}^+_{1,-r}(v)\right] = [2]_q \delta\left(\frac{Cz}{v}\right) \mathbf{X}^+_{1,-(r+1)}(Cq^{-2}z) \wp^+(C^{1/2}q^{-2}z) \,, \tag{3.3.84}$$

where $\boldsymbol{\psi}^-_{1,-1}(z) = \varphi(\boldsymbol{\psi}^+_{1,1}(1/z))$ – see proposition 4.3 in [MZ]. Making use of $\left(\varphi\widehat{\otimes}\varphi\right) \circ \Delta^{\mathrm{cop}} = \Delta \circ \varphi$, we clearly get

$$\left(\widehat{\Psi}\widehat{\otimes}\widehat{\Psi}\right) \circ \Delta(\boldsymbol{\psi}^-_{1,-1}(z)) = \Delta^0(\mathbf{t}^-_{1,-1}(z)) \mod \ddot{\mathrm{U}}^<_q(\mathfrak{a}_1)\widehat{\otimes}\ddot{\mathrm{U}}^>_q(\mathfrak{a}_1)[[z, z^{-1}]] \,.$$

Now, applying $\widehat{\Psi}\widehat{\otimes}\widehat{\Psi}$ to (3.3.76) in definition 3.3.28 clearly proves ii in the case $r=0$. Assuming it holds for $r\in\mathbb{N}$, it suffices to apply $\left(\widehat{\Psi}\widehat{\otimes}\widehat{\Psi}\right)\circ\Delta$ to (3.3.83) above to prove that it also holds for $r+1$. Similarly, if ii holds for some $r\in-\mathbb{N}$, applying $\left(\widehat{\Psi}\widehat{\otimes}\widehat{\Psi}\right)\circ\Delta$ to (3.3.84) to prove that it also holds for $r-1$. This concludes the proof. $\square$

3.4 t-weight $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)$-modules

3.4.1 ℓ-weight modules over $\ddot{\mathrm{U}}_q^0(\mathfrak{a}_1)$

Remember that $\ddot{\mathrm{U}}_q^{0,0}(\mathfrak{a}_1)$ contains a subalgebra that is isomorphic to $\dot{\mathrm{U}}_q^0(\mathfrak{a}_1)$ – see proposition 3.3.5. Hence, in view of remark 3.2.9, we can repeat for modules over $\ddot{\mathrm{U}}_q^0(\mathfrak{a}_1)$ what we did in section 3.2.3 for modules over $\dot{\mathrm{U}}_q(\mathfrak{a}_1)$. We thus make the following

Definition 3.4.1. We shall say that a (topological) $\ddot{\mathrm{U}}_q^0(\mathfrak{a}_1)$-module M is *ℓ-weight* if there exists a countable set $\{M_\alpha : \alpha\in A\}$ of indecomposable locally finite-dimensional $\ddot{\mathrm{U}}_q^{0,0}(\mathfrak{a}_1)$-modules called *$\ell$-weight spaces* of M, such that, as $\ddot{\mathrm{U}}_q^{0,0}(\mathfrak{a}_1)$-modules,

$$M\cong\bigoplus_{\alpha\in A} M_\alpha\,.$$

For every $\alpha\in A$, we let $[-]_{M_\alpha}: M\to M_\alpha$ denote the canonical projection, so that, for every $v\in M$, $[v]_{M_\alpha}$ is the projection of v on M_α. Since for every $\alpha\in A$, M_α is locally finite-dimensional, it is the colimit of its finite-dimensional submodules and we shall refer to the latter as *local ℓ-weight spaces*.

As in section 3.2.3, it follows that

Definition-Proposition 3.4.2. Let M be an ℓ-weight $\ddot{\mathrm{U}}_q^0(\mathfrak{a}_1)$-module. Then:

i. C^2 acts on M by id;

ii. for every ℓ-weight space M_α, $\alpha\in A$, of M, there exist $\kappa_{\alpha,0}\in\mathbb{F}^\times$ and sequences $(\kappa^\pm_{\alpha,\pm m})_{m\in\mathbb{N}^\times}\in\mathbb{F}^{\mathbb{N}^\times}$ such that
$$M_\alpha\subseteq\left\{v\in M:\exists n\in\mathbb{N}^\times\,,\forall m\in\mathbb{N}\quad\left(\mathsf{K}^\pm_{1,0,\pm m}-\kappa^\pm_{\alpha,\pm m}\mathrm{id}\right)^n.v=0\right\}\,,\tag{3.4.1}$$
where we have set $\kappa^\pm_{\alpha,0}=\kappa^{\pm1}_{\alpha,0}$.

We let $\mathrm{Sp}(M)=\{\kappa_{\alpha,0}:\alpha\in A\}$ and we shall refer to

$$\kappa^\pm_\alpha(z)=\sum_{m\in\mathbb{N}}\kappa^\pm_{\alpha,\pm m}z^{\pm m}$$

as the *ℓ-weight* of the ℓ-weight space M_α. We shall say that M is

- of *type* 1 if $\mathsf{C}^{1/2}$ acts by id over M;
- of *type* $(1,N)$ for $N\in\mathbb{N}^\times$ if it is of type 1 and, for every $m\geq N$, $\mathsf{c}^\pm_{\pm m}$ acts by multiplication 0 over M;
- of *type* $(1,0)$ if it is of type $(1,1)$ and $\mathsf{c}^\pm_0$ acts by id over M.

Proof. The proof follows the same arguments as the proof of definition-proposition 3.2.8. □

Proposition 3.4.3. *Let M be a type 1 ℓ-weight $\ddot{\mathrm{U}}_q^0(\mathfrak{a}_1)$-module and let M_α and M_β be two local ℓ-weight spaces of M such that, for some $m \in \mathbb{N}^\times$ and some $n \in \mathbb{Z}$, $M_\alpha \cap \mathsf{K}^\pm_{1,\pm m,n}.M_\beta \neq \{0\}$. Then, there exists a unique $a \in \mathbb{F}^\times$ such that:*

i. the respective ℓ-weights $\kappa^\varepsilon_\alpha(z)$ and $\kappa^\varepsilon_\beta(z)$ of M_α and M_β be related by

$$\kappa^\varepsilon_\alpha(z) = \kappa^\varepsilon_\beta(z) H^\varepsilon_{m,a}(z)^{\pm 1}\,,$$

where $\varepsilon \in \{-,+\}$ and

$$H^\pm_{m,a}(z) = \left(\frac{(1-q^{-2}a/z)(1-q^{-2(m-1)}a/z)}{(1-q^{2}a/z)(1-q^{-2(m+1)}a/z)}\right)_{|z|^{\pm 1}\ll 1}\,; \tag{3.4.2}$$

ii. $(z-a)^N M_\alpha \cap \boldsymbol{K}^\pm_{1,\pm m}(z).M_\beta = \{0\}$ for some $N \in \mathbb{N}^\times$.

Proof. There clearly exist two bases $\{v_i : i = 1, \dots, \dim M_\alpha\}$ and $\{w_i : i = 1, \dots, \dim M_\beta\}$ of M_α and M_β respectively, in which

$$\forall i \in [\![\dim M_\alpha]\!]\,, \qquad \mathsf{K}^\pm_{1,0}(z).v_i = \kappa^\pm_\alpha(z) \sum_{k=i}^{\dim M_\alpha} \eta^\pm_{\alpha,i,k}(z) v_k\,,$$

$$\forall j \in [\![\dim M_\beta]\!]\,, \qquad \mathsf{K}^\pm_{1,0}(z).w_j = \kappa^\pm_\beta(z) \sum_{l=j}^{\dim M_\beta} \eta^\pm_{\beta,j,l}(z) w_l\,,$$

for some $\eta^\pm_{\alpha,i,k}(z), \eta^\pm_{\beta,j,l}(z) \in \mathbb{F}[[z^{\pm 1}]]$, with $i, k \in [\![\dim M_\alpha]\!]$ and $j, l \in [\![\dim M_\beta]\!]$, such that $\eta^\pm_{\alpha,i,i}(z) = 1$ for every $i \in [\![\dim M_\alpha]\!]$ and $\eta^\pm_{\beta,j,j}(z) = 1$ for every $j \in [\![\dim M_\beta]\!]$. Moreover, for every $j \in [\![\dim M_\beta]\!]$,

$$\left[\mathsf{K}^\pm_{1,\pm m}(z).w_j\right]_{M_\alpha} = \sum_{i \in [\![\dim M_\alpha]\!]} \xi^\pm_{m,j,i}(z) v_i\,,$$

for some $\xi^\pm_{m,j,i}(z) \in \mathbb{F}[[z, z^{-1}]]$ – see definition 3.4.1 for the definition of $[-]_{M_\alpha}$.

Now, if $M_\alpha \cap \mathsf{K}^\pm_{1,\pm m,n}.M_\beta \neq \{0\}$, there must exist a largest nonempty subset $J \subseteq [\![\dim M_\beta]\!]$ such that, for every $j \in J$, $\left[\mathsf{K}^\pm_{1,\pm m}(z).w_j\right]_{M_\alpha} \neq \{0\}$. Let $j_* = \max J$. Obviously, for every $j \in J$, there must exist a largest nonempty subset $I(j) \subseteq [\![\dim M_\alpha]\!]$ such that, for every $j \in J$ and every $i \in I(j)$, $\xi^\pm_{m,j,i}(z) \neq 0$. Consequently, for every $j \in J$,

$$\left[\mathsf{K}^\pm_{1,\pm m}(z).w_j\right]_{M_\alpha} = \sum_{i \in I(j)} \xi^\pm_{m,j,i}(z) v_i\,,$$

where $\xi^\pm_{m,j,i}(z) \in \mathbb{F}[[z, z^{-1}]] - \{0\}$, whereas $\xi^\pm_{m,j,i}(z) = 0$ for any (j, i) outside of the set of pairs $\{(j, i) : j \in J, i \in I(j)\}$. For every $j \in J$, we let $i_*(j) = \min I(j)$ and, for simplicity, we let $i_* = i_*(j_*)$. Making use of the relations in $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)$ – namely (3.3.7) and (3.3.8) –, we get, for $\varepsilon \in \{-,+\}$,

$$(z_1 - q^{\pm 2} z_2)(z_1 - q^{2(m\mp 1)} z_2) \mathsf{K}^\pm_{1,\pm m}(z_1) \mathsf{K}^\varepsilon_{1,0}(z_2).w_j = (z_1 q^{\pm 2} - z_2)(z_1 q^{\mp 2} - q^{2m} z_2) \mathsf{K}^\varepsilon_{1,0}(z_2) \mathsf{K}^\pm_{1,\pm m}(z_1).w_j\,.$$

The latter easily implies that, for every $j \in J$ and every $i \in I(j)$,

$$\begin{aligned}(z_1 - q^{\pm 2} z_2)(z_1 - q^{2(m\mp 1)} z_2)\kappa^{\varepsilon}_{\beta}(z_2) \sum_{\substack{l\in J\\ l\geq j}} \eta^{\varepsilon}_{\beta,j,l}(z_2)\xi^{\pm}_{m,l,i}(z_1) \\ = (z_1 q^{\pm 2} - z_2)(z_1 q^{\mp 2} - q^{2m} z_2)\kappa^{\pm}_{\alpha}(z_2) \sum_{\substack{k\in I(j)\\ k\leq i}} \eta^{\varepsilon}_{\alpha,k,i}(z_2)\xi^{\pm}_{m,j,k}(z_1)\end{aligned} \tag{3.4.3}$$

Taking $i = i_*$ and $j = j_*$ in the above equation immediately yields

$$\left[(z_1 - q^{\pm 2} z_2)(z_1 - q^{2(m\mp 1)} z_2)\kappa^{\varepsilon}_{\beta}(z_2) - (z_1 q^{\pm 2} - z_2)(z_1 q^{\mp 2} - q^{2m} z_2)\kappa^{\varepsilon}_{\alpha}(z_2)\right] \xi^{\pm}_{m,j_*,i_*}(z_1) = 0\,.$$

The latter is equivalent to the fact that, for every $p \in \mathbb{Z}$,

$$\begin{aligned}\left(\xi^{\pm}_{m,j_*,i_*,p} q^{2m} z^2 + \xi^{\pm}_{m,j_*,i_*,p+2}\right) \left[\kappa^{\varepsilon}_{\beta}(z) - \kappa^{\varepsilon}_{\alpha}(z)\right] \\ = \xi^{\pm}_{m,j_*,i_*,p+1} z \left[(q^{2(m\mp 1)} + q^{\pm 2})\kappa^{\varepsilon}_{\beta}(z) - (q^{2(m\pm 1)} + q^{\mp 2})\kappa^{\varepsilon}_{\alpha}(z)\right]\end{aligned} \tag{3.4.4}$$

where, as usual, we have set

$$\xi^{\pm}_{m,j_*,i_*,p} = \operatorname*{res}_{z} z^{p-1}\xi^{\pm}_{m,j_*,i_*}(z)\,.$$

Since $\xi^{\pm}_{m,j_*,i_*}(z) \neq 0$, there must exist a $p \in \mathbb{Z}$ such that $\xi^{\pm}_{m,j_*,i_*,p} \neq 0$. Assuming that $\xi^{\pm}_{m,j_*,i_*,p+1} = 0$, one easily obtains that, on one hand $\kappa^{\varepsilon}_{\beta}(z) = \kappa^{\varepsilon}_{\alpha}(z)$ and that, on the other hand,

$$\left[(q^{2(m\mp 1)} + q^{\pm 2})\kappa^{\varepsilon}_{\beta}(z) - (q^{2(m\pm 1)} + q^{\mp 2})\kappa^{\varepsilon}_{\alpha}(z)\right] = 0\,.$$

A contradiction. By similar arguments, one eventually proves that $\xi^{\pm}_{m,j_*,i_*,p} \neq 0$ for every $p \in \mathbb{Z}$. But then, dividing (3.4.4) by $\xi^{\pm}_{m,j_*,i_*,p+1}$, we get

$$\left(a_p^{-1} q^{2m} z^2 + a_{p+1}\right) \left[\kappa^{\varepsilon}_{\beta}(z) - \kappa^{\varepsilon}_{\alpha}(z)\right] = z\left[(q^{2(m\mp 1)} + q^{\pm 2})\kappa^{\varepsilon}_{\beta}(z) - (q^{2(m\pm 1)} + q^{\mp 2})\kappa^{\varepsilon}_{\alpha}(z)\right]\,,$$

for every $p \in \mathbb{Z}$, where $a_p = \xi^{\pm}_{m,j_*,i_*,p+1}/\xi^{\pm}_{m,j_*,i_*,p} \in \mathbb{F}^{\times}$. Since the r.h.s. of the above equation is obviously independent of p, so is its l.h.s. and it easily follows that, for every $p \in \mathbb{Z}$, $a_p = a$ for some $a \in \mathbb{F}^{\times}$, so that, eventually,

$$\left(q^{2m} z^2 + a^2\right) \left[\kappa^{\varepsilon}_{\beta}(z) - \kappa^{\varepsilon}_{\alpha}(z)\right] - az\left[(q^{2(m\mp 1)} + q^{\pm 2})\kappa^{\varepsilon}_{\beta}(z) - (q^{2(m\pm 1)} + q^{\mp 2})\kappa^{\varepsilon}_{\alpha}(z)\right] = 0\,.$$

i follows. Moreover, we clearly have

$$\xi^{\pm}_{m,j_*,i_*}(z) = A^{\pm}_{m,j_*,i_*}\delta(z/a)\,,$$

for some $A^{\pm}_{m,j_*,i_*} \in \mathbb{F}^{\times}$. More generally, we claim that,

$$\forall j \in J\,, \forall i \in I(j)\,, \qquad \xi^{\pm}_{m,j,i}(z) = \sum_{p=0}^{N(i,j)} A^{\pm}_{m,j,i,p}\delta^{(p)}(z/a)\,, \tag{3.4.5}$$

for some $A^{\pm}_{m,j,i,p} \in \mathbb{F}$ and some $N(i,j) \in \mathbb{N}$. This is proven by a finite induction on j and i. Indeed, making use of (3.4.2), we can rewrite (3.4.3) as

$$\begin{aligned}
&(z_1 - q^{\pm2}z_2)(z_1 - q^{2(m\mp1)}z_2)(z_2 - q^{\pm2}a)(z_2 - q^{-2(m\pm1)}a)\sum_{\substack{l\in J\\ l\geq j}} \eta^{\varepsilon}_{\beta,j,l}(z_2)\xi^{\pm}_{m,l,i}(z_1)\\
&\quad = (z_1q^{\pm2} - z_2)(z_1q^{\mp2} - q^{2m}z_2)(z_2 - q^{\mp2}a)(z_2 - q^{-2(m\mp1)}a)\sum_{\substack{k\in I(j)\\ k\leq i}} \eta^{\varepsilon}_{\alpha,k,i}(z_2)\xi^{\pm}_{m,j,k}(z_1)
\end{aligned}\tag{3.4.6}$$

for every $j \in J$ and every $i \in I(j)$. Now, assume that (3.4.5) holds for every pair in

$$\{(j,i) : j\in J,\, i \in I(j), \quad j > j_0\} \cup \{(j_0, i) : i \in I(j_0), \quad i \leq i_0\}\,,$$

for some $j_0 \in J$ and some $i_0 \in I(j_0)$ such that $i_0 < \max I(j_0)$. Let i'_0 be the smallest element of $I(j_0)$ such that $i_0 < i'_0$. It suffices to write (3.4.6) for $j = j_0$ and $i = i'_0$, to get

$$\begin{aligned}
&(z_1 - a)z_2(z_1a - q^{2m}z_2^2)(q^{\mp2} + q^{-2(m\mp1)} - q^{\pm2} - q^{-2(m\pm1)})\xi^{\pm}_{m,j_0,i'_0}(z_1)\\
&\quad = -(z_1 - q^{\pm2}z_2)(z_1 - q^{2(m\mp1)}z_2)(z_2 - q^{\pm2}a)(z_2 - q^{-2(m\pm1)}a)\sum_{\substack{l\in J\\ l> j_0}} \eta^{\varepsilon}_{\beta,j_0,l}(z_2)\xi^{\pm}_{m,l,i'_0}(z_1)\\
&\quad +(z_1q^{\pm2} - z_2)(z_1q^{\mp2} - q^{2m}z_2)(z_2 - q^{\mp2}a)(z_2 - q^{-2(m\mp1)}a)\sum_{\substack{k\in I(j_0)\\ k\leq i_0}} \eta^{\varepsilon}_{\alpha,k,i'_0}(z_2)\xi^{\pm}_{m,j_0,k}(z_1)
\end{aligned}\tag{3.4.7}$$

Combining the recursion hypothesis and lemma 5.1.8 from the appendix, one easily concludes that (3.4.5) holds for the pair (j_0, i'_0). Repeating the argument finitely many times, we get that it actually holds for all the pairs in $\{(j,i) : j \in J,\, i \in I(j), \quad j \geq j_0\}$. Now, either $j_0 = \min J$ and we are done; or $j_0 > \min J$ and there exists a largest $j'_0 \in J$ such that $j_0 > j'_0$. Writing (3.4.6) for $j = j'_0$ and $i = i_*(j'_0)$, we get

$$\begin{aligned}
&(z_1 - a)z_2(z_1a - q^{2m}z_2^2)(q^{\mp2} + q^{-2(m\mp1)} - q^{\pm2} - q^{-2(m\pm1)})\xi^{\pm}_{m,j'_0,i_*(j'_0)}(z_1)\\
&\quad = -(z_1 - q^{\pm2}z_2)(z_1 - q^{2(m\mp1)}z_2)(z_2 - q^{\pm2}a)(z_2 - q^{-2(m\pm1)}a)\sum_{\substack{l\in J\\ l\geq j'_0}} \eta^{\varepsilon}_{\beta,j'_0,l}(z_2)\xi^{\pm}_{m,l,i_*(j'_0)}(z_1)\,.
\end{aligned}$$

Combining again the recursion hypothesis and lemma 5.1.8, we easily get that (3.4.19) holds for $(j'_0, i_*(j'_0))$. It is now clear that the claim holds for every $j \in J$ and every $i \in I(j)$. Letting

$$N = 1 + \max\{N(i,j) : j \in J,\, i \in I(j)\}\,,$$

ii follows. Furthermore, for every $b \in \mathbb{F} - \{a\}$ and every $n \in \mathbb{N}$, we obviously have $(z-b)^n M_\alpha \cap \mathsf{K}^{\pm}_{1,\pm m}(z).M_\beta \neq \{0\}$, thus making a the only element of $\mathbb{F}$ satisfying ii. This concludes the proof. □

Remark 3.4.4. It is worth emphasizing that proposition 3.4.3.i holds for arbitrary pairs of (possibly non-local) ℓ-weight spaces since it must hold for at least one pair of local ℓ-weight spaces therein.

We let ω_1 denote the fundamental weight of $\mathfrak{a}_1$ and we let $P = \mathbb{Z}\omega_1$ be the corresponding weight lattice. In view of proposition 3.4.3, it is natural to make the following

Definition 3.4.5. Let M be a type $(1,0)$ ℓ-weight $\ddot{\mathrm{U}}^0_q(\mathfrak{a}_1)$-module and let $\{M_\alpha : \alpha \in A\}$ be the countable set of its ℓ-weight spaces. We shall say that M is *rational* if, for every $\alpha \in A$, there exist relatively prime monic polynomials $P_\alpha(1/z), Q_\alpha(1/z) \in \mathbb{F}[z^{-1}]$, called Drinfel'd polynomials of M, such that the ℓ-weight $\kappa^\pm_\alpha(z)$ of M_α be given by

$$\kappa^\pm_\alpha(z) = -q^{\deg(P_\alpha)-\deg(Q_\alpha)} \left(\frac{P_\alpha(q^{-2}/z)Q_\alpha(1/z)}{P_\alpha(1/z)Q_\alpha(q^{-2}/z)}\right)_{|z|^{\pm1}\ll1} .$$

With each rational ℓ-weight $\kappa^\pm_\alpha(z)$ of a rational $\ddot{\mathrm{U}}^0_q(\mathfrak{a}_1)$-module M, we associate an integral weight $\lambda_\alpha \in P$, by setting

$$\lambda_\alpha = [\deg(P_\alpha) - \deg(Q_\alpha)]\,\omega_1 .$$

We shall say that M is *ℓ-dominant* (resp. *ℓ-anti-dominant*) if it is rational and there exists $N \in \mathbb{N}^\times$ such that, for every $\alpha \in A$, $\deg(P_\alpha) = N$ and $\deg(Q_\alpha) = 0$ (resp. $\deg(P_\alpha) = 0$ and $\deg(Q_\alpha) = N$).

Remark 3.4.6. The classical weight $N\omega_1$ (resp. $-N\omega_1$) associated with any ℓ-dominant (resp. ℓ-anti-dominant) type 1 ℓ-weight rational $\ddot{\mathrm{U}}^0_q(\mathfrak{a}_1)$-module M is a dominant (resp. anti-dominant) integral weight. Note that the converse need not be true.

Remark 3.4.7. The data of the ℓ-weights of a rational $\ddot{\mathrm{U}}^0_q(\mathfrak{a}_1)$-module is equivalent to the data of its Drinfel'd polynomials $\{(P_\alpha, Q_\alpha) : \alpha \in A\}$ which, in turn, is equivalent to the data of their finite multisets of roots $\{(\nu^+_\alpha, \nu^-_\alpha) : \alpha \in A\}$. The latter are finitely supported maps $\nu^\pm_\alpha : \mathbb{F}^\times \to \mathbb{N}$ such that, for every $\alpha \in A$,

$$P_\alpha(1/z) = \prod_{x\in\mathbb{F}^\times} (1 - x/z)^{\nu^+_\alpha(x)} \qquad \text{and} \qquad Q_\alpha(1/z) = \prod_{x\in\mathbb{F}^\times} (1 - x/z)^{\nu^-_\alpha(x)} .$$

Note that, in the above formulae, since $\nu^\pm_\alpha$ is finitely supported, the products only run through the finitely many numbers in the support $\mathrm{supp}(\nu^\pm_\alpha)$ of $\nu^\pm_\alpha$. Moreover, since P_α and Q_α are relatively prime for every $\alpha \in A$, we have $\mathrm{supp}(\nu^+_\alpha) \cap \mathrm{supp}(\nu^-_\alpha) = \emptyset$. We denote by $\mathbb{N}^{\mathbb{F}^\times}_f$, the set of finitely supported $\mathbb{N}$-valued maps over $\mathbb{F}^\times$. As is customary in the theory of q-characters, we associate with every ℓ-weight given by a pair (P_α, Q_α) of Drinfel'd polynomials or, equivalently, by a pair $(\nu^+_\alpha, \nu^-_\alpha)$ with $\nu^+_\alpha, \nu^-_\alpha \in \mathbb{N}^{\mathbb{F}^\times}_f$ and $\mathrm{supp}(\nu^+_\alpha) \cap \mathrm{supp}(\nu^-_\alpha) = \emptyset$, a monomial

$$m_\alpha = Y^{\nu^+ - \nu^-} = \prod_{x\in\mathbb{F}^\times} Y_x^{\nu^+_\alpha(x) - \nu^-_\alpha(x)} \in \mathbb{Z}[Y_a, Y_a^{-1}]_{a\in\mathbb{F}^\times} .$$

Definition 3.4.8. Let M be an ℓ-dominant $\ddot{\mathrm{U}}^0_q(\mathfrak{a}_1)$-module and let M_α and M_β be any two ℓ-weight spaces of M with respective ℓ-weights

$$\kappa^\pm_\alpha(z) = -q^{\deg(P_\alpha)} \left(\frac{P_\alpha(q^{-2}/z)}{P_\alpha(1/z)}\right)_{|z|^{\pm1}\ll1} \qquad \text{and} \qquad \kappa^\pm_\beta(z) = -q^{\deg(P_\beta)} \left(\frac{P_\beta(q^{-2}/z)}{P_\beta(1/z)}\right)_{|z|^{\pm1}\ll1} ,$$

where $P_\alpha(1/z), P_\beta(1/z) \in \mathbb{F}[z^{-1}]$ are two monic polynomials. By proposition 3.4.3.*i.*, if $M_\alpha \cap \mathsf{K}^\pm_{1,\pm m}(z).M_\beta \neq \{0\}$ for some $m \in \mathbb{N}^\times$, then there exists a unique $a \in \mathbb{F}^\times$ such that

$$\kappa^\varepsilon_\alpha(z) = \kappa^\varepsilon_\beta(z) H^\varepsilon_{m,a}(z)^{\pm1} ,$$

where $\varepsilon \in \{-,+\}$. We shall say that M is *t-dominant* if, for any pair of ℓ-weight spaces satisfying the above assumptions, we have, in addition, that

$$P_\beta(1/aq^{-(m\pm m)}) = P_\beta(1/aq^{2-(m\pm m)}) = 0\,.$$

For every $a \in \mathbb{F}^\times$, we let $\delta_a \in \mathbb{N}_f^{\mathbb{F}^\times}$ be defined by

$$\delta_a(x) = \begin{cases} 1 & \text{if } x = a; \\ 0 & \text{otherwise.} \end{cases}$$

For every $a \in \mathbb{F}^\times$, we let $\mathbb{N}_a^{\mathbb{F}^\times} = \left\{\nu \in \mathbb{N}_f^{\mathbb{F}^\times} : \{a, aq^2\} \subseteq \operatorname{supp}(\nu)\right\}$ and we define, for every $m \in \mathbb{Z}$, an operator $\Gamma_{m,a} : \mathbb{N}_{aq^{-2m}}^{\mathbb{F}^\times} \to \mathbb{N}_a^{\mathbb{F}^\times}$ by letting [1], for every $\nu \in \mathbb{N}_{aq^{-2m}}^{\mathbb{F}^\times}$,

$$\Gamma_{m,a}(\nu) = \nu - \delta_{aq^{-2m}} - \delta_{aq^{2-2m}} + \delta_a + \delta_{aq^2}\,.$$

$\Gamma_{m,a}$ is obviously invertible, with inverse $\Gamma_{m,a}^{-1} : \mathbb{N}_a^{\mathbb{F}^\times} \to \mathbb{N}_{aq^{-2m}}^{\mathbb{F}^\times}$ given by $\Gamma_{m,a}^{-1} = \Gamma_{-m,aq^{-2m}}$. Note that, for every $a \in \mathbb{F}^\times$, $\Gamma_{0,a} = \mathrm{id}$ over $\mathbb{N}_a^{\mathbb{F}^\times}$. Given two finite multisets $\nu, \nu' \in \mathbb{N}_f^{\mathbb{F}^\times}$, we we shall say that they are *equivalent* and write $\nu \sim \nu'$ iff

$$\nu = \Gamma_{m_1,a_1} \circ \cdots \circ \Gamma_{m_n,a_n}(\nu')\,, \tag{3.4.8}$$

for some $n \in \mathbb{N}$, $m_1, \dots, m_n \in \mathbb{Z}^n$ and some $a_1, \dots, a_n \in \mathbb{F}^\times$. In writing (3.4.8), it is assumed that, for every $p = 2, \dots, n$, $\Gamma_{m_p,a_p} \circ \cdots \circ \Gamma_{m_n,a_n}(\nu') \in \mathbb{N}_{a_{p-1}q^{-2m_{p-1}}}^{\mathbb{F}^\times}$. It is clear that $\sim$ is an equivalence relation and we denote by $[\nu] \in \mathbb{N}_f^{\mathbb{F}^\times}/\sim$ the equivalence class of ν in $\mathbb{N}_f^{\mathbb{F}^\times}$. Following remark 3.4.7, we naturally extend the action of $\Gamma_{m,a}$ to $\mathbb{Z}[Y_b, Y_b^{-1}]_{b\in\mathbb{F}^\times}$, by setting

$$\Gamma_{m,a}(Y^\nu) = Y^{\Gamma_{m,a}(\nu)}\,.$$

The equivalence relation $\sim$ similarly extends from $\mathbb{N}_f^{\mathbb{F}^\times}$ to $\mathbb{Z}[Y_b, Y_b^{-1}]_{b\in\mathbb{F}^\times}$. Note that, setting

$$H_{m,a} = Y_{aq^{-2m}}^{-1} Y_{aq^{2-2m}}^{-1} Y_a Y_{aq^2} \in \mathbb{Z}[Y_b, Y_b^{-1}]_{b\in\mathbb{F}^\times}\,,$$

for every $a \in \mathbb{F}^\times$ and every $m \in \mathbb{Z}$, we have, for every $\nu \in \mathbb{N}_a^{\mathbb{F}^\times}$

$$\Gamma_{m,a}(Y^\nu) = H_{m,a} Y^\nu\,.$$

Corollary 3.4.9. *Let M be a simple t-dominant $\ddot{\mathrm{U}}_q^0(\mathfrak{a}_1)$-module. Then there exists a multiset $\nu \in \mathbb{N}_f^{\mathbb{F}^\times}$ such that all the monomials associated with the ℓ-weights of M be in the equivalence class of Y^ν.*

Proof. By proposition 3.4.3, for any two ℓ-weight spaces, M_α and M_β, of an ℓ-dominant $\ddot{\mathrm{U}}_q^0(\mathfrak{a}_1)$-module M, with respective ℓ-weights

$$\kappa_\alpha^\pm(z) = -q^{\deg(P_\alpha)} \left(\frac{P_\alpha(q^{-2}/z)}{P_\alpha(1/z)}\right)_{|z|^{\pm 1} \ll 1} \qquad \text{and} \qquad \kappa_\beta^\pm(z) = -q^{\deg(P_\beta)} \left(\frac{P_\beta(q^{-2}/z)}{P_\beta(1/z)}\right)_{|z|^{\pm 1} \ll 1}\,,$$

---

[1] Although the definition of $\Gamma_{\pm1,a}^{\pm1}$ easily extends to $\left\{\nu \in \mathbb{N}_f^{\mathbb{F}^\times} : aq^{-(1\pm1)} \in \operatorname{supp}(\nu)\right\}$, we will not make use of that extension and exclusively regard $\Gamma_{\pm1,a}^{\pm1}$ as a map $\mathbb{N}_{aq^{-2(1\pm1)}}^{\mathbb{F}^\times} \to \mathbb{N}_{aq^{2(1\mp1)}}^{\mathbb{F}^\times}$.

if $M_\alpha \cap \mathsf{K}^{\pm}_{1,\pm m,n}.M_\beta \neq \{0\}$ for some $m \in \mathbb{N}^\times$ and some $n \in \mathbb{Z}$, then we must have

$$\frac{P_\alpha(q^{-2}/z)}{P_\alpha(1/z)} = \frac{P_\beta(q^{-2}/z)}{P_\beta(1/z)} \left(\frac{(1-q^{-2}a/z)(1-a/z)}{(1-a/z)(1-q^2a/z)} \frac{(1-q^{-2m}a/z)(1-q^{-2(m-1)}a/z)}{(1-q^{-2(m+1)}a/z)(1-q^{-2m}a/z)} \right)^{\pm 1}, \tag{3.4.9}$$

for some $a \in \mathbb{F}^\times$. Now, denote by ν_α (resp. ν_β) the multiset of roots of $P_\alpha(1/z)$ (resp. $P_\beta(1/z)$) and assume $m > 1$, it is clear that:

- for the upper choice of sign on the right hand side of the above equation, we get

  $$(1-q^{-2m}a/z)(1-q^{-2(m-1)}a/z)P_\alpha(1/z) = (1-a/z)(1-q^2a/z)P_\beta(1/z),$$

  implying that $\nu_\alpha + \delta_{aq^{-2m}} + \delta_{aq^{-2(m-1)}} = \nu_\beta + \delta_a + \delta_{aq^2}$;
- for the lower choice of sign,

  $$(1-a/z)(1-q^2a/z)P_\alpha(1/z) = (1-q^{-2m}a/z)(1-q^{-2(m-1)}a/z)P_\beta(1/z),$$

  implying that $\nu_\alpha + \delta_a + \delta_{aq^2} = \nu_\beta + \delta_{aq^{-2m}} + \delta_{aq^{-2(m-1)}}$.

If on the other hand $m = 1$, since M is t-dominant, we have, by definition, that $aq^{\mp 2}$ is a root of $P_\beta(1/z)$. In any case, it is clear that $\nu_\alpha \sim \nu_\beta$ and hence $Y^{\nu_\alpha} \sim Y^{\nu_\beta}$. Since M is simple, there can be no non-zero ℓ-weight space M_β of M such that $M_\alpha \cap \mathsf{K}^{\pm}_{1,\pm m,n}.M_\beta = \{0\}$ for every ℓ-weight space M_α of M, every $m \in \mathbb{N}^\times$ and every $n \in \mathbb{Z}$. □

In view of definition-proposition 3.4.2, we can make the following

Definition 3.4.10. For every monic polynomial $P(1/z) \in \mathbb{F}[z^{-1}]$, denote by $\mathbb{F}_P$ the one-dimensional $\ddot{\mathrm{U}}_q^{0,0}(\mathfrak{a}_1)$-module such that

$$\mathsf{K}^{\pm}_{1,0}(z).v = -q^{\deg(P)} \left(\frac{P(q^{-2}/z)}{P(1/z)} \right)_{|z|^{\pm 1} \ll 1} v,$$

for every $v \in \mathbb{F}_P$. There exists a universal $\ddot{\mathrm{U}}_q^0(\mathfrak{a}_1)$-module $M^0(P) \cong \ddot{\mathrm{U}}_q^0(\mathfrak{a}_1) \underset{\ddot{\mathrm{U}}_q^{0,0}(\mathfrak{a}_1)}{\widehat{\otimes}} \mathbb{F}_P$ that admits the ℓ-weight associated with P. Denoting by $N^0(P)$ the maximal $\ddot{\mathrm{U}}_q^0(\mathfrak{a}_1)$-submodule of $M^0(P)$ such that $N^0(P) \cap \mathbb{F}_P = \{0\}$, we define the unique – up to isomorphisms – simple $\ddot{\mathrm{U}}_q^0(\mathfrak{a}_1)$-module $L^0(P) = M^0(P)/N^0(P)$.

Proposition 3.4.11. *For every simple ℓ-dominant $\ddot{\mathrm{U}}_q^0(\mathfrak{a}_1)$-module M, there exists a monic polynomial $P(1/z) \in \mathbb{F}[z^{-1}]$ such that $M \cong L^0(P)$.*

Proof. Obviously, for every $v \in M - \{0\}$, we have $M \cong \ddot{\mathrm{U}}_q^0(\mathfrak{a}_1).v$. Now since M is ℓ-dominant, v can be chosen as an ℓ-weight vector, i.e.

$$\mathsf{K}^{\pm}_{1,0}(z).v = -q^{\deg(P)} \left(\frac{P(q^{-2}/z)}{P(1/z)} \right)_{|z|^{\pm 1} \ll 1} v$$

for some monic polynomial $P(1/z) \in \mathbb{F}[z^{-1}]$. □

Remark 3.4.12. The above proof makes it clear that if $\{P_\alpha : \alpha \in A\}$ is the set of Drinfel'd polynomials of a simple ℓ-dominant $\ddot{\mathrm{U}}_q^0(\mathfrak{a}_1)$-module M, then, for every $\alpha \in A$, $M \cong L^0(P_\alpha)$.

Theorem 3.4.13. *For every monic polynomial* $P(1/z) \in \mathbb{F}[z^{-1}]$*,* $L^0(P)$ *is* t*-dominant.*

Proof. We postpone the proof of this theorem until section 3.5, where we construct $L^0(P)$ for every P and directly check that it is indeed t-dominant. □

Proposition 3.4.14. *Any topological* $\widehat{\ddot{\mathrm{U}}_q^0(\mathfrak{a}_1)}$*-module pulls back to a module over the elliptic Hall algebra* $\mathcal{E}_{q^{-4},q^2,q^2}$*.*

Proof. It suffices to make use of the Hopf algebra homomorphism

$$\mathcal{E}_{q^{-4},q^2,q^2} \xrightarrow{f} \ddot{\mathrm{U}}_q^{0^+}(\mathfrak{a}_1) \hookrightarrow \widehat{\ddot{\mathrm{U}}_q^0(\mathfrak{a}_1)}\,,$$

where f is defined in proposition 3.3.18 and the second arrow is the canonical injection into $\widehat{\ddot{\mathrm{U}}_q^0(\mathfrak{a}_1)}$ of its Hopf subalgebra $\ddot{\mathrm{U}}_q^{0^+}(\mathfrak{a}_1)$ – see proposition 3.3.12. □

Remark 3.4.15. It is worth mentioning that, as an example of the above proposition, ℓ-anti-dominant $\ddot{\mathrm{U}}_q^0(\mathfrak{a}_1)$-modules pullback to a family of $\mathcal{E}_{q^{-4},q^2,q^2}$-modules that were recently introduced in [DK19]. It might be interesting to investigate further the class of $\mathcal{E}_{q^{-4},q^2,q^2}$-modules obtained by pulling back other (rational) $\ddot{\mathrm{U}}_q^0(\mathfrak{a}_1)$-modules.

We conclude the present subsection by proving the following

Lemma 3.4.16. *Let* M *be an* ℓ*-dominant* $\ddot{\mathrm{U}}_q^0(\mathfrak{a}_1)$*-module. Suppose that, for any two local* ℓ*-weight spaces* M_α *and* M_β *of* M*, with respective* ℓ*-weights* $\kappa_\alpha^\pm(z)$ *and* $\kappa_\beta^\pm(z)$*, such that* $M_\alpha \cap \boldsymbol{K}_{1,\pm 1}^\pm(z).M_\beta \neq \{0\}$*, the unique* $a \in \mathbb{F}^\times$ *such that* $\kappa_\alpha^\varepsilon(z) = \kappa_\beta^\varepsilon(z) H_{1,a}^\varepsilon(z)^{\pm 1}$*, for every* $\varepsilon \in \{-,+\}$*, and* $(z-a)^N M_\alpha \cap \boldsymbol{K}_{1,\pm 1}^\pm(z).M_\beta = \{0\}$ *for some* $N \in \mathbb{N}^\times$ *– see proposition 3.4.3 – also satisfies* $P_\beta(1/a) = 0$*. Then* M *is* t*-dominant.*

Proof. Let M be as above and let M_α and M_β be two local ℓ-weight spaces of M with respective ℓ-weights

$$\kappa_\alpha^\pm(z) = -q^{\deg(P_\alpha)} \left(\frac{P_\alpha(q^{-2}/z)}{P_\alpha(1/z)}\right)_{|z|^{\pm 1} \ll 1} \qquad \text{and} \qquad \kappa_\beta^\pm(z) = -q^{\deg(P_\beta)} \left(\frac{P_\beta(q^{-2}/z)}{P_\beta(1/z)}\right)_{|z|^{\pm 1} \ll 1}.$$

Suppose that $M_\alpha \cap \mathbf{K}_{1,\pm m}^\pm(z).M_\beta \neq \{0\}$ for some $m \in \mathbb{N}^\times$. If $m > 1$, writing down $\kappa_\alpha^\varepsilon(z) = \kappa_\beta^\varepsilon(z) H_{m,a}^\varepsilon(z)^{\pm 1}$, we obtain equation (3.4.9) as in the proof of corollary 3.4.9. By the same discussion as the one following equation (3.4.9), we conclude that $P_\beta(1/aq^{-(m\pm m)}) = P_\beta(1/aq^{2-(m\pm m)}) = 0$, as needed – see definition 3.4.8. Finally, if $m = 1$, writing down $\kappa_\alpha^\varepsilon(z) = \kappa_\beta^\varepsilon(z) H_{1,a}^\varepsilon(z)^{\pm 1}$, we obtain

$$\frac{P_\alpha(q^{-2}/z)}{P_\alpha(1/z)} = \frac{P_\beta(q^{-2}/z)}{P_\beta(1/z)} \left(\frac{(1-a/z)}{(1-q^2a/z)} \frac{(1-q^{-2}a/z)}{(1-q^{-4}a/z)}\right)^{\pm 1}.$$

Then, it is clear that:

- for the upper choice of sign on the right hand side of the above equation, we get

$$(1-q^{-2}a/z) P_\alpha(1/z) = (1-q^2a/z) P_\beta(1/z)\,;$$

- for the lower choice of sign on the right hand side of the above equation, we get

$$(1 - q^2 a/z) P_\alpha(1/z) = (1 - q^{-2} a/z) P_\beta(1/z) \,.$$

In any case, it follows that $P_\beta(1/aq^{\mp 2}) = 0$. But by our assumptions on M, we also have that $P_\beta(1/a) = 0$ and the t-dominance of M follows – see definition 3.4.8. $\square$

3.4.2 t-weight $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)$-modules

Definition 3.4.17. For every $N \in \mathbb{N}^\times$, we shall say that a (topological) module M over $\ddot{\mathrm{U}}'_q(\mathfrak{a}_1)$ is of *type* $(1, N)$ if:

i. $\mathsf{C}^{\pm 1/2}$ acts as id on M;

ii. $\mathsf{c}^\pm_{\pm m}$ acts by multiplication by 0 on M, for every $m \geq N$.

We shall say that M is of *type* $(1, 0)$ if points i. and ii. above hold for every $m > 0$ and, in addition, $\mathsf{c}^\pm_0$ acts as id on M.

Remark 3.4.18. Let $N \in \mathbb{N}$. Then the $\ddot{\mathrm{U}}'_q(\mathfrak{a}_1)$-modules of type $(1, N)$ are in one-to-one correspondence with the $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)^{(N)}/(\mathsf{C}^{1/2} - 1)$-modules – see section 3.3.3 for a definition of $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)^{(N)}$. Obviously $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)$-modules of type $(1, 0)$ descend to modules over the double quantum loop algebra of type $\mathfrak{a}_1$, $\ddot{\mathrm{L}}_q(\mathfrak{a}_1)$.

Definition 3.4.19. We shall say that a (topological) $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)$-module M is a *t-weight* module if there exists a countable set $\{M_\alpha : \alpha \in A\}$ of indecomposable ℓ-weight $\ddot{\mathrm{U}}^0_q(\mathfrak{a}_1)$-modules, called *t-weight spaces* of M, such that, as (topological) $\ddot{\mathrm{U}}^0_q(\mathfrak{a}_1)$-modules,

$$M \cong \bigoplus_{\alpha \in A} M_\alpha \,. \tag{3.4.10}$$

We shall say that M is *weight-finite* if, regarding it as a completely decomposable $\ddot{\mathrm{U}}^0_q(\mathfrak{a}_1)$-module, its $\mathrm{Sp}(M)$ is finite – see definition-proposition 3.4.2 for the definition of Sp. A vector $v \in M - \{0\}$ is a *highest t-weight vector* of M if $v \in M_\alpha$ for some $\alpha \in A$ and, for every $r, s \in \mathbb{Z}$,

$$\mathsf{X}^+_{1,r,s} . v = 0 \,. \tag{3.4.11}$$

We shall say that M is *highest t-weight* if $M \cong \ddot{\mathrm{U}}_q(\mathfrak{a}_1).v$ for some highest t-weight vector $v \in M - \{0\}$.

Definition-Proposition 3.4.20. Let M be a t-weight $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)$-module that admits a highest t-weight vector $v \in M - \{0\}$. Denote by M_0 the t-weight space of M containing v. Then $M_0 = \ddot{\mathrm{U}}^0_q(\mathfrak{a}_1).v$ and, for every $r, s \in \mathbb{Z}$,

$$\mathsf{X}^+_{1,r,s} . M_0 = \{0\} \,. \tag{3.4.12}$$

We shall say that M_0 is a *highest t-weight space* of M. If in addition M is simple, then it admits a unique – up to isomorphisms of $\ddot{\mathrm{U}}^0_q(\mathfrak{a}_1)$-modules – highest t-weight space M_0.

Proof. It is an easy consequence of the triangular decomposition of $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)$ – see proposition 3.3.11 – and of the root grading of $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)$ that, indeed, $\mathsf{X}^+_{1,r,s} . \left(\ddot{\mathrm{U}}^0_q(\mathfrak{a}_1).v\right) = \{0\}$, for every $r, s \in \mathbb{Z}$. Now since M

is highest t-weight, we have $M \cong \ddot{\mathrm{U}}_q(\mathfrak{a}_1).v$. By proposition 3.3.11, $M_0 \subset M \cong \ddot{\mathrm{U}}_q^-(\mathfrak{a}_1)\ddot{\mathrm{U}}_q^0(\mathfrak{a}_1).v$ and it follows that $M_0 \cong \ddot{\mathrm{U}}_q^0(\mathfrak{a}_1).v$. Now, assuming that M is simple and that it admits highest t-weight spaces M_0 and M_0', we have that $\ddot{\mathrm{U}}_q^-(\mathfrak{a}_1).M_0 \cong M \cong \ddot{\mathrm{U}}_q^-(\mathfrak{a}_1).M_0'$ as $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)$-modules. In particular, $M_0 \cong M_0'$ as $\ddot{\mathrm{U}}_q^0(\mathfrak{a}_1)$-modules. □

In view of the triangular decomposition of $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)$ – see proposition 3.3.11 –, the above proposition implies that any highest t-weight $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)$-modules M is entirely determined as $M \cong \ddot{\mathrm{U}}_q^-(\mathfrak{a}_1).M_0$, by the data of its highest t-weight space M_0, a cyclic ℓ-weight $\ddot{\mathrm{U}}_q^0(\mathfrak{a}_1)$-module. Now for any $v \in M_0 - \{0\}$ such that $M_0 \cong \ddot{\mathrm{U}}_q^0(\mathfrak{a}_1).v$, let N_0 be the maximal $\ddot{\mathrm{U}}_q^0(\mathfrak{a}_1)$-submodule of M_0 not containing v and set $L_0 = M_0/N_0$ [2]. Then, by construction, L_0 is a simple $\ddot{\mathrm{U}}_q^0(\mathfrak{a}_1)$-module such that, as $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)$-modules, $M \cong \ddot{\mathrm{U}}_q^-(\mathfrak{a}_1).L_0 \mod \ddot{\mathrm{U}}_q^-(\mathfrak{a}_1).N_0$. We therefore make the following

Definition 3.4.21. We extend every simple (topological) ℓ-weight $\ddot{\mathrm{U}}_q^0(\mathfrak{a}_1)$-module M_0 into a $\ddot{\mathrm{U}}_q^{\geq}(\mathfrak{a}_1)$-module by setting $\mathsf{X}^+_{1,r,s}.M_0 = \{0\}$ for every $r, s \in \mathbb{Z}$. This being understood, we define the *universal* highest t-weight $\ddot{\mathrm{U}}_q'(\mathfrak{a}_1)$-module with highest t-weight space M_0 by setting

$$\mathcal{M}(M_0) = \widehat{\ddot{\mathrm{U}}_q'(\mathfrak{a}_1)} \underset{\ddot{\mathrm{U}}_q^{\geq}(\mathfrak{a}_1)}{\widehat{\otimes}} M_0$$

as $\ddot{\mathrm{U}}_q'(\mathfrak{a}_1)$-modules. Denoting by $\mathcal{N}(M_0)$ the maximal (closed) $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)$-submodule of $\mathcal{M}(M_0)$ such that $M_0 \cap \mathcal{N}(M_0) = \{0\}$, we define the simple highest t-weight $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)$-module $\mathcal{L}(M_0)$ with highest t-weight space M_0 by setting $\mathcal{L}(M_0) \cong \mathcal{M}(M_0)/\mathcal{N}(M_0)$. It is unique up to isomorphisms.

Classifying simple highest t-weight $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)$-modules therefore amounts to classifying those simple ℓ-weight $\ddot{\mathrm{U}}_q^0(\mathfrak{a}_1)$-modules M_0 that appear as their highest t-weight spaces. In the case of weight-finite $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)$-modules, this is achieved by the following

Theorem 3.4.22. *The following hold:*

i. *Every weight-finite simple $\ddot{\mathrm{U}}_q'(\mathfrak{a}_1)$-module M is highest t-weight and can be obtained by twisting a type (1,0) weight-finite simple $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)$-module with an algebra automorphism from the subgroup of* $\mathrm{Aut}(\ddot{\mathrm{U}}_q'(\mathfrak{a}_1))$ *generated by the algebra automorphisms τ and σ of proposition 3.3.27.*

ii. *The type (1,0) simple highest t-weight $\ddot{\mathrm{U}}_q'(\mathfrak{a}_1)$-module $\mathcal{L}(M_0)$ is weight-finite if and only if its highest t-weight space M_0 is a simple t-dominant $\ddot{\mathrm{U}}_q^0(\mathfrak{a}_1)$-module – see proposition-definition 3.4.5.*

Proof. We shall prove ii in section 3.5. We now prove i. Let M be a weight-finite simple t-weight $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)$-module and assume for a contradiction that, for every $w \in M - \{0\}$, there exist $r, s \in \mathbb{Z}$ such that $\mathsf{X}^+_{1,r,s}.w \neq 0$. Then, there must exist two sequences $(r_n)_{n\in\mathbb{N}}, (s_n)_{n\in\mathbb{N}} \in \mathbb{Z}^{\mathbb{N}}$, such that

$$0 \notin \left\{w_n = \mathsf{X}^+_{r_1,s_1} \dots \mathsf{X}^+_{r_n,s_n}.w : n \in \mathbb{N}\right\} .$$

Choosing $w \in M - \{0\}$ to be an eigenvector of $\mathsf{K}^+_{1,0,0}$ with eigenvalue $\lambda \in \mathbb{F}^\times$ – see definition-proposition 3.4.2 for the existence of such a vector –, one easily sees from the relations that, for every $n \in \mathbb{N}$, $\mathsf{K}^+_{1,0,0}.w_n = \lambda q^{2n} w_n$. It follows – see definition-proposition 3.4.2 – that $\{\lambda q^{2n} : n \in \mathbb{N}\} \subseteq \mathrm{Sp}(M)$. A

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[2] N_0 clearly does not depend on the chosen generator v. Indeed, if N_0 contained a generator v' of M_0, it would contain all the others, including v. It follows that N_0 and hence L_0 are both independent of v.

contradiction with the weight-finiteness of M. Thus, we conclude that there exists a highest t-weight vector $v_0 \in M - \{0\}$ such that $\mathsf{K}^{\pm}_{1,0,0}.v_0 = \kappa_0^{\pm 1} v_0$ for some $\kappa_0 \in \mathbb{F}^{\times}$. Obviously, $M \cong \ddot{\mathrm{U}}_q(\mathfrak{a}_1).v_0$, for $\ddot{\mathrm{U}}_q(\mathfrak{a}_1).v_0 \neq \{0\}$ is a submodule of the simple $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)$-module M. Thus M is highest t-weight. Denote by $M_0 = \ddot{\mathrm{U}}^0_q(\mathfrak{a}_1).v_0$ its highest t-weight space. The latter is an ℓ-weight $\ddot{\mathrm{U}}^0_q(\mathfrak{a}_1)$-module. As such, it completely decomposes into countably many locally finite-dimensional indecomposable $\ddot{\mathrm{U}}^{0,0}_q(\mathfrak{a}_1)$-modules that constitute its ℓ-weight spaces. Over any of these, $\mathsf{C}^{1/2}$ must admit an eigenvector. But since M is simple and $\mathsf{C}^{1/2}$ is central, the latter acts over M by a scalar multiple of id. It follows from definition-proposition 3.4.2 that C acts over M by id or $-$id. In the former case, there is nothing to do; whereas in the latter, it is quite clear from proposition 3.3.27 that, twisting the $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)$ action on M by τ, we can ensure that C acts by id. It follows that $\mathsf{C}^{1/2}$ acts by id or $-$id. Again, in the former case, there is nothing to do; whereas in the latter, twisting by σ, we can ensure that $\mathsf{C}^{1/2}$ acts by id. Similarly, for every $m \in \mathbb{N}$, $\mathsf{c}^{\pm}_{\pm m}$ must admit an eigenvector over any locally finite-dimensional ℓ-weight space of M_0. But again, since M is simple and $\mathsf{c}^{\pm}_{\pm m}$ is central, the latter must act over M by a scalar multiple of id.

In any case, in view of (3.3.7) and (3.3.8), $\mathsf{K}^{\pm}_{1,0,0}$ commutes with all the other generators of $\ddot{\mathrm{U}}^0_q(\mathfrak{a}_1)$ and, since $M_0 = \ddot{\mathrm{U}}^0_q(\mathfrak{a}_1).v_0$, we have $\mathsf{K}^{\pm}_{1,0,0}.w = \kappa_0^{\pm 1} w$ for every $w \in M_0$. Moreover, M_0 turns out to be a type 1 ℓ-weight $\ddot{\mathrm{U}}^0_q(\mathfrak{a}_1)$-module and, by definition-proposition 3.4.2,

$$M_0 \subseteq \bigoplus_{\alpha \in A} \left\{ v \in M : \mathsf{K}^{\pm}_{1,0,0}.v = \kappa_0^{\pm 1} v \quad \text{and} \quad \exists n \in \mathbb{N}^{\times}, \forall m \in \mathbb{N}^{\times} \quad \left(\mathsf{K}^{\pm}_{1,0,\pm m} - \kappa^{\pm}_{\alpha,\pm m}\mathrm{id}\right)^n .v = 0 \right\}$$

for some countable set of sequences $\left\{(\kappa^{\pm}_{\alpha,\pm m})_{m \in \mathbb{N}^{\times}} \in \mathbb{F}^{\mathbb{N}^{\times}} : \alpha \in A\right\}$. By proposition 3.4.20,

$$\mathsf{X}^{+}_{1,r,s}.M_0 = \{0\}\,, \tag{3.4.13}$$

for every $r, s \in \mathbb{Z}$. Pulling back with $\iota^{(0)}$ and $\iota^{(1)}$ respectively, we can simultaneously regard M as a $\mathrm{U}_q(\mathrm{L}\mathfrak{a}_1)$-module for both of its Dynkin diagram subalgebras $\mathrm{U}_q(\mathrm{L}\mathfrak{a}_1)^{(0)}$ and $\mathrm{U}_q(\mathrm{L}\mathfrak{a}_1)^{(1)}$ – see discussion before theorem 3.3.22 in section 3.3 for definitions. Let $v \in M_0 - \{0\}$ be a simultaneous eigenvector of the pairwise commuting linear operators in $\left\{\mathsf{K}^{\pm}_{1,0,\pm m} : m \in \mathbb{N}\right\}$. Equation (3.4.13) implies that $\mathbf{x}^{+}_1(z).v = \mathbf{x}^{-}_0(z).v = 0$. Thus v is a highest (resp. lowest) ℓ-weight vector of $\dot{\mathrm{U}}_q(\mathfrak{a}_1)^{(1)}.v$ (resp. $\dot{\mathrm{U}}_q(\mathfrak{a}_1)^{(0)}.v$). The weight finiteness of M now allows us to apply corollary 3.2.12 to prove that the respective simple quotients of $\mathrm{U}_q(\mathrm{L}\mathfrak{a}_1)^{(0)}.v$ and $\mathrm{U}_q(\mathrm{L}\mathfrak{a}_1)^{(1)}.v$ containing v are both finite-dimensional and isomorphic to a unique simple highest (resp. lowest) ℓ-weight module $L(P_1)$ (resp. $\bar{L}(P_0)$). As a consequence of theorem 3.2.5 and of proposition 3.2.6, we conclude that

$$\mathbf{k}^{\pm}_0(z).v = q^{-\deg(P_0)} \left(\frac{P_0(1/z)}{P_0(q^{-2}/z)}\right)_{|z|^{\mp 1} \ll 1} v \qquad \text{and} \qquad \mathbf{k}^{\pm}_1(z).v = q^{\deg(P_1)} \left(\frac{P_1(q^{-2}/z)}{P_1(1/z)}\right)_{|z|^{\mp 1} \ll 1} v\,,$$

for some monic polynomials P_0 and P_1. On the other hand, pulling back with ι_m for every $m \in \mathbb{Z}$ – see proposition 3.3.23 for a definition –, we can regard M as a $\mathrm{U}_q(\mathrm{L}\mathfrak{a}_1)$-module in infinitely many independent ways. Again, for every $m \in \mathbb{Z}$, v turns out to be a highest ℓ-weight vector for a unique simple weight finite, hence finite dimensional $\mathrm{U}_q(\mathrm{L}\mathfrak{a}_1)$-module. As such, it satisfies

$$\iota_m(\mathbf{k}^{\pm}_1(z)).v = q^{\deg(Q_m)} \left(\frac{Q_m(q^{-2}/z)}{Q_m(1/z)}\right)_{|z|^{\mp 1} \ll 1} v\,,$$

for some monic polynomial Q_m. Now since

$$\iota_m(\mathbf{k}_1^\pm(z)) = -\prod_{p=1}^{|m|} \mathsf{c}^\pm\left(q^{(1-2p)\mathrm{sign}(m)-1}z\right)^{\mathrm{sign}(m)} \mathsf{K}^\mp_{1,0}(\mathsf{C}^{-1/2}z)$$

and $\widehat{\Psi}(\mathbf{k}_0^\pm(z)\mathbf{k}_1^\pm(z)) = \mathsf{c}^\pm(z)$, we must have

$$\begin{aligned} q^{\deg(Q_m)}\left(\frac{Q_m(q^{-2}/z)}{Q_m(1/z)}\right)_{|z|^{\mp 1}\ll 1} &= q^{\deg(P_1)+m(\deg(P_1)-\deg(P_0))}\left(\frac{P_1(q^{-2}/z)}{P_1(1/z)}\right)_{|z|^{\mp 1}\ll 1} \\ &\times\prod_{p=1}^{|m|}\left(\frac{P_1(q^{(2p-1)\mathrm{sign}(m)-1}/z)P_0(q^{(2p-1)\mathrm{sign}(m)+1}/z)}{P_1(q^{(2p-1)\mathrm{sign}(m)+1}/z)P_0(q^{(2p-1)\mathrm{sign}(m)-1}/z)}\right)^{\mathrm{sign}(m)}_{|z|^{\mp 1}\ll 1} \end{aligned} \tag{3.4.14}$$

for every $m \in \mathbb{Z}^\times$. In the limit as $z^{-1} \to 0$, this implies $q^{\deg(Q_m)} = q^{\deg(P_1)+m(\deg(P_1)-\deg(P_0))}$ for every $m \in \mathbb{Z}$ and, consequently, $\deg(P_0) = \deg(P_1) = \deg(Q_m)$. After obvious simplifications, (3.4.14) becomes

$$\left(\frac{Q_m(q^{-2}/z)}{Q_m(1/z)}\right)_{|z|^{\mp 1}\ll 1} = \left(\frac{P_1(q^{-1-\mathrm{sign}(m)}/z)}{P_0(q^{\mathrm{sign}(m)-1}/z)}\,\frac{P_0(q^{2m+1-\mathrm{sign}(m)}/z)}{P_1(q^{2m+1-\mathrm{sign}(m)}/z)}\right)^{\mathrm{sign}(m)}_{|z|^{\mp 1}\ll 1} \tag{3.4.15}$$

for every $m \in \mathbb{Z}^\times$. Now, $z^{-1} = 0$ is not a root of $P(1/z)$ for any monic polynomial P. Moreover, q being a formal parameter – in case q is regarded as a complex number, we shall assume that $1 \notin q^{\mathbb{Z}^\times}$ –, it follows that the map $z^{-1} \mapsto q^m z^{-1}$ has no fixed points over the set of roots of a monic polynomial. Thus, for $|m|$ large enough, the respective sets of roots of $P_1(q^{-1-\mathrm{sign}(m)}/z)$ and $P_1(q^{2m+1-\mathrm{sign}(m)}/z)$ are disjoint. Similarly, for $|m|$ large enough, the respective sets of roots of $P_0(q^{\mathrm{sign}(m)-1}/z)$ and $P_0(q^{2m+1-\mathrm{sign}(m)}/z)$ are disjoint. It follows that, for $|m|$ large enough, on the r.h.s. of (3.4.15), cancellations can only occur between factors on opposite sides of the same fraction line. Now, either $P_0 = P_1$, which obviously solves (3.4.15); or $P_0 \neq P_1$. Assume for a contradiction that $P_0 \neq P_1$. In that case, there exist a monic polynomial P with $\deg(P) < \deg(P_0) = \deg(P_1)$, an integer $n \in \mathbb{N}^\times$ with $n \leq \deg(P_0) = \deg(P_1)$ and two n-tuples $(\alpha_p)_{p\in[\![n]\!]}$, $(\beta_p)_{p\in[\![n]\!]} \in \mathbb{F}^n$ with

$$\{\alpha_p : p \in [\![n]\!]\} \cap \{\beta_p : p \in [\![n]\!]\} = \emptyset\,,$$

such that

$$P_0(1/z) = P(1/z)\prod_{p=1}^{n}(1-\beta_p/z) \qquad \text{and} \qquad P_1(1/z) = P(1/z)\prod_{p=1}^{n}(1-\alpha_p/z)\,.$$

Thus, in that case, (3.4.15) yields

$$\begin{aligned} \left(\frac{Q_m(q^{-2}/z)}{Q_m(1/z)}\right)_{|z|^{\mp 1}\ll 1} &= \left(\frac{P(q^{-2}/z)}{P(1/z)}\left(\prod_{p=1}^{n}\frac{1-\alpha_p q^{-\mathrm{sign}(m)-1}/z}{1-\beta_p q^{\mathrm{sign}(m)-1}/z}\right)^{\mathrm{sign}(m)}\right)_{|z|^{\mp 1}\ll 1} \\ &\times\left(\prod_{p=1}^{n}\frac{1-\beta_p q^{2m+1-\mathrm{sign}(m)}/z}{1-\alpha_p q^{2m+1-\mathrm{sign}(m)}/z}\right)^{\mathrm{sign}(m)}_{|z|^{\mp 1}\ll 1}, \end{aligned}$$

where, for $|m|$ large enough, cancellations on the r.h.s. can only involve factors in the numerators and denominators of the leftmost two fraction lines. The third fraction must therefore be a factor in the l.h.s. But this leads to a contradiction since we cannot have simultaneously

$$\{\beta_p : p \in [\![n]\!]\} \subset \bigcup_{p\in[\![n]\!]} \alpha_p q^{-2\mathbb{N}^\times} ,$$

as required when $m > 0$, and

$$\{\beta_p : p \in [\![n]\!]\} \subset \bigcup_{p\in[\![n]\!]} \alpha_p q^{2\mathbb{N}^\times} ,$$

as required when $m < 0$. Hence $P_0 = P_1$ and i follows. □

Although we must postpone the proof of part ii of theorem 3.4.22, the proof above still makes it clear that

Proposition 3.4.23. *If a type (1,0) simple highest t-weight* $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)$*-module* $\mathcal{L}(M_0)$ *is weight-finite, then its highest t-weight space* M_0 *is a simple* ℓ*-dominant* $\dot{\mathrm{U}}_q^0(\mathfrak{a}_1)$*-module.*

Proposition 3.4.24. *Let* M *be a t-weight* $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)$*-module and let* M_α *and* M_β *be two local* ℓ*-weight spaces of* M *such that, for some* $m, n \in \mathbb{Z}$, $M_\alpha \cap \mathsf{X}^{\pm}_{1,m,n}.M_\beta \neq \{0\}$. *Then, there exists a unique* $a \in \mathbb{F}^\times$ *such that:*

i. the respective ℓ*-weights* $\kappa^\varepsilon_\alpha(z)$ *and* $\kappa^\varepsilon_\beta(z)$ *of* M_α *and* M_β *be related by*

$$\kappa^\varepsilon_\alpha(z) = \kappa^\varepsilon_\beta(z) A^\varepsilon_a(z)^{\pm 1} , \tag{3.4.16}$$

where $\varepsilon \in \{-, +\}$ *and*

$$A^{\pm}_a(z) = q^2 \left(\frac{1 - q^{-2}a/z}{1 - q^2 a/z}\right)_{|z|^{\pm1} \ll 1} ;$$

ii. $(z - a)^N M_\alpha \cap \boldsymbol{X}^{\pm}_{1,m}(z).M_\beta = \{0\}$ *for some* $N \in \mathbb{N}^\times$.

Proof. We keep the same notations as in the proof of proposition 3.4.3. More specifically, we have two bases $\{v_i : i = 1, \dots, \dim(M_\alpha)\}$ and $\{w_j : j = 1, \dots, \dim(M_\beta)\}$ of M_α and M_β respectively, in which

$$\forall i \in [\![\dim M_\alpha]\!] , \qquad \mathsf{K}^{\pm}_{1,0}(z).v_i = \kappa^{\pm}_\alpha(z) \sum_{k=i}^{\dim M_\alpha} \eta^{\pm}_{\alpha,i,k}(z) v_k ,$$

$$\forall j \in [\![\dim M_\beta]\!] , \qquad \mathsf{K}^{\pm}_{1,0}(z).w_j = \kappa^{\pm}_\beta(z) \sum_{l=j}^{\dim M_\beta} \eta^{\pm}_{\beta,j,l}(z) w_l ,$$

for some $\eta^{\pm}_{\alpha,i,k}(z), \eta^{\pm}_{\beta,j,l}(z) \in \mathbb{F}[[z^{\pm1}]]$, with $i, k \in [\![\dim M_\alpha]\!]$ and $j, l \in [\![\dim M_\beta]\!]$, such that $\eta^{\pm}_{\alpha,i,i}(z) = 1$ for every $i \in [\![\dim M_\alpha]\!]$ and $\eta^{\pm}_{\beta,j,j}(z) = 1$ for every $j \in [\![\dim M_\beta]\!]$. Moreover, for every $j \in [\![\dim M_\beta]\!]$,

$$\left[\mathsf{X}^{\pm}_{1,m}(z).w_j\right]_{M_\alpha} = \sum_{i\in[\![\dim M_\alpha]\!]} \xi^{\pm}_{m,j,i}(z) v_i ,$$

for some $\xi^{\pm}_{m,j,i}(z) \in \mathbb{F}[[z, z^{-1}]]$ – see definition 3.4.1 for the definition of $[-]_{M_\alpha}$.

Now, if $M_\alpha \cap \mathsf{X}^{\pm}_{1,m,n}.M_\beta \neq \{0\}$, there must exist a largest nonempty subset $J \subseteq [\![\dim M_\beta]\!]$ such that, for every $j \in J$, $\left[\mathsf{X}^{\pm}_{1,m}(z).w_j\right]_{M_\alpha} \neq \{0\}$. Let $j_* = \max J$. Obviously, for every $j \in J$, there must exist a largest nonempty subset $I(j) \subseteq [\![\dim M_\alpha]\!]$ such that, for every $j \in J$ and every $i \in I(j)$, $\xi^\pm_{m,j,i}(z) \neq 0$. Consequently, for every $j \in J$,

$$\left[\mathsf{X}^\pm_{1,m}(z).w_j\right]_{M_\alpha} = \sum_{i\in I(j)} \xi^\pm_{m,j,i}(z) v_i \,,$$

where $\xi^\pm_{m,j,i}(z) \in \mathbb{F}[[z,z^{-1}]] - \{0\}$, whereas $\xi^\pm_{m,j,i}(z) = 0$ for any (j,i) outside of the set of pairs $\{(j,i) : j \in J, i \in I(j)\}$. For every $j \in J$, we let $i_*(j) = \min I(j)$ and, for simplicity, we let $i_* = i_*(j_*)$. Making use of the relations in $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)$ – namely (5.0.1) and (3.3.10) –, we get, for every $j \in J$ and every $\varepsilon \in \{-,+\}$,

$$(z_1 - q^{\pm 2} z_2)\mathsf{X}^\pm_{1,m}(z_1)\mathsf{K}^\varepsilon_{1,0}(z_2).w_j = (z_1 q^{\pm 2} - z_2)\mathsf{K}^\varepsilon_{1,0}(z_2)\mathsf{X}^\pm_{1,m}(z_1).w_j \,.$$

The latter easily implies that, for every $j \in J$ and every $i \in I(j)$,

$$(z_1 - q^{\pm 2} z_2)\kappa^\varepsilon_\beta(z_2) \sum_{\substack{l\in J\\ l\geq j}} \eta^\varepsilon_{\beta,j,l}(z_2)\xi^\pm_{m,l,i}(z_1) = (z_1 q^{\pm 2} - z_2)\kappa^\pm_\alpha(z_2) \sum_{\substack{k\in I(j)\\ k\leq i}} \eta^\varepsilon_{\alpha,k,i}(z_2)\xi^\pm_{m,j,k}(z_1) \,. \tag{3.4.17}$$

Taking $i = i_*$ and $j = j_*$ in the above equation immediately yields

$$\left[(z_1 - q^{\pm 2} z_2)\kappa^\varepsilon_\beta(z_2) - (z_1 q^{\pm 2} - z_2)\kappa^\varepsilon_\alpha(z_2)\right] \xi^\pm_{m,j_*,i_*}(z_1) = 0 \,.$$

The latter is equivalent to the fact that, for every $p \in \mathbb{Z}$,

$$\xi^\pm_{m,j_*,i_*,p} z \left(q^{\pm 2}\kappa^\varepsilon_\beta(z) - \kappa^\varepsilon_\alpha(z)\right) = \xi^\pm_{m,j_*,i_*,p+1}\left(\kappa^\varepsilon_\beta(z) - q^{\pm 2}\kappa^\varepsilon_\alpha(z)\right) \,, \tag{3.4.18}$$

where, as usual, we have set

$$\xi^\pm_{m,j_*,i_*,p} = \operatorname*{res}_z z^{p-1}\xi^\pm_{m,j_*,i_*}(z) \,.$$

Since $\xi^\pm_{m,j_*,i_*}(z) \neq 0$, there exists at least one $p \in \mathbb{Z}$ such that $\xi^\pm_{m,j_*,i_*,p} \neq 0$. Assuming that $\xi^\pm_{m,j_*,i_*,p+1} = 0$, one easily derives a contradiction from (3.4.18) and, repeating the argument, one proves that $\xi^\pm_{m,j_*,i_*,p} \neq 0$ for every $p \in \mathbb{Z}$. Dividing (3.4.18) by $\xi^\pm_{m,j_*,i_*,p}$, one gets

$$z\left(q^{\pm 2}\kappa^\varepsilon_\beta(z) - \kappa^\varepsilon_\alpha(z)\right) = a_p\left(\kappa^\varepsilon_\beta(z) - q^{\pm 2}\kappa^\varepsilon_\alpha(z)\right) \,,$$

where we have set, for every $p \in \mathbb{Z}$, $a_p = \xi^\pm_{m,j_*,i_*,p+1}/\xi^\pm_{m,j_*,i_*,p} \in \mathbb{F}^\times$. Since the l.h.s. of the above equation is independent of p, so it its r.h.s. and there must therefore exist an $a \in \mathbb{F}^\times$ such that $a_p = a$ for every $p \in \mathbb{Z}$, eventually yielding

$$z\left(q^{\pm 2}\kappa^\varepsilon_\beta(z) - \kappa^\varepsilon_\alpha(z)\right) = a\left(\kappa^\varepsilon_\beta(z) - q^{\pm 2}\kappa^\varepsilon_\alpha(z)\right) \,.$$

i. now follows. Moreover, we clearly have

$$\xi^\pm_{m,j_*,i_*}(z) = A^\pm_{m,j_*,i_*}\delta(z/a) \,,$$

for some $A^{\pm}_{m,j_*,i_*} \in \mathbb{F}^{\times}$. More generally, we claim that,

$$\forall j \in J\,, \forall i \in I(j)\,, \qquad \xi^{\pm}_{m,j,i}(z) = \sum_{p=0}^{N(i,j)} A^{\pm}_{m,j,i,p}\delta^{(p)}(z/a)\,, \tag{3.4.19}$$

for some $A^{\pm}_{m,j,i,p} \in \mathbb{F}$ and some $N(i,j) \in \mathbb{N}$. This is proven by a finite induction on j and i. Indeed, making use of (3.4.16), we can rewrite (3.4.17) as

$$(z_1 - q^{\pm 2}z_2)(z_2 - q^{\pm 2}a)\sum_{\substack{l\in J\\ l\geq j}} \eta^{\varepsilon}_{\beta,j,l}(z_2)\xi^{\pm}_{m,l,i}(z_1) = (z_1q^{\pm 2} - z_2)(q^{\pm 2}z_2 - a)\sum_{\substack{k\in I(j)\\ k\leq i}} \eta^{\varepsilon}_{\alpha,k,i}(z_2)\xi^{\pm}_{m,j,k}(z_1)\,, \tag{3.4.20}$$

for every $j \in J$ and every $i \in I(j)$. Now, assume that (3.4.19) holds for every pair in

$$\{(j,i) : j \in J,\, i \in I(j), \quad j > j_0\} \cup \{(j_0, i) : i \in I(j_0), \quad i \leq i_0\}\,,$$

for some $j_0 \in J$ and some $i_0 \in I(j_0)$ such that $i_0 < \max I(j_0)$. Let i_0' be the smallest element of $I(j_0)$ such that $i_0 < i_0'$. It suffices to write (3.4.20) for $j = j_0$ and $i = i_0'$, to get

$$\begin{aligned}(z_1 - a)z_2(1 - q^{\pm 4})\xi^{\pm}_{m,j_0,i_0'}(z_1) \quad &= \quad -(z_1 - q^{\pm 2}z_2)(z_2 - q^{\pm 2}a)\sum_{\substack{l\in J\\ l> j_0}} \eta^{\varepsilon}_{\beta,j_0,l}(z_2)\xi^{\pm}_{m,l,i_0'}(z_1)\\ &\quad +(z_1q^{\pm 2} - z_2)(q^{\pm 2}z_2 - a)\sum_{\substack{k\in I(j_0)\\ k\leq i_0}} \eta^{\varepsilon}_{\alpha,k,i_0'}(z_2)\xi^{\pm}_{m,j_0,k}(z_1)\,.\end{aligned} \tag{3.4.21}$$

Combining the recursion hypothesis and lemma 5.1.8 from the appendix, one easily concludes that (3.4.19) holds for the pair (j_0, i_0'). Repeating the argument finitely many times, we get that it actually holds for all the pairs in $\{(j,i) : j \in J,\, i \in I(j), \quad j \geq j_0\}$. Now, either $j_0 = \min J$ and we are done; or $j_0 > \min J$ and there exists a largest $j_0' \in J$ such that $j_0 > j_0'$. Writing (3.4.20) for $j = j_0'$ and $i = i_*(j_0')$, we get

$$(z_1 - a)z_2(1 - q^{\pm 4})\xi^{\pm}_{m,j_0',i_*(j_0')}(z_1) = -(z_1 - q^{\pm 2}z_2)(z_2 - q^{\pm 2}a)\sum_{\substack{l\in J\\ l\geq j_0'}} \eta^{\varepsilon}_{\beta,j_0',l}(z_2)\xi^{\pm}_{m,l,i_*(j_0')}(z_1)\,.$$

Combining again the recursion hypothesis and lemma 5.1.8, we easily get that (3.4.19) holds for $(j_0', i_*(j_0'))$. It is now clear that the claim holds for every $j \in J$ and every $i \in I(j)$. Letting

$$N = 1 + \max\{N(i,j) : j \in J,\, i \in I(j)\}\,,$$

ii. follows. Furthermore, for every $b \in \mathbb{F} - \{a\}$ and every $n \in \mathbb{N}$, we obviously have $(z-b)^n M_\alpha \cap \mathsf{X}^{\pm}_{1,m}(z).M_\beta \neq \{0\}$, thus making a the unique element of $\mathbb{F}$ satisfying *ii.*. ☐

Remark 3.4.25. Obviously, proposition 3.4.24.i holds for arbitrary pairs of (possibly non-local) ℓ-weight spaces since it must hold for at least one pair of local ℓ-weight spaces therein.

Corollary 3.4.26. *The ℓ-weights of any type $(1,0)$ weight-finite simple $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)$-module are all rational – see definition 3.4.5.*

Proof. Let M be a type $(1,0)$ weight-finite simple $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)$-module. By proposition 3.4.23, its highest t-

weight space M_0 is an ℓ-dominant simple $\ddot{\mathrm{U}}_q^0(\mathfrak{a}_1)$-module. Hence, $M \cong \mathcal{L}(M_0) \cong \ddot{\mathrm{U}}_q^-(\mathfrak{a}_1).M_0$ and it easily follows by proposition 3.4.24 that all the ℓ-weights of $\mathcal{L}(M_0)$ are of the form

$$\kappa_\alpha^\pm(z) \prod_{p=1}^{N} A_{a_p}^\pm(z)^{-1} ,$$

for some $N \in \mathbb{N}$, some $a_1, \dots, a_N \in \mathbb{F}^\times$ and

$$\kappa_\alpha^\pm(z) = -q^{\deg P_\alpha} \left(\frac{P_\alpha(q^{-2}/z)}{P_\alpha(1/z)} \right)_{|z|^{\pm 1} \ll 1} ,$$

for some monic polynomial $P_\alpha(1/z) \in \mathbb{F}[z^{-1}]$. Now, observe that

$$A_a^\pm(z)^{-1} = q^{-2} \left(\frac{1 - q^2 a/z}{1 - q^{-2} a/z} \right)_{|z|^{\pm 1} \ll 1} = q^{-1} \left(\frac{1 - q^2 a/z}{1 - a/z} \right)_{|z|^{\pm 1} \ll 1} q^{-1} \left(\frac{1 - a/z}{1 - q^{-2} a/z} \right)_{|z|^{\pm 1} \ll 1} .$$

Hence, all the ℓ-weights of $\mathcal{L}(M_0)$ are of the form

$$\kappa_\beta^\pm(z) = -q^{\deg(P_\beta) - \deg(Q_\beta)} \left(\frac{P_\beta(q^{-2}/z) Q_\beta(1/z)}{P_\beta(1/z) Q_\beta(q^{-2}/z)} \right)_{|z|^{\pm 1} \ll 1} , \tag{3.4.22}$$

for some relatively prime monic polynomials $P_\beta(1/z), Q_\beta(1/z) \in \mathbb{F}[z^{-1}]$, which concludes the proof. □

In view of remark 3.4.7, we can therefore associate with any weight-finite simple $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)$-module a q-character defined as the (formal) sum of the monomials corresponding to all its rational ℓ-weights.

Proposition 3.4.27. *Let M_0 and N_0 be two t-dominant simple $\ddot{\mathrm{U}}_q^0(\mathfrak{a}_1)$-modules such that $M_0 \widehat{\otimes} N_0$ be simple. Then:*

i. $M_0 \widehat{\otimes} N_0$ is a simple t-dominant $\ddot{\mathrm{U}}_q^0(\mathfrak{a}_1)$-module of type $(1,0)$;

ii. there exists a short exact sequence of $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)$-modules

$$\{0\} \to \mathcal{N} \to \mathcal{L}(M_0) \widehat{\otimes} \mathcal{L}(N_0) \to \mathcal{L}(M_0 \widehat{\otimes} N_0) \to \{0\} ;$$

iii. if, in addition, $\mathcal{L}(M_0) \widehat{\otimes} \mathcal{L}(N_0)$ is simple, then

$$\mathcal{L}(M_0) \widehat{\otimes} \mathcal{L}(N_0) \cong \mathcal{L}(M_0 \widehat{\otimes} N_0) .$$

Proof. Combining eqs. (3.3.19), (3.3.20), (3.3.21), (3.3.23), (3.3.24), (3.3.25) and (3.3.26), we easily prove that

$$\Delta^0(\mathsf{c}^\pm(z)) = \mathsf{c}^\pm(z \mathsf{C}_{(2)}^{\pm 1/2}) \otimes \mathsf{c}^\pm(z \mathsf{C}_{(1)}^{\mp 1/2}) , \tag{3.4.23}$$

$$\Delta^0(\mathsf{K}_{1,m}^+(z)) = -\sum_{k=0}^{m} \prod_{l=k+1}^{m} \mathsf{c}^-(z q^{-2l} \mathsf{C}_{(1)}^{1/2}) \mathsf{K}_{1,k}^+(z) \widehat{\otimes} \mathsf{K}_{1,m-k}^+(z q^{-2k} \mathsf{C}_{(1)}) , \tag{3.4.24}$$

$$\Delta^0(\mathsf{K}_{1,-m}^-(z)) = -\sum_{k=0}^{m} \mathsf{K}_{1,-(m-k)}^-(z q^{-2k} \mathsf{C}_{(2)}) \widehat{\otimes} \mathsf{K}_{1,-k}^-(z) \prod_{l=k+1}^{m} \mathsf{c}^+(z q^{-2l} \mathsf{C}_{(2)}^{1/2}) , \tag{3.4.25}$$

for every $m \in \mathbb{N}$ – the case $m = 0$ being just (3.3.23). Now M_0 and N_0 are both of type $(1,0)$ and (3.3.22) and (3.4.23) respectively imply that so is $M_0 \widehat{\otimes} N_0$. Similarly, they are both ℓ-weight and ℓ-dominant. It follows that, if $\{M_{0,\alpha} : \alpha \in A\}$ and $\{N_{0,\beta} : \beta \in B\}$ are the countable sets of ℓ-weights of M_0 and N_0 respectively, with respective Drinfel'd polynomials $\{P_\alpha : \alpha \in A\}$ and $\{P_\beta : \beta \in B\}$, then $\{M_{0,\alpha} \otimes N_{0,\beta} : \alpha \in A\,, \quad \beta \in B\}$ is the countable set of ℓ-weight spaces of $M_0 \widehat{\otimes} N_0$. Moreover, the latter is obviously ℓ-dominant since its Drinfel'd polynomials are in $\{P_\alpha P_\beta : \alpha \in A \quad \beta \in B\}$. Now let $\alpha, \alpha' \in A$, $\beta, \beta' \in B$ and let P_α, $P_{\alpha'}$, P_β and $P_{\beta'}$ be the Drinfel'd polynomials of $M_{0,\alpha}$, $M_{0,\alpha'}$, $N_{0,\beta}$ and $N_{0,\beta'}$ respectively and assume that

$$(M_{0,\alpha} \otimes N_{0,\beta}) \cap \Delta^0(\mathsf{K}^\pm_{1,\pm1}(z)).\left(M_{0,\alpha'} \otimes N_{0,\beta'}\right) \neq \{0\}\,. \tag{3.4.26}$$

Then, writing (3.4.24) and (3.4.25) above with $m = 1$, we get

$$\Delta^0(\mathsf{K}^+_{1,1}(z)) = -\mathsf{c}^-(zq^{-2}\mathsf{C}^{1/2}_{(1)})\mathsf{K}^+_{1,0}(z)\widehat{\otimes}\mathsf{K}^+_{1,1}(z\mathsf{C}_{(1)}) - \mathsf{K}^+_{1,1}(z)\widehat{\otimes}\mathsf{K}^+_{1,0}(zq^{-2}\mathsf{C}_{(1)})\,,$$

$$\Delta^0(\mathsf{K}^-_{1,-1}(z)) = -\mathsf{K}^-_{1,-1}(z\mathsf{C}_{(2)})\widehat{\otimes}\mathsf{K}^-_{1,0}(z)\mathsf{c}^+(zq^{-2}\mathsf{C}^{1/2}_{(2)}) - \mathsf{K}^-_{1,0}(zq^{-2}\mathsf{C}_{(2)})\widehat{\otimes}\mathsf{K}^-_{1,-1}(z)\,.$$

Since both $M_{0,\alpha'}$ and $N_{0,\beta'}$ are ℓ-weight spaces, it follows that

$$\Delta^0(\mathsf{K}^\pm_{1,\pm1}(z)).\left(M_{0,\alpha'} \otimes N_{0,\beta'}\right) \subseteq \left(\mathsf{K}^\pm_{1,\pm1}(z).M_{0,\alpha'} \otimes N_{0,\beta'}\right) \oplus \left(M_{0,\alpha'} \otimes \mathsf{K}^\pm_{1,\pm1}(z).N_{0,\beta'}\right)\,,$$

Therefore, condition (3.4.26) holds only if the direct sum on the r.h.s. above has a non-vanishing intersection with $M_{0,\alpha} \otimes N_{0,\beta}$. But since the latter is an ℓ-weight space, this happens only if either $M_{0,\alpha} \cap \mathsf{K}^\pm_{1,\pm1}(z).M_{0,\alpha'} \neq \{0\}$ or $N_{0,\beta} \cap \mathsf{K}^\pm_{1,\pm1}(z).N_{0,\beta'} \neq \{0\}$. The t-dominance of M_0 and N_0 implies that, for the only $a \in \mathbb{F}^\times$ such that $(z-a)^m M_{0,\alpha} \cap \mathsf{K}^\pm_{1,\pm1}(z).M_{0,\alpha'} = \{0\}$ for some $m \in \mathbb{N}^\times$, $P_{\alpha'}(1/a) = 0$; or, for the only $b \in \mathbb{F}^\times$ such that $(z-b)^n N_{0,\beta} \cap \mathsf{K}^\pm_{1,\pm1}(z).N_{0,\beta'} = \{0\}$ for some $n \in \mathbb{N}^\times$, $P_{\beta'}(1/b) = 0$. In any case, $P_{\alpha'}(1/a)P_{\beta'}(1/a) = 0$ or $P_{\alpha'}(1/b)P_{\beta'}(1/b) = 0$ and $M_0 \widehat{\otimes} N_0$ is t-dominant. i follows. By lemma 3.3.30, it is clear that $\dot{\Delta}(\mathsf{X}^+_{1,r}(z)).\left(M_0 \widehat{\otimes} N_0\right) = \{0\}$. Hence $M_0 \widehat{\otimes} N_0$ is a highest t-weight space in $\mathcal{L}(M_0)\widehat{\otimes}\mathcal{L}(N_0)$. Let $\mathcal{N}$ denote the largest closed $\widehat{\ddot{\mathrm{U}}'_q(\mathfrak{a}_1)}$-submodule of $\mathcal{L}(M_0)\widehat{\otimes}\mathcal{L}(N_0)$ such that $\mathcal{N} \cap \left(M_0 \widehat{\otimes} N_0\right) = \{0\}$. ii obviously follows. iii is clear. ☐

3.5 An evaluation homomorphism and evaluation modules

In this section, we construct an evaluation algebra $\widehat{\mathcal{A}}_t$ and an F-algebra homomorphism $\mathrm{ev} : \ddot{\mathrm{U}}_q(\mathfrak{a}_1) \to \widehat{\mathcal{A}}_t$, that we shall refer to as the evaluation homomorphism.

3.5.1 The quantum Heisenberg algebras $\mathcal{H}^+_t$ and $\mathcal{H}^-_t$

Definition 3.5.1. The *quantum Heisenberg algebra* $\mathcal{H}^\pm_t$ is the Hopf algebra generated over $\mathbb{K}(t)$ by

$$\left\{\gamma^{1/2}, \gamma^{-1/2}, \alpha_\pm, \alpha_\pm^{-1}, \alpha_{\pm,m} : m \in \mathbb{Z}^\times\right\}\,,$$

subject to the relations,

$$\gamma^{1/2}, \gamma^{-1/2}, \alpha_\pm, \alpha_\pm^{-1} \text{ are central,}$$

$$[\alpha_{\pm,-m}, \alpha_{\pm,n}] = -\frac{\delta_{m,n}}{m}[2m]_t \frac{\gamma^m - \gamma^{-m}}{t - t^{-1}}\,,$$

for every $m, n \in \mathbb{Z}^\times$, with comultiplication Δ defined by setting

$$\Delta(\gamma^{1/2}) = \gamma^{1/2} \otimes \gamma^{1/2}\,, \qquad \Delta(\gamma^{-1/2}) = \gamma^{-1/2} \otimes \gamma^{-1/2}\,,$$

$$\Delta(\alpha_\pm) = \alpha_\pm \otimes \alpha_\pm\,, \qquad \Delta(\alpha_\pm^{-1}) = \alpha_\pm^{-1} \otimes \alpha_\pm^{-1}\,,$$

$$\Delta(\alpha_{\pm,m}) = \alpha_{\pm,m} \otimes \gamma^{|m|/2} + \gamma^{-|m|/2} \otimes \alpha_{\pm,m}\,,$$

for every $m, n \in \mathbb{Z}^\times$, antipode S defined by setting

$$S(\gamma^{1/2}) = \gamma^{-1/2}\,, \qquad S(\gamma^{-1/2}) = \gamma^{1/2}\,,$$

$$S(\alpha_\pm) = \alpha_\pm^{-1}\,, \qquad S(\alpha_\pm^{-1}) = \alpha_\pm$$

$$S(\alpha_{\pm,m}) = -\alpha_{\pm,m}\,,$$

and counit ε defined by setting

$$\varepsilon(\gamma^{1/2}) = \varepsilon(\gamma^{-1/2}) = \varepsilon(\alpha_\pm) = \varepsilon(\alpha_\pm^{-1}) = \varepsilon(1) = 1\,,$$

$$\varepsilon(\alpha_{\pm,m}) = 0\,.$$

Definition 3.5.2. In $\mathcal{H}_t^+$, we let

$$\mathbf{L}^+(z) = 1 + \sum_{m \in \mathbb{N}^\times} L^+_{-m} z^m = \exp\left[-(t - t^{-1}) \sum_{m \in \mathbb{N}^\times} \alpha_{+,-m}(t^2 z)^m\right]\,,$$

$$\mathbf{R}^+(z) = \alpha_+ \left(1 + \sum_{m \in \mathbb{N}^\times} R^+_m z^{-m}\right) = \alpha_+ \exp\left[(t - t^{-1}) \sum_{m \in \mathbb{N}^\times} \alpha_{+,m}(t^{-2} z)^{-m}\right]\,.$$

Similarly, in $\mathcal{H}_t^-$, we let

$$\mathbf{L}^-(z) = \alpha_- \left(1 + \sum_{m \in \mathbb{N}^\times} L^-_{-m} z^m\right) = \alpha_- \exp\left[-(t - t^{-1}) \sum_{m \in \mathbb{N}^\times} \alpha_{-,-m}(t^{-2} z)^m\right]\,,$$

$$\mathbf{R}^-(z) = 1 + \sum_{m \in \mathbb{N}^\times} R^-_m z^{-m} = \exp\left[(t - t^{-1}) \sum_{m \in \mathbb{N}^\times} \alpha_{-,m}(t^2 z)^{-m}\right]\,.$$

Then, we have the following equivalent presentation of $\mathcal{H}_t^\pm$.

Proposition 3.5.3. *$\mathcal{H}_t^\pm$ is the Hopf algebra generated over $\mathbb{K}(t)$ by*

$$\{\gamma^{1/2}, \gamma^{-1/2}, L^\pm_{-m}, R^\pm_m : m \in \mathbb{N}\}$$

subject to the relations

$$[\mathbf{L}^\pm(v), \mathbf{L}^\pm(z)] = [\mathbf{R}^\pm(v), \mathbf{R}^\pm(z)] = 0\,,$$

$$\mathbf{R}^{\pm}(v)\mathbf{L}^{\pm}(z) = \theta^{\pm}(z/v)\mathbf{L}^{\pm}(z)\mathbf{R}^{\pm}(v)\,,$$

where we have defined $\theta^{\pm}(z) \in \mathcal{Z}(\mathcal{H}_t)[[z]]$*, by setting*

$$\theta^{\pm}(z) = \left(\frac{(1-t^{2\pm 4}\gamma z)(1-t^{\pm 4-2}\gamma^{-1}z)}{(1-t^{\pm 4-2}\gamma z)(1-t^{2\pm 4}\gamma^{-1}z)}\right)_{|z|\ll 1}\,.$$

Furthermore, we have

$$\Delta(\mathbf{L}^{\pm}(z)) = \mathbf{L}^{\pm}(z\gamma_{(2)}^{1/2}) \otimes \mathbf{L}^{\pm}(z\gamma_{(1)}^{-1/2})\,,$$

$$\Delta(\mathbf{R}^{\pm}(z)) = \mathbf{R}^{\pm}(z\gamma_{(2)}^{-1/2}) \otimes \mathbf{R}^{\pm}(z\gamma_{(1)}^{1/2})\,,$$

where, by definition,

$$\gamma_{(1)}^{1/2} = \gamma^{1/2} \otimes 1\,, \qquad \gamma_{(1)}^{-1/2} = \gamma^{-1/2} \otimes 1\,, \qquad \gamma_{(2)}^{1/2} = 1 \otimes \gamma^{1/2}\,, \qquad \gamma_{(2)}^{-1/2} = 1 \otimes \gamma^{-1/2}$$

and

$$S(\mathbf{L}^{\pm}(z)) = \mathbf{L}^{\pm}(z)^{-1}\,, \qquad S(\mathbf{R}^{\pm}(z)) = \mathbf{R}^{\pm}(z)^{-1}\,.$$

Finally, $\varepsilon(\mathbf{L}^{\pm}(z)) = \varepsilon(\mathbf{R}^{\pm}(z)) = 1$.

Proof. This is an easy consequence of the definition of $\mathcal{H}_t^{\pm}$. □

Remark 3.5.4. Observe that $\theta^{+}(z)$ and $\theta^{-}(z)$ are not independent and that we actually have $\theta^{-}(z) = \theta^{+}(t^{-8}z)$.

3.5.2 A PBW basis for $\mathcal{H}_t^{\pm}$

For every $n \in \mathbb{N}^{\times}$, we let $\Lambda_n := \{\lambda = (\lambda_1, \dots, \lambda_n) \in (\mathbb{N}^{\times})^n : \lambda_1 \geq \cdots \geq \lambda_n\}$ denote the set of n-partitions. We adopt the convention that $\Lambda_0 = \{\emptyset\}$ reduces to the empty partition and we let $\Lambda = \bigcup_{n\in\mathbb{N}} \Lambda_n$ be the set of all partitions.

Proposition 3.5.5. *Define, for every* $\lambda \in \Lambda$,

$$L_{\lambda}^{\pm} = L_{-\lambda_1}^{\pm} \cdots L_{-\lambda_n}^{\pm}\,, \tag{3.5.1}$$

$$R_{\lambda}^{\pm} = R_{\lambda_1}^{\pm} \cdots R_{\lambda_n}^{\pm}\,, \tag{3.5.2}$$

with the convention that $L_{\emptyset}^{\pm} = R_{\emptyset}^{\pm} = 1$*. Then,*

$$\left\{\Phi_{\lambda,\mu}^{\pm} = L_{\lambda}^{\pm} R_{\mu}^{\pm} : \lambda, \mu \in \Lambda\right\} \tag{3.5.3}$$

is a $\mathbb{K}(t)[\gamma^{1/2}, \gamma^{-1/2}]$*-basis for* $\mathcal{H}_t^{\pm}$*.*

Proof. The relations in $\mathcal{H}_t^{\pm}$ read, for every $m, n \in \mathbb{N}$,

$$[L_{-m}^{\pm}, L_{-n}^{\pm}] = [R_m^{\pm}, R_n^{\pm}] = 0\,,$$

$$R_m^{\pm} L_{-n}^{\pm} = L_{-n}^{\pm} R_m^{\pm} + \sum_{p=1}^{\min(m,n)} \theta_p^{\pm} L_{p-n}^{\pm} R_{m-p}^{\pm}\,,$$

where, for every $p \in \mathbb{N}$, $\theta_p^\pm \in \mathbb{K}(t)[\gamma^{1/2}, \gamma^{-1/2}]$ can be obtained from

$$\theta^\pm(z) = 1 + \sum_{p \in \mathbb{N}^\times} \theta_p^\pm z^p \,.$$

It is clear that any monomial in $\{L_{-m}^\pm, R_m^\pm : m \in \mathbb{N}\}$ can therefore be rewritten as a linear combination with coefficients in $\mathbb{K}(t)[\gamma^{1/2}, \gamma^{-1/2}]$ of elements in $\{\phi_{\lambda,\mu}^\pm : \lambda, \mu \in \Lambda\}$. The independence of the latter is clear. □

A convenient way to encode the above basis elements is through $\mathcal{H}_t^\pm$-valued symmetric formal distributions. Let indeed, for every $n^+, n^-, m^+, m^- \in \mathbb{N}$, every $n^\pm$-tuple $\boldsymbol{z}^\pm = (z_1^\pm, \dots, z_{n^\pm}^\pm)$ and every $m^\pm$-tuple $\boldsymbol{\zeta}^\pm = (\zeta_1^\pm, \dots, \zeta_{m^\pm}^\pm)$ of formal variables,

$$\Phi^\pm(\boldsymbol{z}^\pm, \boldsymbol{\zeta}^\pm) = \mathbf{L}^\pm(\boldsymbol{z}^\pm)\mathbf{R}^\pm(\boldsymbol{\zeta}^\pm) \,,$$

where we have set

$$\mathbf{L}^\pm(\boldsymbol{z}^\pm) = \prod_{p=1}^{n^\pm} \mathbf{L}^\pm(z_p^\pm) \,,$$

$$\mathbf{R}^\pm(\boldsymbol{\zeta}^\pm) = \prod_{p=1}^{m^\pm} \mathbf{R}^\pm(\zeta_p^\pm) \,,$$

with the convention that if $n^\pm$ (resp. $m^\pm = 0$), then $\mathbf{L}^\pm(\emptyset) = 1$ (resp. $\mathbf{R}^\pm(\emptyset) = 1$). It turns out that

$$\Phi^\pm(\boldsymbol{z}^\pm, \boldsymbol{\zeta}^\pm) \in \mathcal{H}_t^\pm[[\boldsymbol{z}^\pm, (\boldsymbol{\zeta}^\pm)^{-1}]]^{S_{n^\pm} \times S_{m^\pm}} \,.$$

Indeed, owing to the commutation relations in $\mathcal{H}_t^\pm$, the formal distribution $\Phi^\pm(\boldsymbol{z}^\pm, \boldsymbol{\zeta}^\pm)$ is symmetric in each of its argument tuples, $\boldsymbol{z}^\pm$ and $\boldsymbol{\zeta}^\pm$ respectively; i.e. it is invariant under the natural action of $S_{n^\pm} \times S_{m^\pm}$ on its arguments. It is also clear that, for every $\lambda^\pm \in \Lambda_{n^\pm}$ and $\mu^\pm \in \Lambda_{m^\pm}$,

$$\Phi_{\lambda^\pm, \mu^\pm}^\pm = \operatorname*{res}_{\boldsymbol{z}^\pm, \boldsymbol{\zeta}^\pm} (\boldsymbol{z}^\pm)^{-1-\lambda^\pm} (\boldsymbol{\zeta}^\pm)^{-1+\mu^\pm} \Phi^\pm(\boldsymbol{z}^\pm, \boldsymbol{\zeta}^\pm) \,,$$

where we have set

$$(\boldsymbol{z}^\pm)^{-1-\lambda^\pm} = \prod_{p=1}^{n^\pm} (z_p^\pm)^{-1-\lambda_p^\pm} \qquad \text{and} \qquad (\boldsymbol{\zeta}^\pm)^{-1+\mu^\pm} = \prod_{p=1}^{m^\pm} (\zeta_p^\pm)^{-1+\mu_p^\pm} \,.$$

3.5.3 The dressing factors $\mathbf{L}_m^\pm(z)$ and $\mathbf{R}_m^\pm(z)$

Definition 3.5.6. For every $m \in \mathbb{Z}^\times$, we let

$$\mathbf{L}_m^\pm(z) = \prod_{p=1}^{|m|} \mathbf{L}^\pm(z t^{\pm 2(1-2p)\mathrm{sign}(m)+2})^{\pm \mathrm{sign}(m)} \tag{3.5.4}$$

$$\mathbf{R}_m^\pm(z) = \prod_{p=1}^{|m|} \mathbf{R}^\pm(z t^{\pm 2(1-2p)\mathrm{sign}(m)+2})^{\pm \mathrm{sign}(m)} \tag{3.5.5}$$

It easily follows that

Proposition 3.5.7. *In $\mathcal{H}_t^{\pm}$, for every $m, n \in \mathbb{Z}^{\times}$, we have*

$$[\mathbf{L}_m^{\pm}(v), \mathbf{L}_n^{\pm}(z)] = [\mathbf{R}_m^{\pm}(v), \mathbf{R}_n^{\pm}(z)] = 0\,,$$

$$\mathbf{R}_m^{\pm}(v)\mathbf{L}_n^{\pm}(z) = \theta_{m,n}^{\pm}(z/v)\mathbf{L}_n^{\pm}(z)\mathbf{R}_m^{\pm}(v)\,,$$

where we have set

$$\theta_{m,n}^{\pm}(z) = \prod_{r=1}^{|m|}\prod_{s=1}^{|n|} \theta^{\pm}(zt^{\pm 2(1-2s)\mathrm{sign}(n) \mp 2(1-2r)\mathrm{sign}(m)})^{\mathrm{sign}(mn)}\,.$$

Furthermore, we have, for every $m \in \mathbb{Z}^{\times}$,

$$\Delta(\mathbf{L}_m^{\pm}(z)) = \mathbf{L}_m^{\pm}(z\gamma_{(2)}^{1/2}) \otimes \mathbf{L}_m^{\pm}(z\gamma_{(1)}^{-1/2})\,,$$

$$\Delta(\mathbf{R}_m^{\pm}(z)) = \mathbf{R}_m^{\pm}(z\gamma_{(2)}^{-1/2}) \otimes \mathbf{R}_m^{\pm}(z\gamma_{(1)}^{1/2})\,.$$

It is worth emphasizing that the $\mathbf{L}_m^{\pm}(z)$ are not indepedent for all values of $m \in \mathbb{Z}^{\times}$ and that neither are the $\mathbf{R}_m^{\pm}(z)$. Indeed, we have

Lemma 3.5.8. *For every $m, n \in \mathbb{Z}^{\times}$,*

$$\mathbf{L}_{-m}^{\pm}(z)^{-1} = \mathbf{L}_m^{\pm}(zt^{\pm 4m}) \tag{3.5.6}$$

$$\mathbf{R}_{-m}^{\pm}(z)^{-1} = \mathbf{R}_m^{\pm}(zt^{\pm 4m}) \tag{3.5.7}$$

$$\mathbf{L}_m^{\pm}(zt^{\pm 4m})\mathbf{L}_n^{\pm}(z) = \mathbf{L}_{m+n}^{\pm}(zt^{\pm 4m}) \tag{3.5.8}$$

$$\mathbf{R}_m^{\pm}(zt^{\pm 4m})\mathbf{R}_n^{\pm}(z) = \mathbf{R}_{m+n}^{\pm}(zt^{\pm 4m}) \tag{3.5.9}$$

3.5.4 The algebra $\mathcal{B}_t$

Remember the Hopf algebra $\breve{\mathrm{U}}_q(\mathrm{L}\mathfrak{a}_1)$ from definition 3.2.1. It is naturally $\mathbb{Z}$-graded and we can endow it with a topology following what was done for $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)$ in section 3.3.2. Let $\widehat{\breve{\mathrm{U}}_q(\mathrm{L}\mathfrak{a}_1)}$ denote the corresponding completion. Then, we have

Definition 3.5.9. We endow the topological $\mathbb{F}$-algebra $\widehat{\breve{\mathrm{U}}_q(\mathrm{L}\mathfrak{a}_1)}$ with:

i. the comultiplication $\Delta : \widehat{\breve{\mathrm{U}}_q(\mathrm{L}\mathfrak{a}_1)} \to \breve{\mathrm{U}}_q(\mathrm{L}\mathfrak{a}_1)\widehat{\otimes}\breve{\mathrm{U}}_q(\mathrm{L}\mathfrak{a}_1)$ defined by

$$\Delta(\mathbf{k}_1^{\pm}(z)) = \mathbf{k}_1^{\pm}(z) \otimes \mathbf{k}_1^{\pm}(z)\,, \tag{3.5.10}$$

$$\Delta(\mathbf{x}_1^{+}(z)) = \mathbf{x}_1^{+}(z) \otimes 1 + \mathbf{k}_1^{-}(z)\widehat{\otimes}\mathbf{x}_1^{+}(z)\,, \tag{3.5.11}$$

$$\Delta(\mathbf{x}_1^{-}(z)) = \mathbf{x}_1^{-}(z)\widehat{\otimes}\mathbf{k}_1^{+}(z) + 1 \otimes \mathbf{x}_1^{-}(z)\,, \tag{3.5.12}$$

ii. the counit $\varepsilon : \widehat{\breve{\mathrm{U}}_q(\mathrm{L}\mathfrak{a}_1)} \to \mathbb{F}$, defined by $\varepsilon(\mathbf{k}_1^{\pm}(z)) = 1$, $\varepsilon(\mathbf{x}_1^{\pm}(z)) = 0$ and;

iii. the antipode $S : \widehat{\breve{\mathrm{U}}_q(\mathrm{L}\mathfrak{a}_1)} \to \widehat{\breve{\mathrm{U}}_q(\mathrm{L}\mathfrak{a}_1)}$, defined by

$$S(\mathbf{k}_1^{\pm}(z)) = \mathbf{k}_1^{\pm}(z)^{-1}\,, \quad S(\mathbf{x}_1^{+}(z)) = -\mathbf{k}_1^{-}(z)^{-1}\mathbf{x}_1^{+}(z)\,, \quad S(\mathbf{x}_1^{-}(z)) = -\mathbf{x}_1^{-}(z)\mathbf{k}_1^{+}(z)^{-1}\,.$$

With the operations so defined, $\widehat{\breve{\mathrm{U}}_q(\mathrm{L}\mathfrak{a}_1)}$ becomes a topological Hopf algebra that we shall denote simply $\breve{\mathrm{U}}_q(\mathrm{L}\mathfrak{a}_1)$. It has an invertible antipode and we denote by $\breve{\mathrm{U}}_q(\mathrm{L}\mathfrak{a}_1)^{\mathrm{cop}}$ its coopposite topological Hopf algebra.

Proposition 3.5.10. *The quantum Heisenberg algebra $\mathcal{H}_t^+$ (resp. $\mathcal{H}_t^-$) is a left $\breve{\mathrm{U}}_{t^2}(\mathrm{L}\mathfrak{a}_1)$-module algebra (resp. a left $\breve{\mathrm{U}}_{t^2}(\mathrm{L}\mathfrak{a}_1)^{\mathrm{cop}}$-module algebra) with*

$$\mathbf{k}_1^{\varepsilon}(v) \triangleright \gamma^{1/2} = \mathbf{k}_1^{\varepsilon}(v) \triangleright \gamma^{-1/2} = 0\,,$$

$$\mathbf{k}_1^{\varepsilon}(v) \triangleright \mathbf{L}^{\pm}(z) = \lambda^{\varepsilon,\pm}(v,z)\mathbf{L}^{\pm}(z)\,, \quad \mathbf{k}_1^{\varepsilon}(v) \triangleright \mathbf{R}^{\pm}(z) = \rho^{\varepsilon,\pm}(v,z)\mathbf{R}^{\pm}(z)\,,$$

$$\mathbf{x}_1^{\varepsilon}(v) \triangleright \gamma^{1/2} = \mathbf{x}_1^{\varepsilon}(v) \triangleright \gamma^{-1/2} = \mathbf{x}_1^{\varepsilon}(v) \triangleright \mathbf{L}^{\pm}(z) = \mathbf{x}_1^{\varepsilon}(v) \triangleright \mathbf{R}^{\pm}(z) = 0\,,$$

for $\varepsilon \in \{-,+\}$ and where we have set

$$\lambda^{\varepsilon,\pm}(v,z) = \left(\frac{t^{2\mp 2}v - t^{-2\pm 2}z}{v - t^{\pm 4}z}\right)_{|z/v|^{\varepsilon 1} \ll 1} \qquad \text{and} \qquad \rho^{\varepsilon,\pm}(v,z) = \left(\frac{t^{\pm 4}v - z}{t^{2\pm 2}v - t^{-(2\pm 2)}z}\right)_{|z/v|^{\varepsilon 1} \ll 1}.$$

Proof. One readily checks the compatibility with the defining relations of $\mathcal{H}_t^{\pm}$ and $\breve{\mathrm{U}}_{t^2}(\mathrm{L}\mathfrak{a}_1)$. □

Proposition 3.5.11. *For every $m \in \mathbb{Z}^{\times}$ and every $\varepsilon \in \{-,+\}$, we have*

$$\mathbf{k}_1^{\varepsilon}(v) \triangleright \mathbf{L}_m^{\pm}(z) = \lambda_m^{\varepsilon,\pm}(v,z)\mathbf{L}_m^{\pm}(z)\,, \qquad \mathbf{k}_1^{\varepsilon}(v) \triangleright \mathbf{R}_m^{\pm}(z) = \rho_m^{\varepsilon,\pm}(v,z)\mathbf{R}_m^{\pm}(z)\,,$$

$$\mathbf{x}_1^{\varepsilon}(v) \triangleright \mathbf{L}_m^{\pm}(z) = \mathbf{x}_1^{\varepsilon}(v) \triangleright \mathbf{R}_m^{\pm}(z) = 0\,,$$

where we have set

$$\lambda_m^{\varepsilon,\pm}(v,z) = \left(\frac{t^{-2(1\mp 1)m}v - t^{\pm 4 - 2(1\pm 1)m}z}{v - t^{\pm 4}z}\right)_{|z/v|^{\varepsilon 1} \ll 1}$$

and

$$\rho_m^{\varepsilon,\pm}(v,z) = \left(\frac{t^{\pm 4}v - z}{t^{\pm 4 - 2(1\mp 1)m}v - t^{-2(1\pm 1)m}z}\right)_{|z/v|^{\varepsilon 1} \ll 1}.$$

Proof. This is readily checked making use of definition 3.5.6, of the Hopf algebraic structures of $\breve{\mathrm{U}}_{t^2}(\mathrm{L}\mathfrak{a}_1)$ and $\breve{\mathrm{U}}_{t^2}(\mathrm{L}\mathfrak{a}_1)^{\mathrm{cop}}$, of the $\breve{\mathrm{U}}_{t^2}(\mathrm{L}\mathfrak{a}_1)$-module algebra structures of $\mathcal{H}_t^+$ and of the $\breve{\mathrm{U}}_{t^2}(\mathrm{L}\mathfrak{a}_1)^{\mathrm{cop}}$-module algebra structure of $\mathcal{H}_t^-$. □

Definition-Proposition 3.5.12. We denote by $\mathcal{H}_t^+ \rtimes \breve{\mathrm{U}}_{t^2}(\mathrm{L}\mathfrak{a}_1) \overset{\mathrm{cop}}{\ltimes} \mathcal{H}_t^-$ the associative $\mathbb{F}$-algebra obtained by endowing $\mathcal{H}_t^+ \otimes \breve{\mathrm{U}}_{t^2}(\mathrm{L}\mathfrak{a}_1) \otimes \mathcal{H}_t^-$ with the multiplication given by setting, for every $h_+, h'_+ \in \mathcal{H}_t^+$, every $h_-, h'_- \in \mathcal{H}_t^-$ and every $x, x' \in \breve{\mathrm{U}}_{t^2}(\mathrm{L}\mathfrak{a}_1)$,

$$(h_+ \otimes x \otimes h_-)\,.\,(h'_+ \otimes x' \otimes h'_-) = \sum h_+\left(x_{(1)} \triangleright h'_+\right) \otimes x_{(2)}x' \otimes h_-\left(x_{(3)} \triangleright h'_-\right)\,,$$

– see proposition 3.5.11 for the definition of the $\breve{\mathrm{U}}_{t^2}(\mathrm{L}\mathfrak{a}_1)$-module algebra structure of $\mathcal{H}_t^+$ and of the $\breve{\mathrm{U}}_{t^2}(\mathrm{L}\mathfrak{a}_1)^{\mathrm{cop}}$-module algebra structure of $\mathcal{H}_t^-$. In that algebra, $\{\gamma^{1/2} - t, \gamma^{-1/2} - t^{-1}\}$ generates a left ideal.

The latter is actually a two-sided ideal since $\gamma^{\pm 1/2}$ is central and, denoting it by $(\gamma^{1/2}-t)$, we can set $\breve{\mathcal{B}}_t = \mathcal{H}_t^+ \rtimes \breve{\mathrm{U}}_{t^2}(\mathrm{L}\mathfrak{a}_1) \overset{\mathrm{cop}}{\ltimes} \mathcal{H}_t^- / (\gamma^{1/2}-t)$.

Proof. Making use of the coassociativity of the comultiplication Δ, it is very easy to prove that, with the above defined multiplication, $\mathcal{H}_t^+ \rtimes \breve{\mathrm{U}}_{t^2}(\mathrm{L}\mathfrak{a}_1) \overset{\mathrm{cop}}{\ltimes} \mathcal{H}_t^-$ is actually an associative $\mathbb{F}$-algebra. $\square$

Proposition 3.5.13. *Setting $x \mapsto 1 \otimes x \otimes 1$, for every $x \in \breve{\mathrm{U}}_{t^2}(\mathrm{L}\mathfrak{a}_1)$, defines a unique injective $\mathbb{K}(t)$-algebra homomorphism $\breve{\mathrm{U}}_{t^2}(\mathrm{L}\mathfrak{a}_1) \hookrightarrow \breve{\mathcal{B}}_t$. Similarly, $h \mapsto h \otimes 1 \otimes 1$ and $h \mapsto 1 \otimes 1 \otimes h$ define unique injective $\mathbb{K}(t)$-algebra homomorphisms $\mathcal{H}_t^+ \hookrightarrow \breve{\mathcal{B}}_t$ and $\mathcal{H}_t^- \hookrightarrow \breve{\mathcal{B}}_t$ respectively.*

Remark 3.5.14. We shall subsequently identify $\breve{\mathrm{U}}_{t^2}(\mathrm{L}\mathfrak{a}_1)$, $\mathcal{H}_t^+$ and $\mathcal{H}_t^-$ with their respective images in $\breve{\mathcal{B}}_t$ under the injective algebra homomorphisms of the above proposition.

Proposition 3.5.15. *In $\breve{\mathcal{B}}_t$, for every $m \in \mathbb{Z}^\times$ and every $\varepsilon \in \{-,+\}$, we have the following relations*

$$(v - t^{\pm 4}z)(v - t^{\pm 4(1+m-n)}z)\mathbf{R}_m^\pm(v)\mathbf{L}_n^\pm(z) = (v - t^{\pm 4(1-n)}z)(v - t^{\pm 4(1+m)}z)\mathbf{L}_n^\pm(z)\mathbf{R}_m^\pm(v)\,, \tag{3.5.13}$$

$$(zt^{\pm 4} - v)\mathbf{k}_1^\varepsilon(v)\mathbf{L}_m^\pm(z) = (zt^{\pm 4 - 2(1\pm 1)m} - vt^{-2(1\mp 1)m})\mathbf{L}_m^\pm(z)\mathbf{k}_1^\varepsilon(v)\,, \tag{3.5.14}$$

$$(zt^{\pm 4} - v)\mathbf{x}_1^\pm(v)\mathbf{L}_m^\pm(z) = (zt^{\pm 4 - 2(1\pm 1)m} - vt^{-2(1\mp 1)m})\mathbf{L}_m^\pm(z)\mathbf{x}_1^\pm(v)\,, \tag{3.5.15}$$

$$\mathbf{x}_1^\pm(v)\mathbf{L}_m^\mp(z) = \mathbf{L}_m^\mp(z)\mathbf{x}_1^\pm(v)\,, \tag{3.5.16}$$

$$(zt^{-2(1\pm 1)m} - vt^{\pm 4 - 2(1\mp 1)m})\mathbf{k}_1^\varepsilon(v)\mathbf{R}_m^\pm(z) = (z - vt^{\pm 4})\mathbf{R}_m^\pm(z)\mathbf{k}_1^\varepsilon(v)\,, \tag{3.5.17}$$

$$(zt^{-2(1\pm 1)m} - vt^{\pm 4 - 2(1\mp 1)m})\mathbf{x}_1^\pm(v)\mathbf{R}_m^\pm(z) = (z - vt^{\pm 4})\mathbf{R}_m^\pm(z)\mathbf{x}_1^\pm(v)\,, \tag{3.5.18}$$

$$\mathbf{x}_1^\pm(v)\mathbf{R}_m^\mp(z) = \mathbf{R}_m^\mp(z)\mathbf{x}_1^\pm(v)\,, \tag{3.5.19}$$

Proof. In order to prove (3.5.13), it suffices to check that

$$\theta^\pm(z) = \left(\frac{(1-z)(1-t^{\pm 8}z)}{(1-t^{\pm 4}z)^2}\right)_{|z|\ll 1} \mod (\gamma^{1/2}-t)$$

and that subsequently, for every $m, n \in \mathbb{Z}^\times$,

$$\theta_{m,n}^\pm(z) = \left(\frac{(1-t^{\pm 4(1-n)}z)(1-t^{\pm 4(1+m)}z)}{(1-t^{\pm 4}z)(1-t^{\pm 4(1+m-n)}z)}\right)_{|z|\ll 1} \mod (\gamma^{1/2}-t)\,.$$

As for the equations (3.5.14 – 3.5.19), they immediately follow from the definitions of $\mathcal{H}_t^+ \rtimes \breve{\mathrm{U}}_{t^2}(\mathrm{L}\mathfrak{a}_1) \overset{\mathrm{cop}}{\ltimes} \mathcal{H}_t^-$ and of the actions $\triangleright$ of $\breve{\mathrm{U}}_{t^2}(\mathrm{L}\mathfrak{a}_1)$ on $\mathcal{H}_t^+$ and $\mathcal{H}_t^-$ – see proposition 3.5.11. E.g., we have, by definition,

$$\begin{aligned}
\mathbf{x}_1^+(v)\mathbf{L}_m^+(z) &= \big(1 \otimes \mathbf{x}_1^+(v) \otimes 1\big)\big(\mathbf{L}_m^+(z) \otimes 1 \otimes 1\big) = \sum \big(\mathbf{x}_1^+(v)_{(1)} \triangleright \mathbf{L}_m^+(z)\big) \otimes \mathbf{x}_1^+(v)_{(2)} \otimes \big(\mathbf{x}_1^+(v)_{(3)} \triangleright 1\big)\\
&= \big(\mathbf{x}_1^+(v) \triangleright \mathbf{L}_m^+(z)\big) \otimes 1 \otimes 1 + \big(\mathbf{k}_1^-(v) \triangleright \mathbf{L}_m^+(z)\big) \otimes \mathbf{x}_1^+(v) \otimes 1\\
&\quad + \big(\mathbf{k}_1^-(v) \triangleright \mathbf{L}_m^+(z)\big) \otimes \mathbf{k}_1^-(v) \otimes \varepsilon(\mathbf{x}_1^+(v))1\\
&= \lambda_m^+(v,z)\mathbf{L}_m^+(z)\mathbf{x}_1^+(v)\,,
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{x}_1^+(v)\mathbf{L}_m^-(z) &= \left(1\otimes\mathbf{x}_1^+(v)\otimes 1\right)\left(1\otimes 1\otimes\mathbf{L}_m^-(z)\right) = \sum\left(\mathbf{x}_1^+(v)_{(1)}\triangleright 1\right)\otimes\mathbf{x}_1^+(v)_{(2)}\otimes\left(\mathbf{x}_1^+(v)_{(3)}\triangleright\mathbf{L}_m^-(z)\right)\\
&= \varepsilon(\mathbf{x}_1^+(v))1\otimes 1\otimes\left(1\triangleright\mathbf{L}_m^-(z)\right)+1\otimes\mathbf{x}_1^+(v)\otimes\left(1\triangleright\mathbf{L}_m^-(z)\right)+1\otimes\left(\mathbf{x}_1^+(v)\triangleright\mathbf{L}_m^-(z)\right)\otimes\mathbf{k}_1^-(v)\\
&= \mathbf{L}_m^-(z)\mathbf{x}_1^+(v)\,,
\end{aligned}$$

as claimed. □

Remark 3.5.16. In addition to the above, we obviously have in $\breve{\mathcal{B}}_t$, all the relations of its subalgebra $\breve{\mathrm{U}}_{t^2}(\mathrm{L}\mathfrak{a}_1)$ and all the relations of its subalgebras $\mathcal{H}_t^+$ and $\mathcal{H}_t^-$ modulo $(\gamma^{1/2}-t)$.

Definition-Proposition 3.5.17. Let $\mathcal{I}$ be the left ideal of $\breve{\mathcal{B}}_t$ generated by

$$\left\{\operatorname*{res}_{z_1,z_2} z_1^{-1+m}z_2^{-1+n}\left([\mathbf{x}_1^+(z_1),\mathbf{x}_1^-(z_2)]-\frac{1}{t^2-t^{-2}}\delta\left(\frac{z_1}{z_2}\right)\left[\mathbf{k}_1^+(z_1)-\mathbf{k}_1^-(z_1)\right]\right):m,n\in\mathbb{Z}\right\}.$$

Then $\mathcal{I}.\breve{\mathcal{B}}_t\subseteq\mathcal{I}$ and $\mathcal{I}$ is a two-sided ideal of $\breve{\mathcal{B}}_t$. Set $\mathcal{B}_t=\breve{\mathcal{B}}_t/\mathcal{I}$.

Proof. In order to prove that $\mathcal{I}.\breve{\mathcal{B}}_t\subseteq\mathcal{I}$, it suffices to prove that, for any $x\in\breve{\mathcal{B}}_t$,

$$\left([\mathbf{x}_1^+(z_1),\mathbf{x}_1^-(z_2)]-\frac{1}{t^2-t^{-2}}\delta\left(\frac{z_1}{z_2}\right)\left[\mathbf{k}_1^+(z_1)-\mathbf{k}_1^-(z_1)\right]\right)x\in\mathcal{I}.$$

The latter easily follows by inspection, making use of the relevant relations in $\breve{\mathcal{B}}_t$ and $\breve{\mathrm{U}}_{t^2}(\mathrm{L}\mathfrak{a}_1)$, namely (3.5.14 - 3.5.19) and (3.2.3 - 3.2.7). □

Remark 3.5.18. Thus, in addition to the relations in $\breve{\mathcal{B}}_t$, we have, in $\mathcal{B}_t$,

$$[\mathbf{x}_1^+(z_1),\mathbf{x}_1^-(z_2)]=\frac{1}{t^2-t^{-2}}\delta\left(\frac{z_1}{z_2}\right)\left[\mathbf{k}_1^+(z_1)-\mathbf{k}_1^-(z_1)\right].$$

3.5.5 The completion $\widehat{\mathcal{B}}_t$ of $\mathcal{B}_t$

Making use of its natural $\mathbb{Z}$-grading, we endow $\mathcal{B}_t$ with a topology, in the same way as we endowed $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)$ with its topology in section 3.3. We denote by $\widehat{\mathcal{B}}_t$ the corresponding completion. Consequently, its subalgebra $\mathcal{H}_t^\pm$ inherits a topology and we denote by $\widehat{\mathcal{H}}_t^\pm$ its corresponding completion in that topology.

3.5.6 The shift factors

Definition 3.5.19. In $\widehat{\mathcal{H}}_t^\pm$, we define,

$$\mathbf{H}^\pm(z)=\mathbf{L}^\pm(z)\mathbf{R}^\pm(z)\,.$$

Similarly, for every $m\in\mathbb{Z}^\times$, we let

$$\mathbf{H}_m^\pm(z)=\overrightarrow{\prod_{p\in[\![|m|]\!]}}\mathbf{H}^\pm(zt^{\pm 2(1-2p)\mathrm{sign}(m)+2})^{\pm\mathrm{sign}(m)}\,.$$

Lemma 3.5.20. *For every* $m, n \in \mathbb{Z}^{\times}$*,*

$$\mathbf{H}^{\pm}_{-m}(z)^{-1} = \mathbf{H}^{\pm}_{m}(zt^{\pm 4m}) \tag{3.5.20}$$

$$\mathbf{H}^{\pm}_{m}(zt^{\pm 4m})\mathbf{H}^{\pm}_{n}(z) = \mathbf{H}^{\pm}_{m+n}(zt^{\pm 4m}) \tag{3.5.21}$$

Proof. Follows directly from the definition in the same way as lemma 3.5.8. ☐

Proposition 3.5.21. *In* $\widehat{\mathcal{H}}^{\pm}_{t}$*, we have, for every* $m, n \in \mathbb{Z}^{\times}$*,*

$$\mathbf{H}^{\pm}_{m}(z)\mathbf{H}^{\pm}_{n}(v) = \Theta^{\pm}_{m,n}(z,v)\mathbf{H}^{\pm}_{n}(v)\mathbf{H}^{\pm}_{m}(z)\,,$$

where

$$\Theta^{\pm}_{m,n}(z,v) = \frac{(v - t^{\pm 4}z)(v - t^{\pm 4(1+n-m)}z)(t^{\pm 4(1-n)}v - z)(t^{\pm 4(1+m)}v - z)}{(z - t^{\pm 4}v)(z - t^{\pm 4(1+m-n)}v)(t^{\pm 4(1-m)}z - v)(t^{\pm 4(1+n)}z - v)}\,.$$

Proof. In view of definition 3.5.19 and of the relations in proposition 3.5.7, it is clear that commuting $\mathbf{H}^{\pm}_{m}(z)$ and $\mathbf{H}^{\pm}_{n}(v)$ amounts to commuting, on one hand $\mathbf{L}^{\pm}_{m}(z)$ and $\mathbf{R}^{\pm}_{n}(v)$ and, on the other hand, $\mathbf{R}^{\pm}_{m}(z)$ and $\mathbf{L}^{\pm}_{n}(v)$. The result follows. ☐

Proposition 3.5.22. *For every* $m \in \mathbb{Z}^{\times}$ *and every* $\varepsilon \in \{-, +\}$*, we have*

$$\mathbf{k}^{\varepsilon}_{1}(v) \triangleright \mathbf{H}^{\pm}_{\pm m}(z) = H^{\varepsilon}_{m,z}(v)^{\pm 1}\mathbf{H}^{\pm}_{\pm m}(z)\,, \tag{3.5.22}$$

$$\mathbf{x}^{\varepsilon}_{1}(v) \triangleright \mathbf{H}^{\pm}_{m}(z) = 0\,. \tag{3.5.23}$$

Proof. The left $\breve{\mathrm{U}}_{t^2}(\mathrm{L}\mathfrak{a}_1)$-module algebra (resp. a left $\breve{\mathrm{U}}_{t^2}(\mathrm{L}\mathfrak{a}_1)^{\mathrm{cop}}$-module algebra) structure of $\mathcal{H}^{+}_{t}$ (resp. $\mathcal{H}^{-}_{t}$) – see proposition 3.5.10 – is extended by continuity to $\widehat{\mathcal{H}}^{+}_{t}$ (resp. $\widehat{\mathcal{H}}^{-}_{t}$) Then, it suffices to check that, for every $m \in \mathbb{Z}^{\times}$ and every $\varepsilon \in \{-, +\}$,

$$\mathbf{k}^{\varepsilon}_{1}(v) \triangleright \mathbf{H}^{\pm}_{\pm m}(z) = \lambda^{\varepsilon,\pm}_{\pm m}(v,z)\rho^{\varepsilon,\pm}_{\pm m}(v,z)\mathbf{H}^{\pm}_{\pm m}(z)\,,$$

and that

$$H^{\varepsilon}_{m,z}(v)^{\pm 1} = \lambda^{\varepsilon,\pm}_{\pm m}(v,z)\rho^{\varepsilon,\pm}_{\pm m}(v,z)\,.$$

☐

Corollary 3.5.23. *For every* $m \in \mathbb{Z}$*, every* $p \in \mathbb{N}$ *and every* $\varepsilon \in \{-, +\}$*, we have*

$$\prod_{k=1}^{p+1} \left[\mathbf{k}^{\varepsilon}_{1}(v_k) - H^{\varepsilon}_{m,z}(v_k)^{\pm 1}\mathrm{id}\right] \triangleright \partial^{p}\mathbf{H}^{\pm}_{\pm m}(z) = 0\,,$$

Proof. It suffices to differentiate (3.5.22) p times with respect to z to obtain

$$\left[\mathbf{k}^{\varepsilon}_{1}(v) - H^{\varepsilon}_{m,z}(v)^{\pm 1}\,\mathrm{id}\right] \triangleright \partial^{p}\mathbf{H}^{\pm}_{\pm m}(z) = \sum_{k=0}^{p-1} \binom{p}{k+1} \frac{\partial^{k+1}}{\partial z^{k+1}}\left[H^{\varepsilon}_{m,z}(v)^{\pm 1}\right]\partial^{p-k-1}\mathbf{H}^{\pm}_{\pm m}(z)\,.$$

The claim immediately follows. ☐

Proposition 3.5.24. *In $\widehat{\mathcal{B}}_t$, we have, for every $m, n \in \mathbb{Z}^\times$,*

$$\mathbf{H}^+_m(z)\mathbf{H}^-_n(v) = \mathbf{H}^-_n(v)\mathbf{H}^+_m(z)\,,$$

$$(zt^{\pm 4} - v)(zt^{-2(1\pm 1)m} - vt^{\pm 4 - 2(1\mp 1)m})\mathbf{k}^{\varepsilon}_1(v)\mathbf{H}^{\pm}_m(z) = (z - vt^{\pm 4})(zt^{\pm 4 - 2(1\pm 1)m} - vt^{-2(1\mp 1)m})\mathbf{H}^{\pm}_m(z)\mathbf{k}^{\varepsilon}_1(v)\,,$$

$$(zt^{\pm 4} - v)(zt^{-2(1\pm 1)m} - vt^{\pm 4 - 2(1\mp 1)m})\mathbf{x}^{\pm}_1(v)\mathbf{H}^{\pm}_m(z) = (z - vt^{\pm 4})(zt^{\pm 4 - 2(1\pm 1)m} - vt^{-2(1\mp 1)m})\mathbf{H}^{\pm}_m(z)\mathbf{x}^{\pm}_1(v)\,,$$

$$\mathbf{x}^{\pm}_1(v)\mathbf{H}^{\mp}_m(z) = \mathbf{H}^{\mp}_m(z)\mathbf{x}^{\pm}_1(v)\,.$$

Proof. This follows immediately from $[\mathbf{L}^{\pm}(z), \mathbf{L}^{\mp}(v)] = [\mathbf{L}^{\pm}(z), \mathbf{R}^{\mp}(v)] = [\mathbf{R}^{\pm}(z), \mathbf{R}^{\mp}(v)] = 0$. ☐

3.5.7 The evaluation algebra $\widehat{\mathcal{A}}_t$

Definition-Proposition 3.5.25. Let $\mathcal{J}$ denote the closed left ideal of $\widehat{\mathcal{B}}_t$ generated by

$$\left\{\operatorname*{res}_z z^m \left[\mathbf{H}^-(z)\left(\mathbf{k}^+_1(zt^{-4}) - \mathbf{k}^-_1(zt^{-4})\right) - \mathbf{H}^+(z)^{-1}\left(\mathbf{k}^+_1(z) - \mathbf{k}^-_1(z)\right)\right] : m \in \mathbb{Z}\right\}\,. \tag{3.5.24}$$

Then, $\mathcal{J}.\widehat{\mathcal{B}}_t \subseteq \mathcal{J}$, making $\mathcal{J}$ a closed two-sided ideal of $\widehat{\mathcal{B}}_t$, and we let $\widehat{\mathcal{A}}_t = \widehat{\mathcal{B}}_t/\mathcal{J}$.

Proof. In order to prove that $\mathcal{J}.\widehat{\mathcal{B}}_t \subseteq \mathcal{J}$, it suffices to check that, for every $x \in \widehat{\mathcal{B}}_t$,

$$\left[\mathbf{H}^-(z)\left(\mathbf{k}^+_1(zt^{-4}) - \mathbf{k}^-_1(zt^{-4})\right) - \mathbf{H}^+(z)^{-1}\left(\mathbf{k}^+_1(z) - \mathbf{k}^-_1(z)\right)\right] x \in \mathcal{J}\,.$$

The latter easily follows by inspection, making use of the relevant relations in $\widehat{\mathcal{B}}_t$, namely (3.5.13–3.5.19) in proposition 3.5.15. ☐

Proposition 3.5.26. *For every $m \in \mathbb{Z}$, the following relation holds in $\widehat{\mathcal{A}}_t$,*

$$\mathbf{H}^-_{-m}(z)\left[\mathbf{k}^+_1(zt^{-4m}) - \mathbf{k}^-_1(zt^{-4m})\right] = \mathbf{H}^+_m(z)^{-1}\left[\mathbf{k}^+_1(z) - \mathbf{k}^-_1(z)\right]\,. \tag{3.5.25}$$

Proof. We prove (3.5.25) for $m \in \mathbb{N}^\times$ by induction. The case $m = 1$ corresponds to the vanishing of the generators of the ideal $\mathcal{J}$, see (3.5.24). Assuming the result holds for some $m \in \mathbb{N}^\times$, we have

$$\begin{aligned}
\mathbf{H}^-_{-(m+1)}(z)\left[\mathbf{k}^+_1(zt^{-4(m+1)}) - \mathbf{k}^-_1(zt^{-4(m+1)})\right] &= \mathbf{H}^-(z)\mathbf{H}^-_{-m}(zt^{-4})\left[\mathbf{k}^+_1(zt^{-4(m+1)}) - \mathbf{k}^-_1(zt^{-4(m+1)})\right] \\
&= \mathbf{H}^-(z)\mathbf{H}^+_m(zt^{-4})^{-1}\left[\mathbf{k}^+_1(zt^{-4}) - \mathbf{k}^-_1(zt^{-4})\right] \\
&= \mathbf{H}^+_m(zt^{-4})^{-1}\mathbf{H}^-(z)\left[\mathbf{k}^+_1(zt^{-4}) - \mathbf{k}^-_1(zt^{-4})\right] \\
&= \mathbf{H}^+_m(zt^{-4})^{-1}\mathbf{H}^+(z)^{-1}\left[\mathbf{k}^+_1(z) - \mathbf{k}^-_1(z)\right] \\
&= \mathbf{H}^+_{m+1}(z)^{-1}\left[\mathbf{k}^+_1(z) - \mathbf{k}^-_1(z)\right]
\end{aligned}$$

The cases with $m \in -\mathbb{N}^\times$ follow by rewriting the above equation for $m \in \mathbb{N}^\times$ as

$$\mathbf{H}^+_m(z)\left[\mathbf{k}^+_1(zt^{-4m}) - \mathbf{k}^-_1(zt^{-4m})\right] = \mathbf{H}^-_{-m}(z)^{-1}\left[\mathbf{k}^+_1(z) - \mathbf{k}^-_1(z)\right]$$

and making use of lemma 3.5.8. ☐

Remark 3.5.27. In addition to the above relation, $\widehat{\mathcal{A}}_t$ obviously inherits the relations in $\widehat{\mathcal{B}}_t$ modulo $\mathcal{J}$. In particular, all the relations in proposition 3.5.15 hold in $\widehat{\mathcal{A}}_t$.

3.5.8 The evaluation homomorphism

Remember $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)^{(-1)}$ from section 3.3.3 and ι_0 from section 3.3.8, proposition 3.3.23.

Proposition 3.5.28. *There exists a unique continuous* $\mathbb{K}$*-algebra homomorphism* $\mathrm{ev} : \ddot{\mathrm{U}}_q(\mathfrak{a}_1)^{(-1)} \to \widehat{\mathcal{A}}_t$ *such that, for every* $m \in \mathbb{N}^\times$ *and every* $n \in \mathbb{Z}$,

$$\mathrm{ev}(q) = t^2 \,, \tag{3.5.26}$$

$$\mathrm{ev}(\boldsymbol{K}^{\pm}_{1,0}(z)) = -\mathbf{k}^{\mp}_1(z) \,, \tag{3.5.27}$$

$$\mathrm{ev}(\boldsymbol{K}^{\pm}_{1,\pm m}(z)) = \mathbf{H}^{\pm}_{\pm m}(z) \left[\mathbf{k}^{\pm}_1(zt^{-4m}) - \mathbf{k}^{\mp}_1(zt^{-4m})\right] \,, \tag{3.5.28}$$

$$\mathrm{ev}(\boldsymbol{X}^{\pm}_{1,n}(z)) = \mathbf{H}^{\pm}_n(z)\mathbf{x}^{\pm}_1(zt^{\mp 4n}) \,. \tag{3.5.29}$$

We shall refer to ev *as the* evaluation homomorphism*. It is such that* $\mathrm{ev} \circ \iota_0 = \mathrm{id}$ *over* $\mathrm{U}_{t^2}(\mathrm{L}\mathfrak{a}_1)$.

Proof. It suffices to check all the defining relations of $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)$. E.g. we have, for every $m, n \in \mathbb{Z}$,

$$\left[\mathrm{ev}(\mathsf{X}^{+}_{1,m}(v)), \mathrm{ev}(\mathsf{X}^{-}_{1,n}(z))\right] = \frac{1}{t^2 - t^{-2}}\delta\left(\frac{v}{zt^{4(m+n)}}\right)\mathbf{H}^{+}_m(v)\mathbf{H}^{-}_n(z)\left[\mathbf{k}^{+}_1(vt^{-4m}) - \mathbf{k}^{-}_1(zt^{4n})\right] \,. \tag{3.5.30}$$

If $m + n = 0$, making use of (3.5.25), we are done. Assuming that $m + n > 0$, lemma 3.5.8 allows us to write

$$\begin{aligned}
\mathbf{H}^{+}_m(zt^{4(m+n)})\mathbf{H}^{-}_n(z)\left[\mathbf{k}^{+}_1(zt^{4n}) - \mathbf{k}^{-}_1(zt^{4n})\right] &= \mathbf{H}^{+}_m(zt^{4(m+n)})\mathbf{H}^{+}_{-n}(z)^{-1}\left[\mathbf{k}^{+}_1(z) - \mathbf{k}^{-}_1(z)\right] \\
&= \mathbf{H}^{+}_m(zt^{4(m+n)})\mathbf{H}^{+}_n(zt^{4n})\left[\mathbf{k}^{+}_1(z) - \mathbf{k}^{-}_1(z)\right] \\
&= \mathbf{H}^{+}_{m+n}(zt^{4(m+n)})\left[\mathbf{k}^{+}_1(z) - \mathbf{k}^{-}_1(z)\right]
\end{aligned}$$

so that, eventually,

$$\left[\mathrm{ev}(\mathsf{X}^{+}_{1,m}(v)), \mathrm{ev}(\mathsf{X}^{-}_{1,n}(z))\right] = \frac{1}{t^2 - t^{-2}}\delta\left(\frac{v}{zt^{4(m+n)}}\right)\mathrm{ev}(\mathsf{K}^{+}_{m+n}(zt^{4(m+n)})) \,.$$

A similar argument proves the case $m + n < 0$. ☐

The following is obvious.

Corollary 3.5.29. *For every* $N \in \mathbb{N}$ *there exists an algebra homomorphism* $\mathrm{ev}_{(N)} : \ddot{\mathrm{U}}_q(\mathfrak{a}_1)^{(N)} \to \widehat{\mathcal{A}}_t$ *making the following diagram commutative.*

$$\begin{tikzcd}
\cdots \arrow[r] & \ddot{\mathrm{U}}_q(\mathfrak{a}_1)^{(N)} \arrow[r] \arrow[rrrd, "\mathrm{ev}_{(N)}"'] & \ddot{\mathrm{U}}_q(\mathfrak{a}_1)^{(N-1)} \arrow[r] \arrow[rrd, "\mathrm{ev}_{(N-1)}"] & \cdots \arrow[r] & \ddot{\mathrm{U}}_q(\mathfrak{a}_1)^{(-1)} \arrow[d, "\mathrm{ev}"] \\
& & & & \widehat{\mathcal{A}}_t
\end{tikzcd}$$

We can furthermore define the algebra homomorphism $\mathrm{ev}_{(\infty)} : \ddot{\mathrm{U}}_q(\mathfrak{a}_1) \to \widehat{\mathcal{A}}_t$ *by*

$$\mathrm{ev}_{(\infty)} = \varprojlim \mathrm{ev}_{(N)} \,.$$

3.5.9 Evaluation modules

Remember the surjective algebra homomorphism $\breve{\mathrm{U}}_q(\mathrm{L}\mathfrak{a}_1) \to \mathrm{U}_q(\mathrm{L}\mathfrak{a}_1)$ from proposition 3.2.2. It allows us to pull back any simple $\mathrm{U}_q(\mathrm{L}\mathfrak{a}_1)$-module M into a simple $\breve{\mathrm{U}}_q(\mathrm{L}\mathfrak{a}_1)$-module. With that construction in mind, we have

Proposition 3.5.30. *Let M be a simple finite dimensional $\mathrm{U}_q(\mathrm{L}\mathfrak{a}_1)$-module. Then,*

i. $\widehat{\mathcal{H}}_t^+ \otimes M \otimes \widehat{\mathcal{H}}_t^-$ *is a* $\mathcal{H}_t^+ \rtimes \breve{\mathrm{U}}_{t^2}(\mathrm{L}\mathfrak{a}_1) \overset{\mathrm{cop}}{\ltimes} \mathcal{H}_t^-$*-module with the action defined by setting, for every* $h_+, h'_+ \in \mathcal{H}_t^+$*, every* $h_-, h'_- \in \mathcal{H}_t^-$*, every* $x \in \breve{\mathrm{U}}_{t^2}(\mathrm{L}\mathfrak{a}_1)$ *and every* $v \in M$*,*

$$(h_+ \otimes x \otimes h_-).(h'_+ \otimes v \otimes h'_-) = \sum h_+ \left(x_{(1)} \triangleright h'_+\right) \otimes x_{(2)}.v \otimes h_- \left(x_{(3)} \triangleright h'_-\right)$$

and extending by continuity.

ii. $\widehat{\mathcal{H}}_t^+ \otimes M \otimes \widehat{\mathcal{H}}_t^-$ *descends to a* $\mathcal{B}_t$*-module.*

iii. $\left(\widehat{\mathcal{H}}_t^+ \otimes M \otimes \widehat{\mathcal{H}}_t^-\right) / \mathcal{J}.\left(\widehat{\mathcal{H}}_t^+ \otimes M \otimes \widehat{\mathcal{H}}_t^-\right)$ *is an* $\widehat{\mathcal{A}}_t$*-module. It pulls back along* ev *to a* $\ddot{\mathrm{U}}'_q(\mathfrak{a}_1)$*-module that we denote by* $\mathrm{ev}^*(M)$*.*

iv. As a $\ddot{\mathrm{U}}'_q(\mathfrak{a}_1)$*-module,* $\mathrm{ev}^*(M)$ *is weight-finite.*

v. For any highest ℓ*-weight vector* $v \in M - \{0\}$*, the* $\ddot{\mathrm{U}}^0_q(\mathfrak{a}_1)$*-module*

$$\tilde{M}_0 \cong \left(\widehat{\mathcal{H}}_t^+ \otimes \mathbb{F}v \otimes \widehat{\mathcal{H}}_t^-\right) / \mathcal{J}.\left(\widehat{\mathcal{H}}_t^+ \otimes \mathbb{F}v \otimes \widehat{\mathcal{H}}_t^-\right),$$

is a highest t*-weight space of* $\mathrm{ev}(M)$*. We denote by* M_0 *the simple quotient of* $\tilde{M}_0$ *containing* v *and we let* $\mathrm{ev}^*(M_0) = \ddot{\mathrm{U}}'_q(\mathfrak{a}_1).M_0$*.*

vi. M_0 *is* t*-dominant.*

Proof. i is readily checked. As for ii, it suffices to check that $\mathcal{I}.\left(\widehat{\mathcal{H}}_t^+ \otimes M \otimes \widehat{\mathcal{H}}_t^-\right) = \{0\}$. But the latter is clear when M is obtained by pulling back a $\mathrm{U}_q(\mathrm{L}\mathfrak{a}_1)$-module over which the relation generating $\mathcal{I}$ is automatically satisfied. iii is obvious. It easily follows from proposition 3.5.24 that, for every $m \in \mathbb{Z}^\times$, $[k^\varepsilon_{1,0}, \mathbf{H}^\pm_m(z)] = 0$. Hence, $\mathrm{Sp}(\mathrm{ev}^*(M)) = \mathrm{Sp}(M)$ and the weight finiteness of $\mathrm{ev}^*(M)$ follows from that of M, which proves iv. It is clear that, for every $r \in \mathbb{Z}$, we have

$$\begin{aligned}
\mathrm{ev}(\mathbf{X}^+_{1,r}(z)).\left(\mathcal{H}_t^+ \otimes v \otimes \mathcal{H}_t^-\right) &= \mathbf{H}^+_r(z)\mathbf{x}^+_1(zt^{-4r}).\left(\widehat{\mathcal{H}}_t^+ \otimes v \otimes \widehat{\mathcal{H}}_t^-\right)\\
&= \sum \mathbf{H}^+_r(z)\left(\mathbf{x}^+_1(zt^{-4r})_{(1)} \triangleright \widehat{\mathcal{H}}_t^+\right) \otimes \mathbf{x}^+_1(zt^{-4r})_{(2)}.v \otimes \left(\mathbf{x}^+_1(zt^{-4r})_{(3)} \triangleright \widehat{\mathcal{H}}_t^-\right)\\
&= 0\,.
\end{aligned}$$

v follows. Denote by $P(1/z) \in \mathbb{F}[z^{-1}]$ the Drinfel'd polynomial associated with v and let $\nu \in \mathbb{N}_f^{\mathbb{F}^\times}$ denote the multiset of its roots. Then,

$$\mathbf{k}^\pm_1(z).v = -\kappa^\mp_0(z)v\,, \qquad \text{where} \qquad \kappa^\mp_0(z) = -t^{2\deg(P)}\left(\frac{P(t^{-4}/z)}{P(1/z)}\right)_{|z|^{\mp 1} \ll 1}. \tag{3.5.31}$$

Moreover, the partial fraction decomposition

$$\frac{P(t^{-4}/z)}{P(1/z)} = \prod_{a\in\mathbb{F}^\times} \frac{1}{(1-a/z)^{\nu(a)-\nu(at^4)}} = C_0 + \sum_{a\in\mathbb{F}^\times} \sum_{p=1}^{\nu(a)-\nu(at^4)} \frac{C_p(a)}{(1-a/z)^p}\,,$$

in which $C_0, C_p(a) \in \mathbb{F}$ and the product and sum over $a \in \mathbb{F}^\times$ are always finite since P only has finitely many roots, allows us to write

$$\left[\mathbf{k}_1^+(z) - \mathbf{k}_1^-(z)\right].v = t^{2\deg(P)} \sum_{a\in\mathbb{F}^\times} \sum_{p=0}^{\nu(a)-\nu(at^4)-1} \frac{(-1)^{p+1}C_{p+1}(a)}{p!\,a^{p+1}} \delta^{(p)}\left(\frac{z}{a}\right) v\,.$$

Letting $\tilde{C}_p(a) = (-1)^{p+1}t^{2\deg(P)}C_{p+1}(a)a^{-p-1}/p!$ for every $a \in \mathbb{F}^\times$ and every $p \in [\![0, \nu(a)-\nu(at^4)-1]\!]$, it follows that, for every $m \in \mathbb{N}^\times$,

$$\mathrm{ev}(\mathsf{K}_{1,m}^+(z)).\,(1\otimes v\otimes 1) = t^{2\deg(P)} \sum_{a\in\mathbb{F}^\times} \sum_{p=0}^{\nu(at^{-4m})-\nu(at^{4(1-m)})-1} \tilde{C}_p(at^{-4m})\delta^{(p)}\left(\frac{z}{a}\right)\left(\mathbf{H}_m^+(z)\otimes v\otimes 1\right)\,, \tag{3.5.32}$$

$$\mathrm{ev}(\mathsf{K}_{1,-m}^-(z)).\,(1\otimes v\otimes 1) = -t^{2\deg(P)} \sum_{a\in\mathbb{F}^\times} \sum_{p=0}^{\nu(at^{-4m})-\nu(at^{4(1-m)})-1} \tilde{C}_p(at^{-4m})\delta^{(p)}\left(\frac{z}{a}\right)\left(1\otimes v\otimes \mathbf{H}_{-m}^-(z)\right)\,. \tag{3.5.33}$$

Now, making use of (3.5.27), (3.5.31) and of corollary 3.5.23, one easily shows that, for every $p \in \mathbb{N}$ and every $a \in \mathbb{F}^\times$,

$$\prod_{k=1}^{p+1}\left[\mathrm{ev}(\mathsf{K}_{1,0}^\pm(z_k)) - H_{m,a}^\mp(z_k)\kappa_0^\pm(z_k)\,\mathrm{id}\right].\left(\partial^p\mathbf{H}_m^+(a)\otimes v\otimes 1\right) = 0\,,$$

$$\prod_{k=1}^{p+1}\left[\mathrm{ev}(\mathsf{K}_{1,0}^\pm(z_k)) - H_{m,a}^\mp(z_k)^{-1}\kappa_0^\pm(z_k)\,\mathrm{id}\right].\left(1\otimes v\otimes \partial^p\mathbf{H}_{-m}^-(a)\right) = 0\,,$$

thus proving that $\partial^p\mathbf{H}_m^+(a)\otimes v\otimes 1$ (resp. $1\otimes v\otimes\partial^p\mathbf{H}_{-m}^-(a)$) is an ℓ-weight vector in the ℓ-weight space $\mathrm{ev}^*(M)_{\kappa_{(+,m,a)}}$ (resp. $\mathrm{ev}^*(M)_{\kappa_{(-,m,a)}}$) of $\mathrm{ev}^*(M)$ with ℓ-weight $\kappa_{(+,m,a)}^\pm(z) = \kappa_0^\pm(z)H_{m,a}^\mp(z)$ (resp. $\kappa_{(-,m,a)}^\pm(z) = \kappa_0^\pm(z)H_{m,a}^\mp(z)^{-1}$), as expected from proposition 3.4.3.

On the other hand,

$$\begin{aligned}
&\left\{\mathbf{H}^-(z)\left[\mathbf{k}_1^+(zt^{-4}) - \mathbf{k}_1^-(zt^{-4})\right] - \mathbf{H}^+(z)^{-1}\left[\mathbf{k}_1^+(z)-\mathbf{k}_1^-(z)\right]\right\}.(1\otimes v\otimes 1)\\
&\qquad = \sum_{a\in\mathbb{F}^\times}\left\{\sum_{p=0}^{\nu(at^{-4})-\nu(a)-1} \tilde{C}_p(at^{-4})\delta^{(p)}\left(\frac{z}{a}\right)\left(1\otimes v\otimes\mathbf{H}^-(z)\right)\right.\\
&\qquad\qquad\left. - \sum_{p=0}^{\nu(a)-\nu(at^4)-1}\tilde{C}_p(a)\delta^{(p)}\left(\frac{z}{a}\right)\left(\mathbf{H}^+(z)^{-1}\otimes v\otimes 1\right)\right\}.
\end{aligned}$$

Thus, modulo $\mathcal{J}$, we have, for every $a \in \mathbb{F}^\times$,

$$\sum_{p=0}^{\nu(at^{-4})-\nu(a)-1} \tilde{C}_p(at^{-4})\delta^{(p)}\left(\frac{z}{a}\right)\left(1\otimes v\otimes\mathbf{H}^-(z)\right) = \sum_{p=0}^{\nu(a)-\nu(at^4)-1}\tilde{C}_p(a)\delta^{(p)}\left(\frac{z}{a}\right)\left(\mathbf{H}^+(z)^{-1}\otimes v\otimes 1\right)\,.$$

The above equation makes it clear that every $a \in \mathbb{F}^\times$ such that $\nu(at^{-4}) > \nu(a)$ is a zero of order at least $\nu(at^{-4}) - 2\nu(a) + \nu(at^4)$ of $1 \otimes v \otimes \mathbf{H}^-(z)$, unless $\nu(at^{-4}) - \nu(a) \leq \nu(a) - \nu(at^4)$. Hence, in view of (3.5.33), we have $\left[\mathrm{ev}(\mathsf{K}^-_{1,-1}(z)).(1 \otimes v \otimes 1)\right]_{\mathrm{ev}^*(M)_{\kappa_{(-,1,a)}}} = 0$ unless $a \in \mathsf{D}_1(\nu) = \{x \in \mathbb{F}^\times : \nu(xt^{-4}) > \nu(x) > \nu(xt^4)\}$. But the latter implies that $P(1/a) = 0$. A similar reasoning applies to any ℓ-weight vector in $\tilde{M}_0$ and $\tilde{M}_0$ is t-dominant by lemma 3.4.16. Taking the quotient of $\tilde{M}_0$ to M_0 clearly preserves t-dominance and vi follows. □

By the universality of $\mathcal{M}(M_0)$ – see definition 3.4.21 – and the above proposition, there must exist a surjective $\ddot{\mathrm{U}}'_q(\mathfrak{a}_1)$-module homomorphism $\pi : \mathcal{M}(M_0) \twoheadrightarrow \mathrm{ev}^*(M_0)$. Restricting the latter to the (closed) $\ddot{\mathrm{U}}'_q(\mathfrak{a}_1)$-submodule $\mathcal{N}(M_0)$ of $\mathcal{M}(M_0)$, we get the surjective $\ddot{\mathrm{U}}'_q(\mathfrak{a}_1)$-module homomorphism $\pi_{|\mathcal{N}(M_0)}$, whose image naturally injects as a $\ddot{\mathrm{U}}'_q(\mathfrak{a}_1)$-submodule in $\mathrm{ev}^*(M_0)$. The canonical short exact sequence involving $\mathcal{N}(M_0)$, $\mathcal{M}(M_0)$ and the simple quotient $\mathcal{L}(M_0)$ – see definition 3.4.21 – allows us to define a surjective $\ddot{\mathrm{U}}'_q(\mathfrak{a}_1)$-module homomorphism $\tilde{\pi}$ to get the following commutative diagram,

$$\begin{tikzcd}
\{0\} \arrow[r] & \mathcal{N}(M_0) \arrow[r] \arrow[d, "\pi_{|\mathcal{N}(M_0)}"] & \mathcal{M}(M_0) \arrow[r] \arrow[d, "\pi"] & \mathcal{L}(M_0) \arrow[r] \arrow[d, "\tilde{\pi}"] & \{0\} \\
\{0\} \arrow[r] & \pi(\mathcal{N}(M_0)) \arrow[r] \arrow[d] & \mathrm{ev}^*(M_0) \arrow[r] \arrow[d] & \mathrm{ev}^*(M_0)/\pi(\mathcal{N}(M_0)) \arrow[r] \arrow[d] & \{0\} \\
 & \{0\} & \{0\} & \{0\} &
\end{tikzcd}$$

where columns and rows are exact. It is obvious that $\tilde{\pi}$ is not identically zero and, by the simplicity of $\mathcal{L}(M_0)$, we must have $\ker(\tilde{\pi}) = \{0\}$. Hence, $\tilde{\pi}$ is a $\ddot{\mathrm{U}}'_q(\mathfrak{a}_1)$-module isomorphism and we have constructed the simple weight-finite $\ddot{\mathrm{U}}'_q(\mathfrak{a}_1)$-modules $\mathcal{L}(M_0)$ as a quotient of the evaluation module $\mathrm{ev}^*(M_0)$. To see that all the simple weight-finite $\ddot{\mathrm{U}}'_q(\mathfrak{a}_1)$-modules $\mathcal{L}(M_0)$ can be obtained in this way, it suffices to observe that, by proposition 3.4.11, all the simple ℓ-dominant $\ddot{\mathrm{U}}^0_q(\mathfrak{a}_1)$-modules are of the form $L^0(P)$ for some monic polynomial P and that, in the construction above, one can choose any P, simply by choosing the corresponding simple finite-dimensional $\mathrm{U}_q(\mathrm{L}\mathfrak{a}_1)$-module M. Therefore, as a consequence of the above proposition, the highest t-weight space of any simple weight-finite $\ddot{\mathrm{U}}'_q(\mathfrak{a}_1)$-modules $\mathcal{L}(M_0)$ is t-dominant. This concludes the proof of part ii of theorem 3.4.22 as well as that of theorem 3.4.13.

Chapter 4

Topological Braid Group Action on $\dot{\mathrm{U}}_q(\dot{\mathfrak{g}})$

4.1 Introduction

Let $\dot{\mathfrak{g}}$ be an untwisted affine Kac-Moody algebra. By using Drinfel'd's quantum affinization we define $\dot{\mathrm{U}}_q(\dot{\mathfrak{g}})$ to be the quantum toroidal algebra associated to $\dot{\mathfrak{g}}$ and $\widehat{\dot{\mathrm{U}}_q(\dot{\mathfrak{g}})}$ its completion.

As in the case of the work of I. Damiani and J.Beck, the purpose of this chapter is to establish a (topological) braid group action on quantum toroidal algebras. This can be seen as a generalization of the work of J. Ding and S. Khoroshkin in [DK00] to all Dynkin diagrams and provides the building blocks for defining an affinized version of the Damiani-Beck isomorphism for $\dot{\mathrm{U}}_q(\dot{\mathfrak{g}})$ thus allowing us to define $\ddot{\mathrm{U}}_q(\mathfrak{g})$ as the double Drinfel'd current presentation. We provide a proof by checking the algebra relations except for the Serre relation when $a_{ij} = a_{ji} = -2$, $a_{ij} = -3, -4$ and $a_{ij} = -1$. This is work still in progress at this stage but we conjecture that it will hold as for the proved cases. The proof of the Serre relation relies on defining $\tilde{r}_k^{\pm}(v)$ which is an affine version of what Lusztig defined as ${}_ir^{\pm}$ – see [Lusztig].

This chapter is organized as follows. First we start by giving several automorphisms of $\dot{\mathrm{U}}_q(\dot{\mathfrak{g}})$, crucial to many of the proofs. We then build our way to defining the braid group action of T_i on the generators of the algebra in order to provide the main theorem. Then, in the remaining part of the chapter we construct the necessary machinery in order to to prove this theorem.

4.2 Definition of $\dot{\mathrm{U}}_q(\dot{\mathfrak{g}})$

Definition 4.2.1. The *quantum toroidal algebra* $\dot{\mathrm{U}}_q(\dot{\mathfrak{g}})$ is the associative $\mathbb{F}$-algebra generated by the generators

$$\left\{D, D^{-1}, C^{1/2}, C^{-1/2}, k_{i,n}^{+}, k_{i,-n}^{-}, x_{i,m}^{+}, x_{i,m}^{-} : i \in \dot{I}, m \in Z, n \in \mathbb{N}\right\}$$

subject to the following relations

$$C^{\pm 1/2} \text{ is central} \qquad C^{\pm 1/2}C^{\mp 1/2} = 1 \qquad D^{\pm 1}D^{\mp 1} = 1 \tag{4.2.1}$$

$$D\mathbf{k}_i^{\pm}(z)D^{-1} = \mathbf{k}_i^{\pm}(zq^{-1}) \qquad D\mathbf{x}_i^{\pm}(z)D^{-1} = \mathbf{x}_i^{\pm}(zq^{-1}) \tag{4.2.2}$$

$$\operatorname*{res}_{z_1,z_2} \frac{1}{z_1z_2}\mathbf{k}_i^{\pm}(z_1)\mathbf{k}_i^{\mp}(z_2) = 1 \tag{4.2.3}$$

$$\mathbf{k}_i^{\pm}(z_1)\mathbf{k}_j^{\pm}(z_2) = \mathbf{k}_j^{\pm}(z_2)\mathbf{k}_i^{\pm}(z_1) \tag{4.2.4}$$

$$\mathbf{k}_i^{-}(z_1)\mathbf{k}_j^{+}(z_2) = G_{ij}^{-}(C^{-1}z_1/z_2)G_{ij}^{+}(Cz_1/z_2)\mathbf{k}_j^{+}(z_2)\mathbf{k}_i^{-}(z_1) \tag{4.2.5}$$

$$G_{ij}^{\mp}(C^{\mp 1/2}z_2/z_1)\mathbf{k}_i^{+}(z_1)\mathbf{x}_j^{\pm}(z_2) = \mathbf{x}_j^{\pm}(z_2)\mathbf{k}_i^{+}(z_1) \tag{4.2.6}$$

$$\mathbf{k}_i^{-}(z_1)\mathbf{x}_j^{\pm}(z_2) = G_{ij}^{\mp}(C^{\mp 1/2}z_1/z_2)\mathbf{x}_j^{\pm}(z_2)\mathbf{k}_i^{-}(z_1) \tag{4.2.7}$$

$$(z_1 - q^{\pm c_{ij}}z_2)\mathbf{x}_i^{\pm}(z_1)\mathbf{x}_j^{\pm}(z_2) = (z_1 q^{\pm c_{ij}} - z_2)\mathbf{x}_j^{\pm}(z_2)\mathbf{x}_i^{\pm}(z_1) \tag{4.2.8}$$

$$[\mathbf{x}_i^{+}(z_1), \mathbf{x}_j^{-}(z_2)] = \frac{\delta_{ij}}{q_i - q_i^{-1}}\left[\delta\left(\frac{z_1}{Cz_2}\right)\mathbf{k}_i^{+}(z_1C^{-1/2}) - \delta\left(\frac{z_1C}{z_2}\right)\mathbf{k}_i^{-}(z_2C^{-1/2})\right] \tag{4.2.9}$$

$$\sum_{\sigma\in S_{1-c_{ij}}}\sum_{k=0}^{1-c_{ij}}(-1)^k\binom{1-c_{ij}}{k}_q \mathbf{x}_i^{\pm}(z_{\sigma(1)})\cdots\mathbf{x}_i^{\pm}(z_{\sigma(k)})\mathbf{x}_j^{\pm}(z)\mathbf{x}_i^{\pm}(z_{\sigma(k+1)})\cdots\mathbf{x}_i^{\pm}(z_{\sigma(1-c_{ij})}) = 0 \tag{4.2.10}$$

where, for every $i \in \dot{I}$, we define the following $\dot{\mathrm{U}}_q(\dot{\mathfrak{g}})$-valued formal distributions

$$\mathbf{x}_i^{\pm}(z) := \sum_{m\in\mathbb{Z}} x_{i,m}^{\pm}z^{-m} \in \dot{\mathrm{U}}_q(\dot{\mathfrak{g}})[[z, z^{-1}]]\,; \tag{4.2.11}$$

$$\mathbf{k}_i^{\pm}(z) := \sum_{n\in\mathbb{N}} k_{i,\pm n}^{\pm}z^{\mp n} \in \dot{\mathrm{U}}_q(\dot{\mathfrak{g}})[[z^{\mp 1}]]\,, \tag{4.2.12}$$

for every $i, j \in \dot{I}$, we define the following $\mathbb{F}$-valued formal power series

$$G_{ij}^{\pm}(z) := q_i^{\pm a_{ij}} + (q_i - q_i^{-1})[\pm a_{ij}]_{q_i}\sum_{m\in\mathbb{N}^{\times}} q_i^{\pm m a_{ij}}z^m \in \mathbb{F}[[z]] \tag{4.2.13}$$

and

$$\delta(z) := \sum_{m\in\mathbb{Z}} z^m \in \mathbb{F}[[z, z^{-1}]] \tag{4.2.14}$$

is an $\mathbb{F}$-valued formal distribution.

4.3 Automorphisms of $\dot{\mathrm{U}}_q(\dot{\mathfrak{g}})$

Proposition 4.3.1. *i. For every Dynkin diagram automorphism $\pi : \dot{I} \xrightarrow{\sim} \dot{I}$, there exists a unique $\mathbb{F}$-algebra automorphism $T_\pi \in \mathrm{Aut}(\dot{\mathrm{U}}_q(\dot{\mathfrak{g}}))$ such that*

$$T_\pi(\mathbf{x}_i^{\pm}(z)) = \mathbf{x}_{\pi(i)}^{\pm}(z)\,, \qquad T_\pi(\mathbf{k}_i^{\pm}(z)) = \mathbf{k}_{\pi(i)}^{\pm}(z)\,, \qquad T_\pi(C) = C\,, \qquad T_\pi(D) = D\,. \tag{4.3.1}$$

ii. For every $i \in \dot{I}$, there exists a unique $\mathbb{F}$-algebra automorphism $T_{\omega_i^\vee} \in \mathrm{Aut}(\dot{\mathrm{U}}_q(\dot{\mathfrak{g}}))$ such that

$$T_{\omega_i^\vee}(\mathbf{x}_j^{\pm}(z)) = z^{\pm\delta_{ij}}\mathbf{x}_j^{\pm}(z) \qquad T_{\omega_i^\vee}(\mathbf{k}_j^{\pm}(z)) = C^{\mp\delta_{ij}}\mathbf{k}_j^{\pm}(z) \qquad T_{\omega_i^\vee}(C) = C \qquad T_{\omega_i^\vee}(D) = D \tag{4.3.2}$$

iii. There exists a unique involutive $\mathbb{F}$-algebra anti-homomorphism $\eta \in \mathrm{Aut}(\dot{\mathrm{U}}_q(\dot{\mathfrak{g}}))$ such that

$$\eta(\mathbf{x}_i^{\pm}(z)) = \mathbf{x}_i^{\pm}(1/z) \qquad \eta(\mathbf{k}_i^{\pm}(z)) = \mathbf{k}_i^{\mp}(1/z) \qquad \eta(C) = C \qquad \eta(D) = D \tag{4.3.3}$$

iv. There exists a unique involutive $\mathbb{K}$-algebra anti-homomorphism φ such that

$$\varphi(\mathbf{x}_i^{\pm}(z)) = \mathbf{x}_i^{\mp}(1/z) \qquad \varphi(\mathbf{k}_i^{\pm}(z)) = \mathbf{k}_i^{\mp}(1/z) \qquad \varphi(C) = C^{-1} \qquad \varphi(D) = D^{-1} \qquad \varphi(q) = q^{-1} \tag{4.3.4}$$

4.4 Braid group action

Definition 4.4.1. For all $i \neq j \in \dot{I}$, we let $\mathbf{x}^{+}_{i^1 j}(z) \in \widehat{\dot{\mathrm{U}}_q(\dot{\mathfrak{g}})}[[z, z^{-1}]]$ be defined by:

$$\Big[\mathbf{x}_i^{+}(z_1); \mathbf{x}_j^{+}(z_2)\Big]_{G^{-}_{ij}(z_1/z_2)} = \delta\left(\frac{z_1}{z_2 q_i^{a_{ij}}}\right) \mathbf{x}^{+}_{i^1 j}(z_1). \tag{4.4.1}$$

Proposition 4.4.2. *For all $i \neq j \in \dot{I}$, we have:*

$$\left(z_1 - q_i^2 z_2\right) \mathbf{x}_i^{+}(z_1)\mathbf{x}^{+}_{i^1 j}(z_2) = \left(z_1 q_i^2 - z_2\right) G^{-}_{ij}(z_1 q_i^2/z_2)\mathbf{x}^{+}_{i^1 j}(z_2)\mathbf{x}_i^{+}(z_1) \tag{4.4.2}$$

Proof. Making use of the previous definition, we can write:

$$\begin{aligned}
&\left(z_1 - q_i^2 z_2\right) \mathbf{x}_i^{+}(z_1) \Big[\mathbf{x}_i^{+}(z_2)\mathbf{x}_j^{+}(z_3) - G^{-}_{ij}(z_2/z_3)\mathbf{x}_j^{+}(z_3)\mathbf{x}_i^{+}(z_2)\Big] \\
&= \left(z_1 q_i^2 - z_2\right) \mathbf{x}_i^{+}(z_1)\mathbf{x}_i^{+}(z_2)\mathbf{x}_j^{+}(z_3) - \left(z_1 - q_i^2 z_2\right) G^{-}_{ij}(z_2/z_3)G^{-}_{ij}(z_1/z_3)\mathbf{x}_j^{+}(z_3)\mathbf{x}_i^{+}(z_1)\mathbf{x}_i^{+}(z_2) \\
&+ \left(z_1 - q_i^2 z_2\right) G^{-}_{ij}(z_2/z_3)\delta\left(\frac{z_2}{z_3 q_i^{a_{ij}}}\right) \mathbf{x}^{+}_{i^1 j}(z_1)\mathbf{x}_i^{+}(z_2) \\
&= \left(z_1 q_i^2 - z_2\right) G^{-}_{ij}(z_1/z_3)\mathbf{x}_i^{+}(z_2)\mathbf{x}_j^{+}(z_3)\mathbf{x}_i^{+}(z_1) + \left(z_1 q_i^2 - z_2\right) \delta\left(\frac{z_1}{z_3 q_i^{a_{ij}}}\right) \mathbf{x}_i^{+}(z_2)\mathbf{x}^{+}_{i^1 j}(z_1) \\
&- \left(z_1 q_i^2 - z_2\right) G^{-}_{ij}(z_1/z_3)G^{-}_{ij}(z_2/z_3)\mathbf{x}_j^{+}(z_3)\mathbf{x}_i^{+}(z_2)\mathbf{x}_i^{+}(z_1) - \left(z_1 - z_2 q_i^2\right) G^{-}_{ij}(z_2/z_3)\mathbf{x}^{+}_{i^1 j}(z_1)\mathbf{x}_i^{+}(z_2) \\
&= \left(z_1 q_i^2 - z_2\right) G^{-}_{ij}(z_1/z_3)G^{-}_{ij}(z_2/z_3)\mathbf{x}_j^{+}(z_3)\mathbf{x}_i^{+}(z_2)\mathbf{x}_i^{+}(z_1) \\
&+ \left(z_1 q_i^2 - z_2\right) G^{-}_{ij}(z_1/z_3)\delta\left(\frac{z_2}{q_i^{a_{ij}} z_3}\right) \mathbf{x}^{+}_{i^1 j}(z_2)\mathbf{x}_i^{+}(z_1) \\
&- \left(z_1 q_i^2 - z_2\right) G^{-}_{ij}(z_1/z_3)G^{-}_{ij}(z_2/z_3)\mathbf{x}_j^{+}(z_3)\mathbf{x}_i^{+}(z_2)\mathbf{x}_i^{+}(z_1) \\
&+ \left(z_1 q_i^2 - z_2\right) \delta\left(\frac{z_1}{q_i^{a_{ij}} z_3}\right) \mathbf{x}_i^{+}(z_2)\mathbf{x}^{+}_{i^1 j}(z_1) - \left(z_1 - z_2 q_i^2\right) G^{-}_{ij}(z_2/z_3)\delta\left(\frac{z_1}{q_i^{a_{ij}} z_3}\right) \mathbf{x}^{+}_{i^1 j}(z_1)\mathbf{x}_i^{+}(z_2).
\end{aligned}$$

Thus, we can conclude that:

$$\begin{aligned}
\delta\left(\frac{z_1}{q_i^{a_{ij}} z_3}\right) \Big(\left(z_1 - z_2 q_i^2\right) \mathbf{x}_i^{+}(z_1)\mathbf{x}^{+}_{i^1 j}(z_2) - \left(z_1 q_i^2 - z_2\right) G^{-}_{ij}(z_1/z_3)\mathbf{x}^{+}_{i^1 j}(z_2)\mathbf{x}_i^{+}(z_1)\Big) &= 0 \\
\delta\left(\frac{z_1}{q_i^{a_{ij}} z_3}\right) \Big(\left(z_2 - z_1 q_i^2\right) \mathbf{x}_i^{+}(z_2)\mathbf{x}^{+}_{i^1 j}(z_1) - \left(z_2 q_i^2 - z_1\right) G^{-}_{ij}(z_2/z_3)\mathbf{x}^{+}_{i^1 j}(z_1)\mathbf{x}_i^{+}(z_2)\Big) &= 0.
\end{aligned}$$

and the result follows. □

Lemma 4.4.3. *Let $i \neq j \in \dot{I}$, then $\mathbf{x}^{+}_{i^1 j}(z_1) = \mathbf{x}^{+}_{i j^1}(z_1 q_i^{-a_{ij}})$.*

Proof. By definition 4.4.1, take $m = 1$, we get:

$$[\mathbf{x}_i^+(z_1), \mathbf{x}_{i^0 j}^+(z_2)]_{G_{ij}^-(z_1/z_2)} = \delta\left(\frac{z_1}{z_2 q_i^{a_{ij}}}\right) \mathbf{x}_{i^1 j}^+(z_1).$$

On the other hand, proposition 4.4.2 for $m = 1$ gives,

$$[\mathbf{x}_{j^0 i}^+(z_1); \mathbf{x}_j^+(z_2)]_{G_{i,i^0 j}^-(z_1/z_2)} = \delta\left(\frac{z_1}{z_2 q_j^{a_{ij}}}\right) \mathbf{x}_{j^1 i}^+(z_2).$$

Upon exchanging i and j in the previous equation, we get:

$$[\mathbf{x}_{ij^0}^+(z_1); \mathbf{x}_j^+(z_2)]_{G_{j,j^0 i}^-(z_1/z_2)} = \delta\left(\frac{z_1}{z_2 q_j^{a_{ij}}}\right) \mathbf{x}_{ij^1}^+(z_2).$$

But

$$G_{j,j^0 i}^-(z_1/z_2) = G_{ji}^-(z_1/z_2) = G_{ij}^-(z_1/z_2) = G_{i,i^0 j}^-(z_1/z_2).$$

Moreover,

$$q_j^{a_{ji}} = q^{d_j a_{ji}} = q^{c_{ji}} = q^{c_{ij}} = q^{c_{ji}} = q^{d_i a_{ij}} = q_i^{a_{ij}}.$$

Therefore, we conclude that

$$[\mathbf{x}_{ij^0}^+(z_1), \mathbf{x}_j^+(z_2)]_{G_{ij}^-(z_1/z_2)} = \delta\left(\frac{z_1}{z_2 q_j^{a_{ij}}}\right) \mathbf{x}_{ij^1}^+(z_2) = \delta\left(\frac{z_1}{z_2 q_j^{a_{ij}}}\right) \mathbf{x}_{ij^1}^+(q_i^{-a_{ij}} z_1).$$

□

Generalizing the above result, we can define the following:

Proposition 4.4.4. *For all $i \neq j \in \dot{I}$ and for all $n \in \mathbb{N}^\times$, we let $\mathbf{x}_{i^n j}^+(z) \in \widehat{\dot{\mathrm{U}}_q(\dot{\mathfrak{g}})}[[z, z^{-1}]]$ be defined by:*

$$\left[\mathbf{x}_i^+(z_1); \mathbf{x}_{i^{n-1} j}^+(z_2)\right]_{G_{i,i^{n-1} j}^-(z_1/z_2)} = \delta\left(\frac{z_1}{z_2 q_i^{a_{n-1}}}\right) \mathbf{x}_{i^n j}^+(z_2) \qquad (4.4.3)$$

where, $a_n = a_{ij}$ if $n = 0$ and $a_n = 2$ otherwise, whereas $G_{i,i^{n-1} j}^-(z_1/z_2) = G_{ii}^-(z_1/z_2) G_{i,i^{n-2} j}^-(z_1/z_2 q^{a_{n-2}})$.

Proof. It suffices to show that

$$\left(z_1 - q_i^2 z_2\right) \mathbf{x}_i^+(z_1) \mathbf{x}_{i^{n-1} j}^+(z_2) = \left(z_1 q_i^2 - z_2\right) G_{i,i^{n-2} j}^-(z_1/z_2 q_i^{a_{n-1}}) \mathbf{x}_{i^{n-1} j}^+(z_2) \mathbf{x}_i^+(z_1) \qquad (4.4.4)$$

The proof is straightforward and follows the same steps as for the previous proposition. □

Proposition 4.4.5. *For all $i \neq j \in \dot{I}$,*

$$\mathrm{ad}(\mathbf{x}_i^+(z_1)) \mathbf{x}_j^+(z_2) = \delta\left(\frac{z_1}{q_i^{a_{ij}} z_2}\right) \mathbf{x}_{i^1 j}^+(z_1)$$

Proof. By definition, we have:

$$\begin{aligned}\mathrm{ad}(\mathbf{x}_i^+(z_1))\mathbf{x}_j^+(z_2) &= \sum_i \mathbf{x}_i^+(z_1)_{(1)}\mathbf{x}_j^+(z_2)S(\mathbf{x}_i^+(z_1)_{(2)}) \\ &= \mathbf{x}_i^+(z_1)\mathbf{x}_j^+(z_2) - \mathbf{k}_i^-(z_1)\mathbf{x}_j^+(z_2)\mathbf{k}_i^-(z_1)^{-1}\mathbf{x}_i^+(z_1) \\ &= \Big[\mathbf{x}_i^+(z_1);\mathbf{x}_j^+(z_2)\Big]_{G_{ij}^-(z_1/z_2)} \\ &= \delta\left(\frac{z_1}{q_i^{a_{ij}}z_2}\right)\mathbf{x}_{i^1j}^+(z_1).\end{aligned}$$

□

Therefore, and more generally,

Proposition 4.4.6. *For all $i \neq j \in \dot{I}$,*

$$\mathrm{ad}(\mathbf{x}_i^+(z_1)...\mathbf{x}_i^+(z_m))\mathbf{x}_j^+(v) = \delta\left(\frac{z_1}{q_i^2 z_2}\right)...\delta\left(\frac{z_{m-1}}{q_i^2 z_m}\right)\delta\left(\frac{z_1}{q_i^{a_{ij}}v}\right)\mathbf{x}_{i^mj}^+(z_1)$$

Proof. We prove this proposition by induction on m. The case $m=1$ is the previous proposition. Now assume the result holds for some $m \in \mathbb{N}^\times$. Then,

$$\begin{aligned}&\mathrm{ad}(\mathbf{x}_i^+(z_0)\mathbf{x}_i^+(z_1)...\mathbf{x}_i^+(z_m))\mathbf{x}_j^+(v) = \mathrm{ad}(\mathbf{x}_i^+(z_0))\mathrm{ad}(\mathbf{x}_i^+(z_1)...\mathbf{x}_i^+(z_m))\mathbf{x}_j^+(v) \\ &= \mathbf{x}_i^+(z_0)\mathrm{ad}(\mathbf{x}_i^+(z_1)...\mathbf{x}_i^+(z_m))\mathbf{x}_j^+(v) - \mathbf{k}_i^-(z_0)\mathrm{ad}(\mathbf{x}_i^+(z_1)...\mathbf{x}_i^+(z_m))\mathbf{x}_j^+(v)\mathbf{k}_i^-(z_0)^{-1}\mathbf{x}_i^+(z_0) \\ &= \Big[\mathbf{x}_i^+(z_0);\mathrm{ad}(\mathbf{x}_i^+(z_1)...\mathbf{x}_i^+(z_m))\mathbf{x}_j^+(v)\Big]_{G_{ii}^-(z_0/z_1)...G_{ii}^-(z_0/z_m)G_{ij}^-(z_0/v)} \\ &= \delta\left(\frac{z_1}{q_i^2 z_2}\right)...\delta\left(\frac{z_{m-1}}{q_i^2 z_m}\right)\delta\left(\frac{z_1}{q_i^{a_{ij}}v}\right)\Big[\mathbf{x}_i^+(z_0);\mathbf{x}_{i^mj}^+(z_1)\Big]_{G_{ii}^-(z_0/z_1)G_{i,i^mj}^-(z_0q_i^{a_{m-1}}/z_1)}\end{aligned}$$

and the result follows. □

Lemma 4.4.7. *Let $i \neq j$. Then,*

$$[\mathbf{x}_{i^1j}^+(z_1);\mathbf{x}_j^-(z_2)] = -[a_{ji}]_{q_j}\delta\left(\frac{z_1q^{-a_{ij}}C}{z_2}\right)\mathbf{k}_j^-(z_1q_i^{-a_{ij}}C^{1/2})\mathbf{x}_i^+(z_1). \tag{4.4.5}$$

Proof. By definition 4.4.1, we have

$$\begin{aligned}\delta\left(\frac{z_1}{z_2q_i^{a_{ij}}}\right)[\mathbf{x}_{i^1j}^+(z_1);\mathbf{x}_j^-(v)] &= \Big[[\mathbf{x}_i^+(z_1);\mathbf{x}_j^+(z_2)]_{G_{ij}^-(z_1/z_2)};\mathbf{x}_j^-(v)\Big] \\ &= \frac{1}{q_j-q_j^{-1}}\left[\mathbf{x}_i^+(z_1);\left(\delta\left(\frac{z_2}{Cv}\right)\mathbf{k}_j^+(z_2C^{-1/2}) - \delta\left(\frac{z_2C}{v}\right)\mathbf{k}_j^-(vC^{-1/2})\right)\right]_{G_{ij}^-(z_1/z_2)} \\ &= -\frac{1}{q_j-q_j^{-1}}\delta\left(\frac{z_2C}{v}\right)\Big(G_{ji}^+(z_2/z_1) - G_{ij}^-(z_1/z_2)\Big)\mathbf{k}_j^-(vC^{-1/2})\mathbf{x}_i^+(z_1) \\ &= -[a_{ji}]_{q_j}\delta\left(\frac{z_1q^{-a_{ij}}C}{v}\right)\delta\left(\frac{z_1}{z_2q_i^{a_{ij}}}\right)\mathbf{k}_j^-(z_1q_i^{-a_{ij}}C^{1/2})\mathbf{x}_i^+(z_1).\end{aligned}$$

□

The result follows after taking the residue with respect to z_2.

Corollary 4.4.8. *$i \neq j$. Then,*

$$\left[\mathbf{x}^+_{ji^1}(z_1); \mathbf{x}^-_i(v)\right] = -[a_{ji}]_{q_i} \delta\left(\frac{z_1 q_j^{-a_{ji}} C}{v}\right) \mathbf{k}^-_i(z_1 C^{1/2}) \mathbf{x}^+_j(z_1 q_i^{a_{ij}}). \tag{4.4.6}$$

Proof. This result follows immediately from the previous lemma upon interchanging i and j. □

Lemma 4.4.9. *Let $i \neq j \in \dot{I}$. Then $\forall n \in \mathbb{N}$*

i)

$$\left[\mathbf{x}^+_{i^n j}(z); \mathbf{x}^-_i(v)\right] = \delta\left(\frac{z}{Cv}\right) A_n \mathbf{x}^+_{i^{n-1}j}(zq_i^{-a_{n-1}}) \mathbf{k}^+_i(zC^{-1/2}) \tag{4.4.7}$$

where,

$$A_n = [n]_{q_i}[a_{ij} + n - 1]_{q_i} \tag{4.4.8}$$

ii)

$$\left[\mathbf{x}^-_i(v); \mathbf{x}^+_{ji^n}(z)\right] = \delta\left(\frac{v}{Cz}\right) A_n \mathbf{k}^-_i(zC^{-1/2}) \mathbf{x}^+_{ji^{n-1}}(zq_i^{a_{n-1}}). \tag{4.4.9}$$

Proof. The case $n = 0$ holds since it is one of the algebra defining relations. Now assume the result holds for some $n \in \mathbb{N}$, therefore, by the definition of $\mathbf{x}^+_{i^n j}(z)$, we can write:

$$\begin{aligned}
&\left[\left[\mathbf{x}^+_i(z_1); \mathbf{x}^+_{i^n j}(z_2)\right]_{G^-_{i,i^{-a_{ij}}j}(v/zq_i^n)}; \mathbf{x}^-_i(v)\right] = \delta\left(\frac{z_1}{z_2 q_i^{a_n}}\right) \left[\mathbf{x}^+_{i^{n+1}j}(z_1); \mathbf{x}^-_i(v)\right] \\
&= \frac{1}{q_i - q_i^{-1}} \left[\delta\left(\frac{z_1}{Cv}\right) \mathbf{k}^+_i(z_1 C^{-1/2}) - \delta\left(\frac{v}{Cz_1}\right) \mathbf{k}^-_i(vC^{-1/2}); \mathbf{x}^+_{i^n j}(z_2)\right]_{G^-_{i,i^n j}(z_1/z_2)} \\
&+ \left[\mathbf{x}^+_i(z_1); \left[\mathbf{x}^+_{i^n j}(z_2); \mathbf{x}^-_i(v)\right]\right]_{G^-_{i,i^n j}(z_1/z_2)} \\
&= \frac{1}{q_i - q_i^{-1}} \delta\left(\frac{z_1}{Cv}\right) \left[G^+_{i,i^n j}(z_2/z_1) - G^-_{i,i^n j}(z_1/z_2)\right] \mathbf{x}^+_{i^n j}(z_2) \mathbf{k}^+_i(z_1 C^{-1/2}) \\
&+ \delta\left(\frac{z_2}{Cv}\right) A_n \left[\mathbf{x}^+_i(z_1); \mathbf{x}^+_{i^{n-1}j}(z_2 q_i^{a_{n-1}}) \mathbf{k}^+_i(z_2 C^{-1/2})\right]_{G^-_{i,i^n j}(z_1/z_2)} \\
&= \frac{1}{q_i - q_i^{-1}} \delta\left(\frac{z_1}{Cv}\right) \left[G^+_{i,i^n j}(z_2/z_1) - G^-_{i,i^n j}(z_1/z_2)\right] \mathbf{x}^+_{i^n j}(z_2) \mathbf{k}^+_i(z_1 C^{-1/2}) \\
&+ \delta\left(\frac{z_2}{Cv}\right) A_n \left[\mathbf{x}^+_i(z_1); \mathbf{x}^+_{i^{n-1}j}(z_2 q_i^{a_{n-1}})\right]_{G^-_{i,i^{n-1}j}(z_1/z_2)} \mathbf{k}^+_i(z_2 C^{-1/2}) \\
&= -\delta\left(\frac{z_1}{Cv}\right) \delta\left(\frac{z_1}{z_2}\right) A_n \mathbf{x}^+_{i^n j}(z_2) \mathbf{k}^+_i(z_2 C^{-1/2}) \\
&+ A_{n+1} \delta\left(\frac{z_1}{Cv}\right) \delta\left(\frac{z_1}{z_2 q_i^2}\right) \mathbf{x}^+_{i^n j}(z_2) \mathbf{k}^+_i(z_1 C^{-1/2}) + \delta\left(\frac{z_1}{Cv}\right) \delta\left(\frac{z_1}{z_2}\right) A_n \mathbf{x}^+_{i^n j}(z_2) \mathbf{k}^+_i(z_2 C^{-1/2}).
\end{aligned}$$

and the result follows. Finally, it suffices to apply η to i) to get ii). □

Lemma 4.4.10. *For all $n \in \mathbb{N}$, we have:*

i. $\left[\mathbf{x}^+_{i^{n+1}j}(z_1); \mathbf{x}^-_{ji^n}(z_2)\right] = \alpha_n \delta\left(\frac{Cz_1 q_i^{\delta_n}}{z_2}\right) \mathbf{k}^-_j(z_1 C^{1/2} q_i^{\gamma_n}) \prod_{p=1}^n \mathbf{k}^-_i(z_1 C^{1/2} q_i^{\gamma_{n,p}}) \mathbf{x}^+_j(z_1 q_i^{\epsilon_n})$

ii. $\left[\mathbf{x}^+_{i^n j}(z_1); \mathbf{x}^-_{ji^{n+1}}(z_2)\right] = \phi(\alpha_n) \delta\left(\frac{Cz_2 q_i^{\delta_n}}{z_2}\right) \mathbf{x}^-_i(z_2 q_i^{\epsilon_n}) \mathbf{k}^+_j(z_2 C^{1/2} q_i^{\gamma_n}) \prod_{p=1}^n \mathbf{k}^+_i(z_2 C^{1/2} q_i^{\gamma_{n,p}})$

iii.

$$\left[\mathbf{x}^{+}_{i^n j}(z_1); \mathbf{x}^{-}_{j i^n}(z_2)\right] = \beta_n \left(\delta\left(\frac{z_1}{C z_2}\right) \prod_{p=1}^{n} \mathbf{k}^{+}_{i}(z_1 C^{1/2} q_i^{\tilde{\gamma}_{n,p}}) \mathbf{k}^{+}_{j}(z_1 C^{1/2} q_i^{\gamma_n}) \right. \\ \left. - \prod_{p=1}^{n} \mathbf{k}^{-}_{i}(z_2 C^{-1/2} q_i^{\tilde{\gamma}_{n,p}}) \mathbf{k}^{-}_{j}(z_2 C^{-1/2} q_i^{\gamma_n}) \right) \tag{4.4.10}$$

Proof. It is clear that $\forall n \in \mathbb{N}$, ii. is a consequence of i. - upon applying ϕ. Therefore, it suffices to prove the first and last equation. This is done by recursion. Both points hold for $n = 0$; i. boils down to the previous lemma if we set $\alpha_0 = -[a_{ji}]_{q_j}$, $\delta_0 = \gamma_0 = -a_{ij}$, and $\epsilon_0 = 0$.

Now let us assume that both i. and iii. hold for the same $n \in \mathbb{N}$ we have:

$$\begin{aligned} \left[\mathbf{x}^{+}_{i^{n+2} j}(z_1); \mathbf{x}^{-}_{j i^{n+1}}(v)\right] \delta\left(\frac{z_1}{z_2 q^{a_{n+1}}}\right) &= \left[\left[\mathbf{x}^{+}_{i}(z_1); \mathbf{x}^{+}_{i^{n+1} j}(z_2)\right]_{G^{-}_{i, i^{n+1} j}\left(\frac{z_1}{z_2}\right)}; \mathbf{x}^{-}_{j i^{n+1}}(v)\right] \\ &= \left[\left[\mathbf{x}^{+}_{i}(z_1); \mathbf{x}^{-}_{j i^{n+1}}(v)\right]; \mathbf{x}^{+}_{i^{n+1} j}(z_2)\right]_{G^{-}_{i, i^{n+1} j}\left(\frac{z_1}{z_2}\right)} \\ &+ \left[\mathbf{x}^{+}_{i}(z_1); \left[\mathbf{x}^{+}_{i^{n+1} j}(z_2); \mathbf{x}^{-}_{j i^{n+1}}(v)\right]\right]_{G^{-}_{i, i^{n+1} j}\left(\frac{z_1}{z_2}\right)} \\ &= \delta\left(\frac{z_1 C}{v}\right) A_{n+1} \left[\mathbf{k}^{-}_{i}(v C^{-1/2}) \mathbf{x}^{-}_{j i^{n+1}}(v q^{-a_n}); \mathbf{x}^{+}_{i^{n+1} j}(z_2)\right]_{G^{-}_{i, i^{n+1} j}\left(\frac{z_1}{z_2}\right)} \\ &+ \left[\mathbf{x}^{+}_{i}(z_1); \left[\mathbf{x}^{+}_{i^{n+1} j}(z_2); \mathbf{x}^{-}_{j i^{n+1}}(v)\right]\right]_{G^{-}_{i, i^{n+1} j}\left(\frac{z_1}{z_2}\right)} \end{aligned}$$

Now we focus on the last term, we can write:

$$\begin{aligned} \left[\mathbf{x}^{+}_{i^{n+1} j}(z_2); \mathbf{x}^{-}_{j i^{n+1}}(v)\right] \delta\left(\frac{z_1}{z_2 q^{a_n}}\right) &= \left[\left[\mathbf{x}^{+}_{i}(z_1); \mathbf{x}^{+}_{i^{n} j}(z_2)\right]_{G^{-}_{i, i^{n} j}\left(\frac{z_1}{z_2}\right)}; \mathbf{x}^{-}_{j i^{n+1}}(v)\right] \\ &= \left[\left[\mathbf{x}^{+}_{i}(z_1); \mathbf{x}^{-}_{j i^{n+1}}(v)\right]; \mathbf{x}^{+}_{i^{n} j}(z_2)\right]_{G^{-}_{i, i^{n} j}\left(\frac{z_1}{z_2}\right)} \\ &+ \left[\mathbf{x}^{+}_{i}(z_1); \left[\mathbf{x}^{+}_{i^{n} j}(z_2); \mathbf{x}^{-}_{j i^{n+1}}(v)\right]\right]_{G^{-}_{i, i^{n} j}\left(\frac{z_1}{z_2}\right)} \\ &= \delta\left(\frac{z_1 C}{v}\right) A_{n} \left[\mathbf{k}^{-}_{i}(v C^{-1/2}) \mathbf{x}^{-}_{j i^{n}}(v q^{-a_{n-1}}); \mathbf{x}^{+}_{i^{n} j}(z_2)\right]_{G^{-}_{i, i^{n} j}\left(\frac{z_1}{z_2}\right)} \\ &+ \left[\mathbf{x}^{+}_{i}(z_1); \left[\mathbf{x}^{+}_{i^{n} j}(z_2); \mathbf{x}^{-}_{j i^{n+1}}(v)\right]\right]_{G^{-}_{i, i^{n} j}\left(\frac{z_1}{z_2}\right)}. \end{aligned}$$

We can now make use of ii. and iii. and the result follows.

Proving iii. follows in the exact similar steps, making use of i. and ii. □

Definition 4.4.11. $\forall i \neq j \in \dot{I}$

$$T_i(\mathbf{x}^{+}_{j}(z)) = \frac{\mathbf{x}^{+}_{i^{-a_{ij}} j}(z q_i^{-a_{ij}})}{[-a_{ij}]_{q_i}} \tag{4.4.11}$$

where we recursiely define $\mathbf{x}^{+}_{i^n j}(z)$ by setting:

$$\left[\mathbf{x}^{+}_{i}(z_1); \mathbf{x}^{+}_{i^{n-1} j}(z_2)\right]_{G^{-}_{i, i^{n-1} j}(z_1/z_2)} = \delta\left(\frac{z_1}{q_i^{a_{m-1}} z_2}\right) \mathbf{x}^{+}_{i^n j}(z_2). \tag{4.4.12}$$

Main theorem

Theorem 4.4.12. *$\forall i \neq j \in \dot{I}$, setting T_i as:*

$$\begin{aligned}
&T_i(C) = C, \qquad T_i(D) = D \\
&T_i(\mathbf{x}_i^+(z)) = -\mathbf{x}_i^-(zC^{-1})\mathbf{k}_i^+(zC^{-1/2})^{-1}, \qquad T_i(\mathbf{x}_i^-(z)) = -\mathbf{k}_i^-(zC^{-1/2})^{-1}\mathbf{x}_i^+(zC^{-1}) \\
&T_i(\mathbf{k}_i^\pm(z)) = \mathbf{k}_i^\pm(z)^{-1}, \qquad T_i(\mathbf{k}_j^\pm(z)) = \prod_{p=1}^{|-a_{ij}|} \mathbf{k}_i^\pm(zq_i^{2-2p-a_{ij}})\mathbf{k}_j^\pm(zq_i^2) \\
&T_i(\mathbf{x}_j^+(z)) = \frac{\mathbf{x}^+_{i^{-a_{ij}}j}(zq_i^{-a_{ij}})}{[-a_{ij}]_{q_i}}, \quad T_i(\mathbf{x}_j^-(z)) = \frac{\mathbf{x}^-_{ji^{-a_{ij}}}(zq_i^{-a_{ij}})}{[-a_{ij}]_{q_i}}.
\end{aligned}$$

makes T_i into an algebra homomorphism on $\widehat{\dot{\mathrm{U}}_q(\dot{\mathfrak{g}})}$.

Moreover, by setting:

$$\begin{aligned}
&T_i^{-1}(C) = C, \qquad T_i^{-1}(D) = D \\
&T_i^{-1}(\mathbf{x}_i^+(z)) = -\mathbf{k}_i^+(zC^{1/2})^{-1}\mathbf{x}_i^-(zC), \qquad T_i^{-1}(\mathbf{x}_i^-(z)) = -\mathbf{x}_i^+(zC)\mathbf{k}_i^-(zC^{1/2})^{-1} \\
&T_i^{-1}(\mathbf{k}_i^\pm(z)) = \mathbf{k}_i^\pm(z)^{-1}, \qquad T_i^{-1}(\mathbf{k}_j^\pm(z)) = \prod_{p=1}^{|-a_{ij}|} \mathbf{k}_i^\pm(zq_i^{2-2p-a_{ij}})\mathbf{k}_j^\pm(zq_i^{-2}) \\
&T_i^{-1}(\mathbf{x}_j^+(z)) = \frac{\mathbf{x}^+_{ji^{-a_{ij}}}(zq^{a_{ij}})}{[-a_{ij}]_{q_i}}, \quad T_i^{-1}(\mathbf{x}_j^-(z)) = \frac{\mathbf{x}^-_{i^{-a_{ij}}j}(zq^{a_{ij}})}{[-a_{ij}]_{q_i}}.
\end{aligned}$$

we get an action of $\ddot{\mathfrak{B}}$ on $\widehat{\dot{\mathrm{U}}_q(\dot{\mathfrak{g}})}$.

Proof. It suffices to check this on the algebra relations. Equations 4.2.1 - 4.2.7 are straightforward and left to the reader. The remaining part of this chapter will be dedicated to developing the needed machinery for proving this theorem on the rest of the algebra relations. Then, we can check T_i^{-1} on the generators. □

4.5 Proof of the main theorem

Proposition 4.5.1. *We have*

i. $\varphi \circ T_\pi = T_\pi \circ \varphi$;

ii. $\varphi \circ T_i = T_i \circ \varphi$;

iii. $T_i^{-1} = \eta \circ T_i \circ \eta$.

Proposition 4.5.2. *For all $i, j \in \dot{I}$ and $a_{ij} \in \{-1, -2, -3, -4\}$ the q-Serre relation is equivalent to*

$$\mathbf{x}^+_{i^{1-a_{ij}}j}(z) = 0 \tag{4.5.1}$$

Proof. The proof is cumbersome but straightforward. However, we will highlight the main steps below for the case $a_{ij} = -3$ and all the other cases are similar.

The q-Serre relations is given by:

$$\sum_{\sigma\in S_{1-a_{ij}}}\sum_{k=0}^{1-a_{ij}}(-1)^k\binom{1-a_{ij}}{k}_{q_i}\mathbf{x}_i^{\pm}(z_{\sigma(1)})\cdots\mathbf{x}_i^{\pm}(z_{\sigma(k)})\mathbf{x}_j^{\pm}(z)\mathbf{x}_i^{\pm}(z_{\sigma(k+1)})\cdots\mathbf{x}_i^{\pm}(z_{\sigma(1-a_{ij})})=0 \tag{4.5.2}$$

We start by expanding the sum over k. Then, we pick a specific ordering and by which we mean that we decide to move all the $\mathbf{x}_j^{+}(z)$ to the leftmost side of each term by using equation 4.4.12. Clearly, this will create terms in $\mathbf{x}_{i^1j}^{+}(z)$ which we decide to move to the leftmost side of every term too. After repeating this process for $\mathbf{x}_{i^2j}^{+}(z)$, and $\mathbf{x}_{i^3j}^{+}(z)$ we now move to canceling all the terms except the ones with $\mathbf{x}_{i^3j}^{+}(z)$. Obviously, the terms in $\mathbf{x}_{i^nj}^{+}(z)$ for $n=0,1,2$ can only be canceled with other terms of the same value of n. In order to do that, we factorize the coefficients of each term so that we can make use of the algebra relation given by equation 4.2.8. Finally, the only reamining terms will combine to give us the result of the proposition. □

Lemma 4.5.3. *For all* $i\neq j\in\dot{I}$,

$$\Big[T_i(\mathbf{x}_j^{+}(z));T_i(\mathbf{x}_j^{-}(v))\Big]=\frac{1}{q_j-q_j^{-1}}\left(\delta\left(\frac{z}{Cv}\right)T_i\left(\mathbf{k}_j^{+}(zC^{-1/2}\right)-\delta\left(\frac{zC}{v}\right)T_i\left(\mathbf{k}_j^{-}(vC^{-1/2})\right)\right) \tag{4.5.3}$$

where,

$$T_i\left(\mathbf{k}_j^{\pm}(z)\right)=\prod_{p=1}^{|-a_{ij}|}\mathbf{k}_i^{\pm}(zq_i^{2-2p-a_{ij}})\mathbf{k}_j^{\pm}(zq_i^{2}) \tag{4.5.4}$$

Proof. It suffices to use the definition of T_i on the generators as well as lemma 4.4.10 to get:

$$\begin{aligned}\Big[T_i(\mathbf{x}_j^{+}(z));T_i(\mathbf{x}_j^{-}(v))\Big]&=\frac{1}{[-a_{ij}]_{q_i}^2}\left[\mathbf{x}_{i^{-a_{ij}}j}^{+}(zq_i^{-a_{ij}});\mathbf{x}_{ji^{-a_{ij}}}^{-}(vq_i^{-a_{ij}})\right]\\&=\frac{\beta_{-a_{ij}}}{[-a_{ij}]_{q_i}^2}\Bigg(\delta\left(\frac{z}{Cv}\right)\prod_{p=1}^{|-a_{ij}|}\mathbf{k}_i^{+}(zq_i^{\tilde{\gamma}_{-a_{ij},p-a_{ij}}}C^{-1/2})\mathbf{k}_j^{+}(zq_i^{\tilde{\gamma}_{-a_{ij}}}C^{-1/2})\\&\qquad-\delta\left(\frac{Cz}{v}\right)\prod_{p=1}^{|-a_{ij}|}\mathbf{k}_i^{-}(vq_i^{\tilde{\gamma}_{-a_{ij},p-a_{ij}}}C^{-1/2})\mathbf{k}_j^{-}(vq_i^{\tilde{\gamma}_{-a_{ij}}}C^{-1/2})\Bigg)\end{aligned}$$

and the result follows provided we have:

$$\frac{\beta_{-a_{ij}}}{[-a_{ij}]_{q_i}^2}=\frac{1}{q_j-q_j^{-1}}. \tag{4.5.5}$$

□

Lemma 4.5.4. $\forall i\neq j\in\dot{I}$,

$$\Big[T_i(\mathbf{x}_j^{+}(z));T_i(\mathbf{x}_i^{-}(v))\Big]=0. \tag{4.5.6}$$

Proof. By definition, the latter is equivalent to

$$\begin{aligned}-\left[\mathbf{x}_{i^{-a_{ij}}j}^{+}(zq_i^{-a_{ij}});\mathbf{k}_i^{-}(vC^{-1/2})^{-1}\mathbf{x}_i^{+}(vC^{-1})\right]&=\mathbf{k}_i^{-}(vC^{-1/2})^{-1}\left[\mathbf{x}_{i^{-a_{ij}}j}^{+}(zq_i^{-a_{ij}});\mathbf{x}_i^{+}(vC^{-1})\right]_{G^{-}_{i,i^{-a_{ij}}j}(v/zq_i^{-a_{ij}})}\\&=\delta\left(\frac{C^{-1}v}{zq_i^{2-a_{ij}}}\right)\mathbf{k}_i^{-}(vC^{-1/2})^{-1}\mathbf{x}_{i^{1-a_{ij}}j}^{+}(zq_i^{-a_{ij}})=0\end{aligned}$$

by virtue of the q-Serre relation. ☐

Lemma 4.5.5. *$\forall i \in \dot{I}$,*

$$\left[T_i(\mathbf{x}_i^+(z)); T_i(\mathbf{x}_i^-(v))\right] = \frac{1}{q_i - q_i^{-1}} \left(\delta\left(\frac{z}{Cv}\right) T_i\left(\mathbf{k}_i^+(zC^{-1/2}\right) - \delta\left(\frac{zC}{v}\right) T_i\left(\mathbf{k}_i^-(vC^{-1/2})\right)\right). \qquad (4.5.7)$$

Proof. The proof is similar to that of the previous two lemmas and follows immediately from the definition of T_i and the algebra relations. ☐

Definition 4.5.6. For every $n \in \mathbb{N}$ and every $i \neq j \in \dot{I}$, we recursively define $\mathbf{x}_{ji^n}^+(z)$ by setting:

$$\mathbf{x}_{ji^n}^+(z) = \eta(\mathbf{x}_{i^n j}^+(1/z) \qquad (4.5.8)$$

Proposition 4.5.7. *$\forall i \neq j \in \dot{I}$, $\forall n \in \mathbb{N}$*

$$\left[\mathbf{x}_{ji^{n-1}}^+(z_1); \mathbf{x}_i^+(z_1)\right]_{G^-_{i,i^{n-1}j}(z_1/z_2)} = \delta\left(\frac{z_1}{q_i^{a_{m-1}} z_2}\right) \mathbf{x}_{ji^n}^+(z_2). \qquad (4.5.9)$$

Definition 4.5.8. $\forall i \in \dot{I}$ we denote by $\dot{\mathrm{U}}_q^+(\dot{\mathfrak{g}})[i]$ the closed subalgebra of $\widehat{\dot{\mathrm{U}}_q^+(\dot{\mathfrak{g}})}$ generated by $\{\mathrm{res}_z\ z^{-1+m}\ \mathbf{x}_{ji^n}^+(z) : m \in \mathbb{Z},\ n \in \mathbb{N},\ j \in \dot{I} - \{i\}\}$, and by $\dot{\mathrm{U}}_q^+(\dot{\mathfrak{g}})[i]^\eta$ the closed subalgebra of $\widehat{\dot{\mathrm{U}}_q^+(\dot{\mathfrak{g}})}$ generated by $\{\mathrm{res}_z\ z^{-1+m}\ \mathbf{x}_{i^n j}^+(z) : m \in \mathbb{Z},\ n \in \mathbb{N},\ j \in \dot{I} - \{i\}\}$. Clearly, $\dot{\mathrm{U}}_q^+(\dot{\mathfrak{g}})[i]^\eta = \eta(\dot{\mathrm{U}}_q^+(\dot{\mathfrak{g}})[i])$.

Lemma 4.5.9. *$\forall i \in \dot{I}$, we have*

i)

$$\widehat{\dot{\mathrm{U}}_q^+(\dot{\mathfrak{g}})} = \widehat{\bigoplus_{p\in\mathbb{N}}} x_{i,m_1}^+ \dots x_{i,m_p}^+ \dot{\mathrm{U}}_q^+(\dot{\mathfrak{g}})[i]; m_1, \dots, m_p \in \mathbb{Z} \qquad (4.5.10)$$

ii)

$$\widehat{\dot{\mathrm{U}}_q^+(\dot{\mathfrak{g}})} = \widehat{\bigoplus_{p\in\mathbb{N}}} \dot{\mathrm{U}}_q^+(\dot{\mathfrak{g}})[i]^\eta x_{i,m_1}^+ \dots x_{i,m_p}^+ \dot{\mathrm{U}}_q^+(\dot{\mathfrak{g}})[i]; m_1, \dots, m_p \in \mathbb{Z}. \qquad (4.5.11)$$

Proof. Any product of elements in $\{\mathrm{res}_z\, z^{-1+p}\mathbf{x}_i^+(z);\ \mathrm{res}_z\, z^{-1+m}\mathbf{x}_{ji^n}^+(z) :\ p, m \in \mathbb{Z}, n \in \mathbb{N}, j \in \dot{I} - \{i\}\}$ can be rewritten with modes of $\mathbf{x}_i^+(z)$ on the left by repeatedly making use of the identity

$$\mathbf{x}_{ji^{m-1}}^+(z_1)\mathbf{x}_i^+(z_2) = G_{i,i^{m-1}j}\left(\frac{z_1}{z_2}\right) \mathbf{x}_i^+(z_2)\mathbf{x}_{ji^{m-1}}^+(z_1). \qquad (4.5.12)$$

It is therefore an element of the r.h.s. of i). Moreover, since $\mathbf{x}_j^+(z) = \mathbf{x}_{ji^0}^+(z)$, any word over $\{\mathrm{res}_z\, z^{-1+m}\mathbf{x}_k^+(z) : m \in \mathbb{Z}, k \in I\}$ is a product of elements in the set $\{\mathrm{res}_z\, z^{-1+p}\mathbf{x}_i^+(z);\ \mathrm{res}_z\, z^{-1+m}\mathbf{x}_{ji^n}^+(z) :\ p, m \in \mathbb{Z}, n \in \mathbb{N}, j \in \dot{I} - \{i\}\}$ and the lemma follows. Part ii) follows by applying η on i). ☐

Lemma 4.5.10. *Let $i \in \dot{I}$. T_i restricts to an algebra isomorphism:*

$$T_i|_{\dot{\mathrm{U}}_q^+(\dot{\mathfrak{g}})[i]} : \dot{\mathrm{U}}_q^+(\dot{\mathfrak{g}})[i] \to \dot{\mathrm{U}}_q^+(\dot{\mathfrak{g}})[i]^\eta \qquad (4.5.13)$$

whose inverse is $T_i^{-1}|_{\dot{\mathrm{U}}_q^+(\dot{\mathfrak{g}})[i]}$.

Proof. It suffices to prove that for every $j \in \dot{I} - \{i\}$ and for every $n \in [\![0, 1 - a_{ij}]\!]$

$$T_i(\mathbf{x}^+_{ji^n}(z)) = \begin{cases} \dfrac{\mathbf{x}^+_{i^{-a_{ij}-n}j}(zq_i^{a_n})}{N_n}, & \text{if } n \in [\![0, -1 - a_{ij}]\!] \\ \dfrac{\mathbf{x}^+_j(zq_i^{-a_{ij}})}{N_{-a_{ij}}}, & \text{if } n = -a_{ij} \\ 0 & \text{if } n = 1 - a_{ij}. \end{cases} \tag{4.5.14}$$

We do this recursively on n. The case $n = 0$ is just the definition and allows us to set $N_0 = [-a_{ij}]_{q_i}$. Now suppose that 4.5.14 holds for some $n \in [\![0, -1 - a_{ij}]\!]$ then it clearly holds for $n \geq 1 - a_{ij}$ provided we set $\mathbf{x}^+_{i^pj}(z) = 0$ since $\mathbf{x}^+_{i^{1-a_{ij}}j}(z) = 0$. Then, we have:

$$\begin{aligned} \delta\left(\frac{z_1}{q_i^{a_n} z_2}\right) T_i(\mathbf{x}^+_{ji^{n+1}}(z_2) &= T_i\left(\left[\mathbf{x}^+_{ji^{n+1}}(z_2), \mathbf{x}^+_i(z_1)\right]_{G^-_{i,i^nj}(z_1/z_2)}\right) \\ &= \left[T_i(\mathbf{x}^+_{ji^{n+1}}(z_2)), T_i(\mathbf{x}^+_i(z_1))\right]_{G^-_{i,i^nj}(z_1/z_2)} \\ &= -\left[\frac{\mathbf{x}^+_{i^{-a_{ij}-n}j}(z_1q_i^{-a_n})}{N_n}, \mathbf{x}^-_i(z_2C^{-1})\mathbf{k}^+_i(z_2C^{-1/2})^{-1}\right]_{G^-_{i,i^nj}(z_1/z_2)} \\ &= -\left[\frac{\mathbf{x}^+_{i^{-a_{ij}-n}j}(z_1q_i^{-a_n})}{N_n}, \mathbf{x}^-_i(z_2C^{-1})\right]_{G^-_{i,i^nj}(z_1/z_2)\tilde{G}^-_{i,i^{-a_{ij}-n}j}(z_1/z_2)} \mathbf{k}^+_i(z_2C^{-1/2})^{-1} \end{aligned}$$

Hence, we get:

$$T_i(\mathbf{x}^+_{ji^{n+1}}(z_2) = \frac{\mathbf{x}^+_{i^{a_{ij}-(n+1)}j}(zq_i^{a_{n+1}})}{N_{n+1}} \tag{4.5.15}$$

This completes the proof provided we set $N_{n+1} = -\frac{N_n}{A_{-a_{ij}}-n}$. ☐

Definition 4.5.11. Let $i \neq j \in \dot{I}$. For every $p \in [\![0; m_{ij} - 1]\!]$, we define:

$$T_{i,j,p} = \underbrace{...T_iT_jT_i}_{\text{p-factors}}, \qquad T'_{i,j,p} = \underbrace{...T_i^{-1}T_j^{-1}T_i^{-1}}_{\text{p-factors}} \tag{4.5.16}$$

where $(m_{ij})_{i,j\in\dot{I}}$ is the Coxeter matrix of the affine Weyl group of $\dot{\mathfrak{g}}$.

Lemma 4.5.12. *Let $i \neq j \in \dot{I}$ and let $p \in [\![0; m_{ij} - 1]\!]$. Then,*

i) $T_{i,j,p}(\mathbf{x}^+_j(z))$, $T_{j,i,p}(\mathbf{x}^+_i(z)) \in U_{ij}[[z, z^{-1}]]$,

ii) $T'_{i,j,p}(\mathbf{x}^+_j(z))$, $T'_{j,i,p}(\mathbf{x}^+_i(z)) \in U_{ij}[[z, z^{-1}]]$,

where we denote by U_{ij} the closed $\widehat{\dot{\mathrm{U}}_q(\dot{\mathfrak{g}})}$ subalgebra generated by

$$\{\operatorname*{res}_z z^{-1+m}\mathbf{x}^+_i(z),\ \operatorname*{res}_z z^{-1+m}\mathbf{x}^+_j(z);\ m, n \in \mathbb{N}\} \tag{4.5.17}$$

Proof. We consider all the values of m_{ij} in $\{2, 3, 4, 6, \infty\}$. It is important to mention that there is no loss of generality in fixing $\left\langle \alpha_i, \alpha_j^\vee \right\rangle = \langle \alpha_j, \alpha_i^\vee \rangle = 0$, or $\langle \alpha_j, \alpha_i^\vee \rangle = -1$, and $\left\langle \alpha_i, \alpha_j^\vee \right\rangle \in \{-1, -2, -3, -4\}$, or finally that $\left\langle \alpha_i, \alpha_j^\vee \right\rangle = \langle \alpha_j, \alpha_i^\vee \rangle = -2$.

i) Case $m_{ij} = 2$, and $\left\langle \alpha_i, \alpha_j^\vee \right\rangle = \langle \alpha_j, \alpha_i^\vee \rangle = 0$. The case $p = 0$ is trivial, and for $p = 1$, we have

$$T_i(\mathbf{x}_j^+(z)) = \mathbf{x}_j^+(z) \in U_{ij}[[z, z^{-1}]] \tag{4.5.18}$$

and similarly for the remaining claims.

ii) Case $m_{ij} = 3$, and $\left\langle \alpha_i, \alpha_j^\vee \right\rangle = \langle \alpha_j, \alpha_i^\vee \rangle = -1$. The case $p = 0$ is obvious.

$$T_i(\mathbf{x}_j^+(z)) = \mathbf{x}_{i^1j}^+(zq_i) \in U_{ij}[[z, z^{-1}]]. \tag{4.5.19}$$

Now, apply T_j to get

$$T_j \circ T_i(\mathbf{x}_j^+(z)) = T_j(\mathbf{x}_{i^1j}^+(zq_i)) = \frac{\mathbf{x}_i^+(zq_i^2q_j)}{N_1} \in U_{ij}[[z, z^{-1}]]. \tag{4.5.20}$$

Similar arguments apply for $T'_{i,j,p}$.

iii) Case $m_{ij} = 4$, and $\left\langle \alpha_i, \alpha_j^\vee \right\rangle = -2, \langle \alpha_j, \alpha_i^\vee \rangle = -1$. The case $p = 0$ is obvious.

$$\begin{aligned} T_i(\mathbf{x}_j^+(z)) &= \frac{\mathbf{x}_{i^2j}^+(zq_i^2)}{[2]_{q_i}} \in U_{ij}[[z, z^{-1}]] \\ T_j(\mathbf{x}_i^+(z)) &= \mathbf{x}_{j^1i}^+(zq_j) \in U_{ij}[[z, z^{-1}]]. \end{aligned}$$

By the previous lemma, we have on one hand

$$\begin{aligned} T_i \circ T_j(\mathbf{x}_i^+(z)) &= \frac{\mathbf{x}_{i^1j}^+(zq_i^2q_j)}{N_1} \in U_{ij}[[z, z^{-1}]], \text{and} \\ T_j \circ T_i \circ T_j(\mathbf{x}_i^+(z)) &= \frac{\mathbf{x}_i^+(zq_i^2q_j^2)}{N_1^2} \in U_{ij}[[z, z^{-1}]] \end{aligned}$$

and on the other hand,

$$\begin{aligned} \delta\left(\frac{z}{vq_i^2}\right) T_j \circ T_i(\mathbf{x}_j^+(z)) &= \delta\left(\frac{z}{vq_i^2}\right) \frac{T_j(\mathbf{x}_{i^2j}^+(zq_i^2))}{[2]_{q_i}} \\ &= \left[T_j(\mathbf{x}_i^+(z), T_j(\mathbf{x}_{ij}^+(v))\right]_{G_{i,ij}^-(z/v)} \\ &= \left[\mathbf{x}_{ji^1}^+(zq_j), \mathbf{x}_i^+(vq_j)\right]_{G_{i,ij}^-(z/v)} \\ &= \delta\left(\frac{z}{vq_i^2}\right) \frac{\mathbf{x}_{ji^2}^+(zq_j^2q_i^{-2})}{N_1} \in U_{ij}[[z, z^{-1}]]. \end{aligned}$$

Eventually,

$$T_i \circ T_j \circ T_i(\mathbf{x}_j^+(z)) = \frac{\mathbf{x}_j^+(zq_j^2q_i^2)}{N_1N_2[2]_{q_i}} \in U_{ij}[[z, z^{-1}]]. \tag{4.5.21}$$

iv) Case $m_{ij} = 6$. The steps are very similar to the previous cases where we make a repeated use of the previous lemma. Finally, we can move to the last case.

v) Case $m_{ij} = \infty$. In that case we must distinguish two subcases:

a) $a_{ij} = \langle \alpha_i, \alpha_j^\vee \rangle = -2 = \langle \alpha_j, \alpha_i^\vee \rangle = a_{ji}$

b) $a_{ij} = \langle \alpha_i, \alpha_j^\vee \rangle = -4$, $a_{ji} = \langle \alpha_j, \alpha_i^\vee \rangle = -1$.

In subcase a), we have:

$$\delta\left(\frac{z}{vq_i^2}\right) T_j \circ T_i(\mathbf{x}_j^+(z)) = \left[\frac{\mathbf{x}_{j^2i}^+(zq_j^2)}{[2]_{q_i}}, \frac{\mathbf{x}_{j^2i}^+(vq_j^{-2})}{N_1}\right]_{G_{i,ij}^-(z/v)} \tag{4.5.22}$$

an induction as in [Lusztig] 40.1.1 shows that: $T_{i,j,p}(\mathbf{x}_j^+(z)) \in U_{ij}[[z, z^{-1}]]$. In subcase b) we have:

$$T_{j,i,2}(\mathbf{x}_i^+(z)) = \frac{\mathbf{x}_{i^3j}^+(zq_jq_i^{-2})}{N_1} \tag{4.5.23}$$

Now,

$$\begin{aligned}\delta\left(\frac{z}{vq_i^2}\right) T_j(\mathbf{x}_{i^2j}^+(z)) &= \left[T_j(\mathbf{x}_i^+(z)), T_j(\mathbf{x}^+i^1j(v))\right]_{G_{i,ij}^-(z/v)}\\ &= \delta\left(\frac{z}{vq_i^2}\right) \mathbf{x}_{ji^2}^+(vq_j).\end{aligned}$$

Similarly,

$$T_j(\mathbf{x}_{i^3j}^+(z)) = \left[\mathbf{x}_{ji}^+zq_j, \mathbf{x}_{ji^2}^+(vq_jq_i^{-2})\right]_{G_{i,i^2j}^-(z/v)} \tag{4.5.24}$$

$$T_j(\mathbf{x}_{i^4j}^+(z)) = \left[\mathbf{x}_{ji}^+zq_j, T_j(\mathbf{x}_{ji^2}^+(v)\right]_{G_{i,i^2j}^-(z/v)} \tag{4.5.25}$$

making it clear that $\forall m = 1, ..., 4$

$$T_i(\mathbf{x}_{i^mj}^+(z)) \in U_{ij}[[z, z^{-1}]] \tag{4.5.26}$$

where $\mathcal{Z}$ is the subalgebra of $\widehat{\dot{\mathrm{U}}_q(\dot{\mathfrak{g}})}$ generated by:

$$\{\operatorname*{res}_z\ z^{-1+m}\mathbf{x}_{ji}^+(z),\ \operatorname*{res}_z\ z^{-1+m}\mathbf{x}_{ji^2}^+(z);\ m, n \in \mathbb{Z}\}. \tag{4.5.27}$$

Clearly $\mathcal{Z} \subset U_{ij}$, and $T_i(\mathcal{Z}) \subset U_{ij}$ thus concluding the proof. ☐

Lemma 4.5.13. *$\forall i, j, k \in \dot{I}$ we have:*

$$T_k\left((z_1 - q^{\pm c_{ij}}z_2)\mathbf{x}_i^\pm(z_1)\mathbf{x}_j^\pm(z_2)\right) = T_k\left((z_1q^{\pm c_{ij}} - z_2)\mathbf{x}_j^\pm(z_2)\mathbf{x}_i^\pm(z_1)\right) \tag{4.5.28}$$

Proof. The case $i \neq j \neq k$ follows from the fact that T_k is an adjoint action.

The remaining cases are proved by using the definition of T_k on the generators and then comparing both sides of the equation after using equation 4.2.7 to move all the $\mathbf{k}_i^\pm(z_l)$ to the left. ☐

Definition 4.5.14. For every $i \in \dot{I}$, and every $m \in \mathbb{Z}$, there exists a unique $\mathbb{F}$-linear homomorphism $r_{i,m}^\pm : \dot{\mathrm{U}}_q(\dot{\mathfrak{g}})^+ \to \dot{\mathrm{U}}_q(\dot{\mathfrak{g}})^+$ such that:

i) $r_{i,m}^\pm(1) = 0$.

ii) $r_{i,m}^\pm(\mathbf{x}_i^+(z_1)\mathbf{x}_i^+(z_2)...\mathbf{x}_i^+(z_n)) = \sum_{p=1}^n \delta_{i_p,i} C^{\mp m} z_p^m \prod_{k=1}^{p-1} G_{i,i_k}\left(\frac{z_p}{z_k}\right)^{\mp 1} \prod_{l \in [\![n]\!]} \mathbf{x}_{i_l}^+(z_1)$

Clearly, for all $i \in \dot{I}$,

$$r_i^{\pm}(v) = \sum_{m\in\mathbb{Z}} r_{i,m}^{\pm} v^{-m} \in \mathrm{Hom}_{\mathbb{F}}(\dot{\mathrm{U}}_q(\dot{\mathfrak{g}})^+)[[v,v^{-1}]] \tag{4.5.29}$$

and we have:

$$r_i^{\pm}(v)(\mathbf{x}_i^+(z_1)\mathbf{x}_i^+(z_2)...\mathbf{x}_i^+(z_n)) = \sum_{p=1}^{n} \delta_{i_p,i}\delta\left(\frac{z_pC^{\mp1}}{v}\right)\prod_{k=1}^{p-1} G_{i,i_k}\left(\frac{z_p}{z_k}\right)^{\mp1}\prod_{l\in[\![n]\!]}\mathbf{x}_{i_l}^+(z_1). \tag{4.5.30}$$

We extend $r_i^{\pm}(v)$ by continuity to $\widehat{\dot{\mathrm{U}}_q(\dot{\mathfrak{g}})}$.

Proposition 4.5.15. *For all $X \in \dot{\mathrm{U}}_q(\dot{\mathfrak{g}})^+$, and all $i \in \dot{I}$, we have:*

$$[X, \mathbf{x}_i^-(v)] = \frac{\mathbf{k}_i^+(vC^{1/2})r_i(v)(X) - \mathbf{k}_i^-(vC^{-1/2})r_i^-(v)(X)}{q_i - q_i^{-1}} \tag{4.5.31}$$

Proof. It suffices to prove this claim for $X = \mathbf{x}_{i_1}^+(z_1)\mathbf{x}_{i_2}^+(z_2)...\mathbf{x}_{i_m}^+(z_m)$. Obviously,

$$\begin{aligned}
&[\mathbf{x}_{i_1}^+(z_1)\mathbf{x}_{i_2}^+(z_2)...\mathbf{x}_{i_m}^+(z_m), \mathbf{x}_i^-(v)] \\
&= \sum_{p=1}^{m} \frac{\delta_{i_p,i}}{q_i - q_i^{-1}} \mathbf{x}_{i_1}^+(z_1)\mathbf{x}_{i_2}^+(z_2)...\mathbf{x}_{i_{p-1}}^+(z_{p-1}) \left(\delta\left(\frac{z_pC}{v}\right)\mathbf{k}_i^+(vC^{-1/2}) - \delta\left(\frac{z_pC^{-1}}{v}\right)\mathbf{k}_i^-(vC^{-1/2})\right) \\
&\times \mathbf{x}_{i_{p+1}}^+(z_{p+1})...\mathbf{x}_{i_m}^+(z_m) \\
&= \sum_{p=1}^{m} \frac{\delta_{i_p,i}}{q_i - q_i^{-1}} \Big[\delta\left(\frac{z_pC^{\mp1}}{v}\right)\prod_{k=1}^{p-1} G_{i,i_k}^-\left(\frac{z_p}{z_k}\right)^{\mp1} \mathbf{k}_i^+(vC^{-1/2}) \prod_{l\in[\![n]\!]-\{p\}} \mathbf{x}_{i_l}^+(z_1) \\
&- \delta\left(\frac{z_pC^{\mp1}}{v}\right)\prod_{k=1}^{p-1} G_{i,i_k}^+\left(\frac{C^{-1}z_p}{z_k}\right)^{\mp1} \mathbf{k}_i^-(vC^{-1/2}) \prod_{l\in[\![n]\!]-\{p\}} \mathbf{x}_{i_l}^+(z_1)\Big].
\end{aligned}$$

□

Lemma 4.5.16. *Let $X \in \widehat{\dot{\mathrm{U}}_q(\dot{\mathfrak{g}})}^+$ and let $i \in \dot{I}$.*

i) If $T_i(X) \in \widehat{\dot{\mathrm{U}}_q(\dot{\mathfrak{g}})}^+$ then $r_i^+(v)(X) = 0$.

ii) If $T_i^{-1}(X) \in \widehat{\dot{\mathrm{U}}_q(\dot{\mathfrak{g}})}^+$ then $r_i^-(v)(X) = 0$.

Proof. Let $X \in \dot{\mathrm{U}}_q(\dot{\mathfrak{g}})^+$. It is a linear combination of

$$\{\operatorname*{res}_{z_1,z_2,...z_n} z_1^{-1+m_1}...z_n^{-1+m_n}\mathbf{x}_{i_1}^+(z_1)\mathbf{x}_{i_2}^+(z_2)...\mathbf{x}_{i_m}^+(z_n)\}. \tag{4.5.32}$$

Without loss of generality we can restrict to cases where X is homogeneous. Assume that $T_i(X) \in \widehat{\dot{\mathrm{U}}_q(\dot{\mathfrak{g}})}^+$.

By the previous proposition

$$[X, \mathbf{x}_i^-(v)] = \frac{\mathbf{k}_i^+(vC^{1/2})r_i(v)(X) - \mathbf{k}_i^-(vC^{-1/2})r_i^-(v)(X)}{q_i - q_i^{-1}} \tag{4.5.33}$$

and lemma 4.5.9,

$$r_i^+(v)(X) = \sum_{p\in\mathbb{N}} \operatorname*{res}_{z_1,...z_p} z_1^{-1}...z_p^{-1}\mathbf{x}_i^+(z_1)...\mathbf{x}_i^+(z_p)Y^{\pm}(v,z_1,...z_p) \tag{4.5.34}$$

for some $Y^{\pm}(v, z_1, ...z_p) \in \dot{\mathrm{U}}_q(\dot{\mathfrak{g}})^+[[v, v^{-1}]]((z_1^{-1}, ..., z_p^{-1}))$. We apply T_i to $[X, \mathbf{x}_i^-(v)]$ and we get:

$$\begin{aligned}
&[T_i(X), \mathbf{k}_i^-(C^{-1/2}v)\mathbf{x}_i^+(C^{-1}v)] \\
&= \frac{1}{q_i - q_i^{-1}} \sum_{p\in\mathbb{N}} \operatorname*{res}_{z_1,...z_p} (-1)^p z_1^{-1}...z_p^{-1} \prod_{k\in[\![(]\!]p)} \mathbf{x}_i^-(C^{-1}z_k)\mathbf{k}_i^-(C^{-1/2}z_k)^{-1}\{\mathbf{k}_i^+(C^{1/2}v)^{-1}T_i(Y^+(v, z_1, ...z_p)) \\
&- \mathbf{k}_i^-(C^{-1/2}v)^{-1}T_i(Y^-(v, z_1, ...z_p))\}.
\end{aligned}$$

The left-hand side is in $\widehat{\dot{\mathrm{U}}_q(\dot{\mathfrak{g}})^{\geq}}$ then so is the right-hand side. Now by the triangular decomposition and the fact that T_i restricts to the subalgebras $\dot{\mathrm{U}}_q(\dot{\mathfrak{g}})^+[i]$, we have

$$\forall p \geq 0, \quad T_i(Y^{\pm}(v, z_1, ...z_p)) = 0 \tag{4.5.35}$$

Since T_i is an automorphism, we have $Y^{\pm}(v, z_1, ...z_p) = 0$. The proof of ii) is similar. □

Lemma 4.5.17. *$\forall X \in \dot{\mathrm{U}}_q(\dot{\mathfrak{g}})^+$, $\forall i \in \dot{I}$, we have:*

i) $\langle X, Y\mathbf{x}_i^-(z)\rangle = \mathrm{res}_v\, v^{-1}\langle \mathbf{k}_i^-(C^{-1/2}v)r_i^-(v)(X)\widehat{\otimes}\mathbf{x}_i^+(v); Y \otimes \mathbf{x}_i^-(z)\rangle$

ii) $\langle X, \mathbf{x}_i^-(z)Y\mathbf{x}_i^-(z)\rangle = \mathrm{res}_v\, v^{-1}\langle \mathbf{x}_i^+(Cv)\widehat{\otimes}r_i^+(v)(X)\widehat{\otimes}\mathbf{x}_i^+(v); \mathbf{x}_i^-(z) \otimes Y\rangle$

Proof. It suffices to prove i) and ii) for any

$$X = \mathbf{x}_{i_1}^-(z_1)...\mathbf{x}_{i_m}^-(z_m) \tag{4.5.36}$$

$$X = \mathbf{x}_{j_1}^-(v_1)...\mathbf{x}_{j_{m-1}}^-(v_{m-1}) \tag{4.5.37}$$

We have

$$\Delta(X) = \sum_{M\subseteq[\![m]\!]} \prod_{k\in[\![m]\!]-M, l\in M} G_{i_l,i_k}^+\left(\frac{z_l}{z_k}\right) \prod_{l\in M} \mathbf{k}_{i_l}^-(z_lC^{-1/2}) \prod_{k\in[\![m]\!]-M} \mathbf{x}_{i_k}^+(z_k)\widehat{\otimes} \prod_{l\in M} \mathbf{x}_{i_l}^+(z_lC) \tag{4.5.38}$$

Hence,

$$\langle X, Y\mathbf{x}_i^-(z)\rangle = \frac{\delta_{i_p,i}}{q_i - q_i^{-1}}\delta\left(\frac{z_pC}{z}\right) \sum_{p\in[\![m]\!]} \prod_{k=1}^{p-1} G_{i_p,i_k}^+\left(\frac{z_p}{z_k}\right) \langle \mathbf{k}_{i_p}^-(z_pC^{1/2}) \prod_{k\in[\![m]\!]-\{p\}} \mathbf{x}_{i_k}^+(z_k); Y\rangle \tag{4.5.39}$$

On the other hand,

$$\begin{aligned}
&\langle \mathbf{k}_i^-(C^{-1/2}v)r_i^-(v)(X)\widehat{\otimes}\mathbf{x}_i^+(v); Y \otimes \mathbf{x}_i^-(z)\rangle \\
&= \frac{-1}{q_i - q_i^{-1}}\delta\left(\frac{z}{v}\right) \sum_{p=1}^{m} \delta_{i,i_p}\delta\left(\frac{z_pC}{v}\right) \prod_{k=1}^{p-1} G_{i,i_k}^+\left(\frac{z_p}{z_k}\right) \langle \mathbf{k}_i^-(vC^{-1/2}) \prod_{l\in[\![m]\!]-\{p\}} \mathbf{x}_{i_l}^+(z_l); Y\rangle
\end{aligned}$$

and i) follows. The proof of ii) is very similar. □

Proposition 4.5.18. *Let $\alpha \in \dot{Q}^+ - \{0\}$ and let $X \in \dot{\mathrm{U}}_q(\dot{\mathfrak{g}})_\alpha^+$. If $r_i^{\pm}(v)(X) = 0$, then $X = 0$.*

Proof. Let X be as above. Either $X = 0$, or $X \neq 0$. In the latter case, the non-degeneracy of the pairing implies that there must exist $Y \in \dot{\mathrm{U}}_q(\dot{\mathfrak{g}})^-$ such that, $\langle X; Y\rangle \neq 0$. In particular, since $X \in \dot{\mathrm{U}}_q(\dot{\mathfrak{g}})_\alpha$, we must have $\langle X; \tilde{Y}\mathbf{x}_i^-(z)\rangle \neq 0$. We also see that $r_i^-(v)(X) = 0$ implies $\langle X; \tilde{Y}\mathbf{x}_i^-(z)\rangle = 0$ which proves the claim for lower choices of signs. The proof for the upper choices of signs is similar. $\square$

Lemma 4.5.19. *Let $i \in \dot{I}$, let $Y \in \dot{\mathrm{U}}_q^+(\dot{\mathfrak{g}})$ and let $X(z) \in \dot{\mathrm{U}}_q^+(\dot{\mathfrak{g}})[[z, z^{-1}]]$ be such that:*

$$X(z)\mathbf{k}_i^\pm(C^{\pm 1/2}v) = \theta_{X,i}^\pm\left(\frac{v}{z}\right)\mathbf{k}_i^\pm(C^{\pm 1/2}v)X(z) \tag{4.5.40}$$

for some $\theta_{X,i}^\pm(z) \in \mathbb{F}_q[[z^{\mp 1}]]$. Then,

$$r_i^\pm(v)(X(z)Y) = r_i^\pm(v)(X(z))Y + \theta_{X,i}^\pm\left(\frac{v}{z}\right)X(z)r_i^\pm(v)(Y) \tag{4.5.41}$$

Proof. We have:

$$\begin{aligned}
&\frac{\mathbf{k}_i^+(vC^{1/2})r_i^+(v)(X(z)Y) - \mathbf{k}_i^-(vC^{-1/2})r_i^-(v)(X(z)Y)}{q_i - q_i^{-1}} = \left[X(z)Y, \mathbf{x}_i^-(v)\right] \\
&= \left[X(z), \mathbf{x}_i^-(v)\right]Y + X(z)\left[Y, \mathbf{x}_i^-(v)\right] \\
&= \frac{1}{q_i - q_i^{-1}}[\Big(\mathbf{k}_i^+(vC^{1/2})r_i^+(v)(X(z)) - \mathbf{k}_i^-(vC^{-1/2})r_i^-(v)(X(z))\Big)Y \\
&+ X(z)\Big(\mathbf{k}_i^+(vC^{1/2})r_i^+(v)(Y) - \mathbf{k}_i^-(vC^{-1/2})r_i^-(v)(Y)\Big)] \\
&= \frac{1}{q_i - q_i^{-1}}\mathbf{k}_i^+(vC^{1/2})[\Big(r_i^+(v)(X(z))Y + \theta_{X,i}^+\left(\frac{z}{v}\right)X(z)r_i^-(v)(Y)\Big) \\
&- \mathbf{k}_i^-(vC^{-1/2})[\Big(r_i^+(v)(X(z))Y + \theta_{X,i}^-\left(\frac{z}{v}\right)X(z)r_i^-(v)(Y)\Big)].
\end{aligned}$$

and the result follows. $\square$

Lemma 4.5.20. *Let $i \in \dot{I}$ and let $X(z), Y(z) \in \dot{\mathrm{U}}_q^+(\dot{\mathfrak{g}})[[z, z^{-1}]]$ be such that:*

$$X(z)\mathbf{k}_i^-(C^{-1/2}v) = \theta_{X,i}^-\left(\frac{v}{z}\right)\mathbf{k}_i^-(C^{-1/2}v)X(z) \tag{4.5.42}$$

$$Y(z)\mathbf{k}_i^-(C^{-1/2}v) = \theta_{Y,i}^-\left(\frac{v}{z}\right)\mathbf{k}_i^-(C^{-1/2}v)Y(z) \tag{4.5.43}$$

for some $\theta_{X,i}^-(z), \theta_{Y,i}^-(z) \in \mathbb{F}_q[[z]]$. Then, $\forall a, b$ for which we can write ${}_a[X(z_1), Y(z_2)]_b$.

$$r_i^-(v)\left({}_a[X(z_1), Y(z_2)]_b\right) = {}_a[r_i^-(v)(X(z_1)), Y(z_2)]_{\theta_{X,i}^-\left(\frac{v}{z_2}\right)b} +_{a\theta_{Y,i}^-\left(\frac{v}{z_1}\right)} [X(z_1), r_i^-(v)(Y(z_2))]_{\theta_{X,i}^-\left(\frac{v}{z_2}\right)b} \tag{4.5.44}$$

Proof.

$$\begin{aligned}
r_i^-(v)\left({}_a[X(z_1), Y(z_2)]_b\right) &= r_i^-(v)\left(aX(z_1)Y(z_2) - bY(z_2)X(z_1)\right) \\
&= a\left[r_i^-(v)(X(z_1))Y(z_2) + \theta_{X,i}^-\left(\frac{v}{z_1}\right)X(z_1)r_i^-(v)(Y(z_2))\right] \\
&- b\left[r_i^-(v)(Y(z_2))X(z_1) + \theta_{Y,i}^-\left(\frac{v}{z_2}\right)Y(z_2)r_i^-(v)(X(z_1))\right]
\end{aligned}$$

and the result follows. $\square$

Lemma 4.5.21. *Let $i \neq j \in \dot{I}$. Then, $\forall k \in \dot{I}$, $\forall n \in \mathbb{N}^\times$,*

$$r_k^+(v)(\mathbf{x}^+_{ji^n}(z)) = \delta_{ki}[n]_{q_i}![a_{ij}+n-1]_{q_i}(q_i - q_i^{-1})\delta\left(\frac{v}{Cz}\right)\mathbf{x}^+_{ji^{n-1}}(zq_i^{a_{n-1}}). \tag{4.5.45}$$

Proof. The case $k = i$ follows from lemma 4.4.9 and prop. 4.5.15 .Moreover, it is clear that for $k \in \dot{I} - \{i, j\}$, we have:

$$[\mathbf{x}^+_{ji^n}(z), \mathbf{x}^-_k(v)] = 0. \tag{4.5.46}$$

We now move to the case $k = j$.

$$\begin{aligned}\delta\left(\frac{z_1}{q_i^{a_{n-1}}z_2}\right)[\mathbf{x}^+_{ji^n}(z_2), \mathbf{x}^-_j(v)] &= [[\mathbf{x}^+_{ji^{n-1}}(z_1), \mathbf{x}^+_i(z_2)]_{G^-_{i,i^{n-1}j}\left(\frac{z_1}{z_2}\right)}, \mathbf{x}^-_j(v)]\\ &= [[\mathbf{x}^+_{ji^{n-1}}(z_1), \mathbf{x}^-_j(v)]_{G^-_{i,i^{n-1}j}\left(\frac{z_1}{z_2}\right)}, \mathbf{x}^+_i(z_2)].\end{aligned}$$

In particular, the case $n = 1$ gives:

$$[[\mathbf{x}^+_{ji^1}(z_1), \mathbf{x}^+_i(z_2)]_{G^-_{ij}\left(\frac{z_1}{z_2}\right)}, \mathbf{x}^-_j(v)] = [a_{ji}]_{q_j}\delta\left(\frac{z_1}{Cv}\right)\delta\left(\frac{z_2 q_j^{a_{ji}}}{z_1}\right)\mathbf{x}^-_i(z_2)\mathbf{k}^+_j(vC^{1/2}). \tag{4.5.47}$$

An easy recursion shows that for every $n \in \mathbb{N}^\times$ this is always a multiple of $\mathbf{k}^+_j(vC^{1/2})$. Therefore, by comparing the result with prop. 4.5.15 we get that $r_i^-(v)(\mathbf{x}^+_{ji^n}(z) = 0$. ☐

Remark 4.5.22. Clearly the case $n = 0$ is nothing but the algebra relation.

4.5.1 Proof of the Serre relations

Definition 4.5.23. For every $i \in \dot{I}$ define $\tilde{r}_i^\pm(v) \in Hom_{\mathbb{F}}(\dot{\mathrm{U}}_q^+(\dot{\mathfrak{g}}))[[v, v^{-1}]]$ by setting

$$\forall X \in \dot{\mathrm{U}}_q^+(\dot{\mathfrak{g}}), \qquad \tilde{r}_i^\pm(v)(X) = \eta\left(r_i^\mp(1/v)(X)\right) \tag{4.5.48}$$

Proposition 4.5.24. $\forall X \in \dot{\mathrm{U}}_q^+(\dot{\mathfrak{g}})$, $\forall i\dot{I}$

$$[X, \mathbf{x}^-_i(v)] = \frac{\tilde{r}_i^+(v)(X)\mathbf{k}^+_i(vC^{1/2}) - \tilde{r}_i^-(v)(X)\mathbf{k}^-_i(vC^{-1/2})}{q_i - q_i^{-1}} \tag{4.5.49}$$

Proof. The proof follows from applying η to prop. 4.5.15. ☐

Proposition 4.5.25. *Let $\alpha \in \dot{Q}^+ - \{0\}$ and let $X \in \dot{\mathrm{U}}_q^+(\dot{\mathfrak{g}})_\alpha$. If $\forall i \in \dot{I}$, $\tilde{r}_i^\pm(v)(X) = 0$, then $X = 0$.*

Proof. The proof is straightforward and follows from the previous definition. ☐

Lemma 4.5.26. *Let $i \in \dot{I}$ and let $X(z), Y(z) \in \dot{\mathrm{U}}_q^+(\dot{\mathfrak{g}})[[z, z^{-1}]]$ be such that:*

$$X(z)\mathbf{k}^-_i(C^{-1/2}v) = \theta^-_{X,i}\left(\frac{v}{z}\right)\mathbf{k}^-_i(C^{-1/2}v)X(z) \tag{4.5.50}$$

$$Y(z)\mathbf{k}^-_i(C^{-1/2}v) = \theta^-_{Y,i}\left(\frac{v}{z}\right)\mathbf{k}^-_i(C^{-1/2}v)Y(z) \tag{4.5.51}$$

Then,

$$\tilde{r}_i^+(v)\left({}_a[X(z_1), Y(z_2)]_b\right) = {}_a[X(z_1), \tilde{r}_i^+(v)(Y(z_2))]_{b\theta_{X,i}\left(\frac{z_1}{v}\right)} + {}_{a\theta_{y,i}\left(\frac{z_2}{v}\right)}[\tilde{r}_i^+(v)(X(z_1)), Y(z_2)]_b \tag{4.5.52}$$

Proof. The proof is similar to that of lemma 4.5.20. □

Lemma 4.5.27. *$\forall l, i, j \in \dot{I}$ and $\forall n \in \mathbb{N}^\times$:*

$$\mathbf{k}_l^+(vC^{1/2})\mathbf{x}_{ji^n}^+(z) = G_{l,ji^n}^-\left(\frac{z}{Cv}\right)\mathbf{x}_{ji^n}^+(z)\mathbf{k}_l^+(vC^{1/2}). \tag{4.5.53}$$

Proof. From the definition of $\mathbf{x}_{ji^1}^+(z)$, we have:

$$\delta\left(\frac{z_1}{z_2 q_i^{a_{ij}}}\right)\mathbf{x}_{ji^1}^+(z_2) = [\mathbf{x}_j^+(z_1), \mathbf{x}_i^+(z_2)]_{G_{ij}^-(z_1/z_2)}. \tag{4.5.54}$$

Then,

$$\begin{aligned}
\delta\left(\frac{z_1}{z_2 q_i^{a_{ij}}}\right)\mathbf{k}_l^+(vC^{1/2})\mathbf{x}_{ji^1}^+(z_2) &= \mathbf{k}_l^+(vC^{1/2})[\mathbf{x}_j^+(z_1), \mathbf{x}_i^+(z_2)]_{G_{ij}^-(z_1/z_2)} \\
&= G_{lj}^-\left(\frac{z_1}{Cv}\right)^{-1} G_{li}^-\left(\frac{z_2}{Cv}\right)^{-1} \mathbf{x}_j^+(z_1), \mathbf{x}_i^+(z_2)]_{G_{ij}^-(z_1/z_2)}\mathbf{k}_l^+(vC^{1/2}) \\
&= \delta\left(\frac{z_1}{z_2 q_i^{a_{ij}}}\right) G_{lj}^-\left(\frac{z_2 q_i^{a_{ij}}}{Cv}\right)^{-1} G_{li}^-\left(\frac{z_2}{Cv}\right)^{-1} \mathbf{x}_{ji^1}^+(z_2)\mathbf{k}_l^+(vC^{1/2})
\end{aligned}$$

Now assuming the result holds for $n \in \mathbb{N}^\times$, we have:

$$\begin{aligned}
\delta\left(\frac{z_1}{z_2 q_i^{a_{ij}}}\right)\mathbf{k}_l^+(vC^{1/2})\mathbf{x}_{ji^{n+1}}^+(z_2) &= \mathbf{k}_l^+(vC^{1/2})[\mathbf{x}_{ji^n}^+(z_1), \mathbf{x}_i^+(z_2)]_{G_{i,i^n j}^-(z_1/z_2)} \\
&= \delta\left(\frac{z_1}{z_2 q_i^{a_{ij}}}\right) G_{l,ji^n}^-\left(\frac{z_2 q_i^{a_{ij}}}{Cv}\right)^{-1} G_{li}^-\left(\frac{z_2}{Cv}\right)^{-1} \mathbf{x}_{ji^{n+1}}^+(z_2)\mathbf{k}_l^+(vC^{1/2})
\end{aligned}$$

which completes the recursion. □

Definition 4.5.28. $\forall n \in \mathbb{N}^\times$, we define $\mathbf{x}_{\square i^n}^+(z_1) \in \dot{\mathrm{U}}_q^+(\dot{\mathfrak{g}})[[z, z^{-1}]]$ recursively by setting :

$$\mathbf{x}_{\square i}^+(z_1) = \mathbf{x}_i^+(z_1) \tag{4.5.55}$$

$$\delta\left(\frac{z_1}{z_2 q_i^2}\right)\mathbf{x}_{\square i^{n+1}}^+(z_1) = [\mathbf{x}_{\square i^n}^+(z_1), \mathbf{x}_i^+(z_2)]_{G_{i,i^n}^-(z_1/z_2)} \tag{4.5.56}$$

Lemma 4.5.29. *Let $i \neq j \in \dot{I}$. Then, $\forall k \in \dot{I}$, $\forall n \in \mathbb{N}^\times$:*

$$\tilde{r}_k^+(v)(\mathbf{x}_{ji^n}^+(z)) = \delta_{jk}\delta\left(\frac{z}{Cvq_i^{\gamma_{i,j,n}}}\right)\beta_{i,j,n}\mathbf{x}_{\square i^n}^+(z) \tag{4.5.57}$$

where,

$$\gamma_{i,j,n} = \begin{cases} - \quad a_{ij} - 2(n-1), & \text{if } n \in \mathbb{N}^\times \\ 0 \quad \text{otherwise} \end{cases} \tag{4.5.58}$$

$$\beta_{i,j,n} = \prod_{p=1}^{n-1} \frac{[p + a_{ij}]}{[p]_{q_i}}. \tag{4.5.59}$$

Proof. By induction on $n \in \mathbb{N}$. The case $n = 0$ is from the defining relations. Now suppose the lemma holds for some $n \in \mathbb{N}^\times$.

$$\begin{aligned}\delta\left(\frac{z_1}{z_2 q_i^{a_n}}\right)\tilde{r}_k^+(v)(\mathbf{x}^+_{ji^{n+1}}(z_2)) &= \tilde{r}_k^+(v)([\mathbf{x}^+_{ji^n}(z_1), \mathbf{x}^+_i(z_1)]_{G^-_{i,i^n j}(z_1/z_2)}) \\ &= [\mathbf{x}^+_{ji^n}(z_1), \tilde{r}_k^+(v)(\mathbf{x}^+_i(z_1))]_{G^-_{i,i^n j}(z_1/z_2) G^-_{k,i^n j}(z_1/Cv)^{-1}} \\ &+ {}_{G^-_{k,i}(z_2/Cv)}[\tilde{r}_k^+(v)(\mathbf{x}^+_{ji^n}(z_1)), \mathbf{x}^+_i(z_1)]_{G^-_{i,i^n j}(z_1/z_2)} \\ &+ \delta_{jk}\beta_{i,j,n}\delta\left(\frac{z_1}{Cvq_i^{\gamma_{i,j,n}}}\right) {}_{G^-_{k,i}(z_2/Cv)}[\mathbf{x}^+_{\square i^n}(z_1), \mathbf{x}^+_i(z_2)]_{G^-_{i,i^n j}(z_1/z_2)}.\end{aligned}$$

After multiplying through by $-q_i^{a_{ij}}(z_2 - q_i^{\gamma_{i,j,n}-a_{ij}} z_1)$ we get:

$$\delta_{jk}\beta_{i,j,n}\delta\left(\frac{z_1}{Cvq_i^{\gamma_{i,j,n}}}\right)(z_1 q_i^{2a_{ij}+2(n-1)} - z_2)\delta\left(\frac{z_1}{q_i^2 z_2}\right)\mathbf{x}^+_{\square i^n}(z_1) \tag{4.5.60}$$

The result follows and thus completing the recursion provided that:

$$\beta_{i,j,n+1} = \beta_{i,j,n}\frac{[n + a_{ij}]_{q_i}}{[n]_{q_i}} \tag{4.5.61}$$

and

$$\gamma_{i,j,n+1} = \gamma_{i,j,n} - 2 \tag{4.5.62}$$

$\square$

Lemma 4.5.30. *$\forall i \neq j \in \dot{I}$, $\forall k \in \dot{I}$ and $\forall n \in \mathbb{N}$:*

$$\tilde{r}_k^+(v)(\mathbf{x}^+_{j^n i}(z)) = \delta_{kj}[n]_{q_j}[a_{ji} + n - 1]_{q_j}(q_j - q_j^{-1})\delta\left(\frac{Cz}{v}\right)\mathbf{x}^+_{j^{n-1}i}(zq_j^{-a_n}) \tag{4.5.63}$$

Proof.

$$\begin{aligned}\tilde{r}_k^+(v)(\mathbf{x}^+_{j^n i}(z)) &= \eta \circ r_k^-(1/v)(\eta(\mathbf{x}^+_{j^n i}(z)) \\ &= \eta[r_k^-(1/v)(\mathbf{x}^+_{ij^n}(1/z))] \\ &= \delta_{kj}[n]_{q_j}[a_{ji} + n - 1]_{q_j}(q_j - q_j^{-1})\delta\left(\frac{Cz}{v}\right)\mathbf{x}^+_{j^{n-1}i}(zq_j^{-a_n})\end{aligned}$$

$\square$

Definition 4.5.31. $\forall i \in \dot{I}$, $\forall n \in \mathbb{N}^\times$ let

$$\mathbf{x}^+_{i^n\square}(z) = \eta(\mathbf{x}^+_{\square i^n}(z)) \tag{4.5.64}$$

Proposition 4.5.32. *$\forall n \in \mathbb{N}^\times$, $\forall i \in \dot{I}$ let*

$$\delta\left(\frac{z_1}{z_2 q_i^2}\right)\mathbf{x}^+_{i^{n+1}\square}(z_1) = [\mathbf{x}^+_i(z_1), \mathbf{x}^+_{i^n\square}(z_2)] \tag{4.5.65}$$

Proof. Apply η to definition 4.5.28. $\square$

Lemma 4.5.33. $\forall l, i, j \in \dot{I}$, $\forall n \in \mathbb{N}^\times$

$$\mathbf{x}^+_{j^n i}(z)\mathbf{k}^-_l(vC^{-1/2}) = G^-_{l,j^n i}\left(\frac{C^{-1}v}{z}\right)^{-1} \mathbf{k}^-_l(vC^{-1/2})\mathbf{x}^+_{j^n i}(z) \tag{4.5.66}$$

Proof. The proof is similar to that of lemma 4.5.27. □

Lemma 4.5.34. *$\forall i \neq j \in \dot{I}$ such that $a_{ij} = -2$ and $a_{ji} = -1$, there exists a unique $\xi_{ij}(z) \in \widehat{\dot{\mathrm{U}}^+_q(\dot{\mathfrak{g}})}[[z, z^{-1}]]$ such that:*

$$ {}_{G^-_{i,i}(z_0 q_i^2/z_1)}[\mathbf{x}^+_{ji^2}(z_0 q_i^2), \mathbf{x}^+_i(z_1)]_{G^-_{i,i^1 j}(z_0/z_1)} \tag{4.5.67}$$

$$[\mathbf{x}^+_{ji^1}(z_0), \mathbf{x}^+_{\square i^2}(z_1 q_i^{-2})]_{G^-_{i,ij}(z_0 q_i^2/z_1) G^-_{i,i^1 j}(z_0/z_1)} \tag{4.5.68}$$

Proof.

$$\begin{aligned}
0 &= \delta\left(\frac{z_1}{z_2 q_i^2}\right)\delta\left(\frac{z_0}{z_1 q_i^2}\right)\mathbf{x}^+_{ji^3}(z_2) = [[\mathbf{x}^+_{ji^1}(z_0), \mathbf{x}^+_i(z_1)]_{G^-_{i,i^1 j}(z_0/z_1)}, \mathbf{x}^+_i(z_2)]_{G^-_{i,i^2 j}(z_1/z_2)} \\
&= [\mathbf{x}^+_{ji^1}(z_0), [\mathbf{x}^+_i(z_1), \mathbf{x}^+_i(z_2)]_{G^-_{i,i} z_1/z_2}]_{G^-_{i,i^1 j}(z_2/z_1) G^-_{i,i^1 j}(z_0/z_2)} \\
&+ {}_{G^-_{i,i}(z_1/z_2)}[[\mathbf{x}^+_{ji^1}(z_0), \mathbf{x}^+_i(z_2)]_{G^-_{i,i^1 j}(z_0/z_2)}, \mathbf{x}^+_i(z_1)]_{G^-_{i,i^1 j}(z_0/z_1)} \\
&= \delta\left(\frac{z_1}{z_2 q_i^2}\right)[\mathbf{x}^+_{ji^1}(z_0), \mathbf{x}^+_{\square i^2}(z_2)]_{G^-_{i,i^1 j}(z_0/z_1) G^-_{i,i^1 j}(z_0/z_2)} + \delta\left(\frac{z_0}{z_2 q_i^2}\right) {}_{G^-_{i,i}(z_1/z_2)}[\mathbf{x}^+_{ji^2}(z_2), \mathbf{x}^+_i(z_1)]_{G^-_{i,i^1 j}(z_0/z_1)}
\end{aligned}$$

multiplying the above equation by $(z_0 - q_i^2 z_2)$, $(z_1 - q_i^2 z_2)$ respectively and taking the residue with respect to z_2, we get:

$$(z_0 - z_1)[\mathbf{x}^+_{ji^1}(z_0), \mathbf{x}^+_{\square i^2}(z_1 q_i^{-2})]_{G^-_{i,i^1 j}(z_0/z_1) G^-_{i,i^1 j}(z_0 q_i^2/z_1)} = 0 \tag{4.5.69}$$

$$(z_1 - z_0)_{G^-_{i,i}(z_1 q_i^2/z_0)}[\mathbf{x}^+_{ji^2}(z_0 q^{-2}), \mathbf{x}^+_i(z_1)]_{G^-_{i,i^1 j}(z_0/z_1)} = 0 \tag{4.5.70}$$

the latter imply that:

$$[\mathbf{x}^+_{ji^1}(z_0), \mathbf{x}^+_i \square i^2(z_1)]_{G^-_{i,i^1 j}(z_0/z_1) G^-_{i,i^1 j}(z_0 q_i^2/z_1)} = \delta\left(\frac{z_0}{z_1}\right)\xi_{ij}(z_0) \tag{4.5.71}$$

$$ {}_{G^-_{i,i}(z_1 q_i^2/z_0)}[\mathbf{x}^+_{ji^2}(z_0 q^{-2}), \mathbf{x}^+_i(z_1)]_{G^-_{i,i^1 j}(z_0/z_1)} = \delta\left(\frac{z_0}{z_1}\right)\tilde{\xi}_{ij}(z_0) \tag{4.5.72}$$

by substituting the last two equations, we get:

$$\delta\left(\frac{z_0}{z_1}\right)\delta\left(\frac{z_1}{z_2 q_i^2}\right)[\tilde{\xi}_{ij}(z_0) + \xi_{ij}(z_0)] = 0 \tag{4.5.73}$$

the result follows. □

Lemma 4.5.35. *$\forall i \neq j \in \dot{I}$ and $a_{ij} = -2$, $a_{ji} = -1$,*

$$[\mathbf{x}^+_{ji^1}(z_1 q_j^2), \mathbf{x}^+_{ji^2}(z_2 q_i^{-2} q_j^2)]_{G^-_{i,i^2 j}(z_1/z_2)} = 0 \tag{4.5.74}$$

Proof. Clearly, the left-hand side of this equation is in $\widehat{\dot{\mathrm{U}}^+_q(\dot{\mathfrak{g}})}_{3\alpha_i + 2\alpha_j}$. Therefore, it suffices to prove that $\forall k \in \dot{I}$:

$$\tilde{r}^+_k(v)([\mathbf{x}^+_{ji^1}(z_1 q_j^2), \mathbf{x}^+_{ji^2}(z_2 q_i^{-2} q_j^2)]_{G^-_{i,i^2 j}(z_1/z_2)}) = 0 \tag{4.5.75}$$

Then,

$$\begin{aligned}
&\tilde{r}_k^+(v)([\mathbf{x}^+_{ji^1}(z_1q_j^2), \mathbf{x}^+_{ji^2}(z_2q_i^{-2}q_j^2)]_{G^-_{i,i^2j}(z_1/z_2)}) \\
&= [\mathbf{x}^+_{ji^1}(z_1q_j^2), \tilde{r}_k^+(v)(\mathbf{x}^+_{ji^2}(z_2q_i^{-2}q_j^2))]_{G^-_{i,i^2j}(z_1/z_2)G^-_{i,i^2j(z_1/z_2)^{-1}}} \\
&+ {}_{G^-_{k,ji^2}(z_2q_i^{-2}q_j^2/Cv)}[\tilde{r}_k^+(v)(\mathbf{x}^+_{ji^1}(z_1q_j^2)), \mathbf{x}^+_{ji^2}(z_2q_i^{-2}q_j^2)]_{G^-_{i,i^2j}(z_1/z_2)} \\
&= \delta_{kj}\delta\left(\frac{z_2q_i^{-2}q_j^2}{Cvq_i^{\gamma_{i,j,2}}}\right)\beta_{i,j,2}[\mathbf{x}^+_{ji^1}(z_1q_j^2), \mathbf{x}^+_{\square i^2}(z_2q_i^{-2}q_j^2)]_{G^-_{i,i^2j}\left(\frac{z_1}{z_2}\right)G^-_{i,i^2j}\left(\frac{z_1}{z_2}\right)^{-1}} \\
&+ \delta_{kj}\delta\left(\frac{z_1q_j^2}{Cvq_i^{\gamma_{i,j,k}}}\right)\beta_{i,j,2}{}_{G^-_{k,ji^2}(z_2q_i^{-2}q_j^2/Cv)}[\mathbf{x}^+_{i}(z_1q_j^2), \mathbf{x}^+_{ji^2}(z_2q_i^{-2}q_j^2)]_{G^-_{i,i^2j}(z_1/z_2)}.
\end{aligned}$$

Observe that:

$$G^-_{i,i^2j}\left(\frac{z_1}{z_2}\right)G^-_{j,ji^1}\left(\frac{z_1q_i^2}{z_2}\right)^{-1} = \frac{(z_1q_i^{-2}-z_2q_i^2)(z_1q_i^2-z_2q_i^{-2})}{(z_1-z_2)^2} = G^-_{i,i^1j}\left(\frac{z_1}{z_2}\right)G^-_{i,i^1j}\left(\frac{z_1q_i^2}{z_2}\right) \tag{4.5.76}$$

Similarly,

$$G^-_{i,i^2j}\left(\frac{z_1}{z_2}\right) = G^-_{i,i}\left(\frac{z_1q_i^2}{z_2}\right) \tag{4.5.77}$$

Finally,

$$G^-_{j,ji^2}\left(\frac{z_2}{z_1}\right)^{-1} = G^-_{i,i^1j}\left(\frac{z_1}{z_2}\right) \tag{4.5.78}$$

Then, we get:

$$\begin{aligned}
\tilde{r}_k^+(v)([\mathbf{x}^+_{ji^1}(z_1q_j^2), \mathbf{x}^+_{ji^2}(z_2q_i^{-2}q_j^2)]_{G^-_{i,i^2j}(z_1/z_2)}) = [2]_{q_i}(q_i-q_i^{-1})\delta_{jk}\Bigg[&-\delta\left(\frac{z_2q_i^2}{Cv}\right)\delta\left(\frac{z_2}{z_1}\right)\xi_{ij}(z_1q_j^2) \\
&+\delta\left(\frac{z_2q_i^2}{Cv}\right)\delta\left(\frac{z_2}{z_1}\right)\xi_{ij}(z_1q_j^2)\Bigg] = 0
\end{aligned} \tag{4.5.79}$$

□

Corollary 4.5.36. *$\forall i \neq j \in \dot{I}$ such that $a_{ij} = -2$, $a_{ji} = -1$, we have:*

$$T_j(\mathbf{x}^+_{i^3j}(z)) = 0. \tag{4.5.80}$$

Proof. By using the definitions of T_j and $\mathbf{x}^+_{i^3j}(z)$, observe that $T_j(\mathbf{x}^+_{i^3j}(z))$ is proportional to $[\mathbf{x}^+_{ji^1}(z_1q_j^2), \mathbf{x}^+_{ji^2}(z_2q_i^{-2}q_j^2)]_{G^-_{i,}}$
Then, by the previous lemma, the result follows. □

Lemma 4.5.37. *$\forall i \neq j \in \dot{I}$ such that $a_{ij} = -1$, $a_{ji} = -1$, we have:*

$$T_j(\mathbf{x}^+_{i^2j}(z)) = 0. \tag{4.5.81}$$

Proof. Using the definition of $\mathbf{x}^{+}_{i^2j}(z)$, the definition of T_i, and 4.5.12, we get that:

$$\begin{aligned}
T_j(\mathbf{x}^{+}_{i^2j}(z))\delta\left(\frac{z}{z_1q_i^2}\right) &= \left[T_j(\mathbf{x}^{+}_{i}(z_1)), T_j(\mathbf{x}^{+}_{i^1j}(z))\right]_{G^{-}_{i,ij}(z_1/z)} \\
&= \left[\mathbf{x}^{+}_{j^1i}(z_1q_j), \mathbf{x}^{+}_{i}(z)\right]_{G^{-}_{i,ij}(z_1/z)} \\
&= \left[\mathbf{x}^{+}_{ji^1}(z_1), \mathbf{x}^{+}_{i}(z)\right]_{G^{-}_{i,ij}(z_1/z)} \\
&= \delta\left(\frac{z}{z_1q_i^2}\right)\mathbf{x}^{+}_{ji^2}(z) = 0.
\end{aligned}$$

□

Proposition 4.5.38. *$\forall i \neq j \in \dot{I}$ such that $m_{ij} < \infty$, we have the following braid group relation:*

$$\underbrace{T_iT_jT_i...}_{\mathrm{m_{ij}times}} = \underbrace{T_jT_iT_j...}_{\mathrm{m_{ij}times}} \tag{4.5.82}$$

Proof. Consider the case $m_{ij} = 4$, and $\left\langle \alpha_i, \alpha_j^{\vee} \right\rangle = -2, \langle \alpha_j, \alpha_i^{\vee} \rangle = -1$.

We will show that

$$T_iT_jT_iT_j(\mathbf{x}^{+}_{j}(z)) = T_jT_iT_jT_i(\mathbf{x}^{+}_{j}(z)). \tag{4.5.83}$$

The right-hand side is already given in the proof of lemma 4.5.12. Therefore, we apply T_j to $\mathbf{x}^{+}_{j}(z)$ and,by making use of the definition of T_j on the algebra generators, we get:

$$T_j(\mathbf{x}^{+}_{j}(z)) = -\mathbf{x}^{-}_{j}(zC^{-1})\mathbf{k}^{+}_{j}(zC^{-1/2})^{-1} \tag{4.5.84}$$

Then,

$$T_i \circ T_j(\mathbf{x}^{+}_{j}(z)) = -T_i(\mathbf{x}^{-}_{j}(zC^{-1}))T_i(\mathbf{k}^{+}_{j}(zC^{-1/2})^{-1}). \tag{4.5.85}$$

Clearly, we can seperate the proof between the part regarding $T_i(\mathbf{x}^{-}_{j}(zC^{-1}))$ and $T_i(\mathbf{k}^{+}_{j}(zC^{-1/2})^{-1})$ and multiply again both results at the end. The part concerning the $\mathbf{k}^{+}_{j}(zC^{-1/2})^{-1}$ is straightforward and follows immediately from the definition of T_i on the generators for all $i \in \dot{I}$. We focus our attention on $T_i(\mathbf{x}^{-}_{j}(zC^{-1}))$. Observe that

$$T_i(\mathbf{x}^{-}_{j}(zC^{-1})) = T_i \circ \varphi(\mathbf{x}^{+}_{j}(C/z)) = \varphi \circ T_i(\mathbf{x}^{+}_{j}(C/z)) = \varphi(\mathbf{x}^{+}_{i^2j}(q_i^{a_{ij}}C/z)) \tag{4.5.86}$$

Now apply T_j to the previous result:

$$T_j \circ \varphi(\mathbf{x}^{+}_{i^2j}(q_i^{a_{ij}}C/z)) = \varphi \circ T_j(\mathbf{x}^{+}_{i^2j}(q_i^{a_{ij}}C/z)). \tag{4.5.87}$$

This allows us to use results from lemma 4.5.12 again. Therefore, we have:

$$\varphi \circ T_j(\mathbf{x}^{+}_{i^2j}(q_i^2C/z)) = \varphi(\mathbf{x}^{+}_{ji^2}(q_jq_i^2/z)) \tag{4.5.88}$$

Finally, we apply T_i one last time and use equation 4.5.14 to get our answer. We then multiply by the result we get from following the same steps on $\mathbf{k}^{+}_{j}(zC^{-1/2})^{-1}$ making us ready to compare with the right-hand side obtained from the proof of lemma 4.5.12 and the result follows. The remaining cases are proven the exact same way. □

Chapter 5

Further Directions

In this brief chapter, we will give some possible directions in which one might decide to venture in light of the results presented in chapter 2-4.

Conjecture 5.0.1. Every weight-finite simple $\mathrm{U}_q(\mathrm{L}\mathfrak{g})$-module is finite dimensional.

An obvious direction is also generalizing to higher rank root systems. We will also conjecture that the Drinfel'd presentation of a quantum toroidal algebra associated to a Lie algebra $\mathfrak{g}$ has the following relations:

Conjecture 5.0.2.

$$\{\mathsf{C}^{1/2}, \mathsf{C}^{-1/2}, \mathsf{c}^+_m, \mathsf{c}^-_{-m}, \mathsf{K}^+_{1,0,m}, \mathsf{K}^-_{i,0,-m}, \mathsf{K}^+_{i,n,r}, \mathsf{K}^-_{i,-n,r}, \mathsf{X}^+_{i,r,s}, \mathsf{X}^-_{i,r,s} : m \in \mathbb{N}, n \in \mathbb{N}^\times, r, s \in \mathbb{Z}, i \in I\}$$

$$(v - q^{\pm c_{ij}} z)\mathbf{K}^\pm_{i,r+t}(v)\mathbf{X}^\pm_{j,s}(z) = (q^{\pm c_{ij}} v - z)\mathbf{X}^\pm_{j,s}(z)\mathbf{K}^\pm_{i,r+t}(v)\,, \tag{5.0.1}$$

$$(\mathsf{C}q_i^{\pm 2(r+t)} v - q_i^{a_{ij}} z)\mathbf{K}^\mp_{i,r+t}(v)\mathbf{X}^\pm_{j,s}(z) = (q_i^{\pm 2(r+t)+a_{ij}} v - z)\mathbf{X}^\pm_{j,s}(z)\mathbf{K}^\mp_{i,r+t}(v)\,, \tag{5.0.2}$$

$$(\mathsf{C}q_i^{2r} z - q_i^{a_{ij}} w)(z - \mathsf{C}q_i^{-a_{ij}} q_j^{-2(s+t)w})\mathbf{K}^-_{i,r}(z)\mathbf{K}^+_{j,s+t}(w) = (\mathsf{C}q_i^{2r+a_{ij}} z - w)(zq_i^{-a_{ij}} - \mathsf{C}q_j^{-2(s+t)w})\mathbf{K}^+_{j,s+t}(w)\mathbf{K}^-_{i,r}(z) \tag{5.0.3}$$

$$(\mathsf{C}q_i^{2r} z - \mathsf{C}q_i^{a_{ij}} q_j^{2(s+t)} w)(z - q_i^{-a_{ij}})\mathbf{K}^-_{i,r}(z)\mathbf{K}^-_{j,s+t}(w) = (\mathsf{C}q_i^{2r+a_{ij}} z - \mathsf{C}q_j^{2(s+t)} w)(zq_i^{-a_{ij}} - w)\mathbf{K}^+_{j,s+t}(w)\mathbf{K}^-_{i,r}(z) \tag{5.0.4}$$

$$(v - q^{\pm 2} w)\mathbf{X}^\pm_{1,r}(v)\mathbf{X}^\pm_{1,s}(w) = (vq^{\pm 2} - w)\mathbf{X}^\pm_{1,s}(w)\mathbf{X}^\pm_{1,r}(v)\,, \tag{5.0.5}$$

$$\begin{aligned}
[\mathbf{X}^+_{i,r}(v), \mathbf{X}^-_{j,s}(z)] \quad = \quad & \frac{\delta_{ij}}{q_i - q_i^{-1}} \left\{ \delta\left(\frac{\mathsf{C}v}{q^{2(r+s)}z}\right) \prod_{p=1}^{|s|} \mathbf{c}^-\left(\mathsf{C}^{-1/2} q^{(2p-1)\mathrm{sign}(s)-1} z\right)^{-\mathrm{sign}(s)} \mathbf{K}^+_{i,r+s}(v) \right. \\
& \left. -\delta\left(\frac{\mathsf{C}^{-1}v}{q^{2(r+s)}z}\right) \prod_{p=1}^{|r|} \mathbf{c}^+\left(\mathsf{C}^{-1/2} q^{(1-2p)\mathrm{sign}(r)-1} v\right)^{\mathrm{sign}(r)} \mathbf{K}^-_{j,r+s}(z) \right\}\,,
\end{aligned} \tag{5.0.6}$$

where $m, n \in \mathbb{N}$, $r, s \in \mathbb{Z}$ and we have set

$$\mathbf{c}^\pm(z) = \sum_{m \in \mathbb{N}} \mathsf{c}^\pm_{\pm m} z^{\mp m}\,, \tag{5.0.7}$$

$$\mathsf{K}^\pm_{i,0}(z) = \sum_{m \in \mathbb{N}} \mathsf{K}^\pm_{i,0,\pm m} z^{\pm m}\,, \tag{5.0.8}$$

and, for every $m \in \mathbb{N}^\times$ and $r \in \mathbb{Z}$,

$$\mathsf{K}^{\pm}_{i,\pm m}(z) = \sum_{s \in \mathbb{Z}} \mathsf{K}^{\pm}_{i,\pm m,s} z^{-s}\,, \tag{5.0.9}$$

$$\mathsf{X}^{\pm}_{i,r}(z) = \sum_{s \in \mathbb{Z}} \mathsf{X}^{\pm}_{i,r,s} z^{-s}\,. \tag{5.0.10}$$

At this moment we do not have anything regarding the Serre relations.

Conjecture 5.0.3. The quiver quantum toroidal algebra admits a similar braid group action.

Appendix

5.1 Formal distributions

5.1.1 Definitions and main properties

Let $\mathbb{K}$ be a field of characteristic 0. For any $\mathbb{K}$-vector space V, we let $V[z,z^{-1}]$ denote the ring of V-valued Laurent polynomials. Writing

$$v(z)=\sum_{n\in\mathbb{Z}} v_n z^n\,,$$

where the sum runs over finitely many terms, for any $v(z)\in V[z,z^{-1}]$, we can define

$$\operatorname{supp}(v(z))=\{n\in\mathbb{Z}:v_n\neq 0\}\ ,$$

and set

$$V_n[z,z^{-1}]:=\left\{v(z)\in V[z,z^{-1}]:\operatorname{supp}(v(z))\subseteq[\![-n,n]\!]\right\}\ .$$

It is clear that, for every $n\in\mathbb{N}$, $V_n[z,z^{-1}]\cong V^{2n+1}$ as $\mathbb{K}$-vector spaces. Now, if in addition V is a topological vector space with topology τ_1, making use of that isomorphism, we can endow $V_n[z,z^{-1}]$ with the box topology of V^{2n+1}, for every $n\in\mathbb{N}$. Denote by τ_n that topology.

The obvious inclusions $V_n[z,z^{-1}]\hookrightarrow V_{n+1}[z,z^{-1}]$ are clearly continuous and we define a topology τ on $V[z,z^{-1}]$ as the inductive limit

$$\tau:=\varinjlim\tau_n\,.$$

We now assume that $\mathbb{K}$ is a topological field.

Definition 5.1.1. The space $V[[z,z^{-1}]]$ of V-valued *formal distributions* is the $\mathbb{K}$-vector space of continuous V-valued linear functions over the ring of $\mathbb{K}$-valued Laurent polynomials $\mathbb{K}[z,z^{-1}]$, the latter being endowed with the final topology induced as above from the topology of $\mathbb{K}$.

Proposition 5.1.2. *Any V-valued formal distribution $v(z)\in V[[z,z^{-1}]]$ reads*

$$v(z)=\sum_{n\in\mathbb{Z}} v_n z^n\,,$$

for some $(v_n)_{n\in\mathbb{Z}}\in V^{\mathbb{Z}}$ and the action of $v(z)$ on any Laurent polynomial $f(z)\in\mathbb{K}[z,z^{-1}]$ is given by

$$\langle v(z),f(z)\rangle=\operatorname*{res}_z\left(v(z)f(z)z^{-1}\right)\ ,$$

where we let

$$\operatorname*{res}_z a(z) = \operatorname*{res}_z \left(\sum_{n\in\mathbb{Z}} a_n z^n \right) = a_{-1}\,,$$

for any $a(z) \in V[[z,z^{-1}]]$*.* $V[[z,z^{-1}]]$ *is given the weak* $*$*-topology. It is actually a module over the ring* $\mathbb{K}[z,z^{-1}]$ *of* $\mathbb{K}$*-valued Laurent polynomials.*

Proof. It is clear that, due to its linearity, any $v(z) \in V[[z,z^{-1}]]$ is entirely characterized by the data, for every $n \in \mathbb{N}$, of

$$v_n = \left\langle v(z), z^{-n} \right\rangle \in V\,.$$

Now, writing $v(z) = \sum_{n\in\mathbb{Z}} v_n z^n$, we also have

$$v_n = \left\langle v(z), z^{-n} \right\rangle = \operatorname*{res}_z v(z) z^{-n-1}\,,$$

for every $n \in \mathbb{N}$ and the claim follows. ☐

Let A be a topological $\mathbb{K}$-algebra. Then $A[[z,z^{-1}]]$ is the space of A-valued formal distributions, i.e. of A-valued linear functions over $A[z,z^{-1}]$. In that case, the action of $a(z) \in A[[z,z^{-1}]]$ on $b(z) \in A[z,z^{-1}]$ is given by

$$\langle a(z), b(z)\rangle = \operatorname*{res}_z a(z)b(z)z^{-1}\,. \tag{5.1.1}$$

Clearly, $A[[z,z^{-1}]]$ is a module over the ring $A[z,z^{-1}]$ of A-valued Laurent polynomials. It is generally impossible to consistently extend that structure into a full-fledged product over $A[[z,z^{-1}]]$. However, since A is a topological algebra, we can set

$$a(z)b(z) = \sum_{p\in\mathbb{Z}} \left(\sum_{m\in\mathbb{Z}} a_m b_{p-m} \right) z^p\,,$$

whenever the series

$$\sum_{m\in\mathbb{Z}} a_m b_{p-m}$$

is convergent for every $p \in \mathbb{Z}$. If A is complete as a topological algebra, it suffices that the above series be Cauchy.

We let similarly $V[[z_1,z_1^{-1},\dots,z_n,z_n^{-1}]]$ denote the space of V-valued formal distributions in $n \in \mathbb{N}$ variables, so that any V-valued formal distribution $v(z_1,\dots,z_n)$ in n variable reads

$$v(z_1,\dots,z_n) = \sum_{p_1,\dots,p_n\in\mathbb{Z}} v_{p_1,\dots,p_n} z_1^{p_1}\cdots z_n^{p_n}\,,$$

for some $(v_{p_1\dots,p_n})_{p_1,\dots,p_n\in\mathbb{Z}} \in V^{\mathbb{Z}^n}$. For every $i = 1,\dots,n$, we define

$$\operatorname*{res}_{z_i} : V[[z_1,z_1^{-1},\dots z_n,z_n^{-1}]] \to V[[z_1,z_1^{-1},\dots,\widehat{z_i},\widehat{z_i^{-1}}\dots,z_n,z_n^{-1}]]\,,$$

where a hat over a variable indicates omission of that variable, by setting

$$\operatorname*{res}_{z_i} v(z_1,\dots,z_n) = \operatorname*{res}_{z_i} \sum_{p_1,\dots,p_n\in\mathbb{Z}} v_{p_1,\dots,p_n} z_1^{p_1}\cdots z_n^{p_n} = \sum_{p_1,\dots,\hat{p}_i,\dots,p_n\in\mathbb{Z}} v_{p_1,\dots,p_{i-1},-1,p_{i+1},\dots,p_n} z_1^{p_1}\cdots \widehat{z_i^{-1}}\cdots z_n^{p_n}\,.$$

For every $i = 1, \dots, n$, we define $\partial_i : V[[z_1, z_1^{-1}, \dots, z_n, z_n^{-1}]] \to V[[z_1, z_1^{-1}, \dots, z_n, z_n^{-1}]]$ by setting

$$\partial_i v(z_1, \dots, z_n) = \sum_{p_1,\dots,p_n \in \mathbb{Z}} p_i v_{p_1,\dots,p_n} z_1^{p_1} \cdots z_i^{p_i - 1} \cdots z_n^{p_n} .$$

If A is a topological $\mathbb{K}$-algebra, then the multiplication in A naturally extends to bilinear maps

$$A[[z_1, z_1^{-1}, \dots, z_m, z_m^{-1}]] \times A[[z_{m+1}, z_{m+1}^{-1}, \dots, z_{m+n}, z_{m+n}^{-1}]] \to A[[z_1, z_1^{-1}, \dots, z_{m+n}, z_{m+n}^{-1}]]$$

by setting

$$a(z_1, \dots, z_m) b(z_{m+1}, \dots, z_{m+n}) = \sum_{p_1,\dots,p_{m+n} \in \mathbb{Z}} a_{p_1,\dots,p_m} b_{p_{m+1},\dots,p_{m+n}} z_1^{p_1} \cdots z_{m+n}^{p_{m+n}} .$$

Let $a(z_1, \dots, z_n) \in A[[z_1, z_1^{-1}, \dots, z_n, z_n^{-1}]]$ be an A-valued formal distribution in n variables. Since A is a topological $\mathbb{K}$-agebra, we can define the *localization* $a_{|z_{n-1}=z_n}(z_1, \dots, z_{n-1}) \in A[[z_1, z_1^{-1}, \dots, z_{n-1}, z_{n-1}^{-1}]]$ of $a(z_1, \dots, z_n)$ at $z_{n-1} = z_n$, by setting

$$a_{|z_{n-1}=z_n}(z_1, \dots, z_{n-1}) = \sum_{p_1,\dots,p_{n-1} \in \mathbb{Z}} \left(\sum_{p \in \mathbb{Z}} a_{p_1,\dots,p_{n-2},p,p_{n-1}-p} \right) z_1^{p_1} \dots z_{n-1}^{p_{n-1}} ,$$

whenever

$$\sum_{p \in \mathbb{Z}} a_{p_1,\dots,p_{n-2},p,p_{n-1}-p}$$

is convergent. If A is complete as a topological algebra, it suffices that the above series be Cauchy.

5.1.2 Laurent expansion and the Dirac formal distribution

One way to obtain formal power series is to take the Laurent expansion of some holomorphic function $f : \mathbb{C} \to \mathbb{C}$. We shall usually write $f(z)_{|z| \ll 1}$ to denote the Laurent expansion around 0. Similarly, we shall denote by $f(z)_{|z| \gg 1}$ the Laurent expansion around ∞.

Let

$$\delta(z) = \sum_{n \in \mathbb{Z}} z^n .$$

Lemma 5.1.3. *For every $n \in \mathbb{N}^\times$, we have*

$$\left(\frac{1}{1-z} \right)^n_{|z| \ll 1} - \left(\frac{1}{1-z} \right)^n_{|z| \gg 1} = \frac{\delta^{(n-1)}(z)}{(n-1)!} .$$

Proof. It is straightforward to check that the result holds for $n = 1$. Assuming it holds for some n, it follows, upon differentiation, that

$$n \left[\left(\frac{1}{1-z} \right)^{n+1}_{|z| \ll 1} - \left(\frac{1}{1-z} \right)^{n+1}_{|z| \gg 1} \right] = \frac{\delta^{(n)}(z)}{(n-1)!} ,$$

which completes te recursion. ☐

Lemma 5.1.4. *For any $n \in \mathbb{N}$ and any A-valued Laurent polynomial $f(z) \in A[z, z^{-1}]$, we have*

$$f(z)\delta^{(n)}(z) = \sum_{p=0}^{n} (-1)^{n-p} \binom{n}{p} f^{(n-p)}(1)\delta^{(p)}(z)\,.$$

Proof. The case $n = 1$ is straightforward. Assuming the results holds for some $n \in \mathbb{N}$, we have, upon differentiation,

$$f'(z)\delta^{(n)}(z) + f(z)\delta^{(n+1)}(z) = \sum_{p=0}^{n} (-1)^{n-p} \binom{n}{p} f^{(n-p)}(1)\delta^{(p+1)}(z)\,,$$

which completes the recursion. ☐

Example 5.1.5. In particular, for any A-valued formal distribution $f(z_1, z_2) \in A[[z_1, z_1^{-1}, z_2, z_2^{-1}]]$ with a well-defined localization $f_{|z_1=z_2}(z_1)$ – see previous subsection for a definition –, we have

$$f(z_1, z_2)\delta\left(\frac{z_1}{z_2}\right) = f_{|z_1=z_2}(z_1)\delta\left(\frac{z_1}{z_2}\right)\,,$$

Assuming that $\mathbb{K}$ is an algebraically closed field, we have

Lemma 5.1.6. *Let $P(z) \in \mathbb{K}[z]$ be a polynomial of degree N, with roots $\{\lambda_i : i \in [\![n]\!]\}$ and respective multiplicities $\{m_i : i \in [\![n]\!]\}$. If $a(z) \in \mathbb{K}[[z, z^{-1}]]$ is a $\mathbb{K}$-valued formal distribution, then*

$$P(z)a(z) = 0 \qquad \Leftrightarrow \qquad a(z) = \sum_{i=1}^{n} \sum_{p_i=0}^{m_i-1} \alpha_{i,p} \delta^{(p_i)}\left(\frac{z}{\lambda_i}\right)\,,$$

for some $\alpha_{i,p} \in \mathbb{K}$.

Proof. The if part is easily checked making use of the previous lemma. The only if part follows by an easy recursion, after writing that $P(z) = \prod_{i \in [\![n]\!]}(z - \lambda_i)^{m_i}$. ☐

Lemma 5.1.7. *Let $P(z), Q(z) \in \mathbb{K}[z]$ be two coprime polynomials. Let $\{\lambda_i : i \in [\![n]\!]\}$ be the set of roots of $Q(z)$ and let $\{m_i : i \in [\![n]\!]\}$ be their respective multiplicities. Then, in $\mathbb{K}[[z, z^{-1}]]$,*

$$\left(\frac{P(z)}{Q(z)}\right)_{|z|\ll 1} - \left(\frac{P(z)}{Q(z)}\right)_{|z|\gg 1} = \sum_{i=1}^{n} \sum_{p_i=0}^{m_i-1} \frac{(-1)^{p_i+1}\alpha_{i,p_i+1}}{(p_i)!\lambda_i^{p_i+1}} \delta^{(p_i)}\left(\frac{z}{\lambda_i}\right)\,, \tag{5.1.2}$$

where, for every $i \in [\![n]\!]$ and every $p_i \in [\![m_i]\!]$, α_{i,p_i} is obtained from the partial fraction decomposition

$$\frac{P(z)}{Q(z)} = A(z) + \sum_{i=1}^{n} \sum_{p_i=1}^{m_i} \frac{\alpha_{i,p_i}}{(z - \lambda_i)^{p_i}}\,, \tag{5.1.3}$$

in which $A(z) \in \mathbb{K}[z]$ is a polynomial of degree $\deg(P) - \deg(Q)$.

Proof. Given the partial fraction decomposition (5.1.3), we can write

$$\begin{aligned}
\left(\frac{P(z)}{Q(z)}\right)_{|z|\ll 1} - \left(\frac{P(z)}{Q(z)}\right)_{|z|\gg 1} &= \sum_{i=1}^{n} \sum_{p_i=1}^{m_i} \alpha_{i,p_i} \left[\left(\frac{1}{(z - \lambda_i)^{p_i}}\right)_{|z|\ll 1} - \left(\frac{1}{(z - \lambda_i)^{p_i}}\right)_{|z|\gg 1}\right] \\
&= \sum_{i=1}^{n} \sum_{p_i=1}^{m_i} \frac{(-1)^{p_i}\alpha_{i,p_i}}{(p_i - 1)!\lambda_i^{p_i}} \delta^{(p_i-1)}\left(\frac{z}{\lambda_i}\right)
\end{aligned}$$

where we have used lemma 5.1.3 to derive the last equality. The claim obviously follows. $\square$

Lemma 5.1.8. *Let* $m \in \{0,1\}$ *and* $n \in \mathbb{N}$, *let* $A(v) \in \mathbb{F}[[v]] - \{0\}$ *be a non-zero formal power series and let* $F(z) \in \mathbb{F}[[z, z^{-1}]]$ *be a formal distribution such that*

$$(z-a)(z-v)^m A(v) F(z) + \sum_{p=0}^{n} B_p(v) \delta^{(p)}(z/a) = 0\,, \tag{5.1.4}$$

for some non-zero scalar $a \in \mathbb{F}^\times$ *and some formal power series* $B_0(v), \dots, B_n(v) \in \mathbb{F}[[v]]$. *Then,*

$$F(z) = \sum_{p=0}^{n+1} f_p \delta^{(p)}(z/a)\,,$$

for some scalars $f_0, \dots, f_{n+1} \in \mathbb{F}$.

Proof. Consider first the case where $m = 0$. Then, multiplying (5.1.4) by $(z-a)^{n+1}$, we get

$$(z-a)^{n+2} A(v) F(z) = 0\,.$$

Since $A(v) \neq 0$, there must exist $k \in \mathbb{N}$ such that $\mathrm{res}_v\, v^{-1-k} A(v) \neq 0$ and, specializing the above equation to its v^k term, it follows that

$$F(z) = \sum_{p=0}^{n+1} f_p \delta^{(p)}(z/a)$$

for some scalars $f_0, \dots, f_{n+1} \in \mathbb{F}$. Now consider the case where $m = 1$. It follows from (5.1.4) that

$$(z-a) A(v) F(z) + \left(\frac{1}{z-v}\right)_{|v/z| \ll 1} \sum_{p=0}^{n} B_p(v) \delta^{(p)}(z/a) = C(z) \delta(z/v)\,,$$

for some formal distribution $C(z) \in \mathbb{F}[[z, z^{-1}]]$. But specializing the above equation to any negative power of v, v^{-p} with $p \in \mathbb{N}^\times$, we immediately get that $C(z) = 0$. We are thus back to the previous case. $\square$

5.1.3 The structure power series $G^{\pm}_{ij}(z)$

In this last subsection, we derive identities involving the structure power series $G^{\pm}_{ij}(z)$ by applying lemma 5.1.7. Remember – see remark 3.3.21 – that in type $\dot{\mathfrak{a}}_1$, we have $G^{\pm}_{10}(z) = G^{\mp}_{11}(z)$.

Proposition 5.1.9. *The following hold true in* $\mathbb{F}[[z, z^{-1}]]$.

i. For every $p \in \mathbb{Z} - \{2\}$,

$$\frac{G^{+}_{10}(zq^{p}) G^{+}_{11}(zq^{-p}) - G^{-}_{10}(z^{-1}q^{-p}) G^{-}_{11}(z^{-1}q^{p})}{q - q^{-1}} = \frac{[2]_q [p]_q}{[p-2]_q} \left[\delta\left(zq^{2-p}\right) - \delta\left(zq^{p-2}\right)\right]\,. \tag{5.1.5}$$

In particular, when $p = 1$, *we have*

$$\frac{G^{+}_{10}(zq) G^{+}_{11}(zq^{-1}) - G^{-}_{10}(z^{-1}q^{-1}) G^{-}_{11}(z^{-1}q)}{q - q^{-1}} = [2]_q \left[\delta\left(zq^{-1}\right) - \delta\left(zq\right)\right]\,. \tag{5.1.6}$$

If $p=2$, we have instead

$$\frac{G^+_{10}(zq^2)G^+_{11}(zq^{-2})-G^-_{10}(z^{-1}q^{-2})G^-_{11}(z^{-1}q^2)}{(q-q^{-1})^2}=[2]_q^2\left[\delta\left(z\right)-\delta'\left(z\right)\right]\ . \tag{5.1.7}$$

ii. Similarly,

$$\frac{G^+_{11}(zq^{-2})^2-G^-_{11}(z^{-1}q^2)^2}{(q-q^{-1})^2}=\frac{2q^{-2}[2]_q}{q-q^{-1}}\delta\left(z\right)+[2]_q^2\delta'\left(z\right)\ . \tag{5.1.8}$$

Proof. In each case, it suffices to determine the partial fraction decomposition of the l.h.s and to apply lemma 5.1.7 to get the desired result. ☐

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