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On geodesics in low regularity

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Abstract. We consider geodesics in both Riemannian and Lorentzian manifolds with metrics of low regularity. We discuss existence of extremal curves for continuous metrics and present several old and new examples that highlight their subtle interrelation with solutions of the geodesic equations. Then we turn to the initial value problem for geodesics for locally Lipschitz continuous metrics and generalize recent results on existence, regularity and uniqueness of solutions in the sense of Filippov.

Keywords: geodesics, extremal curves, low regularity, Filippov solutions

MSC2010: 53B30, 53C22, 34A36, 83C99

1. Introduction

Recently there has been an increased interest in several low regularity aspects of general relativity and of semi-Riemannian geometry. In particular, it has been confirmed that the regularity class of $C^{1,1}$ (the metric possessing locally Lipschitz continuous first derivatives) can rightly be seen as the threshold of classical theory. In addition to the well-known fact that the geodesic equation can be locally uniquely solved by classical ODE-theory, the exponential map retains maximal regularity and convex neighborhoods exist [23, 17]. In the Lorentzian case the bulk of causality theory remains valid [4, 23, 18] as well as the classical singularity theorems [19, 20, 9]. On the other hand studies of geometries of regularity below $C^{1,1}$ (see e.g. [4, 32, 33, 8]) have revealed important differences in many of the well known concepts and have emphasized the significance of regularity issues.

In this contribution we focus on geodesics in low regularity in Riemannian as well as in Lorentzian manifolds. This topic again is of particular interest since many of the facts well known for smooth metrics fail to extend to a regularity below $C^{1,1}$. For example in the classical setting the unique local solutions of the geodesic equation are locally extremal curves for the length functional. Conversely extremal curves are pregeodesics. Moreover Lorentzian geodesics and maximal curves have a causal character.

In case of insufficient regularity the question of existence of extremal curves becomes a separate issue. It has long been answered affirmatively in the case of continuous Riemannian metrics and we will discuss the recent equally positive answer in the Lorentzian case. It might be surprising that already in cases where the geodesic equation is (classically locally)



solvable but not uniquely so (in particular, for metrics of Hölder class $C^{1,\alpha}$ for any $\alpha < 1$) the connection between its solutions and extremal curves becomes subtle. We will discuss some classical and some new examples in that realm at some length. We will supplement them by examples (again old and new) which demonstrate the failure of some of the usual causality properties in Lorentzian manifolds.

Complementing this line of investigation we will transfer some recent results on solutions of the geodesic equation for impulsive gravitational wave spacetimes into a more abstract setting. In particular, we will discuss existence and regularity of solutions to the geodesic equation for locally Lipschitz semi-Riemannian metrics using the solution concept of Filippov [7] for ODEs with discontinuous right hand side. We also present some sufficient conditions for uniqueness, which have proved to be useful in the context of applications. Finally we provide an outlook to open questions and related problems.

We end this introduction by fixing some notions and notations. All manifolds will be assumed to be of class C^∞ (which is no loss of generality, [15, Thm. 2.9]) and we will only lower the regularity of the metric. Hence e.g. by a continuous spacetime (M, g) we will mean a smooth, connected manifold M of dimension $n \geq 2$ equipped with a continuous Lorentzian metric g with a time orientation induced by a (continuous) timelike vector field. Our notations are quite standard and we generally follow [25]. Mindful of the above discussion a geodesic will always mean a solution (of some sort) of the geodesic equation and should be strictly distinguished from extremal curves.

2. Extremal curves

As indicated in the introduction, below a regularity of $C^{1,1}$ the notions of extremal curves and geodesics no longer coincide. We discuss existence of extremal curves for continuous metrics and study their relation to geodesics. We start with the case of Riemannian metrics.

2.1. The Riemannian case

It is a classical result by Hilbert [14] (using the Theorem of Arzela-Ascoli) that for *continuous* Riemannian metrics minimizing curves always exist locally. The global existence of minimizing curves is a corollary of the Hopf-Rinow-Cohn-Vossen Theorem for length spaces [2, Thm. 2.5.28]. To be precise, by [3], a continuous Riemannian metric gives rise to a length space, and hence if the corresponding metric space is complete any two points can be connected via a minimizing curve.

So the situation of existence of minimizing curves for continuous Riemannian metrics is exactly as in the smooth case. However, the relation between such minimizing curves and geodesics turns out to differ drastically from the smooth situation if the regularity of the metric drops below $C^{1,1}$.

Indeed classical examples show the explicit failure of the initial value problem for geodesics to be *uniquely* solvable for $g \in C^{1,\alpha}$ for any fixed $\alpha < 1$ [12], while it is possible that at the same time all the usual properties hold [13]: minimizing curves are locally unique, the boundary value problem for geodesics is locally uniquely solvable and even in ‘singular points’ there is a locally minimizing curve starting off in any direction.

However, another classical example [13] shows that for $g \in C^{1,\alpha}$ again for any fixed $\alpha < 1$ geodesics need not be even minimizing locally. This example also shows that again even locally the boundary value problem for geodesics is non-uniquely solvable. As the same phenomenon

occurs also in the Lorentzian case by a modification of the original example we will discuss it in some detail below.

On the positive side the regularity of minimizing curves is slightly better than one would expect. A classical result in this direction is the following: If the metric is C^1 , then minimizing curves are actually geodesics and are of regularity C^2 . This can be seen by using the trick of Du Bois-Reymond [1, Ch. I, §6], which we briefly recall here.

Let g be a C^1 -Riemannian metric. Since it suffices to argue locally we consider $\gamma: [a, b] \rightarrow \mathbb{R}^n$, a locally Lipschitz continuous and minimizing curve from $p = \gamma(a)$ to $q = \gamma(b)$. We parametrize γ by g -arclength, i.e., $g_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t)) = 1$ almost everywhere. That γ is minimizing implies that

$$\frac{d}{d\varepsilon} \Big|_{\varepsilon=0} L_g(\gamma + \varepsilon\phi) = 0,$$

where L_g is the length functional of g , i.e., $L_g(\gamma) = \int_a^b F(t, \gamma(t), \dot{\gamma}(t)) dt$ with $F(t, \gamma(t), \dot{\gamma}(t)) = \sqrt{g_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t))}$, and moreover ϕ is a smooth function with compact support in $[a, b]$. For simplicity we calculate the following in one dimension, however the general case poses no additional difficulties. Integration by parts yields

$$0 = \frac{d}{d\varepsilon} \Big|_{\varepsilon=0} L_g(\gamma + \varepsilon\phi) = \int_a^b F_{\dot{\gamma}}\phi + F_{\gamma}\phi' dt = \int_a^b (F_{\dot{\gamma}} - \int_a^t F_{\gamma} ds)\phi' dt. \tag{1}$$

Thus, since ϕ is arbitrary, $F_{\dot{\gamma}} - \int_a^t F_{\gamma} ds$ is constant and in the general higher dimensional case this yields that

$$F_{\dot{\gamma}^i} - \int_a^t F_{\gamma^i} ds = c_i = \text{constant}, \tag{2}$$

for $i = 1, \dots, n$. Note that γ minimizes on any subinterval $[a, s]$ ($s \leq b$), and hence we obtain from (2) in coordinates (x^i) ($\gamma^i = x^i \circ \gamma$) that

$$g_{ij} \dot{\gamma}^j - \frac{1}{2} \int_a^s \frac{\partial g_{lj}}{\partial x^i} \dot{\gamma}^l \dot{\gamma}^j dr = c_i,$$

and by multiplying with $g^{-1} = (g^{ij})$ this gives

$$\dot{\gamma}^m = g^{im} \left(\frac{1}{2} \int_a^s \frac{\partial g_{lj}}{\partial x^i} \dot{\gamma}^l \dot{\gamma}^j dr + c_i \right). \tag{3}$$

The right-hand-side of (3) is continuous because $\frac{\partial g_{lj}}{\partial x^i}$ is continuous and $\dot{\gamma}^i$ is bounded. Thus γ is C^1 and using this information one sees again by (3) that γ is in fact C^2 .

Finally it follows that γ actually is a geodesic by the usual argument using integration by parts on the other term in (1).

A recent and enhanced regularity result is given in [21]: If the Riemannian metric is of Hölder-regularity $C^{0,\alpha}$ with $0 < \alpha \leq 1$, then minimizing curves are of regularity $C^{1,\beta}$, where $\beta = \frac{\alpha}{2-\alpha}$. This regularity $C^{1,\beta}$ is optimal as there are examples of minimizers with respect to a $C^{0,\alpha}$ Riemannian metric on \mathbb{R}^2 that are not $C^{1,l}$ for $l > \beta$ [21, Thm. 1.1]

Finally, we review a key example of Hartman and Wintner given in [13].

Example 2.1. (The Hartman-Wintner example [13, Sec. 5]) We consider $M := (-1, 1) \times \mathbb{R}$ with the metric

$$g_{(x,y)} = dx^2 + (1 - |x|^\lambda)dy^2, \tag{4}$$

where $1 < \lambda < 2$. The metric is of Hölder-regularity $C^{1,\lambda-1}$ and smooth off the y -axis. The geodesic equations are

$$x'' + \frac{\lambda}{2}|x|^{\lambda-1} \operatorname{sgn} x (y')^2 = 0, \quad y'' - \frac{\lambda}{2} \frac{|x|^{\lambda-1} \operatorname{sgn}(x)}{1 - |x|^\lambda} x' y' = 0. \tag{5}$$

Moreover, it suffices to consider only the initial value $(x_0, y_0) = (0, 0)$ since the metric does not depend on y at all. Note that since the metric is C^1 , $g_{\gamma(s)}(\dot{\gamma}(s), \dot{\gamma}(s))$ is constant and we parametrize any geodesic by arclength. The y -equation in (5) is equivalent to $(1 - |x|^\lambda)y' = c = \text{constant}$. In case $c = 0$ we have $y = 0$, so the x -equation in (5) is trivial and the geodesic is simply given by $y = 0$. If $c \neq 0$, then $y' \neq 0$ and y is strictly monotonous along any such geodesic. Parametrizing by arclength gives

$$x' = \pm \sqrt{1 - \frac{c^2}{1 - |x|^\lambda}}, \tag{6}$$

with $c \in [-1, 1]$. If $c^2 = 1$, then $x = 0$ is the only solution to (6) and we denote this geodesic by γ_0 . If $c^2 < 1$, then the right-hand-side of (6) is (initially) C^1 and thus there is a unique solution with initial condition $x(0) = 0$. Given this unique solution x , we can uniquely solve the y -equation by integrating, i.e., $y(s) = \int_0^s \frac{c}{1 - |x(r)|^\lambda} dr$. To summarize, this means that at $(0, 0)$ for every initial direction there is a unique geodesic, and by symmetry the initial-value-problem is uniquely solvable for arbitrary data.

To determine the shape of the geodesics set $c = \pm\sqrt{1 - \varepsilon}$ for $\varepsilon \in [0, 1]$ and denote by $\gamma_{\pm\varepsilon}$ the geodesics starting at $(0, 0)$ with initial velocity $(\pm\sqrt{\varepsilon}, \sqrt{1 - \varepsilon})$, see Figure 1. A short calculation shows that γ_ε reaches $x = \sqrt[\lambda]{\varepsilon}$ in finite time. We denote the corresponding parameter value by s_0 and the y -value by y_1 , i.e., $\gamma_\varepsilon(s_0) = (\sqrt[\lambda]{\varepsilon}, y_1)$. Then $\dot{\gamma}_\varepsilon(s_0) = (0, 1/\sqrt{1 - \varepsilon})$ is vertical and since g is independent of y we can reflect γ_ε at $y = y_1$. Thus by symmetry $(0, 0)$ is connected to $(0, 2y_1)$ by three distinct geodesics: $\gamma_{\pm\varepsilon}$ and γ_0 . At this point we vary ε and it is not hard to see that $y_1(\varepsilon) \rightarrow 0$ as $\varepsilon \searrow 0$. This implies that the boundary value problem for geodesics is not uniquely solvable in any neighborhood of any point on the y -axis.

Finally, it is again not hard to see that the geodesic γ_0 is not minimizing between any of its points and that in fact $\gamma_{\pm\varepsilon}$ are minimizing between its endpoints. From here we obtain that even locally minimizing curves are not unique and there is no minimizing curve with initial velocity $(0, 1)$.

2.2. The Lorentzian case

In complete analogy with the Riemannian case local existence of maximizing causal curves holds for *continuous* Lorentzian metrics. The key notion here is *global hyperbolicity*, which for continuous metrics has been studied in [32]. There a spacetime is called globally hyperbolic if it is non-totally imprisoning and the causal diamonds are compact. Recall that a *Cauchy hypersurface* is a set $S \subseteq M$ that is met exactly once by any inextendible (locally Lipschitz) causal curve. The existence of a Cauchy hypersurface is equivalent to global hyperbolicity

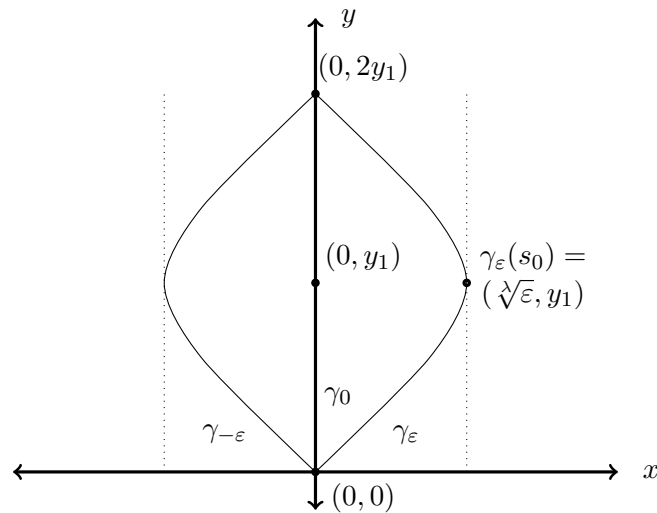


Figure 1: The minimizing geodesics $\gamma_{\pm\epsilon}$ and the non-minimizing one γ_0 for the metric (4).

also in this low regularity [32, Thm. 5.7 and Thm. 5.9]. We may establish without much effort:

Theorem 2.2. (Existence of locally maximizing curves) Every point p in a continuous spacetime (M, g) possesses a neighborhood U such that any two U -causally related points can be joined by a maximizing (in U) causal curve.

Proof: Let \hat{g} be a smooth Lorentzian metric on M , with the property that the lightcones of g are contained in the timelike cones of \hat{g} (see [4]). Let $p \in M$ then there exists a \hat{g} -globally hyperbolic neighborhood (U, x^μ) of p by [24, Thm. 2.14], i.e., in the x^μ -coordinates one has that $x^0 = 0$ is a Cauchy hypersurface in U with respect to \hat{g} . Then obviously this is also a Cauchy hypersurface with respect to g . Hence $(U, g|_U)$ is globally hyperbolic by [32, Thm. 5.7] and thus maximal (in U) causal curves exist between any two (in U) causally related points by [32, Prop. 6.4]. \square

Here we have already made use of the Avez-Seifert result for continuous metrics which we quote below as the global analogue of the above result. Note, however, that Theorem 2.2 can be established without using the global result: one may just use that there are local bounds on the arclength of causal curves with respect to a complete Riemannian metric and then use the Theorem of Arzela-Ascoli in the form of the limit curve theorem [22, 4, 32].

Theorem 2.3. (Avez-Seifert, [32, Prop. 6.4]) Let (M, g) be a globally hyperbolic continuous spacetime. Then there is a maximal causal curve connecting any two points in the spacetime which are causally related.

We summarize the Riemannian and Lorentzian results on the existence of minimal and maximal curves in the following table.

$g \in C^0$	local	global
Riemannian	length structure, Arzela-Ascoli [14]	Hopf-Rinow-Cohn-Vossen in complete length spaces [2, Thm. 2.5.28], [3]
Lorentzian	existence of globally hyperbolic neighborhoods	continuous Avez-Seifert [32]

Again as in the Riemannian case the relation of maximizing causal curves with geodesics becomes subtle for a regularity below $g \in C^{1,1}$. Indeed Example 2.1 can be modified to yield a Lorentzian counterexample.

Example 2.4. (The Lorentzian Hartman-Wintner example) We consider $M := \mathbb{R} \times (-1, 1) \times \mathbb{R}$ with the $C^{1,\lambda-1}$ -metric

$$g_{(t,x,y)} = -dt^2 + dx^2 + (1 - |x|^\lambda)dy^2,$$

where again $1 < \lambda < 2$. Then the geodesic equations for x and y are just equations (5) and in addition we have the trivial equation $t'' = 0$.

Again it suffices to consider geodesics starting at the origin. By Example 2.1 the initial value problem is uniquely solvable and we again consider some special solutions. Defining y_1 as in Example 2.1 we consider the timelike geodesic $\Gamma_0(r) = (2\sqrt{2}s_0r, 0, 2y_1r)$ ($r \in [0, 1]$). To see that Γ_0 is timelike, note that $s_0 < y_1 < 2s_0$. Moreover $L(\Gamma_0) = \sqrt{8s_0^2 - 4y_1^2} < 2s_0$. On the other hand set $\Gamma_{\pm\epsilon}(s) = (\sqrt{2}s, \gamma_{\pm\epsilon}(s))$ ($s \in [0, 2s_0]$), where $\gamma_{\pm\epsilon}$ are the minimizing geodesics of Example 2.1. Then the geodesics $\Gamma_{\pm\epsilon}$ are normalized to eigentime and so the lengths $L(\Gamma_{\pm\epsilon}) = 2s_0$ coincide with parameter length. Therefore we obtain without effort similar results as in the original example.

Indeed $\Gamma_{\pm\epsilon}(s_0) = (\sqrt{2}s_0, \pm\sqrt[3]{\epsilon}, y_1)$. Then the points $(0, 0, 0)$ and $(2\sqrt{2}s_0, 0, 2y_1)$ are connected by the three distinct geodesics $\Gamma_{\pm\epsilon}$ and Γ_0 and again the boundary value problem for geodesics is not uniquely solvable. Moreover, Γ_0 is not maximizing between any of its points since the timelike curves $\Gamma_{\pm\epsilon}$ are longer: $L(\Gamma_0) < 2s_0 = L(\Gamma_{\pm\epsilon})$. Finally there is no maximizing curve with initial velocity $(\sqrt{2}s_0, 0, y_1)$ and by symmetry maximizing curves are not unique.

This example is complemented by the following one from [4] which, in particular, shows that the push-up principle of causality theory fails to hold in Hölder-regularity $C^{0,\lambda}$ with $\lambda < 1$. Recall that the push-up principle states that if two points can be connected via a causal curve that contains a timelike segment, then the points can be connected by a timelike curve. This Hölder-regularity is optimal as the push-up principle holds for (locally) Lipschitz continuous metrics [4, Cor. 1.17 and Prop. 1.23].

Example 2.5. (Bubbling [4, Ex. 1.11]) We consider the manifold \mathbb{R}^2 with the metric

$$g_{(u,x)} = -du^2 + 2(|u|^\lambda - 1)du dx + |u|^\lambda(2 - |u|^\lambda)dx^2, \tag{7}$$

where $0 < \lambda < 1$ and ∂_u gives the time-orientation. The metric is λ -Hölder regular and smooth of the x -axis. Null curves branch off from the x -axis and points in the region between the first branching null curve and the x -axis (which is also null) can be connected to the origin via causal curves (even of positive length) but not by timelike curves, see [4, Fig. 1.1]. This region is called the *causal bubble* and this demonstrates the failure of the push-up principle.

A slight modification of the above example yields the existence of a maximal causal curve in a $C^{0,\lambda}$ -spacetime which does not have a causal character. This example can be found in detail in [16, Cor. 5.5].

Example 2.6. In the above Example 2.5 we put for simplicity $\lambda = \frac{1}{2}$ and consider only the manifold $M := (-1, 1) \times \mathbb{R}$. Let $q := (u_0, x_0)$ be in the (upper right) *bubble region*, i.e., $0 < u_0 < \min(\frac{x_0}{4}, 1)$. As this spacetime is strongly causal and the causal diamond of the origin and q is compact, there is a maximal causal curve from the origin to q . This maximal causal curve cannot have a causal character: The curve has to first go along the x -axis, where it is null and after it leaves the x -axis it is in the smooth part of the spacetime. Moreover, there it has to have positive length and thus must be timelike.

Finally, we remark that the Du Bois-Reymond trick does not work in the Lorentzian setting. If g is a C^1 -Lorentzian metric and γ is a maximal causal curve, then it is not clear that γ has to be timelike even if γ has positive length. Therefore, one cannot ensure that the variation $\gamma + \varepsilon\phi$ (compare the discussion the Riemannian case in Section 2.1) is causal, hence there is no reason that $\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} L_g(\gamma + \varepsilon\phi) = 0$. The argument only works out in case we already know that the maximizing curve is timelike.

3. The geodesic equation for locally Lipschitz metrics

In this section we deal with the geodesic equation for semi-Riemannian manifolds (M, g) with $g \in C^{0,1}$. The motivation for studying this regularity class comes on the one hand from the fact that here some crucial aspects of causality theory remain valid, cf. [4] and Section 2.2 above, and on the other hand that relevant exact solutions of general relativity such as impulsive gravitational wave spacetimes have this regularity.

Recall that in this regularity class the right-hand-side of the geodesic equation is only guaranteed to be locally bounded but *not* to be continuous. It turns out that a fruitful way to deal with these ‘ODEs with discontinuous right hand side’ is to use the Filippov solution concept [7], which we will now briefly review (see [5] for an application-driven introduction).

3.1. Filippov solutions

The key idea of Filippov’s approach is to replace an ODE

$$\dot{x}(t) = f(x(t)) \quad (t \in I), \tag{8}$$

(I is some interval, $f: \mathbb{R}^n \supseteq D \rightarrow \mathbb{R}^n$ possibly of low regularity) by a *differential inclusion*

$$\dot{x}(t) \in \mathcal{F}[f](x(t)), \tag{9}$$

where the *Filippov set-valued map* $\mathcal{F}[f]: D \rightarrow \mathcal{K}_0(\mathbb{R}^n)$ (the collection of all nonempty, closed and convex subsets of \mathbb{R}^n) associated with F is defined as its essential convex hull

$$\mathcal{F}[f](x) := \bigcap_{\delta>0} \bigcap_{\mu(S)=0} \text{co} \left(f(B(x, \delta) \setminus S) \right). \tag{10}$$

Here co denotes the closed convex hull, $B(x, \delta)$ is the closed ball of radius δ centered at x , and μ is the Lebesgue measure, cf. [31, Sec. 2].

The key idea is to look at the set of values f takes near a point of discontinuity. This fact lies also at the heart of the compatibility of this approach with approaches based on

regularization, see [11, Sec. 3.3]. The explicit calculation of a Filippov set-valued map can be non-trivial, however, there exists a calculus to at least bound this set [26]. Also note that a Filippov set-valued map is actually multi-valued only at the points of discontinuity of f .

Now a *Filippov solution* of the ODE (8) is defined to be an absolutely continuous curve $x: J \rightarrow U$, defined on some interval $J \subseteq I$, that satisfies the inclusion relation (9) almost everywhere.

Recall that a curve $x: [a, b] \rightarrow \mathbb{R}^d$ is said to be *absolutely continuous* if for every $\varepsilon > 0$ there is a $\delta > 0$ such that for all collections of non-overlapping intervals $([a_i, b_i])_{i=1}^m$ in $[a, b]$ with $\sum_{i=1}^m (b_i - a_i) < \delta$ we have that $\sum_{i=1}^m \|x(b_i) - x(a_i)\| < \varepsilon$. Moreover, recall that an absolutely continuous curve is differentiable almost everywhere.

Of course, if f is continuous, classical \mathcal{C}^1 -solution and Filippov solutions coincide, but the latter exist under much more general assumptions on f . In fact, Filippov in [7] has developed a complete theory of ordinary differential equations based on this solution concept which has been found to be widely applicable e.g. in non-smooth mechanics. Here we just state the main existence result.

Theorem 3.1. ([7, Thm. 7.1]) The initial value problem

$$\dot{x}(s) \in A(s, x(s)) \text{ a.e.}, \quad x(t_0) = x_0 \quad (t_0, x_0) \in I \times \mathbb{R}^n \quad (11)$$

has an absolutely continuous solution if the set valued map $I \times \mathbb{R}^n \ni (t, x) \mapsto A(t, x)$ satisfies

- (i) $t \mapsto A(t, x)$ is Lebesgue measurable on I for all fixed x ,
- (ii) $x \mapsto A(t, x)$ is upper semi-continuous for almost all t , and
- (iii) $\sup_{x \in \mathbb{R}^n} |A(t, x)| \leq \beta(t) \in L^1_{\text{loc}}(I)$ for almost all t .

Our main interest lies in the following simple consequence, where we denote by L^∞_{loc} the space of all measurable and locally essentially bounded functions.

Corollary 3.2. ([7, Thm. 7.8]) If $f \in L^\infty_{\text{loc}}(D, \mathbb{R}^n)$, then for each $(t_0, x_0) \in I \times D$ there is a Filippov solution x of (8) with $x(t_0) = x_0$.

3.2. Existence of geodesics

This results lends itself to an application to the geodesic equations on semi-Riemannian manifolds M with a $\mathcal{C}^{0,1}$ -metric. First observe that the notion of the essential convex hull is invariantly defined on M . Indeed $\mathcal{F}(f)(\phi(x)) = \mathcal{F}(f \circ \phi)(x)$ for any diffeomorphism ϕ since the respective balls in (10) $B(\phi(x), r)$ and $\phi(B(x, r'))$ can be nested. Now rewriting the geodesic equations as a first order system one locally obtains from the Lipschitz property of the metric an equation of the form (8) with $f \in L^\infty_{\text{loc}}$. Hence we have:

Corollary 3.3. ([34, Thm. 2]) Let (M, g) be a smooth manifold with a $\mathcal{C}^{0,1}$ -semi-Riemannian metric. Then there exist a Filippov solution of the geodesic equation for arbitrary data $p \in M$, $v \in T_p M$. These solutions possess absolutely continuous velocities (hence, in particular, are \mathcal{C}^1 -curves).

Observe that such geodesics do not satisfy the geodesic equation in the classical sense. We only know that for almost all values of an affine parameter we have that $\ddot{\gamma}^i \in \mathcal{F}(-\Gamma^i_{jk} \dot{\gamma}^j \dot{\gamma}^k)$. Consequently, in general the norm of the tangent $|\dot{\gamma}|$ will not be preserved and hence in the Lorentzian case γ will not have a global causal character. For the same reason we do not

know whether these geodesics are locally minimizing resp. maximizing curves. Also concerning regularity we do not know whether the geodesics are $C^{1,1}$, which would match the limiting case $\alpha = 1 = \beta$ in [21], see Section 2.1 above. In fact, we only obtain that the geodesics are C^1 with absolutely continuous tangent.

3.3. Uniqueness of geodesics

Next we turn to uniqueness. Clearly we cannot hope for any general result and we will rather be interested in situations where the $C^{0,1}$ -metric is actually smooth off some hypersurface. This allows us to cover the case of ‘matched spacetimes’ and impulsive gravitational wave geometries, see below. Here we start with a general discussion of uniqueness of Filippov solutions for (8). In principle essentially one-sided Lipschitz conditions, i.e., $(f(x) - f(y))^t(x - y) \leq L\|x - y\|_2^2$ for some L and almost all x, y , provide one-sided uniqueness results (e.g. [5, Prop. 4]) without requiring f to be continuous. However, these results are ill-suited in case f is smooth off some hypersurface in \mathbb{R}^n since piecewise smooth functions generically fail to be essentially one-sided Lipschitz even in cases where the corresponding Filippov solutions are unique, cf. [5, p. 53].

Hence we follow a different route and start by making the situation precise. Suppose that $f \in L_{\text{loc}}^\infty(D)$ and that the domain $D \subseteq \mathbb{R}^n$ is connected and a disjoint union $D = D^- \dot{\cup} N \dot{\cup} D^+$, where D^+, D^- are open sets and their common boundary, $N = \partial D^+ = \partial D^-$ is a smooth hypersurface, see the figures below. Now suppose $f \in C^1(D^\pm)$ up to the boundary N and denote by f^\pm the extensions of $f|_{D^\pm}$ to the closure $\bar{D}^\pm = D^\pm \cup N$. Finally write f_N^\pm for the projections of $f^\pm|_N$ on the unit normal \vec{n} of N , pointing from D^- to D^+ . We will abbreviate this situation by saying that the locally bounded function f is *smooth off the hypersurface* N . Then basic ODE-theory on the domains \bar{D}^\pm yields the following:

Lemma 3.4. (Simple conditions for uniqueness, [7, Lem. 10.2]) Let $f \in L_{\text{loc}}^\infty(D, \mathbb{R}^n)$ be smooth off the hypersurface¹ N . If for $x_0 \in N$ we have $f_N^+(x_0) > 0$, then in \bar{D}^+ there exists a unique C^2 -solution of (8) starting at x_0 . Analogous assertions hold for \bar{D}^- and $f_N^-(x_0) < 0$.

This basic observation also leads to the following conclusions: If at a point $x_0 \in N$ we have that f^+ points into D^+ ($f_N^+ > 0$) and f^- points into D^- ($f_N^- < 0$) then there are C^2 -solutions which proceed into D^+ and such that proceed into D^- and one speaks of *repulsive trajectories*, see Figure 2. Clearly in this case uniqueness of Filippov solutions on D fails.

In all other possible cases uniqueness of Filippov solutions can be secured ([5, Prop. 5]). However, if at $x_0 \in N$ we have that f^+ points into D^- ($f_N^+ < 0$) and f^- points into D^+ ($f_N^- > 0$) then the C^2 -solutions from either side may be trapped in N , a situation referred to as *sliding motion*, see Figure 3. We will be mainly interested in the remaining two cases, i.e., when the Filippov solutions cross from D^- into D^+ or vice versa. Indeed in the cases where f_N^+ and f_N^- share their sign, i.e., $f_N^\pm > 0$ (Figure 4) and $f_N^\pm < 0$ (Figure 5) solutions pass from D^- into D^+ and from D^+ to D^- , respectively. In this case one speaks of *transversally crossing trajectories*. More precisely the following criterion holds:

¹ In the sense defined above.

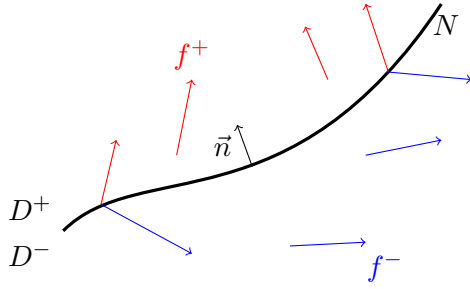


Figure 2: repulsive trajectories

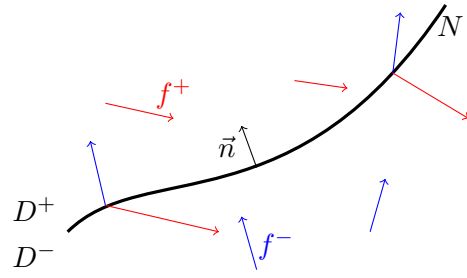


Figure 3: sliding motion

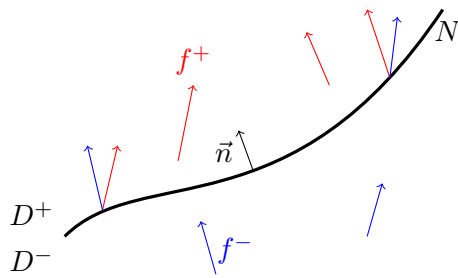


Figure 4: (upward) transversally crossing

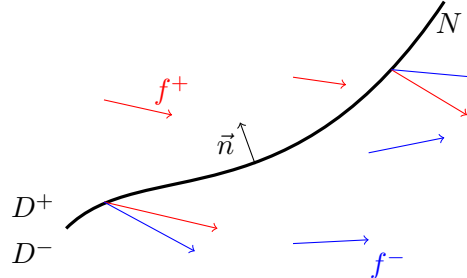


Figure 5: (downward) transversally crossing

Corollary 3.5. (Sufficient conditions for uniqueness [7, Cor. 10.1]) Let f be as above. On the region of the surface N where $f_N^+ > 0$ and $f_N^- > 0$, Filippov solutions that reach N from D^- pass to D^+ and hence uniqueness is not violated. The analogous assertion holds for $f_N^+, f_N^- < 0$ and solutions passing from D^+ to D^- .

We will now apply Corollary 3.5 to a geometric scenario that generically arises e.g. in ‘matched spacetimes’. Essentially the present discussion isolates the abstract core of the one given in [27, Sec. 3.3] by neglecting the specific form of the spacetime metric used there. We start with a semi-Riemannian manifold (M, g) with $g \in C^{0,1}$ and assume that g is smooth off a C^∞ -hypersurface \mathcal{N} in the following sense: M is the disjoint union of some open sets $\mathcal{D}^+, \mathcal{D}^-$ and their common boundary $\mathcal{N} = \partial\mathcal{D}^+ = \partial\mathcal{D}^-$ which we assume to be a smooth hypersurface of M . The metric g is smooth in \mathcal{D}^\pm up to the boundary, that is $g \in C^2(\mathcal{D}^\pm \cup \mathcal{N})$. (Here $\mathcal{D}^\pm \cup \mathcal{N}$ are smooth manifolds with boundary.)

Now a (Filippov solution of the) geodesic (equation) in \mathcal{D}^\pm is a classical geodesic. Let us consider such a curve starting, say in D^- (the case D^+ being analogous) and reaching \mathcal{N} . Locally in coordinates $(\mathcal{U}, (x^1, \dots, x^n))$ we write \mathcal{N} as $\{x^1 = 0\}$ and we rewrite the geodesic equation for $\gamma(t) = (x^1(t), \dots, x^n(t))$ as a first order system of the form

$$\dot{x}^j = \hat{x}^j, \quad \dot{\hat{x}}^j = -\Gamma_{km}^j(x) \hat{x}^k \hat{x}^m, \quad (1 \leq j \leq n). \tag{12}$$

Now the right hand side of (12) is given by the L_{loc}^∞ -vector field

$$f(x^1, \hat{x}^1, \dots, x^n, \hat{x}^n) = (\hat{x}^1, -\Gamma_{km}^1(x) \hat{x}^k \hat{x}^m, \dots, \hat{x}^n, -\Gamma_{km}^n(x) \hat{x}^k \hat{x}^m)$$

defined on some open subset U of \mathbb{R}^{2n} . On U we set $N := \{x^1 = 0\}$ and $D^\pm := \{\pm x^1 > 0\}$. Then the unit normal of N pointing from D^- to D^+ is e_1 , i.e., the first standard unit vector.

It follows that the projection of the limits of $f|_{\mathcal{D}^\pm}$ on N onto the normal coincide and are just given by $f_N^\pm = \hat{x}^1 = \dot{x}^1$. So Corollary 3.5 applies if $\dot{x}^1(t_0) \neq 0$ ($\dot{x}^1(t_0) > 0$ in this case) where t_0 is the parameter value when γ (first) hits \mathcal{N} , i.e., $x^1(t_0) = 0$. But this just means that γ does hit \mathcal{N} transversally and we have the following result:

Proposition 3.6. (Sufficient conditions for uniqueness) Let (M, g) be a $C^{0,1}$ -semi-Riemannian manifold with g smooth off a C^∞ -hypersurface¹ \mathcal{N} . Then the (Filippov) geodesics starting in $M \setminus \mathcal{N}$ that hit \mathcal{N} transversally are globally unique and they cross from \mathcal{D}^- into \mathcal{D}^+ or vice versa.

This proposition especially applies if the hypersurface \mathcal{N} is *totally geodesic* in the following sense: Every (Filippov) geodesic starting in \mathcal{N} tangentially to \mathcal{N} remains initially in \mathcal{N} . Note that the notion of a totally geodesic submanifold in low regularity becomes somewhat subtle. In particular, in the situation at hand the second fundamental form will generically be not defined on all of \mathcal{N} .

Observe that the (Filippov) geodesics γ crossing \mathcal{N} at $\gamma(t_0)$ coincide for all $t \neq t_0$ with the smooth geodesics of \mathcal{D}^\pm . Hence by their C^1 -property $g(\dot{\gamma}, \dot{\gamma})$ is globally constant and these geodesics do have a causal character. Moreover they are extremal curves by the following argument which we only detail in the Riemannian case: Suppose there is a curve λ connecting $\gamma(t_1) \in \mathcal{D}^-$ with $\gamma(t_2) \in \mathcal{D}^+$, which is shorter than $\gamma|_{[t_1, t_2]}$. Then set $t' = \sup\{t > t_1 : \gamma(t) = \lambda(t)\}$ and suppose $t' < t_0$. By continuity $\gamma(t') = \lambda(t')$ and we choose a totally normal neighbourhood $U \subseteq \mathcal{D}^-$ of $\gamma(t')$. Then λ has to be minimizing in U which contradicts the fact that λ is not the radial geodesic γ in U . So $\gamma(t) = \lambda(t)$ for all $t \neq t_0$ and hence everywhere by continuity.

Finally the above result suggests the idea to explicitly obtain the geodesics of M by appropriately matching the geodesics of each ‘side’ \mathcal{D}^\pm across \mathcal{N} . We will discuss these matters in the closing section below.

3.4. The C^1 -matching of geodesics

In this final section we discuss the matching of geodesics in locally Lipschitz semi-Riemannian manifolds with the metric smooth off a hypersurface \mathcal{N} . Indeed such an approach has been frequently applied in the literature on impulsive gravitational waves, see e.g. [6, 29, 30] and the references given in [27, Sec. 3.2]. The idea is to explicitly calculate the geodesics in \mathcal{D}^\pm which is often possible (only) in coordinates which do not extend to the ‘matching hypersurface’ \mathcal{N} . Then one matches the geodesics γ^- of \mathcal{D}^- that hit \mathcal{N} ‘from below’ to the geodesics γ^+ of \mathcal{D}^+ that hit \mathcal{N} ‘from above’ across \mathcal{N} . Explicitly in coordinates which cover \mathcal{N} one sets $\gamma^-(t_0) = \gamma^+(t_0)$ and $\dot{\gamma}^-(t_0) = \dot{\gamma}^+(t_0)$ where t_0 is the parameter value where the respective geodesics hit \mathcal{N} . Obviously one needs to involve the derivatives to obtain the correct number of equations to match all the data and this is why one refers to this approach as C^1 -matching procedure.

Often such calculations were done heuristically without supplying the necessary arguments which we collect here, cf. also [27, Rem. 4.1]. In fact the matching mathematically makes sense only if the following facts on the geodesics of M have been established:

- (i) The geodesics reaching \mathcal{N} cross it (rather than being reflected by or trapped into \mathcal{N}).
- (ii) These geodesics are unique (rather than e.g. branching).

¹ In the sense specified above.

(iii) These geodesics are at least of C^1 -regularity.

Indeed we can extract all this necessary information from the discussion in the previous section to obtain the following result:

Corollary 3.7. (The C^1 -matching) Let (M, g) be a $C^{0,1}$ -semi-Riemannian manifold with g smooth off a smooth hypersurface¹ \mathcal{N} . Let $\gamma^- : (a, b] \rightarrow \mathcal{D}^- \cup \mathcal{N}$ be a smooth geodesic with $\gamma(b) \in \mathcal{N}$ and $\dot{\gamma}(b) \notin T_{\gamma(b)}\mathcal{N}$. Then there exists $c > b$ and a unique (Filippov) geodesic $\gamma : (a, c) \rightarrow M$ such that $\gamma|_{(a,b]} = \gamma^-$.

Now the C^1 -matching of the data can be made explicitly by defining $\gamma^+ := \gamma|_{[b,c)}$. Indeed we then have $\gamma^-(b) = \gamma^+(b)$ and $\dot{\gamma}^-(b) = \dot{\gamma}^+(b)$. Also we remark that this procedure allows to derive the (Filippov) geodesics crossing \mathcal{N} simply by matching the smooth ‘background’ geodesics on either side in a C^1 -manner without the need to go into the details of Filippov’s theory.

This holds true even in case one needs to invoke the fact that \mathcal{N} is totally geodesic to rule out that geodesics hit \mathcal{N} tangentially. Indeed this fact can often be derived purely from knowledge of the ‘background geodesics’ γ^\pm and the C^1 -regularity of the (Filippov) geodesics cf. [27, Sec. 3.6].

Finally in case \mathcal{N} fails to be totally geodesic (as is the case for all classes of expanding impulsive gravitational waves) one might still make use of Proposition 3.6 and Corollary 3.7 (and hence establish the $C^{1,1}$ -matching) by showing ‘by hand’ that the ‘background geodesics’ do meet \mathcal{N} transversally, cf. [28, Sec. 3.3].

4. Outlook and open problems

We conclude with listing some open question and discuss further lines of research.

- Riemannian du Bois-Reymond-trick for Lipschitz continuous metrics
Let g be a Lipschitz continuous Riemannian metric and let γ be a Lipschitz continuous minimizer. Is there a way to see that γ has to be C^2 and that it satisfies the geodesic equations in the sense of Filippov?
- Lorentzian du Bois-Reymond trick
Let g be a C^1 Lorentzian metric and let γ be a Lipschitz continuous maximizer between timelike related points. Does γ have to be timelike? This would imply that one could apply the du Bois-Reymond trick to get that γ is C^2 and that it satisfies the geodesic equations.
- Regularity of maximal causal curves
Is there an analogue of the result by Lytchak and Yaman [21] for Lorentzian Hölder or Lipschitz continuous metrics? Even for C^1 metrics the regularity of maximal curves is unclear (cf. the point above).
- Causal character of maximizing causal curves
What is the minimal regularity of a Lorentzian metric to ensure that maximal causal curves have a causal character? By Example 2.6 the metric has to be at least Lipschitz continuous².

¹ In the sense specified above.

² Note added in proof: In [10] it was meanwhile shown that $g \in C^{0,1}$ is actually sufficient.

- Properties of Filippov geodesics

Can one say more about Filippov geodesics in Riemannian and Lorentzian signature? Do they have to be minimizing or maximizing, respectively, in some sense? Observe that it is not expected that Lorentzian Filippov geodesics have a causal character.

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