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Permutation-symmetric three-particle hyper-spherical harmonics based on the $S_3 \otimes SO(3)_{rot} \subset O(2) \otimes SO(3)_{rot} \subset U(3) \rtimes S_2 \subset O(6)$ subgroup chain

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Abstract

We construct the three-body permutation symmetric hyperspherical harmonics to be used in the non-relativistic three-body Schrödinger equation in three spatial dimensions (3D). We label the state vectors according to the $S_3 \otimes SO(3)_{rot} \subset O(2) \otimes SO(3)_{rot} \subset U(3) \rtimes S_2 \subset O(6)$ subgroup chain, where S_3 is the three-body permutation group and S_2 is its two element subgroup containing transposition of first two particles, $O(2)$ is the "democracy transformation", or "kinematic rotation" group for three particles; $SO(3)_{rot}$ is the 3D rotation group, and $U(3)$, $O(6)$ are the usual Lie groups. We discuss the good quantum numbers implied by the above chain of algebras, as well as their relation to the S_3 permutation properties of the harmonics, particularly in view of the $SO(3)_{rot} \subset SU(3)$ degeneracy. We provide a definite, practically implementable algorithm for the calculation of harmonics with arbitrary finite integer values of the hyper angular momentum K, and show an explicit example of this construction in a specific case with degeneracy, as well as tables of $K \leq 6$ harmonics. All harmonics are expressed as homogeneous polynomials in the Jacobi vectors $(\boldsymbol{\lambda}, \boldsymbol{\rho})$ with coefficients given as algebraic numbers unless the "operator method" is chosen for the lifting of the $SO(3)_{rot} \subset SU(3)$ multiplicity and the dimension of the degenerate subspace is greater than four – in which case one must resort to numerical diagonalization; the latter condition is not met by any $K \leq 15$ harmonic, or by any $L \leq 7$ harmonic with arbitrary K. We also calculate a certain type of matrix elements (the Gaunt integrals of products of three harmonics) in two ways: 1) by explicit evaluation of integrals and 2) by reduction to known $SU(3)$ Clebsch–Gordan coefficients. In this way we complete the

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calculation of the ingredients sufficient for the solution to the quantum-mechanical three-body bound state problem.

1. Introduction

The quantum mechanical three-body bound-state problem is an old one: it has a huge literature in which the hyperspherical harmonics, Refs. [1–23], form one of the most firmly established theoretical tools – for recent reviews, see Refs. [24–29].[1] Classification of wave functions into distinct classes under permutation symmetry is a fundamental property of non-relativistic quantum mechanics with non-trivial consequences in the three-body system. Permutation symmetric three-body hyperspherical harmonics in three dimensions, however, are explicitly known only in a few special cases such as the total orbital angular momentum $L = 0$, 1 ones, cf. Refs. [5,6,8,19]. Hyperspherical harmonics with higher values of L can be constructed by means of a (numerical) recursive procedure that symmetrizes non-permutation-symmetric hyperspherical harmonics, see Refs. [20,21].

In so doing one loses track, however, of a certain dynamical $O(2)$ symmetry that is related to the so-called "kinematic rotation", Ref. [2], or equivalently to the "democracy" transformations, Refs. [6,8,12]. This "kinematic rotation" invariance, or "democracy" symmetry was viewed as mathematical esoterics, until recently Ref. [30] showed it to be the dynamical symmetry of area-dependent potentials, which class includes the so-called Y-string potential in QCD. The Y-string is the leading candidate for the confinement mechanism of three quarks in QCD, Refs. [30–34]. Consequently followed the increased interest in the properties of the "kinematic rotation" invariant, or "democratic" potentials and in their spectra.

In two spatial dimensions (2D) the problem of constructing permutation symmetric hyperspherical harmonics was solved, at first by Smith, Ref. [4], and then in Ref. [35] in a general way that makes the "kinematic rotation" $O(2)$ invariance (or "democracy" symmetry) explicit, following certain fairly abstract internal geometric ("shape space") considerations by Iwai, Refs. [36,37]. These 2D permutation-symmetrized $SO(4)$ hyperspherical harmonics are closely related to the (3D magnetic) monopole harmonics, Ref. [38], and to so-called spin-weighted spherical harmonics, Ref. [39]. In Refs. [40–42] we have used these symmetrized hyperspherical harmonics with the "kinematic rotation" $O(2)$ label to solve the Schrödinger equation for three-body bound states in two spatial dimensions with area-dependent potentials based on the $so(2) \oplus so_L(2) \subset so(3) \oplus so(3) \subset so(4)$ chain of algebras (where $so_L(2)$ is the total angular momentum part and $so(2)$ is the "democracy" transformation, or "kinematic rotation" generator). Those results show explicitly the role of the "kinematic rotation" $O(2)$ invariance in the energy level-degeneracy and/or splitting in area-dependent potentials in 2D.

Similarly to the three-body problem in two dimensions (2D), Refs. [40–42], the knowledge of the three-body permutation symmetric hyperspherical harmonics in three dimensions (3D) with the "kinematic rotation" $O(2)$ label would allow one to calculate the discrete part of the energy spectrum of the three-body problem. A systematic construction of (all) permutation symmetric

[1] It is commonly assumed that Faddeev's work on quantum-mechanical three-body equations has solved the three-body problem – that is only partially true: Faddeev's equations allow one to solve the three-body scattering problem, but do not affect the bound state problem significantly.

hyperspherical harmonics in 3D, based on the $S_3 \otimes SO(3)_{rot} \subset O(2) \otimes SO(3)_{rot} \subset U(3) \rtimes S_2 \subset O(6)$ "chain" of subgroups for labeling purposes, is the first basic contribution of the present paper. This construction is complete in the sense that a definite algorithm is provided for the construction of arbitrary-integer-K harmonics (where K is the hyper angular, or "grand angular" momentum), that has been used to construct harmonics up to some finite value of K. We do not have a simple formula for arbitrary-K harmonics, however. As the second basic contribution of this paper, we provide explicit results of the evaluation of an integral over tri-linear products of harmonics (this is the $SO(6)$ analogon of the Gaunt formula), in terms of (known) $SU(3)$ Clebsch–Gordan coefficients.

The basic idea that we used is not new: we started out by constructing certain homogeneous polynomials of the two Jacobi vectors, just as Simonov did in Ref. [5], but without the restriction to scalars under spatial rotations. In so doing we used the $O(6)$ group labels to classify the hyperspherical harmonics. In this way the three-body problem in three dimensions can be effectively reduced to an $O(6)$ group theoretical problem. By a careful study of the labeling of three-body states, we arrive at the subgroup "chain" $S_3 \otimes SO(3)_{rot} \subset O(2) \otimes SO(3)_{rot} \subset U(3) \rtimes S_2 \subset O(6)$. Here $SO_{rot}(3)$ is the total angular momentum part and $O(2)$ is the subgroup of so-called "democracy" transformations, Ref. [12], or equivalently "kinematic rotations", Ref. [2], and the S_3 and S_2 are the (discrete) three-particle permutation group and the two-particle permutation (of particles 1 and 2) group, respectively, that are also subgroups of these "democracy" transformations.

As the first step, we construct "core polynomials" of order K that have particular predefined transformation properties w.r.t. the $U(1) \otimes SO(3)_{rot}$ subgroup action, where $U(1) \equiv SO(2)$ is the unit determinant subgroup of the $O(2)$ group of "democracy" transformations containing only cyclic, i.e., even particle permutations. In the following step these core polynomials are "filtered out", i.e., projected so as to become harmonic, i.e., to obtain sharp values of the hyperangular momentum K.

The obtained harmonic polynomials, however, are still plagued with what is known as the multiplicity problem in $SU(3)$ group theory, Refs. [43–47]: in general, the labels of the $U(1) \otimes SO(3)_{rot}$ subgroup together with the hyperangular momentum K are insufficient to uniquely specify the $SO(6)$ harmonics. We offer two possible solutions to this "multiplicity problem": i) the traditional approach, Refs. [43–47], based on the introduction of a multiplicity lifting operator, that must be diagonalized, where we discuss several different such operators, primarily in the light of the (so-induced) permutation properties of the harmonics; and ii) a novel (non-traditional) approach, based on a new auxiliary integer label, that is introduced in the process of constructing the harmonics. Both of these choices present definite algorithms for the construction of an arbitrary (positive integer) K-th order $SO(6)$ three-body permutation-symmetric hyperspherical harmonic, albeit with different advantages and drawbacks.

In the first case, the resulting hyperspherical harmonics can be, in general, expressed in closed algebraic form only when $\mathrm{K} \leq 15$ and/or $L \leq 7$, whereas, beyond $\mathrm{K} \geq 16$ and $L \geq 8$ some harmonics have to be expressed numerically, due to restrictions imposed by Galois theory. Consequently, such harmonics cannot be used for the study of arbitrary-K, L properties, e.g. the Regge trajectories, of three-body states. We present here the $SO(6)$ three-body permutation-symmetric hyperspherical harmonics, based on the Racah degeneracy-lifting operator, Ref. [43], together with their transformation properties under permutations, i.e., the irreducible representations of the permutation group S_3.

In the second case, the multiplicity labeling procedure does not rely on solving any operator eigenvalue problem, so that *all hyperspherical harmonics can be expressed in a closed algebraic* *form*. Such a significant simplification comes at a price, however, *viz.* the new auxiliary label does not have a clear $SU(3)$ group-theoretical meaning. Consequently, it has not been used to evaluate the corresponding Clebsch–Gordan coefficients in the literature (see below). For this reason, here we shall merely state some of the implications of this choice and proceed with further discussion based exclusively on the first type of solution to the multiplicity problem.

Hyperspherical harmonics obtained through the described procedure are labeled according to the subgroup chain $U(1) \otimes SO(3)_{rot} \subset U(3) \subset SO(6)$, plus the multiplicity label (that may, or may not be related to the $so(6)$ enveloping algebra). However, odd particle permutations do not belong to this chain, as they correspond to transformations of the six-dimensional configuration space with determinant equal to -1. In the concluding step of construction of the permutation symmetric harmonics we discuss their action (in a separate section on permutation properties) which then extends the symmetry group/chain to $S_3 \otimes SO(3)_{rot} \subset O(2) \otimes SO(3)_{rot} \subset U(3) \rtimes S_2 \subset O(6)$. The final result are hyperspherical harmonics with clearly established and manifest S_3 permutation properties, that are simple linear combinations of the previously derived $SO(6)$ harmonic functions.

Then we calculated a certain class of integrals (matrix elements) over trilinear products of hyperspherical harmonics that appear in standard quantum-mechanical three-body problems, Ref. [48–50]. We did so firstly by explicit evaluation, i.e., by a reduction to certain gamma (Γ) function type of integrals, and secondly by group theoretical techniques, i.e., by reduction, at first to a product of two $SO(6)$ Clebsch–Gordan coefficients, which are not well known, and then we used a one-to-one relation between the $SO(6)$ hyperspherical harmonics and the $SU(3)$ irreducible representations, to express the integrals as a product of two $SU(3)$ Clebsch–Gordan coefficients, which are quite well known, [51–57]. We do not foresee further simplifications of our results, at least not in matters of principle, though we cannot exclude potential improvements of numerical algorithms used for their evaluation. In this way, we have reduced these integrals over trilinear products of hyperspherical harmonics to their simplest form that is also amenable to straightforward numerical evaluation.

Our results are not specific to any particular three-body problem, i.e., they can, and we hope they will, find application in many realistic 3D three-body problems, such as in the three-quark problem in hadronic physics, as well as in atomic and molecular physics.

As stated above, symmetrized three-body hyper-spherical harmonics have been pursued before, albeit without emphasis on the "kinematic rotation" $O(2)$ symmetry label. To our knowledge, aside from the special case ($L = 0$) results of Simonov and of Dragt, Refs. [5–7], and the $L = 1$ results of Lévy-Leblond and of Barnea and Mandelzweig, Refs. [8,19], several other attempts, based on the so-called "tree pruning" techniques, exist in the literature, Refs. [18,23,28], beside the recursively symmetrized N-body hyperspherical harmonics of Barnea and Novoselsky, Refs. [20,21]. The latter are based on the $O(3) \otimes S_N \subset O(3N - 3)$ chain of algebras, which does not include the "kinematic rotation"/"democracy" $O(2)$ symmetry. Moreover, no explicit examples of three-body symmetrized hyperspherical harmonics were given in Refs. [20,21], as they were meant primarily for numerical computations, and not for fundamental studies.

In several early papers, Refs. [6–8], and, somewhat later, also in Refs. [14,15], the same $SU(3)$ group was used to label and construct some three-particle continuum states with $\mathrm{K} \leq 12$, but their applications to bound state problems was not considered. Refs. [14,15] are particularly close in spirit to our approach, albeit not in technical detail. For a fuller discussion of these other approaches and their relation to the present work, see Sect. 9.

This paper consists of nine sections. After providing the necessary preliminaries in Sect. 2, we explain our $SO(6)$ algebraic methods for constructing the core polynomials in Sect. 3. Then, we project out the core polynomials to get harmonic three-body hyper-spherical polynomials in Sect. 4. In Sect. 5 we discuss how to resolve the multiplicity of three-body hyper-spherical harmonic polynomials in general, and in Sect. 6 we illustrate the procedure with a few examples. Then in Sect. 7 we discuss the permutation symmetry and define final expressions for the harmonics that possess simple and manifest transformation properties with respect to the subgroup chain $S_3 \otimes SO(3)_{rot} \subset O(2) \otimes SO(3)_{rot} \subset U(3) \rtimes S_2 \subset O(6)$. In Sect. 8 we show the calculation of a certain type of matrix elements, and discuss their group theoretical ramifications. Finally, in Section 9 we present a summary and discussion of the results and of their relation to other papers in the literature. Some useful integrals are shown in Appendix A, the details of $SO(6)$ Clebsch–Gordan coefficients are shown in Appendix B, and a list of h.s. harmonics with $K \leq 6$ is given in Appendix C.

2. Preliminaries

2.1. Three-body hyper-spherical harmonics

Coordinates of three (identical) particles (with equal masses) in the center-of-mass (c.m.) rest frame are given by two Jacobi three-vectors:

$$\boldsymbol{\lambda} = \frac{1}{\sqrt{6}}(\mathbf{r}_1 + \mathbf{r}_2 - 2\mathbf{r}_3), \tag{1}$$

$$\boldsymbol{\rho} = \frac{1}{\sqrt{2}}(\mathbf{r}_1 - \mathbf{r}_2). \tag{2}$$

The kinetic energy in the rest frame is of the form:

$$T = \frac{m}{2}\left(\dot{\boldsymbol{\lambda}}^2 + \dot{\boldsymbol{\rho}}^2\right), \tag{3}$$

possessing an $O(6)$ symmetry that is made manifest by introducing six-dimensional coordinate hyper-vector $x_\mu = (\boldsymbol{\lambda}, \boldsymbol{\rho})$: the kinetic energy Eq. (3) can be written as

$$T = \frac{m}{2}\dot{R}^2 + \frac{\mathrm{K}_{\mu\nu}^2}{2mR^2} \tag{4}$$

where $R \equiv \sqrt{\boldsymbol{\lambda}^2 + \boldsymbol{\rho}^2}$ is the hyper-radius and the "grand angular", or hyper-angular momentum tensor $\mathrm{K}_{\mu\nu}$, $\mu, \nu = 1, 2, \ldots, 6$ reads

$$\begin{aligned} \mathrm{K}_{\mu\nu} &= m\left(\mathbf{x}_\mu \dot{\mathbf{x}}_\nu - \mathbf{x}_\nu \dot{\mathbf{x}}_\mu\right) \\ &= \left(\mathbf{x}_\mu \mathbf{p}_\nu - \mathbf{x}_\nu \mathbf{p}_\mu\right). \end{aligned} \tag{5}$$

It has 15 linearly independent components and generates an $SO(6)$ group acting in this six-dimensional space. Among these 15 generators are also the three components of the "ordinary" (total) orbital angular momentum: $\mathbf{L} = \mathbf{l}_\rho + \mathbf{l}_\lambda = m\left(\boldsymbol{\rho} \times \dot{\boldsymbol{\rho}} + \boldsymbol{\lambda} \times \dot{\boldsymbol{\lambda}}\right)$. In addition to the $SO(6)$ group action that generates linear transformations of the 6-dimensional space with unit determinant, particle permutations also constitute a part of the symmetries of the kinetic energy, Eq. (3). The odd permutations, however, correspond to six-dimensional linear transformations with determinant equal to -1, thus extending the full symmetry group to $O(6)$.

Once the potential energy V is introduced, this large symmetry is generally broken to some extent, sometimes all the way down to the product of the three-body permutation symmetry S_3

and the rotation symmetry $SO(3)$ (or $O(3)$), and sometimes with an additional remnant dynamical symmetry. This is a motivation to split the three-particle wave-function into a hyper-radial part (which is a function solely of the hyper-radius R) and the hyper-angular part (which is a function of x^{μ}/R), where the natural basis for the latter one are 6-dimensional hyper-spherical harmonics.

Hyper-spherical harmonics (in any dimension D) transform as symmetric tensor representations of $SO(D)$ group, which is (for $D > 3$) only a subset of all tensorial representations. In turn, this means that any hyper-spherical harmonic is labeled by a single integer K that is also an irreducible representation label, matching the order of the symmetric tensor representation (K corresponds to the highest weight $(\mathrm{K}, 0, 0, 0, \dots)$ irreducible representation, following the usual definitions, or to the Young diagram with K boxes in a single row). Ordinary ($D = 3$) spherical harmonics are eigenfunctions of the square of the angular momentum operator K (this operator is, in the $D = 3$ context, usually denoted as L) with the eigenvalue $\mathrm{K}(\mathrm{K} + 1)$, whereas D-dimensional hyper-spherical harmonics are eigenfunctions of the square of the hyper-angular momentum operator K with the eigenvalue $\mathrm{K}(\mathrm{K} + D - 2)$.

In addition to the irreducible representation label K, hyper-spherical harmonics carry additional labels specifying a concrete vector within that representation, usually describing the transformation properties of the hyper-spherical harmonic with respect to (w.r.t.) certain subgroups of the orthogonal group $SO(D)$.

In the context of three-particle wave functions, additional labels ought to be introduced in a way that respects physically important features, i.e., the remnant symmetries of the system in question. As most three-body potentials in physics are rotationally invariant, the hyper-spherical harmonics should have definite transformation properties under rotations, i.e., they should carry labels L and m (the "magnetic" quantum number) of the rotational subgroup $SO(3)$. Permutation symmetry is often a remnant symmetry, so once we construct the $SO(6)$ hyper-spherical harmonics we shall address the question of how the particle transpositions, i.e., the full $O(6)$ group, act upon them.

2.2. *The $SO(6)$ group structure*

The rotational group here appears as the diagonal $SO(3)$ subgroup of the six-dimensional rotations $SO(3)_{rot} = SO(3)_{diag} \subset SO(6)$, i.e., the rotations act equally on the first three coordinates ($\boldsymbol{\lambda}$) and the last three coordinates ($\boldsymbol{\rho}$) of the six-dimensional coordinate x_{μ}. As we shall shortly demonstrate, the space of 15 generators of the $SO(6)$ Lie algebra decomposes as $(3)_{rot} + (3) + (3) + (5) + (1)$ w.r.t. $SO(3)_{rot}$, so there is exactly one "additional" generator of $SO(6)$ that commutes with the rotations. This decomposition becomes manifest upon introduction of complex coordinates:

$$X_i^{\pm} = \lambda_i \pm i\rho_i, \quad i = 1, 2, 3. \tag{6}$$

One basis of the $so(6)$ algebra generating $SO(6)$ transformations of hyper-coordinates x_{μ} is given by the 15 operators $\{\mathrm{K}_{\mu\nu} \equiv i(x_{\mu}\partial_{\nu} - x_{\nu}\partial_{\mu}) | \mu, \nu = 1, \dots 6\}$. Of a greater physical significance is the following basis, written in terms of the new coordinates:

$$L_{ij} \equiv -i\left(X_i^{+}\frac{\partial}{\partial X_j^{+}} + X_i^{-}\frac{\partial}{\partial X_j^{-}} - X_j^{+}\frac{\partial}{\partial X_i^{+}} - X_j^{-}\frac{\partial}{\partial X_i^{-}}\right), \tag{7}$$

$$Q_{ij} \equiv \frac{1}{2}\left(X_i^{+}\frac{\partial}{\partial X_j^{+}} - X_i^{-}\frac{\partial}{\partial X_j^{-}} + X_j^{+}\frac{\partial}{\partial X_i^{+}} - X_j^{-}\frac{\partial}{\partial X_i^{-}}\right), \tag{8}$$

$$\Delta L_{ij} \equiv -i \left(X_i^+ \frac{\partial}{\partial X_j^-} + X_i^- \frac{\partial}{\partial X_j^+} - X_j^+ \frac{\partial}{\partial X_i^-} - X_j^- \frac{\partial}{\partial X_i^+} \right), \tag{9}$$

$$W_{ij} \equiv \left(X_i^+ \frac{\partial}{\partial X_j^-} - X_i^- \frac{\partial}{\partial X_j^+} - X_j^+ \frac{\partial}{\partial X_i^-} + X_j^- \frac{\partial}{\partial X_i^+} \right). \tag{10}$$

Of these, the L_{ij} (corresponding to orbital angular momentum), the ΔL_{ij} (equal to the difference $L_{ij}^{\lambda} - L_{ij}^{\rho}$) and the W_{ij} are antisymmetric tensors, thus having three components each. The ("quadrupole") tensor Q_{ij} is symmetric, thus decomposing into an irreducible second-rank tensor (with five components) and a scalar (with one component) under rotations. The trace of Q_{ij} is the scalar:

$$Q \equiv Q_{ii} = \sum_{i=1}^{3} X_i^+ \frac{\partial}{\partial X_i^+} - \sum_{i=1}^{3} X_i^- \frac{\partial}{\partial X_i^-} \tag{11}$$

which (obviously) commutes with the rotation generators and its eigenvalue is therefore the natural choice for the additional label of the hyper-spherical harmonics.

Apart from its mathematical significance, the operator Q is also physically important, as it generates the so-called "democracy" transformations [2,12] that are closely related to permutations of three particles. Moreover, as mentioned in Introduction, some interactions preserve this quantum number (e.g., due to $[Q, |\boldsymbol{\lambda} \times \boldsymbol{\rho}|] = 0$, such are all three-particle potentials that are functions of the area of the subtended triangle, $|\boldsymbol{\lambda} \times \boldsymbol{\rho}|$, which potentials are of importance in QCD). Note that the polynomials in X can only have integer eigenvalues of Q: these eigenvalues correspond to the difference $d_+ - d_-$, where d_+, d_- are polynomial degrees in X^+ and X^- variables, respectively.

The centralizer of the element Q in the $so(6)$ algebra, i.e., the subalgebra of the $so(6)$ algebra consisting of elements that commute with Q, is larger than the rotational subalgebra $so(3)_{rot}$: Q commutes not only with operators L_{ij}, but also with operators Q_{ij}. The three rotation generators L_{ij} together with five linearly independent components of the traceless $\tilde{Q}_{ij} \equiv Q_{ij} - \frac{1}{3}\delta_{ij} Q$ part of the symmetric (quadrupole) tensor Q_{ij} form eight generators of an $su(3)$ subalgebra of $so(6)$.

Labeling of the $SO(6)$ hyper-spherical harmonics with labels K, Q, L and m thus corresponds to the subgroup chain $U(1) \otimes SO(3)_{rot} \subset U(3) \subset SO(6)$ Note, however, that the $SU(3)$ subgroup does not introduce any new quantum numbers into the hyper-spherical harmonics labels (K, Q, L, m). For more details on $SU(3)$ aspect of the three particle h.s. harmonics, see Section 8.3.

Yet, these four quantum numbers are generally insufficient to uniquely specify an $SO(6)$ hyper-spherical harmonic: it is well known, see Ref. [43,44,47], that $SU(3)$ representations in general have nontrivial multiplicity w.r.t. decomposition into $SO(3)$ subgroup representations, and such a multiplicity also appears here. We shall deal with this multiplicity issue in Sect. 5 in a general way, and then again in Sects. 6 and 7, in more specific ways.

3. Core polynomials

Six-dimensional hyper-spherical harmonics with hyper-angular momentum K can be expressed as harmonic homogeneous polynomials of order K in variables x_μ (when restricted to the unit hyper-sphere). Our first goal is to construct such polynomials (which we shall call "core polynomials") that have pre-determined sharp values of quantum numbers Q, L and m. Once this goal has been achieved, we shall address the problem of how to project out parts of these polynomials that have also well defined values of K.

We begin by considering polynomials:

$$\mathcal{P}^{d_+d_-}_{L_+m_+L_-m_-}(X) = \left(\boldsymbol{X}^+ \cdot \boldsymbol{X}^+\right)^{\frac{d_+-L_+}{2}} \left(\boldsymbol{X}^- \cdot \boldsymbol{X}^-\right)^{\frac{d_--L_-}{2}} \tilde{\mathcal{Y}}^{L_+}_{3,m_+}(X^+)\tilde{\mathcal{Y}}^{L_-}_{3,m_-}(X^-), \tag{12}$$

where $d_\pm, L_\pm, m_\pm$ are integers such that $d_\pm - L_\pm$ are even and non-negative. Here $\tilde{\mathcal{Y}}^{L}_{3,m}(X)$ denotes an $SO(3)$ spherical harmonic function expressed as a homogeneous polynomial (of degree L) in the three coordinates X_i, cf. Ref. [58], i.e. $\tilde{\mathcal{Y}}^{L}_{3,m}(X)/|X|^L = \mathcal{Y}^{L}_{3,m}(X)$, where $\mathcal{Y}^{L}_{3,m}(X)$ is a standard $SO(3)$ spherical harmonic function.

These polynomials are of degree d_+ in variables X^+_i and d_- in variables X^-_i, meaning that they yield a sharp eigen-value $q = d_+ - d_-$ of the operator Q. The polynomials Eq. (12) are homogeneous functions of order $(d_+ + d_-)$ in coordinates x, but they are not harmonic, i.e., they don't have a vanishing Laplacian, which is, in this context, equivalent to stating that they are not eigenfunctions of $\mathrm{K}^2 \equiv \sum \frac{1}{2}\mathrm{K}_{\mu\nu}\mathrm{K}_{\mu\nu}$. This, in turn, implies that the polynomial $\mathcal{P}^{d_+d_-}_{L_+m_+L_-m_-}(X)$ contains components with various values of K, though none larger then $d_+ + d_-$, i.e. $\mathrm{K} \le d_+ + d_-$.

Furthermore, the maximum value of K appearing in the decomposition of $\mathcal{P}^{d_+d_-}_{L_+m_+L_-m_-}$ (restricted to the unit hyper-sphere) into hyper-spherical harmonics is exactly $d_+ + d_-$. The latter statement follows from the fact that $\mathcal{P}^{d_+d_-}_{L_+m_+L_-m_-}$ as a polynomial is not divisible by R^2.

The $SO(3)$ rotational properties of the polynomials Eq. (12) are determined by the coupling of angular momenta L_+ and L_-; therefore $\mathcal{P}^{d_+d_-}_{L_+m_+L_-m_-}(X)$ decomposes into $SO(3)$ spherical harmonics with L ranging from $|L_+ - L_-|$ to $L_+ + L_-$. By forming linear combinations of polynomials Eq. (12) we define the following homogeneous "core polynomials", that have good quantum numbers Q, L and m and maximal 6-dimensional hyper-angular momentum equal to $\overline{\mathrm{K}}$:

$$\mathcal{P}^{\overline{\mathrm{K}}Q}_{(L_+L_-)L,m}(X) \equiv \sum_{m_+,m_-} C^{L_+L_-L}_{m_+m_-m}\,\mathcal{P}^{\frac{\overline{\mathrm{K}}+Q}{2}\,\frac{\overline{\mathrm{K}}-Q}{2}}_{L_+m_+L_-m_-}(X), \tag{13}$$

where $C^{L_+L_-L}_{m_+m_-m}$ is an "ordinary" $SO(3)$ Clebsch–Gordan coefficient.

In addition, in the definition Eq. (13), the following is required to hold (the motivation for this will be given below):

$$L_+ + L_- = L \text{ or } L_+ + L_- = L + 1. \tag{14}$$

These polynomials exist and are nonzero only when all of the exponents appearing in Eqs. (12) and (13), i.e., $\frac{\overline{\mathrm{K}}+Q}{2}$ and $\frac{\overline{\mathrm{K}}-Q}{2}$, are non-negative integers and all the Clebsch–Gordan coefficients and 3-dim spherical harmonics are nonvanishing. In particular, this implies that: $\overline{\mathrm{K}} - Q$, $\frac{\overline{\mathrm{K}}+Q}{2} - L_+$ and $\frac{\overline{\mathrm{K}}-Q}{2} - L_-$ are all even and nonnegative, $m \le L$ and $|L_+ - L_-| \le L \le L_+ + L_-$ (due to (14), the last requirement is relevant only when $L_+ = 0$ or $L_- = 0$). From this, it also follows that $\overline{\mathrm{K}} \equiv L_+ + L_-$ (mod 2).

The core polynomials Eq. (13) have sharp values of quantum numbers Q, L and m irrespectively of the condition Eq. (14). The condition in Eq. (14) is only necessary to ensure that the decomposition of $\mathcal{P}^{\overline{\mathrm{K}}Q}_{(L_+L_-)L,m}(X)$ into $SO(6)$ hyper-spherical harmonics contains a component with the hyper-angular momentum $\mathrm{K} = \overline{\mathrm{K}}$. The argument goes as follows. A 3-dim $SO(3)$ spherical harmonic polynomial $\tilde{\mathcal{Y}}^{L}_{3,m}(X)$ can be related to a symmetric tensor $(\tilde{\mathcal{Y}}^{L}_{3,m})^{i_1i_2\ldots i_L}$ of order

L, that is trace-free in every pair of indices, as $\tilde{\mathcal{Y}}^{L}_{3,m}(X) = \sum_{i_1 i_2 \ldots i_L} (\tilde{\mathcal{Y}}^{L}_{3,m})^{i_1 i_2 \ldots i_L} X_{i_1} X_{i_2} \ldots X_{i_L}$ (again, restricted to the unit hyper-sphere). Coupling of two polynomials $\tilde{\mathcal{Y}}^{L_+}_{3,m_+}(X^+)$ and $\tilde{\mathcal{Y}}^{L_-}_{3,m_-}(X^-)$ to yield a polynomial transforming as a representation with $SO(3)$ angular momentum $L < L_+ + L_-$ involves contracting indices in the product of the corresponding tensors. On the other hand, simply contracting an index from $(\tilde{\mathcal{Y}}^{L_+}_{3,m_+})^{i_1 i_2 \ldots i_{L_+}}$ with an index from $(\tilde{\mathcal{Y}}^{L_-}_{3,m_-})^{j_1 j_2 \ldots j_{L_-}}$ corresponds to a polynomial that is proportional to $\boldsymbol{X}^+ \cdot \boldsymbol{X}^- = R^2$. This, in turn, means that the entire polynomial Eq. (13) would be proportional to R^2 and thus its maximal value K in the decomposition would be less than $\overline{\mathrm{K}}$, which contradicts our original assumption.

The only allowed contraction that would not effectively lower the $\overline{\mathrm{K}}$ value is a contraction with the Levi-Civita tensor, and such a contraction can be applied only once (two successive contractions of this sort are again equivalent to a direct contraction discussed above). Such a single contraction with the Levi-Civita tensor results in a polynomial that transforms w.r.t. spatial rotations as a vector from representation of angular momentum $L = L_+ + L_- - 1$. Therefore, two distinct types of core polynomials exist: those not contracted at all, with $L = L_+ + L_-$, and those once contracted with Levi-Civita tensor, with $L = L_+ + L_- - 1$. Due to $\overline{\mathrm{K}} \equiv L_+ + L_- \pmod 2$ the two possibilities are distinguished by $\overline{\mathrm{K}} - L \equiv 0 \pmod 2$ and $\overline{\mathrm{K}} - L \equiv 1 \pmod 2$, respectively, and in general:

$$L_+ + L_- = L + (\mathrm{K} - L \pmod 2)). \tag{15}$$

The core polynomials $\mathcal{P}^{\overline{\mathrm{K}}Q}_{(L_+L_-)L,m}(X)$, when restricted to a unit hyper-sphere, are thus equal to a linear combination of 6-dim hyper-spherical harmonics $\mathcal{Y}^{\mathrm{K}Qv}_{L,m}(X)$, with v accounting for possible multiplicity:

$$\frac{1}{R^{\overline{\mathrm{K}}}} \mathcal{P}^{\overline{\mathrm{K}}Q}_{(L_+L_-)L,m}(X) = \sum_{\mathrm{K}=0}^{\overline{\mathrm{K}}} \sum_{v} c_{\mathrm{K},v} \mathcal{Y}^{\mathrm{K}Qv}_{L,m}(X), \tag{16}$$

where at least one $c_{\overline{\mathrm{K}},v}$ is nonzero. Let V^{K} denote the space spanned by all spherical harmonics having hyper-angular momentum less or equal to K, and V^{K}_{QLm} denote a subspace of V^{K} with given values of Q,L and m. Then, the functions $\{\frac{1}{R^{\overline{\mathrm{K}}}} \mathcal{P}^{\overline{\mathrm{K}}Q}_{(L_+L_-)L,m}(X) | \overline{\mathrm{K}} = 0, 1, 2, \ldots, \mathrm{K}_{max}\}$, though not orthonormal, span the subspace $V^{\mathrm{K}_{max}}_{QLm}$. (It can be checked that the number of all core polynomials with given $\overline{\mathrm{K}}$ equals $\frac{(\overline{\mathrm{K}}+3)!(\overline{\mathrm{K}}+2)}{12\overline{\mathrm{K}}!}$, equal to the number of spherical harmonics with $\mathrm{K} = \overline{\mathrm{K}}$, Ref. [5].) Conversely, the 6-dim hyper-spherical harmonics can be obtained from the core polynomials by a procedure of orthogonalization and normalization, such as the Gram–Schmidt one.

4. Harmonic polynomials

As mentioned earlier, the core polynomials Eq. (13) are not harmonic, as they contain components with $\mathrm{K} < \overline{\mathrm{K}}$ belonging to some $V^{\overline{\mathrm{K}}-1}$. We introduce a shorthand notation $\mathcal{P}_a(X)$ for the polynomials $\mathcal{P}^{\mathrm{K}Q}_{(L_+L_-)L,m}(X)$ with fixed given values of Q,L,m, with $\mathrm{K} < \overline{\mathrm{K}}$, and L_+, L_- taking all of the allowed values. That is: $V^{\overline{\mathrm{K}}-1}_{QLm} = span\{\mathcal{P}_a(\frac{X}{R}), a = 1, 2, \ldots \dim(V^{\overline{\mathrm{K}}-1}_{QLm})\}$.

The harmonic polynomial $\mathcal{P}H^{\overline{\mathrm{K}}Q}_{(L_+L_-)L,m}(X)$ can be obtained from the core polynomial $\mathcal{P}^{\overline{\mathrm{K}}Q}_{(L_+L_-)L,m}(X)$ by removing the components that belong to $V^{\overline{\mathrm{K}}-1}_{QLm}$:

$$\mathcal{P}H^{\overline{\mathrm{K}}Q}_{(L_+L_-)L,m}(X) = \mathcal{P}^{\overline{\mathrm{K}}Q}_{(L_+L_-)L,m}(X) - \sum_a c_a R^{\overline{\mathrm{K}}-\mathrm{K}_a}\mathcal{P}_a(X), \tag{17}$$

with c_a being the coefficients to be deduced from orthogonality conditions:

$$\left\langle \mathcal{P}_a \middle| \mathcal{P}H^{\overline{\mathrm{K}}Q}_{(L_+L_-)L,m} \right\rangle = 0. \tag{18}$$

These conditions readily lead to:

$$c_a = \sum_b (M^{-1})_{ab} A_b, \quad \text{with:} \quad M_{ab} \equiv \left\langle \mathcal{P}_a \middle| \mathcal{P}_b \right\rangle, \; A_a \equiv \left\langle \mathcal{P}_a \middle| \mathcal{P}^{\overline{\mathrm{K}}Q}_{(L_+L_-)L,m} \right\rangle. \tag{19}$$

The above scalar product is naturally given by the integration over a unit 6-dimensional hypersphere:

$$\left\langle \mathcal{P}_a \middle| \mathcal{P}_b \right\rangle \equiv \int_\Omega \mathcal{P}^*_a(\frac{X}{R})\mathcal{P}_b(\frac{X}{R}) d\Omega. \tag{20}$$

This is, in turn, can be calculated by using the following formula (cf. Eq. (A.4) in Appendix A) for integration over the sphere of monomials in x_μ:

$$\int_\Omega \frac{1}{R^6} x_1^{m_1} x_2^{m_2} \cdots x_6^{m_6} d\Omega = 2\frac{\prod_{\mu=1}^6 \frac{1+(-1)^{m_\mu}}{2}\Gamma(\frac{m_\mu+1}{2})}{\Gamma(3+\sum_\mu m_\mu)}, \tag{21}$$

where $\Gamma(n)$ is the usual gamma function.

It is now convenient to introduce a "spherical" version of the $X^\pm$ coordinates:

$$X_\pm^{(\pm)} \equiv X_1^{(\pm)} \pm X_2^{(\pm)}, \quad X_0^{(\pm)} \equiv X_3^{(\pm)}, \tag{22}$$

as they are particularly suitable for explicit writing of the core polynomials $\mathcal{P}^{\overline{\mathrm{K}}Q}_{(L_+L_-)L,m}(X)$ for $m = L$:

$$\mathcal{P}^{\overline{\mathrm{K}}Q}_{(L_+L_-)L_++L_-,L_++L_-}(X) = |X^+|^{\frac{\overline{\mathrm{K}}+Q}{2}-L_+} |X^-|^{\frac{\overline{\mathrm{K}}-Q}{2}-L_-} (X_+^+)^{L_+}(X_+^-)^{L_-}, \tag{23}$$

$$\begin{aligned} &\mathcal{P}^{\overline{\mathrm{K}}Q}_{(L_+L_-)L_++L_--1,L_++L_--1}(X) \\ &\quad = \sqrt{\frac{2L_+L_-}{L_++L_-}} |X^+|^{\frac{\overline{\mathrm{K}}+Q}{2}-L_+} |X^-|^{\frac{\overline{\mathrm{K}}-Q}{2}-L_-} \Big((X_+^+)^{L_+}(X_+^-)^{L_--1}(X_0^-) \\ &\qquad - (X_+^+)^{L_+-1}(X_0^+)(X_+^-)^{L_-} \Big), \quad L_+, L_- \neq 0, \end{aligned} \tag{24}$$

where $|X^\pm|^2 = \boldsymbol{X}^\pm \cdot \boldsymbol{X}^\pm = X_+^\pm X_-^\pm + (X_0^\pm)^2$. Note that the formula Eq. (23) is only relevant when $\overline{\mathrm{K}} \equiv L \pmod 2$, and the formula Eq. (24) should be used otherwise.

Expressions for the scalar products of core polynomials with forms of Eq. (23) and Eq. (24) turn out to be relatively simple, due to the following identity (derivable from Eq. (21), or Eq. (A.4) in Appendix A):

$$\int_\Omega |X^+|^{k^+}|X^-|^{k^-}(X_+^+)^{k_+^+}(X_-^+)^{k_-^+}(X_0^+)^{k_0^+}(X_+^-)^{k_+^-}(X_-^-)^{k_-^-}(X_0^-)^{k_0^-}\,d\Omega =$$

$$\frac{2\pi^3}{(2+\frac{\sum k}{2})!}\sum_{l=0}^{\frac{k^+}{2}}\binom{\frac{k^+}{2}}{l}\binom{\frac{k^-}{2}}{l+k_+^+-k_-^-}2^{2l+k_+^++k_-^+}(l+k_+^+)!(l+k_-^+)!(k^+-2l+k_0^+)!\cdot \tag{25}$$
$$\delta(\sum k^+,\sum k^-)\cdot\delta(k_+^++k_+^-,k_-^++k_-^-),$$

where $\sum k^\pm = k^\pm + k_+^\pm + k_-^\pm + k_0^\pm$ and $\sum k = \sum k^+ + \sum k^-$, while $\delta(a,b)=\delta_{ab}$ is the Kronecker delta symbol. The formula allows us not only to directly calculate the scalar products of the core polynomials, but also any spherical integral of the product of arbitrarily many core polynomials. The result is particularly simple if all polynomials in the product have the property that $m = L$.

Furthermore, due to the $SO(3)$ rotational symmetry reasons (i.e., due to the Wigner–Eckart theorem) the scalar products of two core polynomials must be independent of the magnetic quantum number m, so that by combining Eqs. (23), (24) and (25) we may write the result in full generality:

$$\left\langle \mathcal{P}^{\overline{K}'Q'}_{(L'_+L'_-)L',m'}\middle|\mathcal{P}^{\overline{K}Q}_{(L_+L_-)L,m}\right\rangle = \delta_{mm'}\left\langle \mathcal{P}^{\overline{K}'Q'}_{(L'_+L'_-)L',L'}\middle|\mathcal{P}^{\overline{K}Q}_{(L_+L_-)L,L}\right\rangle$$
$$= \begin{cases} \frac{2\pi^3\delta_{QQ'}\delta_{JJ'}\delta_{mm'}}{(2+\frac{\overline{K}+\overline{K}'}{2})!}\sum_{l=0}^{\frac{k^+}{2}}2^{2l+L_++L'_-}\binom{\frac{k^+}{2}}{l}\times \\ \binom{\frac{k^+}{2}+L_+-L'_+}{\frac{k^+}{2}-l}(l+L_+)!(l+L'_-)!(k^+-2l)! \\ \quad \text{if } \overline{K}-L\equiv\overline{K}'-L'\equiv 0 \pmod 2 \\ \frac{2\pi^3\delta_{QQ'}\delta_{JJ'}\delta_{mm'}}{(2+\frac{\overline{K}+\overline{K}'}{2})!}\frac{2\sqrt{L_+L_-L'_+L'_-}}{1+L}\sum_{l=0}^{\frac{k^+}{2}}2^{2l+L_++L'_-}\binom{\frac{k^+}{2}}{l}\times \\ \Big[\binom{\frac{k^+}{2}+L_+-L'_+}{\frac{k^+}{2}-l}(l+L_+-1)!(l+L'_--1)!(k^+-2l+1)!\frac{l+L+1}{2} \\ -\binom{\frac{k^+}{2}+L_+-L'_+}{\frac{k^+}{2}-l-1}(l+L_+)!(l+L'_-)!(k^+-2l)!\Big] \\ \quad \text{if } \overline{K}-L\equiv\overline{K}'-L'\equiv 1 \pmod 2 \\ 0 \quad \text{if } \overline{K}-L\not\equiv\overline{K}'-L' \pmod 2, \end{cases} \tag{26}$$

where $k^+ = \frac{\overline{K}+\overline{K}'}{2} + L_+ - L'_-$ and it is implied that $\binom{m}{n}\equiv 0$ whenever $n<0$ or $n>m$. Scalar product of a polynomial of form Eq. (23) with a polynomial of form Eq. (24) always yields zero: this case corresponds to $\overline{K}-L\not\equiv\overline{K}'-L' \pmod 2$ which, combined with requirement that $L=L'$ leads to $\overline{K}+\overline{K}'\equiv 1 \pmod 2$. And, as the integration of any polynomial of odd order over the unit hyper-sphere yields zero, we conclude that the scalar product (26) when $\overline{K}-L\not\equiv\overline{K}'-L' \pmod 2$ is also zero.

Relations Eqs. (17), (19) and (26) combined give us expressions for $\mathcal{P}_{H}{}^{KQ}_{(L_+L_-)L,m}(X)$ – the homogeneous harmonic polynomials of order K, that are eigenfunctions of the 6-dim hyper-angular momentum and that have well defined values of quantum numbers Q, L and m. In addition to these 4 quantum numbers that are eigenvalues of the corresponding Casimir or Cartan subalgebra operators, harmonic polynomials $\mathcal{P}_{H}{}^{KQ}_{(L_+L_-)L,m}(X)$ are also labeled by two numbers L_+ and L_-, only one of which is independent due to the relation Eq. (15). Existence of this additional freedom demonstrates the nontrivial multiplicity of the totally symmetric tensorial representations of $SO(6)$ w.r.t. decomposition into $U(1) \otimes SO(3)_{rot}$ subrepresentations. As the L_+ is non-negative; for any given values of K, Q and L the value of L_+ can only change in steps of 2; it cannot exceed $L+1$; and it cannot take the values of 0 and $L+1$ within the same degenerate subspace, therefore the maximal multiplicity degree that can occur for a given L is $[L/2]+1$, where $[n]$ is the integer part of n. The same holds for L_-. That implies that a non-trivial multiplicity can occur only for harmonics with $L \geq 2$.

Naturally, either L_+ or L_- can be taken to label this multiplicity, though more convenient choices, both mathematically and physically, will be discussed below. A basic option for the multiplicity label is to introduce the difference

$$\Delta l \equiv L_+ - L_-, \tag{27}$$

(not to be confused with ΔL_{ij} in Eq. (9)) as the multiplicity label, which is essentially the same as choosing L_+, or L_-, for that purpose, yet Δl is more convenient, as we shall explain shortly. In this sense, the harmonic polynomials obtained in the previous section would now be labeled as $\mathcal{P}_{H}{}^{\mathrm{K}Q}_{(\Delta l)L,m}(X)$.

5. Multiplicity of degenerate harmonic polynomials

In general, two harmonic polynomials $\mathcal{P}_{H}{}^{\mathrm{K}Q}_{(\Delta l)L,m}(X)$ and $\mathcal{P}_{H}{}^{\mathrm{K}Q}_{(\Delta l')L,m}(X)$, that differ only in the multiplicity label, are not orthogonal. An orthonormal basis has to be introduced in the degenerate subspace of harmonic polynomials with given K, Q, L and m, and this can be done in (infinitely) many ways. For example, the Gram–Schmidt ortho-normalization procedure can be carried out, choosing as the first vector the normalized (in the sense of Eq. (20)) harmonic polynomial with the highest Δl in this subspace, and then taking the polynomial with the next-to-highest value of Δl, subtracting from it a component proportional to the first vector and normalizing it, and so on.

In this process, care should be taken to preserve the symmetry between X_+ and X_- coordinates, *viz.* of complex conjugation, that had been present thus far – we shall demonstrate in Section 7 that this symmetry is directly related to the permutational symmetry S_2. In practice this means that if we begin the orthonormalization procedure with the highest Δl value in the subspaces with $Q > 0$, then we must start with the lowest Δl value in subspaces with $Q < 0$ (this is due to $\Delta l \to -\Delta l$ when $X_+ \leftrightarrow X_-$). In the limiting case of $Q = 0$, the optimal strategy is firstly to introduce symmetric and antisymmetric combinations of harmonic polynomials with opposite values of Δl:

$$\mathcal{P}_{H}{}^{\mathrm{K},0}_{(|\Delta l|,\pm)L,m}(X) \equiv \mathcal{P}_{H}{}^{\mathrm{K},0}_{(\Delta l)L,m}(X) \pm (-1)^{\mathrm{K}-L}\mathcal{P}_{H}{}^{\mathrm{K},0}_{(-\Delta l)L,m}(X). \tag{28}$$

Of these polynomials, those labeled with the plus sign will turn out to be symmetric w.r.t. transposition of particles 1 and 2, whereas those labeled with the minus sign will be asymmetric – and the factor of $(-1)^{\mathrm{K}-L}$ will be necessary to establish this property, see Eq. (65) in Section 7. In turn, this implies that for $Q = 0$ it is sufficient to perform Gram–Schmidt procedure separately on these two subsets – since the polynomials from different subsets are mutually orthogonal, and that no ortho-normalization procedure is necessary when multiplicity degree equals (only) two.

Note that the harmonic polynomials that are nondegenerate w.r.t. numbers K, Q, L and m should also be normalized, as they are already orthogonal to all other harmonic polynomials Eq. (17).

The set of polynomials obtained by such an ortho-normalization procedure, when restricted to a unit hyper-sphere, constitutes a system of $SO(6)$ hyper-spherical harmonics that we will denote as $\mathcal{Y}^{\mathrm{K}Q\Delta l}_{L,m}(X)$, and is labeled by four quantum numbers: K, Q, L, and m, together with an additional multiplicity label Δl. Advantages of this method for multiplicity labeling are the following: i) all of the harmonics can be expressed in analytical form; ii) multiplicity lifting procedure is computationally efficient, since it relies only on Gram–Schmidt ortho-normalization; iii) the label Δl takes only integer values.

Nevertheless, from the physical viewpoint, it is often convenient to choose a basis that diagonalizes some physically significant operator in this degenerate subspace – e.g. the potential energy. Any operator $\mathcal{V}$ that has no degenerate eigenvalues when reduced to this subspace, can be used for this purpose. Moreover, there are certain operators commonly used for multiplicity lifting in the literature (in the context of $SO(3) \subset SU(3)$ multiplicity) and sticking to one of these choices is good from a compatibility aspect (some general results, such as the values of Clebsch–Gordan coefficients, can then be directly used here – cf. Section 8.3).

To address this approach in full generality, we firstly introduce an abbreviated single-letter notation for labeling harmonic polynomials spanning a given degenerate subspace $V^{\mathrm{K},Q}_{L,m}$: $\{\mathcal{P}_{H\,a} | a = 1, 2, \dots \dim V^{\mathrm{K},Q}_{L,m}\}$, and let:

$$\mathcal{V}_{ab} \equiv \langle \mathcal{P}_{H\,a} | \, \mathcal{V} \, | \mathcal{P}_{H\,b} \rangle , \quad M_{ab} \equiv \left\langle \mathcal{P}_{H\,a} \middle| \mathcal{P}_{H\,b} \right\rangle . \tag{29}$$

The goal is to find an orthonormal basis of hyper-spherical harmonic polynomials $\tilde{\mathcal{Y}}_a(X) = \sum_b c_{ab} \mathcal{P}_{H\,b}$ that diagonalizes $\mathcal{V}$:

$$\left\langle \tilde{\mathcal{Y}}_a \middle| \tilde{\mathcal{Y}}_b \right\rangle = \delta_{ab}, \tag{30}$$

$$\left\langle \tilde{\mathcal{Y}}_a \middle| \mathcal{V} \middle| \tilde{\mathcal{Y}}_b \right\rangle = \delta_{ab} v_a . \tag{31}$$

From Eq. (30) it follows:

$$c^{\dagger} M c = I, \tag{32}$$

where I is a unit matrix and $\dagger$ denotes conjugate transpose matrix. As the matrix M is hermitian, it follows that matrix $(\sqrt{M} c)$ is a unitary matrix, that we shall denote as U:

$$U \equiv \sqrt{M} c, \quad U^{\dagger} U = I. \tag{33}$$

From the condition Eq. (31) we know that the matrix $c^{\dagger} \mathcal{V} c = U^{\dagger} (M^{-\frac{1}{2}} \mathcal{V} M^{-\frac{1}{2}}) U$ has to be diagonal, i.e., a unitary matrix U can be found that diagonalizes the hermitian matrix $(M^{-\frac{1}{2}} \mathcal{V} M^{-\frac{1}{2}})$:

$$U^{-1} (M^{-\frac{1}{2}} \mathcal{V} M^{-\frac{1}{2}}) U = diag(v_1, v_2, \dots v_{dim}). \tag{34}$$

Therefore, resolving the multiplicity problem reduces to finding a unitary matrix U that satisfies Eq. (34); thereafter the hyper-spherical harmonic polynomials, labeled by K, Q, L, m and v_a, are calculated as:

$$\tilde{\mathcal{Y}}_a(X) = \sum_b (M^{-\frac{1}{2}} U)_{ab} \mathcal{P}_{H\,b}. \tag{35}$$

Note that the same procedure, when applied to a non-degenerate one-dimensional subspace $V^{\mathrm{K},Q}_{L,m}$ simply normalizes the corresponding harmonic polynomial.

Polynomials $\tilde{\mathcal{Y}}^{\mathrm{K}Qv}_{L,m}(X)$ obtained by this procedure, when reduced to the unit hyper-sphere – that is, when divided by R^{K}, give a set of orthonormal $SO(6)$ hyper-spherical harmonics $\mathcal{Y}^{\mathrm{K}Qv}_{L,m}(X)$, labeled by the 6-dimensional hyper-angular momentum K, the eigenvalue of the Q operator, the total (orbital) angular momentum of the system L, the projection m of the total (orbital) angular momentum and the eigenvalue of the reduced $\mathcal{V}$ operator:

$$\mathcal{Y}^{\mathrm{K}Qv}_{L,m}(X) = \tilde{\mathcal{Y}}^{\mathrm{K}Qv}_{L,m}(X)/R^{\mathrm{K}}. \tag{36}$$

In the context of the $SO(3) \subset SU(3)$ multiplicity, the operator:

$$\mathcal{V}_{JQJ} \equiv \sum_{ij} L_i Q_{ij} L_j \tag{37}$$

(where $L_i = \frac{1}{2}\varepsilon_{ijk} L_{jk}$ and Q_{ij} is given by Eq. (8)) has often been used in the literature, see Refs. [43–47], to label the multiplicity. This operator has the desirable property that it commutes both with the angular momentum L_i, and with the “democracy rotation” generator Q:

$$\left[\mathcal{V}_{JQJ}, L_i\right] = 0; \quad \left[\mathcal{V}_{JQJ}, Q\right] = 0.$$

Of course, this is not the only operator that commutes with L_i, and with Q, so there is a certain degree of freedom left in this choice that can, perhaps, be used so as to optimize the h.s. harmonics to a particular application, see e.g. Ref. [43–47]. For example, the area of the triangle “operator” $|\boldsymbol{\lambda} \times \boldsymbol{\rho}|$ commutes with L_i, and Q: $[Q, |\boldsymbol{\lambda} \times \boldsymbol{\rho}|] = 0$, $[L_i, |\boldsymbol{\lambda} \times \boldsymbol{\rho}|] = 0$, and can also be used for this purpose. We shall show below that these two operators have “opposite” transformation properties under certain permutations (transpositions) and, in some sense, represent the only two possible classes of such operators.

In Appendix C we list the hyper-spherical harmonics labeled by the operator $\mathcal{V}_{JQJ}$, up to $\mathrm{K} \leq 6$, and compare them with the few explicit harmonics that already exist in the literature, Ref. [5]. There is only a handful of harmonics with non-trivial multiplicity in this range of K-values, so they can be readily calculated and examined with the alternative degeneracy-lifting (“area”) operator. The result is that the two multiplicity-lifting operators are for all practical purposes equivalent. Other examples of degeneracy-lifting operators have been discussed in Refs. [45–47], irrespectively of their geometrical meaning in the three-body problem.

We note that no solution to Eq. (34) is unique and that this arbitrariness directly corresponds to the freedom of choosing multiplicative phase factors for the obtained basis functions. This arbitrariness should be fixed by adopting a definite phase convention: e.g. in the explicit calculations in the remainder of this paper, we shall adjust the overall sign of hyper-spherical harmonic in Eq. (35) so that the projection of each vector $\tilde{\mathcal{Y}}_a(X)$ on the sum $\sum_b \mathcal{P}_{Hb}$ is non-negative, i.e.,

$$\sum_b \left\langle \mathcal{P}_{Hb} \middle| \tilde{\mathcal{Y}}_a(X) \right\rangle \geq 0. \tag{38}$$

It should be clear that the process of using an operator to lift degeneracy amounts to the diagonalization of the chosen operator in a finite-dimensional space. That, in turn, boils down to solving an algebraic eigenvalue equation, that can be solved in closed form (“surds”) only so long as the order of the equation is less than five (due to Galois’ theory) and that solutions to higher-order degeneracy-lifting problems must necessarily be numerical.

The choice of optimal degeneracy-lifting operator(s) is a problem in SU(3) group theory that has been essentially solved in Ref. [47], where it was noted that “it is not possible to choose a complete set of operators whose eigenvalues are all integers and whose eigenfunctions can be constructed analytically. The price we have to pay for having orthonormal basis functions and a physically meaningful operator, providing the missing label in the $SU(3) \supset O(3)$ scheme, and thus providing selection rules, etc., is that many of the computations involved will be numerical by necessity." On the other hand, if we give up insisting on a "physically meaningful operator" to label the states, we can account for the multiplicity by the value of Δl, Eq. (27), and retain both the algebraic form of hyper-spherical harmonics and an integer-valued degeneracy-lifting label.

Thus, with the concept of degeneracy-lifting clarified, we see that the entire construction of three particle hyper-spherical harmonics can be automatized/programmed using (several) commercially available computer software codes for symbolic computation, with the understanding that, if using operator approach for multiplicity lifting, then for sufficiently high value(s) of harmonic labels some of the results will necessarily be numerical. More specifically, maximal $SO(3) \subset SU(3)$ multiplicity that occurs for a given K grows as $[\mathrm{K}/4] + 1$ and for a given L grows as $[L/2] + 1$. Effectively, this means that it is unavoidable to resort to numerical solutions only when $\mathrm{K} \geq 16$ and $L \geq 8$, and even then not for all harmonics (to give some impression of these numbers, we note that there are 27132 hyperspherical harmonicas with $\mathrm{K} < 16$).

Now that we have established a mathematical procedure for calculating $SO(6)$ h.s. harmonics, in the next section we will treat in detail a few examples of this procedure. In particular, we shall illustrate the application of two different multiplicity lifting operators, so as to demonstrate the concept and to clarify the limitations on computability of arbitrary h.s. harmonics, imposed by the degeneracy problem, Ref. [46].

6. Examples of harmonic construction

In order to illustrate the procedure for obtaining the hyper-spherical harmonics described in this paper, we shall explicitly calculate several h.s. harmonics with two different degeneracy-lifting operators.

For the purpose of this demonstration we look for hyper-spherical harmonics with quantum numbers $\mathrm{K} = 4$, $Q = 0$ and $L = 2$, as this is the simplest case with nontrivial multiplicity. As for the quantum number m, we will first demonstrate how to obtain the harmonic function that corresponds to maximal value $m = L$, in this particular case $m = 2$. After that we will discuss how to obtain harmonics with arbitrary values of m, $-L \leq m \leq L$.

The first step is to calculate necessary core polynomials Eq. (13). There are two core polynomials $\mathcal{P}^{\overline{\mathrm{K}}Q}_{(L_+L_-)L,m}(X)$ with quantum numbers $Q = 0$, $L = 2$ and $m = 2$, that have $\overline{\mathrm{K}} = 4$:

$$\mathcal{P}^{\overline{4},0}_{(2,0)2,2}(X) = \left(X^+_+\right)^2 \left|X^-\right|^2 \qquad \text{and} \qquad \mathcal{P}^{\overline{4},0}_{(0,2)2,2}(X) = \left(X^-_+\right)^2 \left|X^+\right|^2. \tag{39}$$

It can be easily checked that these are eigenfunctions of the operators Q, $L^2 \equiv \frac{1}{2}\sum L_{ij}L_{ij}$ and $L_3 \equiv L_{12}$. These are not eigenfunctions of the square of hyper-angular momentum, however, due to the appearance of additional terms on the right-hand-side of the Eqs. (40), (41):

$$\mathrm{K}^2\mathcal{P}^{\overline{4},0}_{(2,0)2,2}(X) = 4(4+6-2)\mathcal{P}^{\overline{4},0}_{(2,0)2,2}(X) - 16R^2X^+_+X^-_+, \tag{40}$$

$$\mathrm{K}^2\mathcal{P}^{\overline{4},0}_{(0,2)2,2}(X) = 4(4+6-2)\mathcal{P}^{\overline{4},0}_{(0,2)2,2}(X) - 16R^2X^+_+X^-_+. \tag{41}$$

The additional terms are identical, and proportional to the core polynomial $\mathcal{P}^{\overline{2},0}_{(1,1)2,2}(X)$:

$$\mathcal{P}^{\overline{4},0}_{(1,1)2,2}(X) = X^+_+X^-_+, \tag{42}$$

and this explicitly demonstrates the necessity of the procedure, described in section 4, to obtain the truly harmonic polynomials. The polynomial $\mathcal{P}^{\overline{2},0}_{(1,1)2,2}(X)$ is also the only polynomial with quantum numbers $Q=0$, $L=2$ and $m=2$ that has $\overline{\mathrm{K}}<4$. In the notation of section 4 this means that in this case the space $V^{\overline{\mathrm{K}}-1}_{QLm}$ is one dimensional, and therefore that the calculation (highly) simplifies as the index a takes only one value.

In order to find the harmonic polynomial $\mathcal{P}_H{}^{\overline{4},0}_{(2,0)2,2}(X)$ from the core polynomial $\mathcal{P}^{\overline{4},0}_{(2,0)2,2}(X)$, we follow the projection procedure, Eq. (17) and use Eq. (26), to readily find:

$$M_{11}=\frac{\pi^3}{3},\quad (M^{-1})_{11}=\frac{3}{\pi^3},\quad A_1=\frac{4\pi^3}{15},\quad c_1=\frac{4}{5}, \tag{43}$$

leading to

$$\mathcal{P}_H{}^{\overline{4},0}_{(2,0)2,2}(X)=\mathcal{P}^{\overline{4},0}_{(2,0)2,2}(X)-\frac{4}{5}R^2\mathcal{P}^{\overline{2},0}_{(1,1)2,2}(X)=\left(X^+_+\right)^2\left|X^-\right|^2-\frac{4}{5}R^2X^+_+X^-_+. \tag{44}$$

In an identical manner one obtains:

$$\mathcal{P}_H{}^{\overline{4},0}_{(0,2)2,2}(X)=\mathcal{P}^{\overline{4},0}_{(0,2)2,2}(X)-\frac{4}{5}R^2\mathcal{P}^{\overline{2},0}_{(1,1)2,2}(X)=\left(X^-_+\right)^2\left|X^+\right|^2-\frac{4}{5}R^2X^+_+X^-_+. \tag{45}$$

Now it can be verified that these polynomials are indeed harmonic, in the sense that they satisfy the Laplace equation:

$$\nabla^2\mathcal{P}_H{}^{\overline{4},0}_{(2,0)2,2}(X)=\nabla^2\mathcal{P}_H{}^{\overline{4},0}_{(0,2)2,2}(X)=0 \tag{46}$$

and that they are eigen-functions of the operator K^2:

$$\begin{aligned}\mathrm{K}^2\mathcal{P}_H{}^{\overline{4},0}_{(2,0)2,2}(X)&=4(4+6-2)\mathcal{P}_H{}^{\overline{4},0}_{(2,0)2,2}(X),\\ \mathrm{K}^2\mathcal{P}_H{}^{\overline{4},0}_{(0,2)2,2}(X)&=4(4+6-2)\mathcal{P}_H{}^{\overline{4},0}_{(0,2)2,2}(X).\end{aligned} \tag{47}$$

Being harmonic and having good quantum numbers K, Q, L and m, these polynomials indeed represent the sought-after hyper-spherical harmonics, i.e. functions $\mathcal{P}_H{}^{\overline{4},0}_{(2,0)2,2}(X)/R^4$ and $\mathcal{P}_H{}^{\overline{4},0}_{(0,2)2,2}(X)/R^4$, reduced to the $R=1$ unit sphere. The fact that there are two different polynomials with the same set of numbers K, Q, L, m means that there is nontrivial multiplicity present.

These functions have certain shortcomings, however: first, these two functions are not mutually orthogonal:

$$\left\langle\mathcal{P}_H{}^{\overline{4},0}_{(2,0)2,2}(X)\middle|\mathcal{P}_H{}^{\overline{4},0}_{(0,2)2,2}(X)\right\rangle=-\frac{8\pi^3}{225}. \tag{48}$$

Secondly, these states are not normalized, as yet.

In order to obtain an ortho-normal basis of harmonic functions and to have the multiplicity labeled in some more precise way, we can follow one of the procedures laid out in section 5.

In the following we shall demonstrate three ways to label the multiplicity: i) by the difference Δl in Sect. 6.1; ii) by the transposition-odd operator $\mathcal{V}_{JQJ}$, Eq. (37) in Sect. 6.2; and iii) by using the transposition-even area operator $|\boldsymbol{\rho}\times\boldsymbol{\lambda}|$ in Sect. 6.3. In Section 7 we discuss the particle permutation properties of the harmonics and show that the symmetric (even) and antisymmetric (odd) multiplicity-lifting operators are the only two relevant classes.

6.1. Δl as the multiplicity label

As this is a $Q = 0$ subspace, we will define symmetric and antisymmetric combinations Eq. (28) of the polynomials $\mathcal{P}_{H}{}^{\overline{4},0}_{(2,0)2,2}(X)$ and $\mathcal{P}_{H}{}^{\overline{4},0}_{(0,2)2,2}(X)$, that, after normalization, take form:

$$\mathcal{P}_{H}{}^{4,0}_{(|\Delta l|=2,+)2,2}(X) = \frac{3\left(-8R^2 X_+^+ X_+^- + 5\left(X_+^-\right)^2\left|X^+\right|^2 + 5\left(X_+^+\right)^2\left|X^-\right|^2\right)}{2\sqrt{7}\pi^{3/2}R^4} \tag{49}$$

$$\mathcal{P}_{H}{}^{4,0}_{(|\Delta l|=2,-)2,2}(X) = \frac{\sqrt{15}\left(\left(X_+^-\right)^2\left|X^+\right|^2 - \left(X_+^+\right)^2\left|X^-\right|^2\right)}{2\pi^{3/2}R^4} \tag{50}$$

By virtue of different transformation properties w.r.t. transposition of first two particles, these combinations are now already mutually orthogonal, even without any Gram–Schmidt procedure (however, had the multiplicity degree been larger than 2 such a procedure would have been necessary). Once we have found and normalized these combinations, we can return to the labeling $\mathcal{Y}_{L,m}^{KQ\Delta l}(X)$, where Δl takes both positive and negative values:

$$\begin{aligned}
&\mathcal{Y}_{2,2}^{4,0,\Delta l=2}(X)\\
&\quad\equiv \frac{1}{\sqrt{2}}\left(\mathcal{P}_{H}{}^{4,0}_{(|\Delta l|=2,+)2,2}(X) + \mathcal{P}_{H}{}^{4,0}_{(|\Delta l|=2,-)2,2}(X)\right)\\
&\quad= \frac{-12\sqrt{14}R^2 X_+^+ X_+^- + \sqrt{105\left(11+\sqrt{105}\right)}\left(X_+^-\right)^2\left|X^+\right|^2 + \sqrt{105\left(11-\sqrt{105}\right)}\left(X_+^+\right)^2\left|X^-\right|^2}{14\pi^{3/2}R^4},
\end{aligned} \tag{51}$$

$$\begin{aligned}
&\mathcal{Y}_{2,2}^{4,0,\Delta l=-2}(X)\\
&\quad\equiv \frac{1}{\sqrt{2}}\left(\mathcal{P}_{H}{}^{4,0}_{(|\Delta l|=2,+)2,2}(X) - \mathcal{P}_{H}{}^{4,0}_{(|\Delta l|=2,-)2,2}(X)\right)\\
&\quad= \frac{-12\sqrt{14}R^2 X_+^+ X_+^- + \sqrt{105\left(11-\sqrt{105}\right)}\left(X_+^-\right)^2\left|X^+\right|^2 + \sqrt{105\left(11+\sqrt{105}\right)}\left(X_+^+\right)^2\left|X^-\right|^2}{14\pi^{3/2}R^4}.
\end{aligned} \tag{52}$$

By using relation Eq. (25) we can also explicitly verify orthonormality of the obtained hyperspherical harmonics:

$$\begin{aligned}
&\int_\Omega \mathcal{Y}_{2,2}^{*\,4,0,\Delta l=2}(X)\,\mathcal{Y}_{2,2}^{4,0,\Delta l=-2}(X)\,d\Omega = 0,\\
&\int_\Omega |\mathcal{Y}_{2,2}^{4,0,\Delta l=2}(X)|^2 d\Omega = \int_\Omega |\mathcal{Y}_{2,2}^{4,0,\Delta l=-2}(X)|^2 d\Omega = 1.
\end{aligned} \tag{53}$$

6.2. Harmonics with antisymmetric degeneracy lifting operator

Next we demonstrate the use the operator $\mathcal{V}_{JQJ}$ in Eq. (37) to label the multiplicity. Combining Eqs. (7)–(8) and Eq. (25) we obtain the following values for matrices $\mathcal{V}$ and M, defined by Eqs. (29):

$$\mathcal{V} = \frac{14\pi^3}{15}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad M = \frac{2\pi^3}{225}\begin{pmatrix} 11 & -4 \\ -4 & 11 \end{pmatrix} \tag{54}$$

leading to

$$\begin{aligned} M^{-\frac{1}{2}} &= \frac{1}{2\pi^{3/2}}\sqrt{\frac{15}{7}}\begin{pmatrix} \sqrt{\left(11+\sqrt{105}\right)} & \sqrt{\left(11-\sqrt{105}\right)} \\ \sqrt{\left(11-\sqrt{105}\right)} & \sqrt{\left(11+\sqrt{105}\right)} \end{pmatrix}, \\ M^{-\frac{1}{2}}\mathcal{V}M^{-\frac{1}{2}} &= \sqrt{105}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned} \tag{55}$$

As the matrix $M^{-\frac{1}{2}}\mathcal{V}M^{-\frac{1}{2}}$ is already diagonal (generally this is not so), the U matrix is trivial and from Eqs. (35), (36) we finally obtain the hyper-spherical harmonics:

$$\begin{aligned} &\mathcal{Y}^{4,0,\sqrt{105}}_{2,2}(X) \\ &= \frac{\sqrt{\frac{15}{7}\left(11+\sqrt{105}\right)}}{2\pi^{3/2}}\mathcal{P}H^{\bar{4},0}_{(2,0)2,2}(X)/R^4 + \frac{\sqrt{\frac{15}{7}\left(11-\sqrt{105}\right)}}{2\pi^{3/2}}\mathcal{P}H^{\bar{4},0}_{(0,2)2,2}(X)/R^4 \\ &= \frac{-12\sqrt{14}R^2X^+_+X^-_+ + \sqrt{105\left(11+\sqrt{105}\right)}\left(X^-_+\right)^2\left|X^+\right|^2 + \sqrt{105\left(11-\sqrt{105}\right)}\left(X^+_+\right)^2\left|X^-\right|^2}{14\pi^{3/2}R^4} \end{aligned} \tag{56}$$

$$\begin{aligned} &\mathcal{Y}^{4,0,-\sqrt{105}}_{2,2}(X) \\ &= \frac{\sqrt{\frac{15}{7}\left(11+\sqrt{105}\right)}}{2\pi^{3/2}}\mathcal{P}H^{\bar{4},0}_{(0,2)2,2}(X)/R^4 + \frac{\sqrt{\frac{15}{7}\left(11-\sqrt{105}\right)}}{2\pi^{3/2}}\mathcal{P}H^{\bar{4},0}_{(2,0)2,2}(X)/R^4 \\ &= \frac{-12\sqrt{14}R^2X^+_+X^-_+ + \sqrt{105\left(11-\sqrt{105}\right)}\left(X^-_+\right)^2\left|X^+\right|^2 + \sqrt{105\left(11+\sqrt{105}\right)}\left(X^+_+\right)^2\left|X^-\right|^2}{14\pi^{3/2}R^4} \end{aligned} \tag{57}$$

We observe that two obtained hyper-spherical harmonics are identical to $\mathcal{Y}^{4,0,\Delta l=2}_{2,2}(X)$ and $\mathcal{Y}^{4,0,\Delta l=-2}_{2,2}(X)$.

6.3. Harmonics with symmetric degeneracy lifting operator

By an identical procedure, only taking this time operator $\mathcal{V}$ to be square of area of the subtended triangle, we obtain the only two $\mathrm{K}=4$ h.s. harmonics with degeneracy:

$$\mathcal{Y}^{4,0,\frac{1}{8}}_{2,2}(X) = \frac{\sqrt{15}\left(\left(X^-_+\right)^2\left|X^+\right|^2 - \left(X^+_+\right)^2\left|X^-\right|^2\right)}{2\pi^{3/2}R^4} \tag{58}$$

which turns out to be antisymmetric w.r.t. particle transpositions, and

$$\mathcal{Y}^{4,0,\frac{47}{280}}_{2,2}(X) = \frac{3\left(-8R^2X^+_+X^-_+ + 5\left(X^-_+\right)^2\left|X^+\right|^2 + 5\left(X^+_+\right)^2\left|X^-\right|^2\right)}{2\sqrt{7}\pi^{3/2}R^4} \tag{59}$$

which is symmetric under permutations.

Of course, these two h.s. harmonics have to be equal to the symmetric and antisymmetric combinations $\mathcal{P}H^{4,0}_{(|\Delta l|=2,+)2,2}(\lambda,\rho)$ and $\mathcal{P}H^{4,0}_{(|\Delta l|=2,-)2,2}(\lambda,\rho)$, and thus also equal to linear combinations of harmonics $\mathcal{Y}^{4,0,\sqrt{105}}_{2,2}$ and $\mathcal{Y}^{4,0,-\sqrt{105}}_{2,2}$ (previously obtained by using $\mathcal{V}_{JQJ}$ as the degeneracy-lifting operator). This is indeed the case:

$$\mathcal{Y}^{4,0,\frac{1}{8}}_{2,2}(\lambda,\rho) = \mathcal{P}H^{4,0}_{(|\Delta l|=2,-)2,2}(\lambda,\rho) = \frac{1}{\sqrt{2}}\left(\mathcal{Y}^{4,0,\sqrt{105}}_{2,2}(\lambda,\rho) - \mathcal{Y}^{4,0,-\sqrt{105}}_{2,2}(\lambda,\rho)\right)$$

and

$$\mathcal{Y}^{4,0,\frac{47}{280}}_{2,2}(\lambda,\rho) = \mathcal{P}H^{4,0}_{(|\Delta l|=2,+)2,2}(\lambda,\rho) = \frac{1}{\sqrt{2}}\left(\mathcal{Y}^{4,0,\sqrt{105}}_{2,2}(\lambda,\rho) + \mathcal{Y}^{4,0,-\sqrt{105}}_{2,2}(\lambda,\rho)\right).$$

6.4. Harmonics with $m < L$

Finally, as we have thus far presented an algorithm for the construction only of hyper-spherical harmonics with $m = L$, we should clarify how one can obtain the hyper-spherical harmonics with $m < L$. This can be done in (at least) two ways. One is to repeat the procedure above, this time starting from the core polynomials with some given value of m, such that $m < L$. However, in this case all of the intermediate expressions will be significantly more complicated. More optimal way is to find the corresponding hyper-spherical harmonic with $m = L$ first and than to use lowering operators $L_- \equiv L_1 - iL_2$ to obtain harmonics with lower values of m, following the well known recurrence formula:

$$L_- \mathcal{Y}^{\mathrm{K}Qv}_{L,m}(X) = \sqrt{L(L+1) - m(m-1)}\,\mathcal{Y}^{\mathrm{K}Qv}_{L,m-1}(X). \tag{60}$$

For example:

$$\begin{aligned}\mathcal{Y}^{4,0,-\sqrt{105}}_{2,1}(X) &= L_- \mathcal{Y}^{4,0,-\sqrt{105}}_{2,2}(X)/2\\ &= \frac{1}{35\pi^{3/2}R^4}\left(5\sqrt{\frac{7}{2}}X_0^-\left(\left(\sqrt{105}-15\right)\,X_+^-\left|X^+\right|^2 + 12R^2X_+^+\right)\right.\\ &\quad \left. -5X_0^+\,\left(\sqrt{105\left(11+\sqrt{105}\right)}X_+^+\left|X^-\right|^2 - 6\sqrt{14}R^2X_+^-\right)\right).\end{aligned} \tag{61}$$

Clearly the spherical harmonics can also be expressed in terms of the initial variables, the Jacobi vectors, e.g.:

$$\begin{aligned}&\mathcal{Y}^{4,0,-\sqrt{105}}_{2,2}(\lambda,\rho)\\ &= \frac{1}{14\pi^{3/2}\left(\lambda_1^2+\lambda_2^2+\lambda_3^2+\rho_1^2+\rho_2^2+\rho_3^2\right)^2}\\ &\times\left(\sqrt{105\left(11+\sqrt{105}\right)}\left((\lambda_1-i\rho_1)^2+(\lambda_2-i\rho_2)^2+(\lambda_3-i\rho_3)^2\right)\right.\\ &\times(\lambda_1+i(\lambda_2+\rho_1+i\rho_2))^2 - 12\sqrt{14}(\lambda_1+i\lambda_2-i\rho_1+\rho_2)\\ &\times\left(\lambda_1^2+\lambda_2^2+\lambda_3^2+\rho_1^2+\rho_2^2+\rho_3^2\right)(\lambda_1+i(\lambda_2+\rho_1+i\rho_2)) + \sqrt{105\left(11-\sqrt{105}\right)}\end{aligned}$$

$$\times\,(\lambda_1 + i\lambda_2 - i\rho_1 + \rho_2)^2\left((\lambda_1 + i\rho_1)^2 + (\lambda_2 + i\rho_2)^2 + (\lambda_3 + i\rho_3)^2\right)\Bigg), \tag{62}$$

by inverting definitions Eq. (6) and Eq. (22). By comparing the forms of Eq. (56) and Eq. (62) it becomes clear that the expressions are much more compact when written in terms of spherical complex coordinates $X^{\pm}_{\pm}, X^{\pm}_{0}$, Eq. (22).

Next, we turn to consider the permutation properties of these h.s. harmonics, which, in turn, fixes (some of) the phase ambiguities, and corroborates our basic claim stated in the title of this paper.

7. Permutation properties

From the viewpoint of applications of the three particle hyperspherical harmonics in physics, it is of some importance that the wave functions have simple and manifest transformation properties with respect to both the spatial rotations and permutations of the three particles. Of course, the rotational properties of the functions $\mathcal{Y}^{\mathrm{K}Qv}_{L,m}(\boldsymbol{\lambda}, \boldsymbol{\rho})$ are manifestly given by the values of corresponding $SO(3)_{rot}$ labels L and m, so it is the permutation properties that must be established here.

Properties of functions $\mathcal{Y}^{\mathrm{K}Qv}_{L,m}(\boldsymbol{\lambda}, \boldsymbol{\rho})$ under particle permutations are (readily) inferred from the (simple) transformation properties of the coordinates $X^{\pm}_{i}$. Namely, under the transpositions (two-body permutations) $\{\mathcal{T}_{12}, \mathcal{T}_{23}, \mathcal{T}_{31}\}$ of pairs $(1, 2)$, $(2, 3)$ and $(3, 1)$, the Jacobi coordinates transform as:

$$\begin{aligned}
&\mathcal{T}_{12} : \boldsymbol{\lambda} \to \boldsymbol{\lambda}, \quad \boldsymbol{\rho} \to -\boldsymbol{\rho},\\
&\mathcal{T}_{23} : \boldsymbol{\lambda} \to -\frac{1}{2}\boldsymbol{\lambda} + \frac{\sqrt{3}}{2}\boldsymbol{\rho}, \quad \boldsymbol{\rho} \to \frac{1}{2}\boldsymbol{\rho} + \frac{\sqrt{3}}{2}\boldsymbol{\lambda},\\
&\mathcal{T}_{31} : \boldsymbol{\lambda} \to -\frac{1}{2}\boldsymbol{\lambda} - \frac{\sqrt{3}}{2}\boldsymbol{\rho}, \quad \boldsymbol{\rho} \to \frac{1}{2}\boldsymbol{\rho} - \frac{\sqrt{3}}{2}\boldsymbol{\lambda}.
\end{aligned} \tag{63}$$

That induces the following transformations of complex coordinates $X^{\pm}_{i}$:

$$\mathcal{T}_{12} : X^{\pm}_{i} \to X^{\mp}_{i}, \quad \mathcal{T}_{23} : X^{\pm}_{i} \to e^{\pm\frac{2i\pi}{3}} X^{\mp}_{i}, \quad \mathcal{T}_{31} : X^{\pm}_{i} \to e^{\mp\frac{2i\pi}{3}} X^{\mp}_{i}. \tag{64}$$

Note that Eqs. (63) imply that the transpositions $\mathcal{T}_{ij}$ correspond to $O(6)$ transformations of x_μ with $\det\mathcal{T}_{ij} = -1$, i.e. they form a set of (parity-like in odd-D spaces; though in D=6 the usual parity (i.e., the reflection of all 6 coordinates) transformation's determinant equals $+1$) "reflection transformation" in the 6-D space and as such do not belong to $SO(6)$ group of proper hyper-rotations. Such reflections generally lead to an appearance of phases, see below.

It follows from Eqs. (7)–(10) and Eq. (64) that none of the quantum numbers K, L and m change under permutations of particles, whereas the value of the "democracy label" Q is inverted under all transpositions: $Q \to -Q$. The fact that K is not changed by particle transpositions implies that the set of $SO(6)$ hyper-spherical harmonics with given K also carry an irreducible representation of the entire $O(6)$ group (and, in this sense, these functions are equally $O(6)$ hyper-spherical harmonics). Group-theoretically, change in the label Q is a consequence of the fact that the discrete group of permutations S_3 is not a subgroup of the $U(1)$ group generated by operator Q, but of the group $O(2) = U(1) \rtimes S_2$ instead. The behavior of the multiplicity label v under transpositions manifestly depends on the choice of the multiplicity-lifting operator $\mathcal{V}$, but this choice is effectively reduced to the choice of the sign change of v under transpositions.

To explain this, we first note that the set of even permutations of the three particles constitutes a discrete subgroup of the $SO(6)$ group of hyper-rotations generated by $K_{\mu\nu}$: each transposition corresponds to an orthogonal matrix with determinant -1, so that the combination of even number of transpositions has determinant $+1$. More specifically, by comparing the actions on the 6-dimensional coordinates, it turns out that $\mathcal{T}_{12}\mathcal{T}_{23} = e^{\frac{2i\pi Q}{3}}$ and $\mathcal{T}_{23}\mathcal{T}_{12} = e^{-\frac{2i\pi Q}{3}}$, as (11) and (64) yield $\mathcal{T}_{12}\mathcal{T}_{23}X_i^{\pm}\mathcal{T}_{23}\mathcal{T}_{12} = e^{\frac{2i\pi Q}{3}}X_i^{\pm}e^{-\frac{2i\pi Q}{3}}$. However, due to the requirement that multiplicity lifting operator $\mathcal{V}$ must commute with generator Q, we conclude that even permutations of particles must leave the operator $\mathcal{V}$ invariant: $\mathcal{T}_{12}\mathcal{T}_{23}\mathcal{V}\mathcal{T}_{23}\mathcal{T}_{12} = e^{\frac{2i\pi Q}{3}}\mathcal{V}e^{-\frac{2i\pi Q}{3}} = \mathcal{V}$ and $\mathcal{T}_{23}\mathcal{T}_{12}\mathcal{V}\mathcal{T}_{12}\mathcal{T}_{23} = e^{-\frac{2i\pi Q}{3}}\mathcal{V}e^{\frac{2i\pi Q}{3}} = \mathcal{V}$. On the other hand, only one-dimensional irreducible representations of permutation group S_3 (i.e. symmetric and antisymmetric representation) have this property that the even permutations are mapped onto the unit operator, whereas the remaining two-dimensional (mixed) representation does not have this property.

In other words, this means that multiplicity lifting operator $\mathcal{V}$ itself can transform according to the one-dimensional antisymmetric representation of S_3, or transform as the one-dimensional symmetric representation, or be a nontrivial linear combination of antisymmetric and symmetric components. We dismiss the third option (linear combinations) both on physical grounds, as there can hardly be physical motivation for the introduction of such an operator, and on practical grounds, as that choice would lead to unnecessarily complicated transformation properties. Put together, these two reasons render such a choice of operator inappropriate for multiplicity lifting. Therefore we consider only two choices for the multiplicity-lifting operator: a) operators that are antisymmetric under permutations, i.e., $\mathcal{T}_{ij}\mathcal{V}\mathcal{T}_{ij} = -\mathcal{V}$; and b) the symmetric ones under permutations, i.e., $\mathcal{T}_{ij}\mathcal{V}\mathcal{T}_{ij} = \mathcal{V}$. For example, the $\mathcal{V}_{JQJ}$ operator is of the antisymmetric type, whereas the triangle area operator is a representative of the symmetric type.

In conclusion, the action of a single transposition $\mathcal{T}_{ij}$ on the label v of a permutation symmetric h.s. harmonic can lead at most to a (minus) sign for the multiplicity label: $v \to \pm v$. In the next subsection, we shall also show that only the antisymmetric degeneracy-lifting operators lead to a completely unambiguous set of permutation properties of h.s. harmonics.

As far as the permutation properties are concerned, the choice of Δl, Eq. (27), for the multiplicity label v is no different than using eigenvalues of any antisymmetric multiplicity lifting operator, because the value of $\Delta l = L_+ - L_-$ (obviously) changes the sign upon the interchange of X_+ and X_-.[2] Therefore, the case of using Δl to label multiplicity need not be treated separately, as it is already included in the case of a general antisymmetric multiplicity lifting operator.

Apart from the changes in labels, transpositions of two particles generally also result in the appearance of an additional phase factor multiplying the hyper-spherical harmonic. For values of K, Q, L and m with no multiplicity, the transformation properties of h.s. harmonics under (two-particle) particle transpositions coincide with the corresponding properties of the core polynomials, so that Eq. (13) and Eq. (64) readily lead to:

$$\begin{aligned}
\mathcal{T}_{12} &: \mathcal{Y}^{\mathrm{K}Qv}_{L,m}(\boldsymbol{\lambda}, \boldsymbol{\rho}) \to (-1)^{\mathrm{K}-L}\mathcal{Y}^{\mathrm{K},-Q,v'}_{L,m}(\boldsymbol{\lambda}, \boldsymbol{\rho}), \\
\mathcal{T}_{23} &: \mathcal{Y}^{\mathrm{K}Qv}_{L,m}(\boldsymbol{\lambda}, \boldsymbol{\rho}) \to (-1)^{\mathrm{K}-L}e^{\frac{2Qi\pi}{3}}\mathcal{Y}^{\mathrm{K},-Q,v'}_{L,m}(\boldsymbol{\lambda}, \boldsymbol{\rho}), \\
\mathcal{T}_{31} &: \mathcal{Y}^{\mathrm{K}Qv}_{L,m}(\boldsymbol{\lambda}, \boldsymbol{\rho}) \to (-1)^{\mathrm{K}-L}e^{-\frac{2Qi\pi}{3}}\mathcal{Y}^{\mathrm{K},-Q,v'}_{L,m}(\boldsymbol{\lambda}, \boldsymbol{\rho}),
\end{aligned} \tag{65}$$

[2] The fact that an operator with eigenvalues that exactly match the values Δl cannot be (easily) written down, does not change anything in principle, because such an operator $\mathcal{V}_{\Delta l}$ can always be formally defined by its action on the hyperspherical harmonics: $\mathcal{V}_{\Delta l}\mathcal{Y}^{\mathrm{K}Q\Delta l}_{L,m}(X) = \Delta l\mathcal{Y}^{\mathrm{K}Q\Delta l}_{L,m}(X)$.

where v' is the transposition-transformed value of v: $v' = \mathcal{T}_{ij}(v)$. We note that the phase factor $(-1)^{\mathrm{K}-L}$ in Eqs. (65) comes about from the transformation properties of the Clebsch–Gordan coefficient $C^{L_+L_-L}_{m_+m_-m}$ in Eq. (13) under the replacement $L_+, m_+ \leftrightarrow L_-, m_-$, induced by the transformation $X_i^+ \leftrightarrow X_i^-$: $C^{L_-L_+L}_{m_-m_+m} = (-1)^{L_++L_--L} C^{L_+L_-L}_{m_+m_-m}$ and $(-1)^{L_++L_--L} = (-1)^{\mathrm{K}-L}$ due to Eq. (15).

7.1. *Irreducible representations of the permutation group*

There are three distinct irreducible representations of the S_3 permutation group – two one-dimensional (the symmetric S and the antisymmetric A ones) and a two-dimensional (the mixed M one). In order to determine to which representation of the permutation group any particular h.s. harmonic $\mathcal{Y}^{\mathrm{K}Qv}_{L,m}(\boldsymbol{\lambda}, \boldsymbol{\rho})$ belongs, we start by considering multiplicity free cases.

7.1.1. *Multiplicity-free case*

When $Q = 0$, we can see from Eq. (65) that the action of transpositions reduces to

$$\mathcal{T}_{ij} : \mathcal{Y}^{\mathrm{K},0,v}_{L,m}(\boldsymbol{\lambda}, \boldsymbol{\rho}) \to (-1)^{\mathrm{K}-L} \mathcal{Y}^{\mathrm{K},0,v}_{L,m}(\boldsymbol{\lambda}, \boldsymbol{\rho}) \tag{66}$$

We obtained this relation by replacement $v' = v$, which necessarily follows from the "multiplicity-free" assumption, i.e. the assumption than numbers K, Q, L and m uniquely specify this h.s. harmonic (in particular, $v = v'$ is always true for permutation-symmetric $\mathcal{V}$, whereas here it holds only as $v = v' = 0$ for permutation-antisymmetric $\mathcal{V}$). Thus, multiplicity-free h.s. harmonics $\mathcal{Y}^{\mathrm{K}0v}_{L,m}(\boldsymbol{\lambda}, \boldsymbol{\rho})$ belong either to the symmetric (S) representation of S_3, for even values of $\mathrm{K} - L$, or to the antisymmetric (A) representation, for odd values of $\mathrm{K} - L$.

When $Q \neq 0$, the action of permutations on h.s. harmonics is reduced to two-dimensional subspaces spanned by pairs of harmonics $\left\{\mathcal{Y}^{\mathrm{K}Qv}_{L,m}(\boldsymbol{\lambda}, \boldsymbol{\rho}), \mathcal{Y}^{\mathrm{K},-Q,v'}_{L,m}(\boldsymbol{\lambda}, \boldsymbol{\rho})\right\}$, as can be seen from Eq. (65). In this basis, the three transposition operators of Eq. (65) have the following matrix representations:

$$\mathcal{T}_{12} \to (-1)^{\mathrm{K}-L} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \mathcal{T}_{23} \to (-1)^{\mathrm{K}-L} \begin{pmatrix} 0 & e^{\frac{2i\pi Q}{3}} \\ e^{-\frac{2}{3}i\pi Q} & 0 \end{pmatrix},$$
$$\mathcal{T}_{31} \to (-1)^{\mathrm{K}-L} \begin{pmatrix} 0 & e^{-\frac{2}{3}i\pi Q} \\ e^{\frac{2i\pi Q}{3}} & 0 \end{pmatrix}. \tag{67}$$

For $Q \not\equiv 0 \pmod 3$, this representation of the permutation group S_3 is irreducible, therefore such h.s. harmonics belong to two dimensional mixed representation.

For $Q \equiv 0 \pmod 3$, this 2×2 matrix representation reduces to two one-dimensional representations, one of which is symmetric and the other antisymmetric; the representations are spanned by the following pair of linear combinations of the harmonics:

$$\mathcal{T}_{ij} : \frac{1}{\sqrt{2}} \left(\mathcal{Y}^{\mathrm{K}Qv}_{L,m}(\boldsymbol{\lambda}, \boldsymbol{\rho}) + \mathcal{Y}^{\mathrm{K},-Q,v'}_{L,m}(\boldsymbol{\lambda}, \boldsymbol{\rho})\right)$$
$$\to (-1)^{\mathrm{K}-L} \frac{1}{\sqrt{2}} \left(\mathcal{Y}^{\mathrm{K}Qv}_{L,m}(\boldsymbol{\lambda}, \boldsymbol{\rho}) + \mathcal{Y}^{\mathrm{K},-Q,v'}_{L,m}(\boldsymbol{\lambda}, \boldsymbol{\rho})\right), \tag{68}$$

$$\mathcal{T}_{ij} : \frac{1}{\sqrt{2}} \left(\mathcal{Y}^{\mathrm{K}Qv}_{L,m}(\boldsymbol{\lambda}, \boldsymbol{\rho}) - \mathcal{Y}^{\mathrm{K},-Q,v'}_{L,m}(\boldsymbol{\lambda}, \boldsymbol{\rho}) \right)$$
$$\to (-1)^{\mathrm{K}-L+1} \frac{1}{\sqrt{2}} \left(\mathcal{Y}^{\mathrm{K}Qv}_{L,m}(\boldsymbol{\lambda}, \boldsymbol{\rho}) - \mathcal{Y}^{\mathrm{K},-Q,v'}_{L,m}(\boldsymbol{\lambda}, \boldsymbol{\rho}) \right). \tag{69}$$

7.1.2. Cases with multiplicity

When there is a (nontrivial) multiplicity of h.s. harmonics with a given set of quantum numbers K, Q, L, m, then the transformation properties under permutations clearly depend also on the choice of multiplicity lifting operator $\mathcal{V}$.

As explained above, there are two types of admissible choices for the multiplicity-lifting operator: a) the antisymmetric ones; and b) the symmetric ones under permutations. For both of these types, the relations Eqs. (65) still hold with $v' = -v$ and $v' = v$, respectively, albeit possibly with additional phase factors on the right-hand sides of Eqs. (65). These additional phases can be, in principle, introduced by multiplicity lifting procedure (Section 5), and hence the transformation properties in cases with multiplicity can no longer be inferred from the corresponding properties of the core polynomials.

However, these possible additional phase factors can be absorbed into the definition of h.s. harmonics, i.e. into the phase convention, in all cases, except one: when both the operator $\mathcal{V}$ is symmetric and $Q = 0$. Only in this one case of nontrivial multiplicity with $v' = v$ does the same h.s. harmonic appear on both sides of Eqs. (65), i.e., Eqs. (65) then lead to Eq. (66) with the aforementioned phase factor $e^{i\phi}$ on the righthand side:

$$\mathcal{T}_{ij} : \mathcal{Y}^{\mathrm{K}0v}_{L,m}(\boldsymbol{\lambda}, \boldsymbol{\rho}) \to e^{i\phi}(-1)^{\mathrm{K}-L} \mathcal{Y}^{\mathrm{K}0v}_{L,m}(\boldsymbol{\lambda}, \boldsymbol{\rho}), \tag{70}$$

and it is clear that no redefinition of $\mathcal{Y}^{\mathrm{K}0v}_{L,m}(\boldsymbol{\lambda}, \boldsymbol{\rho})$ can remove this factor. Due to idempotency of transpositions, this phase factor $e^{i\phi}$ can be either $+1$ or -1. In the former case the h.s. harmonic obtains under transpositions a factor of $(-1)^{\mathrm{K}-L}$, and of $(-1)^{\mathrm{K}-L+1}$ in the latter, but which one of the two cannot be established without providing further details about the chosen operator $\mathcal{V}$.

Exactly such an example was illustrated in the Section 6.3: *a priori* – i.e. based solely on the values of the labels – it is not possible to determine which one of the h.s. harmonics $\mathcal{Y}^{4,0,\frac{1}{8}}_{2,2}$ and $\mathcal{Y}^{4,0,\frac{47}{280}}_{2,2}$ in Eqs. (58), (59) belongs to the symmetric and which one to the antisymmetric representations of S_3.

In all other cases (i.e. apart from the case of symmetric degeneracy-lifting operator $\mathcal{V}$ at $Q = 0$) the same reasoning as in Eqs. (67)–(69) holds and we again conclude that for $Q \not\equiv 0$ (mod 3), the h.s. harmonics belong to the mixed representation of S_3, whereas for $Q \equiv 0$ (mod 3) the two linear combinations Eq. (68) and Eq. (69) belong to the one-dimensional representations, acquiring, respectively, factor of $(-1)^{\mathrm{K}-L}$ and $(-1)^{\mathrm{K}-L+1}$ under transpositions.

7.1.3. Summary of the permutation properties

In order to summarize the above results it is convenient to introduce the following linear combinations of the h.s. harmonics, which are no longer eigenfunctions of Q operator, but are instead eigenfunctions of transposition $\mathcal{T}_{12}$:

$$\mathcal{Y}^{\mathrm{K}|Q|v}_{L,m,\pm}(\boldsymbol{\lambda}, \boldsymbol{\rho}) \equiv \frac{1}{\sqrt{2}} \left(\mathcal{Y}^{\mathrm{K}|Q|v}_{L,m}(\boldsymbol{\lambda}, \boldsymbol{\rho}) \pm (-1)^{\mathrm{K}-L} \mathcal{Y}^{\mathrm{K},-|Q|,v'}_{L,m}(\boldsymbol{\lambda}, \boldsymbol{\rho}) \right). \tag{71}$$

The normalization factor $\frac{1}{\sqrt{2}}$ ought to be changed to $\frac{1}{2}$ in the cases when the both terms in the bracket are equal (instead of orthogonal) and do not cancel each out.

Apart from the specific case when $\mathcal{V}$ is symmetric under permutations, the multiplicity is nontrivial, and $Q = 0$ (and when the conclusions further depend on the details of the operator $\mathcal{V}$), the following holds:

1. the transposition $\mathcal{T}_{12}$ is a pure sign: $\mathcal{T}_{12} : \mathcal{Y}^{\mathrm{K}|Q|v}_{L,m,\pm}(\boldsymbol{\lambda}, \boldsymbol{\rho}) \to \pm\mathcal{Y}^{\mathrm{K}|Q|v}_{L,m,\pm}(\boldsymbol{\lambda}, \boldsymbol{\rho})$,
2. for $Q \not\equiv 0 \pmod 3$, the harmonics $\mathcal{Y}^{\mathrm{K}|Q|v}_{L,m,\pm}(\boldsymbol{\lambda}, \boldsymbol{\rho})$ belong to the mixed representation M,
3. for $Q \equiv 0 \pmod 3$, the harmonic $\mathcal{Y}^{\mathrm{K}|Q|v}_{L,m,+}(\boldsymbol{\lambda}, \boldsymbol{\rho})$ belongs to the symmetric representation S and $\mathcal{Y}^{\mathrm{K}|Q|v}_{L,m,-}(\boldsymbol{\lambda}, \boldsymbol{\rho})$ belongs to the antisymmetric representation A.

Note that the above statements also implicitly contain our previous conclusions about the behavior of $Q = 0$ multiplicity-free harmonics Eq. (66), and thus summarize all of the previous results specifying the representation of S_3 to which any given harmonic belongs.

Above, we have tacitly assumed that the phase convention, i.e., the choice of how to fix the otherwise arbitrary phases of harmonics, obeys Eq. (65). This assumption is satisfied in the (specific) case of the multiplicity-resolving operator Eq. (37) together with the phase convention, Eq. (38): $\mathcal{V}_{JQJ}$ is antisymmetric (which is readily derived from Eq. (7) and Eq. (8)), thus any transposition simply flips the sign of v: $v' = -v$, and the relations Eqs. (65) hold in full generality. *q.e.d.*

8. Matrix elements of $SO(6)$ harmonics

In applications to the quantum mechanical three-body problem, Ref. [50], one often needs to know the $SO(6)$ hyper-angular matrix elements of the form

$$\langle \mathcal{Y}^{\mathrm{K}''}_{[\mathrm{m}'']}(\Omega_5) | \mathcal{Y}^{\mathrm{K}Qv}_{00}(\alpha, \phi) | \mathcal{Y}^{\mathrm{K}'}_{[\mathrm{m}']}(\Omega_5) \rangle \tag{72}$$

This kind of integral can be readily evaluated using formulas from Appendix A so long as the HS harmonics $\mathcal{Y}^{\mathrm{K}'}_{[\mathrm{m}']}(\Omega_5)$ are explicitly known as polynomials of the integration variables, with the result expressed in terms of Γ function, see Eq. (A.4), see Tables 1 and 2. By the procedure laid out in the previous sections, it is possible to find the required polynomial expressions and thus to evaluate matrix elements of the type shown in Eq. (72) yielding algebraic numbers whenever the multiplicity of the hyperspherical harmonics with equal K, Q and L is less than five[3] or the integer Δl is used as a multiplicity label, as already explained in Sect. 5.

In a great number of practical applications, however, there is no need to calculate explicitly the h.s. harmonics, apart from evaluating matrix elements of the form shown in Eq. (72). As the calculation of h.s. harmonic functions, as well as the application of the formulas from Appendix A, can be considerably involved for higher values of K, it is very useful to have some more direct method for evaluation of such matrix elements involving three h.s. harmonics. In this section we shall therefore discuss matrix elements Eq. (72) *per se*, both in the $SO(6)$ and $SU(3)$ context, finally evaluating them as a closed form expression, apart from a single $SU(3)$ Clebsch–Gordan coefficient.

[3] There are alleviating circumstances here that sometimes allow algebraic solutions in cases with multiplicities higher than five, e.g. due to discrete S_3 numbers in $Q = 0$ cases, or in cases discussed in Sects. 3. and 4. of Ref. [46].

Table 1
The values of the three-body potential hyper-angular diagonal matrix elements $\langle \mathcal{Y}^{4,|0|,0}_{0,0,+} \rangle_{\rm ang}$, $\langle \mathcal{Y}^{6,|6|,0}_{0,0,+} \rangle_{\rm ang}$ and $\langle \mathcal{Y}^{8,|0|,0}_{0,0,+} \rangle_{\rm ang}$, for K $\le$ 4 states (for all allowed orbital waves L). The correspondence between the S_3 permutation group irreps. and SU(6)$_{FS}$ symmetry multiplets of the three-quark system: $S \leftrightarrow 56$, $A \leftrightarrow 20$ and $M \leftrightarrow 70$. The table values are independent of the angular moment projection m.

K	(K, $\|Q\|$, L, ν, $\pm$)	$[SU(6), L^P]$	$\pi\sqrt{\pi}\langle \mathcal{Y}^{4,\|0\|,0}_{0,0,+} \rangle_{\rm ang}$	$\pi\sqrt{2\pi}\langle \mathcal{Y}^{6,\|6\|,0}_{0,0,+} \rangle_{\rm ang}$	$\pi\sqrt{\pi}\langle \mathcal{Y}^{8,\|0\|,0}_{0,0,+} \rangle_{\rm ang}$
2	$(2, \|2\|, 0, 0, \pm)$	$[70, 0^+]$	$\frac{1}{\sqrt{3}}$	0	0
2	$(2, \|0\|, 2, 0, +)$	$[56, 2^+]$	$\frac{\sqrt{3}}{5}$	0	0
2	$(2, \|2\|, 2, -3, \pm)$	$[70, 2^+]$	$-\frac{1}{5\sqrt{3}}$	0	0
2	$(2, \|0\|, 1, 0, -)$	$[20, 1^+]$	$-\frac{1}{\sqrt{3}}$	0	0
3	$(3, \|3\|, 1, -1, -)$	$[20, 1^-]$	$\frac{1}{\sqrt{3}}$	-1	0
3	$(3, \|3\|, 1, -1, +)$	$[56, 1^-]$	$\frac{1}{\sqrt{3}}$	1	0
3	$(3, \|1\|, 1, 3, \pm)$	$[70, 1^-]$	0	0	0
3	$(3, \|1\|, 2, -5, \pm)$	$[70, 2^-]$	$-\frac{1}{\sqrt{3}}$	0	0
3	$(3, \|1\|, 3, -2, \pm)$	$[70, 3^-]$	$\frac{5}{7\sqrt{3}}$	0	0
3	$(3, \|3\|, 3, -6, +)$	$[56, 3^-]$	$-\frac{\sqrt{3}}{7}$	$\frac{2}{7}$	0
3	$(3, \|3\|, 3, -6, -)$	$[20, 3^-]$	$-\frac{\sqrt{3}}{7}$	$-\frac{2}{7}$	0
4	$(4, \|4\|, 0, 0, \pm)$	$[70, 0^+]$	$\frac{\sqrt{3}}{2}$	0	$\frac{1}{2\sqrt{5}}$
4	$(4, \|0\|, 0, 0, +)$	$[56, 0^+]$	0	0	$\frac{2}{\sqrt{5}}$
4	$(4, \|2\|, 1, 2, \pm)$	$[70, 1^+]$	0	0	$-\frac{1}{\sqrt{5}}$
4	$\left(4, \|0\|, 2, \sqrt{105}, +\right)$	$[56, 2^+]$	$-\frac{12\sqrt{3}}{35}$	0	$\frac{\sqrt{5}}{7}$
4	$\left(4, \|0\|, 2, \sqrt{105}, -\right)$	$[20, 2^+]$	0	0	$-\frac{1}{\sqrt{5}}$
4	$(4, \|2\|, 2, 2, \pm)$	$[70, 2^+]$	$\frac{4\sqrt{3}}{35}$	0	$\frac{\sqrt{5}}{7}$
4	$(4, \|4\|, 2, -3, \pm)$	$[70', 2^+]$	$\frac{2\sqrt{3}}{7}$	0	$-\frac{1}{7\sqrt{5}}$
4	$(4, \|2\|, 3, -13, \pm)$	$[70, 3^+]$	$-\frac{5\sqrt{3}}{14}$	0	$\frac{1}{14\sqrt{5}}$
4	$(4, \|0\|, 3, 0, -)$	$[20, 3^+]$	$-\frac{3\sqrt{3}}{14}$	0	$-\frac{\sqrt{5}}{14}$
4	$(4, \|0\|, 4, 0, +)$	$[56, 4^+]$	$\frac{5\sqrt{3}}{14}$	0	$\frac{3}{14\sqrt{5}}$
4	$(4, \|2\|, 4, -5, \pm)$	$[70, 4^+]$	$\frac{3\sqrt{3}}{14}$	0	$-\frac{\sqrt{5}}{42}$
4	$(4, \|4\|, 4, -10, \pm)$	$[70', 4^+]$	$-\frac{3\sqrt{3}}{14}$	0	$\frac{1}{42\sqrt{5}}$

8.1. Some matrix elements and their properties

Decomposition into hyperspherical harmonics with manifest permutation properties highly simplifies solving of Schrödinger's equation. The benefits are most notable when the three-body potential is permutation symmetric. The decomposition of any such potential into h.s. harmonics has a low number of nonzero components due to the permutation symmetry constraints: e.g. up to K $\le$ 11 the only h.s. harmonics that can appear in such decomposition are $\mathcal{Y}^{0,|0|,0}_{0,0,+}$, $\mathcal{Y}^{4,|0|,0}_{0,0,+}$, $\mathcal{Y}^{8,|0|,0}_{0,0,+}$ and $\mathcal{Y}^{6,|6|,0}_{0,0,+}$. In Tables 1 and 2, we show the nonzero matrix elements between states of

Table 2
The values of the off-diagonal matrix elements of the hyper-angular part of the three-body potential $\pi\sqrt{2\pi}\langle[SU(6)_f, L_f^P]|\mathcal{Y}_{0,0,+}^{6,|6|,0}|[SU(6)_i, L_i^P]\rangle_{\text{ang}}$, for various K = 4 states (for all allowed orbital waves L).

K	$[SU(6)_f, L_f^P]$	$[SU(6)_i, L_i^P]$	$\pi\sqrt{2\pi}\langle\mathcal{Y}_{0,0,+}^{6,\lvert 6\rvert,0}\rangle_{\text{ang}}$
4	$[70, 2^+]$	$[70', 2^+]$	$\frac{6}{7}\sqrt{\frac{6}{5}}$
4	$[70', 2^+]$	$[70, 2^+]$	$\frac{6}{7}\sqrt{\frac{6}{5}}$
4	$[70, 4^+]$	$[70', 4^+]$	$\frac{8}{21}$
4	$[70', 4^+]$	$[70, 4^+]$	$\frac{8}{21}$
4	$[20, L^+]$	$[20, L^+]$	0
4	$[56, L^+]$	$[56, L^+]$	0
4	$[20, L^+]$	$[56, L^+]$	0

same K, K ≤ 4. These matrix elements are sufficient to evaluate the matrix elements of permutation symmetric sums of arbitrary one-, two- and three-body operators, such as the three-body potential, and thus to solve Schrödinger's equation in the first order of perturbation theory (in K ≤ 4 subspace). These harmonics have been applied to three homogeneous confining potentials in Ref. [49].

We observe that generally, the $SO(6)$ matrix elements obey the following selection rules that reduce the number of non-zero values: they are subject to the "triangular" conditions $\mathrm{K}' + \mathrm{K}'' \geq \mathrm{K} \geq |\mathrm{K}' - \mathrm{K}''|$ plus the condition that $\mathrm{K}' + \mathrm{K}'' + \mathrm{K} = 0, 2, 4, \ldots$, and the angular momenta satisfy the selection rules: $L' = L''$, $m' = m''$. Moreover, Q is an Abelian (i.e. additive) quantum number that satisfies the simple selection rule: $Q'' = Q' + Q$.

The aforementioned selection rules naturally follow since the hyper-angular matrix element Eq. (72) can be reduced to a product of two $SO(6)$ group Clebsch–Gordan coefficients.

8.2. *Matrix elements as functions of $SO(6)$ Clebsch–Gordan coefficients*

In the case of $SO(3)$ harmonics holds the Gaunt formula [60]

$$\int Y_{LM}^*(\theta,\phi) Y_{l_1 m_1}(\theta,\phi) Y_{l_2 m_2}(\theta,\phi) \sin\theta d\theta d\phi$$
$$= \left[\frac{(2l_1+1)(2l_2+1)}{4\pi(2L+1)}\right]^{1/2} C_{m_1 m_2 M}^{l_1\ l_2\ L} C_{0\ 0\ 0}^{l_1\ l_2\ L}, \tag{73}$$

where $C_{m_1 m_2 M}^{l_1\ l_2\ L}$ is the $SO(3)$ Clebsch–Gordan coefficient. In the context of the Wigner–Eckart theorem, the Clebsch–Gordan coefficient $C_{0\ 0\ 0}^{l_1\ l_2\ L}$ in Eq. (73) is proportional to/defines the "reduced matrix element" $\langle L||T_{l_1}||l_2\rangle$, in this case of the $SO(3)$ spherical harmonic $T_{l_1} = \mathbf{Y}_{l_1}$: $\langle LM|T_{l_1 m_1}|l_2 m_2\rangle = \langle LM|l_2 l_1 m_2 m_1\rangle\langle L||T_{l_1}||l_2\rangle$. Of course, the precise definition of the reduced matrix element depends on the conventions used, see e.g. Refs. [58,59], but the right-hand side of Eq. (73) is independent of convention.

The equivalent formula holds also for $SO(n)$ groups with higher-values of n (see Appendix B for derivation), and the integral of three $SO(6)$ harmonics is, similarly, proportional to products of two $SO(6)$ group Clebsch–Gordan coefficients:

$$\int_{\mathcal{M}} \mathcal{Y}^{*K}_{[m]}(\Omega_5)\mathcal{Y}^{K_1}_{[m_1]}(\Omega_5)\mathcal{Y}^{K_2}_{[m_2]}(\Omega_5)d\Omega_5$$
$$= \frac{1}{\sqrt{V_{\mathcal{M}}}}\sqrt{\frac{\dim(K_1)\dim(K_2)}{\dim(K)}}\, C^{K_1\ K_2\ K}_{[m_1]\,[m_2]\,[m]} C^{K_1\ \ K_2\ \ K}_{[0_H]\ [0_H]\ [0_H]}, \tag{74}$$

where $V_{\mathcal{M}} = \pi^3$ is the "volume"of the coset space $\mathcal{M} = SO(6)/SO(5)$, $\dim_{SO(6)}(K) = \frac{(K+3)!(K+2)}{12K!} = \frac{(K+3)(K+2)^2(K+1)}{12}$ and $[0_H]$ are labels of the vector that is invariant w.r.t. $SO(5)$ subgroup (because a sphere in 6 dimensions is isomorphic with $SO(6)/SO(5)$ coset space, see Appendix B).

Now, this does not necessarily simplify the problem at hand, because the SO(6) Clebsch–Gordan coefficients are not well known in general. The prospect of evaluating these Clebsch–Gordan coefficients or finding their values in literature is further complicated by the fact that the physical context dictates the choice of the basis, $SO(6) \supset U(3) \supset SO(3) \times U(1)$, which is not the simplest one from the mathematical viewpoint, instead of the simpler one $SO(6) \supset U(3) \supset U(2) \supset U(1)$. It is a (physical) necessity to have manifest transformation properties w.r.t. the (physical) angular momentum $SO(3)$ that spoils such attempts – if it were not for this, the multiplicity problem would not arise and both the construction of $SO(6)$ H.H. and calculation of $SO(6)$ Clebsch–Gordan coefficients would be much simpler.

Nevertheless, certain general properties of the matrix elements Eq. (72) can be inferred based on elemental $SO(6)$ group-theoretical arguments. For example, the following dimensional C.G. series (the reduction of tensor products) immediately give information, for some lower dimensional cases, when the value of the matrix element in Eq. (72) is allowed to be nonzero:

$$\begin{aligned}
\mathbf{6} \otimes \mathbf{1} &= \mathbf{6} \\
\mathbf{6} \otimes \mathbf{6} &= \mathbf{1} \oplus \mathbf{20} \\
\mathbf{6} \otimes \mathbf{20} &= \mathbf{6} \oplus \mathbf{50} \\
\mathbf{6} \otimes \mathbf{50} &= \mathbf{20} \oplus \mathbf{105} \\
&\vdots
\end{aligned} \tag{75}$$

where boldface numbers denote dimensions of $SO(6)$ irreps. In general:

$$[1] \otimes [K'] = [K' - 1] \oplus [K' + 1].$$

This result, and the property that the left-hand and right-hand sides of these equations do not agree arithmetically, e.g., $6 \times 6 = 36 \neq 1 + 20 = 21$, are direct consequences of the fact that h.s. harmonics transform as totally symmetric tensors of $SO(6)$ and that the rest of irreps (e.g. antisymmetric ones) are absent from the right hand side.

Unlike the $SO(6)$ Clebsch–Gordan coefficients, the $SU(3)$ Clebsch–Gordan coefficients are quite well known in various bases, Refs. [51–57,63], including the multiplicity problem of $SU(3)$ to $SO(3)$ reduction, Ref. [43,44,46,47]. We shall exploit this fact in the following subsection.

8.3. Matrix elements as functions of SU(3) Clebsch–Gordan coefficients

We have already shown that the $U(3)$ subgroup appears as an intermediary step in the reduction $SO(6) \supset U(3) \supset U(1) \otimes SO(3)$ that dictates our choice of basis. On the other hand the $SU(3)$ subgroup does not introduce any new quantum numbers into the hyper-spherical harmonics labels (K, Q, L, m) – the reason being that $SU(3)$ irreducible representations contained within the $SO(6)$ harmonics are already fully determined by the integers K and Q. Namely, coordinates X_i^+, Eq. (6) transform as the fundamental $SU(3)$ unitary irreducible representation (UIR) of the $U(3)$ subgroup, i.e., one box Young diagram, whereas the coordinates X_i^- transform as the conjugate representation of $U(3)$, i.e., a two-box column Young diagram. Therefore, an $SU(3)$ representation with given K and Q corresponds to a Young diagram with K boxes in the first row, and $(\mathrm{K}-Q)/2$ boxes in the second one.[4]

Moreover, it easy to see, Ref. [14,15], that three-particle hyper-spherical harmonics can be also viewed as functions on the $SU(3)/SU(2)$ coset space. In decomposition of the Hilbert space of square integrable functions over $SU(3)/SU(2)$ coset space into $SU(3)$ irreducible components, each $SU(3)$ UIR appears exactly once. In other words, there is exactly one three particle harmonic transforming as each of the $SU(3)$ UIR's (and the state vectors within), i.e. there is one set of harmonics for each allowed combination of K and Q (where by "allowed", we mean $|Q| \leq \mathrm{K}$ and $Q \equiv \mathrm{K} \pmod 2$). That much ought to be clear already from our construction of hyper-spherical harmonics, as the polynomials with given degrees K and Q cannot constitute more than one copy of the same $SU(3)$ UIR, and yet there are polynomials for each combination of (K, Q) (one of the ways to verify the first part of this statement is to note that there is only one polynomial with the highest weight for that representation).[5] In accordance with this, an $SO(6)$ symmetric tensor representation of order K decomposes to $SU(3)$ UIR's (K, Q), $Q = -\mathrm{K}, -\mathrm{K}+2, \ldots \mathrm{K}$, each UIR appearing only once in the decomposition – as the sum of dimensions of $SU(3)$ irreducible representations building up the order-K harmonics, confirms:

$$\begin{aligned} n_{\mathrm{K}} &= \sum_{Q=-\mathrm{K},-\mathrm{K}+2,\ldots \mathrm{K}} \dim(\mathrm{K}, Q) = \sum_{Q=-\mathrm{K},-\mathrm{K}+2,\ldots \mathrm{K}} \frac{1}{8}(\mathrm{K}+2)(\mathrm{K}-Q+2)(\mathrm{K}+Q+2) \\ &= \frac{1}{8}(\mathrm{K}+2) \sum_{Q=-\mathrm{K},-\mathrm{K}+2,\ldots \mathrm{K}} (\mathrm{K}+2)^2 - (Q)^2 = \frac{1}{12}(\mathrm{K}+1)(\mathrm{K}+2)^2(\mathrm{K}+3), \end{aligned}$$

where $\dim(\mathrm{K}, Q) = \frac{1}{8}(\mathrm{K}+2)(\mathrm{K}-Q+2)(\mathrm{K}+Q+2)$ and indeed $\dim_{O(6)}(\mathrm{K}) = n_{\mathrm{K}} = \frac{(\mathrm{K}+3)!(\mathrm{K}+2)}{12\mathrm{K}!} = \frac{1}{12}(\mathrm{K}+3)(\mathrm{K}+2)^2(\mathrm{K}+1)$.

The embeddings of $S_3 \otimes SO(3)$ and $SU(3)$ multiplets in $SO(6)$ multiplets is illustrated in Table 3. Note that each (complete) $SU(3)$ irreducible representations appears once and only once among all the H.H.s – there is no repetition. Moreover, one must be careful not to double-count the self-conjugate irreps, such as the $(1, 1) = 8$ one.

The same multiplicity issue that we have dealt with in Sect. 5 has been studied in the $SU(3)$ context. It is well known, see Refs. [43,46,47], that $SU(3)$ representations in general have non-trivial multiplicity w.r.t. decomposition into $SO(3)$ subgroup representations, see e.g. the two $\mathrm{K} = 4$, $L = 2$ states $\in$ **27**-plet in Table 3.[6] Different multiplicity lifting operators were considered in the literature, and the corresponding bases constructed Refs. [43,44,46,47].

4 Notice that giving the pair K, Q differs from the usual $SU(3)$ irreducible representation labeling described by two integers (p, q) that correspond to a Young diagram with $p+q$ boxes in the first row, and q boxes in the second one, see e.g. Ref. [62].)

5 Another way to prove this property is by invoking the Frobenius reciprocity theorem, as in Ref. [14,15].

6 Note that non-trivial multiplicities do not exist for $L = 0, 1$ states, thus also explaining why these two series of states have been explicitly constructed in Refs. [5,8,19].

Table 3
The labels of distinct $\mathrm{K} \leq 4$ h.s. harmonics $\mathcal{Y}^{\mathrm{K},Q,\nu}_{L,m}$ (three-body states, with allowed orbital angular momentum value L; only $L = m$ labels are shown). The correspondence between the S_3 permutation group irreps. and SU(6)$_{FS}$ symmetry multiplets of the three-quark system: $S \leftrightarrow 56$, $A \leftrightarrow 20$ and $M \leftrightarrow 70$. The number of states in an SO(3) irrep is $n_L = 2L + 1$, and the number of states in an O(6) multiplet/H.H. is $n_{\mathrm{K}} = \sum \dim_{S_3} \times n_L = \sum \dim_{SU(3)}$, where the sum goes over all the O(3) multiplets, or over all SU(3) multiplets contained in the O(6) H.H. The sign $\pm$ in front of Q values (second column) denotes mixing of mutually conjugate $SU(3)$ representations, which occurs as a consequence of $S_3 \not\subset U(3)$.

K	$(\mathrm{K}, Q, L, m, \nu)$	$[SU(6), L^P]$	S_3 irrep.	$SU(3)$ irrep.	(λ_1, λ_2)	$\dim_{S_3} \times n_L$	N_{K}
0	$(0,0,0,0,0)$	$[56, 0^+]$	S	$\mathbf{1}$	(0,0)	1×1	1
1	$(1,\pm1,1,1,\mp1)$	$[70, 1^-]$	M	$\mathbf{3}, \bar{\mathbf{3}}$	(1,0), (0,1)	2×3	6
2	$(2,\pm2,0,0,0)$	$[70, 0^+]$	M	$\mathbf{6}, \bar{\mathbf{6}}$	(2,0), (0,2)	2×1	20
2	$(2,\mp2,2,2,\pm3)$	$[70, 2^+]$	M	$\mathbf{6}, \bar{\mathbf{6}}$	(2,0), (0,2)	2×5	20
2	$(2,0,2,2,0)$	$[56, 2^+]$	S	$\mathbf{8}$	(1,1)	1×5	20
2	$(2,0,1,1,0)$	$[20, 1^+]$	A	$\mathbf{8}$	(1,1)	1×3	20
3	$(3,\mp3,1,1,\pm1)$	$[20, 1^-]$	A	$\mathbf{10}, \overline{\mathbf{10}}$	(3,0), (0,3)	1×3	50
3	$(3,\mp3,1,1,\pm1)$	$[56, 1^-]$	S	$\mathbf{10}, \overline{\mathbf{10}}$	(3,0), (0,3)	1×3	50
3	$(3,\pm3,3,3,\mp6)$	$[56, 3^-]$	S	$\mathbf{10}, \overline{\mathbf{10}}$	(3,0), (0,3)	1×7	50
3	$(3,\pm3,3,3,\mp6)$	$[20, 3^-]$	A	$\mathbf{10}, \overline{\mathbf{10}}$	(3,0), (0,3)	1×7	50
3	$(3,\pm1,1,1,\pm3)$	$[70, 1^-]$	M	$\mathbf{15}, \overline{\mathbf{15}}$	(2,1), (1,2)	2×3	50
3	$(3,\mp1,2,2,\pm5)$	$[70, 2^-]$	M	$\mathbf{15}, \overline{\mathbf{15}}$	(2,1), (1,2)	2×5	50
3	$(3,\mp1,3,3,\pm2)$	$[70, 3^-]$	M	$\mathbf{15}, \overline{\mathbf{15}}$	(2,1), (1,2)	2×7	50
4	$(4,\pm4,0,0,0)$	$[70, 0^+]$	M	$\mathbf{15}', \overline{\mathbf{15}}'$	(4,0), (0,4)	2×1	105
4	$(4,\pm4,2,2,\mp3)$	$[70', 2^+]$	M	$\mathbf{15}', \overline{\mathbf{15}}'$	(4,0),(0,4)	2×5	105
4	$(4,\mp4,4,4,\pm10)$	$[70', 4^+]$	M	$\mathbf{15}', \overline{\mathbf{15}}'$	(4,0), (0,4)	2×9	105
4	$(4,\pm2,1,1,\pm2)$	$[70, 1^+]$	M	$\mathbf{24}, \overline{\mathbf{24}}$	(3,1), (1,3)	2×3	105
4	$(4,\pm2,2,2,\pm2)$	$[70, 2^+]$	M	$\mathbf{24}, \overline{\mathbf{24}}$	(3,1),(1,3)	2×5	105
4	$(4,\mp2,3,3,\pm13)$	$[70, 3^+]$	M	$\mathbf{24}, \overline{\mathbf{24}}$	(3,1),(1,3)	2×7	105
4	$(4,\mp2,4,4,\pm5)$	$[70, 4^+]$	M	$\mathbf{24}, \overline{\mathbf{24}}$	(3,1), (1,3)	2×9	105
4	$(4,0,0,0,0)$	$[56, 0^+]$	S	$\mathbf{27}$	(2,2)	1×1	105
4	$\left(4,0,2,2,\mp\sqrt{105}\right)$	$[56, 2^+]$	S	$\mathbf{27}$	(2,2)	1×5	105
4	$\left(4,0,2,2,\mp\sqrt{105}\right)$	$[20, 2^+]$	A	$\mathbf{27}$	(2,2)	1×5	105
4	$(4,0,3,3,0)$	$[20, 3^+]$	A	$\mathbf{27}$	(2,2)	1×7	105
4	$(4,0,4,4,0)$	$[56, 4^+]$	S	$\mathbf{27}$	(2,2)	1×9	105

Of special interest is the fact that the $SU(3)$ Clebsch–Gordan coefficients are known for certain choices of multiplicity lifting operator, [51–57,63]. This means that an $SU(3)$ analogon of formula, Eq. (74), has substantial practical utility (Appendix B.3):

$$\int_{\mathcal{M}} \mathcal{Y}^{*\mathrm{K}Q\nu}_{L,m}(X)\mathcal{Y}^{\mathrm{K}_1Q_1\nu_1}_{L_1,m_1}(X)\mathcal{Y}^{\mathrm{K}_2Q_2\nu_2}_{L_2,m_2}(X)dX^3 =$$

$$= \frac{1}{\sqrt{V_{\mathcal{M}}}}\sqrt{\frac{\dim(\mathrm{K}_1,Q_1)\dim(\mathrm{K}_2,Q_2)}{\dim(\mathrm{K},Q)}}\, C^{\{\mathrm{K}_1,Q_1\}\ \ \{\mathrm{K}_2,Q_2\}\ \ \{\mathrm{K},Q\}}_{\{L_1,m_1,\nu_1\}\{L_2,m_2,\nu_2\}\{L,m,\nu\}} C^{\{\mathrm{K}_1,Q_1\}\ \{\mathrm{K}_2,Q_2\}\ \{\mathrm{K},Q\}}_{\ \ 0_H\ \ \ \ \ \ \ 0_H\ \ \ \ \ \ \ 0_H}, \quad (76)$$

where X_i are the complex coordinates defined in Eq. (6), the integration is over $SU(3)/SU(2)$ coset space which is parameterized by X subjected to constraint $|X| = 1$, 0_H is the unique vector from the given $SU(3)$ UIR that is invariant w.r.t. $SU(2)$ subgroup, $V_{\mathcal{M}} = \pi^3$, and dim(K, Q) $= \frac{1}{8}(\mathrm{K}+2)(\mathrm{K}-Q+2)(\mathrm{K}+Q+2)$. Since $SU(3)$ Clebsch–Gordan coefficients appearing in the above formula are known entities, whose numerical evaluation has been carefully studied in the literature, Refs. [54,55,57], in various $SU(3)$ bases, (and no longer the obscure CG coefficients of the $SO(6)$ group), it means that Eq. (76) can be used to evaluate matrix elements, Eq. (72), in practice.

Furthermore, another nice thing is that this formula can be made even more explicit, that is, less dependent on knowledge of (tables of) $SU(3)$ CG coefficients:

1) The first Clebsch–Gordan coefficient in Eq. (76) factors into an $SO(3)_{\rm rot}$ part and the reduced Clebsch–Gordan coefficient, see pp. 360 in Ref. [63],

$$C^{\{\mathrm{K}_1,Q_1\}\ \ \{\mathrm{K}_2,Q_2\}\ \ \{\mathrm{K},Q\}}_{\{L_1,m_1,v_1\}\{L_2,m_2,v_2\}\{L,m,v\}} = C^{L_1L_2L}_{m_1m_2m} C_r{}^{\{\mathrm{K}_1,Q_1\}\ \ \{\mathrm{K}_2,Q_2\}\ \{\mathrm{K},Q\}}_{\{L_1,v_1\}\ \ \{L_2,v_2\}\ \ \{L,v\}}. \tag{77}$$

When applied to Eq. (72), the $SO(3)$ coefficient becomes simply a product of two Kronecker delta functions: $C^{L'0L''}_{m'0m''} = \delta_{L',L''}\delta_{m',m''}$. The values of the reduced Clebsch–Gordan coefficient can be found in literature, Refs. [54,55,57], at least in their numerical form.

2) The second of the two SU(3) Clebsch–Gordan coefficients (the "reduced matrix element") in Eq. (76) does not depend on the L, m, v labels and can be explicitly evaluated in closed form, as follows,

$$\begin{aligned} C^{\{\mathrm{K}_1,Q_1\}\ \{\mathrm{K}_2,Q_2\}\ \{\mathrm{K},Q\}}_{\ \ 0_H\ \ \ \ \ \ \ \ 0_H\ \ \ \ \ \ \ \ 0_H} &= \Bigg(A_0^{\mathrm{K}_1,Q} A_0^{\mathrm{K}_2,Q} A_0^{\mathrm{K},Q} \sqrt{\frac{\pi^3 \dim(\mathrm{K},Q)}{\dim(\mathrm{K}_1,Q_1)\dim(\mathrm{K}_2,Q_2)}} \\ &\times \sum_{\mathrm{K}_1'=|Q_1|,|Q_1|+2,\ldots}^{\mathrm{K}_1} \ \sum_{\mathrm{K}_2'=|Q_2|,|Q_2|+2,\ldots}^{\mathrm{K}_2} \ \sum_{\mathrm{K}'=|Q|,|Q|+2,\ldots}^{\mathrm{K}} \Pi^{\mathrm{K}_1,Q_1}_{\mathrm{K}_1'} \Pi^{\mathrm{K}_2,Q_2}_{\mathrm{K}_2'} \Pi^{\mathrm{K},-Q}_{\mathrm{K}'} \\ &\times \frac{2\pi^3}{(\frac{\mathrm{K}_1'+\mathrm{K}_2'+\mathrm{K}'}{2}+1)(\frac{\mathrm{K}_1'+\mathrm{K}_2'+\mathrm{K}'}{2}+2)} \delta_{Q_1+Q_2,Q} \Bigg)^{\frac{1}{2}} \end{aligned} \tag{78}$$

where

$$A_0^{\mathrm{K},Q} = (-1)^{\frac{\mathrm{K}-|Q|}{2}} \Bigg(\sum_{\mathrm{K}_1,\mathrm{K}_2=|Q|,|Q|+2,\ldots}^{\mathrm{K}} \Pi^{\mathrm{K},Q}_{\mathrm{K}_1} \Pi^{\mathrm{K},Q}_{\mathrm{K}_2} \frac{2\pi^3}{(\frac{\mathrm{K}_1+\mathrm{K}_2}{2}+1)(\frac{\mathrm{K}_1+\mathrm{K}_2}{2}+2)} \Bigg)^{-\frac{1}{2}} \tag{79}$$

and

$$\Pi^{\mathrm{K},Q}_{\mathrm{K}'} = \prod_{\mathrm{K}''=|Q|,|Q|+2,\ldots}^{\mathrm{K}'-2} \left(1 - \frac{(\mathrm{K}+2)^2 - Q^2}{(\mathrm{K}''+2)^2 - Q^2} \right). \tag{80}$$

Combining the simplifications shown above, we finally obtain:

$$\begin{aligned} \langle \mathcal{Y}^{\mathrm{K}''}_{[\mathrm{m}'']}(\Omega_5) | \mathcal{Y}^{\mathrm{K}Qv}_{00}(\alpha,\phi) | \mathcal{Y}^{\mathrm{K}'}_{[\mathrm{m}']}(\Omega_5) \rangle &= \frac{1}{\sqrt{\pi^3}} \sqrt{\frac{\dim(\mathrm{K},Q)\dim(\mathrm{K}',Q')}{\dim(\mathrm{K}'',Q'')}} \\ &\times \delta_{L',L''}\delta_{m',m''} C_r{}^{\{\mathrm{K},Q\}\ \ \{\mathrm{K}',Q'\}\ \{\mathrm{K}'',Q''\}}_{\{0,0\}\ \ \{L',v'\}\ \ \{L'',v''\}} C^{\{\mathrm{K},Q\}\ \{\mathrm{K}',Q'\}\ \{\mathrm{K}'',Q''\}}_{\ \ 0_H\ \ \ \ \ \ \ \ 0_H\ \ \ \ \ \ \ \ \ 0_H}, \end{aligned} \tag{81}$$

The remaining $SU(3)/SO(3)$ reduced Clebsch–Gordan coefficient cannot be further simplified/analytically evaluated in general, for two reasons: i) its value depends on the choice of multiplicity lifting operator; ii) due to conclusions of Moshinsky et al. [47], irrespectively of the choice of multiplicity lifting operator some of the values inevitably have to be numerical. Therefore, Eq. (81), together with Eqs. (78), (79), (80), represents our final result.

These results can be difficult to interpret without specifying the phase- and other conventions, both for the state vectors (both $SO(6)$ and $SU(3)$) and for the Clebsch–Gordan coefficients.

8.4. Conventions for $SO(6)$ and $SU(3)$ states and Clebsch–Gordan coefficients

It is well known [64] that the Clebsch–Gordan coefficients of $SO(3)$, and/or $SU(2)$ can be chosen to be real under a specific convention on the phase of the state vectors. Our construction of the $SO(6)$ hyperspherical harmonics in Sects. 3, 4 provides a specific set of conventions that lead to the reality of matrix elements in Eq. (72). That, in turn, does not guarantee the reality of the $SO(6)$ Clebsch–Gordan coefficients, but the converse statement [7] is assured. We shall henceforth assume that (all of) the relevant $SO(6)$ Clebsch–Gordan coefficients are real with our choice of phases for the $SO(6)$ hyperspherical harmonics.

Even after such a convention is imposed, however, there is one "remnant" sign ambiguity left over, in the form of the overall sign of the Clebsch–Gordan matrix, which is conventionally fixed, say by the Condon–Shortley definition. Such a remnant sign ambiguity does not affect the Gaunt formula either in the $SO(3)$, or in the $SO(6)$ case, because the right-hand sides of Eqs. (73), (74) are bi-linear in their respective Clebsch–Gordan coefficients. Similarly, we shall assume[8] that the $SU(3)$ Clebsch–Gordan coefficients appearing in Eq. (81) are real, as well, which is a common/standard convention, see Refs. [51,53–55,57].

The above relation between the $SO(6)$ and $SU(3)$ Clebsch–Gordan coefficients calls for yet another comment about the conventions adopted here. When dealing with Clebsch–Gordan coefficients of an $SU(n)$ group with $n > 2$, there is also a certain freedom related to the so called "outer multiplicity", Ref. [62]. This freedom amounts to the fact that not only some of the phase factors depend on the adopted phase conventions (as in the $SU(2)$ case), but that there are other more general phase ambiguities. Namely, the Kronecker product of two irreducible $SU(n)$ representations contains more often than not, a multiplicity, i.e., the reduction of the Kronecker product (the "Clebsch–Gordan series") contains more than one copy of one and the same irreducible representation, Ref. [62]. Consequently, the $SU(3)$ Clebsch–Gordan tables must generally have a number of (different) coefficients for each triplet ($\{\mathrm{K}, Q\}$ $\{\mathrm{K}', Q'\}$ $\{\mathrm{K}'', Q''\}$) of $SU(3)$ state labels.[9] Therefore, in cases when the Clebsch–Gordan series contains outer multiplicity, an additional label, or some other method of identification must be specified to distinguish between otherwise identical copies of irreducible representations. This ambiguity "spills over" into the evaluation of Eq. (81), as follows.

In the case when existing programs (e.g. Refs. [55,57]) for the evaluation ("tables") of $SU(3)$ Clebsch–Gordan coefficients feature more than one coefficient for the given triplet ($\{\mathrm{K}, Q\}$ $\{\mathrm{K}', Q'\}$ $\{\mathrm{K}'', Q''\}$) of $SU(3)$ labels, the question arises how to tell which one corresponds to the decomposition of hyperspherical harmonics and should be plugged into Eq. (81), that is, how does one tell which one (of sometimes many) copies of the same UIR appearing after the reduction of the Kronecker product is relevant to application here?

The answer to this question, and the behavior of the decomposition of the three-particle hyperspherical harmonics product into $SU(3)$ UIR's, are governed by the very value of the coefficient $C^{\{\mathrm{K},Q\}\ \{\mathrm{K}',Q'\}\ \{\mathrm{K}'',Q''\}}_{\ 0_H\quad\ 0_H\quad\ 0_H}$: The value of this coefficient is nonzero in only one, of many, copies of

[7] The reality of the $SO(6)$ Clebsch–Gordan coefficients guarantees the reality of matrix elements in Eq. (72).

[8] As explained earlier, checking this convention would be equivalent to an explicit calculation of all $SO(6)$ coefficients, which is beyond our scope here.

[9] Up till now, we had worked under the assumption that only one well-defined Clebsch–Gordan coefficient exists for each triplet of $SU(3)$ state labels. The basis for this assumption was the fact that such an outer multiplicity does not appear in the context of products of $SO(6)$ hyperspherical harmonics, as we already noted that in the decomposition of the Hilbert space of functions over $SU(3)/SU(2)$ cosets, each $SU(3)$ UIR appears exactly once.

the same $SU(3)$ UIR, and that fact effectively implies that there is no outer multiplicity in the Clebsch–Gordan decomposition of Kronecker products of three-particle hyperspherical harmonics.

In practice, this means the following. Let there be n copies of UIR (K, Q) appearing in the product of (K_1, Q_1) and (K_2, Q_2), distinguished by the value of an additional label $\alpha = 1, 2, \dots n$ (obviously, there is freedom in choosing orthonormal bases within the sum of these irreducible spaces). A look-up in a Clebsch–Gordon table in a such case generally reveals the $n \geq 2$ values for the coefficients $C^{\{\mathrm{K},Q\}\ \{\mathrm{K}',Q'\}\ \{\mathrm{K}'',Q''\}}_{\ 0_H\qquad 0_H\qquad 0_H}$: $C_0^1, C_0^2, \dots C_0^n$. Nevertheless, the "overall magnitude" of these values $\sqrt{(C_0^1)^2 + (C_0^2)^2 + \cdots + (C_0^n)^2}$ is independent of any conventions and must coincide with the value determined by Eq. (78). Let us, for convenience call $C_r^1, C_r^2, \dots C_r^n$ the multiple values of the reduced CG coefficient $C_r{}^{\{\mathrm{K},Q\}\ \{\mathrm{K}',Q'\}\ \{\mathrm{K}'',Q''\}}_{\{0,0\}\ \ \{L',v'\}\ \ \{L'',v''\}}$ needed in Eq. (81). Then, the proper value of the CG coefficient to be plugged in Eq. (81) is the one "projected" on the relevant UIR, where $C^{\{\mathrm{K},Q\}\ \{\mathrm{K}',Q'\}\ \{\mathrm{K}'',Q''\}}_{\ 0_H\qquad 0_H\qquad 0_H}$ coefficient is nonzero:

$$C_r{}^{\{\mathrm{K},Q\}\ \{\mathrm{K}',Q'\}\ \{\mathrm{K}'',Q''\}}_{\{0,0\}\ \ \{L',v'\}\ \ \{L'',v''\}} = \frac{(C_r^1 C_0^1 + C_r^2 C_0^2 + \cdots + C_r^n C_0^n)}{\sqrt{(C_0^1)^2 + (C_0^2)^2 + \cdots + (C_0^n)^2}}.$$

This formula further assumes that the value of $C^{\{\mathrm{K},Q\}\ \{\mathrm{K}',Q'\}\ \{\mathrm{K}'',Q''\}}_{\ 0_H\qquad 0_H\qquad 0_H}$ coefficient is taken to be positive, which is in agreement with our convention, that amounts to fixing the positive sign in Eq. (78), when compared with Eq. (B.21).

Explicit applications of the described method by using programs for the evaluation of $SU(3)$ Clebsch–Gordan coefficients, Refs. [54,55,57], confirm the above analysis (taking into account additional sign conventions used by the authors of these tables) and yield results identical with the values obtained by integration of explicit expressions for HSH (Appendix A).

9. Summary, discussion and conclusions

In summary, we have constructed the three-body permutation symmetric $SO(6)$ hyperspherical harmonics in three spatial dimensions. We used a method of constructing homogeneous harmonic polynomials that are labeled by SO(6) group's indices. In this way we arrived at the subgroup chain $S_3 \otimes SO(3)_{rot} \subset O(2) \otimes SO(3)_{rot} \subset U(3) \rtimes S_2 \subset SO(6)$ (where $SO_{rot}(3)$ is the group of spatial rotations, $O(2)$ is the group of so-called "democracy" transformations where the permutation group S_3 is a (discrete) subgroup of the so-called "kinematic rotations", Ref. [2], or equivalently the "democracy" transformation (continuous) group $O(2)$, Ref. [12]).

The constructed symmetrized hyperspherical harmonics can be used to reformulate the three-body Schrödinger equation in three spatial dimensions, [48,50]. Then we calculated a certain type of integrals that appear in the three-body Schrödinger equation. We reduced these integrals at first to a product of two $SO(6)$ Clebsch–Gordan coefficients, and then to the product of two $SU(3)$ Clebsch–Gordan coefficients, that are readily available in the literature, Refs. [51–57,63], at least in their numerical form.

Next we give a brief discussion of some previous attempts at constructing symmetrized three-body hyperspherical harmonics and their relation to ours.

The first attempts to systematically construct all hyperspherical wave functions with well defined permutational symmetry go back to Aquilanti et al., Ref. [18] and subsequent papers.

They used something they called “tree pruning” technique (that appears to be related to the “tree” method of Vilenkin, Kuznetsov, and Smorodinskii, Ref. [61], see below) to obtain certain partial results in both 2D and 3D. Ultimately, this approach has not yielded a definitive answer – for a recent review of this approach see Ref. [28].

Second, Barnea and Novoselsky constructed hyperspherical wave functions with orthogonal and permutational symmetry in Ref. [20], where they used “a recursive algorithm for the (efficient) construction of N-body wave functions that belong to a given irreducible representation (irrep) of the orthogonal group and are at the same time characterized by a well-defined permutational symmetry.” Whereas, in the final instance, Barnea and Novoselsky’s work ought to be related to ours, we note the following basic differences: a) their work is based on a different subgroup chain, with a missing link in comparison with ours: $O(3) \otimes S_3 \subset O(6)$ vs. our $U(1) \otimes SO(3)_{rot} \subset U(3) \subset SO(6)$; b) theirs is an essentially recursive-numerical method relying on the knowledge of tables of the symmetric group S_3 Clebsch–Gordan coefficients, whereas ours is a group-theoretical approach; c) their S_3 hyperspherical states are expressed in terms of S_2 hyperspherical states, that are coupled, via the “tree” method of Vilenkin, Kuznetsov, and Smorodinskii, Ref. [61]; whereas ours makes no reference to any two-body substate; d) they evaluated only matrix elements of two-body operators, whereas we can treat all kinds of three-body operators.

Third, it ought to be said that Wang and Kuppermann Ref. [23] used symbolic algebra programs to calculate certain three- and four-body hyperspherical harmonics that were used in atomic and molecular physics. Their method does not seem to be based on a clearly defined algorithm, or group structure, however.

Last, but not least, we re-iterate that Dragt, Ref. [6] had used the $SU(3) \subset SO(6)$ chain of algebras to label three-particle scattering states as early as 1965, with follow-up work in Refs. [8,14,15], albeit with an emphasis on the applications to three-body decays, as opposed to our emphasis on applications to the three-body bound-state problem. Of course, these results, particularly those in the all but forgotten/unnoticed Refs. [14,15], must be closely related to ours, but this relation is not straightforward to see, due to their use of different kinematic variables (Dalitz–Fabri coordinates vs. hyperspherical angles) and to different construction methods. Ref. [15] in particular shows tables of some ($L \leq 6$, K ≤ 12) harmonics and their matrix elements. Ref. [14] on the other hand, gives “classification of three particle states according to an orthonormal $SU(3) \supset SO(3)$ basis” and then some. These authors simply could not evaluate the triple-harmonic matrix elements in terms of $SU(3)$ Clebsch–Gordan coefficients without the benefit of more recent developments, such as those in Refs. [54,55].

We conclude that we have provided (all of) the previously missing pieces that are sufficient for a complete reduction and efficient solution of the three-body Schrödinger equation, as in Refs. [48,50], and thus we opened the doors to simplified algebraic and faster numerical solutions to many specific physical three-body problems.

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Appendix A. Integrals over the $SO(6)$ hyper-sphere

Let $f(x_1, x_2, \dots x_n)$ be a homogeneous function of degree K of n coordinates $x_i, i = 1, 2, \dots n$:

$$f_{\mathrm{K}}(ax_1, ax_2, \dots ax_n) = a^{\mathrm{K}} f_{\mathrm{K}}(x_1, x_2, \dots x_n), \quad a \neq 0. \tag{A.1}$$

In particular, it holds $f_{\mathrm{K}}(x_1, x_2, \dots x_n) = R^{\mathrm{K}} f(\frac{x_1}{R}, \frac{x_2}{R}, \dots \frac{x_n}{R})$, where R is the radius in the n-dimensional space: $R = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$. The weighted integral of the function $f_{\mathrm{K}}(x_1, x_2, \dots x_n)$ by the factor e^{-aR^2} over the entire volume of the hyper-space V can be evaluated via an unit-radius hyper-sphere Ω surface integral as follows:

$$\int_V f_{\mathrm{K}}(x_1, x_2, \dots x_n) e^{-aR^2} dV = \int_0^\infty R^{\mathrm{K}} e^{-aR^2} R^{n-1} dR \int_\Omega f(\frac{x_1}{R}, \frac{x_2}{R}, \dots \frac{x_n}{R}) d\Omega. \tag{A.2}$$

This leads to the following connection of hyper-sphere surface and volume integrals:

$$\int_\Omega f(\frac{x_1}{R}, \frac{x_2}{R}, \dots \frac{x_n}{R}) d\Omega = \frac{\int_V f_{\mathrm{K}}(x_1, x_2, \dots x_n) e^{-aR^2} dV}{\frac{1}{2} a^{-\frac{\mathrm{K}+n}{2}} \Gamma(\frac{\mathrm{K}+n}{2})}. \tag{A.3}$$

In the particular case when the function is a homogeneous polynomial $f_{\mathrm{K}}(x_1, x_2, \dots x_n) = x_1^{\mathrm{K}_1} x_2^{\mathrm{K}_2} \cdots x_n^{\mathrm{K}_n}$ with $\sum_i \mathrm{K}_i = \mathrm{K}$ the right-hand-side can be explicitly evaluated:

$$\int_\Omega \frac{1}{R^{\mathrm{K}}} x_1^{\mathrm{K}_1} x_2^{\mathrm{K}_2} \cdots x_n^{\mathrm{K}_n} d\Omega = \frac{\prod_{i=1}^n \frac{1+(-1)^{\mathrm{K}_i}}{2} a^{-\frac{\mathrm{K}_i+1}{2}} \Gamma(\frac{\mathrm{K}_i+1}{2})}{\frac{1}{2} a^{-\frac{\mathrm{K}+n}{2}} \Gamma(\frac{\mathrm{K}+n}{2})} = 2 \frac{\prod_{i=1}^n \frac{1+(-1)^{\mathrm{K}_i}}{2} \Gamma(\frac{\mathrm{K}_i+1}{2})}{\Gamma(\frac{\mathrm{K}+n}{2})}. \tag{A.4}$$

If the hyper-spherical integrand on the left side is evaluated on the unit radius sphere, the $\frac{1}{R^{\mathrm{K}}}$ factor can be obviously left out.

Appendix B. Three-body $SO(6)$ hyperspherical harmonics as Wigner D-functions on the $SU(3)/SU(2)$ coset space

B.1. Spherical harmonics as Wigner D-functions

Spherical (hyperspherical) harmonics in a generalized sense are functions on a given manifold $\mathcal{M}$ of interest that have certain given properties w.r.t. action of some group G that acts transitively on $\mathcal{M}$ (here we will constrain to the cases when G is a compact Lie group). More precisely, (hyper)spherical harmonics are usually required to transform as basis vectors of unitary irreducible representations of G, and are given labels accordingly. E.g. harmonic function $\mathcal{Y}_m^L(\Omega), \Omega \in \mathcal{M}$ is required to transform under action of G as basis vector m of the irreducible representation L of G:

$$g : \mathcal{Y}_m^L(\Omega) \to \sum_{m'} D_{m'm}^L(g) \mathcal{Y}_{m'}^L(\Omega), \qquad g \in G, \tag{B.1}$$

where $D_{m'm}^L(g)$ is matrix representing element g in irreducible representation given by label(s) L:

$$D^L_{m'm}(g) = \langle{}^L_{m'}|D(g)|{}^L_m\rangle. \tag{B.2}$$

Seen as functions of g, matrix elements $D^L_{m'm}(g)$ are known as Wigner D-functions.

We proceed by considering square integrable functions on the G group manifold. Hilbert space of such functions we denote as $\mathcal{L}^2(G)$ and its vectors are of the form:

$$|\phi\rangle = \int_G \phi(g)\,|g\rangle\,dg, \qquad g \in G, \tag{B.3}$$

where ϕ is a square integrable function on the manifold of group G parameters, $|g\rangle$ are (generalized) basis vectors and dg is the normalized Haar measure.

The usual (left) action of the group G elements is given by:

$$g'\,|\phi\rangle = g' \int \phi(g)\,|g\rangle\,dg = \int \phi(g)\,|g'g\rangle dg, \qquad g', g \in G. \tag{B.4}$$

Note that this action induces the following transformation of the function ϕ:

$$g' : \phi(g) \to \phi'(g) = \phi(g'^{-1}g). \tag{B.5}$$

The functions belonging to $\mathcal{L}^2(G)$ that transform according to (B.1) are (complex conjugate) Wigner D-functions $D^{*L}_{mk}(g)$, as can be easily verified:

$$g' : D^{*L}_{mk}(g) \to D^{*L}_{mk}(g'^{-1}g) = \sum_{m'} D^{*L}_{mm'}{}^{-1}(g')D^{*L}_{m'k}(g) = \sum_{m'} D^L_{m'm}(g')D^{*L}_{m'k}(g). \tag{B.6}$$

Notice that index k above, that corresponds to a vector of UIR L, is arbitrary.

If the manifold $\mathcal{M}$ were to coincide with the group G manifold, then it is these functions $D^{*L}_{mk}(g)$ that would play the role of the (hyper) spherical harmonics. However, in most cases of practical interest that is not the case. Instead, each point Ω of the manifold $\mathcal{M}$ has nontrivial stabilizer (isotropy) subgroup $H_\Omega \subset G$ such that $H_\Omega \cdot \Omega = \Omega$. As we already assumed that action of G is transitive on $\mathcal{M}$, all stabilizer subgroups H_Ω are mutually conjugate, isomorphic to some group H, and the manifold $\mathcal{M}$ is homogeneous space G/H.

To obtain hyperspherical harmonics on $\mathcal{M}$ we need functions of Ω that transform according to (B.1). Let $g(\Omega)$ be a mapping from $\mathcal{M}$ to a set of coset representatives in G. Then arbitrary group element g can be written as $g(\Omega)h$ for some $\Omega \in \mathcal{M}$ and $h \in H$. To obtain (hyper)spherical functions on $\mathcal{M}$ in the sense of definition (B.1), we can use the arbitrariness of choice of vector k in Eq. (B.6), by choosing it to be invariant w.r.t. action of subgroup H, i.e. $k = 0_H$ where:

$$D(h)|{}^L_{0_H}\rangle = |{}^L_{0_H}\rangle, \forall h \in H. \tag{B.7}$$

Now the Wigner D-function $D^{*L}_{mk}(g)$ reduces to a function on the coset space $\mathcal{M}$ by:

$$D^{*L}_{m0_H}(g) = D^{*L}_{m0_H}(g(\Omega)h) = \langle{}^L_m|D(g(\Omega))D(h)|{}^L_{0_H}\rangle = D^{*L}_{m0_H}(g(\Omega)) \equiv D^{*L}_{m0_H}(\Omega). \tag{B.8}$$

Therefore, functions $D^{*L}_{m0_H}(\Omega)$, where $|{}^L_{0_H}\rangle$ is invariant w.r.t. H action, are functions on manifold $\mathcal{M} = G/H$ that properly transform, in the sense of (B.1), under the action of group G, i.e. transform as vector $|{}^L_m\rangle$ of the UIR labeled by L. Thus we can establish proportionality between generalized (hyper)spherical harmonics and Wigner D-functions:

$$\mathcal{Y}^L_m(\Omega) = N(L)D^{*L}_{m0_H}(\Omega), \tag{B.9}$$

with $N(L)$ being a proportionality constant possibly dependent on L. In addition to transformation properties (B.1), it is usual to require that a (hyper)spherical harmonic should be normalized to unity. From:

$$\begin{aligned}\int\limits_{\mathcal{M}} \mathcal{Y}^{*L}_m(\Omega)\mathcal{Y}^{L'}_{m'}(\Omega)d\Omega &= \int\limits_{\mathcal{M}} N(L)N(L')D^L_{m0_H}(\Omega)D^{*L}_{m'0_H}(\Omega)d\Omega \\ &= \frac{N(L)N(L')}{V_H}\int\limits_{\mathcal{M}}\int\limits_{H} D^L_{m0_H}(g(\Omega)h)D^{*L}_{m'0_H}(g(\Omega)h)d\Omega dh \\ &= \frac{N(L)N(L')}{V_H}\int\limits_{G} D^L_{m0_H}(g)D^{*L}_{m'0_H}(g)dg \\ &= \frac{N(L)N(L')}{V_H}\frac{V_G}{dim(L)}\delta_{JJ'}\delta_{mm'},\end{aligned} \tag{B.10}$$

where $d\Omega$ is a measure on $\mathcal{M}$, V_H and V_G are volumes of H and G group manifolds (corresponding to measures dh and dg) and $dim(L)$ is the dimension of the UIR L, it follows $N(L) = \sqrt{\frac{dim(L)}{V_{\mathcal{M}}}}$. (Compactness of the subgroup H and of the manifold M follows from the presumed compactness of G.) Finally we conclude:

$$\mathcal{Y}^L_m(\Omega) = \sqrt{\frac{dim(L)}{V_{\mathcal{M}}}}D^{*L}_{m0_H}(\Omega), \tag{B.11}$$

while, of course, arbitrariness of choice of overall complex phase in the definition necessarily remains.

B.2. Integral of three (hyper)spherical harmonics

Expressing (hyper)spherical harmonics via Wigner D-functions allows immediate evaluation of integral of three (or more) of h.s. harmonics in terms of group G Clebsch–Gordan coefficients. Namely:

$$\begin{aligned}&\int\limits_{\mathcal{M}} \mathcal{Y}^{*L}_m(\Omega)\mathcal{Y}^{L_1}_{m_1}(\Omega)\mathcal{Y}^{L_2}_{m_2}(\Omega)d\Omega \\ &\quad= \sqrt{\frac{\dim(L)\dim(L_1)\dim(L_2)}{V^3_{\mathcal{M}}}}\int\limits_{\mathcal{M}} D^L_{m0_H}(\Omega)D^{*L_1}_{m_10_H}(\Omega)D^{*L_2}_{m_20_H}(\Omega)d\Omega \\ &\quad= \frac{1}{V_H}\sqrt{\frac{\dim(L)\dim(L_1)\dim(L_2)}{V^3_{\mathcal{M}}}}\int\limits_{G} D^L_{m0_H}(g)D^{*L_1}_{m_10_H}(g)D^{*L_2}_{m_20_H}(g)dg \\ &\quad= \frac{1}{\sqrt{V_{\mathcal{M}}}}\sqrt{\frac{\dim(L_1)\dim(L_2)}{\dim(L)}}C^{L_1\ L_2\ L}_{m_1\ m_2\ m}C^{L_1\ L_2\ L}_{0_H\ 0_H\ 0_H},\end{aligned} \tag{B.12}$$

where $C^{L_1\ L_2\ L}_{m_1\ m_2\ m}$ denotes a G group Clebsch–Gordan coefficient and the well known formulas for integral of three Wigner D-functions were used.

B.3. Application to three particle systems

It is known, Ref. [14,15], that three-particle hyper-spherical harmonics can be viewed as functions on the $SU(3)/SU(2)$ coset space. Thus all of the previous considerations directly apply to this case. Of particular importance for the evaluation of interaction matrix elements is the formula for integral of three three-particle h.s. harmonics, that now takes the form:

$$\int\limits_{\mathcal{M}} \mathcal{Y}^{*\mathrm{K}Qv}_{L,m}(X)\mathcal{Y}^{\mathrm{K}_1Q_1v_1}_{L_1,m_1}(X)\mathcal{Y}^{\mathrm{K}_2Q_2v_2}_{L_2,m_2}(X)dX^3$$
$$= \frac{1}{\sqrt{V_{\mathcal{M}}}}\sqrt{\frac{\dim(\mathrm{K}_1,Q_1)\dim(\mathrm{K}_2,Q_2)}{\dim(\mathrm{K},Q)}}\, C^{\{\mathrm{K}_1,Q_1\}\ \{\mathrm{K}_2,Q_2\}\ \{\mathrm{K},Q\}}_{\{L_1,m_1,v_1\}\{L_2,m_2,v_2\}\{L,m,v\}} C^{\{\mathrm{K}_1,Q_1\}\ \{\mathrm{K}_2,Q_2\}\ \{\mathrm{K},Q\}}_{0_H\quad\ 0_H\quad\ 0_H}, \tag{B.13}$$

where X_i are the complex coordinates (6), $\dim(\mathrm{K},Q) = \frac{1}{8}(\mathrm{K}+2)(\mathrm{K}-Q+2)(\mathrm{K}+Q+2)$ and $V_{\mathcal{M}} = \pi^3$. The $SU(3)$ Clebsch–Gordan coefficients appearing in the above formula are well known entities, whose (numerical) evaluation is well studied in the literature [57], in various $SU(3)$ bases.

The second of the two Clebsch–Gordan coefficients can be evaluated in the following way. From (B.13) it follows that

$$|C^{\{\mathrm{K}_1,Q_1\}\ \{\mathrm{K}_2,Q_2\}\ \{\mathrm{K},Q\}}_{0_H\quad\ 0_H\quad\ 0_H}|$$
$$= \left(\sqrt{V_{\mathcal{M}}}\sqrt{\frac{\dim(\mathrm{K},Q)}{\dim(\mathrm{K}_1,Q_1)\dim(\mathrm{K}_2,Q_2)}}\int\limits_{\mathcal{M}} \mathcal{Y}^{*\{\mathrm{K},Q\}}_{0_H}(X)\mathcal{Y}^{\{\mathrm{K}_1,Q_1\}}_{0_H}(X)\mathcal{Y}^{\{\mathrm{K}_2,Q_2\}}_{0_H}(X)dX^3\right)^{\frac{1}{2}}. \tag{B.14}$$

Now, $\mathcal{Y}^{\{\mathrm{K},Q\}}_{0_H}(X)$ is an $SU(3)$ harmonic on $SU(3)/SU(2)$ that is invariant w.r.t. $SU(2)$ subgroup, and for the sake of concreteness, let it be the subgroup that nontrivially acts on indices 1 and 2. That means that $\mathcal{Y}^{\{\mathrm{K},Q\}}_{0_H}(X)$ is of the form:

$$\mathcal{Y}^{\{\mathrm{K},Q\}}_{0_H}(X) = \sum_{\mathrm{K}'=|Q|,|Q|+2,\ldots}^{\mathrm{K}} A^{\mathrm{K},Q}_{\mathrm{K}'}\cdot (X_3^+)^{\frac{\mathrm{K}'+Q}{2}}(X_3^-)^{\frac{\mathrm{K}'-Q}{2}}, \tag{B.15}$$

where sum over K' goes only over odd or over even integers and $A^{\mathrm{K},Q}_{\mathrm{K}'}$ are algebraic coefficients that can be determined from two requirements: i) the functions $\mathcal{Y}^{\{\mathrm{K},Q\}}_{0_H}(X)$ must be $SO(6)$ harmonics, i.e., $\Delta\mathcal{Y}^{\{\mathrm{K},Q\}}_{0_H}(X) = 0$ and ii) they have to be normalized to unity, i.e.,

$$\int\limits_{\mathcal{M}} \mathcal{Y}^{*\{\mathrm{K},Q\}}_{0_H}(X)\mathcal{Y}^{\{\mathrm{K},Q\}}_{0_H}(X)dX^3 = 1. \tag{B.16}$$

From the first requirement one obtains:

$$A^{\mathrm{K},Q}_{\mathrm{K}'} = A^{\mathrm{K},Q}_{0}\Pi^{\mathrm{K},Q}_{\mathrm{K}'} \tag{B.17}$$

where $A^{\mathrm{K},Q}_{0}$ is a remaining constant, yet to be determined, and

$$\Pi^{\mathrm{K},Q}_{\mathrm{K}'} = \prod_{\mathrm{K}''=|Q|,|Q|+2,\ldots}^{\mathrm{K}'-2}\left(1 - \frac{(\mathrm{K}+2)^2 - Q^2}{(\mathrm{K}''+2)^2 - Q^2}\right), \tag{B.18}$$

where the product over K″, yet again takes only every other integer value, even or odd, as the case may be.

By plugging Eq. (B.15) into Eq. (B.16) and by using integration formulas from Appendix A one determines the absolute value of the remaining constant as

$$|A_0^{\mathrm{K},Q}| = \left(\sum_{\mathrm{K}_1,\mathrm{K}_2=|Q|,|Q|+2,\dots}^{\mathrm{K}} \Pi^{\mathrm{K},Q}_{\mathrm{K}_1} \Pi^{\mathrm{K},Q}_{\mathrm{K}_2} \frac{2\pi^3}{(\frac{\mathrm{K}_1+\mathrm{K}_2}{2}+1)(\frac{\mathrm{K}_1+\mathrm{K}_2}{2}+2)} \right)^{-\frac{1}{2}}. \tag{B.19}$$

Although the phase factor will turn out to be irrelevant for our purposes, for completeness' sake we note that the phase of the constant $A_0^{\mathrm{K},Q}$ and thus the overall phase of the harmonic $\mathcal{Y}_{0_H}^{\{\mathrm{K},Q\}}(X)$ can be recovered from the consistency requirement obtained from (B.11) by taking Ω to be the coset of the unit group element:

$$\int_{\mathcal{M}} \sqrt{\frac{\dim(\mathrm{K},Q)}{\pi^3}} \mathcal{Y}_{0_H}^{*\{\mathrm{K},Q\}}(X) \mathcal{Y}_{L,m}^{\mathrm{K}'Q'\nu}(X) dX^3 = \mathcal{Y}_{L,m}^{\mathrm{K}'Q'\nu}(X)\Big|_{X_1=0,X_2=0,X_3=1}. \tag{B.20}$$

Thus we conclude that $A_0^{\mathrm{K},Q} = (-1)^{\frac{\mathrm{K}-|Q|}{2}} |A_0^{\mathrm{K},Q}|$.

Finally, we plug the resulting expressions for $\mathcal{Y}_{0_H}^{\{\mathrm{K},Q\}}(X)$ into Eq. (B.14) and obtain:

$$\begin{aligned} |C^{\{\mathrm{K}_1,Q_1\}\ \{\mathrm{K}_2,Q_2\}\ \{\mathrm{K},Q\}}_{\ 0_H\ \ \ \ \ \ \ 0_H\ \ \ \ \ \ \ 0_H}| = & \Bigg(A_0^{\mathrm{K}_1,Q} A_0^{\mathrm{K}_2,Q} A_0^{\mathrm{K},Q} \sqrt{\frac{\pi^3 \dim(\mathrm{K},Q)}{\dim(\mathrm{K}_1,Q_1)\dim(\mathrm{K}_2,Q_2)}} \cdot \\ & \sum_{\mathrm{K}_1'=|Q_1|,|Q_1|+2,\dots}^{\mathrm{K}_1} \sum_{\mathrm{K}_2'=|Q_2|,|Q_2|+2,\dots}^{\mathrm{K}_2} \sum_{\mathrm{K}'=|Q|,|Q|+2,\dots}^{\mathrm{K}} \Pi^{\mathrm{K}_1,Q_1}_{\mathrm{K}_1'} \Pi^{\mathrm{K}_2,Q_2}_{\mathrm{K}_2'} \Pi^{\mathrm{K},-Q}_{\mathrm{K}'} \\ & \times \frac{2\pi^3}{(\frac{\mathrm{K}_1'+\mathrm{K}_2'+\mathrm{K}'}{2}+1)(\frac{\mathrm{K}_1'+\mathrm{K}_2'+\mathrm{K}'}{2}+2)} \delta_{Q_1+Q_2,Q} \Bigg)^{\frac{1}{2}}. \end{aligned} \tag{B.21}$$

Appendix C. Tables of hyper-spherical harmonics

Below we explicitly list all hyper-spherical harmonics up to K = 6, where multiplicity is resolved by using the operator Eq. (37). We list only the harmonics with $m = L$ and $Q \geq 0$, as the rest can be easily obtained by acting on them with standard lowering operators Eq. (60) and by using the permutation symmetry Eqs. (63), (65): $\mathcal{Y}_{L,m}^{\mathrm{K}Q\nu}(\lambda,\rho) = (-1)^{\mathrm{K}-L}\mathcal{Y}_{L,m}^{\mathrm{K}-Q-\nu}(\lambda,-\rho)$. We write the K ≤ 3 harmonics in both complex spherical and Jacobi coordinates.

Of the harmonics listed below, expressions for $\mathcal{Y}_{0,0}^{4,0,0}(X)$ and $\mathcal{Y}_{0,0}^{6,6,0}(X)$ can be compared with the corresponding expressions in [5], where a few particular examples for $L = 0$ are explicitly shown. After taking into account the differences in notation it is easily verified that the expressions coincide.

$$\mathcal{Y}_{0,0}^{0,0,0}(X) = \frac{1}{\pi^{3/2}}$$

$$\mathcal{Y}_{1,1}^{1,1,-1}(X) = \frac{\sqrt{\frac{3}{2}}X_+^+}{\pi^{3/2}R} = \frac{\sqrt{\frac{3}{2}}(\lambda_1 + i(\lambda_2 + \rho_1 + i\rho_2))}{\pi^{3/2}\sqrt{\lambda_1^2+\lambda_2^2+\lambda_3^2+\rho_1^2+\rho_2^2+\rho_3^2}}$$

$$\mathcal{Y}_{1,1}^{2,0,0}(X) = \frac{\sqrt{3}\left(X_+^- X_0^+ - X_+^+ X_0^-\right)}{\pi^{3/2}R^2} = \frac{2\sqrt{3}(\lambda_3(\rho_2 - i\rho_1) + i(\lambda_1 + i\lambda_2)\rho_3)}{\pi^{3/2}\left(\lambda_1^2+\lambda_2^2+\lambda_3^2+\rho_1^2+\rho_2^2+\rho_3^2\right)}$$

$$\mathcal{Y}^{2,0,0}_{2,2}(X) = \frac{\sqrt{3}X^{+}_{+}X^{-}_{+}}{\pi^{3/2}R^2} = \frac{\sqrt{3}\left(\lambda_1 + i\left(\lambda_2 + \rho_1 + i\rho_2\right)\right)\left(\lambda_1 + i\lambda_2 - i\rho_1 + \rho_2\right)}{\pi^{3/2}\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \rho_1^2 + \rho_2^2 + \rho_3^2\right)}$$

$$\mathcal{Y}^{2,2,0}_{0,0}(X) = \frac{\sqrt{2}\left|X^{+}\right|^2}{\pi^{3/2}R^2}$$
$$= \frac{\sqrt{2}\left(2i\lambda_1\rho_1 + 2i\lambda_2\rho_2 + 2i\lambda_3\rho_3 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2 - \rho_1^2 - \rho_2^2 - \rho_3^2\right)}{\pi^{3/2}\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \rho_1^2 + \rho_2^2 + \rho_3^2\right)}$$

$$\mathcal{Y}^{2,2,-3}_{2,2}(X) = \frac{\sqrt{\frac{3}{2}}\left(X^{+}_{+}\right)^2}{\pi^{3/2}R^2} = \frac{\sqrt{\frac{3}{2}}\left(\lambda_1 + i\left(\lambda_2 + \rho_1 + i\rho_2\right)\right)^2}{\pi^{3/2}\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \rho_1^2 + \rho_2^2 + \rho_3^2\right)}$$

$$\mathcal{Y}^{3,1,3}_{1,1}(X) = \frac{\sqrt{6}\left(X^{-}_{+}\left|X^{+}\right|^2 - \frac{1}{2}R^2X^{+}_{+}\right)}{\pi^{3/2}R^3}$$
$$= \frac{\sqrt{6}}{\pi^{3/2}\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \rho_1^2 + \rho_2^2 + \rho_3^2\right)^{3/2}}$$
$$\times\Bigg(\left(\lambda_1 + i\lambda_2 - i\rho_1 + \rho_2\right)\left(\left(\lambda_1 + i\rho_1\right)^2 + \left(\lambda_2 + i\rho_2\right)^2 + \left(\lambda_3 + i\rho_3\right)^2\right)$$
$$- \frac{1}{2}\left(\lambda_1 + i\left(\lambda_2 + \rho_1 + i\rho_2\right)\right)\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \rho_1^2 + \rho_2^2 + \rho_3^2\right)\Bigg)$$

$$\mathcal{Y}^{3,1,-5}_{2,2}(X) = \frac{\sqrt{5}X^{+}_{+}\left(X^{-}_{+}X^{+}_{0} - X^{+}_{+}X^{-}_{0}\right)}{\pi^{3/2}R^3}$$
$$= \frac{2\sqrt{5}\left(\lambda_1 + i\left(\lambda_2 + \rho_1 + i\rho_2\right)\right)\left(\lambda_3\left(\rho_2 - i\rho_1\right) + i\left(\lambda_1 + i\lambda_2\right)\rho_3\right)}{\pi^{3/2}\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \rho_1^2 + \rho_2^2 + \rho_3^2\right)^{3/2}}$$

$$\mathcal{Y}^{3,1,-2}_{3,3}(X) = \frac{\sqrt{15}\left(X^{+}_{+}\right)^2X^{-}_{+}}{2\pi^{3/2}R^3}$$
$$= \frac{\sqrt{15}\left(\lambda_1 + i\left(\lambda_2 + \rho_1 + i\rho_2\right)\right)^2\left(\lambda_1 + i\lambda_2 - i\rho_1 + \rho_2\right)}{2\pi^{3/2}\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \rho_1^2 + \rho_2^2 + \rho_3^2\right)^{3/2}}$$

$$\mathcal{Y}^{3,3,-1}_{1,1}(X) = \frac{\sqrt{3}X^{+}_{+}\left|X^{+}\right|^2}{\pi^{3/2}R^3}$$
$$= \frac{\sqrt{3}\left(\lambda_1 + i\left(\lambda_2 + \rho_1 + i\rho_2\right)\right)\left(2i\lambda_1\rho_1 + 2i\lambda_2\rho_2 + 2i\lambda_3\rho_3 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2 - \rho_1^2 - \rho_2^2 - \rho_3^2\right)}{\pi^{3/2}\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \rho_1^2 + \rho_2^2 + \rho_3^2\right)^{3/2}}$$

$$\mathcal{Y}^{3,3,-6}_{3,3}(X) = \frac{\sqrt{5}\left(X^{+}_{+}\right)^3}{2\pi^{3/2}R^3} = \frac{\sqrt{5}\left(\lambda_1 + i\left(\lambda_2 + \rho_1 + i\rho_2\right)\right)^3}{2\pi^{3/2}\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \rho_1^2 + \rho_2^2 + \rho_3^2\right)^{3/2}}$$

$$\mathcal{Y}^{4,0,0}_{0,0}(X) = -\frac{\sqrt{3}\left(R^4 - 2\left|X^{-}\right|^2\left|X^{+}\right|^2\right)}{\pi^{3/2}R^4}$$

$$\mathcal{Y}^{4,0,-\sqrt{105}}_{2,2}(X) = \frac{-12\sqrt{14}R^2X^+_+X^-_+ + \sqrt{105\left(11-\sqrt{105}\right)}\left(X^-_+\right)^2\left|X^+\right|^2 + \sqrt{105\left(11+\sqrt{105}\right)}\left(X^+_+\right)^2\left|X^-\right|^2}{14\pi^{3/2}R^4}$$

$$\mathcal{Y}^{4,0,\sqrt{105}}_{2,2}(X) = \frac{-12\sqrt{14}R^2X^+_+X^-_+ + \sqrt{105\left(11+\sqrt{105}\right)}\left(X^-_+\right)^2\left|X^+\right|^2 + \sqrt{105\left(11-\sqrt{105}\right)}\left(X^+_+\right)^2\left|X^-\right|^2}{14\pi^{3/2}R^4}$$

$$\mathcal{Y}^{4,0,0}_{3,3}(X) = \frac{3\sqrt{5}X^+_+X^-_+\left(X^-_+X^+_0 - X^+_+X^-_0\right)}{2\pi^{3/2}R^4}$$

$$\mathcal{Y}^{4,0,0}_{4,4}(X) = \frac{3\sqrt{\frac{5}{2}}\left(X^+_+\right)^2\left(X^-_+\right)^2}{2\pi^{3/2}R^4}$$

$$\mathcal{Y}^{4,2,2}_{1,1}(X) = \frac{3\left(X^-_+X^+_0 - X^+_+X^-_0\right)\left|X^+\right|^2}{\pi^{3/2}R^4}$$

$$\mathcal{Y}^{4,2,2}_{2,2}(X) = \frac{\sqrt{\frac{3}{7}}X^+_+\left(5X^-_+\left|X^+\right|^2 - 2R^2X^+_+\right)}{\pi^{3/2}R^4}$$

$$\mathcal{Y}^{4,2,-13}_{3,3}(X) = \frac{3\sqrt{\frac{5}{2}}\left(X^+_+\right)^2\left(X^-_+X^+_0 - X^+_+X^-_0\right)}{2\pi^{3/2}R^4}$$

$$\mathcal{Y}^{4,2,-5}_{4,4}(X) = \frac{\sqrt{15}\left(X^+_+\right)^3X^-_+}{2\pi^{3/2}R^4}$$

$$\mathcal{Y}^{4,4,0}_{0,0}(X) = \frac{\sqrt{3}\left|X^+\right|^4}{\pi^{3/2}R^4}$$

$$\mathcal{Y}^{4,4,-3}_{2,2}(X) = \frac{3\sqrt{\frac{5}{14}}\left(X^+_+\right)^2\left|X^+\right|^2}{\pi^{3/2}R^4}$$

$$\mathcal{Y}^{4,4,-10}_{4,4}(X) = \frac{\sqrt{15}\left(X^+_+\right)^4}{4\pi^{3/2}R^4}$$

$$\mathcal{Y}^{5,1,-3}_{1,1}(X) = -\frac{\sqrt{3}\left(X^+_+\left(R^4 - 3\left|X^-\right|^2\left|X^+\right|^2\right) + R^2X^-_+\left|X^+\right|^2\right)}{\pi^{3/2}R^5}$$

$$\mathcal{Y}^{5,1,13}_{2,2}(X) = \frac{\sqrt{\frac{5}{2}}\left(X^+_+X^-_0 - X^-_+X^+_0\right)\left(R^2X^+_+ - 3X^-_+\left|X^+\right|^2\right)}{\pi^{3/2}R^5}$$

$$\mathcal{Y}^{5,1,-7-\sqrt{241}}_{3,3}(X) = \frac{\sqrt{\frac{35}{723}}X^+_+\left(\sqrt{482-26\sqrt{241}}X^-_+\left(3X^-_+\left|X^+\right|^2 - 2R^2X^+_+\right) + \sqrt{2651+163\sqrt{241}}X^+_+\left(X^+_+\left|X^-\right|^2 - R^2X^-_+\right)\right)}{4\pi^{3/2}R^5}$$

$$\mathcal{Y}^{5,1,-7+\sqrt{241}}_{3,3}(X) = \frac{\sqrt{\frac{35}{723}}X^+_+\left(\sqrt{482+26\sqrt{241}}X^-_+\left(3X^-_+\left|X^+\right|^2 - 2R^2X^+_+\right) + \sqrt{2651-163\sqrt{241}}X^+_+\left(X^+_+\left|X^-\right|^2 - R^2X^-_+\right)\right)}{4\pi^{3/2}R^5}$$

$$\mathcal{Y}_{4,4}^{5,1,-8}(X) = \frac{3\sqrt{7}\left(X_+^+\right)^2 X_+^- \left(X_+^- X_0^+ - X_+^+ X_0^-\right)}{2\pi^{3/2} R^5}$$

$$\mathcal{Y}_{5,5}^{5,1,-3}(X) = \frac{\sqrt{105}\left(X_+^+\right)^3 \left(X_+^-\right)^2}{4\pi^{3/2} R^5}$$

$$\mathcal{Y}_{1,1}^{5,3,5}(X) = \frac{\sqrt{\frac{3}{2}}\left|X^+\right|^2 \left(3X_+^- \left|X^+\right|^2 - 2R^2 X_+^+\right)}{\pi^{3/2} R^5}$$

$$\mathcal{Y}_{2,2}^{5,3,-3}(X) = \frac{\sqrt{15}X_+^+ \left(X_+^- X_0^+ - X_+^+ X_0^-\right)\left|X^+\right|^2}{\pi^{3/2} R^5}$$

$$\mathcal{Y}_{3,3}^{5,3,0}(X) = -\frac{\sqrt{\frac{35}{6}}\left(X_+^+\right)^2 \left(R^2 X_+^+ - 3X_+^- \left|X^+\right|^2\right)}{2\pi^{3/2} R^5}$$

$$\mathcal{Y}_{4,4}^{5,3,-24}(X) = \frac{\sqrt{21}\left(X_+^+\right)^3 \left(X_+^- X_0^+ - X_+^+ X_0^-\right)}{2\pi^{3/2} R^5}$$

$$\mathcal{Y}_{5,5}^{5,3,-9}(X) = \frac{\sqrt{\frac{105}{2}}\left(X_+^+\right)^4 X_+^-}{4\pi^{3/2} R^5}$$

$$\mathcal{Y}_{1,1}^{5,5,-1}(X) = \frac{3X_+^+ \left|X^+\right|^4}{\sqrt{2}\pi^{3/2} R^5}$$

$$\mathcal{Y}_{3,3}^{5,5,-6}(X) = \frac{\sqrt{\frac{35}{3}}\left(X_+^+\right)^3 \left|X^+\right|^2}{2\pi^{3/2} R^5}$$

$$\mathcal{Y}_{5,5}^{5,5,-15}(X) = \frac{\sqrt{\frac{21}{2}}\left(X_+^+\right)^5}{4\pi^{3/2} R^5}$$

$$\mathcal{Y}_{1,1}^{6,0,0}(X) = -\frac{\sqrt{6}\left(X_+^- X_0^+ - X_+^+ X_0^-\right)\left(R^4 - 3\left|X^-\right|^2 \left|X^+\right|^2\right)}{\pi^{3/2} R^6}$$

$$\mathcal{Y}_{2,2}^{6,0,0}(X) = -\frac{\sqrt{\frac{10}{7}}\left(X_+^+ X_+^- \left(R^4 - 7\left|X^-\right|^2 \left|X^+\right|^2\right) + 2R^2 \left(X_+^-\right)^2 \left|X^+\right|^2 + 2R^2 \left(X_+^+\right)^2 \left|X^-\right|^2\right)}{\pi^{3/2} R^6}$$

$$\mathcal{Y}_{3,3}^{6,0,-3\sqrt{105}}(X) = \frac{\left(X_+^- X_0^+ - X_+^+ X_0^-\right)\left(-8\sqrt{5}R^2 X_+^+ X_+^- + \left(7\sqrt{5} - 3\sqrt{21}\right)\left(X_+^-\right)^2 \left|X^+\right|^2 + \left(7\sqrt{5} + 3\sqrt{21}\right)\left(X_+^+\right)^2 \left|X^-\right|^2\right)}{2\sqrt{6}\pi^{3/2} R^6}$$

$$\mathcal{Y}_{3,3}^{6,0,3\sqrt{105}}(X) = \frac{\left(X_+^- X_0^+ - X_+^+ X_0^-\right)\left(-8\sqrt{5}R^2 X_+^+ X_+^- + \left(7\sqrt{5} + 3\sqrt{21}\right)\left(X_+^-\right)^2 \left|X^+\right|^2 + \left(7\sqrt{5} - 3\sqrt{21}\right)\left(X_+^+\right)^2 \left|X^-\right|^2\right)}{2\sqrt{6}\pi^{3/2} R^6}$$

$$\mathcal{Y}_{4,4}^{6,0,-\sqrt{385}}(X) = \frac{\sqrt{\frac{3}{154}}X_+^+ X_+^- \left(-6\sqrt{70}R^2 X_+^+ X_+^- + 7\sqrt{23 - \sqrt{385}}\left(X_+^-\right)^2 \left|X^+\right|^2 + 7\sqrt{23 + \sqrt{385}}\left(X_+^+\right)^2 \left|X^-\right|^2\right)}{\pi^{3/2} R^6}$$

$$\mathcal{Y}_{4,4}^{6,0,\sqrt{385}}(X)$$
$$= \frac{\sqrt{\frac{3}{154}} X_+^+ X_+^- \left(-6\sqrt{70} R^2 X_+^+ X_+^- + 7\sqrt{23+\sqrt{385}} \left(X_+^-\right)^2 \left|X^+\right|^2 + 7\sqrt{23-\sqrt{385}} \left(X_+^+\right)^2 \left|X^-\right|^2\right)}{\pi^{3/2} R^6}$$

$$\mathcal{Y}_{5,5}^{6,0,0}(X) = \frac{\sqrt{105} \left(X_+^+\right)^2 \left(X_+^-\right)^2 \left(X_+^- X_0^+ - X_+^+ X_0^-\right)}{2\pi^{3/2} R^6}$$

$$\mathcal{Y}_{6,6}^{6,0,0}(X) = \frac{\sqrt{35} \left(X_+^+\right)^3 \left(X_+^-\right)^3}{2\pi^{3/2} R^6}$$

$$\mathcal{Y}_{0,0}^{6,2,0}(X) = \frac{6\left|X^-\right|^2 \left|X^+\right|^4 - 4R^4 \left|X^+\right|^2}{\pi^{3/2} R^6}$$

$$\mathcal{Y}_{2,2}^{6,2,4-\sqrt{193}}(X)$$
$$= -\frac{\sqrt{\frac{5}{1158}}}{21\pi^{3/2} R^6} \Bigg(\sqrt{5597+355\sqrt{193}} X_+^+ \left(X_+^+ \left(5R^4 - 21\left|X^-\right|^2 \left|X^+\right|^2\right) + 12R^2 X_+^- \left|X^+\right|^2\right)$$
$$+ \sqrt{4439-305\sqrt{193}} \left(24R^2 X_+^+ X_+^- \left|X^+\right|^2 - 21\left(X_+^-\right)^2 \left|X^+\right|^4 - 4R^4 \left(X_+^+\right)^2\right)\Bigg)$$

$$\mathcal{Y}_{2,2}^{6,2,4+\sqrt{193}}(X)$$
$$= \frac{\sqrt{\frac{5}{1158}}}{21\pi^{3/2} R^6} \Bigg(\sqrt{4439+305\sqrt{193}} \left(-24R^2 X_+^+ X_+^- \left|X^+\right|^2 + 21\left(X_+^-\right)^2 \left|X^+\right|^4 + 4R^4 \left(X_+^+\right)^2\right)$$
$$- \sqrt{5597-355\sqrt{193}} X_+^+ \left(X_+^+ \left(5R^4 - 21\left|X^-\right|^2 \left|X^+\right|^2\right) + 12R^2 X_+^- \left|X^+\right|^2\right)\Bigg)$$

$$\mathcal{Y}_{3,3}^{6,2,8}(X) = \frac{X_+^+ \left(X_+^+ X_0^- - X_+^- X_0^+\right) \left(2R^2 X_+^+ - 7X_+^- \left|X^+\right|^2\right)}{\pi^{3/2} R^6}$$

$$\mathcal{Y}_{4,4}^{6,2,-40}(X) = \frac{3\sqrt{\frac{5}{506}} \left(X_+^+\right)^2 \left(-20R^2 X_+^+ X_+^- + 7\left(X_+^-\right)^2 \left|X^+\right|^2 + 14\left(X_+^+\right)^2 \left|X^-\right|^2\right)}{\pi^{3/2} R^6}$$

$$\mathcal{Y}_{4,4}^{6,2,6}(X) = \frac{\sqrt{\frac{21}{23}} \left(X_+^+\right)^2 \left(-8R^2 X_+^+ X_+^- + 12\left(X_+^-\right)^2 \left|X^+\right|^2 + \left(X_+^+\right)^2 \left|X^-\right|^2\right)}{2\pi^{3/2} R^6}$$

$$\mathcal{Y}_{5,5}^{6,2,-19}(X) = \frac{\sqrt{\frac{35}{2}} \left(X_+^+\right)^3 X_+^- \left(X_+^- X_0^+ - X_+^+ X_0^-\right)}{\pi^{3/2} R^6}$$

$$\mathcal{Y}_{6,6}^{6,2,-7}(X) = \frac{\sqrt{105} \left(X_+^+\right)^4 \left(X_+^-\right)^2}{4\pi^{3/2} R^6}$$

$$\mathcal{Y}_{1,1}^{6,4,4}(X) = \frac{3\sqrt{2} \left(X_+^- X_0^+ - X_+^+ X_0^-\right) \left|X^+\right|^4}{\pi^{3/2} R^6}$$

$$\mathcal{Y}_{2,2}^{6,4,4}(X) = \frac{\sqrt{\frac{10}{21}} X_+^+ \left|X^+\right|^2 \left(7X_+^- \left|X^+\right|^2 - 4R^2 X_+^+\right)}{\pi^{3/2} R^6}$$

$$\mathcal{Y}_{3,3}^{6,4,-11}(X) = \frac{\sqrt{\frac{35}{2}}\left(X_+^+\right)^2\left(X_+^- X_0^+ - X_+^+ X_0^-\right)\left|X^+\right|^2}{\pi^{3/2}R^6}$$

$$\mathcal{Y}_{4,4}^{6,4,-3}(X) = \frac{\sqrt{\frac{3}{11}}\left(X_+^+\right)^3\left(7X_+^-\left|X^+\right|^2 - 2R^2X_+^+\right)}{\pi^{3/2}R^6}$$

$$\mathcal{Y}_{5,5}^{6,4,-38}(X) = \frac{\sqrt{\frac{35}{2}}\left(X_+^+\right)^4\left(X_+^- X_0^+ - X_+^+ X_0^-\right)}{2\pi^{3/2}R^6}$$

$$\mathcal{Y}_{6,6}^{6,4,-14}(X) = \frac{\sqrt{\frac{21}{2}}\left(X_+^+\right)^5 X_+^-}{2\pi^{3/2}R^6}$$

$$\mathcal{Y}_{0,0}^{6,6,0}(X) = \frac{2\left|X^+\right|^6}{\pi^{3/2}R^6}$$

$$\mathcal{Y}_{2,2}^{6,6,-3}(X) = \frac{\sqrt{5}\left(X_+^+\right)^2\left|X^+\right|^4}{\pi^{3/2}R^6}$$

$$\mathcal{Y}_{4,4}^{6,6,-10}(X) = \frac{\sqrt{\frac{105}{11}}\left(X_+^+\right)^4\left|X^+\right|^2}{2\pi^{3/2}R^6}$$

$$\mathcal{Y}_{6,6}^{6,6,-21}(X) = \frac{\sqrt{7}\left(X_+^+\right)^6}{4\pi^{3/2}R^6}$$

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