Yukawa Coupling Unification in Grand Unified Theories

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Abstract

Scalar extensions of the Standard Model of particle physics are well motivated from several theories that attempt to explain phenomena like neutrino oscillations, hypercharge quantization, etc. By adding more scalars, the Renormalization Group Equations of the theory change as they become active species in the interactions. In the case of Grand Unified Theories, a unification of the gauge and Yukawa couplings is expected at a high energy, which could be used as a selection tool in the space of scalar extensions.

Here, the scalar extensions with representations $\{(1,2)_0, (3,1)_0, (\bar{3},1)_0\}$ under the Standard Model gauge group are investigated at 1-loop. By inserting multiple generations at up to two scales between 10^3 GeV and 10^{15} GeV in the Modified Minimal Subtraction scheme, and running the Renormalization Group Equations to the Grand Unified Theory scale $\approx 10^{15}$ GeV, we look for optimal scenarios for gauge and Yukawa unification. A comparison is done against FlexibleSUSY to check that the results are comparable for 2 models; the SM and a model with 2 of each scalar type with pole masses of 10^3 GeV.

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1 Introduction

The Standard Model (SM) of particle physics has recently been completed by the discovery of the Higgs boson with the Large Hadron Collider [1, 2]. While this is a great success of high energy physics, there are still theoretical conundrums and experimental measurements that need mechanisms and explanations. Several theories expect a unification of the forces of the SM, while others extend the SM with new fields, so as to explain observations like neutrino oscillations.

The work done here is to extending the SM by adding new degrees of freedom in the form of extra scalar fields. This is done to see how they contribute towards Yukawa and gauge unification at the Grand Unified Theory (GUT) scale $\approx 10^{15}$ GeV, that is far from what we can achieve with today's experiments. There are numerical tools that are able to do this with a high degree of precision, but are quite slow as they are built with a high degree of flexibility in their design and capabilities. To search a subset of the space of possible extra degrees of freedom quicker, specifically new scalars at 1-loop, a tool is built to rapidly examine new models. With this tool a subset of possible models are investigated.

The outline of the thesis is as follows;

In Section (2) I quickly review basic building blocks of the SM; Quantum Field theory (QFT), Yang-Mills theory, Lie group theory and the Higgs mechanism, leading in the end to a quick formulation of the SM. A quick tour into renormalization is then required to motivate some of the various theoretical questions that face the SM in the current era, and to get the Renormalization Group Equations (RGEs) that are used to investigate these. To motivate this work, we look at some of the problems that are facing the SM in Section (3), and a solution to some of the problems in the form of GUTs. We then go back to renormalization to find the Yukawa beta functions in Section (4), before we look at how the numerical integration of the RGEs is done, uncertainty control and some benchmarks against FlexibleSUSY to check for consistency, in Section (5). The results of some selected scalar extensions are given in Section (6).

Notation

- $\vec{y}$ - 3-dimensional space vector.
- x^μ - 4-dimensional space-time vector.
- $\boldsymbol{Y}$ - represents a matrix.
- $(x, y)_z$ - Gauge group field representation with representations x under SU(3), y under SU(2) and hypercharge z. E.g. the lepton doublet is given as $(1, 2)_{-1/2}$, which is a singlet under SU(3), doublet under SU(2) and has hypercharge $-1/2$.
- $\boldsymbol{N}$ - N-plet representation under a gauge group.
- $\overline{\boldsymbol{N}}$ - conjugate N-plet representation under a gauge group.

Abbreviation

- CKM - Cabibbo–Kobayashi–Maskawa
- GeV - Giga electronvolt
- GUT - Grand Unified Theory
- irrep - Irreducible representation
- $\overline{MS}$ - Modified Minimal Subtraction
- ppm - parts per million
- QCD - Quantum Chromodynamics
- QED - Quantum Electrodynamics
- QTF - Quantum Field Theory
- RGE - Renormalization Group Equation
- RK4 - Runge-Kutta of 4th order
- SM - Standard Model
- SO - Special Orthogonal
- SSM - Spontaneous Symmetry Breaking
- SU - Special Unitary
- SUSY - Supersymmetry

2 Background

The leading theory for getting a grip on the physics at the smallest experimentally accessible scales today is QFT[1]. QFT is a framework which uses at its starting point classical relativistic fields and the known Lagrangian principle to derive the dynamical equations of the system. The typical fields one works with are integer and half-integer spin fields, or more commonly known as **bosons** (or scalars for $s = 0$) ϕ_i and **fermions** ψ_j respectively. Their Lagrangian $\mathcal{L}$ as free, non-interactive, fields is given as

$$\mathcal{L}_\psi = \bar{\psi}(i\partial\!\!\!/ - m_\psi)\psi, \tag{1}$$

$$\mathcal{L}_\phi = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m_\phi^2\phi^2. \tag{2}$$

To get the equations of motion for the fields, the Euler-Lagrange equations are applied which are solved analytically or numerically. To successfully merge this with quantum mechanics, a quantization procedure is done, second canonical quantization, where all the degrees of freedom are expanded into harmonic oscillators and then quantized in the following way

$$\{\psi_a(x), \psi_b^\dagger(y)\} = \delta^{(4)}(x-y)\delta_{ab}, \tag{3}$$

$$\{\psi_a(x), \psi_b(y)\} = \{\psi_a^\dagger(x), \psi_b^\dagger(y\} = 0, \tag{4}$$

and

$$[\phi(x), \pi(y)] = i\delta^{(4)}(x-y), \tag{5}$$

$$[\phi(x), \phi(y)] = [\pi(x), \pi(y)] = 0, \tag{6}$$

where $\pi = \dfrac{\partial \mathcal{L}}{\partial(\partial_0\phi)}$. Again, applying the Euler-Lagrange equation with these considerations we get the well known Klein-Gordon equation for scalars and the Dirac equation for fermions

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2 + m_\phi^2\right)\phi = \left(\partial^2 + m_\phi^2\right)\phi = 0, \tag{7}$$

$$(i\gamma^\mu\partial_\mu - m_\psi)\psi = 0. \tag{8}$$

These respect locality, causality and special relativity, and are together with spin 1 **gauge** fields A^μ responsible for the massive particles that we are today able to experimentally access.

[1]A good amount of the introduction and results in this chapter are based on the QFT bible by Peskin & Schröder [3] .

2.1 Symmetries and Yang-Mills theory

The framework above is powerful, but additional ingredients are needed to build the final theory. Upon making the equations respect special relativity, the Lagrangian is made invariant under space-time transformations, known as Lorentz transformations. This means that if we have a Lagrangian $\mathcal{L}(\psi, \partial\psi)$, with fermion field ψ, and we do a Lorentz transformation $x^\mu \to \Lambda x^\mu$ the fields would also transform in the following way so as to not change the Lagrangian

$$\psi(x^\mu) \to \psi'(x^\mu) = \Lambda\psi(x^\mu), \qquad \mathcal{L} \to \mathcal{L}' = \mathcal{L}. \tag{9}$$

This is known as a symmetry of the Lagrangian and is a central principle in theoretical physics. It stems from Nöethers theorem that for every symmetry of the Lagrangian, there is a conserved quantity. Note that the transformation Λ did not depend on the position, and is known as a global transformation. There are also local transformations that depend on the spacetime position

$$\psi(x^\mu) \to \Lambda(x^\mu)\psi(x^\mu). \tag{10}$$

The transformations Λ above could be a simple phase $exp(i\theta)$, or a more complicated matrix that adhere to a set of rules given by group theory. Let's quickly review the basic group theoretical concepts that we will need to understand and make the calculations down the road.

2.1.1 Group theory

A **group** is a set of elements G together with a binary operation $\cdot$ that have the following properties:

- ***Closure***: For all $a, b \in G$, $a \cdot b = c \in G$
- ***Associativity***: For all $a, b, c \in G$, $a \cdot (b \cdot c) = (a \cdot b) \cdot c$
- ***Identity***: There exists an element $e \in G$ known as the identity such that for all $a \in G$, $a \cdot e = e \cdot a = a$
- ***Inverse***: For each element $a \in G$ there exists an element $b \in G$ such that $a \cdot b = b \cdot a = e$

If we look at the simple phase example above, we see that the set of elements $exp(i\theta)$, with $\theta \in \mathbb{R}$ and the group operation $exp(i\theta_1) \cdot exp(i\theta_2) = exp(i(\theta_1 + \theta_2))$ adhere to the requirements with $e = exp(0i) = 1$ and the inverse of a given element $exp(i\theta)$ is given by $exp(-i\theta)$. If we look at the values of all the $exp(i\theta)$, they form a circle in the complex plane, and this group is known as the **unitary group** of degree 1, U(1) or the circle group.

We can classify groups further based on how the ordering of the binary operation acts, i.e. if

$$[a, b] = a \cdot b - b \cdot a = 0, \tag{11}$$

then the group is **abelian**, otherwise it is **non-abelian**. Again we look at the example of a phase and see that the ordering of two elements does not matter, leading to a classification as a abelian group.

A group can also have a **subgroup**, where there exists a subset $H \subset G$ which under the same binary operation is closed. As an example the even integers under addition form a subgroup of the integers. For all groups, the identity and the group itself form two subgroups, and subgroups that are not equal to these are called **proper** subgroups. Furthermore, we can multiply groups together via direct product and form new groups. A example would be for the groups G, H with the binary operations $(*, \hat{})$we have $G \times H$ in the following way

$$(g_1, h_1) \times (g_2, h_2) = (g_1 * g_2, h_1 \hat{} h_2). \tag{12}$$

The fermion and scalar fields X are not themselves group elements in G, but are acted upon by them. The precise terminology is a **group action** $\phi : G \times X \to X$ such that

$$\begin{aligned} \phi(e, x) &= x, \\ \phi(g, \phi(h, x)) &= \phi(g \cdot h, x). \end{aligned} \tag{13}$$

The smallest set of elements $\{\alpha^a\}$ that can generate the whole group are called the **generators** of the group. For the group of integers under addition $\mathbb{Z}^+$ the element $+1$ is a generator, as by repeated application to itself or repeated application of its inverse gives the whole set of integers. The smallest number of generators of a group is called the **dimension** of a group and has a direct correspondence to the number of gauge bosons in the theory.

The set of generators t^a and the commutation relations between them

$$[t^a, t^b] = f^{abc} t^c, \tag{14}$$

where f^{abc} are known as the **structure constants**, fully describe the properties of the group close to the identity element of the group.

There exists a huge literature of such mathematical structure, and we can make physics theories based on our demands that the Lagrangian should be invariant under this or that group action applied to the fields it contains. If we look at a fermion field that has a Lagrangian that respect a local $U(1)$ symmetry in the form $\psi \to exp(i\alpha(x^\mu)\theta)\psi$, then the extra gauge field that is required to keep the Lagrangian invariant is the vector gauge boson A^μ. It transforms as

$$A^i_\mu \to A^i_\mu + \frac{1}{g}\partial_\mu \alpha^i, \tag{15}$$

where g is the coupling constant. For example, at energies below the SM scale there is one field of this type, which corresponds to the photon. The dynamical field equations of the gauge boson that come out of such a Lagrangian perfectly reproduce the equations of electromagnetism, which is a great success of the theory.

We can further specify the subset of groups of interest to this work and that are also the basis of the SM. These are called Lie Groups.

2.1.2 Lie Groups

A group could, for example, be the set of integers $\mathbb{Z}_n$, a permutation of n elements or a complex phase, as described above. There is a fundamental distinction between the first two and the last, which is the granularity at which one can do the group operation. There is no such thing as a half-permutation, while we can split the phase θ as much as we want. The first are known as **discrete** groups while the latter is known as a **Lie group** and these are the groups that allow infinitesimal transformations[2] of the form

$$\psi \to e^{i\alpha^a t^a}\psi = (1 + i\alpha^a t^a + \mathcal{O}(\alpha^2))\psi, \tag{16}$$

where α^a are the parameters of the hermitian generators t^a. In the example we have already used, the $U(1)$ group, we know that we can expand the exponential on the form

$$exp(i\theta) = 1 + i\theta - \frac{1}{2}\theta^2 + \mathcal{O}(\theta^3). \tag{17}$$

The generators of Lie groups form a vector space called the **Lie Algebra** which contains the properties that we need close to the identity element[3]. If the Lie group is non-abelian and does not have a normal subgroup it is called a **simple group**[4], and if it does but does not contain a $U(1)$ subgroup it is **semi-simple**. There are many interesting simple Lie groups and two common ones are

- **SU(N)** - the groups of $n \times n$ complex unitary matrices with determinant 1. These are

[2]The opposite of discrete groups are continuous groups, of which Lie groups are a subset. These are defined on manifolds with a smooth binary operation.

[3]We are justify studying only the local properties of groups as it is normal not to include terms in the Lagrangian with mass-terms greater that 4 as they are highly suppressed (non-renormalizable), and as the coefficient α^a are small any exponential expansions of Lie groups will contain rather few powers of $\alpha^a t^a$.

[4]This is a simplification, a thorough definition would need a some extra pages.

the ones that are used in the SM, and are of special interest to us. The dimension is $SU(N) = N^2 - 1$. The classical example is the $SU(2)$ group which describe spin 1/2 systems and the generators are the Pauli matrices.

- **SO(N)** - the groups of $n \times n$ real orthogonal matrices with determinant 1. Rotations in Euclidean space are described by the group $SO(3)$, and in the plane by $SO(2)$ as they both leave the length invariant and are orthogonal.

As described above, making the Lagrangian invariant under a abelian group, E.g. $U(1)$, required a single gauge field. In a similar way we can use non-abelian groups as the gauge symmetries giving more structure to the theory. These are known as **Yang-Mills theories** and have a Lagrangian on the form

$$\mathcal{L} = \bar{\psi}(i\not{D})\psi - \frac{1}{4}(F^i_{\mu\nu})^2 - m\bar{\psi}\psi, \tag{18}$$

where $\bar{\psi} = \psi^\dagger\gamma^0$ is the Dirac adjoint, $\not{D} = D_\mu\gamma^\mu$ and $F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu + \epsilon^{ijk}A^j_\mu A^k_\nu$ is the field strength tensor. $D_\mu = \partial_\mu - igA^a_\mu t^a$ is the covariant derivative, where g is a coupling constant between the field ψ and A, and A^i_μ is the gauge field and transform as

$$A^i_\mu \mapsto A^i_\mu + \frac{1}{g}\partial_\mu\alpha^i + f^{abc}A^b_\mu\alpha^c. \tag{19}$$

The most interesting part is the term $\epsilon^{ijk}A^j_\mu A^k_\nu$ which leads to self-interaction among the fields and new phenomena, and was not present for abelian groups.

2.1.3 Representations

A notion that is especially important for model builders are **representations** of groups. It is roughly defined as a map from the group G to the general linear group (GL) on a vector space V

$$\phi : G \to GL(V), \tag{20}$$

where the elements $g \in G$ and $\rho \in GL(V)$ satisfy

$$\rho(g_1 \cdot g_2) = \rho(g_1)\rho(g_2). \tag{21}$$

In physics we often use representations of the groups of interest, and use them to act upon the fields of the theory by the group action defined above. This means that as long as we can keep the structure of (21) we can choose any dimension of V as we would like to. A consequence of this in the group action is the dimensionality of the space X which is acted upon which now needs to be equal to the dimensionality of the vector space V, often called a **n-plet** where

$n = dim(V)$. Groups are often defined in a representation which is called the **fundamental** representation. For $SU(N)$ groups this is the $n \times n$ dimensional matrices as noted above, but one can use larger dimensions also. Looking at $SU(2)$ as an example, in the fundamental representation we have $t^a = \sigma^a/2$, the Pauli matrices

$$t^1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad t^2 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad t^3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{22}$$

which act upon a two dimensional vector called a **doublet** and is often written as **2**. We can also use a 3-dimensional vector space giving the representation matrices

$$t^{1'} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \qquad t^{2'} = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \qquad t^{3'} = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{23}$$

and the fields that are acted upon are now triplets **3**, while the Lagrangian still has the same $SU(2)$ symmetry. If the group is complex[5], then a related representation is the **anti-fundamental** representation. It is the complex conjugate of the fundamental representation and is represented as $\bar{\mathbf{3}}$. In the SM all matter fields are in fundamental or anti-fundamental representations of the gauge groups, but the additional gauge fields reside in the **adjoint** representation in which the generators have the form

$$(t^b)_{ac} = if^{abc}. \tag{24}$$

For simple groups the dimension is then equal to the dimension of this representation on the vector space V. At last there is also the trivial or **singlet** representation **1** which is 1-dimensional, i.e. all $t^a = 0$. It tells us that the fields does not transform under this group, and is also a heavily used representation in the SM.

If we have a representation of a group and look at a subspace $W \subset V$, we call it a **subrepresentation** if $\rho(g)w \in W$ for all $g \in G$, $w \in W$ and $\rho(g)$ is restricted to a subgroup $H \subset G$. In a similar way as we had simple groups, we have **irreducible** representations in which the only subrepresentations are trivial. If we have a reducible representation we can decompose it to a sum of irreducible ones (irreps), which are easier to study as they act upon the fields as block diagonal matrices.

To construct Lagrangian invariants under the symmetries, one can use **Young tableaux**,

[5]The vector space that is acted upon is complex.

which are a set of blocks (irreps) and some rules as to how to combine them to other blocks. As a showcase of this tool we will try to build product which contains a invariant (singlet) from fundamental representations of $SU(3)$. Each fundamental representation is denoted as a

$$\boldsymbol{N} = \square \quad , \tag{25}$$

while anti-fundamental $\overline{\boldsymbol{N}}$ ones are

$$\overline{\boldsymbol{N}} \quad = \quad \begin{array}{|c|}\hline \\ \hline \\ \hline \end{array} \atop {\vdots \atop \begin{array}{|c|}\hline \\ \hline \end{array}} \tag{26}$$

with $N-1$ vertically stacked boxes. In the Lagrangian there are products of fields in irreps, one irrep for each group. Now we multiply them in the following way[6]

$$\mathbf{3} \otimes \bar{\mathbf{3}} = \square \otimes \begin{array}{|c|}\hline \\ \hline \\ \hline \end{array} = \begin{array}{|c|c|}\hline & \\ \hline & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \oplus \begin{array}{|c|}\hline \\ \hline \\ \hline \\ \hline \end{array} \tag{27}$$

To calculate the dimension of the resulting irreps, we introduce the **hook** number for a box, which is the number of boxes below and to the right on the same line, plus itself. Then the diagrams above gives

$$\begin{array}{|c|c|}\hline 3 & 1 \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \oplus \begin{array}{|c|}\hline 3 \\ \hline 2 \\ \hline 1 \\ \hline \end{array} \tag{28}$$

In addition to this, as this is a representation of $SU(N)$ we include another number which has N in the upper left-hand corner, and for each block to the right add 1, and subtract for one down. This gives for $SU(3)$

$$\begin{array}{|c|c|}\hline 3 & 4 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \oplus \begin{array}{|c|}\hline 3 \\ \hline 2 \\ \hline 1 \\ \hline \end{array} \tag{29}$$

Now to get the final dimension of the irreps we divide the product of last numbers we calculated by the product of the hook numbers giving

$$\frac{2}{1}\frac{3}{3}\frac{4}{1} \oplus \frac{3}{3}\frac{2}{2}\frac{1}{1} = \mathbf{8} \oplus \mathbf{1}. \tag{30}$$

[6]For the full set of rules for multiplying together irreps, see e.g. [4].

The result is that from a product of a fundamental and a anti-fundamental representation of $SU(3)$ we get a singlet and a octet representation. The singlet is invariant under the transformation, and is an allowed term in the Lagrangian when we require that it should be invariant under $SU(3)$ transformations. Such a check needs to be done for all the groups the theory is invariant under, but as we have irreducible representations the groups are independent and give simple calculations as the one above.

2.1.4 Group invariants

Finally we will look at some tools that will be useful in calculations down the road. The generators t_r^a in the representation r are usually normalized in the following way

$$tr[t_r^a t_r^b] = C(r)\delta^{ab}, \tag{31}$$

where $C(r)$ is known as the **Dynkin index**. Now for fundamental representations of $SU(N)$, $r = N$ we define $C(N) = 1/2$. This is the reason for the factor 1/2 in front of the Pauli matrices above, Eq. (22). For $U(1)$ t^a are proportional to the identity, so it is common to define the gauge coupling g to be universal for all fields that have such $U(1)$ transformations, and absorb any difference in the $U(1)$ charge q. This gives for $C(U(1), 1) = q^2$. The **Casimir operator** is given by

$$T^2 = t^a t^a, \tag{32}$$

and is an invariant of the Lie algebra. The archetypical example of this is the angular momentum squared, J^2 from quantum mechanics, as it commutes with all the components along the individual axes. Since it is an invariant, it commutes with all the generators of the group and we get

$$t_r^a t_r^a = C_2(r)\mathbf{1}, \tag{33}$$

where r denotes the representation. E.g., for the $SU(2)$ doublet we get 3/4, for the triplet 6. $C_2(r)$ is known as the **Quadratic Casimir** and in the fundamental representation of $SU(N)$ groups it is given by

$$C_2(N) = \frac{N^2 - 1}{2N}, \tag{34}$$

while in the adjoint representation we get

$$G_2(G)\delta^{ab} = f^{acd} f^{bcd}. \tag{35}$$

By combining (31),(33) and the definition of $C(N) = 1/2$ for a given $SU(N)$ group we get the following useful equation for consistency between representations

$$d(r)C_2(r) = d(G)C(r), \tag{36}$$

where $r = G$ is the adjoint representation.[7]

2.2 Standard Model gauge group and content

We are now equipped with enough theoretical concepts to look the SM. Its gauge group is a direct product of several groups

$$G_{SM} = SU(3)_C \times SU(2)_L \times U(1)_Y, \tag{37}$$

where the subgroups are

- $U(1)_Y$ - Hypercharge
- $SU(2)_L$ - Left-handed weak isospin
- $SU(3)_C$ - Color

We see that it is a direct product of 3 groups, which means that each field in the Lagrangian is now in a irreducible representation for each of them and that the transformation is decomposable to a block diagonal matrix. The list of fields are given in Table (1) where all the representations under the various gauge groups are given.

As there are 3 gauge groups, there are also 3 coupling constants g_i by our definitions above. Note that there is a peculiar pattern for the $U(1)_Y$ representations, i.e. all come in multiples of the same underlying quantity. Currently there is no deeper understanding as to why all the SM fields have hypercharges of the form

$$q_{U(1),i} = \frac{n_i}{6}e \qquad\qquad n \in \mathbb{Z}. \tag{38}$$

One could ask why all these seemingly arbitrary gauge groups and field configurations are chosen, and the answer is currently that it is this setup that currently fits the data best. Also note the odd scalar in the mix, which is a complex spin 0 particle compared to all the other fields that have either spin 1/2 or 1. We will pursuit the problem of the hypercharges after a

[7] To get the values at the different representations, some more group theory is needed which we will not go into. Tabulated values are easy to find, see e.g. [5].

Table 1: *The field content of the SM and their representation under the SM gauge group* $G_{SM} = SU(3) \times SU(2) \times U(1)$.

	Field Type	$SU(3)_C$	$SU(2)_L$	$U(1)_Y$	generations
Q_L	Fermion	3	2	1/6	3
u_R	Fermion	$\bar{3}$	1	-2/3	3
d_R	Fermion	$\bar{3}$	1	1/3	3
L_L	Fermion	1	2	-1/2	3
e_R	Fermion	1	1	1	3
H	Scalar	1	2	1/2	1
B	Gauge	1	1	0	1
W	Gauge	1	3	0	1
G	Gauge	8	1	0	1

quick review of the possible interactions in the Lagrangian, and how scalars and the breaking of gauge symmetries are connected.

2.3 Yukawa Couplings and masses

Mass terms in the Lagrangian are the ones that are quadratic in the fields, i.e. we associate the mass m to the parameter ξ in $\xi^2(\phi^\dagger\phi)$ or $\xi(\bar{\psi}\psi)$. For a scalar field it is sufficient to simply add such a parameter and the Lagrangian would stay invariant, the story for fermions is quite different and more complicated as the Dirac mass terms $\overline{\Psi}\Psi$ are not gauge invariant in the SM. The solution to this problem is the Higgs mechanism, which involves using scalars to generate masses for the fields via a **vacuum expectation value** (VEV). We start by looking at all terms of the following type

$$y\bar{\psi}_L\phi\psi_R, \tag{39}$$

which are gauge and Lorentz invariant. These terms are interactions between a scalar and two Weyl fermions, and are known as **Yukawa couplings**. The fermion fields in the SM come in 3 copies, or **generations**, leaving us with the possibility of choosing a $\bar{\psi}_L$ and ψ_R from different generations giving the Cabibbo–Kobayashi–Maskawa (CKM) matrix. If we write the up type quarks (u, c, t) as u_i, the down type quarks (d, s, b) as d_i and the charged leptons (e, μ, τ) as l_i we write the most general Yukawa terms as

$$-\sum_{ij}\left\{(\bar{u}_i, \bar{d}_i)\left[y^d_{ij}\begin{pmatrix}\phi^+\\ \phi^0\end{pmatrix}d_{jR} + y^u_{ij}\begin{pmatrix}\phi^{0*}\\ -\phi^-\end{pmatrix}u_{jR}\right] + (\bar{\nu}_i, \bar{l}_i)y^l_{ij}\begin{pmatrix}\phi^+\\ \phi^0\end{pmatrix}l_{jR}\right\}, \tag{40}$$

where $H = (\phi^+, \phi^0)/\sqrt{2}, H^* = (\phi^{0*}, -\phi^-)/\sqrt{2}$. Now **spontaneous symmetry breaking** (SSB) occurs when the SM gauge group is broken to the subgroup $SU(3)_C \times U(1)_{em}$ at low energies. The driving force for this is the scalar Higgs field that has a non-zero minimum v_0 in its potential, and by re-parameterizing the scalar field with small perturbations around v_0

$$\phi = v_0 + h, \tag{41}$$

we get terms that look like $yv_0\bar{\psi}_L\psi_R + y\bar{\psi}_L h\psi_R$. The first term is recognized as a effective mass term with masses yv_0 and the second gives an interaction of the fermions with the Higgs field. By choosing the VEV to be in the real ϕ^0 component, and using the unitary gauge given by

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ \phi \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v_0 + h(x) \end{pmatrix}, \tag{42}$$

to explicitly show how the Goldstone bosons get absorbed by the gauge bosons. By looking at the covariant derivative of the Higgs field

$$|D_\mu H|^2 = |\left(\partial_\mu - ig_2 A_\mu^a t^a - ig_1 Y_\phi B_\mu\right) H|^2, \tag{43}$$

which after inserting the Pauli matrices gives three massive gauge bosons given by

$$\begin{aligned} W_\mu^\pm &= \frac{1}{\sqrt{2}}(A_\mu^1 \mp A_\mu^2), \\ Z_\mu^0 &= \frac{1}{\sqrt{g_1^2 + g_2^2}}(g_2 A_\mu^3 - g_1 B_\mu), \end{aligned} \tag{44}$$

with masses $g_2 v_0/2$ and $\sqrt{g_1^2 + g_2^2}v_0/2$, respectively. The last gauge boson, the photon γ_μ, becomes massless and is given by

$$\gamma_\mu = \frac{1}{\sqrt{g_2^2 + g_1^2}}(g_1 A_\mu^3 + g_2 B_\mu). \tag{45}$$

At energies lower than the masses of the Z and $W^\pm$ bosons, only the massless γ_μ remain active, corresponding to the unbroken $U(1)_B$ symmetry. We now have the typical mass terms e.g. for the electron as $v_0 y^l \bar{l}_L l_R$ where we can identify the mass with

$$m_i = \frac{1}{\sqrt{2}}v_0 y_i. \tag{46}$$

Notice that the neutrinos do not have a mass in the SM, and is thus conflicting with observations of neutrino oscillations.[8] We see that from the theory side the fundamental parameter of the SM masses are not the masses, but the Yukawa couplings and the VEV. Note the general mechanism that a large symmetry group breaks into a smaller one via scalars that obtain VEVs. When the breaking of $SU(2)_L \times U(1)_Y$ to $U(1)_B$ occurs, we get a relation between the diagonal generators of the gauge groups, in the form of the Gellman-Nishijima relation

$$Q = I_3 + Y, \tag{47}$$

where Y is the hypercharge, I_3 is the third component of the $SU(2)_L$ isospin and Q is the familiar, everyday, electric charge. The isospin for a $SU(2)$ doublet comes in pairs of $\pm 1/2$, leading to electric charges that are related by a factor $1/3$. One could entertain the idea that there might be something similar going on creating the mysterious pattern in the SM hypercharges.

2.4 Renormalization

Now for something completely different, but a important concept nevertheless for understanding quantum field theories. It will also serve as a highly motivating factor for extending the SM and give us the primary equations for investigating its behavior at extremely high energies.[9]

Calculations in QFT are between specific in-states and out-states, and what happens in the middle are the interactions that interest us. In the path-integral formalism we exponentiate the action

$$A = \int \mathcal{D}\phi e^{i \int d^4x \mathcal{L}(\phi, \partial_\mu \phi)} = \int \mathcal{D}\phi e^{iS}, \tag{48}$$

and create observables that are known as n-point functions. An example for a scalar field theory with multiple scalars is showed below

$$\langle \Omega | T \Pi_i \phi_i(\boldsymbol{x}_i) | \Omega \rangle = \frac{\int \mathcal{D}\phi e^{iS} \Pi_i \phi_i(\boldsymbol{x}_i)}{\int \mathcal{D}\phi e^{iS}}, \tag{49}$$

and corresponds to the probability amplitude for the process of finding the fields in the configuration given by $\Pi_i \phi_i(\boldsymbol{x}_i)$. Upon expanding the e^{iS} in powers of the coupling constants we get the analytic amplitude, but we will in reality have to settle with only a couple of terms as the calculations quickly become tedious and the number of terms increases rapidly. Mnemonic

[8] The problem of neutrino masses is not solved in the SM, but there are several ways of solving it, e.g. by adding a sterile (singlet representations $(1,1)_0$) right-handed neutrino would allow Yukawa couplings of the correct type $y_\nu \bar{\nu}_R H L_L$.

[9] What you should get out of this chapter is that *coupling constants* is a bad name, and *coupling coefficients* are more appropriate as they change as a function of the energy scale.

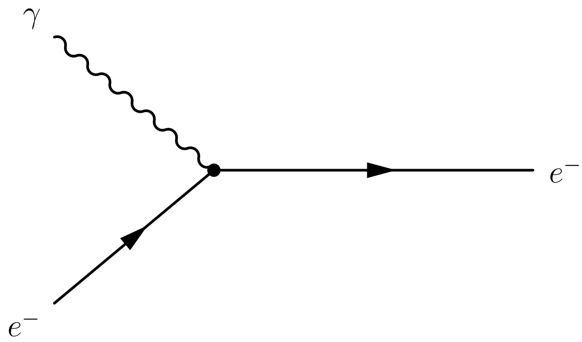


Figure 1: *Tree-level contribution to the process $e^-\gamma \to e^-$, SM electron absorbing a photon.*

Feynman diagrams are used to visualize the interactions, and by doing some calculation one can get the **Feynman rules** which tell us what interaction terms the diagrams represent. If we look at the SM process $e^-\gamma \to e^-$ depicted in Figure (1), the Feynman rules tell us that this diagram in momentum space gives the finite amplitude

$$-ie\gamma^\mu. \tag{50}$$

These leading order contributions to the amplitude without loops are known as **tree-level** diagrams, while higher order diagrams are called **loop-diagrams**.

2.4.1 Loop diagrams

To calculate the same process to a higher precision, we include terms that are at 1-loop known as **leading order** (LO) corrections. Now we have the diagram in Figure (2), which has a significantly uglier equation to solve. The main cause for this is the loop which requires us to integrate over all possible momenta $p \in (-\infty, +\infty)$. In the end it boils down to a logarithmically divergent term

$$\approx -ie\gamma^\mu\alpha \log \Lambda^2, \tag{51}$$

where Λ is a regularization cutoff and α is the QED coupling constant.

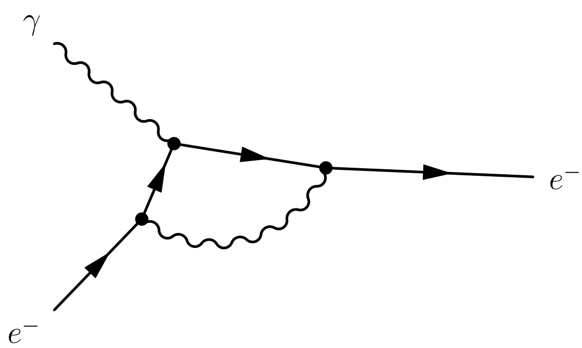


Figure 2: *Divergent 1-loop electron-photon QED interaction.*

Renormalization is needed to fix these problems, which essentially forces us to reinterpret the coefficient in the Lagrangian as effective parameters at a given scale μ, and that they are necessarily not what we observe. In the most common framework **renormalized perturbation theory**, new **counterterms** are needed for each divergent diagram, and we make these absorb the infinite parts of the integrals. For each of these counterterms a arbitrary set of **renormalization conditions** are applied at a scale M so as to connect with measurement. We will use the **modified minimal subtraction** ($\overline{MS}$) scheme where we require that the counterterms should absorb the divergent part of the integral at a scale M (and the term containing the Euler–Mascheroni constant). The theory is then defined by the fact that the n-point correlation function $\langle\Omega| T\phi(\boldsymbol{x}_1)...\phi(\boldsymbol{x}_n) |\Omega\rangle \equiv G^{(n)}$ is still the same, as this is what we measure experimentally. As it is a function of the fields and all the coupling constants, after a infinitesimal shift in the renormalization scale M the correlation function should stay the same. This forces us to infinitesimally shift the values of the field strengths and the coupling constants in the following way

$$\begin{aligned} M &\mapsto M + \delta M, \\ \lambda &\mapsto \lambda + \delta\lambda, \\ \phi &\mapsto \phi + \delta\eta\phi. \end{aligned} \tag{52}$$

For $G^{(n)}$ to stay invariant under this shift, we get an equation of the form

$$dG^{(n)} = \frac{\partial G^{(n)}}{\partial M}\delta M + \frac{\partial G^{(n)}}{\partial \lambda}\delta\lambda = n\delta n G^{(n)}, \tag{53}$$

leading to the Callan-Symanzik equation

$$\left[M\frac{\partial}{\partial M} + \beta\frac{\partial}{\partial \lambda} + n\gamma\right] G^{(n)}(x_1, ..., x_n; M, \lambda) = 0, \tag{54}$$

where $\beta \equiv M\delta\lambda/\delta M = \dfrac{dg}{d\log(\mu/M)}$ is the **beta function** that describes the evolution,[10] of the coupling constant λ and $\gamma \equiv -M\delta\eta/\delta M$ is the **anomalous dimension** and describes the rescaling of the field ϕ. Now it is relatively easy work to find how the coefficient change as a function of scale by looking at explicit examples of the n-point functions. As an example, the QED beta function at 1-loop is

$$\beta(e) = \frac{e^3}{12\pi^2}. \tag{55}$$

[10]Other names are **running** or **RG-flow** for the flow generated by a change in energy scale in coupling-constant space.

Similar expression can be found for all of the coefficients in front of terms in a Lagrangian, and are important tools for evaluating how the theory looks at other energy scales. Note that the example of e at 1-loop is an exception as it only dependent upon itself. Going to higher loop orders one often finds that the beta functions of all the parameters in the theory are strongly coupled, which means that analytic solutions are off the table. An important consequence of renormalization, is that in all the final physical observables any reference to the cutoff Λ disappears. This is a property which is known as separation of scales as the scales at Λ are integrated out in the evolution of the constants.

2.5 SM gauge coupling evolution

Calculating the evolution of the SM gauge couplings is a worthwhile task as experimental detectors work at various scales, but it is also interesting to extrapolate the couplings to high energies.[11] We will use the following equations in numerical integration later on, so we will start showing more intermediate calculations from this point on. The general 1-loop beta function for a gauge coupling g is given by

$$\beta(g) = \frac{dg}{d\log(\mu/M)} = -b\frac{g^3}{(4\pi)^2}, \tag{56}$$

where b is a coefficient determined by the particle content of the theory and some group theory. Then we need the general form of b for a non-abelian SU(N) gauge theory with scalars, fermions and gauge bosons which was found in 1973 by 2 groups independently [6, 7].

$$\begin{aligned} b(g) &= \frac{11}{3}C_2(G) - \sum_{fermions}\frac{4}{3}\frac{C_2(r)d(r)}{d(G)} - \sum_{scalars}\frac{1}{6}\frac{C_2(r)d(r)}{d(G)} \\ &= \frac{11}{3}C_2(G) - \sum_{fermions}\frac{4}{3}C(r) - \sum_{scalars}\frac{1}{6}C(r), \end{aligned} \tag{57}$$

where $C_2(G)$ is the quadratic Casimir invariant of the group, $C(r)$ is the Dynkin index, $d(r)$ is the dimension of the representation, the factor 4/3 is used for Dirac fermions, while 2/3 is the one used for Weyl and Majorana fermions. For the scalars the factor 1/6 is for real representations, and 1/3 is for complex representations.

We also note that the sums only include particles that are smaller than the scale μ, i.e. at the SM scale we do not include the top-quark which has a pole-mass of $172.9^{+2.5}_{-2.6}$ GeV [8]. This low energy theory is a Effective Field Theory (EFT) where the top-quark is integrated out. It should still describe the same physics, and as new fields are available we match the EFT to

[11] It is important to keep in mind that this is extreme extrapolation by many orders of magnitude outside the measurements.

a new EFT where the new fields are included. To do this the running constants are patched together, and a set of threshold corrections are applied to account for the missing information that the low energy EFT has lost from integrating out the new fields. The 1-loop threshold corrections are of the form in the $\overline{MS}$ scheme [9],

$$g_{full}^{-2} = g_{eft}^{-2} - \frac{f}{8\pi^2}, \tag{58}$$

where f is a function similar to (57) which contains group theory and field content parameters. It is however quite common to only keep $(N-1)$-loop threshold corrections when doing N-loop calculations. For this calculation it means that we can neglect them as we keep ourselves to 1-loop. As mentioned in Section (2.1.4) we will use the charge or hypercharge squared for $C_2(r)$, which allows us to use Equation (57) for the abelian $U(1)$ group as well. We start by finding the analytic solution to (56) by first defining $t \equiv \log(p/M)$

$$\begin{aligned} \frac{dg}{dt} &= -b\frac{g^3}{(4\pi)^2}, \\ \frac{dg}{g^3} &= -\frac{b}{(4\pi)^2}dt. \end{aligned} \tag{59}$$

Integrating both sides, assuming that b is constant, with the boundary conditions $g_0 = g(t_0)$ and $g = g(t)$ we get

$$\begin{aligned} -\frac{b}{16\pi^2}[t - t_0] &= \frac{1}{2}\left[\frac{1}{g^2} - \frac{1}{g_0^2}\right] \\ g(t) &= \frac{g_0}{\sqrt{g_0^2\frac{b}{8\pi^2}(t - t_0) + 1}}, \end{aligned} \tag{60}$$

where only b is unknown (assuming that we know $g_0(t_0)$ from measurement). To explicitly show that the result does not depend upon the arbitrary scale M, we can put back the definition of t and see that

$$t - t_0 = \log(\mu/M) - \log(\mu_0/M) = \log(\mu/\mu_0). \tag{61}$$

Since the SM is broken we will need to match the coupling constants $g_{SU(2)}$ and $g_{U(1)_Y}$ at m_z to e. This is done via the equations that show the mixing that occurs during the spontaneous symmetry breaking

$$e = g\sin\theta_W = g'\cos\theta_W, \quad \text{where} \quad \cos\theta_W \equiv \frac{m_W}{m_Z}. \tag{62}$$

For the SM we will start at the scale of the Z-boson mass 91.1876(21) GeV [10], and run the

Table 2: *Initial values of the SM gauge couplings used at the m_Z scale [10, 11].*

α_{QED}^{-1}	$\sin\theta_W^2$	α_s^{-1}
127.916 ± 0.015	0.23120 ± 0.00015	0.1185 ± 0.0006

values up to $m_t = 172.9$ GeV. Then we include the top quark and run up to $M_{GUT} \approx 10^{16}$ GeV scale. In Table (2) the initial values after the symmetry breaking at m_z is shown.

Then we have to calculate $b(g)$ for the theory between m_Z and m_t. When calculating these values we also have to multiply by the multiplicity of the fields in orthogonal groups, e.g. in $SU(2)$ we need to multiply by the number of $SU(3)$ fields, and note that all the fields are complex Weyl fermions (except the complex Higgs boson). Starting with $SU(3)_C$ we have from Equation (57)

$$\begin{aligned} b_3 &= \frac{11}{3}3 - (n_g)(4\,Weyl\,quarks)\frac{2}{3}\frac{1}{2} - 0 \\ &= 11 - n_f\frac{2}{3}, \end{aligned} \tag{63}$$

where the number of generations is given by $n_g = n_f/2$ and n_f is the number of flavors. This is done for later convenience as we can simply switch n_f from 5 to 6 when we pass the m_t pole mass. For $SU(2)_L$ we have

$$\begin{aligned} b_2 &= \frac{11}{3}2 - (n_g)[3\,Weyl\,quarks\;+\;1\,Weyl\,lepton]\frac{2}{3}\frac{1}{2} - \frac{1}{3}\frac{1}{2} \\ &= \frac{19}{3} - \frac{n_f}{2} - \frac{1}{6}. \end{aligned} \tag{64}$$

At last we have $U(1)_Y$ for which $C_2(G) = 0$ as it is an abelian group and the structure constants are 0. Furthermore we have $C(r) = Y^2$ for the hypercharges giving

$$\begin{aligned} b_1 = &-\frac{2}{3}\left[6\left(\frac{1}{6}\right)^2 + 3\left(\frac{2}{3}\right)^2 - 3*\left(\frac{1}{3}\right)^2\right] n_{g,quark} \\ &+\frac{2}{3}\left[2\left(\frac{1}{2}\right)^2 + 1*1^2\right] n_{g,leptons} - \frac{1}{3}2*\left(\frac{1}{2}\right)^2 \\ = &-\frac{11}{18}n_f - n_{g,leptons} - \frac{1}{6}. \end{aligned} \tag{65}$$

Now inserting the correct values for n_f and n_g, we show in Table (3) the beta function coefficient in the intermediate scale ($n_f = 5$) and the desert scale ($n_f = 6$). We then multiply b_1 by a factor of 3/5 to correctly account for possible unification à la $SU(5)$, which we will see later. Is quite normal to represent the running of coupling constants both as g_i, but also in the form of $\alpha^{-1} = 4\pi^2/g^2$ as they form straight lines with respect to the 1-loop evolution, and it is

Table 3: *b_1, b_2 and b_3 represent the b coefficient for the groups $U(1)_Y$, $SU(2)_L$ and $SU(3)_C$ respectively.*

	intermediate scale b	desert scale b
b_3	23/3	7
b_2	22/6	19/6
b_1	-56/9	-62/9

historically used for the $U(1)_{QED}$ fine structure constant. In Figure (3) the running of α^{-1} is showed by using the analytic expression and matching. As the coupling constants get close to the GUT scale $\approx 10^{15}$ they nearly meet and this is one of the motivations for new physics beyond the SM from a theoretical perspective. By this result one might be tempted to try to check if the lines meet by including higher loop corrections, but that is not the case as we will see later. The values at the scale of $3 \cdot 10^{15}$ GeV is given in Table (4).

In the calculation of the b_i of the gauge groups we saw that each field contributed to the running based on their representations. Now one could imagine that an extra field might just be in the correct representation to make this unification perfect. There might also be other normalization factors other that the 3/5 that we used above in the case of $SU(5)$, which we will come back to later.

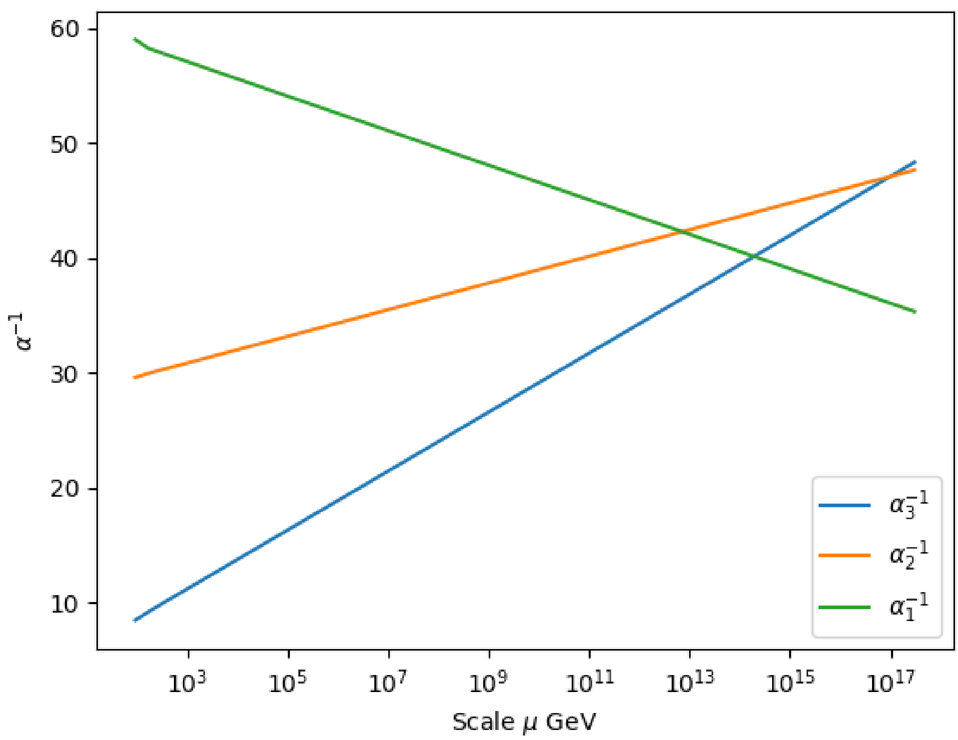


Figure 3: *The 1-loop running of the SM gauge couplings calculated with Equation (60). The SU(5) normalization for α_1 is used.*

Table 4: *The gauge couplings at the Z scale, top mass scale and at the typical GUT scale* $3 \cdot 10^{15}$ *GeV.*

	m_z	m_t	$3 \cdot 10^{15}$ GeV
g_3	1.22	1.18	0.54
g_2	0.65	0.65	0.52
g_1	0.46	0.46	0.57

3 Motivation for extended SM

Now that we have seen the basic tools that are used to construct the SM we will focus on some troublesome experimental and theoretical issues that are the motivation for this work.

3.1 Issues of interest

As seen above, the rather ad hoc choice of gauge groups, number of fields, representations and especially hypercharges in the SM might be unsatisfactory to some. Why is there only a single scalar while there are 5 fermion fields each with 3 generations? The SM also consists of a high number of free parameters, 19 or 26 if one respectively excludes or includes the neutrino mixing matrix and masses. Does the running of the coupling constants we saw in Figure (3) hint at a unification at a higher scale, with a spontaneous symmetry breaking as in the case of the electroweak symmetry? While these are rather theoretical questions, other experimental problems haunt the SM.

The observation of "missing" or dark matter in galaxies as inferred by galactic rotation curves [12] and the CMB power spectrum, leads to estimates of the missing mass by a whopping 80% of the total mass content of the universe. This suggest that Table (1) is missing something, possibly heavy fields that are we are not able to experimentally excite. The observation of neutrino oscillations could also possibly require new fields to generate the corresponding mass, even if they are Majorana fermions and are their own antiparticles. As an example, a new scalar triplet Δ that acquires a VEV ν_Δ could generate the correct gauge invariant mass term [13]

$$\mathcal{L}_\nu \sim \lambda \bar{\nu}_L \Delta \nu_L \sim \lambda v_\Delta \bar{\nu}_L \nu_L, \tag{66}$$

in what is known as the type II seesaw mechanism. To do this, we have to extend the SM by introducing new degrees of freedom in the form of new scalar fields. There exists many theories that attempt to solve this problem with scalar extension e.g. Two Higgs Doublet Models [14], Zee models [15], Higgs triplet [16] etc. There are many other Beyond the SM (BSM) theories that add new fields to solve other problems, and it is a common procedure in model building. We will quickly look at one of the historically main contenders to solve some of the theoretical headaches.

3.2 Grand Unified Theories

Grand Unified Theories are attempts to unify the symmetry groups of the SM into a single larger symmetry group [17], which is theoretically cleaner/simpler and solves a few problems:

- Why do the hypercharges of the SM fermion fields have their current values? The quarks and leptons have fractional values of the same underlying unit, which leads to the cancellation of the electron and proton charge.
- Why does the Weinberg mixing angle θ_W take the arbitrary value $\sin^2(\theta_W) = 0.2223(21)$ [10]
- Is there any mechanism generating the spectrum of fermion masses in the SM? They currently range from 0.5 MeV for the electron up to 172.9 GeV for the Top quark, which has a Yukawa coupling constant of ≈ 1.
- As portrayed above in Figure (3), the running coupling constants seem to almost meet at energies around the scale 10^{15} GeV is the SM, which might suggest that there is some underlying relation of the groups and their coupling constants.

Already in 1974, an extension of the SM into the rank 4 simple Lie Group SU(5) was undertaken by Georgi and Glashow [18], and has become known as the Georgi-Glashow model. This is the smallest Lie group that can contain a generation of fermions, without including new fields, and is thus a minimal GUT. All the gauge coupling constants are unified into the $SU(5)$ coupling constant, and also θ_W is a prediction (which however does miss the observed value slightly). For each generation of the SM, all the fermions can be fitted into the reducible representation

$$\bar{5} \oplus 10, \tag{67}$$

where $\bar{5} \ni d^c, L$ and $10 \ni Q, u^c, e^c$ (d^c is the charge conjugate down quark triplet). As it is a Yang-Mills theory, the gauge bosons reside in the adjoint representation, which has $5^2 - 1 = 24$ generators. We can recognize the SM gauge group generators among them which are under the normalization requirement $Tr[T^a T^b] = \delta^{ab}/2$ as mentioned earlier. This gives specially for the diagonal generator

$$T^{12} = \frac{1}{2\sqrt{15}} \begin{pmatrix} -2 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix}, \tag{68}$$

which we will relate to the hypercharge. The $SU(3)$ generators are embedded in the 3×3 upper-left hand block of the $SU(5)$ generators, while the $SU(2)$ are in the 2×2 lower-right hand block. The electric charge is a diagonal operator which is a linear combination of these. Since all the generators are traceless, we get a requirement on the spinor fields in the $\overline{5}$

$$3q_{d^c} + q_\nu + q_{e^c} = 0 \qquad \rightarrow \qquad q_{d^c} = -\frac{1}{3}q_{e^c} = -\frac{1}{3}e, \tag{69}$$

which is just the explanation of the hypercharges we sought. We can now also see why the factor $\sqrt{5/3}$ enters as we can compare the generator of the electric charge

$$Q = diag\left(-\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, 1, 0\right), \tag{70}$$

with T^{12} giving

$$Q = \sqrt{\frac{5}{3}}T^{12} + T^{11}, \tag{71}$$

where T^{11} is the diagonal generator for the neutral $SU(2)_L$ component. Here we see that based on the unification gauge group, we get some numerical factors like $\sqrt{5/3}$ for the $SU(5)$, and we get a explanation for the hypercharges.

To break the $SU(5)$ gauge symmetry down to the SM gauge group, a extended scalar field is needed in the adjoint representation, which takes an expectation value in the hypercharge direction. There are however several problems with this theory that currently rules it out. The 24 gauge bosons, of which 12 are the SM gluons, $W^\pm, Z^0$ and γ, have 12 extra ones which have dangerous consequences. These are called leptoquarks as they have both isospin, color and hypercharge leading to a violation of baryon and lepton number, B and L, respectively. This leads to couplings in the Lagrangian which allow proton decay. By a simple estimate from such a proton decay process assuming the mass of the leptoquarks lies at the M_{GUT} scale, leads to a prediction of the proton lifetime at $\approx 10^{31}\,years$. This is ruled out as the current experimental lower bound of the proton lifetime at $1.67 \cdot 10^{34}\,years$ [19]. As mentioned earlier there is also the problem with θ_W which does not come out correct.

There is a zoo of possible gauge groups, and a huge possibility of adding new fields to various representations providing a huge theoretical model building freedom. The Lie group $SO(10)$ [17] is another interesting possibility as it allows a combination of all the SM fermions plus a ν_R into a **16**. Here one needs to include further scalars to break the $SO(10)$ to one of the many subgroups, some have intermediate groups on their way to G_{SM} as $SU(5)$. A typical breaking pattern is $SO(10) \rightarrow SU(5) \otimes U(1) \rightarrow G_{SM}$ where we now have two scales at which symmetry breaking occurs. To generate the masses of the fermions a scalar in the representation 10_S gives the Yukawa coupling

$$y 10_S 16_f 16_f, \tag{72}$$

where y is a 3×3 matrix in generation space. Note that the Yukawa coupling of all the fermions in a generation is the same, predicting unification of the SM Yukawas within a generation. GUTs are not the only theories that predict that there should be some sort of Yukawa or gauge unification. Whenever there is a gauge group that have a simple group which the SM gauge groups unify into we expect unification of the coupling constants.

3.3 Other motivating theories

Supersymmetry [20] (SUSY) is a well known candidate for extending the SM, and predicts a superpartner for all each of the SM fields. It is easy to achieve gauge unification in simplified models as the MSSM [21], and Yukawa unification has been studied to a great extent [22] in this framework. One can also combine SUSY with GUT theories, known as SUSY-GUTs, which is a highly active area of research. One of the main problems is, however, that as the intermediate SUSY scale is being probed by Large Hadron Collider, no sign of any new supersymmetric particles has been found. Further up the energy scale there is string theory, which in some cases predict unification at a higher scale. The gauge groups and fields are here an emergent phenomena of the string vacua, and some of these groups might contain the SM.

3.4 This work

By looking at these theories, we see that the motivation for gauge and Yukawa unification is there, but there is no single satisfactory solution to all of the problems yet. The way that these extended gauge groups are broken is often through extra scalars via the spontaneous symmetry breaking mechanism that we have already described lightly for the Higgs, and briefly touched upon for $SU(5)/SO(10)$. This require extra scalars at the various stages that the symmetries are broken. Typically, a high scale gauge group is chosen together with various fields and couplings, which are then tested against the SM after integrating the RGEs, e.g in [23]. Another common approach is to add a few fields/extended gauge symmetry and either start at the GUT scale by postulating unification and evaluating the model at the electroweak scale, or starting at the electroweak scale and evaluating it at the GUT scale.

What we will do here instead is to start with the SM and assume no specific gauge group at high scale, except a $SU(5)$ normalization of the hypercharge. As the space of possible extra fields is gigantic, we will only including some rather minimal scalar extensions. The ones that we will investigate are a triplet of $SU(3)$, a anti-triplet of $SU(3)$ and a doublet of $SU(2)$, which we will denote in the following chapters as $s3$, $\overline{s3}$ and $s2$. See Table (5) for full G_{SM} representations. These extra scalars are added at up to two scales between 10^3 GeV and 10^{15} GeV in 3 different representations. The coupling constants and Yukawas of the third generation are then run from 10^3 GeV up to the GUT scale 10^{15} GeV to look for optimal scenarios for gauge and Yukawa unification. This is done at the 1-loop level and only the gauge couplings and the third generation Yukawas are evaluated for two reasons; the other parameters do not enter the RGEs at 1-loop and these 6 parameters are direct hints for unification if they meet.

We will ignore the exact Lagrangian parameters in the models besides the SM ones, as is not relevant for the analysis at 1-loop. New renormalizable couplings are: $\rho_i \phi_i^2 H^\dagger H$ between the complex Higgs doublet H and all the new scalars ϕ_i, and quartic couplings between the scalars $\rho_{ij}\phi_i^2\phi_j^2$. Based on the coupling constants these could strongly affect other parameters or observables at higher loop order, and a further analysis including these would be enlightening.

Table 5: *Representations under SM gauge groups of the scalar extensions that we investigate.*

	$SU(3)_C$	$SU(2)_L$	$U(1)_Y$
$s2$	1	2	0
$s3$	3	1	0
$\overline{s3}$	$\bar{3}$	1	0

4 Yukawa and Gauge unification

Now that we understand the rationale for extending the SM with extra scalar fields, we will look at how this affects the beta functions of the gauge constants and Yukawa couplings. We will also need a way of checking how good a proposed model is so as to be able to evaluate them. As the gauge couplings need to unify we use a $SU(5)$ normalization of the $U(1)_Y$ coupling and the measure

$$R_g = \frac{\max(g_3, g_2, g_1)}{\min(g_3, g_2, g_1)}. \tag{73}$$

For the Yukawas we will use a similar measure as we expect in a unified theory that there might be a way of including all the fermions in a generation in a single multiplet. For the third generation we have

$$R_y = \frac{\max(y_t, y_b, y_\tau)}{\min(y_t, y_b, y_\tau)}. \tag{74}$$

As we already have the expression for beta function for the gauge couplings, we only need the beta function for the Yukawas.

4.1 Yukawa beta functions

All the parameters of the SM theory are subject to radiative corrections and renormalization. This means that the renormalized masses will change as a function the scale, and we can integrate the RGEs up to a high scale in a similar fashion as the gauge couplings. There are however 3 generations which in itself is hard to explain, and the pattern of masses are quite different across the generations. It is therefore quite common to focus only on one generation as it is quite hard to achieve Yukawa unification at all with any of the generations. The choice of the third generation often stems from the uncertainties in the measurements, which is better at high energy as QCD still is perturbative.

The 1-loop beta function for the Yukawas in a Yang-Mills theory with gauge group $G = G_1 \times G_2 ... G_n$ is given by [24]

$$(4\pi)^2 \boldsymbol{\beta}^a = \frac{1}{2}[\boldsymbol{Y}_2^\dagger(F)\boldsymbol{Y}^a + \boldsymbol{Y}^a\boldsymbol{Y}_2(F)] + 2\boldsymbol{Y}^b\boldsymbol{Y}^{a\dagger}\boldsymbol{Y}^b + 2\kappa\boldsymbol{Y}^b Tr[\boldsymbol{Y}^{\dagger b}\boldsymbol{Y}^{\boldsymbol{a}}] - 3\sum_{k=1}^{n} g_k^2\{\boldsymbol{C}_{2,k}(F), \boldsymbol{Y}^a\}, \tag{75}$$

where $\boldsymbol{Y}_2(F) = \boldsymbol{Y}^{\dagger a}\boldsymbol{Y}^a$, $\boldsymbol{Y}^a = Y_{ij}^a$ is the Yukawa for the spinors ψ_i,ψ_j and the real scalar field ϕ_a, and $\boldsymbol{C}_{k,2}(F)$ is the Casimir of the spinor fields under gauge group k. For the SM most of the terms in this matrix are zero as they break gauge invariance, and the non-zero ones are $\boldsymbol{Y}_u, \boldsymbol{Y}_d$ and $\boldsymbol{Y}_e$ which are matrices in generation space. From the Equation (75) we see that there is no dependence on any scalar factors at 1-loop, as they would only create new Yukawa coupling matrices as $\boldsymbol{Y'_a}$ above or be couplings of the type $\bar{\psi}\psi\phi_i^2$ which are heavily suppressed and therefore not considered. As we are currently limiting the investigation to 1-loop SM and additional scalars, we need the base SM version which is given below [25]

$$
\begin{aligned}
\boldsymbol{Y}_U^{-1}\frac{d\boldsymbol{Y}_U}{dt} &= \frac{1}{16\pi^2}\beta_U^{(1)} + O(\beta_U^2),\\
\boldsymbol{Y}_D^{-1}\frac{d\boldsymbol{Y}_D}{dt} &= \frac{1}{16\pi^2}\beta_D^{(1)} + O(\beta_D^2),\\
\boldsymbol{Y}_L^{-1}\frac{d\boldsymbol{Y}_L}{dt} &= \frac{1}{16\pi^2}\beta_L^{(1)} + O(\beta_L^2),
\end{aligned}
\tag{76}
$$

where

$$
\beta_U^{(1)} = \frac{3}{2}(\boldsymbol{Y}_U^{\dagger}\boldsymbol{Y}_U - \boldsymbol{Y}_D^{\dagger}\boldsymbol{Y}_D) + \boldsymbol{Y}_2(S) - \left(\frac{17}{20}g_1^2 + \frac{9}{4}g_2^2 + 8g_3^2\right), \tag{77}
$$

$$
\beta_D^{(1)} = \frac{3}{2}(\boldsymbol{Y}_D^{\dagger}\boldsymbol{Y}_D - \boldsymbol{Y}_U^{\dagger}\boldsymbol{Y}_U) + \boldsymbol{Y}_2(S) - \left(\frac{1}{4}g_1^2 + \frac{9}{4}g_2^2 + 8g_3^2\right), \tag{78}
$$

$$
\beta_L^{(1)} = \frac{3}{2}\boldsymbol{Y}_L^{\dagger}\boldsymbol{Y}_L + \boldsymbol{Y}_2(S) - \frac{9}{4}(g_1^2 + g_2^2), \tag{79}
$$

and

$$
Y_2(S) = Tr(\boldsymbol{Y}_L^{\dagger}\boldsymbol{Y}_L + 3\boldsymbol{Y}_U^{\dagger}\boldsymbol{Y}_U + 3\boldsymbol{Y}_D^{\dagger}\boldsymbol{Y}_D). \tag{80}
$$

With a field content enriched with scalar fields, further Yukawa coupling matrices like the one above would exist and based on the representation of the scalar and fermion there could be non-zero terms. However with the scalar fields we are investigating, there are no such allowed terms.

We will assume that the mass eigenstates and the $SU(2)$ states are the same, giving only diagonal couplings in the matrices. The CKM-matrix is nearly diagonal, given by [26]

$$
\begin{pmatrix}
|V_{ud}| & |V_{us}| & |V_{ub}|\\
|V_{cd}| & |V_{cs}| & |V_{cb}|\\
|V_{td}| & |V_{ts}| & |V_{tb}|
\end{pmatrix}
=
\begin{pmatrix}
0.97427 \pm 0.00015 & 0.22534 \pm 0.00065 & 0.00347^{+0.00015}_{-0.00014}\\
0.22520 \pm 0.00065 & 0.97344 \pm 0.00016 & 0.0412^{+0.0011}_{-0.0005}\\
0.00867^{+0.00029}_{-0.00031} & 0.0404^{+0.0011}_{-0.0005} & 0.999146^{+0.000021}_{-0.000046}
\end{pmatrix}. \tag{81}
$$

The maximal diagonal element of the third generation is 0.0412 which is the mixing between the charm and bottom quark, which already have a mass difference of ≈ 3. The error in assuming that all the off-diagonal elements are zero when we are only interested in the third generation elements are thus maximally $\approx 0.04/3 \approx 2\%$. We will later see that the theoretical uncertainty of 2-loop calculations are $\approx 2\%$ and $\approx 20\%$ for Gauge and Yukawas respectively, making this assumption acceptable. The matrices Y_L, Y_U and Y_D with the assumptions are then given by

$$\boldsymbol{Y}_L = \begin{pmatrix} y_e & 0 & \\ 0 & y_\mu & 0 \\ 0 & 0 & y_\tau \end{pmatrix}, \quad \boldsymbol{Y}_D = \begin{pmatrix} y_d & 0 & \\ 0 & y_s & 0 \\ 0 & 0 & y_b \end{pmatrix}, \quad \boldsymbol{Y}_U = \begin{pmatrix} y_u & 0 & \\ 0 & y_c & 0 \\ 0 & 0 & y_t \end{pmatrix}. \tag{82}$$

This leads to the simple relation

$$Y_2(S) = \sum_{leptons} y_{lepton}^2 + 3 \sum_{quarks} y_{quarks}^2. \tag{83}$$

The third generation Yukawa betas are then

$$\begin{aligned} \frac{dy_t}{dt} &= \frac{y_t}{16\pi^2}\left(\frac{3}{2}\left[y_t^2 - y_b^2\right] + Y_2(S) - \left(\frac{17}{20}g_1^2 + \frac{9}{4}g_2^2 + 8g_3^2\right)\right), \\ \frac{dy_b}{dt} &= \frac{y_b}{16\pi^2}\left(\frac{3}{2}\left[y_b^2 - y_t^2\right] + Y_2(S) - \left(\frac{1}{4}g_1^2 + \frac{9}{4}g_2^2 + 8g_3^2\right)\right), \\ \frac{dy_\tau}{dt} &= \frac{y_\tau}{16\pi^2}\left(\frac{3}{2}y_\tau^2 + Y_2(S) - \frac{9}{4}(g_1^2 + g_2^2)\right). \end{aligned} \tag{84}$$

While we could analytically integrate the gauge running at 1-loop, we would struggle to do this here with Yukawa couplings as there are too many variables that depend on the scale t. We then have to turn to numerical methods to get anywhere with these equations.

5 Numerical tools

We now have a set of differential equations for each of the relevant parameters in the theory, in the form of beta functions for the coupling constants and the third generation Yukawas. As we are only looking at the 1-loop RGEs, only these 6 of the 19 SM parameters are needed,[12] which is a huge simplification compared to higher loop orders. The set of equations we now have are of the form

$$\frac{dp_i}{d\log(\mu)} = \beta_i(\mu; p_1, p_2, ..., p_n) = f_i(\mu, \vec{p}), \tag{85}$$

where $(p_1, p_2, ..., p_n)$ is the set of all the parameters of the theory, and $i \in \{1, ..., m\}$ is the third generation Yukawa and gauge parameters. As already showed in the background, an analytic solution is available for the gauge couplings. This is not the case for the Yukawas, as they form a set of equations which are strongly coupled first order ordinary differential equations (ODE). To get any progress we turn to numerical integration. The initial values as $\overline{MS}$ masses are known at the electroweak scale, and only an integration upwards in energy is needed. Several methods for numerical integration exists, from the Euler methods which is a first order method to the often used family of Runge-Kutta (RK) methods which has multiple intermediate stages and achieves a higher order. A stage is a evaluation of $f_i(\mu, \vec{p})$ at a set of $(\mu, \vec{p})$.

5.0.1 Runge-Kutta 4

For the RK methods the number of stages is equal to the order up to order 4. After this the number of stages increases more than a linear function of the order [27], and the fourth order is usually chosen to achieve a balance between low error and computational intensity. It uses a weighted set of 4 vectorized evaluations in the following intermediate stages

$$\begin{aligned}
\vec{k}_1 &= f\left(\mu, \vec{p}(\mu)\right), \\
\vec{k}_2 &= f\left(\mu + \frac{\delta\mu}{2}, \vec{p}(\mu) + \frac{\delta\mu}{2}\vec{k}_1\right), \\
\vec{k}_3 &= f\left(\mu + \frac{\delta\mu}{2}, \vec{p}(\mu) + \frac{\delta\mu}{2}\vec{k}_2\right), \\
\vec{k}_4 &= f\left(\mu + \delta\mu, \vec{p}(\mu) + \delta\mu\vec{k}_3\right),
\end{aligned} \tag{86}$$

and at the end we calculate the change to the next $\vec{p}$ as

$$\vec{p}(\mu + \delta\mu) = \vec{p}(\mu) + \frac{\delta\mu}{6}\left(\vec{k}_1 + 2\vec{k}_2 + 2\vec{k}_3 + \vec{k}_4\right). \tag{87}$$

[12] This means that in the determination of $Y_2(S)$ there will be a small error as the first and second generation Yukawas do not run. The largest contribution is $3y_t^2 \approx 3$, while the largest one we neglect as not changing is $3y_c^2 \approx 0.0002$, so the error is not that large.

By using this set of equations is can be shown that the total accumulated error of $\mathcal{O}(\delta\mu^4)$ which is a huge improvement over the much simpler Euler method $\mathcal{O}(\delta\mu)$, but notice that we need 4 evaluations of the function f. The following sections describe how this was done numerically and compared/sanity checked against FlexibleSUSY.

5.1 FlexibleSUSY

FlexibleSUSY version 2.0.0 was released on October 2017 [28,29], and was used as a benchmark for the SM running and a special scalar extension of the SM. It is a spectrum generator for both SUSY and non-SUSY theories with general gauge groups and fields at up to 3-loop and 4-loop in special cases (e.g. the SM Higgs). It builds upon SARAH [30] which takes in a theory where the gauge group, fields and Lagrangian are defined and calculates the beta functions, tadpole equations, self-energy and more. FlexibleSUSY then compiles it into C++ code with a modified RK4 integrator which then works as a spectrum generator. When it generates a spectrum it runs the SM at a mix of 2 and 3-loop order up to a scale where it matches to the model requested by the user. It then continues on at either 1 or 2-loop as requested by the user, and the pole masses, $\overline{MS}$ masses and Yukawas are given at a desired scale.

The initial conditions at the electroweak scale are not straightforward to get, e.g. the Yukawas are usually measured at the pole masses of the fields and then translated to $\overline{MS}$ at that scale. To get a set of initial values at a single scale ready for integrating towards the GUT scale, one would have to start with some rough guesses of the parameters at a scale. Then one would loop through the scales of the pole masses where e.g. the running top Yukawa would be set equal to the experimental $y_{\overline{MS},top}(m_{top})$. The loop would go on until the values have converged, and the initial values would then be ready for integration. However this would not lend itself to a good comparison with FlexibleSUSY as they use a higher loop order before matching to the extended theory. To make any accurate comparison with it, I did not calculate these initial values myself, but made FlexibleSUSY output the following SM parameters at the scale 10^3 GeV

$$\vec{p}(10^3\,GeV) = (g_1, g_2, g_3, y_t, y_b, y_\tau) = (0.4641,\ 0.6476,\ 1.0457,\ 0.9956,\ 0.01378,\ 0.01021). \quad (88)$$

5.2 1-loop tool

The numerical tool that was written to do the numerical integration is done in python initially and the slow parts are sped up with Cython [31]. Cython is a superset of the python language which allows static annotation of variables and functions, and then generates C code which is compiled. This mix allows for quick prototyping and development time, and during optimization to speed up the hot loops which gave a factor 20 improvement in speedup this work. The general structure is to start with a struct of a model

```
cdef struct TheoryC:
    FieldC[12] fields
    int n_fields
```

where FieldC is roughly

```
cdef struct FieldC:
    float spin
    float rep_su3
    float rep_su2
    float rep_u1
    int generations
    double pole_mass_scale
    float su3_casimir
    float su2_casimir
    float u1_casimir
    float su3_dynkin
    float su2_dynkin
    float u1_dynkin
```

Then the SM field content in Table (1) is added and the values of the Casimir and Dynkin are either calculated or looked up from a table. E.g., for the quark doublet a FieldC with 3 generations at the top pole mass is added, which only allows for accurate calculation above this scale. As the initial values are given at 10^3 GeV this is no issue. For each scalar extension, a set of extra fields was added with the correct representations. As they are added in up to 2 scales for each of the three scalars $\overline{s3}, s3, s2$, the number of each type is simply the number of generations which is nothing but a multiplicative factor. The scalar fields are assumed to be real.

To determine the contribution to the gauge coupling beta functions, all the fields in the TheoryC are looped over and their contribution to the b_i are added. The contribution is added if the evaluation scale of b_i is above or below the FieldC pole mass. This method would only work at 1-loop, and is therefore not very well suited for more precise work. It does however allow a quick evaluation of the contribution to the unification measures R_y and R_g for a given theory which was the target of this work. Compared to FlexibleSUSY which is a hugely more accurate, versatile and flexible framework for doing these calculations, it offers a factor $\approx 60\,000$ speedup at 1-loop for the SM.[13]

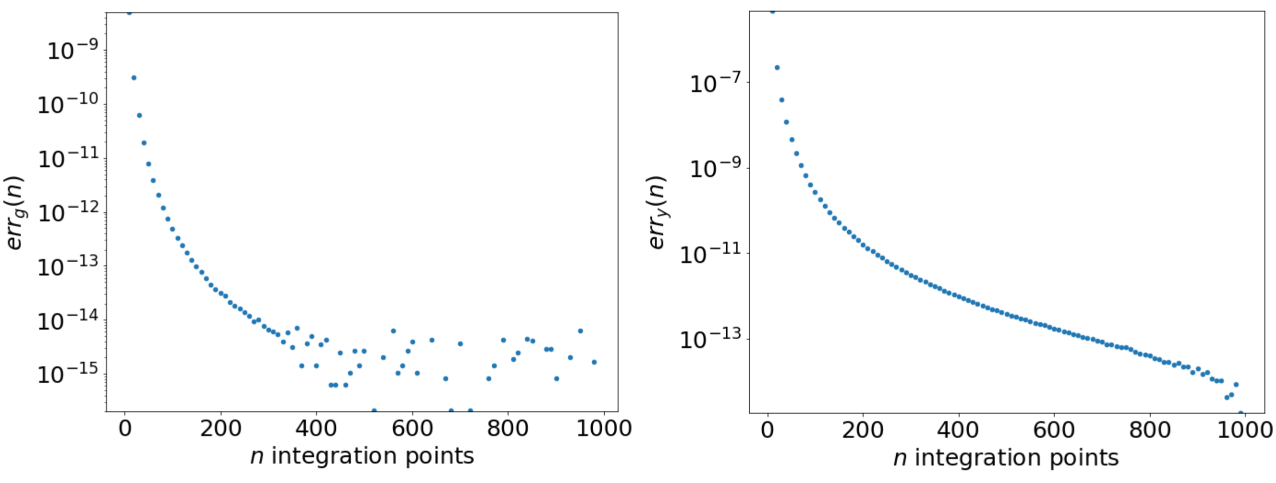
The final evaluation speed of running the SM up to the m_{GUT} scale were ≈ 5 ms on a i5-7200U @ 2.50GHz with 1 physical core and is embarrassingly parallel (I used 2 cores for a perfect speedup factor of 2). As the space of possible combinations of scalars at various scales is so large, the storage of the results became the main limitation.[14]

5.2.1 Uncertainty

To control the integration error, a comparison of the unification measures of the SM as a function of the integration steps was done.

$$
\begin{aligned}
err_g(n) &= \frac{R_g(n) - R_g(n=10^3)}{R_g(n=10^3)}, \\
err_y(n) &= \frac{R_y(n) - R_y(n=10^3)}{R_y(n=10^3)}.
\end{aligned}
\tag{89}
$$

In Figure (4) the equations (89) are plotted as a function of the integration steps with a 1-loop SM RG-flow evaluated at 10^{16} GeV. Here we see that the integration error is negligible even at small $n \approx 100$. In the work below $n = 300$ integration steps are used, which is overkill compared to other sources of uncertainty, and reducing this number to 100 would be sufficient and at the same time give a huge speedup in runtime.



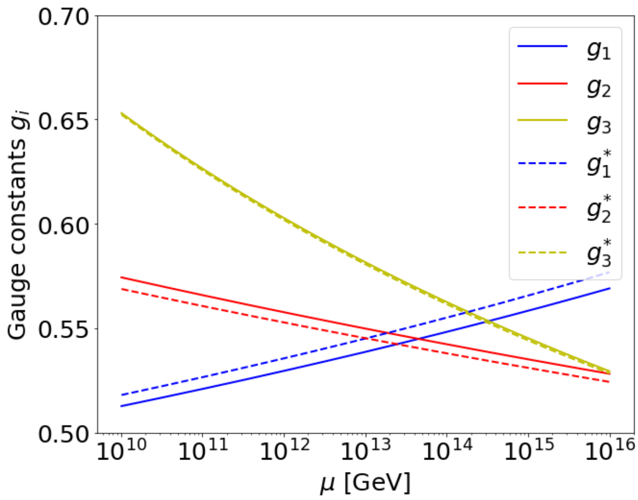
(a) R_g (b) R_y

Figure 4: *Absolute error of R_g and R_y for the 1-loop SM as a function of the integration points n compared to $n = 10^3$ integration points.*

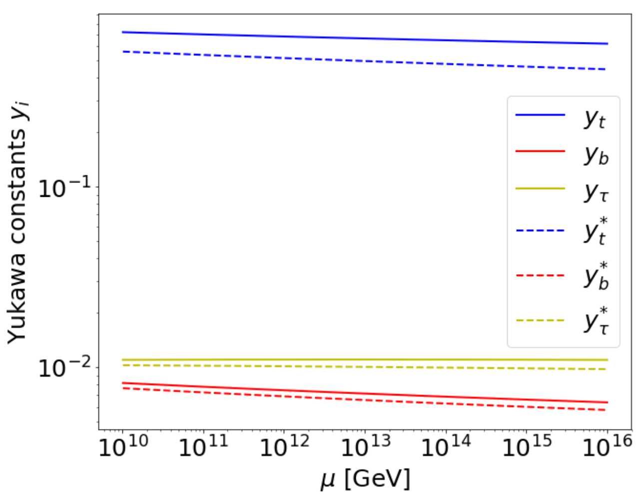
[13] Here I have included the building and compiling of the RGE equations that FlexibleSUSY does which is the primary time-taker for a grid search. A single evaluation with a precompiled C++ model in FlexibleSUSY takes on the order of 1 second.

[14] I did not include any persistent database for this, so a limitation of 8 GB RAM was hit quite early as python used 6 times more memory than the underlying *float* or *double*, and kept everything in memory.

Other sources of uncertainty are the theoretical uncertainty of 1-loop approximations that is done here, and initial values. The initial values are known relatively accurate, with the highest uncertainty in the $\overline{MS}$ top mass 175.3 ± 5.1 GeV [32] at the m_W scale which corresponds to a 3% uncertainty. We will see that the theory error of 1-loop approximation are the main source of uncertainty, so we can quickly forget the uncertainty of the initial values. To put a number on the 1-loop approximation, a comparison of the third generation Yukawa- and gauge couplings are seen in Table (6) and visually in Figure (5) . These are calculated with FlexibleSUSY, which runs a mixed 2/3-loop model up to 10^3 GeV and then pure 1 or 2-loop up to 10^{16} GeV. Looking at the values there is a clear grouping of gauge coupling and Yukawa couplings with respect to how they run, which is reflected in the change in R_g and R_y respectively. The gauge couplings are relatively stable from 2-loop corrections, which will greatly help when we do extra scalars extension. The same cannot be said about R_y which has a difference of 21%, but an improvement in the extremal Yukawas y_t or y_b would still be an improvement as the lines "have not crossed" as a result of the 2-loop corrections.



(a)



(b)

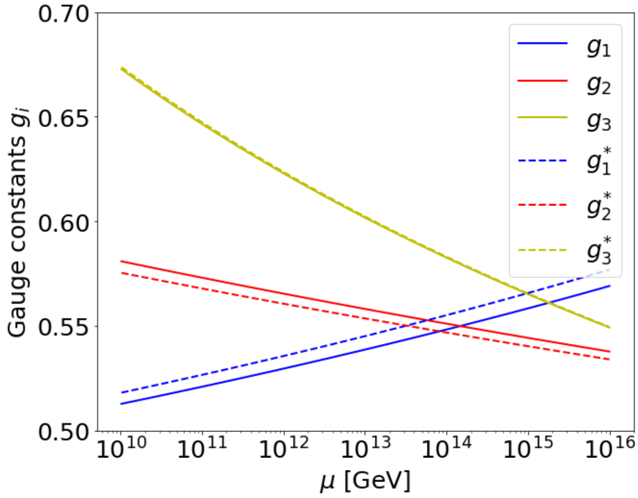
Figure 5: *a) The gauge coupling and b) the third Yukawa running of the SM at 1 (solid) and 2 (dotted)-loops calculated with FlexibleSUSY. The erratic behavior of* $err_g(n)$ *with* $n \gtrsim 300$ *does most likely stem from numerical issues.*

Table 6: *Comparison of SM parameters at 1- and 2-loop evaluated at* 10^{16} *GeV with the same initial conditions at* 10^3 *GeV.*

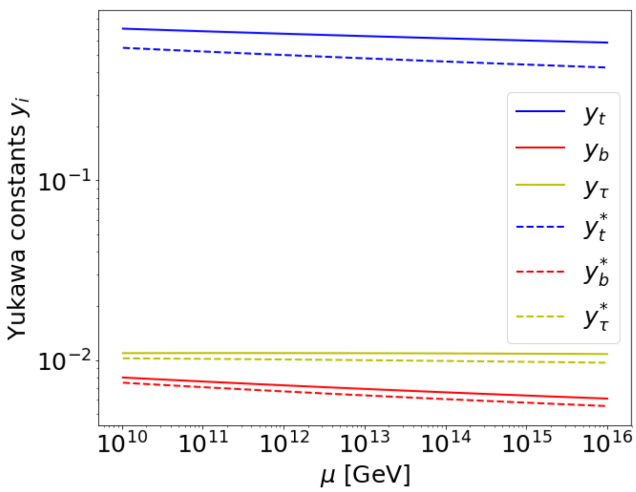
	g_3	g_2	g_1	y_t	y_b	y_τ	R_g	R_y
1-loop	0.5294	0.5282	0.569	0.62	0.0064	0.0110	1.08	97
2-loop	0.5284	0.5243	0.577	0.45	0.0058	0.0097	1.10	77
% diff	0.2	0.7	−1.3	28	9	11	−2	21

A similar analysis is done in a model where the SM is extended to include 2 of each of the extra scalars in Table (5) with a pole mass at 10^3 GeV. The relative difference is here −2%

and 20% for R_g and R_y respectively, and there is still the same uncertainty as seen in Figure (6). As the models we will use extension are an of this type, some with more extreme numbers of scalars, it is comforting to see that the uncertainty is relatively stable as a function of their number.



(a)



(b)

Figure 6: *a) The gauge coupling and b) the third Yukawa running of a BSM model with 2 of each scalar in Table (5) with a pole mass of* 10^3 *GeV. Lines are 1 (solid) and 2 (dotted)-loops calculated with FlexibleSUSY.*

5.3 Replicating FlexibleSUSY at 1-loop

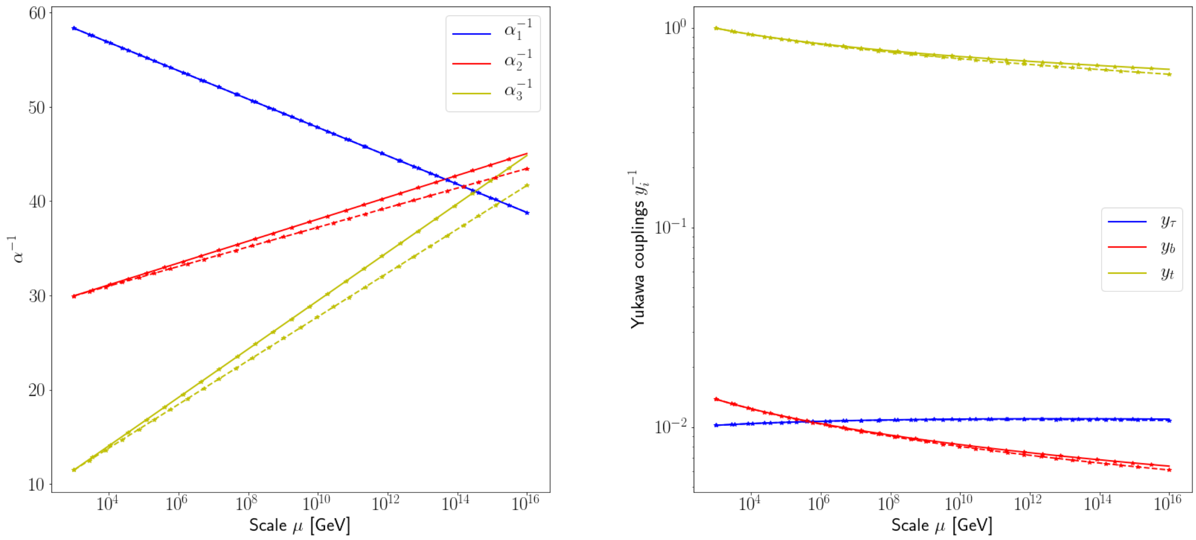
Before we start investigating models with an extended field content, we want to make sure that we can trust that the 1-loop integration done here is reasonable and comparable to FlexibleSUSY.

5.3.1 Matching the SM

Here the SM gauge and 3rd generation Yukawa couplings are integrated from 10^3 GeV to up to 10^{16} GeV in both tools. The running as seen for the gauge couplings in Figure (7) shows that the results are comparable to a high degree, which is verified by looking at the final values in Table (7). The maximal absolute deviation between any of the values is 0.004%.

Table 7: *Comparison of SM parameters after integrating the SM RGE with both FlexibleSUSY and my own calculations from* 10^3 *GeV with the same initial conditions up to* 10^{16} *GeV. n=300 integration point in RK4.*

	g_3	g_2	g_1	y_t	y_b	y_τ
FS	0.5294	0.5282	0.5690	0.6202	0.00637	0.01096
Here	0.5293	0.5282	0.5690	0.6202	0.00637	0.01096



(a) (b)

Figure 7: *RG-flow of a) gauge couplings, b) Yukawa couplings, with both the SM (solid) and a extended model (stripes), with FlexibleSUSY (stars) and my own calculation (lines). FlexibleSUSY's RG-flow overlap with with my own calculations at both the SM and the scalar extension of choice.*

1

5.3.2 Scalar extension

As we are going to check extensions of the SM, it is important that the additional contributions from extra scalars to the beta functions are correct. As there are 3 types of scalars that we are going to do a grid search over later, we will use a model where we place 2 of each scalar in Table (5) with pole masses of 10^3 GeV. This should give a change in the running in both tools, and allows us to see how the uncertainty evolves in the vector space of possible scalar extensions. In Table (8) we see that at the scale of 10^{16} GeV we have reasonably similar values with a maximal difference of $0.004\% = 40ppm$. This running is also showed in Figure (7) where the dots perfectly overlap with the lines, confirming the last values across the energy interval (10^3 GeV, 10^{16} GeV) .

Table 8: *Comparison of SM extension after running both FlexibleSUSY and my own tool from 10^3 GeV with the same initial conditions up to 10^{16} GeV. N=300 integration points in RK4.*

	g_3	g_2	g_1	y_t	y_b	y_τ
FS	0.5492	0.5377	0.5690	0.5857	0.006094	0.01079
OWN	0.5492	0.5377	0.5690	0.5857	0.006095	0.01079

By comparing the values in the Table (8) to (7) one sees a clear difference between the values. The g_1 stayed constant, which is expected as none of the scalars had a non-zero hypercharge, while both g_2 and g_3 increased in values. As a consequence of these changes, the Yukawas are shifted towards lower values as their RGEs are dependent on the gauge couplings.

The general picture we now have is that the fit to FlexibleSUSY is achieved for the SM and with 3 specific scalar extensions at 1-loop. While we have only added 6 scalars, we see that the uncertainty stays approximately the same, which are dominated by the 1-loop approximation. We will use this as a basis to add even more scalars, up to 72 in total. It is hard to quantify the uncertainty at this high number of scalars as the way I added scalars in FlexibleSUSY made the time to calculate the various functions grow polynomially/exponentially as a function of the number of fields in the theory. We will also see that models with a high number of scalars start to move away from the asymptotic freedom of $SU(2)_L$ and $SU(3)_C$ as their b_i change sign.

6 Results and discussion

6.1 Individual scalar type effects

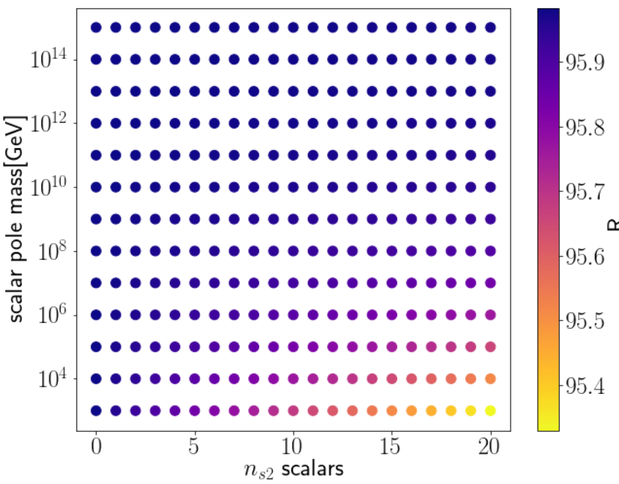
We will look at some selected searches for improvements in R_y and R_g with the real scalars $s3$, $\overline{s3}$ and $s2$ from Table (5). The variables we will use for visualization are

- $\mathbb{Z}^3$: number of scalars at the first scale
- $\mathbb{Z}^3$: number of scalars at the second scale
- $\mathbb{R}^2$: two scales for the scalar pole masses
- $\mathbb{R}^2$: third gen Yukawas condensed to R_y, and gauge couplings condensed to R_g
- $\mathbb{R}$: energy scale at which we look at the values of R_g and R_y

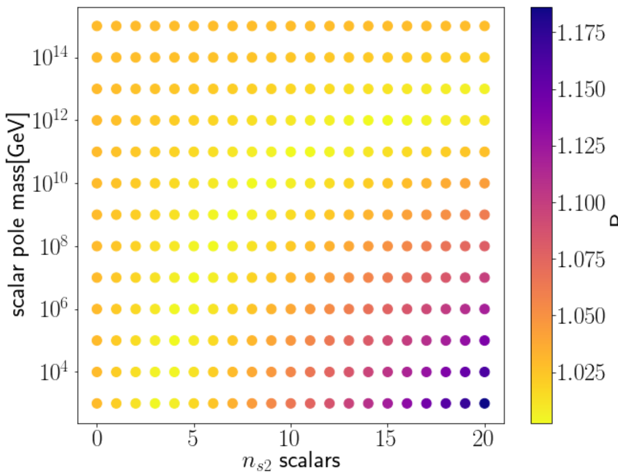
In the first part we will look at each of these extension individually to see how they might affect the running, since we saw from the test results that we got an improvement by including all of them. This data stems from a search where all possible combinations of 0 to 20 of each scalar type at one varying pole mass scale, which run from 10^3 GeV to 10^{15} GeV.

6.1.1 Scalar doublet $s2$ $(1, 2)_0$

For each model including only scalars of type s_2, R_y and R_g is plotted in Figure (8) at the scale 10^{15} GeV.



(a) R_y



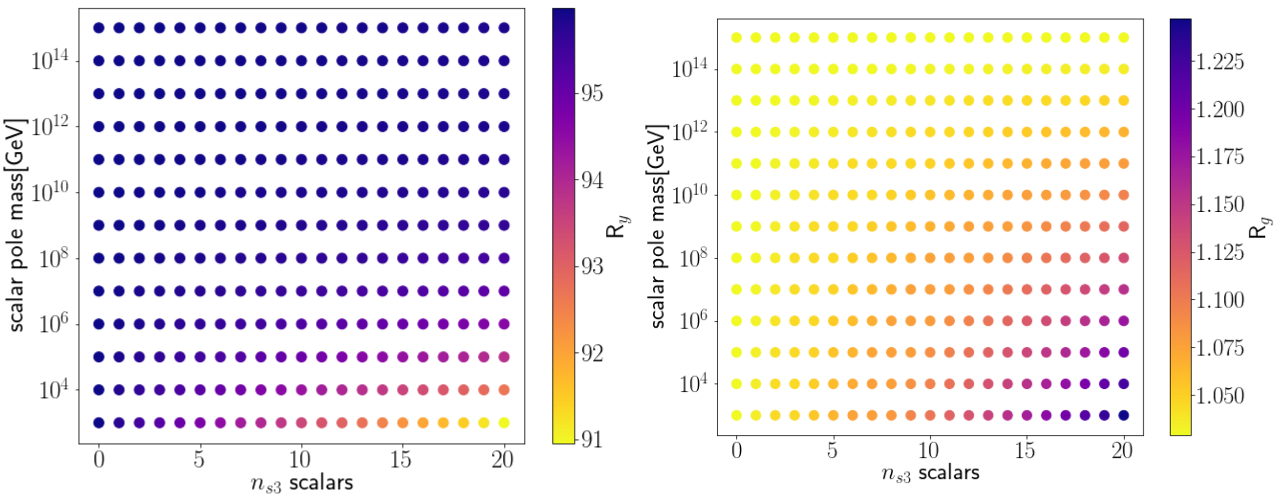
(b) R_g

Figure 8: *The unification measures R_g and R_y as a function of the number of s2 scalars and their pole mass. The R_y has a slight preference for a high number of scalars at a low pole mass, while there is a valley of points that are acceptable for R_g.*

Here we see that as we increase the number of scalars at a low scale, we get a minuscule improvement in R_y from 96 to 95.5. The preference for low scales comes from the fact that it then has a longer scale-interval where it can change the integration. The rate of change is, however, rather low at this point, and for any noticeable change to take place the new scalars need to be at a very low scale. For R_g the picture is more interesting, as there seems to be a valley of points that are viable at this scale which gives good unification within the uncertainty $R_g \approx 1.00 \pm 0.02$. Adding too many scalars at a low scale makes the unification worse compared to the SM, which in the plot is the vertical line with $n_{s2} = 0$.

6.1.2 Scalar triplets $s3$ $(3,1)_0$ and $\overline{s3}$ $(\overline{3},1)_0$

A identical search as the one for $s2$ is done for $s3$ as well, where the results are displayed in Figure (9). The contributions from $s3$ and $\overline{s3}$ were the same, so no separate analysis for both is done. A similar effect on R_y is found, where a high number of scalars at a low scale is preferable. R_g gets worse by adding a high number of low pole mass scalars, but is not affected to any noticeable extent by high pole mass scalars. We do not get any improvement from the SM here as we got for $s2$.



(a) R_y (b) R_g

Figure 9: *The unification measures R_g and R_y as a function of the number of $s3$ scalars and their pole mass. The optimal choice for R_y is to have at least 6 scalars at as low scale as possible, while R_g is closest to 1 with few scalars at a high pole mass, essentially the SM.*

6.2 Models with all scalar types

6.2.1 One scale search

We will then look at some searches where we include both type of scalars at the same scale. We know that by including $SU(2)$ doublets, we get gauge unification, so we will study what combinations of $s2$ and $s3$ scalars get $R_g < 1.02$ at the GUT scale. In Figure (10) a) we see that it is beneficial for R_g to include both types of scalars. As we include more $s2$ fields, the space of allowed $s3$ scalars increase, which is good for R_y which is at this stage far from $\mathcal{O}(1)$. Figure (10) also includes in red the models that have $R_y \leq \min(R_y) + 1 = 93.9$ which generally reside in areas with a high number of $s3$ scalars. The best model at the scale $1.11 \cdot 10^{15}$ GeV, with respect to R_y, is a model with 12 $s3$ scalars and 6 $s2$ scalars at a pole mass of 10^3 GeV. In Figure (10) b) R_y as a function of the energy scale is plotted for this model, where we see that it is far from being any form of unification at all. The separation into two regimes comes from the crossing of the y_τ with y_b at $\approx 10^{5-6}$ GeV.

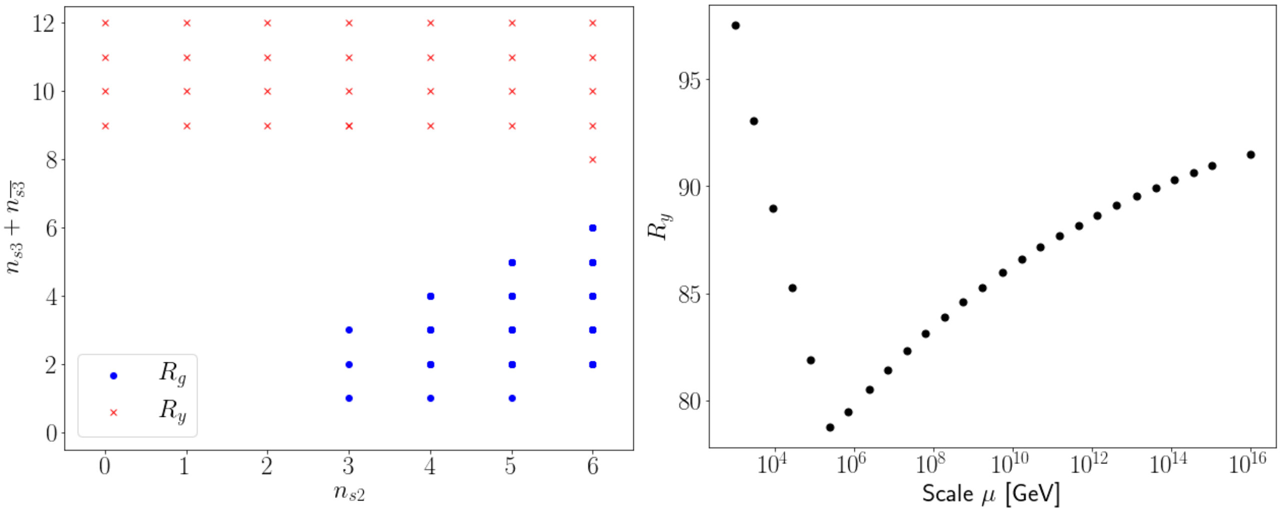


Figure 10: *a) All models with blue:* $R_g < 1.02$*, red:* $R_y \leq \min(R_y) + 1 = 93.9$*. Both as a function of* $s2$ *scalars and* $s3 + \overline{s3}$ *scalars at a scale of* $1.23 \cdot 10^{15}$ *GeV and where the scalar pole masses lie between* $(10^3, 10^{15})$ *GeV. b)* R_y *running of optimal* $R_y(1.11 \cdot 10^{15}$ *GeV) model.*

6.2.2 Two scale search

In the motivational part about GUTs, it was mentioned briefly that the symmetry group of the theory might break at multiple scales on the way down to the SM. Assuming that for each symmetry breaking scale the scalars which break the symmetry have a pole mass at that scale, we add scalars at multiple scales. Even though evaluating a model takes ≈ 5 ms, the model space becomes gigantic when multiple scales are included. As an example, a search with the 3 scalar types up to 10 generations each spread out over 2 different scales, gives roughly $2.6 * 10^7$ models, which again corresponds to around 70 CPU hours. This search is done on a coarse grid where only 0, 4, 8 or 12 of each scalar type are included. In Figure (11) we see again the distribution of the best models in the space of the number of scalars. Again, a similar pattern as Figure (10) emerge, where the preferred models for R_y is at the maximal extension of 40

$s3+\overline{s3}$ and 20 $s2$ scalars with a $R_y = 85$, and the gauge unification has a wider area of allowed scalar combinations.

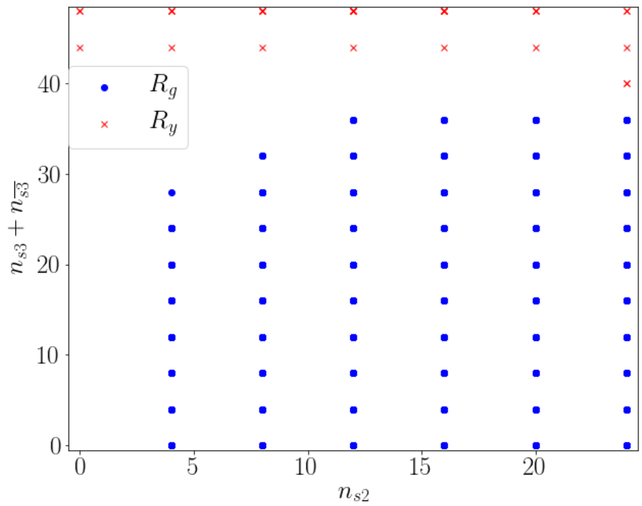


Figure 11: *All models with blue:* $R_g < 1.02$*, red:* $R_y \leq \min(R_y) + 1 = 93.9$*. Both as a function of* $s2$ *scalars and* $s3 + \overline{s3}$ *scalars at a scale of* $1.23 \cdot 10^{15}$ *GeV and where the scalar pole masses lie between* $(10^3, 10^{15})$ *GeV.*

One might be tempted to continue to add more scalars, as we got a reduction in R_y of $96 \to 85$ with the current best model at the GUT scale, but there is a catch to this solution. As hinted at in Section (5.2.1), we will look at when the values of b_3 or b_2 change sign as a function of the active number of scalars of type $\overline{s3} + s3$ or $s2$ respectively. As soon as they change sign, g_3 and g_2 will grow towards higher energies, making the 1-loop approximation worse. The number of scalars when this sign change happens is

$$n'_{s2} = 19 \qquad\qquad n'_{s3+\overline{s3}} = 43. \tag{90}$$

The best way of seeing this is to look at the running of the best model with respect to R_y at the GUT scale. We see in Figure (12) the gauge couplings diverge as we run the model towards higher energies, making the gauge unification worse. It is clear that we need to take the models with a high number of low-mass non-singlet $SU(3)$ representations with a pinch of extra uncertainty.

6.3 Outlook

We see that we are efficiently able to investigate some specific scalar extensions at 1-loop, and achieve gauge unification in a broad set of models. Yukawa unification is harder to achieve as the starting point, the SM, is far from unification, at $R_y = 96$, and that we are only able to get it to $R_y = 85$ by including 72 new scalars. Including more scalars makes the gauge couplings

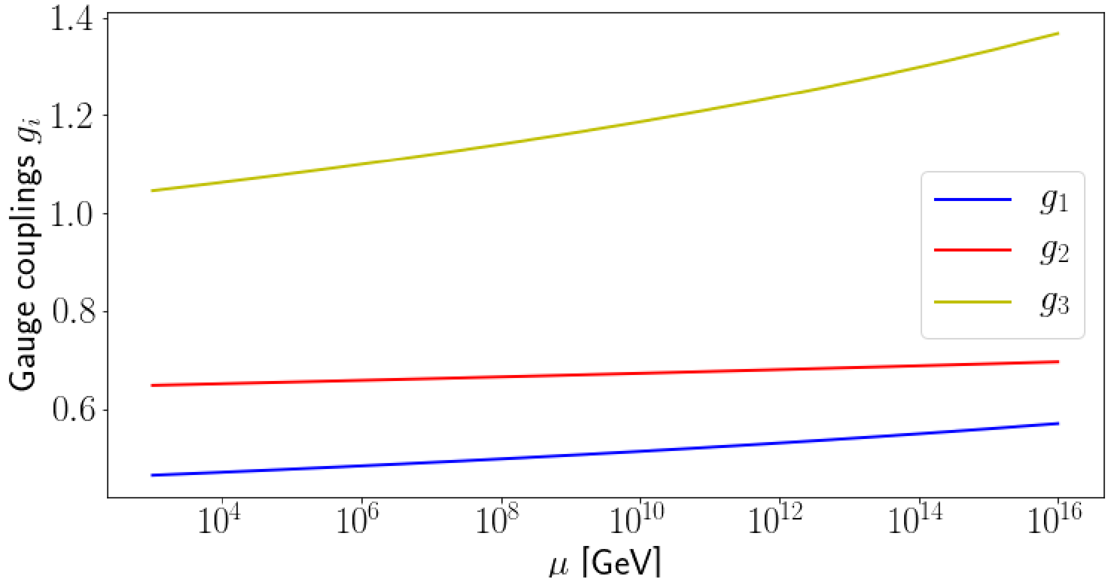


Figure 12: *Gauge running of a model with 24 of each* $s3$, $\overline{s3}$ *and* $s2$ *scalars with pole mass of* 10^3 *GeV.*

get closer to the non-perturbative limit, so this is not a promising way forwards. The most obvious steps one could take to extend the analysis are

- New gauge groups
- 2-loop
- New scalars in other representations
- New fermions
- Flavor symmetry

or a combination of them. The 2-loop might also reveal some additional problems in this analysis, as it would require, in general, all the parameters of the model. As mentioned earlier, there are quartic couplings associated with scalars, which are of the form

$$\mathcal{L}_{\phi,I} \sim \lambda_{ijkl}\phi_i\phi_j\phi_k\phi_l. \tag{91}$$

If we include only 1 field type, there is still the multiplicity of the generations, $n_{\phi,g}$, making λ contain $n_{\phi,g}^4$ free parameters. For our models with 72 scalars, only the quartic interactions alone of type ϕ_{s2}^4 would contain $\approx 3 \cdot 10^5$ free parameters. We would likely see a difference in R_g and R_y, if λ_{ijkl} would all be $\mathcal{O}(1)$ or $\mathcal{O}(0.0001)$, and it would be enlightening to see how the set of unifying models changes. This is quite an uneconomical model and other, more minimalistic approaches might be preferred.

Including new fields and NLO corrections would require new calculations of the RGEs. Tools that do these calculations automatically are not to be underestimated, as the amount of equation juggling that would be needed for e.g. a model with 4 extra scalars, 3 extra fermions, all in different representations at 2 loop, is tedious at minimum. A brute force search in model-space would be even worse.

Upon adding new fields, gauge, scalar or fermion, there will be new interactions that could affect commonly used observables to evaluate these models, e.g. proton decay. Here one should do a study of the effects that the new interactions have, as this could be used as an effective way of ruling the models out.

7 Conclusion

The SM is a hugely successful framework for explaining most of the experimental observations in high energy physics that we have today. There are measurements that are not explainable by the theory, and together with some theoretical conundrums, the quantization of the hypercharges and the near gauge unification at high scale, leads to the expectation of new physics at higher energies. Scalar extensions are well motivated from several theories, and the RG-flow towards higher energies is investigated in search for models that allow unification of the Yukawas and the gauge couplings.

In this work we have seen that by adding scalars in the SM representations $(3,1)_0$, $(\overline{3},1)_0$ and $(1,2)_0$, we are able at 1-loop to unify the gauge couplings at the GUT scale with a high number of models. Yukawa unification is harder to achieve with these extra degrees of freedom, as y_t is much larger than y_b and y_τ, and there is no explicit 1-loop effect on the Yukawa RGEs. The indirect effects give an improvement in R_y from $96 \to 85$ at the GUT scale, at the cost of adding 24 of each scalar type representation to the model with a pole mass of 10^3 GeV.

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