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Abstract

We consider a quantum field theory that creates at any point of the space-time particles described by a q-deformed Heisenberg algebra which is interpreted as a phenomenological quantum theory describing the scattering of spin-0 composed particles. We discuss the generalization of Wick's expansion for this case and we compute perturbatively the scattering $1+2 \to 1'+2'$ to second order in the coupling constant. The result we find shows that the structure of a composed particle, described here phenomenologically by the deformed algebraic structure, can modify in a simple but non-trivial way the perturbation expansion for the process under consideration.

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1 Introduction

Heisenberg algebra is an essential tool in the second quantization formalism since its generators are interpreted as creating or annihilating particle states. In [1] it was shown that a class of quantum systems characterized by having successive energy eigenvalues obeying $\epsilon_{n+1} = f(\epsilon_n)$ is described by a generalized Heisenberg algebra (GHA) [2]-[3] with the function $f(x)$ being a characteristic function of the algebra.

The representations of the GHA were studied in [3] and they were constructed by studying the stability of the fixed points of the characteristic function of the algebra and of their composed functions. Moreover, the non-relativistic [1] and relativistic [4] square-well potentials in one-dimension and the harmonic oscillator on a circle [5] were shown to be described by a GHA with an appropriated function $f(x)$ for each physical system.

When the characteristic function of the algebra $f(x)$ is linear with slope $\tan^{-1}(q^2)$ the algebra turns into [3] the well known q-oscillator algebra [6]. The algebra of q-oscillators has found applications in different areas of physics [7] and in particular in [8] it was shown that creation and annihilation operators of correlated fermion pairs, in simple many body systems, satisfy a deformed Heisenberg algebra that can be approximated by q-oscillators. Since the combined pairs of fermions can be viewed as a composed system it seems reasonable to explore the consequences of using q-oscillators as an approximated way of considering composed particles in the context of the formalism of second quantization.

Along these lines we constructed in [9] a quantum field theory (QFT) that creates at any point of the space-time particles described by a q-deformed Heisenberg algebra interpreting it as a phenomenological QFT describing the interaction of composed particles. In that paper we constructed the propagator, defined as the Dyson-Wick contraction of two fields, for the free theory and computed perturbatively the scattering $1 + 2 \to 1' + 2'$ to first order in the coupling constant.

In order to compute perturbatively this scattering process to higher orders in the coupling constant it is necessary to extend the Wick's expansion since, in this case, the propagator is not a c-number. In this paper we discuss the extension of the Wick's expansion adapted to this case and we compute perturbatively the scattering under consideration to second order in the coupling constant. The result we find shows that the structure of a composed particle, described here phenomenologically by the algebraic structure, can modify in a simple but non-trivial way the perturbation expansion for the process under consideration.

In section 2, we summarize the GHA, we realize the linear case of the algebra in terms of physical variables and we discuss its interpretation in terms of a phenomenological description of composed particles in the context of the formalism of second quantization. In section 3, after a brief review of the scattering to first order in the coupling constant we discuss the necessity to extend the Wick's expansion in order to compute scattering processes to higher orders in the coupling constant and we present the main result of the paper, i.e., the scattering $1 + 2 \to 1' + 2'$ to second order in the coupling constant. In section 4, we discuss our results and present two conjectures. Finally, we present in appendix A the result for the Wick's expansion of four fields which contains the main modifications necessary to construct a general Wick's expansion for the product of any number of fields based on a GHA.

2 Generalized Heisenberg algebras

Let us consider a class of quantum systems characterized by having energy eigenvalues obeying

$$\epsilon_{n+1} = f(\epsilon_n) \, . \tag{1}$$

where ϵ_{n+1} and ϵ_n are successive energy levels of the physical system under consideration and $f(x)$ is a different function for each physical system.

We showed in [1] that this class of quantum systems is described by a generalized Heisenberg algebra (GHA). The GHA is generated by J_0, A and $A^\dagger$ and described by the relations [3]

$$\begin{aligned} J_0 \, A^\dagger &= A^\dagger f(J_0), & (2) \\ A \, J_0 &= f(J_0) \, A, & (3) \\ \left[A, A^\dagger\right] &= f(J_0) - J_0, & (4) \end{aligned}$$

where † means the Hermitian conjugate and, by hypothesis, $J_0^\dagger = J_0$ and $f(x)$ is a general analytic function. J_0 is the Hamiltonian operator of the system. $A^\dagger$ and A are the ladder operators of the physical system. We stress that the functions $f(x)$ which appear in eq. (1) characterizing the physical system and in eqs. (2-4) characterizing the algebraic structure are the same.

This algebra has a Casimir operator given by

$$C = A^\dagger A - J_0 = A A^\dagger - f(J_0). \tag{5}$$

As proved in [3], under the hypothesis that there is a vacuum state represented by $|0\rangle$, for a general function f we obtain

$$J_0\,|m\rangle = f^m(\alpha_0)\,|m\rangle, \quad m = 0, 1, 2, \cdots . \tag{6}$$
$$A^\dagger\,|m-1\rangle = N_{m-1}\,|m\rangle, \tag{7}$$
$$A\,|m\rangle = N_{m-1}\,|m-1\rangle, \tag{8}$$

where $N_{m-1}^2 = f^m(\alpha_0) - \alpha_0$, α_0 is the lowest J_0 eigenvalue and $f^m(\alpha_0)$ is the mth iteration of α_0 through function f. For each function $f(x)$ the representations of the algebra are constructed by the analysis of the above equations as done in [3] for the linear and quadratic $f(x)$.

Let us now focus on the linear case, $f(x) = q^2x + s$, since it will be used in the next section to construct a QFT. The algebraic relations for the linear case can be rewritten, after the rescaling $J_0 \to sJ_0$, $A \to A/\sqrt{s}$ and $A^\dagger \to A^\dagger/\sqrt{s}$, as

$$\left[J_0, A^\dagger\right]_{q^2} = A^\dagger, \tag{9}$$
$$\left[J_0, A\right]_{q^{-2}} = -\frac{1}{q^2}A, \tag{10}$$
$$\left[A^\dagger, A\right] = (1 - q^2)\,J_0 - 1, \tag{11}$$

where $[a, b]_r \equiv a\,b - r\,b\,a$ is the r-deformed commutation of two operators a and b.

In this linear case we can see that

$$N_{m-1}^2 = f^m(\alpha_0) - \alpha_0 = [m]_{q^2}\,N_0^2\ , \tag{12}$$

where $f^m(\alpha_0)$ denotes the m-th iterate of f, $[m]_r \equiv (r^m - 1)/(r - 1)$ is the Gauss number of m and $N_0^2 = \alpha_0\,(q^2 - 1) + 1$. Moreover, defining $b^\dagger = A^\dagger/(\pm N_0)$, $b = A/(\pm N_0)$ and N such that $N|m\rangle = m|m\rangle$, we can also see that these operators satisfy

$$b\,b^\dagger - q^2\,b^\dagger\,b = 1\ , \tag{13}$$
$$[N, b] = -b\ ,\quad \left[N, b^\dagger\right] = b^\dagger\ .$$

which corresponds to the usual q-oscillator relations [6].

Let us consider a one-dimensional lattice in momentum space. As it is well known, the two possible definitions of discrete derivatives on this lattice are

$$
\begin{aligned}
(\partial_p f)(p) &= \frac{1}{a}[f(p+a) - f(p)] , & (14)\\
(\bar{\partial}_p f)(p) &= \frac{1}{a}[f(p) - f(p-a)] , & (15)
\end{aligned}
$$

where a is the lattice spacing. It is also possible to introduce the momentum shift operators

$$
\begin{aligned}
T &= 1 + a\,\partial_p , & (16)\\
\bar{T} &= 1 - a\,\bar{\partial}_p , & (17)
\end{aligned}
$$

that move the momentum value by a

$$
\begin{aligned}
(Tf)(p) &= f(p+a) , & (18)\\
(\bar{T}f)(p) &= f(p-a) & (19)
\end{aligned}
$$

and satisfy

$$T\bar{T} = \bar{T}T = \hat{1} . \qquad (20)$$

where $\hat{1}$ means the identity on the algebra of functions of p. Finally, we also introduce the momentum operator P [10]

$$(Pf)(p) = p\,f(p) . \qquad (21)$$

Now, we are going to present the realization of the deformed Heisenberg algebra eqs. (9-11) in terms of physical operators. We can associate to the one-parameter deformed Heisenberg algebra in eqs. (9-11) the one-dimensional lattice we have just presented. Observe that we can write J_0 in this case as

$$J_0 = q^{2P/a}\,\alpha_0 + [P/a]_{q^2} \ , \qquad (22)$$

with P given by eq. (21). The application of the operator P to the vector states $|m\rangle$ appearing in (6-8) gives [1]

$$P\,|m\rangle = m\,a\,|m\rangle \ , m = 0, 1, \cdots \ , \qquad (23)$$

which can be written as $N = P/a$ with $N|m\rangle = m|m\rangle$. Moreover,

$$\bar{T}\,|m\rangle = |m+1\rangle \;, m = 0, 1, \cdots \;, \tag{24}$$

where $\bar{T}$ and $T = \bar{T}^\dagger$ are defined in eqs. (16-17).

With the definition of J_0 given in eq. (22) we see that $f^m(\alpha_0)$ given in eq. (6) is the J_0 eigenvalue of the state $|n\rangle$ as we wish. Let us now define

$$A^\dagger = S(P)\,\bar{T} \;, \tag{25}$$
$$A = T\,S(P) \;, \tag{26}$$

where,

$$S(P)^2 = J_0 - \alpha_0 \;, \tag{27}$$

α_0 being the lowest J_0 eigenvalue. It was proven in [9] that the realization given in eqs. (22 and 25-26) really satisfies eqs. (9-11).

Now, we are going to discuss an interpretation of the deformed Heisenberg algebra that will be used in the next section. It is well known that Heisenberg algebra is an essential tool in the second quantization formalism because their generators create and annihilate point particles. As in the generalized case the energy difference of any two successive levels is not equal, one can still consider that the ladder operators of the deformed Heisenberg algebra create and annihilate particles with the difference that the total energy of n particles is not equal to n times the energy of each particle. The next question to be answered is what kind of free physical particle can have this non-additive energy.

In [8] it was shown that the algebra of fermion pairs of zero angular momentum can be approximated by the q-oscillator algebra, eq. (13). Moreover, the pairing Hamiltonian has the above mentioned non-additivity property. Let us briefly focus on the shell model of nuclear collective motion. Fermion pairs of angular momentum $J = 0$ in the theory of pairing in a single-j shell are created by the pair-creation operator

$$B^\dagger = \frac{1}{\sqrt{\Omega}} \sum_{m>0} (-1)^{j+m}\, f^\dagger_{j,m} f^\dagger_{j,-m} \;, \tag{28}$$

with $-j \le m \le j$, where $f^\dagger_{j,m}$ are fermion creation operators and $2\Omega = 2j+1$ is the degeneracy of the shell. The pair creation operator just defined and the annihilation operator satisfy a deformed Heisenberg algebra given by

$$\left[B, B^\dagger\right] = 1 - \frac{N_F}{\Omega} \;, \tag{29}$$

with $N_F = \sum_{m>0}(f^\dagger_{j,m} f_{j,m} + f^\dagger_{j,-m} f_{j,-m})$, the fermion number operator while the pairing Hamiltonian is $H = -G\Omega B^\dagger B$. In [8] it was shown that the deformed algebra of composed operators given in eq. (29) can be approximated by the q-oscillator algebra given in eq. (13) with $q = \exp(-1/\Omega)$ and the pairing Hamiltonian being approximated by the q-oscillator Hamiltonian $H = -G\Omega[N]_{q^2}$.

From the fact that the combined pairs of fermions created by the operator $B^\dagger$ can be viewed as a composed system that is approximated by the q-oscillator algebra eq. (13) with $q = \exp(-1/\Omega)$, it seems reasonable to explore the consequences of using q-oscillators as an approximated way to describe composed particles in the context of the formalism of second quantization.

3 Perturbative computation in a deformed quantum field theory

We are going to discuss in this section a QFT having as excitations objects described by the one-parameter deformed algebra given in eqs. (9-11). In this QFT the mass spectrum consists of only one particle with mass m. In this case the energy of n particles is not equal to n times the energy of one particle and therefore the energy does not obey the additivity rule. This non-additivity comes from the fact that q-oscillators are seen as an approximated way to describe composed particles in the context of the formalism of second quantization.

3.1 First order analysis

In [9], following similar steps to those used to construct a standard spin-0 quantum field theory [12] we analyzed a deformed QFT to first order in the coupling constant. The initial observation is that the analog of the Heisenberg algebra obeyed by the quantum excitations of a standard QFT is in this case

$$[\chi, P] = iaQ\,, \tag{30}$$

$$[P, Q] = ia\chi\,, \tag{31}$$

$$[\chi, Q] = 2iS(P)(S(P+a) - S(P-a))\,, \tag{32}$$

where

$$\chi \equiv i\left(S(P)(1-a\bar{\partial}_p)-(1+a\partial_p)S(P)\right) = -i(A-A^\dagger) \,. \qquad (33)$$
$$Q \equiv S(P)(1-a\bar{\partial}_p)+(1+a\partial_p)S(P) = A+A^\dagger \,, \qquad (34)$$

P is defined in eq. (21) and ∂_p and $\bar{\partial}_p$ are the left and right discrete derivatives defined in eqs. (14, 15).

With the help of χ and Q it is possible to define

$$\phi(\vec{r},t) = \sum_{\vec{k}} \frac{1}{\sqrt{2\Omega\omega(\vec{k})}}\left(A^\dagger_{\vec{k}}\, e^{-i\vec{k}.\vec{r}} + A_{\vec{k}}\, e^{i\vec{k}.\vec{r}}\right) . \qquad (35)$$

$$\Pi(\vec{r},t) = \sum_{\vec{k}} \frac{i\omega(\vec{k})}{\sqrt{2\Omega\omega(\vec{k})}}\left(A^\dagger_{\vec{k}}\, e^{-i\vec{k}.\vec{r}} - A_{\vec{k}}\, e^{i\vec{k}.\vec{r}}\right) . \qquad (36)$$

where $\omega(\vec{k}) = \sqrt{\vec{k}^2+m^2}$, m is a real parameter and Ω is the volume of a rectangular box and

$$\varphi(\vec{r},t) = \sum_{\vec{k}} \sqrt{\frac{\omega(\vec{k})}{2\Omega}}\; S_{\vec{k}}\, e^{i\vec{k}.\vec{r}} . \qquad (37)$$

We stress that an independent copy of the one-dimensional momentum lattice defined in the previous section was introduced in each point of this $\vec{k}$-lattice so that $P^\dagger_{\vec{k}} = P_{\vec{k}}$ and $T_{\vec{k}}$, $\bar{T}_{\vec{k}}$ and $S_{\vec{k}}$ have the same definitions given previously, eqs. (16-17) and (27), through the substitution $P \to P_{\vec{k}}$. Moreover,

$$A^\dagger_{\vec{k}} = S_{\vec{k}}\,\bar{T}_{\vec{k}} \,, \qquad (38)$$
$$A_{\vec{k}} = T_{\vec{k}}\,S_{\vec{k}} \,, \qquad (39)$$
$$J_0(\vec{k}) = q^{2P_{\vec{k}}/a}\,\alpha_0 + \left[P_{\vec{k}}/a\right]_{q^2} \,, \qquad (40)$$

satisfy the same algebra given in eqs. (9-11) for each point of this $\vec{k}$-lattice and the operators $A^\dagger_{\vec{k}}$, $A_{\vec{k}}$ and $J_0(\vec{k})$ commute among them for different points of this $\vec{k}$-lattice.

By a straightforward computation the Hamiltonian

$$H = \int d^3r \left(\Pi(\vec{r},t)^2 + u|\varphi(\vec{r},t)|^2 + \phi(\vec{r},t)(-\vec{\nabla}^2+m^2)\phi(\vec{r},t)\right) . \qquad (41)$$

where u is an arbitrary number, can be written as

$$H = \frac{1}{2}\sum_{\vec{k}} \omega(\vec{k}) \left(S_{\vec{k}}(N+1)^2 + (1+u)\, S_{\vec{k}}(N)^2 - (q^2-1)\alpha_0 - 1 \right) \ . \qquad (42)$$

Note that in the limit $q \to 1$ ($u \to 0$), the above Hamiltonian is proportional to the number operator.

The eigenvectors of H form a complete set and span the Hilbert space of this system, they are

$$|0\rangle,\ A^{\dagger}_{\vec{k}}|0\rangle,\ A^{\dagger}_{\vec{k}}A^{\dagger}_{\vec{k}'}|0\rangle \text{ for } \vec{k} \neq \vec{k}',\ (A^{\dagger}_{\vec{k}})^2|0\rangle,\ \cdots , \qquad (43)$$

where the state $|0\rangle$ satisfies as usual $A_{\vec{k}}|0\rangle = 0$ for all $\vec{k}$ and $A_{\vec{k}}$, $A^{\dagger}_{\vec{k}}$ for each $\vec{k}$ satisfy the q-deformed Heisenberg algebra given by eqs. (9-11).

Let us define $E^{(n)}(\vec{k})$ as the energy eigenvalue of the state $(A^{\dagger}_{\vec{k}})^n|0\rangle$. Note that for the Hamiltonian in eq. (42) we have $E^{(2)}(\vec{k}) \neq 2E^{(1)}(\vec{k})$ which is a property of non-additivity similar to that we commented just above of eq. (28) for the composed system made with fermions pairs.

The time evolution of the fields can be studied by means of Heisenberg's equation for $A^{\dagger}_{\vec{k}}$, $A_{\vec{k}}$ and $S_{\vec{k}}$. Now, let us define

$$\begin{aligned} h(N_{\vec{k}}) &\equiv \frac{1}{2}(1+u+q^2)\left(S^2(N_{\vec{k}}+1) - S^2(N_{\vec{k}})\right) \\ &\equiv \frac{1}{2}(1+u+q^2)\Delta E(N_{\vec{k}}) \ . \end{aligned} \qquad (44)$$

Thus, using eqs. (42) and (9-11) we obtain

$$\left[H, A^{\dagger}_{\vec{k}}\right] = \omega(\vec{k})\, A^{\dagger}_{\vec{k}}\; h(N_{\vec{k}}) \ . \qquad (45)$$

The Heisenberg equation can be solved and the Fourier transformation shown in eq. (35) can thus be written as

$$\phi(\vec{r},t) = \alpha(\vec{r},t) + \alpha(\vec{r},t)^{\dagger} \ , \qquad (46)$$

where

$$\alpha(\vec{r},t) = \sum_{\vec{k}} \frac{1}{\sqrt{2\Omega\omega(\vec{k})}}\; A_{\vec{k}}\, e^{i\vec{k}.\vec{r} - iq^{-2}\omega(\vec{k})\, h(N_{\vec{k}})\, t} \ . \qquad (47)$$

$A_{\vec{k}}$ given in eq. (47) is time-independent and $\alpha(\vec{r},t)^{\dagger}$ is the Hermitian conjugate of $\alpha(\vec{r},t)$.

The Feynman propagator $D^N_F(x_1,x_2)$ defined, as usual, as the Dyson-Wick contraction between [1] $\phi(x_1)$ and $\phi(x_2)$, can be computed using eqs.

[1] Our notation means that $x_i \equiv (\vec{r}_i, t_i)$

(9-11) and (46-47) and is given in the integral representation as

$$D_F^N(x) = \frac{-i}{(2\pi)^4}\int \frac{S(N_{\vec{k}}+1)^2 e^{i\vec{k}.\vec{r}-ik_0\,h(N_{\vec{k}})\,t}\, d^4k}{k^2+m^2} - (N \to N-1)\,, \qquad (48)$$

where, in the second part of the right hand side of the above equation can be obtained just doing $N \to N-1$. Note that when $q \to 1$, $h(N_{\vec{k}}) \to 1$ and $S_{\vec{k}}(N+1)^2 - S_{\vec{k}}(N)^2 \to 1$, the standard result for the propagator is recovered. It is interesting to point out that this propagator is not a simple c-number since it depends on the number operator N.

We shall now present the result of the first order scattering process $1+2 \to 1'+2'$ for $p_1 \neq p_2 \neq p_1' \neq p_2'$ with the initial state

$$|1,2\rangle \equiv A^\dagger_{p_1} A^\dagger_{p_2}|0\rangle\ , \qquad (49)$$

and the final state

$$|1',2'\rangle \equiv A^\dagger_{p_1'} A^\dagger_{p_2'}|0\rangle\ , \qquad (50)$$

where A_{p_i} and $A^\dagger_{p_i}$ satisfy the algebraic relations in eqs. (9-11).These particles are supposed to be described by the Hamiltonian given in eq. (41) with an interaction given by $\lambda\int : \phi(\vec{r},t)^4 : d^3r$. To the lowest order in λ, we have (now S means the standard S-matrix)

$$\langle 1',2'|S|1,2\rangle = -i\lambda\int d^4x\langle 1',2'| : \phi^4(x) : |1,2\rangle\ . \qquad (51)$$

In [9] we computed the first order scattering process and we obtained the following result

$$\langle 1',2'|S|1,2\rangle = \frac{-6(2\pi)^4 i {N_0}^6\lambda}{Q\Omega^2\sqrt{\omega_{\vec{p}_1}\omega_{\vec{p}_2}\omega_{\vec{p'}_1}\omega_{\vec{p'}_2}}}\delta^4(P_1+P_2-P_1'-P_2')\ , \qquad (52)$$

where

$$P_i = (\vec{p}_i, \omega_{\vec{p}_i}),\ P_i' = (\vec{p'}_i, \omega_{\vec{p'}_i}) \qquad (53)$$

and $Q = (1+u+q^2)/2$. Note that when $q \to 1$ we have $N_0 \to 1$, $u = 0$, $Q \to 1$ and eq. (52) becomes the standard undeformed result [12].

3.2 Wick's expansion and second order computation

In order to compute the scattering process $1+2 \to 1'+2'$ to second order in the coupling constant we must use the Wick's expansion. The propagator in the present case, see eq. (48), is not a simple c-number since it depends on the number operator N. This fact induces to modifications in the standard Wick expansion.

The consequences of the propagator being not a c-number can already be seen in the Wick's expansion for three fields. After standard calculations we obtain

$$T(\phi(x_1)\phi(x_2)\phi(x_3)) =: \phi(x_1)\phi(x_2)\phi(x_3) : + : \underbrace{\phi(x_1)\phi(x_2)}\phi(x_3) : \\ + : \phi(x_1)\underbrace{\phi(x_2)\phi(x_3)} : + : \underbrace{\phi(x_1)\phi(x_2)\phi(x_3)} : . \quad (54)$$

where $:\phi(x_1)\phi(x_2)\phi(x_3):$ is the standard normal order of the product of three fields and

$$: \underbrace{\phi(x_1)\phi(x_2)}\phi(x_3) : = D_F^N(x_1, x_2)\, \phi(x_3) . \quad (55)$$

$$: \phi(x_1)\underbrace{\phi(x_2)\phi(x_3)} : = \phi(x_1) D_F^N(x_2, x_3) . \quad (56)$$

$$: \underbrace{\phi(x_1)\phi(x_2)\phi(x_3)} : = D_F^N(x_1, x_3)\alpha(x_2) + \alpha^\dagger(x_2) D_F^N(x_1, x_3) . \quad (57)$$

Note that

$$\phi(x_2) D_F^N(x_1, x_3) \neq D_F^N(x_1, x_3)\phi(x_2) \neq D_F^N(x_1, x_3)\alpha(x_2) + \alpha^\dagger(x_2) D_F^N(x_1, x_3) . \quad (58)$$

since the propagator is not a c-number. As will be shown in appendix **A** for the Wick's expansion of four fields, for four fields or more an additional difference appears. In certain terms of this expansion the order the number operator appearing in the propagator is altered by an integer.

We are going now to compute the scattering process of the previous subsection to second order in λ

$$\langle 1', 2' | S | 1, 2 \rangle_2 = \frac{(-i)^2}{2} \lambda^2 \int\int d^4x\, d^4y \langle 1', 2' | T(: \phi^4(x) :: \phi^4(y) :) | 1, 2 \rangle . \quad (59)$$

where T denotes the time-ordered product. In order to convert the time-ordered product into a normal product we use the Wick's expansion taking into account that the propagator is not a c-number. This was done using a program of algebraic computation. Coming from the Wick's expansion of $T(:\phi^4(x)::\phi^4(y):)$ there are three representative terms that contibute to the scattering process of eq. (59) up to second order in λ. they are:

$$\alpha^\dagger(x)\alpha^\dagger(x)\alpha(y)\alpha(y)D_F^N(x,y)D_F^N(x,y)\,, \tag{60}$$

$$\alpha^\dagger(y)\alpha^\dagger(y)\alpha(x)\alpha(x)D_F^N(x,y)D_F^N(x,y)\,. \tag{61}$$

$$\alpha^\dagger(x)\alpha^\dagger(y)\alpha(x)\alpha(y)D_F^N(x,y)D_F^N(x,y)\,. \tag{62}$$

All the other terms contributing to the second order scattering process are different from the above terms only by the position of the propagators in eqs. (60-62) or by a shift of the type $N_{\vec{k}_i} \to N_{\vec{k}_i} + \delta_{\vec{k}_i,\vec{p}_j}$ in the propagators appearing in eqs. (60-62).

Let us firstly compute the second order contribution to the scattering under consideration coming from the term given in eq. (60). As seen in eq. (48) the propagator has two terms and we start considering only the first term of the propagator since, as it will be clear in what follows, the second term from the propagator gives a trivial contribution. Thus, putting the representative term given in eq. (60) into eq. (59). taking the exponentials and $S(N)$ outside the matrix element. using

$$\begin{aligned}\langle 0|A_{\vec{p'}_1}A_{\vec{p'}_1}A^\dagger_{\vec{k}_1}A^\dagger_{\vec{k}_2}A_{\vec{k}_3}A_{\vec{k}_4}A^\dagger_{\vec{p}_1}A^\dagger_{\vec{p}_2}|0\rangle = N_0^4\Big(N_0^4\delta_{\vec{k}_3\vec{p}_1}\delta_{\vec{k}_4\vec{p}_2}\delta_{\vec{k}_1\vec{p'}_1}\delta_{\vec{k}_2\vec{p'}_2}+\\ N_0^2\Delta E(\delta_{\vec{k}_2\vec{p'}_2})\delta_{\vec{k}_3\vec{p}_1}\delta_{\vec{k}_4\vec{p}_2}\delta_{\vec{k}_2\vec{p'}_1}\delta_{\vec{k}_1\vec{p'}_2}+N_0^2\Delta E(\delta_{\vec{k}_4\vec{p}_2})\delta_{\vec{k}_3\vec{p}_2}\delta_{\vec{k}_4\vec{p}_1}\delta_{\vec{k}_1\vec{p'}_1}\delta_{\vec{k}_2\vec{p'}_2}+\\ \Delta E(\delta_{\vec{k}_4\vec{p}_2})\Delta E(\delta_{\vec{k}_2\vec{p'}_2})\delta_{\vec{k}_3\vec{p}_2}\delta_{\vec{k}_4\vec{p}_1}\delta_{\vec{k}_2\vec{p'}_1}\delta_{\vec{k}_1\vec{p'}_2}\Big)\end{aligned} \tag{63}$$

we can sum over the $\vec{k}$'s coming from the Fourier expansion of $\alpha(x)$ given in eq. (47) obtaining, after a redefinition of the time as $t \to t/h(0)$,

$$\begin{aligned}\langle 1',2'|S|1,2\rangle_2^a = \frac{N_0^4\lambda^2}{2\Omega^2Q^2(2\pi)^8\sqrt{\omega_{\vec{p}_1}\omega_{\vec{p}_2}\omega_{\vec{p'}_1}\omega_{\vec{p'}_2}}}\int\frac{d^4xd^4yd^4k_1d^4k_2}{(k_1^2+m^2)(k_2^2+m^2)}\\ S(1+\delta_{\vec{k}_1,\vec{p}_1}+\delta_{\vec{k}_1,\vec{p}_2})^2S(1+\delta_{\vec{k}_2,\vec{p}_1}+\delta_{\vec{k}_2,\vec{p}_2})^2exp[i(\kappa_1+\kappa_2-P'_1-P'_2).x\\ +i(P_1+P_2-\kappa_1-\kappa_2).y]\;.\end{aligned} \tag{64}$$

where

$$\kappa_i = \left(\vec{k}_i, h_{i,0}\, k_i^0\right) \ , i = 1, 2 \ , \tag{65}$$
$$k_i = \left(\vec{k}_i, k_i^0\right) \ . \tag{66}$$
$$h_{i,0} = h\left(\delta_{\vec{k}_i,\vec{p}_1} + \delta_{\vec{k}_i,\vec{p}_2}\right)/h(0) \ . \tag{67}$$

Using the property $\int_{-\infty}^{\infty} dx\, f(x+\delta_{x,x_0}) = \int_{-\infty}^{\infty} dx\, f(x)$ we can integrate eq. (64) over x, y and using standard properties of delta functions we can also integrate over k_1 or k_2 obtaining

$$\langle 1', 2'|S|1, 2\rangle_2^a = \frac{N_0^8 \lambda^2}{2\Omega^2 Q^2 \sqrt{\omega_{\vec{p}_1}\omega_{\vec{p}_2}\omega_{\vec{p'}_1}\omega_{\vec{p'}_2}}}\delta^4\left(P_1 + P_2 - P'_1 - P'_2\right) \ I \ , \tag{68}$$

where I is the standard one loop divergent integral that appears in the usual λ-ϕ^4 model with value

$$I = \int d^4k \, \frac{1}{(k^2 + m^2)\left[(-k+s)^2 + m^2\right]} \ . \tag{69}$$

where $s = P_1 + P_2$. As usual the finite part of this integral can be computed using for example the method of dimensional regularization [13] giving the standard result.

We recall again that the propagator (see eq. (48)) has two terms and in the above computation we considered only the first term of the propagator. Now let us discuss the consequence of using the second term of the propagator in eq. (48) for the first propagator appearing in eq. (60) and the first term of the propagator for the second propagator appearing in eq. (60). After a similar computation as the one described above that resulted in eq. (68) we obtain

$$\frac{N_0^6 S(0)^2 \lambda^2}{2\Omega^2 Q^2 \sqrt{\omega_{\vec{p}_1}\omega_{\vec{p}_2}\omega_{\vec{p'}_1}\omega_{\vec{p'}_2}}}\delta^4\left(P_1 + P_2 - P'_1 - P'_2\right) \ I \ , \tag{70}$$

which gives a trivial result since $S(0)^2 = 0$. We will have this trivial result every time the second term of the propagator enters the game. Thus, the second order contribution to the scattering under consideration coming from the term given in eq. (60) is given in eq. (68). We note that the difference from the standard spin-0 comes only from the constant N_0^8/Q^2 which goes to one when $q \to 1$.

Now, let us compute the contribution to the scattering in eq. (59) which arises from the term shown in eq. (61). This computation goes along the same lines as the previous computation and the result is

$$\langle 1', 2'|S|1,2\rangle_2^b = \frac{N_0^8\lambda^2}{2\Omega^2 Q^2\sqrt{\omega_{\vec{p}_1}\omega_{\vec{p}_2}\omega_{\vec{p'}_1}\omega_{\vec{p'}_2}}}\delta^4(P_1 + P_2 - P'_1 - P'_2)\ I' \ . \quad (71)$$

where $I' = I(s \to -s)$.

Finally, we discus now the contribution to the scattering in eq. (59) coming from the term shown in eq. (62). The first thing to do is to put the representative term given in eq. (62) into eq. (59), next, performing the following steps:

1. Take the exponentials and $S(N)$ outside the matrix element.
2. use eq. (63),
3. sum over the $\vec{k}$'s coming from the Fourier expansion of $\alpha(x)$,
4. redefine $t \to t/h(0)$,
5. use the property given just below eq. (67),
6. integrate over d^4x and d^4y,

we obtain

$$\langle 1', 2'|S|1,2\rangle_2^c = \frac{N_0^8\lambda^2}{8\Omega^2 Q^2\sqrt{\omega_{\vec{p}_1}\omega_{\vec{p}_2}\omega_{\vec{p'}_1}\omega_{\vec{p'}_2}}}\int\frac{d^4xd^4yd^4k_1d^4k_2}{(k_1^2+m^2)(k_2^2+m^2)}$$
$$\Big[\delta^4(k_1+k_2+P_1-P'_1)\delta^4(-k_1-k_2+P_2-P'_2)+$$
$$\delta^4(k_1+k_2+P_2-P'_2)\delta^4(-k_1-k_2+P_1-P'_1)+$$
$$\delta^4(k_1+k_2+P_1-P'_2)\delta^4(-k_1-k_2+P_2-P'_1)+$$
$$\delta^4(k_1+k_2+P_2-P'_1)\delta^4(-k_1-k_2+P_1-P'_2)\Big] \ . \quad (72)$$

Note that the first two terms in the main bracket correspond to a contribution in the t-channel while the last two in the u- channel. Considering separately the contributions in the two channels we have

$$\langle 1',2'|S|1,2\rangle_2^{c,t} = \frac{N_0^8\lambda^2}{8\Omega^2 Q^2\sqrt{\omega_{\vec{p}_1}\omega_{\vec{p}_2}\omega_{\vec{p'}_1}\omega_{\vec{p'}_2}}}\delta^4(P_1+P_2-P'_1-P'_2)\ I'' \ . \quad (73)$$

$$\langle 1',2'|S|1,2\rangle_2^{c,u} = \frac{N_0^8\lambda^2}{8\Omega^2 Q^2\sqrt{\omega_{\vec{p}_1}\omega_{\vec{p}_2}\omega_{\vec{p'}_1}\omega_{\vec{p'}_2}}}\delta^4(P_1+P_2-P'_1-P'_2)\ I''' \ . \quad (74)$$

where $I'' = I(s \to t)$ and $I''' = I(s \to u)$ with $t = P_1 - P'_1$ and $t = P_1 - P'_2$.

We have computed so far the contribution to the scattering process $1 + 2 \to 1' + 2'$ to second order in λ coming from the representative terms given in eqs. (60- 62). The other terms appearing in the generalized Wick's expansion of $T(:\phi^4(x)::\phi^4(y):)$ that contribute to the scattering are of the form given in eqs. (60-62) having the propagator in different positions of the product. Moreover, in these terms N has possible shifts of the type $N_{\vec{q_i}} \to N_{\vec{q_i}} + n_1\delta_{\vec{q_i},\vec{k_1}} + n_2\delta_{\vec{q_i},\vec{k_2}} + n_3\delta_{\vec{q_i},\vec{k_3}} + n_4\delta_{\vec{q_i},\vec{k_4}}$, where $n_j = 0,1,2,3$, $\vec{q_i}$ is the momentum associated with the propagator and $\vec{k_j}$ the momenta of the fields. But, since we have always a finite number of deltas in this shift and the functions $S(x)$ and $h(x)$ that will carry these shifts are finite at the shifted points then it will be possible to exclude the finite number of the shifted points. The final result will be independent on the position where the propagator is inside the product shown in eqs. (60-62) and it is also independent of the shifts. Thus the result of using any other term of the Wick's expansion of $T(:\phi^4(x)::\phi^4(y):)$ in eq. (59) if it is different from zero, it will be necessarily one of the three results we presented in eqs. (68), (73) and (74).

In summary, the scattering process $1 + 2 \to 1' + 2'$ for $p_1 \neq p_2 \neq p'_1 \neq p'_2$ with the initial state and the final state given in eqs. (49, 50) respectively, where A_{p_i}, $A^\dagger_{p_i}$ satisfy the algebraic relations in eqs. (9-11) and the particles are supposed to be described by the Hamiltonian given in eq. (41) with an interaction given by $\lambda\int :\phi(\vec{r},t)^4: d^3r$ is given up to second order in the coupling constant λ as

$$\langle 1',2'|S|1,2\rangle = \frac{\lambda N_0^6}{Q}A_1 + \frac{\lambda^2 N_0^8}{Q^2}\left(A_2^s + A_2^t + A_2^u\right) \ , \quad (75)$$

where A_1, A_2^s, A_2^t and A_2^u are the same contributions that we find in the *standard* λ-ϕ^4 (non-deformed) model corresponding to the tree level, the s, t and u channels for one-loop level respectively. Then, the contribution we find in the perturbation series due to the phenomenological way we consider the structure of a particle appears only as non-trivial factors depending on the parameters of the algebra, namely the factor N_0^6/Q for the tree part and N_0^8/Q^2 for the one-loop level.

These algebraic contributions are non-trivial since we cannot get rid of them by a simple redefinition of λ. Note that, depending on the values of the parameters of the algebra we could improve or even destroy the convergence of the perturbation series. This result shows that the structure of a particle can change in a non-trivial way the behavior of the perturbation series corresponding to the physical process involving these particles.

4 Final comments

Motivated by a result given in reference [8] we have used the q-oscillator algebra as an approximated description of composed particles in the context of the formalism of second quantization. We have constructed a QFT which creates at any space-time point, particles described by a q-deformed Heisenberg algebra. Apart a non-propagating term the free Hamiltonian is the standard Klein-Gordon Hamiltonian describing spin-0 particles. The propagator for the free theory, defined as the Dyson-Wick contraction between $\phi(x_1)$ and $\phi(x_2)$, depends on the number operator, thus being not a c-number anymore and this fact introduces differences in the Wick's expansion as explained in the paper.

We have computed the scattering process $1+2 \to 1'+2'$ to second order in the coupling constant and the final result given in eq. (75) shows that the structure of a composed particle, viewed here by the algebraic structure, can modify non-trivially the perturbation expansion of a specific process. The modification we find due to our phenomenological way of treating a scattering of composed particles results as follows. The perturbative expansion corresponding to the scattering under consideration, computed with the standard λ-ϕ^4 interacting term, is given term by term by a factor coming from the algebraic structure multiplying the non-deformed result. This fact may provide interesting surprises when implementing the ideas developed here to specific phenomenological models. Note that depending on the values that appear in the scattering coming from the algebra the convergence of the perturbation series can be changed.

We suspect that the above mentioned rule we found, to second order in the coupling constant, to construct the deformed scattering $1+2 \to 1'+2'$

in terms of the non-deformed one and the algebraic structure is general. The relevant differences from the standard computation are the algebraic part of the matrix element inside the integrals (as in the integrands in eqs. (51) and (59)) and moreover comes from the factor $S(N)$ in the first part of the propagator.

We have developed here a QFT for the linear case of the generalized Heisenberg algebra (GHA) we summarized in section 2. Since a realization similar to the one presented here in eqs. (25)-(27) was presented in [1]-[5] for three non-linear particular cases of the GHA we also suspect that a consistent QFT can also, besides the QFT considered here, be developed for a subclass of the GHA.

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A Wick's expansion of four fields

The Wick's expansion of four fields is the simplest case in which we can see the differences from the standard case. In order to establish our notations, we define the field $\phi(x_i)$ as given in eqs. (46-47) with $\vec{k} \to \vec{k}_i$, and $\vec{k}$ being the momentum to be integrated in the propagator.

With the notations defined above we obtain after standard manipulations

$$T(\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)) =: \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) : + : \underbrace{\phi(x_1)\phi(x_2)}\phi(x_3)\phi(x_4) :$$

$$+ : \underbrace{\phi(x_1)\phi(x_2)\phi(x_3)}\phi(x_4) : + : \underbrace{\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)} : + : \phi(x_1)\underbrace{\phi(x_2)\phi(x_3)}\phi(x_4) :$$

$$+ : \phi(x_1)\underbrace{\phi(x_2)\phi(x_3)\phi(x_4)} : + : \phi(x_1)\phi(x_2)\underbrace{\phi(x_3)\phi(x_4)} : + : \underbrace{\phi(x_1)\phi(x_2)}\underbrace{\phi(x_3)\phi(x_4)} :$$

$$+ : \underbrace{\phi(x_1)\phi(x_2)\phi(x_3)}\!\!\!\!\!\!\!\underbrace{\phi(x_4)} : + : \underbrace{\phi(x_1)\underbrace{\phi(x_2)\phi(x_3)}\phi(x_4)} : , \qquad (76)$$

where

$$: \underbrace{\phi(x_1)\phi(x_2)}\phi(x_3)\phi(x_4) := D_N(x_1, x_2) : \phi(x_3)\phi(x_4) : , \qquad (77)$$

$$: \underbrace{\phi(x_1)\phi(x_2)\phi(x_3)}\phi(x_4) := D_N(x_1, x_3) \left[\alpha(x_2)\alpha(x_4) + \alpha^\dagger(x_4)\alpha(x_2)\right]$$

$$+\alpha^\dagger(x_2) D_N(x_1, x_3)\phi(x_4) , \qquad (78)$$

$$: \underbrace{\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)} := D_N(x_1, x_4)\alpha(x_2)\alpha(x_3) + \alpha^\dagger(x_2) D_N(x_1, x_4)\alpha(x_3) +$$

$$\alpha^\dagger(x_3) D_N(x_1, x_4)\alpha(x_2) + \alpha^\dagger(x_2)\alpha^\dagger D_N(x_1, x_4) , \qquad (79)$$

$$: \phi(x_1)\underbrace{\phi(x_2)\phi(x_3)}\phi(x_4) := \alpha(x_1) D_N(x_2, x_3)\alpha(x_4) + \alpha^\dagger(x_1) D_N(x_2, x_3)\alpha(x_4) +$$

$$\alpha^\dagger(x_4)\alpha(x_1) D_{N+\delta_{\vec{k},\vec{k}_4}}(x_2, x_3) + \alpha^\dagger(x_1) D_N(x_2, x_3)\alpha^\dagger(x_4) . \qquad (80)$$

$$: \phi(x_1)\underbrace{\phi(x_2)\phi(x_3)}\phi(x_4) : = a(x_1)D_N(x_2,x_4)a(x_3) + a^\dagger(x_1)D_N(x_2,x_4)a(x_3) +$$

$$a^\dagger(x_3)a(x_1)D_N(x_2,x_4) + a^\dagger(x_1)a(x_3)D_N(x_2,x_4) \; . \tag{81}$$

$$: \phi(x_1)\phi(x_2)\underbrace{\phi(x_3)\phi(x_4)} : = : \phi(x_1)\phi(x_2) : D_N(x_3,x_4) \; . \tag{82}$$

$$: \underbrace{\phi(x_1)\phi(x_2)}\underbrace{\phi(x_3)\phi(x_4)} : = D_N(x_1,x_2)D_N(x_3,x_4) \; , \tag{83}$$

$$: \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) : = D_N(x_1,x_3)D_N(x_2,x_4) \; . \tag{84}$$

$$: \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) : = D_N(x_1,x_4)D_{N+\delta_{\vec{k},\vec{k}_4}}(x_2,x_3) \; . \tag{85}$$

with $D_{N+\delta_{\vec{k},\vec{k}_4}}$ in eqs. (80) and (85) meaning that we substitute $N_{\vec{k}} + \delta_{\vec{k},\vec{k}_4}$ in place of $N_{\vec{k}}$ in the expression for the propagator being $\vec{k}$ the momentum to be integrated in the propagator. We note that from the operator dependence of the propagator its position in the Wick's expansion is important.

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