

A Brief Review on Canonical Loop Quantum Gravity: The Kinematical Part

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Abstract. In this article, we briefly review the kinematical part of canonical loop quantum gravity. This article starts with tetradic formulation of gravity both in the covariant approach and canonical approach. The next step is to introduce Ashtekar new variables and to apply the Dirac canonical quantization procedure of gravity. By the holonomy representation, one obtains the loop representation of quantum gravity.

1. Introduction

The search for quantum gravity as a quantization of General Relativity has been carried for a long time and had not been completed recently. The research has evolved in various ways, ranging from the perturbative to the non-perturbative theory. The perturbative quantization meets several obstacles, one of them being the non-renormalizable property of the gravitational field. In the other hand, the non-perturbative theories provide possible ways to quantize gravity; they could be broadly categorized into two main branches: string and non-string approach. One of the candidates of non-string approach of quantum gravity is loop quantum gravity (LQG), which is based rigorously on Dirac quantization procedure.

In this article, we review the basic of canonical loop quantum gravity. Due to time and length constraint, we discuss only on the kinematical part. Section II consists of tetradic formulation of gravity as an attempt to treat gravity as a gauge theory. It contains three subsections, reviewing on the covariant approach, canonical approach, and Ashtekar new variables. In this section, we briefly review the Hamiltonian formulation of general relativity. The third section consists the Dirac canonical quantization procedure of gravity. It contains two subsections, which are the quantization via connection and holonomy representation. The previous are problematic, while the latter leads to loop quantum gravity. Section 4 consist the main subject of this article, namely a review of the kinematical part of loop quantum gravity. It consist four subsections which respectively review the cylindrical functions and the Hilbert space of quantum gravity, the Gauss constraint and the kinematical Hilbert space, the graph and spin-network states, and the geometric operators on quanta of space.

The review on this article is mainly based on reviews of the kinematical part of quantum gravity in [1, 2, 3, 4], and can be viewed as a summary of these articles.



2. First Order Formulation Of Gravity

2.1. Covariant Approach

The original and standard metric formulation of General Relativity is usually known as the second order formulation. One could start from the Einstein-Hilbert Lagrangian density $\mathcal{L}_{EH}[g] = R(g)$, such that Einstein-Hilbert action of gravity is:

$$S_{EH}[g] = \Delta R(g) \text{ } ^4\text{vol} = \Delta \star R(g),$$

with R is the Ricci scalar, and metric g as the dynamical variable. Minimizing the variation of action δS with respect to a variation on the metric δg , gives the Euler-Lagrange equation which is equivalent to the vacuum Einstein Field Equation:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 0.$$

It is already mentioned in the begining of this article that gravity could be described as a gauge field, and it would be convenient to have General Relativity rewritten in a form closer to gauge theory. In the latter, a 'physical' field is defined as a section of a bundle. In the case of GR, the bundle is a tangent bundle. The first attempt on this approach is the Einstein-Cartan formulation, or non-coordinate base version of GR, where one has bundle $\mathcal{E} \simeq \mathcal{M} \times \mathcal{F}$, equipped with a (spin) connection $\omega \in \mathfrak{sl}(2, \mathfrak{c}) \times \wedge^1(\mathcal{M})$ on spacetime \mathcal{M} . One defines a local trivialization \mathbf{e} to map the vector bundle to tangent bundle $T\mathcal{M}$. Using this map, one could send all properties on $T\mathcal{M}$ to \mathcal{E} and vice-versa, for example, the frame field or tetrad $e_I = \mathbf{e}(\xi_I) = e_I^\mu \partial_\mu$, satisfying the orthogonality condition $\langle e_I, e_J \rangle = e_I^\mu e_J^\nu q_{\mu\nu} = \eta_{IJ}$. The Riemann tensor on \mathcal{M} is related to the curvature 2-form of E by $R = \mathbf{e}(F)$, so that the Einstein-Hilbert action can be written as:

$$S_{EH}[e] = \Delta e_I^\mu e^{\nu J} F_{\mu\nu}^I \text{ } ^4\text{vol} = \frac{1}{2} \varepsilon_{IJKL} \Delta e^I \wedge e^J \wedge F^{KL}, \quad (1)$$

where now the tetrad e_I is the dynamical variable. In fact, it is more convenient to consider both of the tetrad and spin connection ω as the dynamical variables, such that the action becomes:

$$S_P[e, \omega] = \Delta \mathcal{L}_P[e, \omega] = \frac{1}{2} \varepsilon_{IJKL} \Delta e^I \wedge e^J \wedge F^{KL}(\omega),$$

known as the Palatini action [5]. The connection ω enters from the curvature 2-form $F = d_D\omega$. This is known as the first order formulation of gravity.

Minimizing the variation of δS with respect to δe and $\delta \omega$ gives the following Euler-Lagrange equations:

$$\varepsilon_{IJKL} e^J \wedge d_D e^I = 0, \quad \varepsilon_{IJKL} e^I \wedge F^{KL}(\omega) = 0. \quad (2)$$

If $e^I \neq 0$, the first equation gives the torsionless condition $d_D e^I = 0$, where the solution gives the Levi-Civita connection. The second is the equivalent of the vacuum Einstein Field Equation. Both relations provide the standard dynamics of GR.

2.2. Canonical Approach

As a first step towards the quantization of a system, one needs to obtain the canonical Hamiltonian. For a covariant system, the first step is to apply the 3+1 ADM formulation. The original 3+1 decomposition is done in the metric formulation, this can be seen in [6]. In this article we will only review the tetradic ADM formulation. Let \mathcal{M} be a globally hyperbolic manifold such that local diffeomorphism $\mathcal{M} \rightarrow \Sigma \times \mathbb{R}$ exists for each point on \mathcal{M} . Let us define \vec{n} as the vector field normal to Σ , the 'time' direction for spatial space Σ . The split $\mathcal{M} \rightarrow \Sigma \times \mathbb{R}$

induces the splitting on the fibre \mathcal{F} , such that the 3-connection on Σ is ${}^3\omega \in \mathfrak{su}(2) \times \bigwedge^1(\Sigma)$. The derivative of connection and its corresponding momentum conjugate are, respectively :

$$\frac{d^3\omega}{d\tau} = {}^3\dot{\omega}, \quad \frac{\delta\mathcal{L}_{EH}}{\delta^3\dot{\omega}} = E,$$

with $E = \sqrt{\det q} {}^3e$ is the densitized triads and q is the 3-metric of Σ . Using the time gauge as a gauge fixing, and applying the Hodge star operator on the internal indices of the 3-connections and triads, one obtains:

$$\frac{\delta\mathcal{L}}{\delta^3\dot{\omega}_a^i} = E_i^a = e = \frac{1}{2}\varepsilon_{ijk}\varepsilon^{abc}e_b^je_c^k.$$

The 3-connection and densitized triads are canonically conjugate to each other, such that they satisfy the symplectic structure of the phase space variables from Poisson Brackets:

$$\left\{ {}^3\omega_a^i(x), E_j^b(x') \right\} = \delta_j^i \delta_a^b \delta^3(x - x').$$

The corresponding Hamiltonian density could be obtained from the Legendre transformation:

$$\mathcal{H} = e_I^\mu e^{\nu J} F_{\mu\nu J}^I - E_i^{a3} \dot{\omega}_a^i = \omega_0^i \mathcal{G}_i + NC + N^a \mathcal{C}_a \approx 0.$$

Gravity is a totally constrained system; the Hamiltonian is a summation of constraints density, and therefore $\mathcal{H} \approx 0$. Writing the constraint explicitly [8]:

$$\mathcal{G}_i = \varepsilon_{ij}^k K_a^j E_k^a \approx 0, \tag{3}$$

$$\mathcal{C}_a = D_{[b} \left(K_{a]}^i E_i^a \right) \approx 0. \tag{4}$$

$$\mathcal{C} = \frac{1}{\sqrt{q}} \left(E_i^b E_j^a - E_i^a E_j^b \right) K_a^i K_b^j - \sqrt{q}^3 R \approx 0. \tag{5}$$

\mathcal{H} is the Hamiltonian density; to obtain the Hamiltonian of the system, the corresponding constraint are smeared on the spatial hypersurface:

$$G(\vec{\omega}) = \Delta_\Sigma \omega_0^i \mathcal{G}_i q^{12} d^3x, \tag{6}$$

$$C(N) = \Delta_\Sigma N \mathcal{C} q^{12} d^3x, \tag{7}$$

$$C(\vec{N}) = \Delta_\Sigma N^a \mathcal{C}_a q^{12} d^3x. \tag{8}$$

$G(\vec{\omega})$, $C(\vec{N})$, and $C(N)$ are, respectively, the Gauss, diffeomorphism, and Hamiltonian constraints.

For constrained system, it is important to check that the constraints are consistent to each other. This lead Dirac and Bergmann to propose an algorithm to check the consistency of the constraints [7]. Constraints on a Hamiltonian system are classified into the following conditions: (1) primary constraint, if a constraint is independent from the dynamical equation, otherwise it is called secondary; (2) first class, if a constraint commutes with all other existing constraints on the constraint hypersurface, otherwise it is called second class. First class constraints generates gauge transformation [7, 8].

The constraints in GR (6)-(8) are primary, but second class. To do a quantization on each foliation of the constraint hypersurface, it is more convenient to have a set of first class constraint, which is done in the next subsection.

2.3. Ashtekar New Variables

The procedure proposed by Ashtekar, is an attempt to make the constraints (6)-(8) first class [9]. As another advantage to this, the constraints can be rewritten in nice polynomial forms, which make them easier to quantize.

2.3.1. Holst Term and Holst Action. A Lagrangian of a system is not unique, in the sense that it can be modified as long as the Euler-Lagrange equation is invariant. Using this fact, one could add an additional term labeled as Holst term in the Palatini action [10]:

$$S_H [e, \omega] = \Delta \mathcal{L}_H = \Delta \left[\frac{1}{2} \varepsilon_{IJKL} e^I \wedge e^J \wedge F^{KL} (\omega) + \frac{1}{\gamma} \underbrace{\delta_{IJKL} e^I \wedge e^J \wedge F^{KL} (\omega)}_{\text{Holst term}} \right], \quad (9)$$

with $\delta_{IJKL} = \delta_{I[K} \delta_{L]J}$. The factor γ is a constant known as the Barbero-Immirzi parameter, and the action is called as Holst action [10].

Minimizing the variation of δS for any δe and $\delta \omega$ gives the following Euler-Lagrange equations:

$$\left(\frac{1}{2} \varepsilon_{IJKL} + \frac{1}{\gamma} \delta_{IJKL} \right) e^J \wedge F^{KL} (\omega) = 0, \quad \left(\frac{1}{2} \varepsilon_{IJKL} + \frac{1}{\gamma} \delta_{IJKL} \right) e^I \wedge d_D e^J = 0.$$

The dynamical equation derived from the Holst action are still equivalent to EFE (2) as long as the tetrads are non-degenerate, $e^I \neq 0$.

In the original version, one starts with a complexification of $\mathfrak{sl}(2, \mathbb{C})$ by setting $\gamma = i$, thus gauge group of general relativity is the complexified Lorentz group $SO_{\mathbb{C}}(3, 1)$. With $SO_{\mathbb{C}}(3, 1)$, one has two distinct $SO(3, 1)$ elements (which consist the real $SO(3, 1)$ group, its complex conjugate part, and their dual groups). But due to the reality issue, it is more convenient to use real variables, that is, restricting γ to be real.

2.3.2. Ashtekar Variables. As already been discussed in the previous subsection, the triads ${}^3\omega$ is an element of $\mathfrak{su}(2) \times \bigwedge^1(\Sigma)$. Let us construct another variables which is also an element of $\mathfrak{su}(2)$, with the following transformation:

$$A = {}^3\omega \pm \gamma \star {}^3\omega \in \mathfrak{su}(2) \times \bigwedge^1(\Sigma) \simeq \mathcal{A}, \quad (10)$$

$\star {}^3\omega$ is the Hodge-dual part (in the internal/fibre space) of ${}^3\omega$. A is the half of the Ashtekar 'new' variables [9]. Taking the derivative of connection, and using the Holst Lagrangian density (9), the corresponding momentum conjugate is exactly the densitized triads of the previous case:

$$\frac{dA}{d\tau} = \dot{A}, \quad \frac{\delta \mathcal{L}_H}{\delta {}^3\dot{A}} = E.$$

Using the time gauge and applying the Hodge star operator on the 3-connections and triads, one obtains:

$$\frac{\delta \mathcal{L}_H}{\delta \dot{A}_a^i} = E_i^a = \frac{1}{2} \varepsilon_{ijk} \varepsilon^{abc} e_b^j e_c^k.$$

The symplectic structure is satisfied by $\tilde{E}_j^b = \frac{E_j^b}{\gamma}$ (or by E_j^b , but with a factor γ in the RHS of the following Poisson Bracket):

$$\left\{ A_a^i(x), \tilde{E}_j^b(x') \right\} = \delta_j^i \delta_a^b \delta^3(x - x'), \quad (11)$$

The symplectic pair are known as Ashtekar new variables.

The Hamiltonian density is:

$$\mathcal{H} = e_I^\mu e^{\nu J} F_{\mu\nu J}^I(A) - E_i^{a3} \dot{A}_a^i = A_0^i \mathcal{G}_i + N\mathcal{C} + N^a \mathcal{C}_a \approx 0$$

with the following constraint density in terms of the new variables:

$$\mathcal{G}_i = D_a \tilde{E}_i^a = D_a \tilde{E}_i^a + \varepsilon_{ij}^k A_a^j \tilde{E}_k^a, \quad (12)$$

$$\mathcal{C}_a = F_{ab}^i \tilde{E}_i^b - \frac{1 + \gamma^2}{\gamma} K_a^i \mathcal{G}_i, \quad (13)$$

$$\mathcal{C} = \left(F_{ab}^j - (1 + \gamma^2) \varepsilon^j{}_{mn} K_a^m K_b^n \right) \frac{\varepsilon_j{}^{kl} \tilde{E}_k^a \tilde{E}_l^b}{\det \tilde{E}} + (1 + \gamma^2) \mathcal{G}^i \partial_a \frac{\tilde{E}_i^a}{\det \tilde{E}}. \quad (14)$$

For a special condition $\gamma = i$, the constraint greatly simplifies to a compact form as follows:

$$\mathcal{G}_i = d_a \tilde{E}_i^a, \quad (15)$$

$$\mathcal{C}_a = F_{ab}^i \tilde{E}_i^b, \quad (16)$$

$$\mathcal{C} = F_{ab}^j \frac{\varepsilon_j{}^{kl} \tilde{E}_k^a \tilde{E}_l^b}{\det \tilde{E}}. \quad (17)$$

Smearing the constraints as in the previous subsection:

$$G(\vec{A}) = \Delta_\Sigma A_0^i \mathcal{G}_i q^{12} d^3 x, \quad (18)$$

$$C(N) = \Delta_\Sigma N \mathcal{C} q^{12} d^3 x, \quad (19)$$

$$C(\vec{N}) = \Delta_\Sigma N^a \mathcal{C}_a q^{12} d^3 x, \quad (20)$$

one could obtain the following constraints algebra:

$$\begin{aligned} \left\{ G(\vec{A}), G(\vec{A}') \right\} &= G\left([A_0^i, A_0^{i'}] \right), & \left\{ C(N), C(N') \right\} &= C(\vec{K}) - G(A_a^i K^a), \quad (21) \\ \left\{ G(\vec{A}), C(N') \right\} &= 0, & \left\{ C(N), C(\vec{N}') \right\} &= -C(L_{\vec{N}'} N), \\ \left\{ G(\vec{A}), C(\vec{N}') \right\} &= -G(L_{\vec{N}'} \vec{N}), & \left\{ C(\vec{N}), C(\vec{N}') \right\} &= C(L_{\vec{N}} \vec{N}'), \end{aligned}$$

with vector \vec{K} is defined by $K^a = \tilde{E}_i^a \tilde{E}_j^b \delta^{ij} (N \partial_b N' - N' \partial_b N)$. The algebra, although they do not construct a true Lie algebra since RHS of (21) is not a structure constant, is closed. Therefore, by restricting on the constraint hypersurface, where $G(\vec{A}) \approx 0$, $C(\vec{N}) \approx 0$, and $C(N) \approx 0$, all the primary constraint commutes with themselves on the constraint hypersurface. This means the sets of constraint arising from the Ashtekar formulation of GR are primary first class constraint. As a consequence to this, these constraints generate gauge transformation: Gauss constraint generates SU(2) gauge transformation, diffeomorphism constraint generates evolution in space, Hamiltonian constraint generates evolution in time.

3. Canonical Quantization

Since the set of constraints are primary and first class, one could proceed to the quantization procedure, based rigorously on [7]. The steps are:

- (i) To promote Poisson bracket $\{\dots\}$ to commutator $[\dots]$.
- (ii) To associate every dynamical variable \mathcal{O} with an operator $\hat{\mathcal{O}}$.
- (iii) To construct space of states where the the operators act to: the target/representation space.

3.1. Quantization via Connection Representation

One could immediately quantized the phase space, the Poisson bracket (11) becomes:

$$\left[A_a^i(x), \tilde{E}_j^b(x') \right] = i\hbar \delta_j^i \delta_a^b \delta^3(x - x').$$

The next step is to promote the fundamental variables to the following operators:

$$\begin{aligned} A_a^i &\rightarrow \hat{A}_a^i = A_a^i, \\ E_i^a &\rightarrow \hat{E}_i^a = \frac{\delta}{\delta A_a^i}, \end{aligned}$$

$\frac{\delta}{\delta A}$ is the functional derivatives with respect to A_a^i . The last step is to obtain the representation space. Normally, the representation space is the square-integrable space over the configuration space of the system. In our case, the configuration space is the space of connection $\mathcal{A} \simeq \mathfrak{su}(2) \times \bigwedge^1(\Sigma)$, therefore the representation space should be the functional space $C^\infty[\mathcal{A}] \simeq L_2[\mathcal{A}]$. Moreover, one could equip the space with a well-defined inner product and completeness requirement; in this case, the functional space is a Hilbert space.

At this point, several problems arise: the properties of the representation space are not well-defined: $C^\infty[\mathcal{A}] \ni \psi[A]$ has infinite (uncountable) dimensions, the measure and the inner product in this space is unknown. Let us neglect this problem for a while. Nevertheless, there exist a remaining problem when one tries to promote constraints (18)-(20) to an operator: the ordering of triads \tilde{E} and curvature $F(A)$ matters. Let us check what can be obtained from each ordering.

3.1.1. Triads on the right: Wilson Loops. In this ordering, the constraint becomes the following operator:

$$\hat{\mathcal{G}}_i \psi[A] = d_a \frac{\delta}{\delta A_a^i} \psi[A], \quad (22)$$

$$\hat{\mathcal{C}}_a \psi[A] = F_{ab}^i \frac{\delta}{\delta A_b^i} \psi[A], \quad (23)$$

$$\hat{\mathcal{C}} \psi[A] = \varepsilon_j^{kl} F_{ab}^j \frac{\delta}{\delta A_a^k} \frac{\delta}{\delta A_b^l} \psi[A]. \quad (24)$$

The Gauss constraint becomes an infinitesimal generator of gauge transformation for $SU(2)$ and the diffeomorphism constraint becomes infinitesimal generator of 3D diffeomorphism. The wavefunctional that satisfies the Gauss constraint needs to be invariant under $SU(2)$ transformation, thus the candidate for the solution is the Wilson loop of an $SU(2)$ holonomy:

$$\psi[A] = W[A, \gamma] = \text{tr} U[A, \gamma], \quad U[A, \gamma] = \hat{P} \exp_{\gamma} A.$$

In fact, the Wilson loop is also a solution to the Hamiltonian constraint. On the other hand, (1) $W[A, \gamma]$ is not a solution to the diffeomorphism constraint (23), (2) $W[A, \gamma]$ is not a solution to Hamiltonian constraint if the loops contains kinks or intersections, (3) given a metric operator, the wavefunction represents a space with degenerate metric [4]. Unfortunately, research in this direction might be not really useful. Nevertheless, it is historically important since this gives motivational research to the loop direction.

3.1.2. *Triads on the left: Chern-Simon Forms.* In this ordering, the constraint becomes the following operator:

$$\hat{\mathcal{G}}_i \psi [A] = \frac{\delta}{\delta A_a^i} d_a \psi [A], \quad (25)$$

$$\hat{\mathcal{C}}_a \psi [A] = \frac{\delta}{\delta A_b^i} F_{ab}^i \psi [A], \quad (26)$$

$$\hat{\mathcal{C}} \psi [A] = \varepsilon_j^{kl} \frac{\delta}{\delta A_a^k} \frac{\delta}{\delta A_b^l} F_{ab}^j \psi [A]. \quad (27)$$

For a reason which will be clear later, let us modify the Hamiltonian constraint by the existence of the cosmological constant:

$$\hat{\mathcal{C}}_\Lambda \psi [A] = \left(\varepsilon_j^{kl} \frac{\delta}{\delta A_a^k} \frac{\delta}{\delta A_b^l} F_{ab}^j - \frac{\Lambda}{6} \varepsilon_{jkl} \varepsilon^{abc} \frac{\delta}{\delta A_a^j} \frac{\delta}{\delta A_b^k} \frac{\delta}{\delta A_c^l} \right) \psi [A].$$

Chern-Simon Theory. At this point, let us review a different subject: the Yang-Mills field. The action of a Yang-Mills fields is given as follows:

$$S_{YM} = \Delta_{\mathcal{M}} \mathcal{L}_{YM} = \Delta_{\mathcal{M}} \text{tr} (F \wedge \star F). \quad (28)$$

For a case without source, as an example, in the electromagnetism case, the electromagnetic field satisfy self-duality $F = \star F$, such that the action (28) becomes:

$$S_{YM} = \Delta_{\mathcal{M}} \text{tr} (F \wedge F) = \Delta_{\mathcal{M}} \text{tr} (F^2). \quad (29)$$

It is clear that the action is independent from the use of metric, which is included in the definition of the Hodge-dual. This is favourable in the background independence perspective. In fact, one could generalize the action for arbitrary even $2n$ -dimension of \mathcal{M} , n integers:

$$S_{CS} = \Delta_{\mathcal{M}} \mathcal{L}_{CS} = \Delta_{\mathcal{M}} \text{tr} (F^n). \quad (30)$$

The term F^n is known as the n^{th} Chern form, and (30) is the Chern-Simon action. One can prove that $d_D F^n = 0$. Since F^n is closed, it is reasonable to ask if one could obtain a solution to $d_D F^n = 0$, namely a $(2n - 1)$ -form potential ϕ such that $d_D \phi = F^n$. This is equivalent with requiring the n^{th} Chern form F^n to be exact. An equivalence class of exact F^n is known as the n^{th} Chern class, and the corresponding potential ϕ as the Chern-Simon form. This solution will be important if one is interested to consider the bulk/boundary correspondence of the field. For $n = 4$ case, one could prove that $\phi = A \wedge dA + \frac{2}{3} A \wedge A \wedge A$, a Chern-Simon 3-form. Therefore, using Stokes theorem, one could write (29) as:

$$S_{YM} = \Delta_{\mathcal{M}} \text{tr} (F \wedge F) = \Delta_{\partial \mathcal{M}} \text{tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right),$$

with $\partial \mathcal{M}$ is the boundary of \mathcal{M} .

Now, let us construct a Chern-Simon state as follows:

$$\psi [A] = e^{-\frac{i6}{\Lambda} S_{YM}[A]} = e^{-\frac{i6}{\Lambda} \Delta_{\Sigma} \text{tr} (A \wedge dA + \frac{2}{3} A \wedge A \wedge A)}.$$

It is widely known as Kodama state [11]. Remarkably, it is the solution to these three constraints, but with an existence of a positive cosmological constant. The Kodama state is interpreted as the ground state wavefunction for a deSitter space. Nevertheless, it is not a state for a gravitational field, one could attempt to find a solution based on the action of GR, and the research in this direction is still on progress.

3.2. Quantization via Holonomy Representation

It had been explained in the previous chapters that there exist a wavefunction of GR in the form of Kodama state. Nevertheless, the Hilbert space of this state, had not been obtained in a rigorous manner. One could argue that the Ashtekar new variables are not the best phase-space variables to quantize GR; one needs to search for another phase-space variables. Implementing the main idea of the construction of Lattice Gauge Theories, it turns out that the good variables compatible with the quantization procedure are the regularized version of the Ashtekar variables.

3.2.1. Regularization. The 3-connections A , which is an $\mathfrak{su}(2)$ -valued 1-form, is smeared along a curve (line, or link) l , and is known as *holonomy*:

$$U_l[A, \gamma] = \hat{P} \exp \Delta_l {}^3A. \quad (31)$$

This is also a solution to the following parallel transport equation along curve γ , with $\gamma'(\tau) = \frac{d\gamma(\tau)}{d\tau}$:

$$D_{\gamma'(\tau)} U[A, \gamma(\tau)] = \frac{dU[A, \gamma(\tau)]}{d\tau} + A(\gamma'(\tau)) U[A, \gamma(\tau)] = 0.$$

Meanwhile the 'electric' triads $\tilde{E} = \frac{E}{\gamma}$, which is a matrix-valued 2-form, is smeared on a portion of surface area S , known as the *flux*:

$$\tilde{E}_l = \tilde{E}_l(S) = \Delta_S \tilde{E}. \quad (32)$$

The index l in the flux indicates that the (infinitesimal) surface S in 3D hypersurface Σ always posses a link l which is defined as its (lattice) dual, thus it is convenient to label the flux crossing the surface with the index of the link. The smeared phase-space variable associated on link l is now (U_l, E_l) , where the U_l is smeared along the link, and \tilde{E}_l is smeared along the surface (lattice)-dual to the link. They satisfy the symplectic structure through the Poisson bracket:

$$\begin{aligned} \{U_l, U_{l'}\} &= 0, \\ \{U_l, \tilde{E}_{l'}^a\} &= \delta_{ll'} \bar{\sigma}^a U_l, \\ \{\tilde{E}_l^a, \tilde{E}_{l'}^b\} &= \delta_{ll'} \varepsilon_c^{ab} \tilde{E}_l^c, \end{aligned}$$

this algebra is known as the holonomy-flux algebra, labeled by \mathcal{J}_Γ , with Γ is a collection of links and node known as a graph. $\bar{\sigma}^a$ is the Pauli matrices. There are some subtleties concerning the orientation of the flux \tilde{E} with respect to the surface S . In this article, we always take the flux to cross surface S positively. For a more subtle derivation consult [3]. Clearly, U_l is an element of $SU(2)$, since it is an 'exponential map' of ${}^3A \in \mathfrak{su}(2)$, while \tilde{E}_l^a , which are components of \tilde{E}_l , from (35), is an element of $\mathfrak{su}(2)$, since they satisfy the $\mathfrak{su}(2)$ algebra. Therefore, these variables are conjugate to each other, forming an elements of 'regularized' phase space $(U_l, J_l^a) \in T^*SU(2) \simeq SU(2) \times \mathfrak{su}(2)^*$ (since $\mathfrak{su}(2) \sim \mathfrak{su}(2)^*$).

The smearing used for this regularization is different from the standard smearing use in QFT, namely, the smearing over a portion of a 3D volume. This happens because the smearing is defined with respect to a specific background metric. For general relativity, one use different type of smearing such that it does not assume a fix background metric, since the theory is expected to be background independent.

3.2.2. *Quantization.* Repeating the quantization on the regularized variables, the holonomy-flux algebra naturally satisfies the following algebra relation:

$$[U_l, U_{l'}] = 0, \tag{33}$$

$$[U_l, E_{l'}^a] = i\ell^2 \gamma \delta_{ll'} \bar{\sigma}^a U_l, \tag{34}$$

$$[E_l^a, E_{l'}^b] = i\ell^2 \gamma \delta_{ll'} \varepsilon_c^{ab} E_l^c. \tag{35}$$

ℓ is a constant equal to $\sqrt{\frac{\hbar G}{c^3}}$, known as *Planck length*. Notice that we use E instead of \tilde{E} , which give rise to the Immirzi factor γ on the RHS of (34)-(35). Promoting the phase-space elements as operators, one obtains:

$$U_l \rightarrow \hat{U}_l \psi [A] = U_l \psi [A], \tag{36}$$

$$E_l \rightarrow \hat{E}_l \psi [A] = \vec{J}_l \psi [A], \tag{37}$$

with \vec{J}_l is a left invariant vector field on $SU(2)$ generated by Pauli matrices $\bar{\sigma}^a$ if the orientation of the link points outward from the node and right invariant if it points inward:

$$\vec{J}_l : \mathcal{G} \rightarrow \mathfrak{g}, \quad \vec{J}_l : g_l \mapsto \vec{J}_l(g_l) = \begin{cases} L_l^a(g_l) = i \frac{d}{d\tau} (e^{\bar{\sigma}^a \tau} g_l) \Big|_{\tau=0} \\ R_l^a(g_l) = i \frac{d}{d\tau} (g_l e^{\bar{\sigma}^a \tau}) \Big|_{\tau=0} \end{cases}. \tag{38}$$

The flux is defined as an invariant vector field acting on $SU(2)$ with a reason which will be clear later. The phase-space variables are automatically the basic operators of the canonical theory; any operator \hat{O} , including the constraints, can be written in the form of these basic operators.

The last step of quantization is constructing the Hilbert space. Since the regularized phase space variables is now (U, J) , the regularized configuration space is $SU(2) \ni U_l$. This made us possible to define the flux operator in (38) as an invariant vector field working on $SU(2)$. Therefore, the representation space of quantum gravity could be build over $SU(2)$, instead of \mathcal{A} . To be precise, the representation space is exactly the space of square-integrable function of $SU(2)$: $C^\infty[SU(2)] \simeq L_2[SU(2)]$. This allows us to define the Hilbert space, which will be explained in detail in the next section.

4. Kinematical Part of Loop Quantum Gravity

4.1. Cylindrical Functions

4.1.1. *Hilbert space of quantum gravity.* As explained in the previous section, the representation space over the connection $C^\infty[\mathcal{A}]$ is problematic, i.e., one can not define an inner product, measure, and completeness, which are possessed by a Hilbert space. Therefore, with the connection representation, one cannot obtain the Hilbert space of the theory. Nevertheless, there is a possible way to obtain the Hilbert space using the holonomy representation.

Let a functional over connection A be written as:

$$\psi[A] = \langle A | \psi \rangle \in C^\infty[\mathcal{A}].$$

Now let us consider a subset of $C^\infty[\mathcal{A}]$, say $\text{Cyl}[\mathcal{A}]$, such that each element is a function of the holonomy $U = U[\gamma, A]$, namely, the dependence on A enters through U . Such functional which are build from some subset of the field are generally known as cylindrical functions. We label this kind of function as:

$$\psi_{\Gamma, \varphi}[A] = \langle A | \psi_{\Gamma, \varphi} \rangle = \varphi(U_{l_1}[A], \dots, U_{l_L}[A]) \in \text{Cyl}[\mathcal{A}] \simeq C^\infty[SU(2)] \subset C^\infty[\mathcal{A}].$$

Γ is a graph constructed from intersecting loops, where each link l contains the information of the holonomy U_l . Notice that since $\mathcal{A} \sim \mathfrak{su}(2) \times \Lambda^1(\Sigma)$, then the space of holonomies, new configuration space, is clearly $SU(2)$ which is compact. Because of this, one has completeness, and could define the following inner product on $\text{Cyl}[\mathcal{A}] \simeq C^\infty[SU(2)]$:

$$\langle \psi_{\Gamma, \varphi} | \psi_{\Gamma, \varphi'} \rangle = \int_{SU(2)} \prod_l dU_l \varphi^*(U_{l_1}[A], \dots, U_{l_L}[A]) \varphi'(U_{l_1}[A], \dots, U_{l_L}[A]),$$

with dU_l is the Haar measure on $SU(2)$. Moreover, Ashtekar and Lewandowski shows that $\text{Cyl}[\mathcal{A}]$ could be extended to define a Hilbert space over a connection, $\mathcal{H}_{AL} \simeq L_2[\mathcal{A}, d\mu_{AL}]$, using the Ashtekar-Lewandowski measure $d\mu_{AL}$ [12, 13]. Therefore, one obtains a candidate of Hilbert space for quantum gravity.

4.1.2. The basis on Ashtekar-Lewandowski Hilbert space. Since now the cylindrical function $|\psi_{\Gamma, \varphi}\rangle$ is an element of Hilbert space \mathcal{H}_{AL} , one can use Peter-Weyl theorem: *A basis in $\mathcal{H} \simeq L_2[\mathcal{G}, d\mu_{\text{Haar}}]$, that is, a space of function over a compact group \mathcal{G} , is given by the matrix element of unitary, irreducible representations of \mathcal{G} .* In the LQG case, $\mathcal{G} \simeq SU(2)$, therefore, one could obtain the basis of $\mathcal{H}_{AL} \simeq L_2[\mathcal{A}, d\mu_{AL}]$ in the following steps.

Firstly, let us construct the Hilbert space for a single link of graph Γ . A link $l = (n, n')$ connecting node n to n' of the graph is equipped by an element of phase space on the boundary, $(U_l, J_l) \in T^*SU(2)$, together with the label of spin- j , the irreducible representation of $SU(2)$ in $(2j + 1)$ -dimension. The representation space for each link is build over $SU(2)$, and from the theorem one has:

$$\mathcal{H}_l = L_2[SU(2), d\mu_{\text{Haar}}] \simeq \bigoplus^{j_l} (\mathcal{H}_{j_l} \otimes \mathcal{H}_{j_l}^*), \quad \mathcal{H}_{j_l} \simeq L^2[\mathbb{R}^n, d\mu]. \tag{39}$$

Thus, the basis in $\mathcal{H}_l \simeq L_2[SU(2)]$ is:

$$|j_l, m_l, n_l\rangle = |j_l, m_l\rangle \langle j_l, n_l| \in \mathcal{H}_l \simeq \bigoplus^{j_l} (\mathcal{H}_{j_l} \otimes \mathcal{H}_{j_l}^*), \tag{40}$$

with each spin basis is related to one end of the link, see Figure. 1.

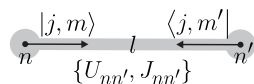


Figure 1. Variables and spin quantum numbers attached to one link of a graph.

For a graph containing n -links, the Hilbert space is the product of (39), namely:

$$\mathcal{H}_\Gamma = L_2[SU(2)^{|\mathcal{L}|}, d\mu_{\text{Haar}}] = \bigotimes_l^{|\mathcal{L}|} \mathcal{H}_l = \bigotimes_l^{|\mathcal{L}|} \bigoplus^{j_l} (\mathcal{H}_{j_l} \otimes \mathcal{H}_{j_l}^*), \tag{41}$$

and the basis, which could be represented with n non-connected strains, is:

$$\begin{array}{c} \text{---} j_1 \\ \vdots \\ \text{---} j_n \end{array} = \bigotimes_{l=1}^n |j_l, m_l, n_l\rangle$$

$\otimes |j_l, m_l, n_l\rangle$ labels the basis of matrix representation of the wavefunctional.

4.1.3. *General states.* Let us write the vector state $|\psi_{\Gamma,\varphi}\rangle \in \mathcal{H}_{AL}$ of an arbitrary graph Γ having $|\mathcal{N}|$ nodes and $|\mathcal{L}|$ links, in the representation of holonomy as follows:

$$|\varphi\rangle = \int_{SU(2)} \prod_l dU_l \left\langle \otimes^l U_{l_1} [A] |\varphi\rangle \left| \otimes^l U_{l_1} [A] \right\rangle = \int_{SU(2)} \prod_l dU_l \varphi(U_{l_1} [A], \dots, U_{l_L} [A]) \left| \otimes^l U_{l_1} [A] \right\rangle,$$

where we have $\varphi(U_{l_1} [A], \dots, U_{l_L} [A]) = \langle \otimes^l U_{l_1} [A] |\varphi\rangle$ which is a function over $SU(2)$. Using the completeness and orthogonality of (40), one could write $\varphi(U_{l_1} [A], \dots, U_{l_L} [A])$ as a linear combination of irreducible representation of $SU(2)$, as follows:

$$\varphi(U_{l_1} [A], \dots, U_{l_L} [A]) = \sum_{j_l, m_l, n_l} \left\langle \otimes^l U_{l_1} [A] | \otimes_l j_l, m_l, n_l \right\rangle \left\langle \otimes_l j_l, m_l, n_l | \varphi \right\rangle = \sum_{j_l, m_l, n_l} \varphi_{m_l n_l}^{j_l} \prod_l D_{m_l n_l}^{j_l}(U_l), \quad (42)$$

with $D_{m_l n_l}^{j_l}(U_l) = \langle U_l [A] | j_l, m_l, n_l \rangle$, is the component of the *Wigner-D matrix*, or the *rotator*.

4.2. Gauss Constraint and the Kinematical Hilbert Space

4.2.1. Gauss constraint in flux-representation: closure constraint of a quantum polyhedron.

Having the well-defined Ashtekar-Lewandowski Hilbert space and its complete basis, one could start to promote the constraints as operators and find the wavefunctional solving the constraints. In this review, we only consider the Gauss constraint. The Gauss constraint (12) is a divergence theorem, it can be interpreted as the 'incompressibility' of the flux E_i^a . We can write the Gauss constraint without indices as follows:

$$\mathcal{G} = d_{\Sigma} E = 0. \quad (43)$$

with d_{Σ} is the exterior-covariant derivative on the spatial slice Σ . The quantity (43) is a 3-form, therefore, to define the regularization, one needs to smeared the quantity over a single finite region with volume R_i , and by using Stokes theorem, one obtains:

$$\Delta_{R_i} d_{\Sigma} E = \Delta_{\partial R_i} E = 0,$$

with ∂R is the closed surface enclosing the finite volume R_i . Using the smeared variable of the infinitesimal momentum over a finite surface, one obtains:

$$\Delta_{\partial R_i} E = \lim_{\mathbf{a}_l \rightarrow 0} \sum_{\mathbf{a}_l \subset \partial R_i} E_l(S).$$

Defining a regulator or a cut-off which prevent $\mathbf{a}_l \rightarrow 0$, the regularized Gauss constraint is obtained as follows:

$$\mathcal{G}_{\Delta} = \sum_{\mathbf{a}_l \subset \partial R} E_l(S).$$

By promoting the Gauss constraint to an operator using (37), one has:

$$\mathcal{G}_{\Delta} \rightarrow \hat{G} |\varphi\rangle = \sum_l \hat{E}_l(S) |\varphi\rangle = \sum_l \hat{J}_l |\varphi\rangle = 0, \quad (44)$$

with the index l labels the link describing the area \mathbf{a}_l . (44) can be interpreted as the quantum operator version of the classical closure constraint of a polyhedron, where flux J_l correspond to the area \mathbf{a}_l of the polygons enclosing the polyhedron, see Figure 2. For now, let us choose for

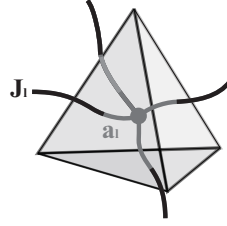


Figure 2. A quantum tetrahedron.

simplicity an $|\mathcal{L}|$ -valent graph, that is, a graph containing only a single node n with $|\mathcal{L}|$ -links $l = 1, \dots, |\mathcal{L}|$, pointing outward from n . (44) clearly becomes:

$$\sum_{l=1}^{|\mathcal{L}|} J_l |\varphi\rangle = J_{1\dots n} |\varphi\rangle = 0. \quad (45)$$

The eigenvalue equation of (45) is translated as:

$$\begin{aligned} |J_{1\dots n}| |\varphi\rangle &= 0, \\ J_{1\dots n}^{(z)} |\varphi\rangle &= 0, \end{aligned} \quad (46)$$

requiring the total spin and magnetic quantum number $j_{1\dots n} = 0$ and $m_{1\dots n} = 0$ (in fact, $m_{1\dots n}$ is automatically zero if the first relation is satisfied, since $-j \leq m \leq j$). The state $|\varphi\rangle \in \mathcal{H}_\Gamma$ satisfying (46) automatically satisfies the quantum Gauss constraint. The *kinematical Hilbert space*:

$$\mathcal{K}_\Gamma \subset \mathcal{H}_\Gamma, \quad \mathcal{K}_\Gamma \sim L_2 \left[SU(2)^{|\mathcal{L}|} / SU(2)^{|\mathcal{M}|} \right]$$

is an invariant subspace of \mathcal{H} which satisfy the quantum Gauss constraint on each node, and the basis which spans \mathcal{K} is called as the *spin network* basis, which will be discussed in detail in the next subsections.

4.2.2. Gauss constraint in holonomy basis: group averaging procedure. The construction of quantum Gauss operator in the previous subsection provides a clear geometrical interpretation. Nevertheless, a formal derivation on states satisfying the Gauss constraint use the procedure borrowed from Lattice Gauge Theory, namely, the group averaging procedure. From (15), it is known that the Gauss constraint is the (infinitesimal) generator of $SU(2)$ gauge transformation.

The finite gauge transformation acts on the holonomy in the following way:

$$U_l[A] \rightarrow U'_l[A] = h_l(\gamma(s)) U_l[A] h_l^{-1}(\gamma(t)), \quad (47)$$

where *source* s is the origin of curve γ (or link l), and *target* t is the endpoint of curve γ (s and t are clearly nodes of the graph Γ). In the matrix representation, (47) acts as:

$$D_{mn}^{(j)}(U_l[A]) \rightarrow D_{mn}^{(j)}(U'_l[A]) = D_{ma}^{(j)}(h_l(\gamma(s))) D_b^{(j)a}(U_l[A]) D_n^{(j)b}(h_l^{-1}(\gamma(t))).$$

Any wavefunction $\varphi(U[A]) \in \mathcal{H}$ can be written as a linear combination of products of irreducible matrix representation of $SU(2)$ as in (42). One would like to obtain a set of wavefunction $\varphi_{\text{inv}}(U[A]) \in \mathcal{K} \subset \mathcal{H}$, such that the following condition holds:

$$\varphi(U_{l_1}[A], \dots, U_{l_L}[A]) \equiv \varphi\left(h_{l_1}(\gamma(s)) U_{l_1}[A] h_{l_1}^{-1}(\gamma(t)), \dots, h_{l_L}(\gamma(s)) U_{l_L}[A] h_{l_L}^{-1}(\gamma(t))\right),$$

That is, the wavefunction needs to be invariant under finite SU(2) gauge transformation. Firstly, one has:

$$\varphi_{\text{inv}}(U_{l_1}[A], \dots, U_{l_L}[A]) = \sum_{j_l, m_l, n_l} \varphi_{m_l n_l}^{j_l} \left(\prod_l D_{m_l n_l}^{j_l}(U_l) \right)_{\text{inv}}. \quad (48)$$

Let us focus to the last term containing product of the rotation matrices. Let us collect the product of matrices by the nodes of graph Γ as follows:

$$\prod_l D_{m_l n_l}^{(j_l)}(U_l[A]) = \prod_{n \in \Gamma} \prod_{\substack{l \in \Gamma, \\ n \cap l \neq \bar{A}}} D_{m_l n_l}^{(j_l)}(U_l[A]).$$

The gauge transformation only acts on the nodes, so let us concentrate on a single node where all of the links are pointing outward from node n . The gauge transformation at source node s acts on the *half* of the rotation matrix, i.e on the first index; this is clearly the left action by gauge group $h_l(\gamma(s))$:

$$\prod_l D_{m_l n_l}^{(j_l)}(U'_l[A]) \Big|_{n_t} = \prod_{\substack{l \in \Gamma, \\ n \cap l \neq \bar{A}}} D_{m_l a}^{(j_l)}(h_l(\gamma(s))) D_{n_l}^{(j_l)a}(U_l[A]).$$

The invariance on the node n means the following condition is satisfied:

$$\prod_l D_{m_l n_l}^{(j_l)}(U'_l[A]) \Big|_n = \prod_{\substack{l \in \Gamma, \\ n \cap l \neq \bar{A}}} D_{m_l a}^{(j_l)}(h_l(\gamma(s))) D_{n_l}^{(j_l)a}(U_l[A]) \equiv \prod_{\substack{l \in \Gamma, \\ n \cap l \neq \bar{A}}} D_{m_l n_l}^{(j_l)}(U_l[A]) \Big|_n. \quad (49)$$

The product of matrices satisfying this property can be obtained by group averaging procedure, or averaging the matrices by a measure i such that condition (49) is satisfied:

$$\left(\prod_l D_{m_l n_l}^{(j_l)}(U_l[A]) \Big|_n \right)_{\text{inv}} = \left(\prod_{\substack{l \in \Gamma, \\ n \cap l \neq \bar{A}}} D_{m_l n_l}^{(j_l)}(U_l[A]) \Big|_n \right)_{\text{inv}} = \sum_{\alpha_l, \beta_l} i_{(n)m_l n_l}^{\alpha_1 \beta_1 \dots \alpha_l \beta_l} \prod_{\substack{l \in \Gamma, \\ n \cap l \neq \bar{A}}} D_{\alpha_l \beta_l}^{(j_l)}(U_l[A]) \Big|_n.$$

One could define the gauge transformation at target node t acting on the second index; which is the right action $h_l^{-1}(\gamma(t))$ in a similar way. Now taking all the nodes on the graph Γ , one obtains:

$$\left(\prod_l D_{m_l n_l}^{(j_l)}(U_l) \right)_{\text{inv}} = \prod_{n \in \Gamma} \left(\prod_l D_{m_l n_l}^{(j_l)}(U_l[A]) \Big|_n \right)_{\text{inv}} = \sum_{\alpha_l, \beta_l} \prod_{n \in \Gamma} i_{(n)m_l n_l}^{\alpha_1 \beta_1 \dots \alpha_l \beta_l} \prod_{\substack{l \in \Gamma, \\ n \cap l \neq \bar{A}}} D_{\alpha_l \beta_l}^{(j_l)}(U_l[A]) \Big|_n, \quad (50)$$

and moreover inserting (50) to (48) gives:

$$\varphi_{\text{inv}}(U_{l_1}[A], \dots, U_{l_L}[A]) = \sum_{j_l, m_l, n_l} \varphi_{m_l n_l}^{j_l} \sum_{\alpha_l, \beta_l} \prod_{n \in \Gamma} i_{(n)m_l n_l}^{\alpha_1 \beta_1 \dots \alpha_l \beta_l} \prod_{\substack{l \in \Gamma, \\ n \cap l \neq \bar{A}}} D_{\alpha_l \beta_l}^{(j_l)}(U_l[A]) \Big|_n. \quad (51)$$

The components of the equivariant map i is known as the *intertwiner*, which could be obtained from a contraction of $\{3j\}$ -symbols. Written in a more compact manner, the state satisfying Gauss constraint can be written as:

$$\varphi_{j_l, v_n, \Gamma}^{\text{inv}}(U_{l_1}[A], \dots, U_{l_L}[A]) = \left\langle \bigotimes_n \mathbf{i}_{(n)} \Big| \bigotimes_l \mathbf{D}^{(j_l)} \right\rangle_{\Gamma}.$$

4.3. Lattice-graph and Spin-network states

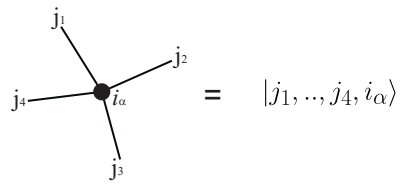
Constructing spin network states. The graphical interpretation of spin network is more clearer in the basis of matrix representation $|j_l, m_l, n_l\rangle$ instead of the holonomy basis $|U_l\rangle$. The steps for constructing a spin network states can be written in the following, based on [14]:

- (i) Having an arbitrary connected graph Γ , with $|\mathcal{N}|$ nodes and $|\mathcal{L}|$ links, construct the Ashtekar-Lewandowski Hilbert space as the product of the Hilbert space of the link by relation (41). The basis functional on this space is $\otimes |j_l, m_l, n_l\rangle$, by Peter-Weyl theorem. We called this the \otimes -product basis.
- (ii) On each node, collect half of the $|j_l, m_l, n_l\rangle = |j_l, m_l\rangle \langle j_l, n_l|$ basis, that is, either $|j_l, m_l\rangle$ or $\langle j_l, n_l|$ depending on the graph, and transform them to the \oplus -sum basis, that is, sum all the half-spins on the node. The Hilbert space satisfies $\otimes^j \mathcal{H}_j = \oplus_{j_{\min}}^{j_{\max}} \mathcal{H}$. The transformation matrix components are the Clebsch-Gordan coefficients, or more general, the recoupling coefficient i_{CG} . This \oplus -sum basis equally spans the Hilbert space (41).
- (iii) Implement the Gauss constraint on each node, that is, by requiring the total spins on each node to satisfy (46). This procedure selects only the singlet spaces $\mathcal{H}_{j_{\min}} = \mathcal{H}_0$ (if they exist). This is equal with giving a condition to the recoupling coefficient such that they are equal with the intertwiner, i.e., $i_{CG} = i$. The singlet \oplus -sum basis is the spin network basis.
- (iv) Contract the spin network basis with the dual-basis in holonomy representation $\langle U_l|$ to obtain the rotation matrices as in (42). One obtains the basis of spin network in terms of (50). Any general spin network states are a linear combination of the spin network basis, as in (51).

Example: 4-valent graph. Taking the invariant subspace of the spin network is implementing the quantum Gauss constraint on each node. In the \oplus -basis, implementing this constraint will set the quantum number $j_{1234} = 0$, which is the Clebsch-Gordan condition. This will cause $m_{1234} = 0$, and $j_{123} = j_4$ for the 4-valent graph case:

$$\begin{aligned} \pi_n \left| j_{1234}, m_{1234}, j_{123}, j_{12}, \prod_{l=1}^{n=4} (j_l, m_l) \right\rangle &= \left| 0, 0, j_{123} = j_4, j_{12}, \prod_{l=1}^{n=4} (j_l, m_l) \right\rangle, \\ &= \left| j_{12}, \prod_{l=1}^{n=4} (j_l, m_l) \right\rangle = |j_l, i_\alpha\rangle, \end{aligned}$$

with $|j_l, i_\alpha\rangle$ is the basis of \mathcal{K}_γ for 4-valent graph. Written diagrammatically:



with the full-coloured node labels the gauge-invariant basis.

4.4. The geometric operators and quanta of space

The spin network state describes the state of the graph, which is embedded on the spatial hypersurface Σ . It contains the informations of the hypersurface. Since the graph has finite nodes and links, it describes a discretized space, where the nodes correspond to a polyhedra, and the links correspond to flat surfaces enclosing each polyhedron, see Figure 2. Therefore, one could construct the geometric operators such as area and volume operators.

4.4.1. *Area operator.* The classical area of an arbitrary surface S on Σ is defined as:

$$A(S) = \Delta_S ds_1 ds_2 \sqrt{E_i^a E^{bi} n_a n_b}.$$

A regularization gives:

$$A_N(S) = \sum_{I=1}^N \sqrt{E_i(S_I) E^i(S_I)},$$

with $A(S) = \lim_{N \rightarrow \infty} A_N(S)$. Promoting the classical area to operator by (37) gives immediately the area operator of LQG:

$$A_N(S) \rightarrow \hat{A}(S), \quad \hat{A}(S) |j, m, n\rangle = \sum_{I=1}^N \sqrt{\hat{E}_i(S_I) \hat{E}^i(S_I)} |j, m, n\rangle, \quad (52)$$

acting on $|j, m, n\rangle$. Working on the basis of matrix representation, the spectrum of the area operator are:

$$\hat{A}(S) |j, m, n\rangle = \sum_{I=1}^N \sqrt{\hat{E}_i(S_I) \hat{E}^i(S_I)} |j, m, n\rangle = \gamma^2 \ell^2 j(j+1) |j, m, n\rangle,$$

which are discrete.

4.4.2. *Volume operator.* The classical volume of region R on Σ is defined as:

$$V_R(x) = \Delta_R \text{vol}_\Sigma = \Delta_R d^3x \sqrt{q(x)},$$

with q is the determinant of metric q_{ij} of Σ . Using the relation between the 3-metric and the triads, one obtains: $q = E$. Therefore:

$$V_R(x) = \Delta_R d^3x \sqrt{E(x)} = \Delta_R d^3x \sqrt{\left| \frac{1}{3!} \varepsilon^{ijk} \varepsilon_{abc} E_i^a(x) E_j^b(x) E_k^c(x) \right|},$$

Promoting the classical volume to operator gives:

$$V_R(x) \rightarrow \hat{V}_R(x) = \Delta_R d^3x \sqrt{\left| \frac{1}{3!} \varepsilon^{ijk} \varepsilon_{abc} \hat{E}_i^a(x) \hat{E}_j^b(x) \hat{E}_k^c(x) \right|},$$

By a specific regularization defined by [13], the Ashtekar-Lewandowski volume operator is:

$$\hat{V}_R(x) = \ell_P^3 \sum_{v \in V(\Gamma)} \delta^3(x, v) V_{v, \Gamma},$$

$$V_{v, \Gamma} = V_{AL} = \sqrt{\left| z \sum_{a>b>c} \epsilon(e_a, e_b, e_c) \varepsilon^{ijk} \hat{j}_i^a \hat{j}_j^b \hat{j}_k^c \right|},$$

or

$$V_{v, \Gamma} = V_{AL} = \sqrt{\left| Z \sum_{a,b,c} \epsilon(e_a, e_b, e_c) \hat{q}_{abc} \right|},$$

The spectrum of the volume operator is quite complicated to obtain since they are in general, not analytical. For a complete derivation, consult [15]. Nevertheless one can conclude that the volume are discrete. The spin-network graph describe quantum polyhedron, which could be viewed as the quanta of space, having discrete surface areas and volumes and fuzzy shapes.

5. Conclusions

In this article, we have given a brief review on the kinematical part of canonical loop quantum gravity. The kinematical part is complete and well-understood. One of the consequences of LQG coming from the kinematical part is the discreteness of space in the Planck scale, namely the existence of quanta of space. The quanta, having discrete surface areas and volumes, as well as fuzzy shapes, are described by spin-networks: a lattice-graph labeled by spin representations of $SU(2)$. The graph may contain loops, where the $SU(2)$ holonomies describing the intrinsic curvature of the discrete space are located.

The second part of this article will briefly review the dynamical part of canonical loop quantum gravity.

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