On the Geometry of pp-Wave Type Spacetimes

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Abstract. Global geometric properties of product manifolds $\mathcal{M} = M \times \mathbb{R}^2$, endowed with a metric type $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_R + 2dudv + H(x,u)du^2$ (where $\langle\cdot,\cdot\rangle_R$ is a Riemannian metric on M and $H : M \times \mathbb{R} \to \mathbb{R}$ a function), which generalize classical plane waves, are revisited. Our study covers causality (causal ladder, non-existence of horizons), geodesic completeness, geodesic connectedness and existence of conjugate points. Appropriate mathematical tools for each problem are emphasized and the necessity to improve several Riemannian (positive definite) results is claimed.

The behaviour of $H(x,u)$ for large spatial component x becomes essential, with a spatial quadratic behaviour being critical for many geometrical properties. In particular, when M is complete, if $-H(x,u)$ is spatially subquadratic, the spacetime becomes globally hyperbolic and geodesically connected. But if a quadratic behaviour is allowed (as happens in plane waves), then both global hyperbolicity and geodesic connectedness may be lost.

From the viewpoint of classical general relativity, the properties which remain true under generic hypotheses on $\mathcal{M}$ (as subquadraticity for H) become meaningful. Natural assumptions on the wave – finiteness or asymptotic flatness of the front – imply the spatial subquadratic behaviour of $|H(x,u)|$ and, thus, strong results for the geometry of the wave. These results do not always hold for plane waves, which appear as an idealized non-generic limit case.

1 Introduction

Among the reasons which contribute to the recent interest in pp-wave type spacetimes, we mention, on the one hand, classical geometrical properties and, on the other, applications to string theory[1]. About the former, pp-waves

[1] Of course, there is also another very influential reason: the possibility of direct detection of gravitational waves. Hulse and Taylor were awarded the Nobel prize in 1993 for the discovery, in the 1970s, of indirect evidence of their existence – a binary

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spacetimes, and especially plane waves, [12,23,35] have curious and intriguing properties, which led to questions still open or only recently solved. The well-known Penrose limit [39] (see also [9,10]) associates to every spacetime and every choice of an (unparametrized) lightlike geodesic a plane-wave metric. Penrose [37] also emphasized that, in spite of being geodesically complete, plane waves are not globally hyperbolic (see Sect. 2 for definitions). Ehlers and Kundt [19] conjectured that gravitational plane waves are the only complete gravitational pp-waves. As we will see, by now the lack of global hyperbolicity is well understood, but the Ehlers-Kundt conjecture still remains open. Applications to string theory have highlights such as: (a) gravitational pp-waves are relevant spacetimes with vanishing scalar invariants (VSI, see [17,40] for a classification), and such spacetimes yield exact backgrounds for string theory (vanishing of α' corrections, see [2,28]), (b) Berenstein, Maldacena and Nastase [5] have recently proposed an influential solvable model for string theory by taking the Penrose limit in $\mathrm{AdS}_5 \times S^5$ spacetimes, or (c) after realizing that Gödel–like universes can be supersymmetrically embedded into string theory, it was realized and emphasized that these solutions were T-dual to compactified plane-wave backgrounds [11,26,33].

The necessity to better understand the geometry of waves and their potential applications to string theory justifies to study pp-waves from a wider perspective, where new mathematical tools appear naturally. The authors, in collaboration with A.M. Candela [15], considered the following class of spacetimes $(\mathcal{M}, \langle\cdot,\cdot\rangle)$, called *plane-fronted waves* (PFW) or Mp-*waves*, see Sect. 2, which essentially include the classical pp-waves and, thus, plane waves:

$$\begin{aligned} \mathcal{M} &= M \times \mathbb{R}^2 \\ \langle\cdot,\cdot\rangle &= \langle\cdot,\cdot\rangle_R + 2\ du\ dv + H(x,u)\ du^2\ . \end{aligned} \tag{1}$$

Here $(M, \langle\cdot,\cdot\rangle_R)$ is any smooth Riemannian (C^∞, positive-definite, connected) n-manifold, the variables (v,u) are the natural coordinates of $\mathbb{R}^2$ and the smooth scalar field $H : M \times \mathbb{R} \to \mathbb{R}$ is not identically 0.

Our initial motivation to study such metrics came from some work by two contributors to this meeting, R. Penrose and P.E. Ehrlich. Penrose [37] showed that, even though plane waves are strongly causal, they are not globally hyperbolic. Moreover, they present a property of focusing lightlike geodesics which forbids not only global hyperbolicity but also the possibility to embed them isometrically in higher-dimensional semi-Euclidean spaces

system loses an exact amount of rotational energy which can be interpreted only as originating from the radiation of gravitational waves. Nowadays, experimentalists look for direct evidence, and a generation of large scale interferometers is close to be operative in various places on the Earth (VIRGO, LIGO, GEO300, TAMA300 ...) and even in space (LISA). Although experimentalists' problems are very different from the ones in this paper, if they succeed an excellent stimulus on waves for the whole relativistic community (and even for the curiosity of the general public) will be achieved.

(Fig. 1). This is a *remarkable* property of plane waves but, as he pointed out, it is also interesting to know "whether the somewhat strange properties of plane waves encountered here will be present for waves which approximate plane waves, but for which the spacetime is asymptotically flat, or asymptotically cosmological in some sense". From our viewpoint, this is a relevant question because the geometrical properties of an exact solution to Einstein's equation (as plane waves) are physically meaningful only if they are "stable" in some sense – surely, this is not fulfilled by a term as H in formula (2) below. Even more, in the setting of Penrose's *Strong Cosmic Censorship Hypothesis* [38], generic solutions to Einstein's equation with reasonable matter and behaviour at infinity must be globally hyperbolic. And, obviously, plane waves fail to be generic and well behaved at infinity because of the many symmetries of the term H (as well as the part $M = (\mathbb{R}^2, dx^2 + dy^2)$).

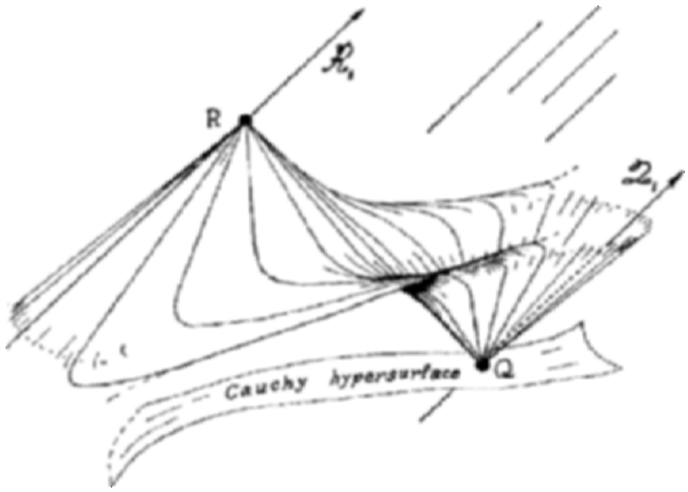

Fig. 1. Focusing of lightlike geodesics in plane waves; the picture is taken from [37]

Ehrlich and Emch, in a series of papers [20–22] (see also [4, Ch. 13]), carried out a detailed investigation of the behaviour of all the geodesics emanating from a (suitably chosen) point p in a gravitational plane wave. Then, they showed that gravitational plane waves are causally continuous but not causally simple, and characterized points necessarily connectable by geodesics (see Subsect. 5.1). However, again all of the study relies on the "non-generic nor stable" conditions of symmetry of the gravitational wave and on the very special form of $H(x, u)$: independence of the choice of the point p, explicit integrability of geodesic equations, identical equations for Killing fields, Jacobi fields and geodesics.

In this framework, our goals in [15, 24] were, essentially: (i) to introduce the class of reasonably generic waves (1), (ii) to justify that, for a physically reasonable asymptotic behaviour of the wave, $|H(\cdot, u)|$ must be "subquadratic" (plane waves lie in the limiting quadratic case), and (iii) to show that, in this case, the geometry of the wave presents good global properties, including global hyperbolicity [24] and geodesic connectedness [15]. Even more, the instability of these geometric properties in the quadratic case leads to interesting questions in Riemannian Geometry, studied in [13].

In the present article, we explain the role of the mathematical tools introduced in [13,15,24] in relation to classical papers on waves such as [19,37], and to later developments [29–33, 44]. For proofs we refer to the original articles or, in the case of more recent results, we provide sketches.

This paper is organised as follows:

In Sect. 2, some general properties of PFWs are explained, including questions related to curvature and the energy conditions. Remarkably, we justify that the behaviour of $|H(x, u)|$ for large x must be subquadratic if the wave is assumed to be finite or with fronts asymptotically flat in any reasonable sense. This becomes relevant from the viewpoint of classical general relativity, and the global geometrical properties of PFWs will depend dramatically on the possible quadratic behaviour of H or $-H$ (for any dimension $n \geq 1$ of M).

In Sect. 3, we show that the behaviour of all the causal curves can be essentially controlled in a PFW (the more accurate control for existence of causal geodesics is postponed to Sect. 5). In Subsect. 3.1 a detailed study of the causal hierarchy of PFWs is carried out. In particular, Penrose's above-mentioned question is answered by showing that the causal hierarchy of plane waves is "unstable" or "critical": deviations in the superquadratic direction of $-H$ may transform them in non-distinguishing spacetimes, but deviations in the (more realistic) subquadratic direction yield global hyperbolicity. Subsequent results by Hubeny, Rangamani and Ross [32] are also discussed. In Subsect. 3.2 the criterion for the non-existence of horizons posed by Hubeny and Rangamani [30, 31] is explained, and a simple proof showing that it holds for any PFW is given.

In Sect. 4, geodesic completeness is studied. We claim that this problem is equivalent to a purely Riemannian problem (Theorem 2), which has been solved satisfactorily only for autonomous H, i.e., $H(x, u) \equiv H(x)$. The power of the known autonomous results (which yield completeness for at most quadratic $H(x)$, Theorem 3) is illustrated by comparison with the examples in [29]. Then, we claim the necessity to improve the non-autonomous ones. Moreover, the Ehlers–Kundt conjecture deserves a special discussion. Even though easily solvable under at most quadraticity for x (Theorem 4), it remains open in general.

In Sect. 5, the problems related to geodesic connectedness are studied. The key is to reduce the problem to a purely Riemannian problem, concretely, the classical variational problem of finding critical points for a Lagrangian type kinetic energy minus (time-dependent) potential energy. That is, solving this classical problem becomes equivalent to solving the geodesic connectedness problem in PFWs. Remarkably, in order to obtain the optimal results on waves (extending Ehrlich-Emch's ones) we had to improve the known Riemannian results; in the Appendix, this Riemannian problem is explained. Finally, the existence of conjugate points is discussed, and reduced again to a purely Riemannian problem. Energy conditions tend to yield conjugate points for causal geodesics. But, in agreement with the other results of the present paper, the focusing property of lighlike geodesics in plane waves (Fig. 1) becomes highly non-generic.

2 General Properties of the Class of Waves

2.1 Definitions

Let us start with some simple properties of the metric (1). The assumed geometrical background material can be found in well-known books such as [4, 27, 36]. Following [36], vector 0 will be regarded as spacelike instead of lightlike.

The vector field ∂_v is absolutely parallel (i.e., covariantly constant) and lightlike, and the time-orientation will be chosen to make it past-directed. Thus, for any future-directed causal curve $z(s) = (x(s), v(s), u(s))$,

$$\dot{u}(s) = \langle \dot{z}(s), \partial_v \rangle \geq 0 \ ,$$

and the inequality is strict if $z(s)$ is timelike. As $\mathrm{grad} u = \partial_v$, coordinate $u : \mathcal{M} \to \mathbb{R}$ plays the role of a "quasi-time" function [4, Def. 13.4], i.e., its gradient is everywhere causal and any causal segment γ with $u \circ \gamma$ constant (necessarily a lightlike pregeodesic without conjugate points) is injective. In particular, the spacetime is causal (see also Sect. 3.1). The hypersurfaces $u \equiv$ constant are degenerate, with the kernel of the metric spanned by ∂_v. The hypersurfaces (non-degenerate n-submanifolds) of these degenerate hypersurfaces which are transverse to ∂_v, must be isometric to open subsets of M. The *fronts* of the wave (1) will be defined as the (whole) n-submanifolds at constant u, v.

According to Ehlers and Kundt [19] (see also [8]) a vacuum spacetime is a plane-fronted gravitational wave if it contains a shearfree geodesic lightlike vector field V, and admits "plane waves" – spacelike (two-)surfaces orthogonal to V. The best known subclass of these waves are the (gravitational) "plane-fronted waves with parallel rays" or pp-waves, which are characterized by the condition that V is covariantly constant, $\nabla V = 0$. Ehlers and Kundt gave several characterizations of these waves in coordinates, and they obtained that (at least locally) the metric can be written as in (1) with $M = \mathbb{R}^2$. Nowadays, pp-wave means any spacetime which admits a covariantly constant lightlike vector field [45, p. 383]. Even though, in general, their fronts may be "non-plane", this happens in the most relevant cases (four dimensional spacetimes which are either vacuum, or solutions to Einstein-Maxwell equations, or pure radiation fields), where expression (1) holds with $M = \mathbb{R}^2$. In what follows, we will use the name "pp-waves" to denote the classical spacetimes (1) with $M = \mathbb{R}^2$. The pp-wave is gravitational (i.e., vacuum, see Subsect. 2.2) if and only if the "spatial" (transverse) Laplacian $\Delta_x H(x,u)$ vanishes. Plane waves constitute the (highly symmetric) subclass of pp-waves with H exactly quadratic in x for appropriate global coordinates on each front; that is, when we can assume:

$$H(x,u) = (x^1, x^2) \begin{pmatrix} f_1(u) & g(u) \\ g(u) & -f_2(u) \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \end{pmatrix} \tag{2}$$

where f_1, f_2, g are arbitrary (smooth) functions. When $f_1 \equiv f_2$, the plane wave is gravitational, and there are other well-known subclasses ("sandwich plane wave" if f_1, f_2, g have compact support; "purely electromagnetic plane wave" if $f_1 \equiv -f_2$, $g \equiv 0$, etc.)

Recall that, in our type of metrics (1), no restriction on the Riemannian part $(M, \langle\cdot,\cdot\rangle_R)$ is imposed. This seems convenient for different reasons as, for example: (i) the generality in the dimension n, for applications to strings, (ii) the generality in the topology, for discussions of horizons, or (iii) the generality in the metric, to obtain "generic results", not crucially dependent on very special particular properties of the metric. In this context, a name such as "M-fronted wave with parallel rays" (Mp-wave) seems natural for our spacetimes (1). Nevertheless, we will maintain the name PFW (plane-fronted wave) in agreement with previous references [15, 24] and the nomenclature in [4], but with no further pretension.

2.2 Curvature and Matter

Fixing some local coordinates $x^1, \ldots, x^n$ for the Riemannian part M, it is straightforward to compute the Christoffel symbols of $\langle\cdot,\cdot\rangle$ and, thus, to relate the Levi–Civita connections ∇, ∇^R for $\mathcal{M}$ and M, respectively (see [15]). We remark the following facts:

- M is totally geodesic, i.e., $\nabla_{\partial_i}\partial_j = \nabla^R_{\partial_i}\partial_j$, $i,j = 1,\ldots,n$.
- $2\,\nabla_{\partial_u}\partial_u = -\mathrm{grad}_x H(x,u) + \partial_u H(x,u)\partial_v$; $\quad 2\nabla_{\partial i}\partial_u = \partial_i H(x,u)\partial_v$; thus, the curvature tensor satisfies

$$-\ 2\,R(\cdot,\partial_u,\partial_u,\cdot) = \mathrm{Hess}_x H(x,u)(\cdot,\cdot)\ . \tag{3}$$

  Here $\mathrm{grad}_x H$ and $\mathrm{Hess}_x H$ denote the spatial (or "transverse") gradient and Hessian of H, respectively.
- The Ricci tensors of $\mathcal{M}$ and M satisfy

$$\mathrm{Ric} = \sum_{i,j=1}^{n} R_{ij}^{(R)} dx^i \otimes dx^j - \frac{1}{2}\Delta_x H du \otimes du\ .$$

Thus, Ric is zero if and only if both the Riemannian Ricci tensor $\mathrm{Ric}^{(R)}$ and the spatial Laplacian $\Delta_x H$ vanish.

From the last item, it is easy to check that the timelike convergence condition holds if and only if

$$\mathrm{Ric}^{(R)}(\xi,\xi) \geq 0, \quad \Delta_x H \leq 0, \quad \text{for all } x \in M,\ \xi \in T_x M\ .$$

Even more, in dimension 4 all the energy conditions are equivalent and easily characterized [24, Proposition 5.1]:

Proposition 1. *Let $\mathcal{M} = M \times \mathbb{R}^2$ be a 4-dimensional PFW, and let $K(x)$ be the (Gauss) curvature of the 2-manifold M. The following conditions are equivalent:*

(A) *The strong energy condition ($Ric(\xi,\xi) \geq 0$ for all timelike ξ).*
(B) *The weak energy condition ($T(\xi,\xi) \geq 0$ for all timelike ξ).*
(C) *The dominant energy condition ($-T^a_b \xi^b$ is either 0 or causal and future-pointing, for all future-pointing timelike $\xi \equiv \xi^b$).*
(D) *Both inequalities:*

$$K(x) \geq 0, \quad \Delta_x H(x,u) \leq 0, \qquad \forall (x,u) \in M \times \mathbb{R}\ .$$

2.3 Finiteness of the Wave and Decay of H at Infinity

Now, let us discuss *minimal necessary* assumptions which must be satisfied by a PFW to be “finite” or “asymptotically vanishing” in any reasonable sense. In principle, one could think that M should be asymptotically flat, but we will not impose this strong condition a priori (i.e., non-trivial fronts at a “cosmological scale” are admitted). In any case, it would not be too relevant for our problem: plane waves have flat fronts and are not finite by any means.

As we have said, all the scalar curvature invariants of a gravitational pp-wave vanish. Thus, instead of such scalars, we will focus on the spatial Hessian $\mathrm{Hess}_x H$. In the case of plane waves, $\mathrm{Hess}_x H$ is essentially the matrix in (2) – it can be viewed as the *transverse frequency matrix* of the lightlike geodesic deviation [37]. By equality (3), $\mathrm{Hess}_x H$ is related to the most “characteristic” curvatures of the wave; these curvatures – taken along a lightlike geodesic – admit an intrinsic interpretation in terms of the Penrose limit (see [9], especially the discussions around formulas (1.2), (2.13)). According to [30,31], “to go arbitrarily far” in a pp-wave can be thought of as taking v, x large for each fixed u (see also Subsect. 3.2). Therefore, any sensible definition of finiteness or asymptotic vanishing of the PFW seems to imply that $\mathrm{Hess}_x H(x,u)$ must go (fast) to 0 for large x.

Rigourously, let $\lambda_i(x,u), i = 1,\ldots,n$, be the eigenvalues of $\mathrm{Hess}_x H(x,u)$, $d(\cdot,\cdot)$ the Riemannian distance on M and fix any $\bar{x} \in M$. From the above discussion, if the wave vanishes asymptotically then $\lim_{d(x,\bar{x})\to\infty} \lambda_i(x,u)$ must vanish fast for each u. Therefore, putting $|\lambda(x,u)|$ equal to the maximum of the $|\lambda_i(x,u)|$'s, we can assume as definition of asymptotic vanishing for a PFW:

$$|\lambda(x,u)| \leq \frac{A(u)}{d(x,\bar{x})^{q(u)}} \tag{4}$$

for some continuous functions $A(u)$ and $q(u) > 0$.

Inequality (4) implies bounds for the spatial growth of $|H|$, as the next proposition shows. But, first, let us introduce the following definition. Let $V(x,u)$ be a continuous function $V : M \times \mathbb{R} \to \mathbb{R}$. We will say that $V(x,u)$ behaves *subquadratically at spatial infinity* if

$$V(x,u) \leq R_1(u) d^{p(u)}(x,\bar{x}) + R_2(u) \quad \forall (x,u) \in M \times \mathbb{R}\ ,$$

for some continuous functions $R_1(u), R_2(u) (\geq 0)$ and $p(u) < 2$. If the last inequality is relaxed into $p(u) \leq 2, \forall u \in \mathbb{R}$ then $V(x,u)$ behaves *at most quadratically at spatial infinity*. Now, we can assert the following result (see [24, Proposition 5.3] for the idea of the proof – notice that the completeness of M is not necessary now, as any curve can be approximated by broken geodesics).

Proposition 2. *If the PFW vanishes asymptotically as in* (4)*, then* $|H(x,u)|$ *behaves subquadratically at spatial infinity.*

The following must be emphasized:

1. The condition of asymptotic vanishing (4) implies subquadraticity for $|H(x,u)|$, but the converse is not true. In the remainder of this paper, we will use only this more general subquadratic condition or, even, only the subquadraticity (or at most quadraticity) of H or $-H$. So, the range of application of our results will be wider.
2. Of course, inequality (4) is compatible with the energy conditions. A simple explicit example can be constructed by taking $H(x^1, x^2, u) \equiv H_u(x^1)$, and, putting $x \equiv x^1$:

$$-\frac{A(u)}{|x|^{q(u)}} \leq \frac{d^2 H_u}{dx^2}(x) \leq 0$$

for some $A(u), q(u) > 0$.
3. For plane waves, neither H nor $-H$ behaves subquadratically. In fact, the eigenvalues of $\mathrm{Hess}_x H(x,u)$ are independent of x, and the fronts of the wave are not “finite”. This is a consequece of the idealized symmetries of the front waves. Nevertheless, $|H(x,u)|$ behaves at most quadratically at infinity and, thus, plane waves lie in the limiting quadratic situation.

3 Causality

3.1 Positions in the Causal Ladder

Recall first the causal hierarchy of spacetimes [4]:

$$\begin{gathered}\text{Globally hyperbolic} \Rightarrow \text{Causally simple} \Rightarrow \text{Causally continuous}\\ \Rightarrow \text{Stably causal} \Rightarrow \text{Strongly causal}\\ \Rightarrow \text{Distinguishing} \Rightarrow \text{Causal} \Rightarrow \text{Chronological}\end{gathered}$$

Roughly, a spacetime is causal if it does not contain closed causal curves, strongly causal if there are no "almost closed" causal curves and stably causal if, after opening slightly the light cones, the spacetime remains causal. It is commonly accepted that stable causality is equivalent to the existence of a *continuous* time function (see [4, 27], and also [43, Sect. 4]), but only recently has the existence of a *smooth* time function with nowhere lightlike gradient – i.e., a "temporal" function – been proven [7]. Globally hyperbolic spacetimes can be defined as the strongly causal ones with compact diamonds $J^+(p) \cap J^-(q)$ for any p, q. They were characterized by Geroch as those possessing a Cauchy hypersurface (which can be also chosen smooth and spacelike [6]). PFWs are always causal (Sect. 2) and the following result was proven in [24]:

Theorem 1. *Any PFW with M complete and $-H$ spatially subquadratic is globally hyperbolic.*

The following points must be emphasized:

1. The proof is carried out by showing strong causality and the compactness of the diamonds. From the technique, one can also check that, if $-H$ is at most quadratic at spatial infinity, then the spacetime is strongly causal (with no assumption on the completeness of M).
2. Hubeny, Rangamani and Ross [32] constructed explicitly a temporal function for plane waves. As the light cones of an at most quadratic pp-wave can be bounded by the cones of a plane wave, they claim that *any pp-wave with $-H$ at most quadratic at spatial infinity is stably causal.* (They also use the temporal function to study quotients of the wave by the isometry group generated by a spacelike Killing field, which may be stably causal or non-chronological, see also [33]).

   Recall from the Introduction that gravitational plane waves are causally continuous (the set valued maps $I^{\pm}$ are outer continuous) but not causally simple (because the causal future or past of a point may be non-closed).
3. If $-H(x, u)$ were not at most quadratic, then the spacetime may be even non-distinguishing (the chronological future or past of two distinct points are equal). In fact, a wide family of non-distinguishing examples with

$-H$ "arbitrarily close" to at most quadratic (and M complete) is constructed in [24, Proposition 2.1]; in these PFWs, the chronological futures $I^+(x, v, u)$ depend only on u. In particular, any pp-wave such that $-H$ behaves as $|x|^{2+\epsilon}, \epsilon > 0$ for large $|x|$ is non-distinguishing [24, Example 2.2].

Nevertheless, the spatially subquadratic or at most quadratic behaviour of $-H$ is not necessary for global hyperbolicity or strong/stable causality, as explicit counterexamples [24, Example 4.5] show (additional hypotheses must be assumed, see [32, Sect. 4]).

4. A curious phenomenon, suggested in [32, Sect. 4], is that the class of distinguishing but non-stably causal pp-waves (or even PFWs) might be empty. In this respect, our technique in [24] suggests that, if non-empty, the class would be scarcely significant.

The technique involved for Theorem 1 can be understood as follows. Any future-directed timelike curve α can be reparametrized by the quasi-time u: $\alpha(u) = (x(u), v(u), u), u \in [u_0, u_1]$. The proof is based on inequalities which relate the distance covered by $x(u)$ with the extreme points of $v(u)$. For fixed $\epsilon > 0$ and $0 < u_1 - u_0 \leq \epsilon$, $u \in [u_0, u_1]$:

$$\frac{1}{\epsilon^2}\int_{u_0}^{u} d^2(x(s), x(u_0))ds \leq \int_{u_0}^{u} \langle \dot{x}(s), \dot{x}(s)\rangle_R ds$$

$$< 4\left(R_2' \cdot (u - u_0) - (v(u) - v(u_0))\right)$$

where the constant $R_2' = R_2'(u_0, \epsilon)$ is independent of $x(u_0)$ in the subquadratic case (in the finer proof of strong causality for the at most quadratic case, R_2' is allowed to depend on a compact subset where $x(u_0)$ lies, and $\epsilon > 0$ is not fixed a priori). Then, such an inequality is used:

– For the proof of strong causality, to show that, fixed a small neighborhood $\mathcal{N}$ of a point z_0 (which can be chosen "rectangular" in the coordinates v, u, i.e., $\mathcal{N} = N \times\,]\,v^-, v^+\,[\, \times\,]\,u^-, u^+\,[\,$), and any causal curve with extremes in this neighborhood, the restrictions on the extremes for $v(u), u$ ($v_0, v_1 \in\,]\,v^-, v^+\,[\,$, $u_0, u_1 \in\,]\,u^-, u^+\,[\,$) also imply restrictions on the distance between $x(u_0), x(u_1)$. This forces the whole curve $x(u)$ to remain in a controlled neighborhood of N.
– For global hyperbolicity, to prove also that the projections of each diamond $J^+(p) \cap J^-(q) \subset M \times \mathbb{R}^2$ on each factor $M, \mathbb{R}^2$ are bounded for the natural (complete) Riemannian distances d on M and $du^2 + dv^2$ on $\mathbb{R}^2$. Therefore $J^+(p) \cap J^-(q)$ will be included in a compact subset, which in turn yields its compactness.

3.2 Causal Connectivity to Infinity and Horizons

Next, let us comment on the applicability of these techniques to the study of horizons in PFWs. The possible existence of horizons in gravitational pp-waves and, in general, in vanishing scalar curvature invariant (VSI) spacetimes, have attracted interest recently, especially motivated by potential applications to string theory. Hubeny and Rangamani [30,31] proposed a criterion for the existence of horizons in pp-waves, and they proved the *non-nexistence* of such horizons. In a more standard framework, Senovilla [44] proved the non-existence of closed trapped or nearly trapped surfaces (or submanifolds in any dimension) in VSI spacetimes. Next, we will give a simple proof of the non-existence of horizons, in the sense of Hubeny and Rangamani, for an arbitrary PFW.

Hubeny-Rangamani's criterion [30, Sects. 2.2, 4] can be reformulated as follows[2]: *A pp-wave spacetime (or, in general, any PFW) $\mathcal{M}$ does not admit an event horizon if and only if, given any points $z_0 = (x_0, v_0, u_0), z_1 = (x_1, v_1, u_1) \in \mathcal{M}$ with $u_0 < u_1$, there is $-v_\infty > -v_1$ such that a future-directed causal curve from z_0 to $z_\infty = (x_1, v_\infty, u_1)$ exists.* According to the authors, this criterion tries to formalize the intuitive idea that any point of the spacetime is visible to an observer who is "arbitrarily far". In fact, one may think of u_1 as being close to u_0 and of x_1 as being far[3] from x_0.

To check that this criterion is satisfied for any PFW, choose any curve α starting at z_0 parametrized by u, $\alpha(u) = (x(u), v(u), u), u \in [u_0, u_1]$ such that $x(u_1) = x_1$. Putting $E_\alpha(u) = \langle \dot\alpha(u), \dot\alpha(u)\rangle = \langle \dot x(u), \dot x(u)\rangle_R + 2\dot v(u) + H(x(u), u)$, the function $v(u)$ can be re-obtained from $E_\alpha(u)$ as:

$$v(u) - v_0 = \frac{1}{2}\int_{u_0}^{u} \left(E_\alpha(\bar u) - \langle \dot x(\bar u), \dot x(\bar u)\rangle_R - H(x(\bar u), \bar u)\right) d\bar u, \quad \forall u \in [u_0, u_1]\,.$$

Choosing $E_\alpha < 0$ the curve α becomes timelike and future directed and, as $|E_\alpha|$ can be chosen arbitrarily big (and even constant, if preferred), the value of $-v(u_1)$ can be taken arbitrarily big, as required.

4 Geodesic Completeness

4.1 Generic Results

From the direct computation of Christoffel symbols of a PFW, it is straightforward to write the geodesic equations in local coordinates. Remarkably, the three geodesic equations for a curve $z(s) = (x(s), v(s), u(s))$, $s \in\,]a, b[$, can be solved in the following three steps [15, Proposition 3.1]:

(a) $u(s)$ is any affine function, $u(s) = u_0 + s\Delta u$, for some $\Delta u \in \mathbb{R}$.

[2] There is a change of sign for v in comparison to this reference because our convention for the metric uses $du\,dv$ instead of $-du\,dv$.

[3] Nevertheless, in our reformulation x_1 may also be "non-far" from x_0 (in fact, "to be far" would make no sense if M were compact or bounded). The criterion will hold in this strong sense.

(b) Then $x = x(s)$ is a solution on M of

$$D_s \dot{x} = -\mathrm{grad}_x V_\Delta(x(s), s) \quad \text{for all } s \in \,]\, a, b\, [\,,$$

where D_s denotes the covariant derivative and V_Δ is defined as[4] :

$$V_\Delta(x, s) = -\,\frac{(\Delta u)^2}{2}\, H(x, u_0 + s\Delta u)\ ;$$

(c) Finally, with a fixed v_0 and an $s_0 \in \,]\, a, b\, [$, $v(s)$ can be computed from:

$$v(s) = v_0 + \frac{1}{2\Delta u} \int_{s_0}^{s} (E_z - \langle \dot{x}(\sigma), \dot{x}(\sigma)\rangle_R + 2V_\Delta(x(\sigma), \sigma))\, d\sigma$$

where $E_z = \langle \dot{z}(s), \dot{z}(s)\rangle$ is a constant (if $\Delta u = 0$ then $v = v(s)$ is also affine).

In particular, geodesic completeness is reduced, essentially, to the completeness of trajectories for (non-autonomous) potentials on M, and one can prove [15, Theorem 3.2]:

Theorem 2. *A PFW is geodesically complete if and only if the Riemannian manifold M is complete and the trajectories of*

$$D_s \dot{x} = \frac{1}{2}\, \mathrm{grad}_x H(x, s)$$

are also complete.

Recall that the completeness of M is an obvious necessary condition (the wave fronts are totally geodesic) and, then, the question is fully reduced to a purely Riemanian problem: the completeness of the trajectories of the potential $V = -H/2$. This problem was studied by several authors in the 1970s [18, 25, 46] and they obtained very accurate results when the potential is autonomous, i.e., H independent of u. For example, a result by Weinstein and Marsden [46] (see also [1, Theorem 3.7.15] or [15, Sect. 3]), formulated in terms of positively complete functions, yields as a straightforward consequence:

Theorem 3. *Any PFW with M complete and $H(x, u) \equiv H(x)$ at most quadratic is geodesically complete.*

Recall that here only H (and not $-H$) needs to be controlled. As an example of the power of this result, one can check that the explicit examples of pp-waves exhibited in [29], which were proven to be complete (by integrating

[4] The subscript Δ for V (and similarly in formula (5) below, for $\mathcal{J}$) is only an abbreviation to keep track of Δu. Notice that the dependence of V on Δu will be important for geodesic connectedness and the existence of conjugate points, but we will get rid of it for geodesic completeness.

– decoupled – geodesic equations), lie under the hypotheses[5] of Theorem 3. For example, for PP1 (see [29, Sect. 5.2]), $H(x) = \cos x^2 - \cosh x^1$; for PP2, $H(x) = -\sum_j f_j(x^j)$ with the f_j's bounded from below; in both cases, H is upper bounded. On the other hand, their incomplete examples strongly violate the conditions of Theorem 3. For example, for the monopole pp-waves in PP3 the Riemannian part M may be incomplete, and for the example PP4 one has the highly violating coefficient $H(x) = -e^{x^2} \sin x^1$.

Nevertheless, the results for non-autonomous potentials are not so accurate [25]. But this is the case of plane waves, which are geodesically complete in any dimension (see [15, Proposition 3.5]) and, then, to find general and accurate criteria seems an interesting topic for further research.

4.2 Ehlers–Kundt Question

From a fundamental viewpoint, the following question on pp-waves ($M = \mathbb{R}^2$) was posed by Ehlers and Kundt [19] (see also [8] or [29]):

> Is any *complete* gravitational pp-wave a plane wave?

As they pointed out, complete gravitational pp-waves represent graviton fields generated independently of matter (vacuum) or external sources (completeness). Thus, they are analogous to source-free photons in electrodynamics.

Notice that the hypotheses become relevant for both the physical interpretation and the involved mathematical problem. In fact, from Theorem 2 (with $V = -H/2$) and the fact that linear terms in the expression of H in (2) can be dropped by choosing appropriate coordinates, the previous question is equivalent to:

> Let $V((x,y),s)$, $V : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ be an harmonic function in (x,y). If the trajectories for V as a (non-autonomous) potential on $\mathbb{R}^2$ are complete, must V be a (harmonic) polynomial of degree ≤ 2 for each fixed s (i.e., $V((x,y),s) = f(s)(x^2 - y^2) + 2g(s)xy + c(s)x + d(s)y + e(s)$)?

Notice that the harmonicity of H allows us to use techniques from complex analysis. In fact, it is easy to show:

Theorem 4. *Any gravitational pp-wave such that $H(x,u)$ behaves at most quadratically at spatial infinity is a (necessarily complete) plane wave.*

To prove this, put $\zeta = x + iy$, $H \equiv H(\zeta, u)$ and consider the complex function $f(\zeta, u)$ which is holomorphic in ζ with real part equal to H. Then, $f(\zeta,u)/\zeta^2$ is meromorphic for $\zeta \in \mathbb{C}$ and bounded for big ζ. Thus, for each u, whenever $f(\cdot, u)$ is not constant, it presents a pole at infinity of order 1 or 2. That is,

[5] For the comparison of hypotheses, recall that their function $F(x,u)$ plays the role of our $-H(x,u)$.

$f(\cdot, u)$ is a complex polynomial of degree at most 2, and the result follows directly.

Even though Theorem 4 covers the most meaningful cases from the physical viewpoint (and is free of hypotheses on completeness), in general the above question remains open as a mathematical problem with roots in the foundations of the theory of gravitational waves.

5 Geodesic Connectedness and Conjugate Points

5.1 The Lorentzian Problem

Next, we will study geodesic connectedness of PFWs, that is, we will ask: fixed any $z_0 = (x_0, v_0, u_0), z_1 = (x_1, v_1, u_1) \in \mathcal{M}$, is there any geodesic connecting z_0, z_1? This problem becomes relevant from different viewpoints (see [41] for a survey): (a) the connectivity of a point z_0 with any point $z_1 \in I^+(z_0)$ through a timelike geodesic admits an obvious physical interpretation, and is satisfied by all globally hyperbolic spacetimes (Avez–Seifert result), (b) the geodesic connectedness of a Lorentzian manifold – through geodesics of any causal type – is a desirable geometrical property[6], which admits a natural variational interpretation and, then, yields an excellent motivation to study critical points of indefinite functionals from a mathematical viewpoint [34], (c) the possible multiplicity of connecting geodesics is related to the existence of conjugate points.

These questions were studied by Penrose [37] and Ehrlich and Emch [20–22] for plane waves, by integrating geodesic equations. They proved that there exists a natural concept of *conjugacy for pairs* $u_0, u_1 \in \mathbb{R}$, $u_0 < u_1$, and obtained the following results:

1. (Penrose). Lightlike geodesics are focused when u_0, u_1 are conjugate (at least for "weak" sandwich waves). In this case, all the lightlike geodesics starting at z_0 (except one)
   - either cross a fixed point with $u = u_1$ (anastigmatic conjugacy, in electromagnetic plane waves),
   - or cross a fixed line (astigmatic conjugacy, in gravitational or mixed plane waves).
2. (Ehrlich-Emch). The connectable points for astigmatic gravitational plane waves can be characterized in an accurate way:
   - if u_1 lies before the first conjugate point of u_0, then there exists a unique geodesic between z_0 and z_1, which is causal if $z_0 < z_1$;
   - otherwise, connecting geodesics may not exist and, in fact, *gravitational plane waves are not geodesically connected.*

[6] This is trivially satisfied for complete Riemannian manifolds but not necessarily for complete Lorentzian ones, such as de Sitter spacetime.

5.2 Relation with a Purely Riemannian Variational Problem

From the study of geodesic equations in Sect. 4, and the classical relation between connecting trajectories for a potential and extremal of Lagrangians, it is not difficult to prove [15]:

Proposition 3. *For fixed $z_0, z_1 \in \mathcal{M}$, the following statements are equivalent:*

(a) *z_0 and z_1 can be connected by a geodesic.*
(b) *There exists a solution for the Riemannian problem*

$$\begin{cases} D_s\dot{x}(s) = -\mathrm{grad}_x V_\Delta(x(s), s) & \text{for all } s \in [0,1] \\ x(0) = x_0, \ \ x(1) = x_1 \ , \end{cases}$$

where $V_\Delta(x,s) = -\ \frac{(\Delta u)^2}{2}\ H(x, u_0 + s\Delta u),\ \Delta u = u_1 - u_0$.
(c) *There exists a critical point for the action functional $\mathcal{J}_\Delta$ defined on the space of absolutely continuous curves $x : [0,1] \to M$ which connect x_0, x_1,*

$$\mathcal{J}_\Delta(x) = \frac{1}{2}\int_0^1 \langle \dot{x}, \dot{x}\rangle_R \ ds \ - \int_0^1 V_\Delta(x,s)\ ds \ . \tag{5}$$

Of course, (c) is *the most classical problem in Lagrangian Mechanics.* Nevertheless (as a surprise for us), it had not been fully solved in the quadratic case. This case corresponds to plane waves and, thus, in order to obtain optimal Lorentzian results (re-obtaining in particular Ehrlich-Emch's), we had to improve the known Riemannian ones. The final Riemannian result [13] is the following (see the Appendix for a discussion of the problem):

Theorem 5. *Let $(M, \langle\cdot,\cdot\rangle_R)$ be a complete (connected) n-dimensional Riemannian manifold. Assume that $V \in C^1(M \times [0,1], \mathbb{R})$ is at most quadratic in x in the following way:*

$$V(x,s) \leq \lambda d^2(x,\bar{x}) + \mu d^p(x,\bar{x}) + k \quad \forall (x,s) \in M \times [0,1] \ ,$$

for some fixed point $\bar{x} \in M$ and constants $p < 2,\ \lambda, \mu, k \geq 0$.

If $\lambda < \pi^2/2$ then, for all $x_0, x_1 \in M$, there exists at least one critical point (in fact, an absolute minimum) of $\mathcal{J}_\Delta$ in (5). In particular, this happens if V is subquadratic, i.e., when $\lambda = 0$.

If, additionally, M is not contractible, then there exists a sequence of critical points $\{x_m\}_m$ such that

$$\lim_{m\to+\infty} \mathcal{J}_\Delta(x_m) \to +\infty \ .$$

One can also assume that λ (as well as p, μ, k) depend on s, and then take the maximum of $\lambda([0,1])$ (and the others) for the conclusion.

5.3 Optimal Results for Connectedness of PFWs

Now, the application of Proposition 3 and Theorem 5 (plus a further discussion for the case of causal geodesics) directly yields the following result. Notice that the limit value $\lambda = \pi^2/2$ in Theorem 5 is related to $(\Delta u)^2$ in the expression of V_Δ in Proposition 3(b); thus, in the at most quadratic case, stronger conclusions hold when $(\Delta u)^2$ is smaller than a critical value.

Theorem 6. *Let $\mathcal{M}$ be a PFW with M complete, and fix $\bar{x} \in M$. Then:*

(1) *If $-H(x,u)$ is spatially subquadratic, then $\mathcal{M}$ is geodesically connected.*
(2) *If $-H(x,u)$ is at most quadratic with*

$$-H(x,u) \leq R_0(u)\, d^2(x,\bar{x}) + R_1(u)\, d^{p(u)}(x,\bar{x}) + R_2(u)$$

$\forall (x,u) \in M \times \mathbb{R}$, $p(u) < 2$, then $z_0 = (x_0, v_0, u_0)$, $z_1 = (x_1, v_1, u_1) \in \mathcal{M}$, $u_0 \leq u_1$ can be connected by means of a geodesic whenever

$$R_0[u_0,u_1]\,(u_1 - u_0)^2 < \pi^2 \ ,$$

where

$$R_0[u_0,u_1] = \mathrm{Max}\{R_0(u) : u \in [u_0,u_1]\} \ .$$

Moreover, in any of the previous cases (1), (2)*:*

(a) *If $z_0 < z_1$ there exists a length-maximizing causal geodesic connecting z_0 and z_1;*
(b) *If M is not contractible:*
   (i) *There exist infinitely many spacelike geodesics connecting z_0 and z_1,*
   (ii) *The number of timelike geodesics from z_0 to $z_v = (x_1, v, u_1)$ goes to infinity when $-v \to \infty$.*

It must be emphasized that these results are optimal because the Riemannian results are optimal. In fact:

- There are explicit counterexamples if any of the hypotheses is dropped.
- In the case of gravitational plane waves, the conclusions of Theorem 6 not only generalize Ehrlich-Emch's ones, but also yield bounds for the appearance of the first astigmatic conjugate pair – a lower bound is the value u_+ ($u_+ > u_0$) such that $R_0[u_0,u_+](u_+ - u_0)^2 = \pi^2$.
- All the results can be naturally extended to the case of M being non-complete with convex boundary.

5.4 Conjugate Points

From the above approach to geodesic connectedness it is also clear that, now, the existence of conjugate points for geodesics on a PFW is equivalent to the existence of conjugate points for the action $\mathcal{J}_\Delta$. More precisely, following [24, Sect. 6], we can define:

Definition 1. *Fix $\overline{z}_0 = (x_0, u_0)$, $\overline{z}_1 = (x_1, u_1) \in M \times \mathbb{R}$, and let $x(s)$ be a critical point of $\mathcal{J}_\Delta$ in (5) with endpoints x_0, x_1 and $\Delta u = u_1 - u_0$. We say that $\overline{z}_0$, $\overline{z}_1$ are conjugate points along $x(s)$ of multiplicity $\overline{m}$ if the dimension of the nullity of the Hessian of $\mathcal{J}_\Delta$ on $x(s)$ is $\overline{m}$ (if $\overline{m} = 0$ we say that $\overline{z}_0$, $\overline{z}_1$ are not conjugate).*

Then, one obtains the following equivalence between conjugate points for Lorentzian geodesics and conjugate points for Riemannian trajectories of a potential [24, Proposition 6.2]:

Proposition 4. *The pairs $\overline{z}_0 = (x_0, u_0)$, $\overline{z}_1 = (x_1, u_1)$ are conjugate of multiplicity $\overline{m}$ along $x(s)$ if and only if for any geodesic $z : [0, 1] \to \mathcal{M}$ with $z(s) = (x(s), v(s), \Delta u \cdot s + u_0)$ the corresponding endpoints $z_0 = (x_0, v_0, u_0)$, $z_1 = (x_1, v_1, u_1)$ are conjugate with the same multiplicity $m = \overline{m}$.*

As we mentioned in Subsect. 5.1, in the particular case of gravitational plane waves, conjugate pairs are defined for u_0, u_1. For general PFWs, the lack of symmetries of the fronts makes it necessary to take care of the M part. Nevertheless, the dependence on v is still dropped.

Now, studying the conjugate points for $\mathcal{J}_\Delta$, one can obtain easily results as [24, Proposition 6.4]: *If H is spatially convex* (i.e. $\mathrm{Hess}_x H(x, u)(w, w) \geq 0$, $\forall w \in T_x M$) *and the sectional curvature of M is non-positive, then no geodesic admits conjugate points.* Of course, the hypotheses of this result go in the *wrong* direction with respect to the energy conditions ($\Delta_x H \leq 0, K \geq 0$), which tend to yield conjugate points. Nevertheless, this focusing of geodesics is, in general, qualitatively different from the focusing in the plane-wave case and, as we have seen, it does not forbid global hyperbolicity.

Appendix: The Riemannian Problem of Connectedness by the Trajectories of a Lagrangian

In Sect. 5 we showed that geodesic connectedness of PFWs depends crucially on the Riemannian variational result Theorem 5. This result is an answer to the classical Bolza problem, which can be stated as:

> **Bolza problem.** For fixed x_0, x_1 in a Riemannian manifold M and some $T > 0$, determine the existence of critical points for the functional
> $$J_T(x) = \frac{1}{2} \int_0^T \langle \dot{x}, \dot{x} \rangle_R \, ds - \int_0^T V(x, s) \, ds$$
> on the set of absolutely continuous curves with $x(0) = x_0, x(T) = x_1$.

In our case, $T = 1$, V is smooth and at most quadratic, and M is complete. For this problem, it is well-known that two abstract conditions on J_T, namely, boundedness from below and coercivity, imply the existence of a critical point

– in fact a minimum. Even more, by using Ljusternik–Schnirelmann theory one can ensure the existence of a sequence of critical points such that J_T diverges.

The following results were known:

1. If $V(x,s)$ is bounded from above or subquadratic in x, then the two abstract conditions hold and J_T attains a minimum.
2. If $V(x,s)$ is at most quadratic, with $V(x,s) \leq \lambda d^2(x,\bar{x}) + \mu d^{p(s)}(x,\bar{x}) + k(s),\ p(s) < 2, \forall s \in [0,T]$ then:
   – Clarke and Ekeland [16] proved that, if $T < 1/\sqrt{\lambda}$, then J_T still admits a minimum.
   – If $T \geq \pi/\sqrt{2\lambda}$ there are simple counterexamples to the existence of critical points (harmonic oscillator).

Therefore, there was a gap for the values of T,

$$T \in \ [\, 1/\sqrt{\lambda}, \pi/\sqrt{2\lambda}\, [\ ,$$

which was covered only in some particular cases (for example, if $\mathrm{Hess}_x V \geq 2\lambda$, then J_T still admits a minimum). Our results in [13] (essentially, Theorem 5) fill this gap, by showing that, even in the case $T \in [\, 1/\sqrt{\lambda}, \pi/\sqrt{2\lambda}\, [$, the functional J_T is bounded from below and coercive and, thus, admits a minimum.

The proof was carried out in three steps:

– Step 1. The essential term to prove the abstract conditions for J_T is $d^2(x(s),\bar{x})$. Then, consider the new functional

$$F_T^\lambda(x) = \frac{1}{2} \int_0^T \langle \dot{x}, \dot{x} \rangle_R \ ds \ - \ \lambda \int_0^T d^2(x(s),\bar{x}) \ ds \ .$$

J_T is essentially greater than F_T^λ and, if F_T^λ is bounded from below and coercive, then so is J_T (recall that the expression of F_T^λ contains $d^2(\cdot,\bar{x})$, which is only continuous, but we are not looking for critical points of this functional).

– Step 2. Reduction to a problem in one variable. For each curve $x(s)$ in the domain of J_T, one can find a continuous curve $y(s)$, $s \in [0,T]$, almost everywhere differentiable, such that (assuming $\bar{x} = x_0$ without loss of generality) $y(0) = 0$, $y(T) = d(x_0,x_1)$ and:

$$\dot{y}(s) = |\dot{x}(s)| \text{ a.e. in } [0,s_0], \qquad \dot{y}(s) = -|\dot{x}(s)| \text{ a.e. in }]\, s_0, T\,]\ ,$$

for some suitable s_0. For this curve $y(s)$,

$$F_T^\lambda(x) \geq \frac{1}{2} \int_0^T |\dot{y}|^2 \ ds \ - \ \lambda \int_0^T |y|^2 \ ds \ . \tag{6}$$

And, then, one has just to prove that the new (1-*dimensional*) functional $G_T^\lambda(y)$, equal to the right hand side of (6), is coercive and bounded from below.

– Step 3. Solution of the 1-variable problem for $G_T^\lambda(y)$ by elementary methods (Fourier series, Wirtinger's inequality).

The technique also works for manifolds with boundary [14]. Remarkably, the procedure has also been used to prove the geodesic connectedness of static spacetimes under critical quadratic hypotheses [3] (see also [42]), and other related problems.

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