LONGITUDINAL INSTABILITIES IN INTENSE RELATIVISTIC BEAMS

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(presented by K. R. Symon)

I. INTRODUCTION

The present paper is concerned with the influence of space charge forces upon the azimuthal stability of circulating ion beams. The problem treated is the one dimensional azimuthal motion of circulating beams with distribution in momentum and coordinate differing only slightly from certain stationary distributions.

A solution of the problem is most readily obtained for distributions characterized by a density in two dimensional phase space that is uniform within sharp boundaries and zero elsewhere, and consequently this special case is analyzed in detail and illustrated by numerical examples. This analysis is supplemented by results drawn from a general treatment of arbitrary distributions (which will be published in full elsewhere) in order to extend the discussion to two or more intersecting beams of relativistic ions.

In developing a small amplitude theory of stability we first find a stationary distribution and then ask whether a small perturbation of this distribution leads to bounded oscillation or to growth : the first we call stable and the second unstable. It is of course, impossible to say from such analysis based on small amplitude approximations how the perturbation continues to grow after it becomes large, or even whether it decays again after reaching some finite amplitude, and the characterization " unstable " must therefore be understood to mean " initially unstable in small amplitude approximation ".

II. QUALITATIVE ARGUMENTS FOR A SINGLE CIRCULATING BEAM

The simplest possible stationary distribution is a uniform ring of circulating monoenergetic ions. From linearized wave equations obtained by hydrodynamic analysis [1,2] it is easy to see that small density perturbations in such a ring are either oscillatory or growing depending upon whether the ions are below or above transition energy. The instability above transition derives from the fact that a force acting on an ion in the direction of motion so as to increase its energy thereby decreases its revolution frequency; in angular coordinate θ the acceleration is in a direction opposite to the direction of applied torque. Thus, electrostatic force away from regions of greater charge density produces acceleration toward such regions. Since this behavior leads in the wave equation describing the motion of the perturbation to a negative inertia term, we refer to the instability as the negative mass instability.

The stability properties of circulating rings of interacting particles were first analyzed by Maxwell [3] in his famous essay on the stability of Saturn's rings. In this case also the individual particles have a negative angular inertia, but as their mutual interactions are attractive, a single ring is stable (providing that the mass of the ring is small enough compared to the mass of Saturn) and instabilities may arise only from the interaction of adjacent rings.

A distribution of particles with some spread in energy cannot be dealt with in the same way as mono-

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energetic ions because it is impossible to find a coordinate system in which all particles are initially at rest. It is clear, however, that the spread of revolution frequencies associated with energy spread results in a mixing that tends to destroy density perturbations; and we are consequently led to believe that a circulating ring of particles will be made less unstable, if not absolutely stable by an energy spread. We can even foresee that the stabilizing effect will be at least approximately independent of the wavelength of the perturbation, because the characteristic time for growth increases with wavelength and the " mixing time " with a given energy spread likewise increases with wavelength; ratio of growth time to mixing time tends to be invariant. These qualitative arguments are confirmed by the more quantitative analysis below.

III. PHASE SPACE REPRESENTATION

Any assembly of ions may be represented by a set of representative points in an appropriate phase space (μ—space). In the absence of interactions the motion of phase points is described by canonical equations derived from the single particle Hamiltonian, and this motion preserves phase density by Louiville's theorem[4, 5]. The system of present interest is an assembly of interacting particles; but to the extent that short-range interactions are negligible, to the extent that the interactions are dominated by the collective Coulomb field, the interactions are simply equivalent to additional external field and their effect upon the particle motion is accounted for by addition of a collective potential term to the single particle Hamiltonian[6]. Thus with interactions that are only of long-range character phase density is preserved, and motion in two dimensional W—ϕ phase space of a general distribution $\Psi(W, \phi, t)$ satisfies the Liouville equation

$$\frac{d\Psi}{dt} = \frac{\partial\Psi}{\partial W}\dot{W} + \frac{\partial\Psi}{\partial\phi}\dot{\phi} + \frac{\partial\Psi}{\partial t} = 0\,. \qquad (1)$$

(This is the same as the reduced, i.e. collision-free, Boltzmann equation if, as in the present problem, the only velocity-dependent forces are electromagnetic. It is here more convenient, however, to begin with the Liouville equation since the equations of motion are already written in canonical coordinates.) The values of $\dot{W}$ and $\dot{\phi}$ are to be obtained from the time-dependent Hamiltonian (cf. Eq. (6) of the paper by Nielsen and Sessler[5])

$$H(W,\phi,t) = \pi h\left(f\frac{df}{dE}\right)_s W^2 + eV(t)\cos\phi + \dot{W}_s\phi + $$
$$+ 2\pi eh\, U(\phi,t)\,, \qquad (2)$$

where $U(\phi, t)$ describes the collective Coulomb field, $V(t)$ is the amplitude of an applied RF voltage, h is the harmonic number, and the subscript s refers to an energy $E_s(t)$ at which the frequency $f(E_s)$ is synchronous with the RF. The cannonical momentum W is

$$W = \frac{E - E_s}{f_s} \qquad (3)$$

A distribution is stationary if grad Ψ. $\mathbf{V} = 0$, $\mathbf{V}$ being the phase velocity vector at any point.

If the distribution $\Psi(W, \phi, t)$ is specialized to a uniform phase density σ between two boundaries $W_1(\phi, t)$ and $W_2(\phi, t)$, the Liouville equation for Ψ may be replaced by two coupled partial differential equations giving W_1 and W_2 as functions of the two variables ϕ and t. These equations can be obtained from Eq. (1) by development into a series of equations in successive moments of W and suitable transformations of variables[7], but it is simpler and perhaps more instructive to deal (as in the paper by Nielsen[1]) directly with the phase boundary motion. Evidently the behavior of the assembly represented by the uniform density between boundaries is completely determined by the boundary motion, since the distribution is unaltered by any motion of phase points within the boundaries.

IV. BOUNDARY EQUATIONS

Boundary motion is determined by the velocity of the phase points defining the boundary; in particular there is no boundary motion and the distribution is stationary if the boundaries lie along phase trajectories, and in general

$$\frac{dW(\phi,t)}{dt} = \frac{\partial W(\phi,t)}{\partial\phi}\frac{d\phi}{dt} + \frac{\partial W(\phi,t)}{\partial t} \qquad (4)$$

for any curve $W(\phi, t)$. Let $W(\phi, t)$ be a boundary, for example $W_2(\phi, t)$, so that this equation describes the boundary motion; then for a point (W_2, ϕ) on the boundary

$$\frac{dW_2}{dt} = -\left.\frac{\partial H(W,\phi,t)}{\partial \phi}\right]_{W_2,\phi},$$

$$\frac{d\phi}{dt} = \left.\frac{\partial H(W,\phi,t)}{\partial W}\right]_{W_2,\phi}, \tag{5}$$

and similarly for the boundary curve $W_1(\phi, t)$.

The meaning of Eq. (4) is shown geometrically in Fig. 1 which is drawn to represent a single region bounded by W_1 and W_2. Time rate of change of boundary position $\frac{\partial W_2(\phi,t)}{\partial t}$ is the difference between the W component of phase velocity and the ϕ component multiplied by the boundary slope $\frac{\partial W_2(\phi,t)}{\partial \phi}$.

To obtain a distribution that is stationary in the absence of space charge effects we may omit the term $\dot{W}_s$ from the Hamiltonian, thus insuring that $f\frac{df}{dE}$ is a constant (except for a presumably negligible variation over the energy interval of the distribution) and we may make V constant. The canonical equations then become

$$\begin{aligned} \dot{W} &= eV\sin\phi - 2\pi e h\frac{\partial U(\phi,t)}{\partial \phi} \\ &\equiv L(\phi,t)\,, \\ \dot{\phi} &= 2\pi h\left(f\frac{df}{dE}\right)_s W \\ &\equiv MW\,. \end{aligned} \tag{6}$$

With $\dot{W}$ and $\dot{\phi}$ thus evaluated, Eq. (4) leads to the two boundary equations,

$$\frac{\partial W_1}{\partial t} - L(\phi,t) + MW_1\frac{\partial W_1}{\partial \phi} = 0\,,$$

$$\frac{\partial W_2}{\partial t} - L(\phi,t) + MW_2\frac{\partial W_2}{\partial \phi} = 0\,. \tag{7}$$

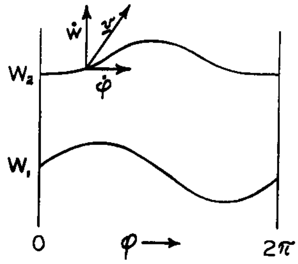

Fig. 1 Geometrical meaning of equations of motion for phase boundaries.

These are coupled by the term L which contains the collective field and is therefore a function of both W_1 and W_2.

In general the collective field involves an integral over ϕ and these are coupled integro-differential equations; but if the radio-frequency field is small enough so that the functions W are continuous with finite first derivatives it is possible even above transition energy to use (as in the paper by Nielsen and Sessler [5]) the approximation that the collective potential is simply proportional to the linear charge density at every point. Then

$$U(\phi,t) = \int_{-h\pi}^{h\pi} K(\phi'-\phi)\lambda(\phi',t)d\phi'$$

$$\doteq g_0\lambda(\phi,t)\,, \tag{8}$$

where, for a tube of charge of radius a midway between two parallel grounded conducting planes, separated a distance G [5],

$$g_0 = 1 + 2\ln(2G/\pi a)\,. \tag{9}$$

The above restriction on W insures that the boundaries are single-valued functions of ϕ, in which case the charge per unit length is

$$\lambda(\phi,t) = \frac{e\sigma h}{R}(W_2 - W_1)\,. \tag{10}$$

Up to this point no account has been taken of the fact that the ions may be moving at nearly the speed of light. Although it will be recalled [5] that W and ϕ are defined in terms of a rotating coordinate system, it must be noted that coordinates in this rotating system differ from those in the laboratory system only by a shift of origin. The transformation is a Galilean transformation and description in W, ϕ coordinates is, from a relativistic point of view, description in laboratory coordinates.

The simplest way to see how to correct self fields for relativistics effects is to observe that two charges moving with linear speed v and separated by distance d in the laboratory system are separated $d/[1-v^2/c^2]^{1/2} \equiv \gamma d$ in their own coordinate system. It follows that the force between them is $1/\gamma^2$ less than between similar charges at rest in the laboratory system and separated the same laboratory distance d. Longitudinal ε being relativistically invariant, the mutual longitudinal interaction of the moving charges is decreased by

$1/\gamma^2$ in both the moving and the laboratory W, ϕ coordinates. Total self field is merely the sum of the fields of all charges, including image charges in the conducting walls enclosing the beam, and consequently the relativistic value of self field is obtained by use of the same $1/\gamma^2$ factor.

This argument applies to the collective field of a circulating ion beam, only if the orbit radius of curvature is large enough so that an instantaneous Lorentz transformation is a good approximation, only if the distance between beam and walls is small enough as compared with the orbit radius and only if the ion velocity spread and time rate of change of the distribution (perturbation wave velocity relative to the beam) are small as compared with the mean ion velocity. Rigorous analysis confirms the $1/\gamma^2$ correction within these limitations. We therefore write for a relativistic beam

$$L(\phi,t) = eV\sin\phi - \frac{2\pi e^2\sigma g_0 h^2}{\gamma^2 R}\frac{\partial}{\partial\phi}(W_2 - W_1)$$

$$\equiv eV\sin\phi - K\frac{\partial}{\partial\phi}(W_2 - W_1)\,. \tag{11}$$

It is instructive to describe the behavior of the distribution representing a single circulating beam by means of variables

$$\Delta \equiv W_2 - W_1\,,$$

$$\overline{W} \equiv \frac{W_2 + W_1}{2}\,, \tag{12}$$

which satisfy the equations

$$\frac{\partial\Delta}{\partial t} + M\frac{\partial}{\partial\phi}(\overline{W}\Delta) = 0\,,$$

$$\frac{\partial\overline{W}}{\partial t} + K\frac{\partial\Delta}{\partial\phi} + \frac{M}{2}\frac{\partial}{\partial\phi}\left(\overline{W}^2 + \frac{\Delta^2}{4}\right) = eV\sin\phi\,. \tag{13}$$

From the first of these we can derive certain general symmetry properties :

1. If $\overline{W} = 0$ at all time and all ϕ, $\partial\Delta/\partial t = 0$ at all time and all ϕ : All permanently symmetrical distributions are stationary.
2. If $\partial\Delta/\partial t = 0$ and $\partial\overline{W}/\partial t = 0$ at all time and all ϕ, $\overline{W}\Delta$ = const., whence if (by choice of coordinates) $\overline{W} = 0$ at some ϕ it is zero at all time and all ϕ : All stationary distributions are permanently symmetrical.
3. Conversely, all unsymmetrical distributions are non-stationary and all non-stationary distributions are (or become) unsymmetrical.

If $\overline{W} = 0$ and $\partial\overline{W}/\partial t = 0$, integration of the second of Eqs. (10) gives the stationary solution obtained earlier for beams below transition (Symon and Sessler[4]), Eq. (18) with $\dot{W}_s = 0$) and valid also above transition within the limits set by the potential approximation.

V. SINGLE BEAM

The behavior of the single beam can be determined from the solution of boundary Eqs. (7) or alternatively from Eqs. (13). In the latter case it is sufficient to find the solution for Δ, which by Eq. (10) is proportional to linear charge density λ, since it turns out that the condition for stability of Δ is the same as the condition for stability of W_1 and W_2 obtained by solution of the boundary equations.

Let Δ_0 be the stationary value of Δ satisfying the second of Eqs. (13) with $\overline{W} = 0$ and $\partial\overline{W}/\partial t = 0$, i.e.

$$K\frac{\partial\Delta_0}{\partial\phi} + \frac{M}{8}\frac{\partial}{\partial\phi}(\Delta_0{}^2) - eV\sin\phi = 0\,. \tag{14}$$

For small amplitude perturbations from this stationary solution,

$$\overline{W}\frac{\partial\Delta}{\partial t} \ll \Delta\frac{\partial\overline{W}}{\partial t} \text{ and } \overline{W}^2 \ll \Delta^2\,, \tag{15}$$

(since $\overline{W}$ is initially small or zero while Δ is initially finite and remains so), and elimination of $\overline{W}$ from Eqs. (13) then yields the linearized equation in $\Delta - \Delta_0$:

$$\frac{\partial^2}{\partial t^2}(\Delta - \Delta_0) =$$

$$= M\Delta_0 K\frac{\partial^2(\Delta-\Delta_0)}{\partial\phi^2} + \tfrac{1}{4}M^2\Delta_0\frac{\partial^2}{\partial\phi^2}[\Delta_0(\Delta-\Delta)_0]\,. \tag{16}$$

Consideration will be limited to the circulating beam in the absence of radio-frequency; (we then set $h = 1$ and the phase coordinate ϕ becomes the space angle). The stationary distribution then reduces to Δ_0 = const., and Eq. (16) reduces to the wave equation with the solution

$$\Delta(\phi,t) = F(\phi + \Omega t)\,. \tag{17}$$

The angular velocity (relative to the circulating beam) is

$$\Omega = \pm\left[M\Delta_0\left(K+\tfrac{1}{4}M\Delta_0\right)\right]^{1/2}, \qquad (18)$$

and is real, giving propagating waves when

$$M\Delta_0 K + \tfrac{1}{4}M^2\Delta_0^2 > 0\,. \qquad (19)$$

This is always true below transition, while above transition M is negative and the criterion for stable waves is, if we substitute for M, K from Eqs. (6), (10) and (11)

$$\left(\frac{\Delta E}{f}\right)^2 = \Delta_0^2 > \frac{4eg_0\lambda_0}{-f\dfrac{df}{dE}\gamma^2}\,. \qquad (20)$$

It follows that for a given charge density λ_o in a beam above transition, one can always find an energy spread great enough to make Ω real and small perturbations of arbitrary wavelength oscillatory.

When condition (20) is not satisfied, Ω becomes imaginary, and it is convenient to resolve the wave (17) into sinusoidal components with the dependence

$$e^{\pm in(\phi-\omega t)}. \qquad (21)$$

With Ω imaginary, the wave travels with the beam and contains damped and anti-damped components which grow exponentially at a rate which in the limit of a monoenergetic beam is

$$nI_m(\Omega) = \frac{2\pi n}{\gamma}\left[eg_0\lambda_0 f\left|\frac{df}{dE}\right|\right]^{1/2}. \qquad (22)$$

It is convenient to write these results in dimensionless form in terms of the Budker parameter [8)]

$$\nu = \frac{N}{2\pi R}\cdot\frac{e^2}{m_0c^2}, \qquad (23)$$

where N is the total number of particles.
Then,

$$\Delta_0 K = \frac{Ne^2 g_0}{\gamma^2 R} = \frac{2\pi\nu g_0 m_0 c^2}{\gamma^2}\,. \qquad (24)$$

In the relativistic limit when $E^2 \gg E_0^2$

$$\frac{df}{dE} = \frac{-f}{(k+1)E}, \qquad (25)$$

so that the stability criterion becomes

$$\frac{\Delta E}{E_0} > \left[\frac{4(k+1)\nu g_0}{\gamma}\right]^{1/2}, \qquad (26)$$

and Ω in units of particle angular velocity $\omega = 2\pi f$ is

$$\frac{\Omega}{\omega} = \left[\frac{\nu g}{(k+1)\gamma^3}\right]^{1/2}. \qquad (27)$$

As a numerical example we may consider an accelerator (the MURA 40 MeV two-way electron accelerator) in which 3×10^{11} electrons are injected (to give approximately 1 ampere circulating current) and accelerated by betatron action to 2 MeV. They are then above transition energy although not far enough to make the approximation used above for df/dE very good) and we use

$$N = 3\times10^{11}$$
$$R = 156\ \text{cm}$$
$$\frac{e^2}{m_0c^2} = 2.82\times10^{-13}\ \text{cm}$$
$$E_0 = 0.51\ \text{MeV}$$
$$g_0 = 2.5$$
$$k = 9.3$$
$$\gamma = 5$$
$$\nu = 0.86\times10^{-4}$$

These numbers put into Eq. (26) and Eq. (27) give

$$\frac{\Delta E}{E_0} > 4.7\times10^{-2}, \quad \Delta E > 24\ \text{keV for stability}$$

and $\Omega/\omega = 5.7\times10^{-4}$.

Since the energy spread at injection and at 2 MeV is of the order of 1 keV, growth of azimuthal inhomogeneities is to be anticipated. If we let $n = 1$, and $\omega = 2\pi\times30$ Mc, growth by the factor e occurs in about 8 μsec at the slowest; and if $n = 100$ the rate is 100 times faster. An evaluation of growth rate made with an expression for potential more accurate at short wavelength than that used here has shown (Nielsen and Sessler [7)]) that the growth rate approaches a limiting value as n increases. In this accelerator the limiting value is of the order of that obtained from Eq. (27) with $n = 100$.

We may consider also the same accelerator with 100 injected pulses " stacked " in a circulating beam at 40 MeV. The value of ν is then 76 times greater, and γ is 16 times greater, increasing the required ΔE by a factor of 4.8 to 53 keV. Since the total ΔE will then be at least 100 times the ΔE of each pulse, it will exceed the spread required for stability. (But individual pulses will remain unstable at 40 MeV.)

It must be borne in mind that these computations refer only to circulating beams in the absence of radio-frequency voltages. The stability problem with radio-frequency present has not yet been solved.

In principle it is possible to extend the boundary equation analysis to distributions defined by more than two boundaries and with more than one value of phase density. This has been done for two separate interacting distributions described by a coupled set of four boundary equations [1]. A dispersion relation results, which is the same as will be obtained later by a more general method. Since it is not clear, however, that this simple procedure can be extended correctly to two interacting relativistic beams we shall here discuss multiple beams only in the context of the more general method outlined in the next section.

VI. SOLUTION OF THE RELATIVISTIC BOLTZMANN EQUATION

We now develop a completely relativistic treatment of the linearized longitudinal Boltzmann equation for the special case when no external forces act except the static focusing and guiding forces in the accelerator, and when the cross-sectional distribution is uniform azimuthally. We further generalize the preceding treatment by allowing a non-uniform density function Ψ. Fig. 2 illustrates the geometry we wish to consider. The beam is contained in a conducting donut with conducting walls of arbitrary cross-section, uniform in azimuth. The cross-sectional distribution of particles in the beam is assumed to be a known function $Q(x, y)$ of the cross-sectional coordinates, also independent of azimuth. In all examples, we shall take the geometry shown in Fig. 2, i.e. a beam uniform inside a circular cylinder of radius A midway between conducting planes of separation G. We will neglect betatron oscillations except insofar as they contribute to the cross sectional distribution $Q(x, y)$. We will assume also for simplicity that Q is independent of W. This latter assumption is not essential to our treatment, and is valid when the beam cross section is predominantly due to betatron oscillations. Likewise, we shall for the moment neglect the dependence of R on W. Instead of the coordinate ϕ used earlier which is measured from an origin rotating

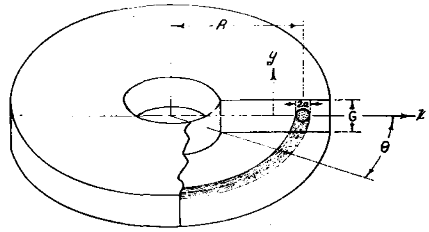

Fig. 2 Beam geometry.

with the beam, we use an angular coordinate θ measured from a fixed origin. The properties of the guide field insofar as we shall need them are specified by giving the angular velocity $\omega(W) = 2\pi f$ of the ions as a function of their energy coordinate W. The radius of curvature of the orbit will be assumed to be large in comparison with the cross-sectional dimensions of the beam, and with at least the minimum cross sectional dimension of the vacuum chamber. We may then treat $x, y, R\theta$ as rectangular coordinates in calculating the electric field due to the beam. This is a legitimate approximation for our problem with one exception which we shall note later.

The Boltzmann equation is just Eq. (1) with ϕ replaced by θ, and where

$$\dot{\theta} = \omega(W), \tag{28}$$

$$\dot{W} = \dot{E}/f = 2\pi e R \bar{\mathcal{E}}, \tag{29}$$

where $\bar{\mathcal{E}}$ is the longitudinal electric field averaged over the beam cross section, and we now take

$$W = \int \frac{dE}{f(E)}, \tag{30}$$

so that we are no longer restricted to a small range of W. Ncte that f, and hence W (and ω) have opposite signs for beams travelling in opposite directions. We may therefore encompass the case of any number of constituent beams travelling with various energies in either direction simply by taking the appropriate distribution $\Psi(W)$.

The azimuthal field $\mathcal{E}$ is determied from Maxwell's equations, and satisfies in a rectangular coordinate system $x, y, R\theta$,

$$\nabla^2 \mathcal{E} + \frac{1}{R^2}\frac{\partial^2 \mathcal{E}}{\partial \theta^2} - \frac{1}{c^2}\frac{\partial^2 \mathcal{E}}{\partial t^2} = \frac{4\pi}{R}\frac{\partial \rho}{\partial \theta} + \frac{4\pi}{c^2}\frac{\partial j}{\partial t}, \tag{31}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}. \tag{32}$$

There will be a boundary condition, $\mathcal{E} = 0$, at the walls of the vacuum chamber. Let $Q(x, y)$ be normalized so that

$$\iint_S Q \, dS = 1 , \tag{33}$$

where S is the cross-section of the chamber. The charge and azimuthal current densities are then

$$\rho(x,y,R\theta,t) = eQ(x,y)\int R^{-1}\Psi(W,\theta,t)\, dW , \tag{34}$$

$$j(x,y,R\theta,t) = eQ(x,y)\int \omega\Psi(W,\theta,t)\, dW , \tag{35}$$

and

$$\bar{\mathcal{E}}(\theta,t) = \iint_S \mathcal{E}Q \, dS . \tag{36}$$

An obvious solution of Eqs. (1) and (31) is

$$\Psi = \Psi^o(W) , \quad \mathcal{E} = 0 . \tag{37}$$

We will call this stationary, azimuthally uniform solution the unperturbed solution, where $\Psi^o(W)$ is some given distribution in energy of the beam. We now set

$$\Psi(W,\theta,t) = \Psi^o(W) + \Psi(W,\theta,t) , \tag{38}$$

where ψ represents a small perturbation. The electric field $\mathcal{E}$ is then also small, and if we neglect second order terms and use Eqs. (28), (29), (34), and (35), Eqs. (1) and (31) become

$$\frac{\partial\psi}{\partial t} + \omega\frac{\partial\psi}{\partial\theta} + 2\pi eR\frac{\partial\Psi^o}{\partial W}\bar{\mathcal{E}} = 0 , \tag{39}$$

$$\nabla^2\mathcal{E} + \frac{1}{R^2}\frac{\partial^2\mathcal{E}}{\partial\theta^2} - \frac{1}{c^2}\frac{\partial^2\mathcal{E}}{\partial t^2} = 4\pi eQ\int\left[\frac{1}{R^2}\frac{\partial\psi}{\partial\theta} + \frac{\omega}{c^2}\frac{\partial\psi}{\partial t}\right]dW . \tag{40}$$

In order to solve these equations, we first make a Fourier transform to eliminate the variable θ, taking advantage of the uniformity in azimuth. For any function $G(\theta)$, we define a Fourier transform $G_F(n)$ by

$$G(\theta) = \sum_{n=-\infty}^{\infty} G_F(n)e^{in\theta},$$

$$G_L(n) = \frac{1}{2\pi}\int_0^{2\pi} G(\theta)e^{-in\theta}\,\delta\theta . \tag{41}$$

We next eliminate the time dependence by taking a Laplace transform of these equations. For any function $G(t)$, we define the Laplace transform $G_L(p)$ by

$$G_L(p) = \int_0^{\infty} G(t)e^{-pt}dt ,$$

$$G(t) = \frac{1}{2\pi i}\int_{\zeta-i\alpha}^{\zeta+i\alpha} G_E(p)e^{pt}\,dp , \tag{42}$$

where the integral over p is to be taken along a Bromwich line lying to the right of all singularities of $G_L(p)$. The transform of a time derivative is

$$\dot{G}_L = pG_L(p) - G(0) . \tag{43}$$

The Fourier-Laplace transforms of Eqs. (39) and (40) are

$$(p+in\omega)\psi_{FL} + 2\pi eR\frac{\partial\Psi^o}{\partial W}\bar{\mathcal{E}}_{FL} = \psi_F(W,n,0) = \psi_{F0} , \tag{44}$$

$$\nabla^2\varepsilon_{FL} - \frac{1}{R^2}\left(\frac{p^2}{\omega_0^2}+n^2\right)\mathcal{E}_{FL} = = \frac{4\pi eQ}{R^2}\int\left(in+\frac{\omega p}{\omega_0^2}\right)\psi_{FL}dW + A_{F0} , \tag{45}$$

where

$$A_{F0}(n,p,x,y) = -\frac{1}{c^2}(p\varepsilon_{F0}+\dot{\mathcal{E}}_{F0}) - \frac{4\pi eQ}{R^2}\int\frac{\omega}{c^2}\psi_{F0}dW , \tag{46}$$

and

$$\omega_0 = c/R \tag{47}$$

is the angular velocity of a particle at the speed of light on a circle of radius R. Note that the initial conditions are explicitly introduced into the solution by this method.

Let $g_j(x, y)$, $-K_j/R^2$ be the eigenfunctions and eigenvalues of the operator ∇^2 in the region bounded by the vacuum tank, and subject to the appropriate boundary condition on the walls (i.e., $g = 0$)

$$\nabla^2 g_j = -\frac{K_j}{R^2}g_j . \tag{48}$$

These functions are normalized so that

$$\iint_S g_j g_l dS = \delta_{jl} \tag{49}$$

Let us first find the solution $\mathcal{E}$ of Eq .(45) for an empty tank ($\Psi = 0$). We may expand

$$\dot{\mathcal{E}}_{F0} + p\mathcal{E}_{F0} = \sum_j X_{j0}(n,p) g_j(x,y) , \qquad (50)$$

and

$$\mathcal{E}_{FL} = \sum_j a_j g_j(x,y) . \qquad (51)$$

We substitute in Eq. (45), using Eq. (48), and solve for

$$a_j = \frac{X_{j0}}{\omega_0^2 K_j + \omega_0^2 n^2 + p^2} . \qquad (52)$$

We substitute in Eq. (51) and invert the Laplace transform

$$\mathcal{E}_F = \sum_j \frac{g_j}{2\pi i} \int_{\zeta - i\infty}^{\zeta + i\infty} \frac{X_{j0} e^{pt} dp}{\omega_0^2 K_j + \omega_0^2 n^2 + p^2} . \qquad (53)$$

The integral is easily evaluated by contour integration, and we have finally for the part of the field with wave number n

$$\mathcal{E}_F e^{in\theta} = \sum_j \frac{g_j(x,y) X_{j0}(n, i\Omega_j)}{2i\Omega_j} [e^{i(n\theta + \Omega_j t)} - e^{i(n\theta - \Omega_j t)}] \qquad (54)$$

where

$$\Omega_j = \omega_0 [K_j + n^2]^{1/2}. \qquad (55)$$

We see that K_j and $g_j(x, y)$ determine the normal modes of propagation of electromagnetic waves in the vacuum tank. There are two normal modes for each n and each eigenfunction g_j which propagate with angular velocity

$$\omega_j = \pm \frac{\Omega_j}{n} = \pm \omega_0 \left[1 + \frac{K_j}{n^2} \right]^{1/2} . \qquad (56)$$

For a straight waveguide, the normal modes propagate with velocities greater than c, hence K_j would always be positive. For a waveguide bent in a circle, however, it is not clear that the angular velocity is necessarily greater than $\omega_0 = c/R$. Note that R is presumably the mean radius of the beam, which may lie anywhere in the vacuum tank, whereas ω_j depends only on the vacuum tank and not on where the beam is. Indeed, it appears that, for a circular chamber of rectangular cross section at any rate, there are always modes that propagate with $\omega_j < \omega_0$ for any R inside the chamber. (D. L. Judd and V. K. Neil, private communication.) This point is very important, because as we shall see later, if some ω_j is smaller than ω_o, the electromagnetic modes can become unstable. For this reason, the eigenvalues K_j, should be calculated taking proper account of the curvature of the vacuum tank; this is the exception noted earlier to the statement that the electric field can be calculated as if the tank were straight. We should perhaps redefine K_j more precisely as the value given by Eq. (56) where ω_j is the exact angular velocity of propagation of the wave when proper account is taken of the curvature of the vacuum tank:

$$K_j = n^2 \left(\frac{\omega_j^2 R^2}{c^2} \right) - 1 . \qquad (57)$$

In order to solve Eq. (45), let us consider the equation

$$\nabla^2 u - \frac{A}{R^2} u = \frac{B}{R^2} Q(x,y) , \qquad (58)$$

where A, B are independent of x, y. We expand

$$Q(x,y) = \sum_j Q_j g_j(x,y) , \qquad (59)$$

where

$$Q_j = \iint_s g_j Q \, dS = \bar{g}_j . \qquad (60)$$

If we set

$$u(x,y) = \sum_j a_j g_j(x,y) , \qquad (61)$$

then we get, after substitution in Eq. (58),

$$a_j = -\frac{B\bar{g}_j}{K_j + A} , \qquad (62)$$

so that

$$u = -\sum_j \frac{B\bar{g}_j g_j}{K_j + A} , \qquad (63)$$

and

$$\bar{u} = -\sum_j \frac{B\bar{g}_j^2}{K_j + A} . \qquad (64)$$

Combining this result with the previous solution for an empty tank, we have, for the solution of Eq. (45),

$$\bar{\mathcal{E}}_{FL} = -\frac{eI}{c^2} g(n,p) - c^2 \sum_j \frac{A_{j0}}{n^2 \omega_j^2 + p^2} , \qquad (65)$$

where

$$I = \int [in\omega_0^2 + \omega p] \psi_{FL} \, dW , \qquad (66)$$

$$g(n,p) = 4\pi c^2 \sum_j \frac{\bar{g}_j^2}{n^2 \omega_j^2 + p^2} , \qquad (67)$$

$$A_{j0}(n,p) = \iint_s A_{F0}(n,p,x,y) g_j(x,y) \, dS . \qquad (68)$$

We substitute in Eq. (36):

$$(p+in\omega)\psi_{FL}-\frac{2\pi e^2 IRg}{c^2}\frac{\partial\Psi^o}{\partial W}$$
$$=\psi_{F0}+2\pi ec^2 R\frac{\partial\Psi^o}{\partial W}\sum_j\frac{A_{j0}}{n^2\omega_j^2+p^2}=F_0(n,p,W)\,. \quad (69)$$

We can now see how to modify this solution to take into account the dependence of R on W. We must solve Eq. (40) for $\delta\mathcal{E}$ due to each component ψdW of beam separately. The solution proceeds as above. Note that ω_0 given by Eq. (47) now depends on W. We arrive at an equation like (65) in which the integrals over W in I and A_{F0} are replaced by their integrands. We now sum over all beam components ψdW and arrive exactly at Eq. (65), since only the integrands in I, A_{F0} are functions of R or W. Note that ω_j depends on n, but not on W, since it refers to the propagation of waves in the empty donut. If we were to take account of the dependence of $Q(x,y)$ on W, then $\bar{g}_j$ depends on W, and $g(n,p)$ would have to be included in the integral I. The dependence of R (or ω_0) on W is not a trivial matter, since we shall see later that cancellations may occur in integrands like that in Eq. (66) which could make small variations in ω_0 important.

To solve Eq. (69), we multiply by

$$(in\omega_0^2+\omega p)/(p+in\omega)$$

and integrate over W, to obtain

$$I=\frac{1}{D(p)}\int F_0(n,p,W)\frac{in\omega_0^2+\omega p}{p+in\omega}\,dW, \quad (70)$$

where

$$D(n,p)=1-\frac{2\pi e^2}{c^2}g(n,p)\int R\frac{\partial\Psi^o}{\partial W}\frac{in\omega_0^2+\omega p}{p+in\omega}\,dW\,. \quad (71)$$

The solution of Eq. (69) is now

$$\psi_{FL}(n,p,W)=\frac{F_0(n,p,W)}{p+in\omega}+\frac{2\pi e^2 RgI}{D(p)(p+in\omega)c^2}\frac{\partial\Psi^o}{\partial W}\times$$
$$\times\int F_0(n,p,W')\frac{in\omega_0^2+\omega' p}{p+in\omega'}\,dW'\,, \quad (72)$$

where $\omega'=\omega(W')$.

The solution for the perturbation at wave number n is now obtained by inverting the Laplace transform:

$$\psi_F(n,W)e^{in\theta}=\frac{1}{2\pi i}\int_{\zeta-i\infty}^{\zeta+i\infty}\psi_{FL}(n,p,W)e^{pt+in\theta}\,dp\,. \quad (73)$$

Without carrying through the details of the integration, (which will be presented in a later publication) we can see the general character of the result. Since ψ_{FL} vanishes as $1/p$ as $|p|\to\infty$, we may close the contour in the integral (Eq. (73)) by an infinite semicircle from the Bromwich line around the left half p-plane. Now ψ_{FL} has poles at $p=-in\omega$ and at the roots p_n of the dispersion relation

$$D(n,p_{nl})=0\,. \quad (74)$$

Examination of Eq. (72) will show that the poles of $F_0(n,p,W)$ at $p=\pm in\omega_j$ cancel in the two terms on the right (recall Eqs. (69) and (67)) as they should since they correspond to normal modes of the empty donut. We will show in the next section that $D(p)$ has branch cuts along that part of the imaginary axis $p=-in\omega(W)$ corresponding to values of W at which $\partial\Psi^o/\partial W\neq 0$, and that no roots of Eq. (74) can lie on these cuts. The inversion of the second term in Eq. (72) is accomplished by interchanging the order of integration over p and W'. (In carrying through the details, it becomes necessary to insure also that $D(p)$ has no roots $p_{nl}=-in\omega(W)$ at points where $\psi_{F0}(n,W)\neq 0$. Presumably in such a case, the linear perturbation theory is inadequate, particularly in view of the result quoted above that there is no root where $\partial\Psi^o/\partial W\neq 0$.) Hence, when we shrink the contour in the integral (73), we are left with residues from the poles at $p=-in\omega$, and $p=p_{nl}$, an integral over residues from $p=-in\omega'$, and an integral over the branch cuts which becomes also an integral over W'. The result has therefore the form

$$\psi_F(n,W)e^{in\theta}=A(n,W)e^{in(\theta-\omega t)}$$
$$+\int dW' B(n,W,W')e^{in(\theta-\omega' t)}$$
$$+\sum_l C_l(n)F_l(n,W)e^{in\theta+p_{nl}t}\,, \quad (75)$$

where A, B, C depend on F_0 and hence on the initial perturbation, and

$$F_l(n,W)=\frac{2\pi e^2}{c^2}\frac{g(n,p)}{D'(n,p_{nl})(p_{nl}+in\omega)}R\frac{\partial\Psi^o}{\partial W}, \quad (76)$$

$$D'(n,p_{nl})=\lim_{p\to p_{ln}}\frac{D(n,p)}{p-p_{nl}}\,. \quad (77)$$

The first term in Eq. (75) represents a perturbation in each component of beam at energy W which moves with the same angular velocity $\omega(W)$ as the beam.

If there were no interaction between beam particles, this would be the only term, with

$$A(n,W) = \psi_{F0}(n,W) ;$$

that is the initial perturbation is simply carried around by the motion of the beam. When the particles interact, the second term appears, which represents a disturbance in the component W moving with the angular velocities ω' of the other components W'. The first two terms we will call the *streaming disturbance*. For each component W' of the beam, there is a streaming disturbance which moves with angular velocity ω', and which consists of a perturbation in the component W' which streams around with that component (first term) together with a co-moving disturbance in all other components W due to the interaction between components (second term). It is clear that the streaming motion cannot give rise to any instabilities. In fact it is easy to show that in general a continuous superposition of waves of different angular velocities as in the second term of Eq. (75) will damp out in the course of time at a rate and with a time dependence depending on the nature of the function $B(n,W,W')$. If B is analytic in ω', as it cannot be for a physically realizable case, the damping is exponential (Landau damping) [9].

The various terms in the sum over l in Eq. (75) we call *normal modes of propagation*. A normal mode has a characteristic (and discrete) time dependence p_{nl} and a characteristic W dependence $F_l\,(n, W)$, which depend on n and on the properties of the unperturbed beam, but do not depend on the nature of the initial perturbation. Only the amplitude (and phase) $C_l\,(n)$ depends on the initial perturbation. If p_{nl} has a positive real part, the corresponding normal mode is anti-damped, and the beam is unstable. Hence the question whether the beam is stable may be answered by a study of the roots of the dispersion relation (74).

VII. THE DISPERSION RELATION

In discussing the dispersion relation (74), it is convenient to introduce, in place of p_{nl} the angular velocity ω_{nl} of the normal mode:

$$p_{nl} = -in\omega_{nl} \, . \tag{78}$$

The dispersion relation (63) then becomes

$$D(\omega_{nl}) = 1-\frac{2\pi e^2}{c^2}g(n,\omega_{nl})\int R\frac{\partial\Psi^o}{\partial W}\,\frac{\omega_0^2-\omega\omega_{nl}}{\omega-\omega_{nl}}dW = 0 \, . \tag{79}$$

The integral is to be evaluated for p_{nl} lying to the right of all singularities in the p-plane, i.e. for ω_{nl} lying above all singularities in the complex ω_{nl}-plane, and continued analytically to other parts of the ω_{nl}-plane, going out around all singularities on the real axis.

We will remove the factor R from the integrand in Eq. (79) since the slight dependence of R on W may be neglected. The integral can then be simplified by adding a term $\omega_{nl}\partial\Psi^o/\partial W$ to the integrand, which does not affect its value. We then have

$$D(\omega_{nl}) = 1-\frac{2\pi e^2 R}{c^2}g(n,\omega_{nl})\int\frac{\omega_0^2-\omega_{nl}^2}{\omega-\omega_{nl}}\,\frac{\partial\Psi^o}{\partial W}dW = 0 \, . \tag{80}$$

Since ω_0 is the angular velocity of a particle of speed c at radius R, and since we shall find that ω_{nl} is near the angular velocity of the beam, the factor $\omega_0^2-\omega_{nl}^2$ will be small in the relativistic case. The dependence of ω_0 on R, and hence W, may therefore be important. We will for convenience remove the factor $\omega_0^2-\omega_{nl}^2$ from the integrand with the understanding that ω_0^2 is to be given a suitable average value, presumably the value corresponding to the value of W which makes the major contribution to the integral in Eq. (80). This value may be somewhat different for different roots ω_{nl}. We can then write the dispersion relation

$$D(\omega_{nl}) = 1-\frac{2\pi e^2}{R\gamma_{nl}^2}g(n,\omega_{nl})\int\frac{1}{\omega-\omega_{nl}}\,\frac{\partial\Psi^o}{\partial W}\,dW = 0, \tag{81}$$

where

$$\gamma_{nl} = [1-\omega_{nl}^2/\omega_0^2]^{-1/2} \tag{82}$$

is just $(1-v_{nl}^2/c^2)^{-1/2}$ where $v_{nl} = R\omega_{nl}$ is the velocity of wave propagation around a circle of radius R.

The factor $g\,(n,\omega_{nl})$ defined by Eq. (67) can be rewritten, utilizing Eq. (57) to bring out its behavior in the long or short wavelength limits as follows:

$$g(n,\omega_{nl}) = 2\pi R^2\sum_j\frac{g_j^{-2}}{K_j+n^2/\gamma_{nl}^2} \, . \tag{83}$$

In the limit of very short wavelengths, this becomes

$$g_\infty \doteq \frac{2\pi R^2\gamma_{nl}^2}{n^2}\sum_j \bar{g}_j^2 \, , \quad \left(n \gg \frac{R}{a}\gamma_{nl}\right) , \tag{84}$$

since $\bar{g}_j$ is very small for $K_j \gg R^2/a^2$. The relativistic factor γ_{nl}^2 cancels out in the dispersion relation (81), as we might expect in this case, except insofar as it determines for what n the short-wavelength approximation is valid. For long wavelengths, g reduces to a constant, independent of n or γ_{nl}:

$$g \doteq g_0 = 2\pi R^2 \sum_j \frac{\bar{g}_j^2}{K_j}. \tag{85}$$

If we follow through the solution of Eq. (40) in the tatic cases $\left(\frac{\partial \mathcal{E}}{\partial t} = \frac{\partial \psi}{\partial t} = 0\right)$, we obtain the first term in Eq. (65) with $p = 0$, so that the field component of order n is

$$\bar{\mathcal{E}}_F e^{in\theta} = -\frac{e g_0}{R^2} \int in e^{in\theta} \psi_F \, dW. \tag{86}$$

Hence

$$\bar{\mathcal{E}}_{\text{static}} = -\frac{e g_0}{R^2} \frac{\partial}{\partial \theta} \int \psi(W,\theta) \, dW. \tag{87}$$

We see that the definition (85) of g_0 agrees with the previous definition (Eq. (8)). (Formula (9) gives not $\bar{\mathcal{E}}$, but $\mathcal{E}$ at the center of the beam, but the difference is not important in most cases.)

We now note that all quantities in the expression (79) or (81) for $D(\omega_{nl})$ (except ω_{nl} itself) are real. Moreover, if $\partial\Psi^o/\partial W = 0$ except over a finite segment of the $\omega(W)$-axis, as will be true in all practical applications (since $|\omega| < \omega_0$ in any case), then the result of analytic continuation of the integral on a path around the line of singularities where $\partial\Psi^o/\partial W \neq 0$, to a point ω_{nl} below the real axis, will be the same as if the integral is evaluated directly at that point. The dispersion relation (79) or (81) is therefore invariant under the transformation $i \to -i$, and complex roots for ω_{nl} can only occur in complex conjugate pairs, one corresponding to damped and the other to anti-damped waves (*).

We next show that Eq. (81) cannot have a (real) root at $\omega_{nl} = \omega\ (W)$ where $\partial\Psi^o/\partial W \neq 0$. For if we let $\omega_{nl} \to \omega$ on the real axis, we have

$$\int \frac{1}{\omega - \omega_{nl}} \frac{\partial \Psi^o}{\partial W} dW = $$

$$= P \int \frac{1}{\omega - \omega_{nl}} \frac{\partial \Psi^o}{\partial W} dW \pm i\pi \left. \frac{\partial\Psi^o/\partial W}{\partial\omega/\partial W} \right|_{W = W(\omega_{nl})}, \tag{88}$$

where 'P' denotes principal part, and the upper and lower signs correspond to whether $\omega_{nl} \to \omega$ from the upper or lower half plane. Therefore if $\partial\Psi^o/\partial W \neq 0$, $D(\omega_{nl})$ has a finite imaginary part, and the relation (81) cannot be satisfied. In fact, we see from Eq. (88) that $D(\omega_{nl})$ has branch cuts along the real axis where $\partial\Psi^o/\partial W \neq 0$. This implies that there can be no stable normal mode propagating with the same angular velocity as any part of the beam for which $\partial\Psi^o/\partial W \neq 0$. This is not surprising, since on physical grounds one might expect the coupling between beam and wave would make such a situation unstable.

VIII. SINGLE STREAM AND TWO-STREAM CASES

Let us first consider the problem solved earlier, in which we have a single stream $\Psi^o(W)$ with uniform phase density between fixed limits W_1, W_2:

$$\Psi^o(W) = \begin{cases} \dfrac{N}{W_2 - W_1}, & W_1 < W < W_2, \\ 0, & \text{otherwise.} \end{cases} \tag{89}$$

where N is the total number of particles.
We substitute into the dispersion relation (81) to obtain in the long wave limit

$$1 - K\left[\frac{1}{\omega_1 - \omega_{nl}} - \frac{1}{\omega_2 - \omega_{nl}}\right] = 0, \tag{90}$$

where $\omega_1 = \omega(W_1)$, $\omega_2 = \omega(W_2)$. The solution for ω_{nl} is

(*) This result is a consequence of the way in which we have chosen to shrink the contour in the Lapace inversion integral (73), i.e. so as to leave a loop around the segment on the imaginary p-axis corresponding to values of $p = -ik\,\omega(W)$ for which $\partial\Psi^o/\partial W \neq 0$. Other ways of shrinking the contour, for example, by leaving loops extending to $p = -\infty + i\alpha$ around all branch points $p = i\alpha$ on the imaginary p-axis, lead to other rules for continuing $D(\omega_{nl})$ into the lower half ω_{nl}-plane, and hence to different normal mode frequencies, as well as to different forms for the second term in Eq. (75). The solutions are of course equivalent, but the separation into streaming disturbances and normal modes of propagation is different. The roots of the dispersion relation corresponding to anti-damped modes lie in the upper half ω_{nl}-plane and are unaffected by this choice; hence questions of stability are unaffected. It appears that the convention adopted here is most convenient for the present purpose both because of the physical significance of the terms in the solution (75) and because of the mathematical convenience in the symmetry of our dispersion relation relative to the upper and lower half ω_{nl}-planes. One consequence of our convention is that the Landau damped modes of propagation, if they exist, are included in the streaming disturbance.

$$\omega_{nl} = \tfrac{1}{2}(\omega_1+\omega_2) \pm \tfrac{1}{2}(\omega_2-\omega_1)\left[1+\frac{4K}{\omega_2-\omega_1}\right]^{1/2}, \qquad (91)$$

where K is again defined as in Eq. (11):

$$K = \frac{2\pi e^2 g_0 N}{\gamma_{nl}^2 R(W_2-W_1)}, \qquad (92)$$

except that γ_{nl} is the γ for the wave velocity. There are two normal modes with angular velocities independent of n. If we set

$$\omega_2-\omega_1 = 2\pi f \frac{df}{dE}(W_2-W_1) = M\Delta_0, \qquad (93)$$

then Eq. (91) agrees with Eq. (18), and all the previous results for this case then follow. If we replace g_0 by g_∞ (Eq. (84)), we see that the minimum energy spread for stability decreases with increasing n; the wave propagation is always stable for large enough n. Strictly, we should not have treated γ_{nl}^2 in Eq. (90) as a constant. If we take $\gamma_{nl}^2 = 1-\omega_{nl}^2/\omega_0^2$, the number of roots is not affected, and the relation is not significantly changed except for very large N. Above transition, the criterion for stability is less stringent when $N \gtrsim c^2/(4\pi^2 e^2 R\gamma^2 f|df/dE|)$.

Above the transition energy, the roots (91) for ω_{nl} lie between ω_1 and ω_2 when the criterion for stability is satisfied. The theorem proved earlier shows that real roots cannot occur at angular velocities represented in the beam except where $\partial\Psi^o/\partial W = 0$, which is true for this special case of constant phase density. We therefore investigate the case of a triangular distribution where such roots cannot exist:

$$\Psi^0(W)\begin{cases} = \dfrac{4N(W-W_1)}{(W_2-W_1)^2}, & W_1 \le W < \dfrac{W_1+W_2}{2}, \\ = -\dfrac{4N(W_2-W)}{(W_2-W_1)^2}, & \dfrac{W_1+W_2}{2} \le W \le W_2, \\ = 0, & \text{otherwise} \end{cases} \qquad (94)$$

where N is the total number of particles. The dispersion relation is now (we assume $d\omega/dW$ is constant in the range W_1 to W_2),

$$1-\frac{4K}{\omega_2-\omega_1}\ln\frac{\left(\dfrac{\omega_1+\omega_2}{2}-\omega_{nl}\right)^2}{(\omega_1-\omega_{nl})(\omega_2-\omega_{nl})} = 0, \qquad (95)$$

where K is again defined by Eq. (92). (The logarithm is to be taken as real for ω_{nl} outside the range between ω_1 and ω_2, and continued analytically into the upper and lower half planes.) The solution is

$$\omega_{nl} = \tfrac{1}{2}(\omega_1+\omega_2) \pm \tfrac{1}{2}(\omega_2-\omega_1)\left[1-e^{-\frac{\omega_2-\omega_1}{4K}}\right]^{-1/2} \qquad (96)$$

Below transition ($\omega_2 > \omega_1$) the solutions are again stable and lie outside the region ($\omega_1 \le \omega \le \omega_2$) occupied by the beam. For $K \gg \frac{1}{4}(\omega_2-\omega_1)$, Eqs. (96) and (91) give the same result, namely that for a δ-function distribution. Above transition ($\omega_1 > \omega_2$) ω_{nl} is always complex, and hence there is always instability. The lapse rate at wave number n is given by

$$nI_m(\omega_{nl}) = \tfrac{1}{2}n(\omega_1-\omega_2)\left[e^{\frac{\omega_1-\omega_2}{4K}}-1\right]^{-1/2}. \qquad (97)$$

The limiting energy spread given by the criterion (20) which is obtained by setting the exponent equal to unity in Eq. (97), now becomes the dividing line between very rapid and very slow growth of the instability. That is, if the exponent is small, then the instability is very fast:

$$nI_m(\omega_{nl}) \doteq nK^{1/2}(\omega_1-\omega_2)^{1/2}, \qquad \frac{\omega_1-\omega_2}{4K} \ll 1, \qquad (98)$$

which agrees with Eq. (22) for a monoenergetic beam. If the exponent is large, the lapse rate is very small:

$$nI_m(\omega_{nl}) \doteq \tfrac{1}{2}n(\omega_1-\omega_2)e^{-\frac{\omega_1-\omega_2}{8K}}, \qquad \frac{\omega_1-\omega_2}{4K} \gg 1. \qquad (99)$$

This residual slow instability may be regarded as due to the discontinuity in slope at the center of our distribution, as may indeed be shown by taking a rectangular distribution with a triangular roof, whereupon the rapid and slow instabilities become separated, and the rapid part vanishes when criterion (20) is satisfied.

Let us now consider the case of two equal and oppositely directed beams, each with the density distribution given by Eq. (89), except that W and ω have opposite signs for the two beams. Proceeding as above, we find

$$\omega_{nl}^2 = \left[\frac{\omega_1^2+\omega_2^2+K(\omega_2-\omega_1)}{2}\right] \pm$$
$$\pm \{[\;]^2-\omega_1^2\omega_2^2+K\omega_1\omega_2(\omega_2-\omega_1)\}^{1/2}, \qquad (100)$$

where the square bracket under the square root is the same as the first term. We get a complex root if the quantity in curly brackets is negative. This is the negative mass instability and the criterion for stability is identical with condition (20) except for an always negligible factor $\left(1-\frac{1}{8}\frac{N}{N_0}\right)$ on the right, where N_0 is defined below. There remains the case when the second term is real and greater than the first, which can only occur below transition. This is the two-stream instability and the criterion for stability is, if we neglect the energy spread of the beam

$$N < N_0 = \frac{\gamma^2 f R}{g_0 e^2 df/dE} \,. \tag{101}$$

Since most intersecting beam machines which have been proposed operate above the transition energy, this two-stream instability cannot arise. The number N_0 is in any case very large for any reasonable choice of accelerator parameters. We may estimate its order of magnitude, since $f(E)$ is nearly always concave toward the E-axis,

$$N_0 \leq \gamma^3 R/(e^2/mc^2) \,. \tag{102}$$

Of course the case of two beams interpenetrating uniformly around the accelerator does not occur in any actual machine, but for, say a two-way accelerator it might be expected to give a conservative estimate of the condition for stability. For stability in a single intersection region in a straight section, one should presumably take condition (101) with f replaced by $v/2\pi\, R$:

$$N < N_0 = \frac{\beta^2\gamma^5}{g_0}\frac{R}{(e^2/mc^2)} \,, \tag{102}$$

again a criterion which is satisfied in all intersecting beam devices so far proposed.

Let us now consider the dispersion relation (81) in the region $\omega_{nl} \gtrsim \omega_0$, where the denominators in g may become small. Since $\partial\Psi^0/\partial W = 0$ for $\omega(W) > \omega_0$, the integrand in Eq. (81) has no singularities in this region and we may integrate by parts. The dispersion relation then becomes

$$8\pi^2 e^2 R(\omega_0^2-\omega_{nl}^2)\sum_j \frac{\bar{g}_j^2}{n^2(\omega_j^2-\omega_{nl}^2)}\int \frac{\Psi^o}{(\omega-\omega_{nl})^2}\frac{\partial\omega}{\partial W}\,dW = 1 \,. \tag{103}$$

Let us assume first that the beam lies entirely below the transition energy, so that $\partial\omega/\partial W > 0$ for all values of W for which $\Psi^o(W) \neq 0$, and assume that all ω_j are greater than ω_0. Then, on the real ω_{nl}-axis above ω_0, the left member of Eq. (103) behaves as shown in Fig. 3, where ω_1, ω_2, ω_3, ω_4 are the values of ω_j. In order to be able to count roots,

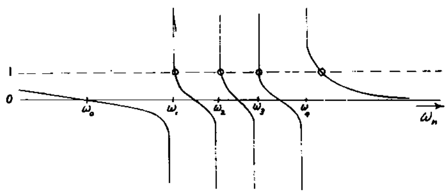

Fig. 3 Dispersion relation for $\partial\omega/\partial W > 0$.

it will be convenient to assume a finite number of terms in the sum over j (in this case, four). It is clear that to each mode of propagation ω_j in the empty vacuum tank corresponds a real positive root of the dispersion relation (103) somewhat larger than ω_j. One can of course show in an exactly similar way that there is also a real negative root ω_{nl} below $-\omega_j$. The effect of the beam is to increase the angular velocity of the electromagnetic modes of propagation, but they remain stable. One can readily see that the larger Ψ^0, the more ω_{nl} is increased above ω_j, but that it never exceeds ω_{j+1}. Moreover, one can see that the modes which propagate in the same sense as the beam (ω_{nl} same sign as ω) are affected most.

In case the beam is all above the transition energy, $\partial\omega/\partial W < 0$, the left member of Eq. (103) behaves as shown in Fig. 4. There is again a real root ω_{nl} corresponding to each mode ω_j of the empty tank, only now the roots are reduced in angular velocity. We see that in either case, the presence of the beam does not make any electromagnetic mode unstable.

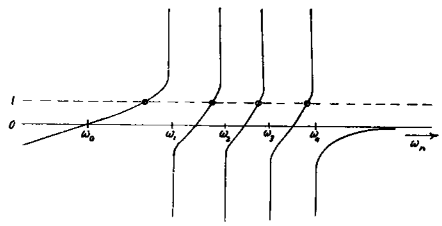

Fig. 4 Dispersion relation for $\partial\omega/\partial W < 0$.

It is now rather easy to see that if any ω_j lie below ω_0 (as some certainly do), then the above argument cannot be carried through, and hence instabilities may be associated with the electromagnetic modes. So long as no part of the beam has an ω above the lowest ω_j, a similar argument leads again to stable electromagnetic modes if the beam is all above transition. If part or all of the beam is below transition and if part of the beam has ω near enough to ω_j

(even though still less than ω_j), then the electromagnetic modes can become unstable. Calculations in a few special examples suggest that instabilities arise for reasonable beam intensities only when part of the beam is very nearly in resonance with some ω_j. The importance of determining the modes ω_j in a conducting donut is therefore clear.

IX. INADEQUACIES IN OUR PRESENT KNOWLEDGE OF LONGITUDINAL INSTABILITIES.

We have given an analysis of the stability of circulating ion beams of nearly uniform charge density. The approximations made are probably sufficiently good for application of the results to beams in most strong-focusing accelerators. We have ignored possible coupling with betatron oscillations, radial and axial motions entering the present theory only in their influence upon beam cross section. We have also ignored the effect of stationary ions of opposite sign which may collect around the beam although the effect of these on betatron oscillations is known to be important [10]. When the cyclotron radii of the stationary ions is small, it is easy to show that they have only a static effect and do not play any role in the stability of a nearly uniform beam.

The frequency function $f(E)$ depends slightly on the amplitude of betatron oscillations, and this effect can be important in the applications we have considered. It is easy to take this into account in the general treatment if we neglect scattering so that the betatron oscillations remain constant in amplitude (or change adiabatically with E). Then we assign a betatron amplitude A to each particle, and include A as a parameter in Ψ. It is easy to show that the result is that an integration over A occurs in our results coincident with each integration over W. The result is equivalent to smoothing out and spreading out the function $\Psi^o(W)$, so that, for example, even a monoenergetic beam may be stable above transition if the frequency spread due to betatron oscillations is large enough.

The influence of particle energy loss mechanisms upon beam stability, which has been omitted from the analysis, is probably negligible except when rate of energy loss is large enough to change the energy spread during the life of the beam; and the effect then appears to be largely interpretable as a consequence of the change in energy spread.

Experimental confirmation of the predicted instabilities is lacking (except as the existence of Saturn's rings may be viewed as a confirmation of the inverse); it is hoped that observation of beams in the MURA 40 MeV electron accelerator will yield relevant information.

The following questions have as yet only conjectural answers :

1. What occurs when stationary configurations of ions confined in radio-frequency buckets are subjected to small perturbations?
2. How do space charge forces influence the capture of ions into buckets? (This appears likely to be especially important above transition.)
3. How do growing perturbations develop after they have grown too large to be considered small?

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(*) See note on reports, p. 696.