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Abstract. Universal spacetimes are vacuum solutions to all theories of gravity with the Lagrangian $L = L(g_{ab}, R_{abcd}, \nabla_{a_1} R_{bcde}, \ldots, \nabla_{a_1 \ldots a_p} R_{bcde})$. Well known examples of universal spacetimes are plane waves which are of the Weyl type N. Here, we discuss recent results on necessary and sufficient conditions for all Weyl type N spacetimes in arbitrary dimension and we conclude that a type N spacetime is universal if and only if it is an Einstein Kundt spacetime. We also summarize the main points of the proof of this result.

1. Introduction

Due to the diffeomorphism invariance, correction terms in perturbative quantum gravity which are added to the Einstein action consist of curvature invariants constructed from the Riemann tensor and its covariant derivatives. The resulting modified field equations are in general very complicated. In the case where only correction terms quadratic in the Riemann tensor are added to the Einstein-Hilbert action - the so called quadratic gravity,

$$S = \int \mathrm{d}^n x \sqrt{-g} \left(\frac{1}{\kappa} \left(R - 2\Lambda_0 \right) + \alpha R^2 + \beta R_{ab}^2 + \gamma \left(R_{abcd}^2 - 4R_{ab}^2 + R^2 \right) \right), \tag{1}$$

the field equations read [1]

$$\begin{aligned} &\frac{1}{\kappa} \left(R_{ab} - \frac{1}{2} R g_{ab} + \Lambda_0 g_{ab} \right) + 2\alpha R \left(R_{ab} - \frac{1}{4} R g_{ab} \right) + (2\alpha + \beta) \left(g_{ab} \Box - \nabla_a \nabla_b \right) R \\ &+2\gamma \left(R R_{ab} - 2R_{acbd} R^{cd} + R_{acde} R_b{}^{cde} - 2R_{ac} R_b{}^c - \frac{1}{4} g_{ab} \left(R_{cdef}^2 - 4R_{cd}^2 + R^2 \right) \right) \\ &\qquad + \beta \Box \left(R_{ab} - \frac{1}{2} R g_{ab} \right) + 2\beta \left(R_{acbd} - \frac{1}{4} g_{ab} R_{cd} \right) R^{cd} = 0. \end{aligned} \tag{2}$$

Interestingly, there exist vacuum solutions of Einstein's gravity (with possibly non-zero cosmological constant) that are "immune" to these corrections, i.e they are vacuum solutions to both theories, the Einstein gravity and the quadratic gravity.

For instance, it has been shown in [2] that for type N[1] Einstein spacetimes, all terms in (2) are proportional to the metric and thus in arbitrary dimension *all Weyl type N Einstein*

[1] Type N spacetimes in the algebraic classification of tensors introduced in [3], recently reviewed in [4] and briefly discussed in section 2.

spacetimes with appropriately chosen effective cosmological constant Λ *are exact solutions of quadratic gravity* (2).

It is a natural question to ask whether some of the type N Einstein spacetimes solve also more general theories than (2). Here, we will be interested in the so called universal spacetimes first introduced in [5].

Definition 1.1. A metric is called *universal* if all conserved symmetric rank-2 tensors constructed from the metric, the Riemann tensor and its covariant derivatives of arbitrary order[2] are multiples of the metric.

Vacuum field equations of modified gravities obtained by varying a diffeomorphism invariant Lagrangian with respect to the metric are conserved, rank-2, and symmetric and therefore universal spacetimes solve vacuum equations of all theories with the Lagrangian being a polynomial curvature invariant in the form

$$L = L(g_{ab}, R_{abcd}, \nabla_{a_1} R_{bcde}, \dots, \nabla_{a_1 \dots a_p} R_{bcde}). \tag{3}$$

Obviously, all universal spacetimes are Einstein (i.e. $R_{ab} = (R/n) g_{ab}$), where n is the dimension of the spacetime. So far, necessary and sufficient geometrical conditions for universality are unknown. However, when considering only type N spacetimes, such conditions can be found[3].

Theorem 1.2. *A type N spacetime is universal if and only if it is an Einstein Kundt spacetime.*

Note that special examples of Ricci-flat type N Kundt spacetimes are plane waves that were identified as vacuum solutions of all theories given by (3) already in [8] and [9].

So far, we did not properly define type N spacetimes in arbitrary dimension and Kundt spacetimes. These terms together with other basic definitions will be briefly reviewed in the next section. In section 3, we outline the key points of the necessity and sufficiency parts of the proof of the theorem 1.2 and in section 4, we give explicit examples of type N universal metrics.

2. Preliminaries

We employ the algebraic classification of the Weyl tensor [3] and higher dimensional generalizations of the Newman-Penrose [10, 11] and the Geroch-Held-Penrose formalisms [12]. We will follow the notation of [4, 12]. Here, let us give basic definitions just very briefly and refer to [4] for more information.

We will work in a null frame with two null vectors $\boldsymbol{\ell}$ and $\boldsymbol{n}$ and $n-2$ spacelike vectors $\boldsymbol{m^{(i)}}$ obeying

$$\ell^a \ell_a = n^a n_a = 0, \qquad \ell^a n_a = 1, \qquad m^{(i)a} m^{(j)}_a = \delta_{ij}, \tag{4}$$

where coordinate indices $a, b, \dots$ and frame indices $i, j, \dots$ take values from 0 to $n-1$ and 2 to $n-1$, respectively.

We say that a quantity q has a boost weight b if it transforms as

$$\hat{q} = \lambda^{\mathrm{b}} q \tag{5}$$

under boosts

$$\hat{\boldsymbol{\ell}} = \lambda \boldsymbol{\ell}, \qquad \hat{\boldsymbol{n}} = \lambda^{-1} \boldsymbol{n}, \qquad \hat{\boldsymbol{m}}^{(i)} = \boldsymbol{m}^{(i)}. \tag{6}$$

[2] We consider only symmetric rank-2 tensors constructed as contractions of *polynomials* from the metric, the Riemann tensor and its covariant derivatives of arbitrary order, however, most of our results hold also for analytic functions of such invariants.

[3] The necessary conditions for universality were found in [6], sufficient conditions were also first proven in [6], however, without a proof the result was stated already in [5] and brief general comments about some points of the proof were also included by one of us in v2 of [7].

By definition [3, 4], type N spacetimes are spacetimes for which the Weyl tensor (in an appropriately chosen frame (4)) admits only components of boost weight -2 and thus can be expressed as

$$C_{abcd} = 4\Omega'_{ij}\, \ell_{\{a} m^{(i)}_{b} \ell_c m^{(j)}_{d\}}, \tag{7}$$

where Ω'_{ij} is symmetric and traceless and for an arbitrary tensor T_{abcd}

$$T_{\{abcd\}} \equiv \frac{1}{2}(T_{[ab][cd]} + T_{[cd][ab]}), \tag{8}$$

so that $C_{abcd} = C_{\{abcd\}}$.

It can be shown [10] that for type N Einstein spacetimes, the multiple WAND[4] $\boldsymbol{\ell}$ is geodetic. If we choose an affine parameterization we can express the covariant derivative of $\boldsymbol{\ell}$ as

$$\ell_{a;b} = L_{11}\ell_a\ell_b + L_{1i}\ell_a m^{(i)}_b + \tau_i m^{(i)}_a \ell_b + \rho_{ij} m^{(i)}_a m^{(j)}_b. \tag{9}$$

Optical scalars of $\boldsymbol{\ell}$, shear σ^2, expansion θ and twist ω^2 can be expressed as

$$\sigma^2 = \ell_{(a;b)}\ell^{(a;b)} - \tfrac{1}{n-2}\left(\ell^a_{\ ;a}\right)^2, \qquad \theta = \tfrac{1}{n-2}\ell^a_{\ ;a}, \qquad \omega^2 = \ell_{[a;b]}\ell^{a;b}. \tag{10}$$

Now, we are ready to define Kundt spacetimes.

Definition 2.1. *Kundt* spacetimes are spacetimes admitting a null geodetic congruence $\boldsymbol{\ell}$ with vanishing shear, expansion and twist.

Kundt metrics in higher dimensions were introduced in [13, 14].

3. Main points of the proof of theorem 1.2

Let us briefly mention the main points of the proof [6] of the theorem 1.2.

3.1. Sufficiency

First, let us discuss the proof of the sufficiency part of the theorem 1.2 , i.e. the proof of the statement that *all Einstein type N Kundt spacetimes are universal.* Thus, we want to show that in this case, all rank-2 tensors constructed from the Weyl[5] tensor and its covariant derivatives are proportional to the metric (in fact, they vanish).

For rank-2 tensors constructed from the Weyl tensor only (without covariant derivatives), the proof is very simple. Any rank-2 tensor has only terms of boost weight $\geq$ -2 and the type N Weyl tensor admits only boost weight -2 terms. Therefore, all rank-2 tensors constructed from the type N Weyl tensor which are quadratic or of a higher order in the Weyl tensor vanish and, due to the tracelessness of the Weyl tensor, rank-2 tensors linear in the Weyl tensor vanish as well. Thus, it is not possible to construct a non-vanishing rank-2 tensor from the type N Weyl tensor.

For covariant derivatives of the Weyl tensor, the proof is more involved. The key point is

Proposition 3.1. *For type N Einstein Kundt spacetimes, the boost order of $\nabla^{(k)}C$ (a covariant derivative of an arbitrary order of the Weyl tensor) with respect to the multiple WAND is at most -2.*

[4] Weyl aligned null direction [3].
[5] Obviously, for Einstein spacetimes, the Ricci tensor is proportional to the metric and its covariant derivatives vanish.

The proof of the above proposition [6] using balanced scalar approach introduced in [15] is rather technical, and it relies on the special form of the Bianchi and Ricci identities for this class of spacetimes.

A direct consequence is

Lemma 3.2. *For type N Einstein Kundt spacetimes, rank-2 tensors constructed from $\nabla^{(k)}C$, which are quadratic or of higher order in $\nabla^{(k)}C$, vanish.*

Using the expression for the commutator of covariant derivatives, the above results and the Bianchi identities, one can generalize the above lemma also to the case of rank-2 tensors constructed from $\nabla^{(k)}C$, which are linear in $\nabla^{(k)}C$ (see [6]). This completes the sufficient part of the proof of the theorem 1.2.

3.2. Necessity

The proof of the necessity part of the theorem 1.2, i.e. the statement that all type N universal spacetimes are Einstein and Kundt is based on another result of [6] that will be discussed in more detail elsewhere in this volume

Theorem 3.3. *A universal spacetime is necessarily a CSI spacetime.*

CSI (constant curvature invariant) spacetimes are spacetimes for which all curvature invariants constructed from the metric, the Riemann tensor and its derivatives of arbitrary order are constant, see e.g. [14].

Let us study the simplest non-trivial curvature invariant for type N spacetimes [16]

$$I_N \equiv C^{a_1b_1a_2b_2;c_1c_2}C_{a_1d_1a_2d_2;c_1c_2}C^{e_1d_1e_2d_2;f_1f_2}C_{e_1b_1e_2b_2;f_1f_2}. \tag{11}$$

In terms of higher dimensional GHP quantities, it can be shown [15] that I_N is proportional (via a numerical constant) to

$$I_N \propto \left[(\Omega'_{22})^2 + (\Omega'_{23})^2\right]^2 (S^2 + A^2)^4, \tag{12}$$

where S and A are closely related to the optical scalars (see [6]). The invariant above is non-constant unless the type N Einstein spacetime is Kundt [15] and thus, in this class of spacetimes, only Kundt spacetimes are CSI.

From theorem 3.3, it follows that type N universal spacetimes are Kundt.

4. Explicit examples of universal type N Kundt metrics

By theorem 1.2, all type N Einstein Kundt metrics are universal. In four dimensions, all type N Einstein Kundt metrics can be expressed as [17]

$$\mathrm{d}s^2 = 2\frac{Q^2}{P^2}\mathrm{d}u\mathrm{d}v + \left(2k\frac{Q^2}{P^2}v^2 + \frac{(Q^2)_{,u}}{P^2}v - \frac{Q}{P}H\right)\mathrm{d}u^2 + \frac{1}{P^2}\left(\mathrm{d}x^2 + \mathrm{d}y^2\right), \tag{13}$$

where

$$P = 1 + \frac{\Lambda}{12}(x^2+y^2), \quad k = \frac{\Lambda}{6}\alpha(u)^2 + \frac{1}{2}\left(\beta(u)^2 + \gamma(u)^2\right),$$
$$Q = \left(1 - \frac{\Lambda}{12}(x^2+y^2)\right)\alpha(u) + \beta(u)x + \gamma(u)y, \quad H = 2f_{1,x} - \frac{\Lambda}{3P}(xf_1 + yf_2),$$

where $\alpha(u), \beta(u), \gamma(u)$ are free functions (see [18] for the canonical forms) and $f_1 = f_1(u,x,y)$ and $f_2 = f_2(u,x,y)$ obey $f_{1,x} = f_{2,y}$, $f_{1,y} = -f_{2,x}$.

Higher dimensional examples of type N universal metrics can be obtained by warping (13). Another higher-dimensional example is (A)dS-wave [6]

$$\mathrm{d}s^2 = e^{-pw}\left(2\mathrm{d}u\mathrm{d}v + H(u,w,x^M)\mathrm{d}u^2 + \delta_{MN}\mathrm{d}x^M\mathrm{d}x^N\right) + \mathrm{d}w^2, \tag{14}$$

with p being a constant and H obeying $H_{,KL}\,\delta^{KL} + \left(H_{,ww} - \frac{n-1}{2}pH_{,w}\right)\mathrm{e}^{-pw} = 0$. Further explicit examples can be found in [6].

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