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A Derived Equivalence of the Libgober–Teitelbaum and the Batyrev–Borisov Mirror Constructions

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In this paper, we study a particular mirror construction to the complete intersection of two cubics in $\mathbb{P}^5$, due to Libgober and Teitelbaum. Using variations of geometric invariant theory and methods of Favero and Kelly, we prove a derived equivalence of this mirror to the Batyrev–Borisov mirror of the complete intersection.

1 Introduction

Libgober and Teitelbaum [20] proposed a mirror to a Calabi–Yau complete intersection V_λ of two cubics in $\mathbb{P}^5$ defined as the zero locus for the two polynomials

$$Q_{1,\lambda} = x_0^3 + x_1^3 + x_2^3 - 3\lambda x_3x_4x_5, \quad Q_{2,\lambda} = x_3^3 + x_4^3 + x_5^3 - 3\lambda x_0x_1x_2.$$

Their proposed mirror $W_{LT,\lambda}$ is a (minimal) resolution of singularities of the variety $V_{LT,\lambda}$ with defining equations $Q_{1,\lambda}, Q_{2,\lambda}$ but in the quotient space $\mathbb{P}^5/G_{81}$, where G_{81} is a specified order 81 subgroup of $PGL(5,\mathbb{C})$. They showed topological evidence that V_λ and $W_{LT,\lambda}$ are a mirror pair, proving on the level of Euler characteristics that $\chi(V_\lambda) = -\chi(W_{LT,\lambda})$. In [13], Filipazzi and Rota verify a state space isomorphism between the two Calabi–Yau varieties by providing an explicit mirror map.

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Batyrev and Borisov in [3] introduced a mirror construction for Calabi–Yau intersections in Fano toric varieties using polytopes, showing mirror duality for $(1, q)$-Hodge numbers. This mirror construction agrees with constructions by Green–Plesser [16] and Berglund–Hübsch [4] for Fermat hypersurfaces. However, the Batyrev–Borisov mirror to two cubics in $\mathbb{P}^5$ differs from the one given above by Libgober and Teitelbaum.

In this paper, we establish a connection between the mirrors of Libgober–Teitelbaum and Batyrev–Borisov for two cubics in $\mathbb{P}^5$ in the context of Homological Mirror Symmetry, using variations of geometric invariant theory (VGIT). In particular, we show that the bounded derived category of coherent sheaves of the Libgober–Teitelbaum mirror is derived equivalent to that of a complete intersection $Z \subseteq X_\nabla$ in the Batyrev–Borisov mirror family. Note that there exists a toric stack $\mathcal{X}_\nabla$ with coarse moduli space X_∇ (see 2.1 for the toric stack construction and 3.1 for the fan associated to this toric stack). On the level of stacks, we will prove the following result.

Theorem 1.1. Let $\lambda \in \mathbb{C}$ such that $\lambda^6 \neq 0, 1$. Consider the two polynomials

$$p_{1,\lambda} = x_0^3x_6^3 + x_1^3x_7^3 + x_2^3x_8^3 - 3\lambda x_3x_4x_5x_6x_7x_8,$$

$$p_{2,\lambda} = x_3^3x_9^3 + x_4^3x_{10}^3 + x_5^3x_{11}^3 - 3\lambda x_0x_1x_2x_9x_{10}x_{11}.$$

Let $\mathcal{Z}_\lambda = Z(p_{1,\lambda}, p_{2,\lambda}) \subseteq \mathcal{X}_\nabla$ and $\mathcal{V}_{LT,\lambda} = Z(Q_{1,\lambda}, Q_{2,\lambda}) \subseteq [\mathbb{P}^5/G_{81}]$. Then

$$\mathrm{D}^{\mathrm{b}}(\mathrm{coh}\,\mathcal{V}_{LT,\lambda}) \simeq \mathrm{D}^{\mathrm{b}}(\mathrm{coh}\,\mathcal{Z}_\lambda).$$

This result is expected in the context of Kontsevich's Homological Mirror Symmetry Conjecture. As both $\mathcal{V}_{LT,\lambda}$ and $\mathcal{Z}_\lambda$ are conjectured to be (homological) mirrors of the complete intersection of two cubics, we expect their corresponding derived categories to be equivalent to each other and to the Fukaya category of the zero locus $Z(Q_{1,\lambda}, Q_{2,\lambda}) \subseteq \mathbb{P}^5$.

There has been work to unify various (toric) mirror constructions [1–4, 6] in the literature via derived equivalence [8, 9]. This paper adds a new construction to this that has been elusive in the past. In particular, this is the first application of partial compactifications in VGIT quotients to prove the equivalence of derived categories for complete intersections, and not hypersurfaces, for Calabi–Yau varieties.

We start by giving some background on the mathematical tools necessary to prove Theorem 1.1 in Section 2. This includes a short introduction to the relevant tools in toric geometry, the Batyrev–Borisov mirror construction, and VGIT quotients as outlined in [11]. In Section 4, we then study the link between the Batyrev–Borisov mirror construction and the mirror given by Libgober–Teitelbaum, proving Theorem 1.1.

2 Background

In this section, we give the necessary background on the Batyrev–Borisov mirror construction, the Libgober–Teitelbaum construction, and the tools used to connect those two. All the varieties considered in this paper will be defined over the complex numbers. More detailed expositions can for example be found in [5], [7], [9], and [20].

2.1 The Cox construction for toric stacks

Let M be a lattice of rank d and N its dual lattice, with the pairing

$$\langle\ \rangle : M \times N \to \mathbb{Z}\,.$$

We extend this to a pairing between $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$ and $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$ in the natural way.

To associate a variety X_Σ to a fan Σ, we can use the Cox construction (see §5 of [7]). Start by noting that each ray ρ of the fan Σ corresponds to a divisor D_ρ on X_Σ (see §4 of [7]). Then we have the following exact sequence:

$$0 \to M \xrightarrow{\iota} \bigoplus_{\rho\in\Sigma(1)} \mathbb{Z} D_\rho \to \operatorname{coker}\iota \to 0, \tag{1}$$

where $\iota(m) := \operatorname{div}(\chi^m) = \sum_{\rho\in\Sigma(1)} \langle m, u_\rho\rangle D_\rho$.

We will write $\mathbb{Z}^{\Sigma(1)} := \bigoplus_{\rho\in\Sigma(1)} \mathbb{Z} D_\rho$. Since $\mathbb{C}^*$ is a divisible group and hence an injective $\mathbb{Z}$-module, the functor $\operatorname{Hom}_{\mathbb{Z}}(-, \mathbb{C}^*)$ is exact, so applying it to (1) yields the exact sequence:

$$1 \to \operatorname{Hom}_{\mathbb{Z}}(\operatorname{coker}\iota, \mathbb{C}^*) \to \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}^{\Sigma(1)}, \mathbb{C}^*) \to \operatorname{Hom}_{\mathbb{Z}}(M, \mathbb{C}^*) \to 1. \tag{2}$$

Define

$$G_\Sigma := \operatorname{Hom}_{\mathbb{Z}}(\operatorname{coker}\iota, \mathbb{C}^*). \tag{3}$$

Note that $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}^{\Sigma(1)}, \mathbb{C}^*) \simeq (\mathbb{C}^*)^{\Sigma(1)}$ and $\operatorname{Hom}_{\mathbb{Z}}(M, \mathbb{C}^*) \simeq T_N$, where T_N is the torus of the variety. Hence, we may rewrite (2) as

$$1 \to G_\Sigma \to (\mathbb{C}^*)^{\Sigma(1)} \to T_N \to 1. \tag{4}$$

When describing G_Σ explicitly, the following lemma is useful.

Lemma 2.1 (Lemma 5.1.1(c) in [7]). Let $G_\Sigma \subseteq (\mathbb{C}^*)^{\Sigma(1)}$ be as in (4). Given a basis $e_1, \dots, e_n$ of M, we have

$$G_\Sigma = \left\{ (t_\rho) \in (\mathbb{C}^*)^{\Sigma(1)} \;\middle|\; \prod_\rho t_\rho^{\langle e_i, u_\rho \rangle} = 1 \text{ for} 1 \leq i \leq n \right\}.$$

We now have both an affine space $\mathbb{C}^{\Sigma(1)}$ and a group G_Σ, which can be shown to be reductive, thus only further require an exceptional set Z in order to construct the toric variety X_Σ as a geometric quotient. For each ray $\rho \in \Sigma(1)$, introduce a variable x_ρ and consider the total coordinate ring of X_Σ,

$$S := \mathbb{C}[x_\rho \mid \rho \in \Sigma(1)].$$

For each cone $\sigma \in \Sigma$, let $x^{\hat{\sigma}} = \prod_{\rho \notin \sigma(1)} x_\rho$. We define the irrelevant ideal

$$B(\Sigma) = \langle x^{\hat{\sigma}} \mid \sigma \in \Sigma \rangle \subseteq S.$$

Since $\tau \preceq \sigma$, we have that $x^{\hat{\tau}}$ is a multiple of $x^{\hat{\sigma}}$. Thus, we only need to consider maximal cones to generate the irrelevant ideal. Define $Z(\Sigma) = Z(B(\Sigma)) \subseteq \mathbb{C}^{\Sigma(1)}$. We then have:

Theorem 2.2 (Theorem 5.1.11 in [7]). Let X_Σ be a toric variety without torus factors, associated to a fan Σ. Then

$$X_\Sigma \simeq (\mathbb{C}^{\Sigma(1)} \setminus Z(\Sigma)) /\!\!/ G_\Sigma.$$

Most of the discussion to follow happens on the level of stacks, so we define the toric stacks relevant for us here.

Definition 2.3. Let Σ be a fan. Define the Cox fan $\mathrm{Cox}(\Sigma) \subseteq \mathbb{R}^{\Sigma(1)}$ to be

$$\mathrm{Cox}(\Sigma) := \{\mathrm{Cone}(e_\rho \mid \rho \in \sigma) \big| \sigma \in \Sigma\}.$$

Denote by n the number of rays in the fan Σ. Then the Cox fan of Σ is a subfan of the standard fan corresponding to the toric variety $\mathbb{A}^n$. Thus, $U_\Sigma := X_{\mathrm{Cox}(\Sigma)}$ is an open subset of $\mathbb{A}^n$. Consider the group G_Σ as defined in Equation (3).

Definition 2.4. We call U_Σ the Cox open set associated to Σ and define the Cox stack associated to Σ to be

$$\mathcal{X}_\Sigma := \left[U_\Sigma / G_\Sigma\right]$$

In the smooth and orbifold cases, we have the following result relating $\mathcal{X}_\Sigma$ to X_Σ.

Theorem 2.5 ([12]). If Σ is simplicial, then $\mathcal{X}_\Sigma$ is a smooth Deligne–Mumford stack with coarse moduli space X_Σ. When Σ is smooth (or equivalently X_Σ is smooth) $\mathcal{X}_\Sigma \cong X_\Sigma$.

2.2 Polytopes and the Batyrev–Borisov construction

We now define reflexive polytopes and nef partitions. We can use them to introduce the Batyrev–Borisov duality, following [3, 5].

Definition 2.6. A polytope Δ in $M_\mathbb{R}$ is a convex hull of a finite set of points in $M_\mathbb{R}$. If this finite set can be chosen to only consist of lattice points of M, we call Δ a lattice polytope.

Definition 2.7. Let Δ be a full dimensional lattice polytope in $M_\mathbb{R}$ with 0 an interior lattice point. Its dual polytope $\Delta^\vee$ is given by

$$\Delta^\vee := \{n \in N_\mathbb{R} \mid \langle m, n\rangle \geq -1 \ \forall m \in \Delta\}$$

We call Δ reflexive if the dual polytope is also a lattice polytope.

Given a lattice polytope Δ, we can associate a toric variety to it by considering its normal fan Σ_Δ with its corresponding toric variety X_{Σ_Δ}.

The polytope Δ corresponds to the anticanonical divisor of X_{Σ_Δ} in that the lattice points of Δ correspond to the global sections of the anticanonical divisor. This in turn allows one to construct a Calabi–Yau hypersurface in X_{Σ_Δ} by considering the zero section of the global section; however, we want to construct Calabi–Yau complete intersections. To do so, we must construct a nef partition of the polytope Δ.

Definition 2.8. Let $\Delta \subseteq M_\mathbb{R}$ be a reflexive lattice polytope. A nef partition of length r of Δ is a Minkowski sum decomposition $\Delta = \Delta_1 + \cdots + \Delta_r$ where $\Delta_1, \ldots, \Delta_r$ are lattice polytopes with $0 \in \Delta_i$.

Consider a reflexive polytope $\Delta \subseteq M_\mathbb{R}$ with nef partition $\Delta = \Delta_1 + \cdots + \Delta_r$. Then, for $1 \leq j \leq r$, we define

$$\nabla_j := \{n \in N_\mathbb{R} \mid \langle m, n\rangle \geq -\delta_{ij} \text{ for all } m \in \Delta_i, \text{ for } 1 \leq i \leq r\}.$$

We note that these polytopes are all lattice polytopes, and define the polytope ∇ as their Minkowski sum $\nabla := \nabla_1 + \cdots + \nabla_r$. We call $\nabla_1, \ldots, \nabla_r$ the dual nef partition to $\Delta_1, \ldots, \Delta_r$.

To understand the statement of Batyrev–Borisov duality, we note that a lattice polytope Δ corresponds to a d-dimensional Gorenstein Fano toric variety X_Δ. Each of the polytopes Δ_i corresponds to a divisor D_i on X_Δ. The nef partition $\Delta = \Delta_1 + \cdots + \Delta_r$ decomposes the anticanonical sheaf $\mathcal{O}(-K_{X_\Delta})$ as tensor product $\bigotimes_{i=1}^r \mathcal{O}_{X_\Delta}(D_i)$. Now the lattice points inside the Δ_i correspond to global sections of these line bundles. Taking the zero sets of such sections, we can associate to each polytope a family of hypersurfaces. By intersecting these, a nef partition corresponds to a family of $(d-r)$-dimensional Calabi–Yau complete intersections in X_Δ. Similarly, the dual nef partition $\nabla = \nabla_1 + \cdots + \nabla_r$ gives a family of $(d-r)$-dimensional Calabi–Yau complete intersections in X_∇.

Remark 2.9. The generic complete intersection in the family associated to the dual nef partition $\nabla_1, \ldots, \nabla_r$ may be singular.

In [2], Batyrev formulates the original construction in a way that fixes this problem. In this case, one uses a maximal projective crepant partial desingularization (MPCP-desingularization), which reduces to a combinatorial manipulation of the normal fan to ∇.

For every maximal cone of the normal fan, we choose a regular triangulation of it. Therefore, all maximal cones should contain exactly the minimal number of rays dictated by the dimension, since a triangulation uses simplices. Doing this for all maximal cones gives exactly a maximal projective triangulation. When speaking of X_∇, we will thus think of a MPCP-desingularization of the variety associated to the normal fan of ∇, obtained in this way.

Batyrev and Borisov prove the following result, showing that their construction produces topological mirror duality for $(1,q)$-Hodge numbers.

Theorem 2.10 (Theorem 9.6 in [3]). Let V be a Calabi–Yau complete intersection of r hypersurfaces in $\mathbb{P}^d$ and $d-r \geq 3$ and $\widehat{W}$ be a MPCP-desingularization of the Calabi–Yau complete intersection $W \subseteq X_\nabla$. Then

$$h^q(\Omega^1_{\widehat{W}}) = h^{d-r-q}(\Omega^1_V) \text{ for } 0 \leq q \leq d-r.$$

2.3 Toric vector bundles and GIT quotients

We first discuss how to construct toric vector bundles. Recall that a Cartier divisor $D = \sum_\rho a_\rho D_\rho$ on a toric variety X_Σ corresponds to the line bundle $\mathcal{L} = \mathcal{O}_{X_\Sigma}(D)$, which is the sheaf of sections of a rank 1 vector bundle $\pi : V_{\mathcal{L}} \to X_\Sigma$. The variety $V_{\mathcal{L}}$ is toric and π is a toric morphism. This is shown by directly constructing the fan of $V_{\mathcal{L}}$ in terms of Σ and D, which we do now. Given a cone $\sigma \in \Sigma$, set

$$\tilde{\sigma} = \mathrm{Cone}\,((0,1),(u_\rho, -a_\rho) \mid \rho \in \sigma(1)).$$

Then $\tilde{\sigma}$ is a strongly convex rational polyhedral cone in $N_{\mathbb{R}} \times \mathbb{R}$ for all cones $\sigma \in \Sigma$. Now let $\Sigma \times D$ be the collection consisting of cones $\tilde{\sigma}$ for $\sigma \in \Sigma$ and their faces. This is a fan in $N_{\mathbb{R}} \times \mathbb{R}$ and the projection $\overline{\pi} : N \times \mathbb{Z} \to N$ is compatible with $\Sigma \times D$ and Σ, thus inducing a toric morphism

$$\pi : X_{\Sigma \times D} \to X_\Sigma.$$

Proposition 2.11 (Proposition 7.3.1 in [7]). $\pi : X_{\Sigma\times D} \to X_\Sigma$ is a rank 1 vector bundle whose sheaf of sections is $\mathcal{O}_{X_\Sigma}(D)$.

The variety $X_{\Sigma\times D}$ is sometimes also denoted by $X_{\Sigma,D}$.

For decomposable vector bundles of rank higher than 1, we can repeatedly apply Proposition 2.11 to construct the total space of the vector bundle, following [10]. Taking r torus-invariant Weil divisors $D_i = \sum_{\rho\in\Sigma} a_{i\rho} D_\rho$, we define

$$\sigma_{D_1,\ldots,D_r} := \mathrm{Cone}\left(\{u_\rho - a_{1\rho}e_1 - \cdots - a_{r\rho}e_r \mid \rho \in \sigma(1)\} \cup \{e_i | i \in \{1,\ldots,r\}\}\right) \subset N_{\mathbb{R}} \oplus \mathbb{R}^r.$$

Let $\Sigma_{D_1,\ldots,D_r}$ be the fan generated by the cones $\sigma_{D_1,\ldots,D_r}$ and their proper faces, and call $\mathcal{X}_{\Sigma,D_1,\ldots,D_r}$ the associated stack. We obtain the following result.

Proposition 2.12 (Proposition 4.13 in [10]). Let $D_1,\ldots,D_r$ be divisors on X_Σ. There is an isomorphism of stacks

$$\mathcal{X}_{\Sigma,D_1,\ldots,D_r} \cong \mathrm{tot}\left(\mathcal{O}_{\mathcal{X}_\Sigma}(D_i)\right).$$

Geometric invariant theory (GIT), developed by Mumford, is a powerful tool in modern algebraic geometry. We will here discuss the toric version of it, following §14 of [7].

Roughly speaking, GIT deals with ways to take almost geometric quotients of spaces by some reductive groups acting on them. As a model for this, recall the Cox construction in §2.1. It gives a toric variety as almost geometric quotient $X_\Sigma \simeq (\mathbb{C}^{\Sigma(1)} \setminus Z(\Sigma)) /\!\!/ G_\Sigma$. Fundamentally, we start with $\mathbb{C}^{\Sigma(1)}$ and remove a special Zariski closed subset in order to obtain an almost geometric quotient. GIT provides the machinery to do so, but the way is often not unique. The subsets that are removed depend on a choice of stability parameterised by a choice of line bundle. The different choices can give different quotients that are birational.

In GIT, deciding which points are removed is done by a lifting of the G-action on $\mathbb{C}^r$ to the rank 1 trivial vector bundle $\mathbb{C}^r \times \mathbb{C} \to \mathbb{C}^r$. Define the character group of G to be

$$\widehat{G} = \{\chi : G \to \mathbb{C}^* \,|\, \chi \text{ is a homomorphism of algebraic groups}\}.$$

A character $\chi \in \widehat{G}$ then gives the action of G on $\mathbb{C}^r \times \mathbb{C}$ defined by

$$g \cdot (p,t) = (g \cdot p, \chi(g)t), \;\; g \in G, \; (p,t) \in \mathbb{C}^r \times \mathbb{C}.$$

This lifts the G-action on $\mathbb{C}^r$ and furthermore all possible liftings arise this way.

Let $\mathcal{L}_\chi$ or $\mathcal{O}(\chi)$ denote the sheaf of sections of $\mathbb{C}^r \times \mathbb{C}$ with this G-action. It is called the linearised line bundle with character χ. For $d \in \mathbb{Z}$, the tensor product $\mathcal{O}(\chi)^{\otimes d}$ is the linearised line bundle with character χ^d. Note that, if one forgets the G-action, then $\mathcal{O}(\chi) \simeq \mathcal{O}_{\mathbb{C}^r}$ as line bundles on $\mathbb{C}^r$. Thus, a global section $s \in \Gamma(\mathbb{C}^r, \mathcal{O}(\chi))$ can be written as

$$s : \mathbb{C}^r \to \mathbb{C}^r \times \mathbb{C}$$
$$p \mapsto (p, F_s(p)),$$

for some unique $F_s \in \mathbb{C}[x_1, \ldots, x_r]$.

Definition 2.13. Fix $G \subseteq (\mathbb{C}^*)^r$ and $\chi \in \widehat{G}$, with linearised line bundle $\mathcal{O}(\chi)$. Given a global section s of $\mathcal{O}(\chi)$, we denote

$$(\mathbb{C}^r)_s := \{p \in \mathbb{C}^r \;\mid s(p) \neq 0\}.$$

This is an affine open subset of $\mathbb{C}^r$, as $s(p) \neq 0$ means $F_s(p) \neq 0$. Furthermore, G acts on $(\mathbb{C}^r)_s$ when s is G-invariant. We define:

A. $p \in \mathbb{C}^r$ is semistable with respect to χ if there exist $d > 0$ and $s \in \Gamma(\mathbb{C}^r, \mathcal{O}(\chi^d))^G$ such that $p \in (\mathbb{C}^r)_s$.

B. $p \in \mathbb{C}^r$ is stable with respect to χ if there exist $d > 0$ and $s \in \Gamma(\mathbb{C}^r, \mathcal{O}(\chi^d))^G$ such that $p \in (\mathbb{C}^r)_s$, the isotropy subgroup G_p is finite, and all G-orbits in $(\mathbb{C}^r)_s$ are closed in $(\mathbb{C}^r)_s$.

C. The set of all semistable (resp. stable) points with respect to χ is denoted $(\mathbb{C}^r)^{ss}_\chi$ (resp. $(\mathbb{C}^r)^s_\chi$).

Given a group $G \subseteq (\mathbb{C}^*)^r$ and $\chi \in \widehat{G}$, we next need to define the GIT quotient $\mathbb{C}^r /\!\!/_\chi G$. Consider the graded ring $R_\chi = \bigoplus_{d=0}^{\infty} \Gamma(\mathbb{C}^r, \mathcal{O}(\chi^d))^G$.

Definition 2.14. For $G \subseteq (\mathbb{C}^*)^r$ and $\chi \in \widehat{G}$, the GIT quotient $\mathbb{C}^r /\!\!/_\chi G$ is

$$\mathbb{C}^r /\!\!/_\chi G = \mathrm{Proj}(R_\chi).$$

An important property of GIT quotients is that in principle, this is the same as taking the quotient of $(\mathbb{C}^r)^{ss}_\chi$ under the action of G.

Proposition 2.15 (Proposition 14.1.12.c) in [7]). For $G \subseteq (\mathbb{C}^*)^r$ and $\chi \in \widehat{G}$, the GIT quotient $\mathbb{C}^r /\!\!/_\chi G$ is a good categorical quotient of $(\mathbb{C}^r)^{ss}_\chi$ under the action of G, that is, $\mathbb{C}^r /\!\!/_\chi G \simeq (\mathbb{C}^r)^{ss}_\chi /\!\!/ G$.

Theorem 14.2.13 of [7] shows, using a polyhedron associated to the character χ, that the GIT quotient $\mathbb{C}^r /\!\!/_\chi G$ is a toric variety.

2.4 GKZ Fans

Let $G \subseteq (\mathbb{C}^*)^r$. Studying the GIT quotient $\mathbb{C}^r /\!\!/_\chi G$ as χ varies gives rise to the GKZ fan of a toric variety, which has the structure of a generalised fan.

Definition 2.16. A generalised fan Σ in $N_\mathbb{R}$ is a finite collection of cones $\sigma \subseteq N_\mathbb{R}$ such that:

A. Every $\sigma \in \Sigma$ is a rational polyhedral cone.
B. For all $\sigma \in \Sigma$, each face of σ is also in Σ.
C. For all $\sigma_1, \sigma_2 \in \Sigma$, the intersection $\sigma_1 \cap \sigma_2$ is a face of each.

This agrees with the usual definition of a fan, with the exception that cones are not necessarily strongly convex. Consider the cone $\sigma_0 = \bigcap_{\sigma \in \Sigma} \sigma$. It has no proper faces and is thus a subspace of $N_\mathbb{R}$. We consider the lattice $\overline{N} = N/(\sigma_0 \cap N)$. To associate a toric variety for the generalised fan Σ, one constructs the fan $\overline{\Sigma}$ where each cone comes from a cone of Σ quotiented by σ_0. This is a fan in the usual sense, and hence we can associate a toric variety to it as usual. Then $X_\Sigma := X_{\overline{\Sigma}}$.

We will now discuss the notion of a GKZ fan, following both [7] and [11]. Consider a toric variety X. It can be written as a GIT quotient $(\mathbb{C}^r \setminus Z) /\!\!/_\chi G$. Recall the character group $\widehat{G}$ of G. Each choice of character $\chi \in \widehat{G}$ determines an open subset $U_\chi := (\mathbb{C}^r)^{ss}_\chi$, the semi-stable locus of X with respect to χ. Several different characters can give the same semi-stable locus. Thinking of the vector space $\mathrm{Hom}(\widehat{G}, T_N) \otimes_{\mathbb{Z}} \mathbb{Q}$ as parameter space for linearisations, we investigate where the semi-stable locus U_ψ is the same as U_χ for a given character χ. It turns out that dividing the vector space into chambers where U_χ remains the same gives the space a natural fan structure. This fan-structure Σ_{GKZ} is called the GKZ fan. Maximal cones are called chambers and codimension one cones are called walls.

Consider an arbitrary fan Σ, we can construct the GKZ fan as follows. Take the group $G = G_\Sigma \subseteq (\mathbb{C}^*)^r$ acting on X_Σ to be the group in Equation (3). There is a well-known bijection between chambers of GKZ fans and regular triangulations of a certain set of points, constructed as follows. In the general setting, apply $\mathrm{Hom}(-, \mathbb{C}^*)$ to the sequence

$$0 \to G \xrightarrow{i_G} (\mathbb{C}^*)^r \xrightarrow{proj} \mathrm{coker}(i_G) \to 0$$

to obtain the sequence

$$\mathrm{Hom}(\mathrm{coker}(i_G), \mathbb{C}^*) \xrightarrow{\widehat{proj}} \mathbb{Z}^r \xrightarrow{\widehat{i_G}} \mathrm{Hom}(G, \mathbb{C}^*) \to 0.$$

Let $\nu_i(G)$ be the element of $\mathrm{Hom}(\mathrm{coker}(i_G), \mathbb{C}^*)^\vee$ given by composing $\widehat{proj}$ with the projection of $\mathbb{Z}^r$ onto its i^{th} factor. Compare this sequence with the sequence (1). We in fact reversed the process of obtaining (4) from (1). Starting with the correct group acting on the space, we thus recover the map corresponding to ι as $\widehat{proj}$. Hence, the $\nu(G)$ correspond to the primitive generators u_ρ of the rays of Σ. Then the set we will triangulate is the convex hull of the set $\nu(G) = \{\nu_1(G), \dots, \nu_r(G)\}$.

Theorem 2.17 (Proposition 15.2.9 in [7]). There is a bijection between chambers of the GKZ fan for the action of G on $\mathbb{C}^r$ and regular triangulations of the set $\mathrm{Conv}(\nu(G))$. In particular, there are only finitely many chambers of the GKZ fan.

Thus we can enumerate the chambers of the GKZ fan, say by $\sigma_1, \dots, \sigma_k$. For any of those chambers, we can choose a character in its interior and consider the semi-stable locus with respect to it. As this locus does not depend on the choice of character, but solely on the choice of chamber, denote the open affine associated to chamber σ_p by U_p. By the above theorem, it will also correspond to a specific triangulation $\mathcal{T}_p$ of $\mathrm{Conv}(\{\nu_1(G), \dots, \nu_r(G)\})$.

2.5 Categories of singularities and some results on the equivalence of derived categories

In this section, we introduce the categories of singularities (as outlined in [21]) and their equivalences to derived categories through VGIT, reviewing §4 of [11].

Let X be a variety and G an algebraic group acting on X (on the left).

Definition 2.18. An object of $\mathrm{D^b}(\mathrm{coh}[X/G])$ is called perfect if it is locally quasi-isomorphic to a bounded complex of vector bundles. We denote the full subcategory of perfect objects by $\mathrm{Perf}([X/G])$. The Verdier quotient of $\mathrm{D^b}(\mathrm{coh}[X/G])$ by $\mathrm{Perf}([X/G])$ is called the category of singularities and denoted

$$\mathrm{D_{sg}}([X/G]) := \mathrm{D^b}(\mathrm{coh}[X/G])/\,\mathrm{Perf}([X/G]).$$

By the following observation of Orlov's, the category can be viewed as studying the geometry of the singular locus.

Proposition 2.19 (Orlov, [21]). Assume that $\mathrm{coh}[X/G]$ has enough locally free sheaves. Let $i : U \to X$ be a G-equivariant open immersion such that the singular locus of X is contained in $i(U)$. Then the restriction,

$$i^* : \mathrm{D_{sg}}([X/G]) \to \mathrm{D_{sg}}([U/G]),$$

is an equivalence of categories.

Next, consider a G-equivariant vector bundle $\mathcal{E}$ on X. Denote by Z the zero locus of a G-invariant section $s \in H^0(X, \mathcal{E})$. Then $\langle -, s\rangle$ induces a global function on $\mathrm{tot}\,\mathcal{E}^\vee$. Let Y be the zero section of this pairing and consider the fibrewise dilation action on the torus $\mathbb{G}_m$. Then we have the following result.

Theorem 2.20 (Isik [19], Shipman [25], Hirano [17]). Suppose the Koszul complex on s is exact. Then there is an equivalence of categories

$$\mathrm{D_{sg}}([Y/(G \times \mathbb{G}_m)]) \cong \mathrm{D^b}(\mathrm{coh}[Z/G]).$$

Combining the previous two results gives the following.

Corollary 2.21 (Corollary 3.4 in [11]). Let V be an algebraic variety with a $G \times \mathbb{G}_m$ action. Suppose there is an open subset $U \subseteq V$ such that U is $G \times \mathbb{G}_m$ equivariantly isomorphic to Y as above and that U contains the singular locus of X. Then

$$\mathrm{D}_{\mathrm{sg}}([V/(G\times\mathbb{G}_m)])\cong\mathrm{D}^{\mathrm{b}}(\mathrm{coh}[Z/G]).$$

We will move towards making these results applicable to the objects studied in this paper, adapting [11]. Consider an affine space $X := \mathbb{A}^{n+t}$ with coordinates x_i, u_j for $1 \le i \le n, 1 \le j \le t$. Let T denote the standard open torus $\mathbb{G}_m^{n+t}$ and consider a subgroup $S \subseteq T$, with $\tilde{S}$ the connected component that contains the identity.

Recall the notion of GKZ fans from §2.4. We adjust the notation so that S above corresponds to the group G from §2.4. We will now explain how to construct varieties corresponding to the chambers of the GKZ fan, and the goal of this setup is to apply Corollary 2.21 and VGIT to provide equivalences between derived categories.

Definition 2.22. Let G be a group acting on a space X and f a global function on X. f is said to be semi-invariant with respect to a character χ if, for any $g \in G, f(g\cdot x) = \chi(g)f(x)$.

To apply Corollary 2.21, we will add a $\mathbb{G}_m$-action, which is S-invariant and $\mathbb{G}_m$-semi-invariant, acting with weight 0 on the x_i and 1 on the u_j. We refer to this action as R-charge. Consider the action of S on the scheme Spec $\mathbb{C}[u_j]$. It corresponds to a character γ_j of S. Let $f_1, \dots, f_t$ be a collection of S-semi-invariant functions in the x_i with respect to γ_j^{-1}. Then define a function, called superpotential, by

$$w := \sum_{j=1}^{t} u_j f_j.$$

The superpotential w is S-invariant and χ-semi-invariant with respect to the projection character $\chi : S \times \mathbb{G}_m \to \mathbb{G}_m$, hence w is homogeneous of degree 0 with respect to the S-action and of degree 1 with respect to the R-charge. Let $Z(w) \subseteq X$ be its zerolocus and define $Y_p := Z(w) \cap U_p$. Then we have the following result.

Theorem 2.23 (Theorem 3 in [18]). If S is quasi-Calabi–Yau, there is an equivalence of categories

$$\mathrm{D}_{\mathrm{sg}}([Y_p/S\times\mathbb{G}_m])\cong\mathrm{D}_{\mathrm{sg}}([Y_q/S\times\mathbb{G}_m])$$

for all $1 \le p, q \le k$, where k is the number of chambers in the GKZ fan.

We will use this result to show a useful equivalence of derived categories. We start by explicitly describing the open sets U_p corresponding to a chamber σ_p of the

GKZ fan, defined in §2.4. For $1 \leq p \leq k$, we associate an irrelevant ideal $\mathcal{I}_p$ to σ_p by considering the (regular) triangulation $\mathcal{T}_p$ that the chamber corresponds to. So, let

$$\mathcal{I}_p := \left\langle \prod_{i \notin I} x_i \prod_{j \notin J} u_j \;\middle|\; \bigcup_{i \in I} \nu_i(S) \cup \bigcup_{j \in J} \nu_{n+j}(S) \text{ is the set of vertices of a simplex in } \mathcal{T}_p \right\rangle.$$

Then $U_p = X \setminus Z(\mathcal{I}_p)$. Another ideal we will need is a subideal of $\mathcal{I}_p$, given similarly to $\mathcal{I}_p$ by requiring J to be the full set $\{1, \ldots, t\}$, that is,

$$\mathcal{J}_p := \left\langle \prod_{i \notin I} x_i \;\middle|\; \bigcup_{i \in I} \nu_i(S) \cup \bigcup_{j=1}^{t} \nu_{n+j}(S) \text{ is the set of vertices of a simplex in } \mathcal{T}_p \right\rangle.$$

This ideal is therefore generated by those simplices whose sets of vertices contain all ν_{n+j} for $1 \leq j \leq t$. Using this subideal, we get a new open set $V_p := X \setminus Z(\mathcal{J}_p) \subseteq U_p$. Since $\mathcal{J}_p$ has no u_j in its generators, we can see it as ideal $\mathcal{J}_p^x$ in $\mathbb{C}[x_1, \ldots, x_n]$, giving an open subset of $\mathbb{A}^n$ by $V_p^x := \mathbb{A}^n \setminus Z(\mathcal{J}_p^x)$. This set gives us a toric stack $X_p := [V_p^x/S]$. Now suppose $\mathcal{J}_p$ is non-zero. Then the last two quantities defined are nonempty, and one can show $[V_p/S]$ is a vector bundle over X_p, with the inclusion of rings $\mathbb{C}[x_1, \ldots, x_n] \to \mathbb{C}[x_1, \ldots, x_n, u_1, \ldots, u_t]$ restricting to a S-equivariant morphism

$$[V_p/S] \to [V_p^x/S] = X_p.$$

This morphism gives the following proposition.

Proposition 2.24 (Proposition 4.6 in [11]). Suppose $\mathcal{J}_p$ is non-zero. The morphism $[V_p/S] \to X_p$ realizes $[V_p/S]$ as the total space of a vector bundle

$$[V_p/S] \cong \operatorname{tot} \bigoplus_{j=1}^{t} \mathcal{O}(\gamma_j).$$

Furthermore, the R-charge action of $\mathbb{G}_m$ is the dilation action along fibers. Finally, for each j, the function f_j gives a section of $\mathcal{O}(\gamma_j^{-1})$ and the superpotential $w = \sum u_j f_j$ restricts to the pairing with the section $\oplus f_j$.

In particular, from this we can view the function $\oplus f_j$ as a section of V_p, which defines, for all p, a complete intersection $Z_p := Z(\oplus f_j) \subseteq X_p$. Finally, we introduce the Jacobian ideal ∂w, generated by the partial derivatives of w with respect to the coordinates x_i, u_j.

Proposition 2.25 (Proposition 4.7 in [11]). Suppose $\mathcal{J}_p$ is non-zero. If $\mathcal{I}_p \subseteq \sqrt{\partial w, \mathcal{J}_p}$, then

$$\mathrm{D_{sg}}([Y_p/S \times \mathbb{G}_m]) \cong \mathrm{D^b}(\mathrm{coh}\, Z_p).$$

This finally leads us to the following result, which we will use in §4.

Corollary 2.26. Assume S satisfies the quasi-Calabi–Yau condition and that $\mathcal{J}_p$ and $\mathcal{J}_q$ are non-zero. If $\mathcal{I}_p \subseteq \sqrt{\partial w, \mathcal{J}_p}$ and $\mathcal{I}_q \subseteq \sqrt{\partial w, \mathcal{J}_q}$ for some $1 \leq p, q \leq r$, then

$$\mathrm{D^b}(\mathrm{coh}\, Z_p) \cong \mathrm{D^b}(\mathrm{coh}\, Z_q).$$

3 The Libgober–Teitelbaum and the Batyrev–Borisov Constructions

3.1 The Batyrev–Borisov construction in $\mathbb{P}^5$

We now construct a Batyrev–Borisov mirror to a complete intersection of two cubics in $\mathbb{P}^5$. We will do this by giving a nef partition of the anticanonical polytope of $\mathbb{P}^5$, which corresponds to a complete intersection. Then we will apply the Batyrev–Borisov construction to that nef partition, obtaining a polytope ∇ corresponding to the mirror. Fix the lattice $M \cong \mathbb{Z}^5$ and its dual lattice N.

Remark 3.1. Due to the way we will derive certain fans in this section via methods inspired by mirror symmetry (see § 3.3.1) our first fan lives in $M_\mathbb{R}$ and not in the conventional $N_\mathbb{R}$.

Define the rays $\overline{\rho_0}, \ldots, \overline{\rho_{11}}$ in $M_\mathbb{R} \oplus \mathbb{R}^2$ with primitive generators

$$\begin{aligned}
u_{\overline{\rho_0}} &= (3,0,0,-1,-1,0,1), & u_{\overline{\rho_6}} &= (2,-1,-1,0,0,1,0),\\
u_{\overline{\rho_1}} &= (0,3,0,-1,-1,0,1), & u_{\overline{\rho_7}} &= (-1,2,-1,0,0,1,0),\\
u_{\overline{\rho_2}} &= (0,0,3,-1,-1,0,1), & u_{\overline{\rho_8}} &= (-1,-1,2,0,0,1,0),\\
u_{\overline{\rho_3}} &= (-1,-1,-1,3,0,1,0), & u_{\overline{\rho_9}} &= (0,0,0,2,-1,0,1),\\
u_{\overline{\rho_4}} &= (-1,-1,-1,0,3,1,0), & u_{\overline{\rho_{10}}} &= (0,0,0,-1,2,0,1),\\
u_{\overline{\rho_5}} &= (-1,-1,-1,0,0,1,0), & u_{\overline{\rho_{11}}} &= (0,0,0,-1,-1,0,1),\\
u_{\tau_1} &= (0,0,0,0,0,1,0), & u_{\tau_2} &= (0,0,0,0,0,0,1).
\end{aligned}$$

Notation 3.2. For $0 \leq j \leq 11$, we denote by u_{ρ_j} the lattice point in M obtained from $u_{\overline{\rho_j}}$ by projecting onto the first 5 coordinates. Denote by ρ_j the ray generated by u_{ρ_j} in $M_\mathbb{R}$.

Table 1 Maximal cones of X_∇

$\rho_0\rho_1\rho_2\rho_9\rho_{10}$	$\rho_0\rho_1\rho_6\rho_9\rho_{10}$	$\rho_3\rho_4\rho_6\rho_7\rho_9$	$\rho_1\rho_6\rho_7\rho_9\rho_{10}$	$\rho_4\rho_6\rho_7\rho_9\rho_{10}$	$\rho_0\rho_2\rho_6\rho_9\rho_{10}$
$\rho_2\rho_6\rho_8\rho_9\rho_{10}$	$\rho_3\rho_4\rho_6\rho_8\rho_9$	$\rho_4\rho_6\rho_8\rho_9\rho_{10}$	$\rho_1\rho_2\rho_7\rho_9\rho_{10}$	$\rho_2\rho_7\rho_8\rho_9\rho_{10}$	$\rho_3\rho_4\rho_7\rho_8\rho_{10}$
$\rho_5\rho_7\rho_8\rho_9\rho_{10}$	$\rho_1\rho_2\rho_7\rho_{10}\rho_{11}$	$\rho_2\rho_7\rho_8\rho_{10}\rho_{11}$	$\rho_4\rho_5\rho_7\rho_8\rho_{10}$	$\rho_5\rho_7\rho_8\rho_{10}\rho_{11}$	$\rho_0\rho_1\rho_2\rho_9\rho_{11}$
$\rho_0\rho_1\rho_2\rho_{10}\rho_{11}$	$\rho_3\rho_4\rho_5\rho_6\rho_7$	$\rho_0\rho_1\rho_6\rho_9\rho_{11}$	$\rho_1\rho_6\rho_7\rho_9\rho_{11}$	$\rho_3\rho_5\rho_6\rho_7\rho_9$	$\rho_5\rho_6\rho_7\rho_9\rho_{11}$
$\rho_0\rho_1\rho_6\rho_{10}\rho_{11}$	$\rho_1\rho_6\rho_7\rho_{10}\rho_{11}$	$\rho_4\rho_5\rho_6\rho_7\rho_{10}$	$\rho_5\rho_6\rho_7\rho_{10}\rho_{11}$	$\rho_3\rho_4\rho_5\rho_6\rho_8$	$\rho_0\rho_2\rho_6\rho_9\rho_{11}$
$\rho_2\rho_6\rho_8\rho_9\rho_{11}$	$\rho_3\rho_5\rho_6\rho_8\rho_9$	$\rho_5\rho_6\rho_8\rho_9\rho_{11}$	$\rho_0\rho_2\rho_6\rho_{10}\rho_{11}$	$\rho_2\rho_6\rho_8\rho_{10}\rho_{11}$	$\rho_4\rho_5\rho_6\rho_8\rho_{10}$
$\rho_5\rho_6\rho_8\rho_{10}\rho_{11}$	$\rho_3\rho_4\rho_5\rho_7\rho_8$	$\rho_1\rho_2\rho_7\rho_9\rho_{11}$	$\rho_2\rho_7\rho_8\rho_9\rho_{11}$	$\rho_3\rho_5\rho_7\rho_8\rho_9$	$\rho_5\rho_7\rho_8\rho_9\rho_{11}$

Proposition 3.3. Consider the fan Σ_∇ with rays $\rho_0, \ldots, \rho_{11}$ defined above and maximal cones listed in Table 1. Then a general complete intersection in the toric variety X_∇ corresponding to the fan Σ_∇ is a Batyrev–Borisov mirror to a complete intersection of two cubics in $\mathbb{P}^5$.

Proof. The anticanonical sheaf of $\mathbb{P}^5$ is $\mathcal{O}_{\mathbb{P}^5}(6)$, corresponding to the divisor class

$$-K_{\mathbb{P}^5} = T_0 + \cdots + T_5 = (T_0 + T_1 + T_2) + (T_3 + T_4 + T_5).$$

The anticanonical polytope for $\mathbb{P}^5$ is given by

$$\Delta_{-K_{\mathbb{P}5}} = \{m \in M_{\mathbb{R}} | \langle m, u_\rho \rangle \geq -1 \text{ for} \rho \in \Sigma_{\mathbb{P}^5}(1)\} \subseteq M_{\mathbb{R}},$$

which is the convex hull of the six points

$$(5, -1, -1, -1, -1), \quad (-1, 5, -1, -1, -1), \quad (-1, -1, 5, -1, -1),$$
$$(-1, -1, -1, 5, -1), \quad (-1, -1, -1, -1, 5), \quad (-1, -1, -1, -1, -1).$$

A nef partition with respect to the origin of the polytope $\Delta_{-K_{\mathbb{P}5}}$ is given by the polytopes Δ_1, Δ_2 associated to the divisors $T_0 + T_1 + T_2$ and $T_3 + T_4 + T_5$, since the Minkowski sum $\Delta_1 + \Delta_2$ is equal to $\Delta_{-K_{\mathbb{P}5}}$. These polytopes are

$$\begin{aligned}
\Delta_1 = \ & \mathrm{Conv}((2, -1, -1, 0, 0), (-1, 2, -1, 0, 0), (-1, -1, 2, 0, 0), \\
& (-1, -1, -1, 3, 0), (-1, -1, -1, 0, 3), (-1, -1, -1, 0, 0)), \\
\Delta_2 = \ & \mathrm{Conv}((0, 0, 0, -1, 2), (0, 0, 0, 2, -1), (0, 0, 3, -1, -1), \\
& (0, 3, 0, -1, -1), (3, 0, 0, -1, -1), (0, 0, 0, -1, -1)).
\end{aligned}$$

Next, we shall compute the dual nef partition, as defined in §2. We have:

$$\nabla_1 = \text{Conv}((1,0,0,0,0),(0,1,0,0,0),(0,0,1,0,0),(0,0,0,0,0))$$
$$\nabla_2 = \text{Conv}((0,0,0,0,1),(0,0,0,1,0),(0,0,0,0,0),(-1,-1,-1,-1,-1)).$$

Their Minkowski sum $\nabla \subseteq N_{\mathbb{R}}$ is then the convex hull of the 15 points

$(1,0,0,0,0)$, $(0,1,0,0,0)$, $(0,0,1,0,0)$, $(0,0,0,1,0)$,
$(0,0,0,0,1)$, $(1,0,0,1,0)$, $(1,0,0,0,1)$, $(0,1,0,1,0)$,
$(0,1,0,0,1)$, $(0,0,1,1,0)$, $(0,0,1,0,1)$, $(-1,-1,-1,-1,-1)$,
$(0,-1,-1,-1,-1)$, $(-1,0,-1,-1,-1)$, $(-1,-1,0,-1,-1)$.

A SAGE computation shows the normal fan of ∇, $\Sigma'_\nabla \subseteq M_{\mathbb{R}}$, has rays $\rho_0,\dots,\rho_{11}$ from Notation 3.2. The maximal-dimensional cones are the following 15 cones:

$\rho_0\rho_1\rho_2\rho_9\rho_{10}$, $\rho_0\rho_1\rho_3\rho_4\rho_6\rho_7\rho_9\rho_{10}$, $\rho_0\rho_2\rho_3\rho_4\rho_6\rho_8\rho_9\rho_{10}$, $\rho_1\rho_2\rho_3\rho_4\rho_7\rho_8\rho_9\rho_{10}$,
$\rho_1\rho_2\rho_4\rho_5\rho_7\rho_8\rho_{10}\rho_{11}$, $\rho_0\rho_1\rho_2\rho_9\rho_{11}$, $\rho_0\rho_1\rho_2\rho_{10}\rho_{11}$, $\rho_3\rho_4\rho_5\rho_6\rho_7$,
$\rho_0\rho_1\rho_3\rho_5\rho_6\rho_7\rho_9\rho_{11}$, $\rho_0\rho_1\rho_2\rho_5\rho_6\rho_7\rho_{10}\rho_{11}$, $\rho_3\rho_4\rho_5\rho_6\rho_8$, $\rho_0\rho_2\rho_3\rho_5\rho_6\rho_8\rho_9\rho_{11}$,
$\rho_0\rho_2\rho_4\rho_5\rho_6\rho_8\rho_{10}\rho_{11}$, $\rho_3\rho_4\rho_5\rho_7\rho_8$, $\rho_1\rho_2\rho_3\rho_5\rho_7\rho_8\rho_9\rho_{10}\rho_{11}$.

We listed the cones by giving the rays generating them. For instance, $\rho_0\rho_1\rho_2\rho_9\rho_{10}$ stands for the cone Cone $(\rho_0,\rho_1,\rho_2,\rho_9,\rho_{10})$. Note here that some of these maximal cones contain more rays than the others. So, as described in Remark 2.9, we want a MPCP-resolution of the variety associated to the above fan. To do this, we subdivide each of the maximal cones that has more than 5 rays. This procedure involves choice, as each cone can be subdivided in 24 ways (being a total of 24^9 possible choices!). However, all these choices are related by GIT, so any choice gives us a mirror family, all of which are birational. Following this procedure, the Table 1 (see below) gives the 42 maximal cones in the fan corresponding to a MPCP-resolution of the variety associated to the fan Σ'_∇. Define the fan Σ_∇ to be the fan consisting of those 42 5-dimensional cones and all of their faces. Determining the variety X_∇ explicitly is not straightforward, but also not necessary for our purposes, so long as we have the fan Σ_∇. ■

For $i = 0,\dots,11$, call D'_i the torus-invariant divisor on X_∇ corresponding to the ray ρ_i of Σ_Δ. Let $D'_a = D'_0 + D'_1 + D'_2 + D'_9 + D'_{10} + D'_{11}$ and $D'_b = D'_3 + D'_4 + D'_5 + D'_6 + D'_7 + D'_8$.

Corollary 3.4. Let $\Sigma_{\nabla,D'_a,D'_b}$ be the fan with rays $\overline{\rho_0},\dots,\overline{\rho_{11}},\tau_1,\tau_2$, and cones over those rays inherited from Σ_∇. Then $\Sigma_{\nabla,D'_a,D'_b}$ is a fan corresponding to $\text{tot}(\mathcal{O}_{X_\nabla}(-D'_b) \oplus \mathcal{O}_{X_\nabla}(-D'_a))$.

Proof. Apply Proposition 2.11 twice to get the result (recalling that we can do this by Proposition 2.12). ■

3.2 Libgober and Teitelbaum's Mirror

We now recall the family Libgober and Teitelbaum give as a mirror to the generic complete intersection of two cubics in $\mathbb{P}^5$. To start, define $V_\lambda \subseteq \mathbb{P}^5$ to be the vanishing set of the following two polynomials:

$$Q_{1,\lambda} = x_0^3 + x_1^3 + x_2^3 - 3\lambda x_3 x_4 x_5, \qquad Q_{2,\lambda} = x_3^3 + x_4^3 + x_5^3 - 3\lambda x_0 x_1 x_2. \tag{5}$$

For generic λ, this gives a smooth complete intersection in $\mathbb{P}^5$, which is a Calabi–Yau threefold.

Let ζ_n denote a primitive n-th root of unity. Let $\alpha, \beta, \delta, \varepsilon \in \mathbb{Z} \pmod 3$ and $\mu \in \mathbb{Z} \pmod 9$ with $3\mu = \alpha + \beta = \delta + \varepsilon$. Define the diagonal matrix

$$g_{\alpha,\beta,\delta,\epsilon,\mu} := \operatorname{diag}\left(\zeta_3^\alpha \zeta_9^\mu,\ \zeta_3^\beta \zeta_9^\mu,\ \zeta_9^\mu,\ \zeta_3^{-\delta}\zeta_9^{-\mu},\ \zeta_3^{-\epsilon}\zeta_9^{-\mu},\ \zeta_9^{-\mu}\right)$$

and let $G_{81} \subset PGL(5,\mathbb{C})$ denote the order 81 group generated by the $g_{\alpha,\beta,\delta,\varepsilon,\mu}$. Note that G_{81} acts on $\mathbb{P}^5$ by restricting the natural action of $PGL(5,\mathbb{C})$ on $\mathbb{P}^5$. The polynomials $Q_{1,\lambda}, Q_{2,\lambda}$ are invariant with respect to the action of G_{81}, hence G_{81} acts on V_λ.

Note that G_{81} is of isomorphism type $(\mathbb{Z}/3\mathbb{Z})^2 \times (\mathbb{Z}/9\mathbb{Z})$ and can be generated by $(\zeta_3, \zeta_3^{-1}, 1, 1, 1, 1)$, $(1, 1, 1, \zeta_3^{-1}, \zeta_3, 1)$ and $(\zeta_9, \zeta_9^4, \zeta_9, \zeta_9^{-1}, \zeta_9^{-4}, \zeta_9^{-1})$.

Let $V_{LT,\lambda}$ be the quotient of V_λ by the action of G_{81} and let $W_{LT,\lambda}$ be a minimal resolution of singularities of $V_{LT,\lambda}$, which is a Calabi–Yau manifold.

3.3 Expressing Libgober–Teitelbaum torically

In the following, we aim to give a toric description of $V_{LT,\lambda}$. First we give a fan for the toric variety $X_{LT} := \mathbb{P}^5/G_{81}$ and then employ methods of §7.3 of [7] to construct a vector bundle over X_{LT} that has the global section $Q_{1,\lambda} \oplus Q_{2,\lambda}$.

Proposition 3.5. Consider the 1-dimensional cones $\rho_0, \ldots, \rho_5$ with corresponding primitive generators

$$\begin{aligned} u_{\rho_0} &= (3,0,0,-1,-1), & u_{\rho_1} &= (0,3,0,-1,-1), & u_{\rho_2} &= (0,0,3,-1,-1),\\ u_{\rho_3} &= (-1,-1,-1,3,0), & u_{\rho_4} &= (-1,-1,-1,0,3), & u_{\rho_5} &= (-1,-1,-1,0,0). \end{aligned}$$

Consider the collection $\mathcal{C}$ of sets of the form

$$\{\rho_i \mid i \in I, I \subseteq \{0,\ldots,5\}, |I| = 5\}.$$

Let $\Sigma_{LT} \subseteq M_{\mathbb{R}}$ be the fan consisting of maximal cones

$$\{\mathrm{Cone}(C) | C \in \mathcal{C}\}$$

and all their faces.

Then the toric stack associated to Σ_{LT} is the stack corresponding to the Libgober–Teitelbaum construction, $\mathcal{X}_{LT} = [\mathbb{C}^6 \setminus \{0\} / \left(\mathbb{C}^* \times G_{81}\right)]$, with $\mathbb{C}^*$ acting by $(\lambda x_0, \ldots, \lambda x_5) \sim (x_0, \ldots, x_5)$ and G_{81} acting as described above in § 3.2.

Proof. We use the Cox construction described in §2.1. By Lemma 2.1, we obtain the following system of equations characterising elements of $G := G_\Sigma$

$$t_3 t_4 t_5 = t_0^3 \tag{6}$$

$$t_3 t_4 t_5 = t_1^3 \tag{7}$$

$$t_3 t_4 t_5 = t_2^3 \tag{8}$$

$$t_0 t_1 t_2 = t_3^3 \tag{9}$$

$$t_0 t_1 t_2 = t_4^3 \tag{10}$$

First, we note that we have a copy of $\mathbb{C}^*$ in G, given by $\{t \cdot (1, 1, 1, 1, 1, 1) \mid t \in \mathbb{C}^*\}$, so to compute G we consider the group H of cosets of $\mathbb{C}^*$. We will explicitly describe H and subsequently use the direct product theorem to compute G. Consider an element $(t_0, \ldots, t_5) \in G$. By an appropriate choice of coset representative of $(t_0, t_1, t_2, t_3, t_4, t_5) \cdot \mathbb{C}^*$, we may assume $\prod_{i=0}^{5} t_i = 1$.

Using equations (6), (7), and (8), we have $t_0^3 = t_1^3 = t_2^3$, and thus $t_0 = \zeta_3^{\alpha} t_2, t_1 = \zeta_3^{\beta} t_2$ for some $\alpha, \beta \in \mathbb{Z}_3$. Using equations (6)–(10), we have that $t_3^3 t_4^3 t_5^3 = t_0^3 t_1^3 t_2^3 = t_3^6 t_4^3 = t_3^3 t_4^6$, which implies

$$t_5^3 = t_3^3 = t_4^3. \tag{11}$$

Hence, similarly to above, we obtain $t_3 = \zeta_3^{-\delta} t_5, t_4 = \zeta_3^{-\varepsilon} t_5$ for some $\delta, \varepsilon \in \mathbb{Z}_3$.
By combining (8), (9), and (10), we obtain

$$t_2^3 t_5^3 = t_0 t_1 t_2 t_3 t_4 t_5 = 1. \tag{12}$$

Equation (12) implies $t_5^3 = (t_2^{-1})^3$, thus $t_5 = \zeta_3^{\nu} \cdot t_2^{-1}$ for some $\nu \in \mathbb{Z}_3$. Using $t_3 = \zeta_3^{-\delta} t_5$ and $t_4 = \zeta_3^{-\varepsilon} t_5$ and equation (8), we obtain

$$t_2^3 = t_3 t_4 t_5 = t_5^3 \zeta_3^{-(\delta+\varepsilon)} = t_2^{-3} \zeta_3^{-(\delta+\varepsilon)}.$$

Hence $t_2^{18} = 1$. So we can write $t_2 = \zeta_{18}^{l}$ for some $l \in \mathbb{Z}_{18}$.

We now claim that t_2 can be assumed to be a ninth root of unity and t_5 to be its inverse, that is, $t_2 = \zeta_9^{\mu}, t_5 = \zeta_9^{-\mu}$ for some $\mu \in \mathbb{Z}_9$. Indeed, note that $(\zeta_6, \ldots, \zeta_6) \in (1,1,1,1,1,1) \cdot \mathbb{C}^* \subseteq G$, so we can scale an element $(t_0, \ldots, t_5) \in G$ by sixth roots of unity, leaving the product $\prod_{i=1}^{6} t_i$ invariant. The claim follows by multiplication with an appropriate sixth root of unity.

Expressing all the t_i in terms of t_2, the assumption $1 = \prod_{i=1}^{6} t_i$ implies $1 = \zeta_3^{\alpha+\beta-\delta-\varepsilon}$, or, equivalently,

$$\alpha + \beta = \delta + \varepsilon \pmod 3.$$

Finally, using (8) gives $\zeta_9^{3\mu} = \zeta_3^{-\delta+\varepsilon} \zeta_9^{-3\mu}$ and therefore $\zeta_9^{3\mu} = \zeta_3^{\delta+\varepsilon}$. Thus H, the group of cosets of $\mathbb{C}^*$, is isomorphic to G_{81}, where G_{81} is the same group described in §2. In particular, all elements of G are of the form $g \cdot \lambda$ with $g \in G_{81}$, $\lambda \in (1,1,1,1,1,1) \cdot \mathbb{C}^*$ and $G_{81} \cap \{(1,1,1,1,1,1) \cdot \lambda | \lambda \in \mathbb{C}^*\} = \{(1,1,1,1,1,1)\}$. Hence, by the direct product theorem, $G \cong \mathbb{C}^* \times G_{81}$.

The Cox fan of Σ_{LT} can be described as follows. It has six rays $e_{\rho_0}, \ldots, e_{\rho_5}$. It is straightforward to see that the maximal cones are all 5-dimensional cones generated by any 5 of the rays above. Therefore, we obtain $U_{\Sigma_{LT}} = \mathbb{A}^6 \setminus \{0\}$. Thus, the Cox stack associated to Σ_{LT} is

$$\mathcal{X}_{LT} = [U_{\Sigma_{LT}}/G] = [\mathbb{C}^6 \setminus \{0\} / \left(\mathbb{C}^* \times G_{81}\right)],$$

with the prescribed action, as required. ∎

Remark 3.6. We note that by Theorem 2.5 the coarse moduli space of the stack $\mathcal{X}_{LT}$ is X_{LT}, since Σ_{LT} is simplicial.

Starting with the fan Σ_{LT} of X_{LT}, we apply Proposition 2.11 twice to construct a vector bundle. Let D_i be the Weil divisor corresponding to the ray ρ_i in Σ_{LT}. Let $D_a = D_0 + D_1 + D_2$ and $D_b = D_3 + D_4 + D_5$.

Corollary 3.7. Denote by the rays $\overline{\rho_0}, \dots, \overline{\rho_5}, \tau_1$ and τ_2 the rays (These are the same as on page 40.) generated by the primitive generators:

$$\begin{aligned}
&u_{\overline{\rho_0}} = (3,0,0,-1,-1,0,1), && u_{\overline{\rho_1}} = (0,3,0,-1,-1,0,1), && u_{\overline{\rho_2}} = (0,0,3,-1,-1,0,1),\\
&u_{\overline{\rho_3}} = (-1,-1,-1,3,0,1,0), && u_{\overline{\rho_4}} = (-1,-1,-1,0,3,1,0), && u_{\overline{\rho_5}} = (-1,-1,-1,0,0,1,0),\\
&u_{\tau_1} = (0,0,0,0,0,1,0), && u_{\tau_2} = (0,0,0,0,0,0,1).
\end{aligned}$$

Consider the collection $\mathcal{S}$ of sets of the form

$$\{\overline{\rho_i} \mid i \in I, I \subseteq \{0,\dots,5\}, |I| = 5\} \cup \{\tau_1, \tau_2\}.$$

Let Σ_{LT,D_a,D_b} be the fan in $M_{\mathbb{R}} \oplus \mathbb{R}^2$ consisting of the maximal cones

$$\{\mathrm{Cone}(S) | S \in \mathcal{S}\}$$

and all their faces. Then:

(a) Σ_{LT,D_a,D_b} is a fan corresponding to $\mathrm{tot}(\mathcal{O}_{X_{LT}}(-D_b) \oplus \mathcal{O}_{X_{LT}}(-D_a))$;
(b) The vector bundle $\mathcal{O}_{X_{LT}}(D_b) \oplus \mathcal{O}_{X_{LT}}(D_a)$ has the global section $Q_{1,\lambda} \oplus Q_{2,\lambda}$.

Proof. Applying Proposition 2.11 twice yields (a).

We now turn to (b) and show that $Q_{1,\lambda} \in \Gamma(X_{LT}, \mathcal{O}_{X_{LT}}(D_b))$ and $Q_{2,\lambda} \in \Gamma(X_{LT}, \mathcal{O}_{X_{LT}}(D_a))$. We start by noting that on X_{LT} we have $\mathrm{div}(x_i^3) = 3D_i$, so $\mathrm{div}(x_i^3) - 3D_i \geq 0$, that is, $x_i^3 \in \Gamma(X_{LT}, \mathcal{O}_{X_{LT}}(3D_i))$. Similarly, $x_0x_1x_2 \in \Gamma(X_{LT}, \mathcal{O}_{X_{LT}}(D_a))$ and $x_3x_4x_5 \in \Gamma(X_{LT}, \mathcal{O}_{X_{LT}}(D_b))$.

To show the linear equivalence of two divisors, it suffices to consider their difference and show it is principal. We recall that $\mathrm{div}(\chi^n) = \sum_{\rho \in \Sigma(1)} \langle u_\rho, n \rangle D_\rho$, corresponding to the map ι in the exact sequence (1). So, for instance $3D_1 - 3D_0 = \mathrm{div}(x_0^{-3}x_1^3)$, which is the character associated to the lattice point $(-1,1,0,0,0)$. Hence, $3D_1 - 3D_0 = 0$ in $\mathrm{Cl}(X_{LT})$, that is, $3D_0 \sim 3D_1$. Similarly, $3D_1 \sim 3D_2$ and $3D_3 \sim 3D_4 \sim 3D_5$. Using the lattice points $(-1,0,0,0,0)$ and $(0,0,0,-1,0)$, respectively, we also see that $3D_0 \sim D_b$ and $3D_3 \sim D_a$.

Thus

$$\mathcal{O}_{X_{LT}}(3D_0) \simeq \mathcal{O}_{X_{LT}}(3D_1) \simeq \mathcal{O}_{X_{LT}}(3D_2) \simeq \mathcal{O}_{X_{LT}}(D_b)$$

and

$$\mathcal{O}_{X_{LT}}(3D_3) \simeq \mathcal{O}_{X_{LT}}(3D_4) \simeq \mathcal{O}_{X_{LT}}(3D_5) \simeq \mathcal{O}_{X_{LT}}(D_a),$$

implying $Q_{2,\lambda} \in \Gamma(X_{LT}, \mathcal{O}_{X_{LT}}(D_a))$ and $Q_{1,\lambda} \in \Gamma(X_{LT}, \mathcal{O}_{X_{LT}}(D_b))$, as required. ■

3.3.1 *Intuition for constructing X_{LT} torically*

We now explain how we found an explicit description for the fan Σ_{LT}. We start by considering the standard fan $\Sigma_{\mathbb{P}^5} \subseteq N_{\mathbb{R}}$ for $\mathbb{P}^5$ in the standard basis. It is the fan consisting of the cones generated by any proper subset of the six rays $\nu_0, \ldots, \nu_5$ with primitive generators

$$\begin{aligned} u_{\nu_0} &= (1,0,0,0,0), & u_{\nu_1} &= (0,1,0,0,0), & u_{\nu_2} &= (0,0,1,0,0),\\ u_{\nu_3} &= (0,0,0,1,0), & u_{\nu_4} &= (0,0,0,0,1), & u_{\nu_5} &= (-1,-1,-1,-1,-1). \end{aligned}$$

Denote by $T_0, \ldots, T_5$ the six primitive Weil divisors corresponding to the rays $u_{\nu_0}, \ldots, u_{\nu_5}$, respectively. Then

$$\mathcal{O}(-\underbrace{(T_0+T_1+T_2)}_{:=T_a}) = \mathcal{O}(-\underbrace{(T_3+T_4+T_5)}_{:=T_b}) = \mathcal{O}(-3),$$

and we can use the methods of §7.3 of [7] again to construct a fan of $\operatorname{tot}(\mathcal{O}_{\mathbb{P}^5}(-3) \oplus \mathcal{O}_{\mathbb{P}^5}(-3))$. This yields the fan $\Sigma_{\mathbb{P}^5,T_a,T_b}$ in $N_{\mathbb{R}} \oplus \mathbb{R}^2$ with the 8 rays $\overline{\nu_0}, \ldots, \overline{\nu_5}, \tau_1$ and τ_2 having primitive ray generators

$$\begin{aligned} u_{\overline{\nu_0}} &= (1,0,0,0,0,1,0), & u_{\overline{\nu_4}} &= (0,0,0,0,1,0,1),\\ u_{\overline{\nu_1}} &= (0,1,0,0,0,1,0), & u_{\overline{\nu_5}} &= (-1,-1,-1,-1,-1,0,1),\\ u_{\overline{\nu_2}} &= (0,0,1,0,0,1,0), & u_{\tau_1} &= (0,0,0,0,0,1,0),\\ u_{\overline{\nu_3}} &= (0,0,0,1,0,0,1), & u_{\tau_2} &= (0,0,0,0,0,0,1). \end{aligned} \tag{13}$$

The fan $\Sigma_{\mathbb{P}^5,T_a,T_b}$ is the star subdivision of $\operatorname{Cone}(u_{\overline{\nu_0}}, \ldots, u_{\overline{\nu_5}}, u_{\tau_1}, u_{\tau_2})$ along u_{τ_1} and u_{τ_2} (noting the abuse of notation by which u_{τ_i} represent the same vector in both lattices M, N). The dual cone to $\Sigma_{\mathbb{P}^5,T_a,T_b}$ in $M_{\mathbb{R}} \oplus \mathbb{R}^2$ is spanned by the 12 rays $\overline{\rho_0}, \ldots, \overline{\rho_{11}}$ defined in § 3.1 (page 39).

We recall that each lattice point in the interior of the dual cone corresponds to a global function of $X_{\Sigma_{\mathbb{P}^5},T_a,T_b}$ by associating m to the monomial

$$x^m := \prod_{\rho \in \Sigma_{\mathbb{P}^5,T_1,T_2}(1)} x_\rho^{\langle m, u_\rho \rangle}.$$

Now a section $s_1 \oplus s_2 \in \Gamma(\mathbb{P}^5, \mathcal{O}(3) \oplus \mathcal{O}(3))$ will correspond to a global function on $\operatorname{tot}(\mathcal{O}(-3) \oplus \mathcal{O}(-3))$ of the form $u_1 s_1 + u_2 s_2$, where u_i is the variable corresponding to u_{τ_i}. Recalling the polynomials Q_i from (5) in §3.2, we would like to express the global function $F := u_2 Q_{1,\lambda} + u_1 Q_{2,\lambda}$ as a linear combination of global functions of the form x^m.

We do this by finding the lattice points in the dual cone corresponding to each monomial in F.

By splitting it up into its monomials, $u_2 Q_{1,\lambda}$ corresponds to the 4 points $(3,0,0,-1,-1,0,1)$, $(0,3,0,-1,-1,0,1)$, $(0,0,3,-1,-1,0,1)$, and $(0,0,0,0,0,0,1)$. Similarly, $u_1 Q_{2,\lambda}$ corresponds to the points $(-1,-1,-1,3,0,1,0)$, $(-1,-1,-1,0,3,1,0)$, $(-1,-1,-1,0,0,1,0)$, and $(0,0,0,0,0,1,0)$.

We find that these 8 points are the primitive generators for the rays of Σ_{LT,D_a,D_b} (see Corollary 3.7).

Quotienting $M_{\mathbb{R}} \oplus \mathbb{R}^2$ by the rays associated to the bundle coordinates (i.e., the lattice points that are the elements of the dual basis dual to u_{τ_1} and u_{τ_2}) corresponds to a toric morphism $X_{\Sigma_{LT,D_a,D_b}} \to X_{\Sigma_{LT}}$. We emphasize that the dual cone to $\mathrm{Cone}(\Sigma_{\mathbb{P}^5,T_a,T_b}(1))$ is given by $\mathrm{Conv}(u_{\overline{\rho 0}}, \dots, u_{\overline{\rho 11}})$. Here, we take a subcone generated by a subset of $\{u_{\overline{\rho 0}}, \dots, u_{\overline{\rho 11}}\}$.

3.3.2 *Expressing the zero locus of $Q_{1,\lambda}, Q_{2,\lambda}$*

We remark that the cone $|\Sigma_{LT,D_a,D_b}|$ is not a reflexive Gorenstein cone, hence the Batyrev–Borisov construction does not apply to it.

The variety $V_{LT,\lambda} \subseteq X_{LT}$ is the zero locus of the polynomials $Q_{1,\lambda}, Q_{2,\lambda}$, where $Q_{1,\lambda} \oplus Q_{2,\lambda}$ is a section of the vector bundle constructed above in Corollary 3.7. Proceeding in the same way as in §3.3.1, we consider lattice points on the cone $|\Sigma_{LT,D_a,D_b}|^\vee \subseteq N_{\mathbb{R}} \oplus \mathbb{R}^2$ to get global functions of $\mathcal{X}_{\Sigma_{LT,D_a,D_b}}$. The cone $|\Sigma_{LT,D_a,D_b}|^\vee$ is the cone over the convex hull of the following 12 points:

$$\begin{array}{lll}
(1,0,0,0,0,1,0), & (0,1,0,0,0,1,0), & (0,0,1,0,0,1,0), \\
(0,0,0,1,0,0,1), & (0,0,0,0,1,0,1), & (2,-1,-1,0,0,0,3), \\
(-1,2,-1,0,0,0,3), & (-1,-1,2,0,0,0,3), & (1,1,1,3,0,3,0), \\
(1,1,1,0,3,3,0), & (-1,-1,-1,-1,-1,0,1), & (-2,-2,-2,-3,-3,3,0).
\end{array}$$

The points corresponding to the monomials in $u_1 Q_{1,\lambda} + u_2 Q_{2,\lambda}$, and hence to the section $Q_{1,\lambda} \oplus Q_{2,\lambda}$, are the lattice points $u_{\overline{\nu_i}}$ and u_{τ_i} in (13). Later on, describing V_{LT} by these 8 points will allow us to work with $\mathrm{D^b}(\mathrm{coh}\, V_{LT})$, using results in [11].

Remark 3.8. In their recent work [23, 24], Rossi proposes a generalisation of the Batyrev–Borisov mirror construction, called framed duality (f-duality). f-duality gives an algorithm to obtain mirror candidates of hypersurfaces and complete intersections in toric varieties. Applying f-duality to $V_{LT} \subset \mathbb{P}^5/G_{81}$ produces $V_\lambda \subset \mathbb{P}^5$, which in turn gives the same mirror as the Batyrev–Borisov construction when applying f-duality to it.

Theorem 1.1 suggests that different mirror candidates obtained via f-duality may be derived equivalent and prompts the question under what conditions this is the case.

4 A Derived Equivalence Between the Constructions by Libgober–Teitelbaum and Batyrev–Borisov

Here, we will prove the main result, Theorem 1.1.

4.1 Picking a partial compactification

Looking at the dual of the fan Σ_{LT,D_a,D_b} as in Corollary 3.7, we recall from §3.3.2 that the global function $u_1 Q_{1,\lambda} + u_2 Q_{2,\lambda}$ corresponds to the points

$$\begin{aligned}&(1,0,0,0,0,1,0), \quad (0,1,0,0,0,1,0), \quad (0,0,1,0,0,1,0),\\&(0,0,0,1,0,0,1), \quad (0,0,0,0,1,0,1), \quad (-1,-1,-1,-1,-1,0,1),\\&(0,0,0,0,0,0,1), \quad (0,0,0,0,0,1,0).\end{aligned}$$

Consider the GKZ fan of $\mathrm{tot}(\mathcal{O}_{X_\nabla}(-D'_b) \oplus \mathcal{O}_{X_\nabla}(-D'_a))$. We note that the chambers of this GKZ fan correspond to regular triangulations of the polytope $\mathfrak{P} = \mathrm{Conv}(\mathfrak{C})$, where $\mathfrak{C}$ is the collection of the following 14 points:

$$\begin{aligned}P_0 &= (3,0,0,-1,-1,0,1), & P_6 &= (2,-1,-1,0,0,1,0),\\P_1 &= (0,3,0,-1,-1,0,1), & P_7 &= (-1,2,-1,0,0,1,0),\\P_2 &= (0,0,3,-1,-1,0,1), & P_8 &= (-1,-1,2,0,0,1,0),\\P_3 &= (-1,-1,-1,3,0,1,0), & P_9 &= (0,0,0,2,-1,0,1),\\P_4 &= (-1,-1,-1,0,3,1,0), & P_{10} &= (0,0,0,-1,2,0,1),\\P_5 &= (-1,-1,-1,0,0,1,0), & P_{11} &= (0,0,0,-1,-1,0,1),\\S_1 &= (0,0,0,0,0,1,0), & S_2 &= (0,0,0,0,0,0,1).\end{aligned}$$

In the (regular) triangulations of $\mathfrak{P}$, we look for a subtriangulation corresponding to Σ_{LT,D_a,D_b}, as then we obtain a partial compactification of $\mathrm{tot}(\mathcal{O}_{X_{LT}}(-D_b) \oplus \mathcal{O}_{X_{LT}}(-D_a))$ from Corollary 3.7.

Proposition 4.1. There exists a chamber σ_{LT} in the GKZ fan of $\mathrm{tot}(\mathcal{O}_{X_\nabla}(-D'_b) \oplus \mathcal{O}_{X_\nabla}(-D'_a))$ (from Corollary 3.4) so that the triangulation $\mathcal{T}$ corresponding to the chamber σ_{LT} (in the sense of 2.17) has the following properties:

- $\mathcal{T}$ contains the following set of simplices, listed via their vertices:

$$\mathcal{T}_0 := \left\{ \{P_i, S_1, S_2 \mid i \in I\} \mid I \subset \{0,2,\dots,5\}, |I| = 5 \right\}.$$

- Any simplex $T \in \mathcal{T} \setminus \mathcal{T}_0$ fulfills either of the two following conditions:
  A. $S_1, P_6, P_7, P_8 \notin T$ and $\exists\, 3 \le j \le 6$ such that $P_j, P_{6+j} \notin T$.
  B. $S_2, P_9, P_{10}, P_{11} \notin T$ and $\exists\, 0 \le j \le 2$ such that $P_j, P_{6+j} \notin T$.

Moreover, the toric variety X_Σ corresponding to the chamber σ_{LT} is a partial compactification of the variety $\mathrm{tot}(\mathcal{O}_{X_{LT}}(-D_b) \oplus \mathcal{O}_{X_{LT}}(-D_a))$ from Corollary 3.7.

The first property of $\mathcal{T}$ means that the associated variety X_Σ is a partial compactification of X_{LT}, so proving the existence of the triangulation $\mathcal{T}$ is sufficient to prove the Proposition. The second property of $\mathcal{T}$ is not a natural one to consider, but will become necessary to apply results from §2.5.

The proposition can be checked via a simple SAGE program [26] using the TOPCOM package [22]; however, we include an explicit proof on how such a triangulation can be constructed.

To prove the proposition, we break the statement up into 3 steps.

Step 1: We start by defining an explicit regular polyhedral subdivision $\mathcal{S}$ of $\mathfrak{P}$ containing $\mathcal{T}_0$.
Step 2: We prove that the polyhedral subdivision $\mathcal{S}$ can be refined to a regular triangulation $\mathcal{T}$ of $\mathfrak{P}$ containing $\mathcal{T}_0$.
Step 3: We show that any regular triangulation obtained this way fulfills the conditions outlined in the Proposition.

4.1.1 *Step 1:*

We note that $\mathcal{T}_0$ is a regular triangulation of the set of points $P_0, \dots, P_5, S_1, S_2$. It is in fact a star subdivision with respect to S_1, S_2 of the convex hull $\mathrm{Conv}(P_0, \dots, P_5, S_1, S_2)$. Indeed, an example of an explicit weight function w giving the triangulation $\mathcal{T}_0$ is $w(S_1) = w(S_2) = 1, w(P_i) = 2$ for $0 \le i \le 5$. To complete Step 1, we extend this weight function to all 14 points of $\mathfrak{C}$.

Consider the weight function $w(P_i) = 2$ for $0 \le i \le 5$, $w(S_1) = w(S_2) = 1$ and $w(P_j) = 5$ for $6 \le j \le 11$. The convex hull of the points

$$Z_i = (P_i, w(P_i)), \quad R_j = (S_j, w(S_j)), \quad (0 \le i \le 11,\ j = 1, 2)$$

then forms a polyhedron $\mathcal{Q}$ in $\mathbb{R}^8$. To obtain the regular subdivision of $\mathfrak{P}$ corresponding to the weight function w, we need to project the lower facets of the polyhedron $\mathcal{Q}$ down to $\mathbb{R}^7$ along the last coordinate. A lower facet is defined to be a facet of $\mathcal{Q}$ where the inward pointing normal has a positive last coordinate.

Table 2 Dictionary of points contained in each lower facet of $\mathcal{Q}$

Facet	contains
F_0	$Z_1, \dots, Z_5, R_1, R_2$
$\vdots$	$\vdots$
F_5	$Z_0, \dots, Z_4, R_1, R_2$
F_6	$Z_1, \dots, Z_5, R_1, Z_7, Z_8$
F_7	$Z_0, Z_2, \dots, Z_5, R_1, Z_6, Z_8$
F_8	$Z_0, Z_1, Z_3, Z_4, Z_5, R_1, Z_6, Z_7$
F_9	$Z_0, Z_1, Z_2, Z_4, Z_5, R_2, Z_{10}, Z_{11}$
F_{10}	$Z_0, Z_1, Z_2, Z_3, Z_5, R_2, Z_9, Z_{11}$
F_{11}	$Z_0, \dots, Z_4, R_2, Z_9, Z_{10}$.

We claim there are exactly 12 lower facets of $\mathcal{Q}$. We write each lower facet F_i in the form $u_i \cdot x + a_i = 0$ where u_i is the inward pointing normal of the i^{th} facet. Take H_i to be the halfspace corresponding to the lower facet F_i, that is, the halfspace given by $u_i \cdot x + a \geq 0$. The normals and additive constants are:

- $H_0 : (5, -1, -1, 0, 0, 0, 0, 3)x - 3 \geq 0$
- $H_1 : (-1, 5, -1, 0, 0, 0, 0, 3)x - 3 \geq 0$
- $H_2 : (-1, -1, 5, 0, 0, 0, 0, 3)x - 3 \geq 0$
- $H_3 : (1, 1, 1, 6, 0, 0, 0, 3)x - 3 \geq 0$
- $H_4 : (1, 1, 1, 0, 6, 0, 0, 3)x - 3 \geq 0$
- $H_5 : (-5, -5, -5, -6, -6, 0, 0, 3)x - 3 \geq 0$
- $H_6 : (3, -1, -1, 0, 0, 0, 2, 1)x - 1 \geq 0$
- $H_7 : (-1, 3, -1, 0, 0, 0, 2, 1)x - 1 \geq 0$
- $H_8 : (-1, -1, 3, 0, 0, 0, 2, 1)x - 1 \geq 0$
- $H_9 : (1, 1, 1, 4, 0, 0, -2, 1)x + 1 \geq 0$
- $H_{10} : (1, 1, 1, 0, 4, 0, -2, 1)x + 1 \geq 0$
- $H_{11} : (-3, -3, -3, -4, -4, 0, -2, 1)x + 1 \geq 0$.

An easy computation shows that all 14 points lie in the intersection of the relevant half-spaces. This is a direct consequence of the fact that $\mathcal{Q} \subseteq H_i$ for $i = 0, \dots, 11$. Table 2 shows which points lie on each lower facet.

To obtain the polyhedral subdivision $\mathcal{S}$ of $\mathfrak{P}$ corresponding to the weight function w,

Table 3 Polyhedra in the regular subdivision

$\widehat{F}_0 =$	$\mathrm{Conv}(P_1, \ldots, P_5, S_1, S_2)$
$\vdots$	$\vdots$
$\widehat{F}_5 =$	$\mathrm{Conv}(P_0, \ldots, P_4, S_1, S_2)$
$\widehat{F}_6 =$	$\mathrm{Conv}(P_1, \ldots, P_5, S_1, P_7, P_8)$
$\widehat{F}_7 =$	$\mathrm{Conv}(P_0, P_2, \ldots, P_5, S_1, P_6, P_8)$
$\widehat{F}_8 =$	$\mathrm{Conv}(P_0, P_1, P_3, P_4, P_5, S_1, P_6, P_7)$
$\widehat{F}_9 =$	$\mathrm{Conv}(P_0, P_1, P_2, P_4, P_5, S_2, P_{10}, P_{11})$
$\widehat{F}_{10} =$	$\mathrm{Conv}(P_0, \ldots, P_3, P_5, S_2, P_9, P_{11})$
$\widehat{F}_{11} =$	$\mathrm{Conv}(P_0, \ldots, P_4, S_2, P_9, P_{10})$.

we now project these facets down to $\mathbb{R}^7$ along the last coordinate. Denoting by $\widehat{F}_i$ the polyhedron obtained by projecting the facet F_i, we obtain the set of 12 polyhedra given in Table 3. We note here that when projecting, all points that lied on the facet F_i lie in the polyhedron $\widehat{F}_i$, by convexity of the polyhedron $\mathcal{Q}$ in $\mathbb{R}^8$.

It remains to show that the above collection F_i contains all the lower facets of $\mathcal{Q}$. Showing that there is no other lower facet of $\mathcal{Q}$ apart from $F_0, \ldots, F_{11}$ is equivalent to showing that $\bigcup F_i + \langle (0, \ldots, 0, 1) \rangle_{\mathbb{R}_{\geq 0}}$ contains the entire polyhedron $\mathcal{Q}$. Since all vertices of $\mathcal{Q}$ lie inside each half-space H_i, it suffices to show that the union of the projections $\widehat{F}_i$ contains the convex hull of $P_0, \ldots, P_{11}, S_1, S_2$, that is, contains $\mathfrak{P}$. This is equivalent to saying that they give a polyhedral subdivision (regularity is given by construction).

So we aim to prove the following claim.

Lemma 4.2. For $\widehat{F}_i$ and $\mathfrak{P}$ as above, we have $\bigcup_{i=0}^{11} \widehat{F}_i = \mathfrak{P}$.

To prove Lemma 4.2, we will need the following result.

Lemma 4.3. Suppose we are given a set of m inequalities $L_j \leq R_j$ with $\sum_{j=1}^m L_j \leq C \leq \sum_{j=1}^m R_j$, then there exists an m-tuple of real numbers a_j such that $L_j \leq a_j \leq R_j$ and $\sum_{j=1}^m a_j = C$.

Proof. To show that the claim holds, we define $a_j(x) = L_j + x(R_j - L_j)$. This is a linear function such that, for all $x \in [0, 1]$, $L_j \leq a_j(x) \leq R_j$. Define $f(x) = \sum a_j(x)$. f is itself linear and thus continuous in x, with $f(0) = \sum_{j=1}^{m} L_j \leq C \leq \sum_{j=1}^{m} R_j = f(1)$. By the intermediate value theorem, there is an $x_C \in [0,1]$ such that $f(x) = \sum_{j=1}^{m} a_j(x_C) = C$. Setting $a_j = a_j(x_C)$ gives the m-tuple, proving the claim. ■

Proof of Lemma 4.2. The first thing to note is that

$$\mathfrak{P} = \mathrm{Conv}(P_0, \dots, P_{11}, S_1, S_2) = \mathrm{Conv}(P_0, \dots, P_{11}).$$

So we will show that $\bigcup_{i=0}^{11} \widehat{F}_i = \mathrm{Conv}(P_0, \dots, P_{11})$.

We start by showing that $\bigcup_{i=0}^{5} \widehat{F}_i = \mathrm{Conv}(P_0, \dots, P_5, S_1, S_2)$, which is equivalent to saying that $\widehat{F}_0, \dots, \widehat{F}_5$ form a polyhedral subdivision of $\mathrm{Conv}(P_0, \dots, P_5, S_1, S_2)$.

The inclusion $\subseteq$ is immediate from Table 3, so it remains to check the opposite inclusion. Any point $X \in \mathrm{Conv}(P_0, \dots, P_5, S_1, S_2)$ can be written as $X = \sum_{i=0}^{5} \lambda_i P_i + \mu_1 S_1 + \mu_2 S_2$ for some $\lambda_i, \mu_j \in \mathbb{R}_{\geq 0}$ with $\sum \lambda_i + \mu_1 + \mu_2 = 1$. Note also that $\sum_{i=0}^{5} P_i = 3(S_1 + S_2)$. Now define j such that $\lambda_j = \min_{0 \leq i \leq 5}\{\lambda_i\}$. Then

$$\begin{aligned} X &= \sum_{i=0}^{5} (\lambda_i - \lambda_j) P_i + (3\lambda_j + \mu_1) S_1 + (3\lambda_j + \mu_2) S_2 \\ &= \sum_{\substack{0 \leq i \leq 5 \\ i \neq j}} (\lambda_i - \lambda_j) P_i + (3\lambda_j + \mu_1) S_1 + (3\lambda_j + \mu_2) S_2. \end{aligned}$$

Since $\lambda_j = \min_{0 \leq i \leq 5}\{\lambda_i\} \leq \lambda_i$ for $0 \leq i \leq 5$, we have that $(\lambda_i - \lambda_j) \geq 0$ for $0 \leq i \leq 5$. As $\lambda_i, \mu_1, \mu_2 \geq 0$, we also have $3\lambda_j + \mu_1, 3\lambda_j + \mu_2 \geq 0$. Also,

$$\sum_{\substack{0 \leq i \leq 5 \\ i \neq j}} (\lambda_i - \lambda_j) + (3\lambda_j + \mu_1) + (3\lambda_j + \mu_2) = \sum_{i=0}^{5} \lambda_i + \mu_1 + \mu_2 = 1,$$

and thus $X \in \widehat{F}_j$. This shows $\bigcup_{i=0}^{5} \widehat{F}_i = \mathrm{Conv}(P_0, \dots, P_5, S_1, S_2)$.

To show $\bigcup_{i=0}^{11} \widehat{F}_i = \mathfrak{P}$, we note again that the inclusion $\subseteq$ is immediate. For the opposite inclusion $\supseteq$, take a general point X in $\mathfrak{P}$. Then X can be written as $X = \sum_{i=0}^{11} \lambda_i P_i$ with $\lambda_i \geq 0$ for $0 \leq i \leq 11$ and $\sum_{i=0}^{11} \lambda_i = 1$.

Without loss of generality, assume that $(\lambda_6 + \lambda_7 + \lambda_8) \geq (\lambda_9 + \lambda_{10} + \lambda_{11})$ (the case where the inequality is reversed is analogous). We will now show that if $X \notin \bigcup_{i=0}^{5} \widehat{F}_i = \mathrm{Conv}(P_0, \dots, P_5, S_1, S_2)$, then $X \in \bigcup_{i=6}^{8} \widehat{F}_i$ (if the inequality had been reversed, then X would be in $\bigcup_{i=9}^{11} \widehat{F}_i$).

Let

$$\begin{aligned}
\nu_i = &\quad \lambda_i + \lambda_{6+i} - \tfrac{1}{3}\left((\lambda_6 + \lambda_7 + \lambda_8) - (\lambda_9 + \lambda_{10} + \lambda_{11})\right) && \text{for} 0 \le i \le 2, \\
\nu_i = &\quad \lambda_i + \lambda_{6+i} && \text{for} 3 \le i \le 5, \\
\mu_1 = &\quad \left((\lambda_6 + \lambda_7 + \lambda_8) - (\lambda_9 + \lambda_{10} + \lambda_{11})\right), \\
\mu_2 = &\quad 0.
\end{aligned}$$

Then

$$\sum_{i=0}^{5} \nu_i P_i + \mu_1 S_1 + \mu_2 S_2 = \sum_{i=0}^{11} \lambda_i P_i$$

and

$$\sum_{i=0}^{5} \nu_i + \mu_1 + \mu_2 = \sum_{i=0}^{11} \lambda_i = 1.$$

Note $\mu_1 \ge 0$ by assumption and $\mu_2 = 0$. Thus, if $\nu_i \ge 0$ for $0 \le i \le 5, X$ is expressed as an element of $\text{Conv}(P_0, \ldots, P_5, S_1, S_2) = \bigcup_{i=0}^{5} \widehat{F}_i$ using the above equations. Otherwise, we will claim that $X \in \bigcup_{i=6}^{8} \widehat{F}_i$. For $3 \le i \le 5$, we have $\nu_i \ge 0$ as both λ_i and λ_{6+i} are ≥ 0. We turn our attention to the ν_i for $i = 0, 1, 2$.

For $0 \le i \le 2$, $\nu_i \ge 0$ is equivalent to

$$\frac{1}{3}\left((\lambda_6 + \lambda_7 + \lambda_8) - (\lambda_9 + \lambda_{10} + \lambda_{11})\right) \le \lambda_i + \lambda_{6+i},$$

so the condition that all ν_i are non-negative is equivalent to

$$\frac{1}{3}\left(\left(\lambda_6 + \lambda_7 + \lambda_8\right) - (\lambda_9 + \lambda_{10} + \lambda_{11}\right)\Big) \le \min_{0 \le i \le 2} \{\lambda_i + \lambda_{6+i}\}.$$

Therefore, $X \in \bigcup_{i=0}^{5} \widehat{F}_i = \text{Conv}(P_0, \ldots, P_5, S_1, S_2)$ if

$$\frac{1}{3}\left(\left(\lambda_6 + \lambda_7 + \lambda_8\right) - (\lambda_9 + \lambda_{10} + \lambda_{11}\right)\Big) \le \min_{0 \le i \le 2} \{\lambda_i + \lambda_{6+i}\}.$$

Suppose this condition does not hold, that is,

$$\min_{0 \le i \le 2} \{\lambda_i + \lambda_{6+i}\} < \frac{1}{3}\left(\left(\lambda_6 + \lambda_7 + \lambda_8\right) - (\lambda_9 + \lambda_{10} + \lambda_{11}\right)\Big). \tag{14}$$

Without loss of generality, we may assume that $\lambda_0 + \lambda_6 = \min_{0 \le i \le 2}\{\lambda_i + \lambda_{6+i}\}$ (by symmetry, the other cases are analogous). We will show that $X \in \widehat{F}_6$. Any point Y in $\widehat{F}_6 = \text{Conv}\,(P_1, \ldots, P_5, S_1, P_7, P_8)$ can be written as

$$Y = \sum_{i=1}^{5} \nu_i P_i + \sum_{i=7}^{8} \nu_i P_i + \mu_1 S_1.$$

If we find ν_i, μ_1 such that this sum is equal to $\sum_{i=0}^{11} \lambda_i P_i = X$, we are done as we will have expressed X as an element of $\widehat{F}_6$.

Given a choice of real numbers α_1, α_2 with $\alpha_1 + \alpha_2 = 1$, define

$$\begin{aligned}
\nu_i &= \lambda_i + \alpha_i(3\lambda_0 + 2\lambda_6 + (\lambda_9 + \lambda_{10} + \lambda_{11})) - (\lambda_0 + \lambda_6) && \text{for } 1 \le i \le 2, \\
\nu_i &= \lambda_i + \lambda_{6+i} && \text{for } 3 \le i \le 5, \\
\mu_1 &= 3\lambda_0 + 3\lambda_6, \\
\nu_{6+i} &= \lambda_{6+i} + \alpha_i(-3\lambda_0 - 2\lambda_6 - (\lambda_9 + \lambda_{10} + \lambda_{11})) && \text{for } 1 \le i \le 2.
\end{aligned}$$

Substituting these values into the expression for Y gives

$$Y = \sum_{i=1}^{5} \nu_i P_i + \sum_{i=7}^{8} \nu_i P_i + \mu_1 S_1 = \sum_{i=0}^{11} \lambda_i P_i = X,$$

as well as

$$\sum \nu_i + \mu_1 = \sum_{i=0}^{11} \lambda_i = 1.$$

For this choice of ν_i's and μ_1 to define an element $Y \in \widehat{F}_6$, we require $\nu_i \ge 0$ for all i and $\mu_1 \ge 0$. We note that, as $\lambda_0, \lambda_6 \ge 0$, we have $\mu_1 \ge 0$.

Therefore, what remains to prove is that there exist $\alpha_1, \alpha_2 \in \mathbb{R}$ with $\alpha_1 + \alpha_2 = 1$ such that $\nu_i \ge 0$ for $i \in \{1, \ldots, 5, 7, 8\}$. For $i = 1, 2$, we can arrange the inequalities $\nu_i \ge 0$ and $\nu_{6+i} \ge 0$ to give

$$\frac{\lambda_0 + \lambda_6 - \lambda_i}{3\lambda_0 + 2\lambda_6 + (\lambda_9 + \lambda_{10} + \lambda_{11})} \le \alpha_i \le \frac{\lambda_{6+i}}{3\lambda_0 + 2\lambda_6 + (\lambda_9 + \lambda_{10} + \lambda_{11})}. \tag{15}$$

This works provided $3\lambda_0 + 2\lambda_6 + (\lambda_9 + \lambda_{10} + \lambda_{11}) \neq 0$ but if that term was zero, then by non-negativity of the λ_i we would have $\lambda_0 = \lambda_6 = \lambda_9 = \cdots = \lambda_{11} = 0$ and thus $X \in \widehat{F}_6$. So if there exists a pair (α_1, α_2) with (15) holding for $i = 1, 2$ and $\alpha_1 + \alpha_2 = 1$, then $X \in \widehat{F}_6$. Note that for all i,

$$\frac{\lambda_0 + \lambda_6 - \lambda_i}{3\lambda_0 + 2\lambda_6 + (\lambda_9 + \lambda_{10} + \lambda_{11})} \le \frac{\lambda_{6+i}}{3\lambda_0 + 2\lambda_6 + (\lambda_9 + \lambda_{10} + \lambda_{11})},$$

as $\lambda_0 + \lambda_6 = \min_{0 \le i \le 2}\{\lambda_i + \lambda_{6+i}\}$.

Furthermore,

$$\begin{aligned} 0 &\le \lambda_0+\lambda_1+\lambda_2+\lambda_9+\lambda_{10}+\lambda_{11} \\ \Leftrightarrow \quad 2\lambda_0+2\lambda_6 &\le 3\lambda_0+\lambda_1+\lambda_2+2\lambda_6+\lambda_9+\lambda_{10}+\lambda_{11} \\ \Leftrightarrow \quad \textstyle\sum_{i=1}^2 \lambda_0+\lambda_6-\lambda_i &\le 3\lambda_0+2\lambda_6+(\lambda_9+\lambda_{10}+\lambda_{11}) \\ \Leftrightarrow \quad \textstyle\sum_{i=1}^2 \frac{\lambda_0+\lambda_6-\lambda_i}{3\lambda_0+2\lambda_6+(\lambda_9+\lambda_{10}+\lambda_{11})} &\le 1. \end{aligned}$$

Lastly, we are given that $\lambda_0+\lambda_6 = \min_{0\le i\le 2}\{\lambda_i+\lambda_{6+i}\} \le \frac{1}{3}\left((\lambda_6+\lambda_7+\lambda_8)-(\lambda_9+\lambda_{10}+\lambda_{11})\right)$. This leads to the following sequence of implications:

$$\begin{aligned} \lambda_0+\lambda_6 &\le \tfrac{1}{3}\left((\lambda_6+\lambda_7+\lambda_8)-(\lambda_9+\lambda_{10}+\lambda_{11})\right) \\ \Leftrightarrow \quad 3\lambda_0+2\lambda_6+\lambda_9+\lambda_{10}+\lambda_{11} &\le \lambda_7+\lambda_8 \\ \Leftrightarrow \quad 1 &\le \textstyle\sum_{i=1}^2 \frac{\lambda_{6+i}}{3\lambda_0+2\lambda_6+(\lambda_9+\lambda_{10}+\lambda_{11})}. \end{aligned}$$

In summary, we have shown that for $i = 1, 2$, we have

$$\frac{\lambda_0+\lambda_6-\lambda_i}{3\lambda_0+2\lambda_6+(\lambda_9+\lambda_{10}+\lambda_{11})} \le \frac{\lambda_{6+i}}{3\lambda_0+2\lambda_6+(\lambda_9+\lambda_{10}+\lambda_{11})},$$

and that

$$\sum_{i=1}^{2} \frac{\lambda_0+\lambda_6-\lambda_i}{3\lambda_0+2\lambda_6+(\lambda_9+\lambda_{10}+\lambda_{11})} \le 1 \le \sum_{i=1}^{2} \frac{\lambda_{6+i}}{3\lambda_0+2\lambda_6+(\lambda_9+\lambda_{10}+\lambda_{11})}.$$

Applying Lemma 4.3 gives us the existence of a pair α_1, α_2 as required, concluding the proof to Lemma 4.2. ■

The Lemma 4.2 shows that we have indeed found all lower facets of the polyhedron $\mathcal{Q}$, meaning that the collection $\widehat{F}_1, \ldots, \widehat{F}_{11}$ gives a regular polyhedral subdivision $\mathcal{S}$ of $\mathfrak{P}$, thus concluding Step 1.

4.1.2 *Step 2:*

This is true by general convex geometry (using the poset of refinements and the secondary polytope). By Theorem 2.4 in Chapter 7 of [14], the poset of (non-empty) faces of the secondary polytope $\Sigma(\mathfrak{P})$ is isomorphic to the poset of all regular subdivisions of $\mathfrak{P}$, partially ordered by refinement (see also Theorem 16.4.1 in [15]). The vertices of $\Sigma(\mathfrak{P})$ correspond to regular triangulations. Thus, our regular subdivision obtained by projection must correspond to some face of $\Sigma(\mathfrak{P})$ and any vertex of that face will correspond to a regular triangulation refining it.

4.1.3 *Step 3:*

Consider a regular triangulation $\mathcal{T}$ obtained by refining $\mathcal{S}$. By definition, it is a regular triangulation of $\mathfrak{P}$. Recall Table 3. Denote by C_i the collection of points used to define the polyhedron $\widehat{F}_i$ in the table. Note that $\widehat{F}_0, \dots, \widehat{F}_5$ are the simplices in $\mathcal{T}_0$, and therefore any simplices in $\mathcal{T} \setminus \mathcal{T}_0$ do not originate from refining any of $\widehat{F}_0, \dots, \widehat{F}_5$.

Thus, the last step of the proof reduces to showing that none of the polyhedra $\widehat{F}_i$, $0 \le i \le 11$, contain any of the points we did not define it by, that is, $\widehat{F}_i \cap \mathfrak{C} = C_i$. Indeed, in that case we note that, by consulting Table 3, the polyhedra $\widehat{F}_i$ each fulfill at least one of the conditions A or B in the proposition. If $\widehat{F}_i \cap \mathfrak{C} = C_i$, then all simplices in a refinement of $\widehat{F}_i$ are defined as the convex hull of a subset of C_i (as there is no interior point to refine upon), thus inheriting the properties A or B from $\widehat{F}_i$.

Showing that $\widehat{F}_i \cap \mathfrak{C} = C_i$ for $6 \le i \le 11$ reduces to a simple computation. We shall do the computation for $\widehat{F}_6$, as the remaining cases are analogous by symmetry.

We need to show that $P_0, P_6, P_9, P_{10}, P_{11}, S_1 \notin \widehat{F}_6$. Any point X in $\widehat{F}_6$ can be written as

$$\begin{aligned} &\lambda_1 P_1 + \cdots + \lambda_5 P_5 + \mu_1 S_1 + \lambda_7 P_7 + \lambda_8 P_8 = (-\lambda_3 - \lambda_4 - \lambda_5 - \lambda_7 - \lambda_8, \\ &3\lambda_1 - \lambda_3 - \lambda_4 - \lambda_5 + 2\lambda_7 - \lambda_8, 3\lambda_2 - \lambda_3 - \lambda_4 - \lambda_5 - \lambda_7 + 2\lambda_8, -\lambda_1 - \lambda_2 + 3\lambda_3, \\ &\lambda_1 + \lambda_2 + 3\lambda_4, \lambda_3 + \lambda_4 + \lambda_5 + \mu_1 + \lambda_7 + \lambda_8, \lambda_1 + \lambda_2), \end{aligned} \tag{16}$$

with $\lambda_i, \mu_1 \ge 0$ and $\sum \lambda_i + \mu_1 = 1$. We note that the last two coordinates of X are $\lambda_3 + \lambda_4 + \lambda_5 + \mu_1 + \lambda_7 + \lambda_8$ and $\lambda_1 + \lambda_2$, respectively. Assume $P_0 \in \widehat{F}_6$ and had an expression as in Equation (16). Then, as $\lambda_i, \mu_1 \ge 0$, we can see by looking at the last two coordinates that $\lambda_3 = \lambda_4 = \lambda_5 = \mu_1 = \lambda_7 = \lambda_8 = 0$ and $\lambda_1 + \lambda_2 = 1$. But then the first coordinate is $\lambda_1 \cdot 0 + \lambda_2 \cdot 0 = 0 \ne 2$, hence we get a contradiction and $P_0 \notin \widehat{F}_6$. By an analogous reasoning, for S_2, P_9, P_{10}, P_{11} we obtain that all but λ_1, λ_2 would need to be 0 again and the sum of these two would need to be 1, which means that not both the second and third coordinate (being $3\lambda_1, 3\lambda_2$) can be 0. Hence $S_2, P_9, P_{10}, P_{11} \notin \widehat{F}_6$.

Finally, we need to show $P_6 \notin \widehat{F}_6$. Assume we had an expression for P_6 as in Equation (16). Since $\lambda_i \ge 0$, considering the last two coordinates gives $\lambda_1 = \lambda_2 = 0$ (since $\lambda_i \ge 0$) and $\lambda_3 + \lambda_4 + \lambda_5 + \mu_1 + \lambda_7 + \lambda_8 = 1$. But then the first coordinate is $-(\lambda_3 + \lambda_4 + \lambda_5 + \lambda_7 + \lambda_8) \le 0 < 2$, a contradiction. Thus $P_6 \notin \widehat{F}_6$, and thus $\widehat{F}_6 \cap \mathfrak{C} = C_6$ as claimed.

The other cases are analogous by symmetry. Thus we finished Step 3, hence proving Proposition 4.1.

4.2 The ideals associated to the partial compactification

Recalling the notation from section 2.5, we denote by x_i the variable in $\mathbb{C}[x_0,\ldots,x_{11},u_1,u_2]$ corresponding to the point P_i and by u_j the variable corresponding to S_j. These 14 variables correspond to the rays of the fan $\Sigma_{\nabla,D'_a,D'_a}$ from Corollary 3.4.

Lemma 4.4. There exists a global function on X_{∇,D'_a,D'_b} that has the form

$$\begin{aligned} w = u_1(c_0x_0^3x_6^3 + c_1x_1^3x_7^3 + c_2x_2^3x_8^3 - 3\lambda_1x_3x_4x_5x_6x_7x_8) \\ + u_2(c_3x_3^3x_9^3 + c_4x_4^3x_{10}^3 + c_5x_5^3x_{11}^3 - 3\lambda_2x_0x_1x_2x_9x_{10}x_{11}), \end{aligned} \tag{17}$$

for some $c_i, \lambda_j \in \mathbb{C}$.

Proof. Consider the hyperplane

$$H := \{(m,t_1,t_2) \in M_{\mathbb{R}} \oplus \mathbb{R}^2 \mid t_1 + t_2 = 1\} \tag{18}$$

in $M_{\mathbb{R}} \oplus \mathbb{R}^2$. The cone $|\Sigma_{\nabla,D'_a,D'_b}|^\vee$ is given by the cone over the convex hull of the following 8 points on H:

$$\begin{array}{llll} (0,0,0,0,1,0,1), & (-1,-1,-1,-1,-1,0,1), & (0,0,0,1,0,0,1), & (0,0,1,0,0,1,0), \\ (0,1,0,0,0,1,0), & (0,0,0,0,0,0,1), & (1,0,0,0,0,1,0), & (0,0,0,0,0,1,0). \end{array}$$

Recall that there is a correspondence between points in the dual cone $|\Sigma_{LT,D_a,D_b}|^\vee$ and global functions on X_{LT,D_a,D_b}. Note that, when one constructs a superpotential for X_{LT,D_a,D_b} using this correspondence with the 8 points above, one obtains the superpotential $w = u_1Q_{1,\lambda} + u_2Q_{2,\lambda}$. To see the global function on X_{∇,D'_a,D'_b}, it suffices to compute what monomials these 8 points correspond to on X_{∇,D'_a,D'_b}. This gives the 8 monomials in (17). ■

For our purposes of comparing the Batyrev–Borisov construction with the one by Libgober–Teitelbaum, we choose $c_i = 1$ and $\lambda_1 = \lambda_2 =: \lambda$.

We fix a triangulation $\mathcal{T}$ fulfilling the properties of Proposition 4.1. Let $X = \mathbb{C}^{14}$ and consider the group G_Σ corresponding to the fan $\Sigma_{\nabla,D'_a,D'_b}$ with its action on X. From the triangulation $\mathcal{T}$, we obtain the ideals:

$$\mathcal{I} := \left\langle \prod_{i\notin I} x_i \prod_{j\notin J} u_j \;\middle|\; \bigcup_{i\in I} u_{\overline{\rho}_i} \cup \bigcup_{j\in J} u_{\tau_j} \text{ give the set of vertices of a simplex in } \mathcal{T} \right\rangle,$$

$$\mathcal{J} := \left\langle \prod_{i\notin I} x_i \;\middle|\; \bigcup_{i\in I} u_{\overline{\rho}_i} \cup \bigcup_{j=1}^{2} u_{\tau_j} \text{ give the set of vertices of a simplex in } \mathcal{T} \right\rangle.$$

Before we can apply Proposition 2.25 and Corollary 2.26, we need to ensure the condition $\mathcal{I} \subseteq \sqrt{\partial w, \mathcal{J}}$ holds.

Lemma 4.5. For any triangulation $\mathcal{T}$ as in Proposition 4.1, defining $\mathcal{I}, \mathcal{J}$ and w as above with $\lambda^6 \neq 0, 1$, we have $\mathcal{I} \subseteq \sqrt{\mathcal{J}, \partial w}$. Therefore, this choice of superpotential fulfills the condition of Proposition 2.25.

Proof. To show the containment $\mathcal{I} \subseteq \sqrt{\partial w, \mathcal{J}}$, we prove that all the generators of $\mathcal{I}$ are in $\sqrt{\partial w, \mathcal{J}}$. The ideal $\mathcal{I}$ is, by definition, generated by the monomials that correspond to the simplices in the triangulation $\mathcal{T}$. For a simplex $T \in \mathcal{T}_0$, both S_1 and S_2 are vertices. Thus, by definition, the monomial associated to T is in $\mathcal{J}$ and hence in $\sqrt{\partial w, \mathcal{J}}$.

For any simplex $T \in \mathcal{T} \setminus \mathcal{T}_0$, either condition (A) or (B) of Proposition 4.1 holds. We claim that the monomial associated to a simplex T fulfilling either of those two conditions is an element of $\sqrt{\partial w}$ and therefore an element of $\sqrt{\partial w, \mathcal{J}}$.

We note that if $T \in \mathcal{T} \setminus \mathcal{T}_0$ fulfills condition (A), that is, does not contain any of the points $S_1.P_6, P_7, P_8$ and there is a pair of points of the form P_j, P_{6+j} with $3 \leq j \leq 5$ also not contained, then by definition $u_1 x_j x_{6+j} x_6 x_7 x_8$ divides the monomial associated to T. Similarly, if T fulfilled condition (B) instead, $u_2 x_j x_{6+j} x_9 x_{10} x_{11}$ (for some $0 \leq j \leq 2$) would divide the monomial generator of $\mathcal{I}$ associated to T.

To show that any monomial associated to a simplex in $\mathcal{T} \setminus \mathcal{T}_0$ is in $\sqrt{\partial w} \subseteq \sqrt{\partial w, \mathcal{J}}$, it is thus sufficient to prove that the six monomials $u_2 x_0 x_6 x_9 x_{10} x_{11}$, $u_2 x_1 x_7 x_9 x_{10} x_{11}$, $u_2 x_2 x_8 x_9 x_{10} x_{11}$, $u_1 x_3 x_9 x_6 x_7 x_8$, $u_1 x_4 x_{10} x_6 x_7 x_8$ and $u_1 x_5 x_{11} x_6 x_7 x_8$ are elements of $\sqrt{\partial w}$.

By symmetry of the x_i in w, we note that it is sufficient to show that $u_2 x_0 x_6 x_9 x_{10} x_{11} \in \sqrt{\partial w}$. Start by explicitly writing down the ideal $\langle \partial w \rangle$, that is, the ideal generated by the partial derivatives of w.

$$\begin{aligned}
\langle \partial w \rangle = \quad & \langle 3u_1 x_0^2 x_6^3 - 3\lambda u_2 x_1 x_2 x_9 x_{10} x_{11}, 3u_1 x_1^2 x_7^3 - 3\lambda u_2 x_0 x_2 x_9 x_{10} x_{11}, \\
& 3u_1 x_2^2 x_8^3 - 3\lambda u_2 x_0 x_1 x_9 x_{10} x_{11}, 3u_2 x_3^2 x_9^3 - 3\lambda u_1 x_4 x_5 x_6 x_7 x_8, \\
& 3u_2 x_4^2 x_{10}^3 - 3\lambda u_1 x_3 x_5 x_6 x_7 x_8, 3u_2 x_5^2 x_{11}^3 - 3\lambda u_1 x_3 x_4 x_6 x_7 x_8, \\
& 3u_1 x_0^3 x_6^2 - 3\lambda u_1 x_3 x_4 x_5 x_7 x_8, 3u_1 x_1^3 x_7^2 - 3\lambda u_1 x_3 x_4 x_5 x_6 x_8, \\
& 3u_1 x_2^3 x_8^2 - 3\lambda u_1 x_3 x_4 x_5 x_6 x_7, 3u_2 x_3^3 x_9^2 - 3\lambda u_2 x_0 x_1 x_2 x_{10} x_{11}, \\
& 3u_2 x_4^3 x_{10}^2 - 3\lambda u_2 x_0 x_1 x_2 x_9 x_{11}, 3u_2 x_5^3 x_{11}^2 - 3\lambda u_2 x_0 x_1 x_2 x_9 x_{10}, \\
& x_0^3 x_6^3 + x_1^3 x_7^3 + x_2^3 x_8^3 - 3\lambda x_3 x_4 x_5 x_6 x_7 x_8, x_3^3 x_9^3 + x_4^3 x_{10}^3 + x_5^3 x_{11}^3 - 3\lambda x_0 x_1 x_2 x_9 x_{10} x_{11} \rangle
\end{aligned}$$

We see that $3u_1 x_i^2 x_{6+i}^3 - 3\lambda u_2 \frac{x_0 x_1 x_2 x_9 x_{10} x_{11}}{x_i} \in \langle \partial w \rangle$ for $0 \leq i \leq 2$. Notice that since $ac - bd = c(a - b) + b(c - d)$, if $a - b, c - d$ are elements in an ideal, then so is $ac - bd$.

Hence by iterating this we obtain that

$$27u_1^3x_0^2x_1^2x_2^2x_6^3x_7^3x_8^3 - 27\lambda^3u_2^3x_0^2x_1^2x_2^2x_9^3x_{10}^3x_{11}^3 \in \langle \partial w \rangle.$$

Similarly,

$$27u_2^3x_3^2x_4^2x_5^2x_9^3x_{10}^3x_{11}^3 - 27\lambda^3u_1^3x_3^2x_4^2x_5^2x_6^3x_7^3x_8^3 \in \langle \partial w \rangle.$$

Therefore,

$$\begin{aligned}
&(27)^2u_1^3u_2^3x_0^2 \dots x_5^2x_6^3 \dots x_{11}^3 - (27)^2\lambda^6u_1^3u_2^3x_0^2 \dots x_5^2x_6^3 \dots x_{11}^3 \in \langle \partial w \rangle \\
&\Rightarrow 27^2(1-\lambda^6)u_1^3u_2^3x_0^2 \dots x_5^2x_6^3 \dots x_{11}^3 \in \langle \partial w \rangle \\
&\Rightarrow u_1^3u_2^3x_0^2 \dots x_5^2x_6^3 \dots x_{11}^3 \in \langle \partial w \rangle \\
&\Rightarrow (u_1u_2x_0 \dots x_{11})^3 \in \langle \partial w \rangle. \qquad (19)
\end{aligned}$$

Consider $\frac{\partial w}{\partial u_1}$, giving

$$x_0^3x_6^3 + x_1^3x_7^3 + x_2^3x_8^3 - 3\lambda x_3x_4x_5x_6x_7x_8 \in \langle \partial w \rangle. \qquad (20)$$

Furthermore, we note that $\sum_{i=0}^2 x_i \frac{\partial w}{\partial x_i} \in \langle \partial w \rangle$, and thus

$$\sum_{i=0}^{2}(3u_1x_i^3x_{6+i}^3 - 3\lambda u_2x_0x_1x_2x_9x_{10}x_{11}) \in \langle \partial w \rangle. \qquad (21)$$

By (20), we have that $x_3x_4x_5x_6x_7x_8 + \langle \partial w \rangle = \frac{1}{3\lambda}(x_0^3x_6^3 + x_1^3x_7^3 + x_2^3x_8^3) + \langle \partial w \rangle$. We use this to substitute into (19) to obtain that

$$\frac{1}{27\lambda^3}u_2^3x_0^3x_1^3x_2^3x_9^3x_{10}^3x_{11}^3(u_1(x_0^3x_6^3 + x_1^3x_7^3 + x_2^3x_8^3))^3 \in \langle \partial w \rangle.$$

Performing the same style of substitution with (21), we obtain

$$u_2^3x_0^3x_1^3x_2^3x_9^3x_{10}^3x_{11}^3u_2^3x_0^3x_1^3x_2^3x_9^3x_{10}^3x_{11}^3 = (u_2x_0x_1x_2x_9x_{10}x_{11})^6 \in \langle \partial w \rangle.$$

Thus $u_2x_0x_1x_2x_9x_{10}x_{11} \in \sqrt{\partial w}$.

By comparing the elements $x_2\frac{\partial w}{\partial x_2}, u_2x_0x_1x_2x_9x_{10}x_{11} \in \sqrt{\partial w}$, we obtain that $u_1x_2^3x_8^3 \in \sqrt{\partial w}$, implying $u_1x_2x_8 \in \sqrt{\partial w}$. This, in turn, implies that $u_2x_0x_1x_9x_{10}x_{11} \in$

$\sqrt{\partial w}$, by inspection of $\frac{\partial w}{\partial x_2} \in \langle \partial w \rangle \subseteq \sqrt{\partial w}$. Similarly, $u_2 x_0 x_2 x_9 x_{10} x_{11} \in \sqrt{\partial w}$. We also have $u_2 x_5 x_{11} \in \sqrt{\partial w}$ by an analogous computation. Finally, $\frac{\partial w}{\partial u_1} = x_0^3 x_6^3 + x_1^3 x_7^3 + x_2^3 x_8^3 - 3\lambda x_3 x_4 x_5 x_6 x_7 x_8 \in \sqrt{\partial w}$.

Therefore, one can intuit and then compute that

$$\begin{aligned}(u_2 x_0 x_6 x_9 x_{10} x_{11})^4 = &-u_2^3 x_1^2 x_6 x_7^3 x_9^3 x_{10}^3 x_{11}^3 \cdot (u_2 x_0 x_1 x_9 x_{10} x_{11}) \\ &- u_2^3 x_2^2 x_6 x_8^3 x_9^3 x_{10}^3 x_{11}^3 \cdot (u_2 x_0 x_2 x_9 x_{10} x_{11}) \\ &+ 3\lambda u_2^3 x_0 x_3 x_4 x_6^2 x_7 x_8 x_9^4 x_{10}^4 x_{11}^4 \cdot (u_2 x_5 x_{11}) \\ &+ u_2^4 x_0 x_6 x_9^4 x_{10}^4 x_{11}^4 \cdot (x_0^3 x_6^3 + x_1^3 x_7^3 + x_2^3 x_8^3 - 3\lambda x_3 x_4 x_5 x_6 x_7 x_8) \in \sqrt{\partial w}.\end{aligned} \tag{22}$$

Thus, we have shown that $u_2 x_0 x_6 x_9 x_{10} x_{11} \in \sqrt{\partial w}$. By symmetry, any simplex fulfilling properties (A) or (B) corresponds to a monomial in $\sqrt{\partial w, \mathcal{J}}$. Hence, any monomial associated to a simplex $T \in \mathcal{T} \setminus \mathcal{T}_0$ is an element of $\sqrt{\partial w, \mathcal{J}}$, concluding the proof that $\mathcal{I} \subseteq \sqrt{\partial w, \mathcal{J}}$. ■

Corollary 4.6. Consider the GKZ fan of $\mathrm{tot}\left(\mathcal{O}_{X_\nabla}(-D'_b) \oplus \mathcal{O}_{X_\nabla}(-D'_a)\right)$ and the group G_Σ from above. There is a chamber σ_p with affine open U_p such that:

- (i) $[U_p/G_\Sigma]$ is a partial compactification of $\mathrm{tot}\left(\mathcal{O}_{\mathcal{X}_{LT}}(-D_b) \oplus \mathcal{O}_{\mathcal{X}_{LT}}(-D_a)\right)$.
- (ii) There is a superpotential corresponding to the eight points in $|\Sigma_{\nabla, D'_a, D'_b}|^\vee \cap H$ taking the form $w = u_1(x_0^3 x_6^3 + x_1^3 x_7^3 + x_2^3 x_8^3 - 3\lambda x_3 x_4 x_5 x_6 x_7 x_8) + u_2(x_3^3 x_9^3 + x_4^3 x_{10}^3 + x_5^3 x_{11}^3 - 3\lambda x_0 x_1 x_2 x_9 x_{10} x_{11})$.
- (iii) With $\mathcal{I}_p, \mathcal{J}_p$ as defined in §2.5, we have $\mathcal{I}_p \subseteq \sqrt{\partial w, \mathcal{J}_p}$.

Proof. Proposition 4.1 proves (i), Lemma 4.4 proves (ii), and finally Lemma 4.5 shows (iii). ■

4.3 Relating X_∇ and X_{LT}

Recall that the partial compactification of the total space $\mathrm{tot}\left(\mathcal{O}_{X_{LT}}(-D_b) \oplus \mathcal{O}_{X_{LT}}(-D_a)\right)$ in Corollary 4.6 corresponds to a chamber σ_p of the GKZ fan of $\mathrm{tot}(\mathcal{O}_{X_\nabla}(-D'_b) \oplus \mathcal{O}_{X_\nabla}(-D'_a))$. We then know that it is birationally equivalent to $\mathrm{tot}(\mathcal{O}_{X_\nabla}(-D'_b) \oplus \mathcal{O}_{X_\nabla}(-D'_a))$.

Thus, we want to now explicitly find a triangulation of $\mathfrak{P}$ corresponding to the Batyrev–Borisov mirror family. There, the superpotential will take the form

$$\begin{aligned}w = u_1(x_0^3 x_6^3 + x_1^3 x_7^3 + x_2^3 x_8^3 - 3\lambda x_3 x_4 x_5 x_6 x_7 x_8) + u_2(x_3^3 x_9^3 + x_4^3 x_{10}^3 + x_5^3 x_{11}^3 \\ - 3\lambda x_0 x_1 x_2 x_9 x_{10} x_{11}).\end{aligned}$$

Note that, by Lemma 4.4, this is the form the superpotential should take in the Batyrev–Borisov mirror. In other words, we need a chamber σ_q in the GKZ fan corresponding to $\mathrm{tot}(\mathcal{O}_{X_\nabla}(-D'_b) \oplus \mathcal{O}_{X_\nabla}(-D'_a))$, where a general section of $\mathcal{O}_{X_\nabla}(-D'_b) \oplus \mathcal{O}_{X_\nabla}(-D'_a)$ will yield a complete intersection in X_∇, and thus a Batyrev–Borisov mirror.

Lemma 4.7. Consider the GKZ fan of $\mathrm{tot}\left(\mathcal{O}_{X_\nabla}(-D'_b) \oplus \mathcal{O}_{X_\nabla}(-D'_a)\right)$ and recall the group G_Σ from above. There is a chamber σ_q with affine open U_q such that:

(i) $[U_q/G_\Sigma] = \mathrm{tot}\left(\mathcal{O}_{\mathcal{X}_\nabla}(-D'_b) \oplus \mathcal{O}_{\mathcal{X}_\nabla}(-D'_a)\right)$.
(ii) A superpotential corresponding to the eight lattice points of $|\Sigma_{\nabla,D'_a,D'_b}|^\vee \cap H$ is of the form $w = u_1(x_0^3x_6^3 + x_1^3x_7^3 + x_2^3x_8^3 - 3\lambda x_3x_4x_5x_6x_7x_8) + u_2(x_3^3x_9^3 + x_4^3x_{10}^3 + x_5^3x_{11}^3 - 3\lambda x_0x_1x_2x_9x_{10}x_{11})$.
(iii) For $\mathcal{I}_q, \mathcal{J}_q$ as defined in §2.5, $\mathcal{I}_q \subseteq \sqrt{\partial w, \mathcal{J}_q}$.

Proof. This proof will construct the triangulation $\mathcal{T}_q$ corresponding to the chamber σ_q. We consider the 42 maximal cones from Table 1. For each of those cones $\sigma_i, 1 \leq i \leq 42$, we associate a simplex given as convex hull of the 5 vertices corresponding to the 5 rays of σ_i plus the two vertices corresponding to the bundle coordinates, that is, $(0,0,0,0,0,1,0)$ and $(0,0,0,0,0,0,1)$. So for example the first cone, with rays $\rho_0, \rho_1, \rho_2, \rho_9, \rho_{10}$, will correspond to the simplex with vertices $(3,0,0,-1,-1,1,0)$, $(0,3,0,-1,-1,1,0)$, $(0,0,3,-1,-1,1,0)$, $(0,0,0,0,0,1,0)$, $(0,0,0,0,0,0,1)$, $(0,0,0,2,-1,1,0)$, $(0,0,0,-1,2,1,0)$. Another way to formulate this is that we take the star subdivision of the cones from Table 1 on the two bundle points S_1, S_2.

Regularity of this triangulation of the 14 points is an easy consequence of its construction as a star subdivision, hence it corresponds to some chamber σ_q in the GKZ-fan. Indeed, the star subdivision can be obtained by giving the points S_1, S_2 a weight of 1 and giving all other points the same weight of $w = 2$ and then refining the resulting regular polyhedral subdivision into a triangulation. Alternatively, one can check the regularity of this triangulation by using SAGE.

The third item follows from the fact that we do not partially compactify, hence $\mathcal{J}_q = \mathcal{I}_q$ and therefore $\mathcal{I}_q \subseteq \sqrt{\partial w, \mathcal{J}_q}$, as required. ■

We now have all the necessary tools to prove the main result of this paper, Theorem 1.1.

Proof of Theorem 1.1. Recall the chambers σ_p and σ_q in the GKZ fan of the toric variety $\mathrm{tot}\left(\mathcal{O}_{X_\nabla}(-D'_b) \oplus \mathcal{O}_{X_\nabla}(-D'_a)\right)$ given in Corollary 4.6 and Lemma 4.7. By applying Corollary 2.26, we have $\mathrm{D^b}(\mathrm{coh}\,\mathcal{Z}_\lambda) \cong \mathrm{D^b}(\mathrm{coh}\,\mathcal{V}_{LT,\lambda})$, as required. ■

We note that analogous computations to the ones displayed in this paper can yield the following result in lower dimension.

Theorem 4.8. Let $Q_1 = x_1^2 + x_2^2 - x_3x_4$, $Q_2 = x_3^2 + x_4^2 - x_1x_2$ and let $p_1 = x_1^2x_5^2 + x_2^2x_6^2 - x_3x_4x_5x_6$, $p_2 = x_3^2x_7^2 + x_4^2x_8^2 - x_1x_2x_7x_8$. We define the group $G_4 \subseteq PGL(3, \mathbb{C})$ given by the four automorphisms

$$\mathrm{diag}(1,1,1,1),\ \mathrm{diag}(\zeta_8, -\zeta_8, -\zeta_8^{-1}, \zeta_8^{-1}),\ \mathrm{diag}(\zeta_4, \zeta_4, \zeta_4^{-1}, \zeta_4^{-1}),\ \mathrm{diag}(\zeta_8^3, -\zeta_8^3, -\zeta_8^{-3}, \zeta_8^{-3}),$$

where ζ_k is a primitive k^{th} root of unity.

The Batyrev–Borisov mirror to $Z(Q_1, Q_2) \subseteq \mathbb{P}^3$ can be computed to be a complete intersection $\mathcal{Z}_2$ in a 3-dimensional toric stack $\mathcal{X}_{BB}$ given as the zero locus $Z_2 = Z(p_1, p_2) \subseteq \mathcal{X}_{BB}$. Take the stacky complete intersection $\mathcal{V}_2 := Z(Q_1, Q_2) \subseteq [(\mathbb{C}^4 \setminus \{0\})/(\mathbb{C}^* \times G_4)]$. Then

$$\mathrm{D^b}(\mathrm{coh}\,\mathcal{V}_2) \cong \mathrm{D^b}(\mathrm{coh}\,\mathcal{Z}_2).$$

Remark 4.9. One can aim to generalise this to higher dimensions by looking at the zeroset of the two polynomials

$$Q_{1,n} = x_1^n + \cdots + x_n^n - x_{n+1} \dots x_{2n} \text{ and } Q_{2,n} = x_{n+1}^n + \cdots + x_{2n}^n - x_1 \dots x_n$$

in $\mathbb{P}^{2n-1}$.

Unfortunately, $Z(Q_{1,n}, Q_{2,n}) \subseteq \mathbb{P}^{2n-1}$ is itself singular for $n \geq 4$, which poses problems for the required ideal containment condition $\mathcal{I} \subseteq \sqrt{\partial w, \mathcal{J}}$ to hold. However, using these methods of VGIT is still interesting in the context of categorical resolutions. Indeed, the direct generalisation of the Libgober–Teitelbaum construction above can be categorically resolved. This technique and its generalisations are a subject of future work.

Remark 4.10. The notion of f-duality introduced by Rossi in [23] and [24] gives an efficient method of computing and extending the Batyrev–Borisov mirror construction. In particular, applying f-duality to the variety $V_{LT,\lambda} \subseteq \mathbb{P}^5/G_{81}$ yields $V_\lambda \subseteq \mathbb{P}^5$.

The generalisations looked at in the Remark 4.9 were inspired by f-duality and it seems to be an interesting question when, in general, one can use the methods of variations of GIT employed in this paper to strengthen the notion of f-duality.

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