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RESEARCH ARTICLE

The Calogero–Moser derivative nonlinear Schrödinger equation

Patrick Gérard[1] | **Enno Lenzmann**[2]

[1]Laboratoire de Mathématiques d'Orsay, CNRS, Université Paris-Saclay, Orsay, France

[2]Department of Mathematics and Computer Science, University of Basel, Basel, Switzerland

Correspondence
Patrick Gérard, Laboratoire de Mathématiques d'Orsay, CNRS, Université Paris-Saclay, 91405 Orsay, France.
Email: patrick.gerard@universite-paris-saclay.fr

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Abstract

We study the Calogero–Moser derivative nonlinear Schrödinger NLS equation

$$i\partial_t u + \partial_{xx} u + (D + |D|)(|u|^2)u = 0$$

posed on the Hardy–Sobolev space $H^s_+(\mathbb{R})$ with suitable $s > 0$. By using a Lax pair structure for this L^2-critical equation, we prove global well-posedness for $s \geq 1$ and initial data with sub-critical or critical L^2-mass $\|u_0\|^2_{L^2} \leq 2\pi$. Moreover, we prove uniqueness of ground states and also classify all traveling solitary waves. Finally, we study in detail the class of multi-soliton solutions $u(t)$ and we prove that they exhibit energy cascades in the following strong sense such that $\|u(t)\|_{H^s} \sim_s |t|^{2s}$ as $t \to \pm\infty$ for every $s > 0$.

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CONTENTS

1 | INTRODUCTION AND MAIN RESULTS

This paper is devoted to the study of the *Calogero–Moser derivative nonlinear Schrödinger equation*, which can be written as

$$i\partial_t u + \partial_{xx} u + (D + |D|)(|u|^2)u = 0 \qquad \text{(CM-DNLS)}$$

for $u : I \times \mathbb{R} \to \mathbb{C}$ with some time interval $I \subset \mathbb{R}$. Here and in what follows, we use the standard notation $D = -i\partial_x$ and hence $|D|$ denotes the Fourier multiplier with symbol $|\xi|$.

We remark that Equation (CM-DNLS) was introduced in [1] as a formal continuum limit of classical Calogero–Moser systems [27, 28]. Also, prior to [1], a defocusing version given by

$$i\partial_t u + \partial_{xx} u - (D + |D|)(|u|^2) = 0 \qquad \text{(INLS)}$$

was introduced in [30] under the name *intermediate nonlinear Schrödinger equation (INLS)*, as describing envelope waves in a deep stratified fluid. We will concentrate on (CM-DNLS), because it offers richer dynamics, for example, multi-soliton solutions with turbulence in Sobolev norms (see Theorem 1.3 below). However, part of our results can be extended to this defocusing version above.

1.1 | Symmetries, phase space, and Hamiltonian features

We observe that Equation (CM-DNLS) admits the invariance by phase, scaling and translation,

$$u(t,x) \mapsto \mathrm{e}^{i\theta}\lambda^{1/2}u(\lambda^2 t, \lambda x + x_0), \quad x_0 \in \mathbb{R}, \theta \in \mathbb{R}, \lambda > 0,$$

which makes it a L^2-critical equation on the line. It also enjoys the Galilean invariance

$$u(t,x) \mapsto \mathrm{e}^{i\eta x - it\eta^2}u(t, x - 2t\eta), \quad \eta \in \mathbb{R},$$

as well as the pseudo–conformal symmetry found by Ginibre and Velo for the L^2-critical NLS. Recall that a special case of this space-time transform reads

$$u(t,x) \mapsto \frac{1}{t^{1/2}}\mathrm{e}^{i\frac{x^2}{4t}}u\left(-\frac{1}{t}, \frac{x}{t}\right).$$

In what follows, we are interested in solutions of (CM-DNLS) satisfying the additional condition that

$$u(t) \in H^s_+(\mathbb{R}) := \{f \in H^s(\mathbb{R}) : \operatorname{supp}(\hat{f}) \subset [0, +\infty)\},$$

where $H^s(\mathbb{R})$ denotes the usual Sobolev space based on $L^2(\mathbb{R})$. The spaces $H^s_+(\mathbb{R})$ will serve as phase spaces on which we study (CM-DNLS) as a Hamiltonian system.

Recall that $H^0_+(\mathbb{R}) = L^2_+(\mathbb{R})$ denotes the Hardy space of holomorphic functions on the complex upper half-plane. If we let $\Pi_+ : L^2(\mathbb{R}) \to L^2_+(\mathbb{R})$ denote Cauchy–Szegő orthogonal projection onto $L^2_+(\mathbb{R})$ given by

$$\Pi_+(f)(x) := \frac{1}{2\pi}\int_0^\infty e^{i\xi x}\hat{f}(\xi)\,d\xi,$$

then Equation (CM-DNLS) can be written as

$$i\partial_t u + \partial_{xx}u + 2D_+(|u|^2)u = 0.$$

Here $D_+ := D\Pi_+$ can be seen as the compression of $D = -i\partial_x$ onto the Hardy space $L^2_+(\mathbb{R})$. The positive Fourier frequency condition $\mathrm{supp}(\hat{f}) \subset [0, +\infty)$ is interpreted as a chirality condition in [1]. In fact, such a condition naturally appears if one thinks of the Benjamin–Ono equation,

$$\partial_t v + \partial_x |D| v - \partial_x(v^2) = 0, \tag{BO}$$

which is known to be well-posed for real valued functions v; see [22, 26]. Introducing the new unknown $u = \Pi_+ v$, the condition $v = \overline{v}$ reads $v = u + \overline{u}$, so that (BO) is equivalent to

$$i\partial_t u + \partial_{xx}u + D(u^2) + 2D_+(|u|^2) = 0.$$

This way, (CM-DNLS) and its defocusing sibling can be seen as L^2-critical versions of (BO).

Notice that the pseudo-conformal symmetry does not preserve chirality and that the Galilean transformation acts on chiral solutions of (CM-DNLS) only if $\eta \geq 0$.

Now, let us come to the Hamiltonian properties of (CM-DNLS). To this end, we introduce the following gauge transformation

$$v(x) := u(x)\,\mathrm{e}^{-\frac{i}{2}\int_{-\infty}^x |u(y)|^2\,dy}, \tag{1.1}$$

which turns out to be a diffeomorphism of $H^s(\mathbb{R})$ into itself for every $s \geq 0$. An elementary calculation shows that (CM-DNLS) is equivalent to the equation

$$i\partial_t v + \partial_{xx}v + |D|(|v|^2)v - \frac{1}{4}|v|^4 v = 0. \tag{1.2}$$

This is a Hamiltonian PDE with the standard symplectic form $\omega(h_1, h_2) = \mathrm{Im}\langle h_1, h_2\rangle_{L^2}$ and the energy functional

$$\widetilde{E}(v) := \frac{1}{2}\|\partial_x v\|^2_{L^2} - \frac{1}{4}\langle |D|(|v|^2), |v|^2\rangle_{L^2} + \frac{1}{24}\|v\|^6_{L^6}.$$

By classical product identities for the Hilbert transform H (see Appendix C for details), the energy functional $\widetilde{E}$ can be written as

$$\widetilde{E}(v) = \frac{1}{2}\int_{\mathbb{R}} |\partial_x v + \frac{1}{2}\mathsf{H}(|v|^2)v|^2\,dx \geq 0.$$

Inverting the gauge transformation (1.1) and in view of $\Pi_+ = \frac{1}{2}(1 + i\mathsf{H})$, we find that $E(u) = \widetilde{E}(v)$ is an energy functional for (CM-DNLS) given by

$$E(u) = \frac{1}{2} \int_{\mathbb{R}} \left| \partial_x u - i\Pi_+(|u|^2)u \right|^2 dx. \tag{1.3}$$

In summary, we deduce that (CM-DNLS) is a Hamiltonian equation generated by the energy functional $E(u)$ and the symplectic form

$$\omega_u^\sharp(h_1, h_2) := \operatorname{Im}\langle h_1, h_2 \rangle_{L^2} + \iint_{\mathbb{R}\times\mathbb{R}} \operatorname{Re}(\overline{u}h_1)(x)\operatorname{Re}(\overline{u}h_2)(y)\operatorname{sgn}(x-y)\, dx\, dy,$$

which is just the pullback of the standard symplectic ω under the gauge transformation $u \mapsto v$ defined in (1.1). Recall that we will study (CM-DNLS) as Hamiltonian PDE on the phase spaces $H^s_+(\mathbb{R})$ corresponding to chiral solutions. It is interesting to note that $\omega_u^\sharp$ provides a non-standard symplectic form on the spaces $H^s_+(\mathbb{R})$ with $s \geq 0$.

Next, we discuss the conservation laws exhibited by (CM-DNLS). Due to symmetry by complex phase shifts, spatial translations and its Hamiltonian nature, we easily obtain the following conserved quantities:

$$M(u) = \int_{\mathbb{R}} |u|^2\, dx \quad (L^2\text{-mass}), \qquad P(u) = \int_{\mathbb{R}} \left(Du\overline{u} - \frac{1}{2}|u|^4\right) dx \quad (\text{Momentum}),$$

$$E(u) = \frac{1}{2} \int_{\mathbb{R}} |\partial_x u - i\Pi_+(|u|^2)u|^2\, dx \quad (\text{Energy}).$$

In the expression for the conserved momentum $P(u)$, the nonlinear term $|u|^4$ arises due to the non-standard symplectic structure $\omega_u^\sharp$. In fact, we will show below that $M(u)$, $P(u)$ and $E(u)$ belong to an *infinite hierarchy of conservation laws* $\{I_k(u)\}_{k=0}^\infty$ based on a Lax pair structure for (CM-DNLS); see Section 2.

Finally, we briefly comment on the L^2-critical nature of (CM-DNLS). As one may expect, there exists a special solution which separates solutions into small and large data. Indeed, we will prove that the energy $E(u)$ has a *unique* (up to symmetries) minimizer given by the rational function

$$\mathcal{R}(x) = \frac{\sqrt{2}}{x+i} \in H^1_+(\mathbb{R}),$$

which we refer to as the **ground state** for (CM-DNLS); see Section 4. An elementary calculation shows that $u(t,x) = \mathcal{R}(x)$ provides a *static solution* of (CM-DNLS) and its L^2-mass is found to be

$$M(\mathcal{R}) = \int_{\mathbb{R}} \frac{2}{1+x^2}\, dx = 2\pi.$$

As we will see below, this number 2π provides a threshold in the analysis of (CM-DNLS). Consequently, we shall refer to solutions $u(t) \in H^s_+(\mathbb{R})$ with

$$M(u_0) < M(\mathcal{R}), \quad M(u_0) = M(\mathcal{R}), \quad M(u_0) > M(\mathcal{R})$$

as having sub-critical, critical, and super-critical L^2-masses, respectively. The main results of this paper will address these various regimes.

1.2 Main results

As a starting point, we first establish local well-posedness of (CM-DNLS) for initial data in $H^s_+(\mathbb{R})$ with $s > 1/2$. For sufficiently regular initial data in $H^s_+(\mathbb{R})$ with $s > 3/2$, this follows from Kato's classical iteration scheme for quasilinear evolution equations. Extending the local well-posedness to less regular data in $H^s_+(\mathbb{R})$ with $1/2 < s \leq 3/2$ can then be achieved by adapting arguments from [13], which in turn is inspired by Tao's gauge trick for the Benjamin-Ono equation [35]. However, for the rest of the paper, we will be mainly be concerned with solutions of (CM-DNLS) such that $u(t) \in H^s_+(\mathbb{R})$ with some integer $s \geq 1$.

The general question of global well-posedness for (CM-DNLS) seems to be rather delicate because of the focusing L^2-criticality of the nonlinearity, which might generate blowup of solutions in finite time. The following result establishes global well-posedness for initial data with finite energy and L^2-mass that is less or equal to the ground state mass.

Theorem 1.1 (Global Well-Posedness Result). *Let $s \geq 1$ be an integer. Then (CM-DNLS) is globally well-posed for initial data in $u_0 \in H^s_+(\mathbb{R})$ with L^2-mass*

$$M(u_0) \leq M(\mathcal{R}) = 2\pi.$$

Moreover, we have the a-priori bound

$$\sup_{t\in\mathbb{R}} \|u(t)\|_{H^s} < +\infty,$$

provided the strict inequality $M(u_0) < M(\mathcal{R})$ holds.

Remarks 1.1.

(1) In the case of sub-critical L^2-mass, the a-priori bounds on $\|u(t)\|_{H^s}$ will follow from exploiting an infinite hierarchy of conservation laws for (CM-DNLS). We refer to Section 5 for a detailed discussion.
(2) The case of critical L^2-mass when $M(u_0) = M(\mathcal{R})$ is rather delicate to handle and will follow from ruling out the so-called minimal mass blowup solutions for (CM-DNLS) with finite energy. A key element in the proof will be the slow algebraic decay of the ground states $\mathcal{R}$.
(3) It is an interesting open question whether global-in-time existence holds for large initial data in $H^s_+(\mathbb{R})$ with $s \geq 1$ and L^2-mass $M(u_0) > M(\mathcal{R})$. As a striking example below, there exist smooth global-in-time solutions for (CM-DNLS) given by multi-solitons, which always blowup in infinite time due to unbounded growth of all Sobolev norms $\|u(t)\|_{H^s}$ for any $s > 0$.
(4) By applying the pseudo-conformal transformation to the static solution $\mathcal{R}(x)$, we obtain the explicit solution

$$u_{\text{sing}}(t,x) = \frac{1}{t^{1/2}} e^{ix^2/4t} \mathcal{R}\left(\frac{x}{t}\right) \in L^2(\mathbb{R}) \quad \text{for all } t > 0,$$

which solves (CM-DNLS) and becomes singular as $t \to 0^-$. Due to slow algebraic decay of $\mathcal{R}(x)$, we find that $u_{\text{sing}}(t) \notin H^1(\mathbb{R})$ has no finite energy[1] and, moreover, we see that the solution $u_{\text{sing}}(t) \notin L^2_+(\mathbb{R})$ fails to be chiral. Still, this explicit example shows that we cannot expect global well-posedness for (CM-DNLS) with arbitrary initial data in the scaling-critical

[1] A closer inspection shows that $u_{\text{sing}}(t,\cdot) \in H^s(\mathbb{R})$ for all $0 \leq s < 1/2$.

space $L^2(\mathbb{R})$. It remains an intriguing open question if initial data in $L^2_+(\mathbb{R})$ will always lead to global-in-time solutions for (CM-DNLS).

Next, we turn our attention to sufficiently regular solutions $u(t) \in H^s_+(\mathbb{R})$ of (CM-DNLS) with initial data having critical or super-critical L^2-mass:

$$M(u_0) \geq M(\mathcal{R}) = 2\pi.$$

In this regime of sufficiently large data, we expect (CM-DNLS) to possess traveling ground state solitons as well as multi-soliton solutions. As a main result in this setting, we completely classify all traveling solitary waves for the Calogero–Moser DNLS with finite energy by showing that are given by the ground state $\mathcal{R}(x)$ up to scaling, phase, translation, and Galilean boosts preserving the chirality condition.

Theorem 1.2 (Classification of Traveling Solitary Waves). *Every traveling solitary wave for Equation (CM-DNLS) in $H^1_+(\mathbb{R})$ is of the form*

$$u(t,x) = e^{i\theta + i\eta x - i\eta^2 t}\lambda^{1/2}\mathcal{R}(\lambda(x - 2\eta t) + y)$$

with some $\theta \in [0, 2\pi)$, $y \in \mathbb{R}$, $\lambda > 0$, and $\eta \geq 0$.

In particular, every traveling solitary waves $u(t) \in H^1_+(\mathbb{R})$ for (CM-DNLS) have critical L^2-mass $M(u) = M(\mathcal{R})$.

Remarks 1.2.

(1) The condition $\eta \geq 0$ enters through the chirality condition $u(t) \in H^1_+(\mathbb{R})$ and thus traveling solitary waves can only move to right. If we take negative values $\eta < 0$ above, we obtain left-moving traveling solitary waves $u(t) \in H^1(\mathbb{R})$ solving (CM-DNLS); see Section 4 for the definition of traveling solitary waves.

(2) A key step in the complete classification above is to establish *uniqueness* of (non-trivial) minimizers of the energy $E(u)$, which is equivalent to classifying all solutions $u \in H^1_+(\mathbb{R})$ of the nonlinear equation

$$Du - \Pi_+(|u|^2)u = 0.$$

We refer to Section 4 below for details including a more general result assuming only that $u \in H^1(\mathbb{R})$.

As our final main result, we study the dynamics of *multi-soliton solutions* for (CM-DNLS); see Section 6 below for a precise definition using the Lax pair structure. For the Calogero–Moser DNLS, it turns out that multi-solitons $u = u(t,x)$ are rational functions of $x \in \mathbb{R}$ in the Hardy spaces $L^2_+(\mathbb{R})$. As an interesting fact, we remark that they necessarily have a *quantized* L^2-mass given by

$$M(u) = 2\pi N \quad \text{with } N = 1, 2, 3, \ldots$$

In the special case when $N = 1$, the multi-solitons are given by the ground state $\mathcal{R}(x)$ up to symmetries. For $N \geq 2$, we note that multi-solitons have super-critical L^2-mass. As a consequence, the proof of their global-in-time existence is far from trivial and will follow from the analysis of a suitable inverse spectral formula based on the Lax structure. As an outcome, we obtain a detailed dynamical description in the long-time limit. Here, a surprising feature is the general ”turbulent” behavior of multi-solitons with $N \geq 2$, leading to unbounded growth of higher Sobolev norms (energy cascades) as follows.

Theorem 1.3 (Growth of Sobolev Norms). *For every $N \geq 2$, every multi-soliton $u = u(t,x)$ for (CM-DNLS) exists for all times $t \in \mathbb{R}$ and it exhibits growth of Sobolev norms such that*

$$\|u(t)\|_{H^s} \sim_s |t|^{2s} \quad as \quad t \to \pm\infty,$$

for any real number $s > 0$.

Remarks.

(1) In Section 6, we make a detailed analysis of the dynamics of multi-solitons. After having established their global-in-time existence, we show that there exists a sufficiently large time $T = T(u_0) \gg 1$ such that a multi-soliton reads

$$u(t,x) = \sum_{k=1}^{N} \frac{a_k(t)}{x - z_k(t)} \quad \text{for } t \geq T, \tag{1.4}$$

with residues $a_1(t), \dots, a_N(t) \in \mathbb{C}$ and pairwise distinct poles $z_1(t), \dots, z_N(t) \in \mathbb{C}_-$ that satisfy a complexified version of the rational Calogero–Moser system for N classical particles. A detailed investigation (exploiting on the Lax pair structure) then yields that the poles—except for $z_1(t)$ – will all approach the real axis asymptotically, that is,

$$\operatorname{Im} z_k(t) \to 0 \quad \text{as} \quad t \to +\infty \quad \text{for } 2 \leq k \leq N.$$

A careful analysis of this fact then leads to the precise growth bound in Theorem 1.3. The limit $t \to -\infty$ can be handled in the same way.

(2) It is a subtle fact that multi-solitons $u(t,x)$ may fail to be of the form (1.4) for *all* times $t \in \mathbb{R}$. That is, we can have collisions of poles in finite time, which renders the form (1.4) invalid. To handle this collision scenario (see explicit examples for $N = 2$ in Section 6), we will make use of a general representation formula of $u(t,x)$ in terms of an inverse spectral formula. See Section 6 for details.

(3) It is an interesting open question whether the growth phenomenon in Theorem 1.3 is stable under perturbations of multi-solitons.

1.3 | Comments on the Lax structure

A central feature of (CM-DNLS) is the fact that it admits a Lax pair. That is, as detailed in Section 2, we can recast the dynamical evolution into commutator form

$$\frac{d}{dt} L_u = [B_u, L_u], \tag{1.5}$$

where the Lax operator L_u is given by

$$L_u = D - T_u T_{\bar{u}}. \tag{1.6}$$

This defines an unbounded self-adjoint operator acting the Hardy space $L^2_+(\mathbb{R})$ with a suitable operator domain, depending on the regularity of u. Here $T_b(f) = \Pi_+(bf)$ denotes the Toeplitz operator on $L^2_+(\mathbb{R})$ with symbol b.

As an important consequence of (1.5), we will find an *infinite hierarchy of conservation laws* in terms of expressions

$$I_k(u) = \langle L_u^k u, u\rangle \quad \text{with} \quad k = 0, 1, 2, \dots$$

provided that $u(t) \in H^s_+(\mathbb{R})$ is a sufficiently regular solution of (CM-DNLS). It is an intriguing feature, due to the L^2-criticality of the problem, that the hierarchy $\{I_k(u)\}_{k\in\mathbb{N}}$ will generally provide a-priori bounds on solutions only if we have sub-critical L^2-mass; see Theorem 1.1. This is in striking contrast to many other completely integrable PDEs (e. g., KdV, Benjamin-Ono, cubic NLS etc.) where the corresponding hierarchy of conservation laws yields control over any smooth solutions without any assumption on its size.

In Section 5, we examine the spectral properties of the Lax operator L_u in more detail. Based on a key commutator formula, we find a sharp bound on the number of eigenvalues N of the form

$$N \leq \frac{\|u\|^2_{L^2}}{2\pi}. \tag{1.7}$$

Furthermore, we prove that every eigenvalue of L_u is simple. As interesting aside, we remark that this bound not only applies to isolated eigenvalues, but also to eigenvalues which are embedded in the essential spectrum $\sigma_{\mathrm{ess}}(L) = [0,\infty)$. Moreover, we emphasize the fact that we can easily generate embedded eigenvalues of L_u by action of the Beurling–Lax semigroup $\{e^{i\eta x}\}_{\eta\geq 0}$ acting on $L^2_+(\mathbb{R})$; see Section 5 again.

In terms of spectral theory, it is a natural question to study which potentials $u \in L^2_+(\mathbb{R})$ will lead to equality in the general bound (1.7). Here we will find a distinguished class of potentials given by rational functions of the form

$$u(x) = \frac{P(x)}{Q(x)} \in H^1_+(\mathbb{R}),$$

where $Q, P \in \mathbb{C}[x]$ are suitable polynomials with $\deg Q = N$ and $\deg P \leq N-1$; see Proposition 5.2. We will refer to these $u(x) = P(x)/Q(x)$ as above as *multi-soliton potentials* and the corresponding solutions will be called *multi-solitons* for (CM-DNLS). As an immediate consequence of saturating the bound (1.7), we obtain the multi-solitons $u(t,x)$ have quantized L^2-mass with

$$M(u) = 2\pi N.$$

Another noteworthy feature of any multi-soliton solution $u(t,x)$ is that it is *completely* supported in the pure point spectrum of the Lax operator, that is, we have

$$u(t) \in \mathcal{E}_{pp}(L_{u(t)}),$$

where $\mathcal{E}_{pp}$ denotes the N-dimensional space spanned the eigenfunctions of L_u. This fact will allow us to derive a very explicit inverse spectral formula representing a multi-soliton. This will enable us to prove global-in-time existence and, more strikingly, the growth bounds states in Theorem 1.3. In the future, we plan to further refine the spectral analysis of L_u in order to study the long-time behavior of solutions of (CM-DNLS) beyond the case of multi-solitons.

Finally, we remark that the Lax structure for (CM-DNLS) bears some resemblance to the Lax structure for (BO), which is known to have the Lax operator

$$L_u^{(\mathrm{BO})} = D - T_u$$

acting on the Hardy space $L^2_+(\mathbb{R})$; see for example [16]. Note that the occurrence of T_u in $L_u^{(\mathrm{BO})}$ instead of $T_u T_{\bar{u}}$ in L_u is consistent with the different degrees of the nonlinearity (quadratic vs. cubic).

1.4 | Comparison to other PDEs

Let us comment on the energy cascade phenomenon in Theorem 1.3 in comparison to other Hamiltonian PDEs on the line. Among the recently studied Hamiltonian PDEs on the line, the closest one to (CM-DNLS) is certainly the derivative nonlinear Schrödinger equation,

$$i\partial_t u + \partial_{xx} u + i\partial_x(|u|^2 u) = 0, \tag{DNLS}$$

which—like (CM-DNLS) – is L^2-mass critical with a Lax pair structure. Using the Lax pair structure, global existence was first proved in [20] in the space $H^2(\mathbb{R}) \cap \hat{H}^2(\mathbb{R})$. Then global well-posedness with uniform bounds in H^s, $s \geq 1/2$ was obtained in [4, 5]. Quite recently, the flow map was extended to the whole of $L^2(\mathbb{R})$ in [18], proving that all trajectories of (DNLS) are uniformly equicontinuous with values in $L^2(\mathbb{R})$. All these results prevent any kind of energy cascade and are therefore in strong contrast with the dynamics of (CM-DNLS), which turns out to be much richer.

Another integrable Hamiltonian PDE on the line is the cubic Szegő equation, see [31],

$$i\partial_t u = \Pi_+(|u|^2 u), \tag{1.8}$$

where multi-solitons were recently studied in [17], and where energy cascades were displayed under some degeneracy assumption of the spectrum of the corresponding Lax operator. There is definitely some similarity in the approaches to multi-solitons in (1.8) and (CM-DNLS), particularly in the inverse spectral formulae. However, let us emphasize that the spectral properties of the Lax operators are very different, and that energy cascades for multi-solitons in (1.8) only occur under some degeneracy assumption, while they always occur for multi-solitons in (CM-DNLS). This suggests that the dynamics of (CM-DNLS) is particularly turbulent, even compared to the non-dispersive Equation (1.8). We hope to explore other aspects of this dynamics in the near future.

1.5 | Notation

We denote by $\langle f, g\rangle = \int_{\mathbb{R}} f\overline{g}$ the L^2-inner product of functions f, g on the line. We recall that Π_+ denotes the orthogonal projector from $L^2(\mathbb{R})$ onto the Hardy space $L^2_+(\mathbb{R})$. Sometimes, we will also use the notation $\Pi_- = 1 - \Pi_+$. Notice that, for every L^2 function f,

$$\Pi_-(f) = \overline{\Pi_+(\overline{f})}\,.$$

Finally, observe that the Hilbert transform $\mathsf{H} = -i\,\mathrm{sgn}(D)$ is related to $\Pi_\pm$ by the identities

$$\Pi_+ = \frac{1}{2}(1 + i\mathsf{H}), \; \Pi_- = \frac{1}{2}(1 - i\mathsf{H}).$$

2 | WELL-POSEDNESS, LAX STRUCTURE, AND CONSERVATION LAWS

In this section, we study the Cauchy problem for (CM-DNLS) in $H^s_+(\mathbb{R})$ with suitable s. As a key element for obtaining global-in-time solutions, we will find a Lax pair structure on the Hardy-type space H^s_+, which will generate an infinite hierarchy of conservation laws. With this at hand, we will derive a-priori bounds for initial $u_0 \in H^s_+(\mathbb{R})$ with integer $s \geq 1$ and sub-critical L^2-mass $M(u_0) < M(\mathcal{R})$.

2.1 | Local well-posedness

As starting point for local well-posedness, we consider the case of initial data in $H^s_+(\mathbb{R})$ with $s > 3/2$, where Kato's classical iterative scheme for quasilinear evolution equations can be utilized. We remark that the presence of the derivative term $D_+(|u|^2)u$ raises some analytic challenges that need to be addressed.

Proposition 2.1. *Let $s > 3/2$. For any $R > 0$, there is some $T(R) > 0$ such that, for every $u_0 \in H^s_+(\mathbb{R})$ with $\|u_0\|_{H^s} \leq R$, there exists a unique solution $u \in C([-T,T]; H^s_+(\mathbb{R}))$ of (CM-DNLS) with $u(0) = u_0$.*

Furthermore, the H^σ-regularity of u_0 for $\sigma > s$ is propagated on the whole maximal interval of existence of u, and the flow map $u_0 \mapsto u(t)$ is continuous on H^s.

Proof. As already mentioned above, we apply a Kato-type iterative scheme to obtain this result. For concreteness, we shall consider the case

$$s = 2$$

in what follows. The general case $s > 3/2$ can be handled in an analogous way.

We first write (CM-DNLS) as

$$\partial_t u = i\partial_{xx} u + 2T_u T_{\overline{u}} \partial_x u + 2u H_u \partial_x u, \tag{2.1}$$

where

$$T_a f := \Pi_+(af), \quad H_b f := \Pi_+(b\overline{f}) \tag{2.2}$$

denote the *Toeplitz* and *Hankel operators* acting on $L^2_+(\mathbb{R})$ with symbols a and b, respectively. Our first observation is that the term $H_u \partial_x u$ is of order 0 if u is smooth enough.

Lemma 2.1. *If $u \in H_+^{\frac{3}{2}}(\mathbb{R})$, then $H_u \partial_x : L_+^2(\mathbb{R}) \to L_+^2(\mathbb{R})$ is bounded with $\|H_u(\partial_x f)\|_{L^2} \le \frac{1}{\sqrt{2\pi}} \|u\|_{\dot H^{3/2}} \|f\|_{L^2}$. If $u, v \in H_+^2(\mathbb{R})$, then $H_u \partial_x v \in H_+^2(\mathbb{R})$ with*

$$\|H_u \partial_x v\|_{H^2} \le C \|u\|_{H^2} \|v\|_{H^2}$$

with some constant $C > 0$.

Proof. Let $f \in H_+^1(\mathbb{R})$. Then

$$\widehat{H_u \partial_x f}(\xi) = -\int_0^\infty \hat{u}(\xi+\eta) \overline{\hat{f}(\eta)} \frac{d\eta}{2\pi} \quad \text{for} \quad \xi \ge 0.$$

Consequently,

$$\begin{aligned} |\widehat{H_u \partial_x f}(\xi)|^2 &\le \left| \int_0^\infty |\hat{u}(\xi+\eta)| |\xi+\eta| |\hat{f}(\eta)| \frac{d\eta}{2\pi} \right|^2 \\ &\le \int_0^\infty |\hat{u}(\xi+\eta)|^2 (\xi+\eta)^2 \frac{d\eta}{2\pi} \cdot \int_0^\infty |\hat{f}(\eta)|^2 \frac{d\eta}{2\pi}. \end{aligned}$$

Thus

$$\begin{aligned} \|H_u \partial_x f\|_{L^2}^2 &\le \int_0^\infty \int_0^\infty |\hat{u}(\xi+\eta)|^2 (\xi+\eta)^2 \frac{d\eta}{2\pi} \frac{d\xi}{2\pi} \|f\|_{L^2} \\ &\le \int_0^\infty |\hat{u}(\zeta)|^2 \zeta^3 \frac{d\zeta}{4\pi^2} \|f\|_{L^2}^2 = \frac{1}{2\pi} \|u\|_{\dot H^{3/2}}^2 \|f\|_{L^2}^2. \end{aligned}$$

By density, this bound extends to all $f \in L_+^2(\mathbb{R})$. This proves the first claim of the lemma.

For the second statement, we note that this follows from the first statement combined with Sobolev embeddings and the identity

$$\partial_{xx}(H_u \partial_x v) = H_u \partial_x (\partial_{xx} v) + 2 H_{\partial_x u} \partial_x v + H_{\partial_{xx} u} \partial_x v.$$

This completes the proof of Lemma 2.1. □

In view of (2.1), we consider the following iteration scheme

$$\partial_t u^{k+1} = i \partial_{xx} u^{k+1} + 2 T_{u^k} T_{\overline{u}^k} \partial_x u^{k+1} + 2 u^k H_{u^k} \partial_x u^k \tag{2.3}$$

with initial datum $u^{k+1}(0,x) = u_0(x) \in H_+^2(\mathbb{R})$. Notice that $T_u T_{\overline{u}}$ is a self-adjoint operator. Hence a standard energy methods yields the following result.

Lemma 2.2. *Let $u \in C([-T,T], H_+^2(\mathbb{R}))$ with some $T > 0$, $p \in \{0,1,2\}$, and $w_0 \in H_+^p(\mathbb{R})$, $f \in L^1([-T,T]; H_+^p(\mathbb{R}))$. Then there exists a unique $w \in C([-T,T]; H_+^p(\mathbb{R}))$ such that*

$$\partial_t w = i \partial_{xx} w + 2 T_u T_{\overline{u}} w + f, \quad w(0,x) = w_0(x).$$

Furthermore,

$$\sup_{|t|\leq T} \|w(t)\|_{H^p} \leq C e^{C\int_{-T}^{T} \|u(t)\|_{H^2}^2 \, dt} \left(\|w_0\|_{H^p} + \|f\|_{L_t^1 H^p} \right).$$

Coming back to the scheme (2.3), we see that Lemmas 2.1 and 2.2 allow us to construct by induction a sequence (u^k) in $C(\mathbb{R}; H_+^2(\mathbb{R}))$ with $u^0(t,x) = u_0(x)$. We are now going to prove that if $\|u_0\|_{H^2} \leq R$ and $T(R) > 0$ suitably chosen, then the sequence (u^k) is bounded in $H_+^2(\mathbb{R})$ and uniformly convergent in $L_+^2(\mathbb{R})$ for $|t| \leq T(R)$.

Let us first prove that (u^k) is bounded in $H_+^2(\mathbb{R})$ for $|t| \leq T(R)$ with suitably chosen $T(R) > 0$. Indeed, by using the second estimate in Lemma 2.1 together with the bound in Lemma 2.2 for $p = 2$, we obtain

$$\sup_{|t|\leq T} \|u^{k+1}(t)\|_{H^2} \leq C e^{C\int_{-T}^{T} \|u^k(t)\|_{H^2}^2 \, dt} \left(\|u_0\|_{H^2} + \int_{-T}^{T} \|u^k(t)\|_{H^2}^3 \, dt \right).$$

Assume $\|u_0\|_{H^2} \leq R$ and let $R_1 = (1 + C)R$. Since $R_1 > CR$, we can choose $T = T(R) > 0$ such that

$$C e^{2CTR_1^2} (R + 2TR_1^3) \leq R_1.$$

By an elementary induction argument, we find that $\sup_{|t|\leq T(R)} \|u^k(t)\|_{H^2} \leq R_1$ for all k.

Next, we show that we have a contraction property of the sequence (u^k) in $L_+^2(\mathbb{R})$ for $|t| \leq T(R)$ as follows. Observe that

$$\begin{aligned} \partial_t (u^{k+1} - u^k) &= i\partial_{xx}(u^{k+1} - u^k) + 2T_{u^k} T_{\overline{u}^k} \partial_x (u^{k+1} - u^k) + \\ &+ 2(T_{u^k} T_{\overline{u}^k} - 2T_{u^{k-1}} T_{\overline{u}^{k-1}}) \partial_x u^k + 2u^k H_{u^k} \partial_x u^k - 2u^{k-1} H_{u^{k-1}} \partial_x u^{k-1}. \end{aligned}$$

Using the estimate of Lemma 2.2 with $p = 0$ and the bound on u^k in H^2, we infer

$$\sup_{|t|\leq T} \|u^{k+1}(t) - u^k(t)\|_{L^2} \leq KT \sup_{|t|\leq T} \|u^k(t) - u^{k-1}(t)\|_{L^2}$$

with some constant $K > 0$. If we choose $T = T(R) > 0$ from above small enough to ensure that $KT < 1$, then the series $\sup_{|t|\leq T(R)} \|u^{k+1}(t) - u^k(t)\|_{L^2}$ is geometrically convergent.

Finally, the sequence (u^k) is uniformly weakly convergent in $C([-T,T]; H_+^2(\mathbb{R}))$ and strongly convergent in $C([-T,T]; L_+^2(\mathbb{R}))$. Hence its limit $u(t)$ solves (2.1) – and therefore (CM-DNLS). (To prove that the limit $u(t)$ actually belongs to $C([-T,T]; H_+^2(\mathbb{R}))$, we can invoke Tao's frequency envelope method [35] or adapt an argument due to Bona-Smith [8].)

Uniqueness follows along the same lines as the contraction property in $L_+^2(\mathbb{R})$. The proof of Proposition 2.1 is now complete. □

Following the analysis in [13], we can further lower the regularity for local well-posedness to initial data in $H_+^s(\mathbb{R})$ with $s > 1/2$. In particular, we can reach the energy space $H_+^1(\mathbb{R})$ for (CM-DNLS). In fact, the arguments adapt Tao's frequency localized gauge transform introduced to treat low regularity solutions for the Benjamin-Ono equation. For (CM-DNLS), we obtain the following local well-posedness result.

Theorem 2.1 (Local Well-Posedness in H^s_+ with $s > 1/2$). *Let $u_0 \in H^s_+(\mathbb{R})$ with some $s > 1/2$. Then there exist a time $T = T(\|u_0\|_{H^s}) > 0$ and some Banach space $Z_{s,T} \subset C([-T,T]; H^s_+(\mathbb{R}))$ and a unique solution $u \in Z_{s,T}$ of (CM-DNLS) with initial datum $u(0) = u_0$.*

Furthermore, the H^σ-regularity of u_0 for $\sigma > s$ is propagated on the whole maximal interval of existence of u, and the flow map $u_0 \mapsto u(t)$ is continuous on H^s.

Remark. Recall that (CM-DNLS) is L^2-critical with respect to scaling. It remains a fundamental open problem to understand the case $u \in H^s_+(\mathbb{R})$ when $0 \le s \le 1/2$.

Proof. We can adapt the estimates proven in [13] to our case. Suppose that $u_0 \in H^s_+(\mathbb{R})$ with some $s > 1/2$. For $\varepsilon > 0$, let $\eta_\varepsilon(x) = \sqrt{\frac{1}{\varepsilon\pi}}\mathrm{e}^{-|x|^2/\varepsilon}$ be a Gaussian mollifier. Then $u_{0,\varepsilon}(x) = (\eta_\varepsilon * u_0)(x)$ satisfies $u_{0,\varepsilon} \in H^\infty_+(\mathbb{R}) \subset H^2_+(\mathbb{R})$. By Proposition 2.1, there exists a unique solution $u_\varepsilon \in C([-T_\varepsilon, T_\varepsilon]; H^2_+(\mathbb{R}))$ with $u_\varepsilon(0) = u_{0,\varepsilon}$. We can now apply the arguments in the proof of Theorem 1.1 in [13]. First, we can show that there exists $T = T(\|u_0\|_{H^s}) > 0$ satisfying $T_\varepsilon \ge T$ for all $\varepsilon > 0$. Then following Proposition 3.2 in [13] we see that (u_ε) is Cauchy in $Z_{s,T}$ as $\varepsilon \to 0$; we refer to [13] for the definition of the Banach space $Z_{s,T}$. Finally, the uniqueness of the limit of (u_ε) can be proven by the estimate (3.40) in [13]. □

2.2 | Lax pair and conservation laws

In this subsection, we will show that (CM-DNLS) admits a *Lax pair* with certain densely defined operators L_u and B_u acting on the Hardy space $L^2_+(\mathbb{R})$. Here we will exploit this fact to derive an infinite hierachy of conservation laws. For an analysis of the spectral properties of L_u, we refer to Section 6 below.

For $u \in H^s_+(\mathbb{R})$ with some $s \ge 0$, we formally define the operators L_u and B_u acting on $L^2_+(\mathbb{R})$ by setting

$$\boxed{L_u = D - T_u T_{\bar{u}} \quad \text{and} \quad B_u = T_u T_{\partial_x \bar{u}} - T_{\partial_x u} T_{\bar{u}} + i(T_u T_{\bar{u}})^2} \tag{2.4}$$

Here $T_b(f) = \Pi_+(bf)$ denotes the *Toeplitz operator* on $L^2_+(\mathbb{R})$ with symbol $b \in L^2(\mathbb{R}) + L^\infty(\mathbb{R})$. For $u \in H^1_+(\mathbb{R}) \subset L^\infty(\mathbb{R})$, we readily check that T_u and $T_{\bar{u}}$ are bounded operators on $L^2_+(\mathbb{R})$. Thus, for $u \in H^1_+(\mathbb{R})$, it is straightforward to verify that L_u is semibounded and a self-adjoint operator, that is,

$$L_u^* = L_u$$

with operator domain $\mathrm{dom}(L_u) = H^1_+(\mathbb{R})$. For $u \in H^2_+(\mathbb{R})$, we readily check that $B_u = -B_u^*$ is a skew-adjoint and bounded operator on $L^2_+(\mathbb{R})$.

Remark 2.1. In Appendix A below, we detail how L_u can be defined via quadratic forms if we only assume that $u \in L^2_+(\mathbb{R})$, which is a natural class in view of the L^2-criticality of (CM-DNLS).

Next, we see that L_u and B_u form indeed a Lax pair for the Calogero–Moser DNLS.

Lemma 2.3 (Lax Equation). *If $u \in C([0,T]; H^s_+(\mathbb{R}))$ solves (CM-DNLS) with $s \geq 0$ sufficiently large (e.g., with $s = 2$), then it holds*

$$\frac{d}{dt} L_u = [B_u, L_u].$$

Proof. We divide the proof into the following steps.

Step 1. We first calculate the commutators

$$I := [T_u T_{\partial_x \bar{u}}, D], \quad II := [T_{\partial_x u} T_{\bar{u}}, D], \quad III := i[(T_u T_{\bar{u}})^2, D].$$

We find

$$I = T_u [T_{\partial_x \bar{u}}, D] + [T_u, D] T_{\partial_x \bar{u}} = T_u T_{i\partial_{xx} \bar{u}} + T_{i\partial_x u} T_{\partial_x \bar{u}},$$

$$II = T_{\partial_x u}[T_{\bar{u}}, D] + [T_{\partial_x u}, D] T_{\bar{u}} = T_{\partial_x u} T_{i\partial_x \bar{u}} + T_{i\partial_{xx} u} T_{\bar{u}}.$$

In addition, we see

$$\begin{aligned} III &= i T_u T_{\bar{u}} [T_u T_{\bar{u}}, D] + i [T_u T_{\bar{u}}, D] T_u T_{\bar{u}} \\ &= i T_u T_{\bar{u}} \big(T_u T_{i\partial_x \bar{u}} + T_{i\partial_x u} T_{\bar{u}} \big) + i \big(T_u T_{i\partial_x \bar{u}} + T_{i\partial_x u} T_{\bar{u}} \big) T_u T_{\bar{u}}. \end{aligned}$$

As a next step, we consider the terms

$$IV := [T_u T_{\partial_x \bar{u}}, T_u T_{\bar{u}}], \quad V := [T_{\partial_x u} T_{\bar{u}}, T_u T_{\bar{u}}], \quad VI = i[(T_u T_{\bar{u}})^2, T_u T_{\bar{u}}].$$

We find

$$IV = T_u T_{\partial_x \bar{u}} T_u T_{\bar{u}} - T_u T_{\bar{u}} T_u T_{\partial_x \bar{u}}, \quad V = T_{\partial_x u} T_{\bar{u}} T_u T_{\bar{u}} - T_u T_{\bar{u}} T_{\partial_x u} T_{\bar{u}},$$

and, clearly, we have $VI = 0$. If we combine all commutator terms, we see

$$\begin{aligned} [B_u, L_u] &= I - II + III - IV + V \\ &= T_u T_{i\partial_x x \bar{u}} - T_{i\partial_{xx} u} T_{\bar{u}} - 2 T_u T_{\bar{u}} T_{\partial_x u} T_{\bar{u}} - 2 T_u T_{\partial_x \bar{u}} T_u T_{\bar{u}} \\ &= T_u T_{i\partial_{xx} \bar{u}} - T_{i\partial_{xx} u} T_{\bar{u}} - 2 T_u T_{\partial_x |u|^2} T_{\bar{u}}, \end{aligned}$$

where in the last step we used that $u \in H^1_+(\mathbb{R})$, which implies that

$$T_{\bar{u}} T_{\partial_x u} + T_{\partial_x \bar{u}} T_u = T_{\bar{u}\partial_x u + \partial_x \bar{u} u} = T_{\partial_x |u|^2}$$

holds on $L^2_+(\mathbb{R})$.

Step 2. We now calculate

$$\begin{aligned} \frac{d}{dt} L_u &= -T_{\dot{u}} T_{\bar{u}} - T_u T_{\dot{\bar{u}}} \\ &= -T_{i\partial_{xx} u} T_{\bar{u}} - 2 T_{\Pi_+(\partial_x |u|^2) u} T_{\bar{u}} - T_u T_{-i\partial_{xx} \bar{u}} - 2 T_u T_{\overline{\Pi_+(\partial_x |u|^2)} \bar{u}} \end{aligned}$$

In view of the expression for $[B_u, L_u]$ derived in Step 1, it remains to show the identity

$$T_{\Pi_+(\partial_x |u|^2) u} T_{\bar{u}} + T_u T_{\overline{\Pi_+(\partial_x |u|^2)} \bar{u}} = T_u T_{\partial_x |u|^2} T_{\bar{u}} \tag{2.5}$$

Indeed, using that $u \in H^1_+(\mathbb{R})$, we find

$$T_{\Pi_+(\partial_x|u|^2)u} = T_u T_{\Pi_+(\partial_x|u|^2)} \quad \text{and} \quad T_{\overline{\Pi_+(\partial_x|u|^2)\bar{u}}} = T_{\overline{\Pi_+(\partial_x|u|^2)}} T_{\bar{u}}.$$

Since $\partial_x|u|^2 = \Pi_+(\partial_x|u|^2) + \overline{\Pi_+(\partial_x|u|^2)}$, we deduce that (2.5) holds true. This completes the proof of Lemma 2.3. ☐

Remark 2.2. Since we have that $[L_u, L_u^2] = 0$, the skew-adjoint operator

$$\widetilde{B}_u = B_u - iL_u^2$$

also satisfies the Lax equation $\frac{d}{dt}L_u = [\widetilde{B}_u, L_u]$. A direct calculation shows that

$$\widetilde{B}_u = -iD^2 + 2iT_u D T_{\bar{u}} \tag{2.6}$$

Note that $\widetilde{B}_u$ is an unbounded skew-adjoint operator on $L^2_+(\mathbb{R})$ with operator domain $\mathrm{dom}(\widetilde{B}_u) = H^2_+(\mathbb{R})$. Actually, we first found the operator $\widetilde{B}_u$ in the analysis of the Lax structure of (CM-DNLS). We also note that the relation between B_u and $\widetilde{B}_u$ is reminiscent to the Lax structure for the Benjamin-Ono equation used in [16].

As a consequence of the Lax equation, we obtain an infinite hierarchy of conservation of laws for (CM-DNLS) as follows

Lemma 2.4 (Hierarchy of Conservation Laws). *Let $u \in C([0,T], H^s_+(\mathbb{R}))$ be a solution of (CM-DNLS) with sufficiently large $s \geq 0$. Then, for every $\lambda \notin \sigma(L_{u(0)})$, we have the conserved quantity*

$$\mathcal{H}_\lambda(u) := \langle (L_u - \lambda I)^{-1} u, u \rangle.$$

As a consequence, if $u \in C([0,T]; H^{n/2}_+(\mathbb{R}))$ with some $n \in \mathbb{N}$, the quantities

$$I_k(u) := \langle L_u^k u, u \rangle \quad \text{with } k = 0, \dots, n$$

are conserved, where $\langle \cdot, \cdot \rangle$ denotes the dual pairing of $H^{-n/2}_+$ and $H^{n/2}_+$.

Proof. Let $\lambda \notin \sigma(L_{u_0})$, which by the Lax equation implies that $\lambda \notin \sigma(L_u)$ for all $u = u(t)$ with $t \in [0,T]$. A quick calculation reveals that (CM-DNLS) can be written as

$$\partial_t u = \widetilde{B}_u u$$

with the operator $\widetilde{B}_u = B_u - iL_u^2$; see also (2.6) above. Using that $\frac{d}{dt}L_u = [\widetilde{B}_u, L_u]$, it is elementary to verify that

$$\frac{d}{dt}\mathcal{H}_\lambda(u) = 0.$$

Finally, we note the expansion

$$\mathcal{H}_\lambda(u) = \sum_{k=0}^{\infty} \lambda^{-(k+1)} \langle L_u^k u, u \rangle$$

for all real $\lambda < 0$ sufficiently negative (using that L_u is bounded below) and with $u \in H^\infty_+(\mathbb{R}) = \cap_{s\geq 0} H^s_+(\mathbb{R})$. Thus we deduce that $I_k(u) = \langle L_u^k u, u\rangle$ are constant in time for solutions in $H^\infty_+(\mathbb{R})$. The conservation laws $I_0(u), \dots, I_n(u)$ for solutions $u \in C([0,T]; H^{n/2}_+(\mathbb{R}))$ follow from an approximation argument, which we omit. □

2.3 | Global existence for sub-critical L^2-mass

As an application of Lemma 2.4, we deduce the following global-in-time existence result.

Corollary 2.1 (Global Existence and Bounds for sub-critical L^2-Mass). *Let $u_0 \in H^s_+(\mathbb{R})$ with some integer $s \geq 1$ and suppose that*

$$M(u_0) < M(\mathcal{R}) = 2\pi.$$

Then the corresponding solution $u(t) \in H^s_+(\mathbb{R})$ of (CM-DNLS) exists for all times $t \in \mathbb{R}$ with the a-priori bound

$$\sup_{t\in\mathbb{R}} \|u(t)\|_{H^s} < +\infty.$$

Remarks.

(1) The condition $s \geq 1$ arises from the current state of the local well-posedness theory for (CM-DNLS). It is conceivable that, with some great effort though, that this result extends to initial data in $H^0_+(\mathbb{R}) = L^2_+(\mathbb{R})$ satisfying $\|u_0\|^2_{L^2} < 2\pi$.
(2) From Theorem 1.3 we deduce that the infinite hierarchy of conservation laws

$$I_k(u(t)) = I_k(u_0) \quad \text{with } k = 1, 2, \dots$$

fail in general to produce a-priori bounds on H^s-norms for $s > 0$ for solutions with initial data with L^2-mass $M(u_0) \geq M(\mathcal{R}) = 2\pi$.

Proof. We first consider the case $s = 1$. By the local well-posedness theory in $H^1_+(\mathbb{R})$, we need to find an a-priori bound on $\sup_{t\in I} \|u(t)\|_{H^1}$, where $I \subset \mathbb{R}$ denotes the maximal time interval of existence. Indeed, we have the conservation laws

$$I_0(u) = \langle u, u\rangle = \|u\|^2_{L^2},$$

$$I_1(u) = \langle L_u u, u\rangle = \langle Du, u\rangle - \|T_{\bar{u}} u\|^2_{L^2}.$$

Now, we use the sharp inequality (see Lemma A.1):

$$\|T_{\bar{u}} u\|^2_{L^2} \leq \frac{1}{2\pi} \|u\|^2_{L^2} \langle Du, u\rangle.$$

Therefore if we assume that $\|u_0\|^2_{L^2} < 2\pi$, we deduce the a-priori bound

$$\sup_{t\in I} \|u(t)\|_{H^{1/2}} \leq C(u_0).$$

Next, we use the conservation of energy together with a standard Gagliardo–Nirenberg interpolation and Sobolev inequality:

$$E(u) = 2I_2(u) = \langle L_u^2 u, u\rangle = \langle Du, Du\rangle - \frac{1}{2}\langle |u|^2, |D||u|^2\rangle + \frac{1}{12}\|u\|_{L^6}^6$$
$$\geq \|\partial_x u\|_{L^2}^2 - C\|u\|_{L^6}^3\|\partial_x u\|_{L^2} \geq \|\partial_x u\|_{L^2}^2 - C\|u\|_{H^{1/2}}^3\|\partial_x u\|_{L^2}.$$

From the a-priori bound on $\|u(t)\|_{H^{1/2}}$, we readily infer that $\sup_{t\in I}\|\partial_x u(t)\|_{L^2} \leq C(u_0)$. This completes the proof for $s = 1$.

The remaining case of integer $s \geq 2$ follows by iteration and using the conserved quantities

$$I_k(u) = \langle L^k u, u\rangle = \|u\|_{\dot{H}^{k/2}}^2 + \text{lower order terms}$$

with $k = 0, \dots, 2s$. We omit the details. □

2.4 | Proof of theorem 1.1 for $M(u_0) < M(\mathcal{R})$

The assertions in Theorem 1.1 in the case $M(u_0) < M(\mathcal{R})$ follow directly from Corollary 2.1.

The critical case $M(u_0) = M(\mathcal{R})$ will be discussed in the following section.

3 | NONEXISTENCE OF MINIMAL MASS BLOWUP

The goal of this section is to rule out finite-time minimal mass blowup for (CM-DNLS) with finite energy. As a consequence, we obtain that initial data $u(0) \in H_+^s(\mathbb{R})$ with some $s \geq 1$ with critical L^2-mass

$$M(u(0)) = M(\mathcal{R}) = 2\pi$$

will always lead to global-in-time solutions $u \in C(\mathbb{R}; H_+^s(\mathbb{R}))$, completing the proof of Theorem 1.1.

Notice that the absence of minimal mass blowup is in striking contrast to focusing L^2-critical NLS, where the existence of minimal mass blowup is a direct consequence of applying the pseudo-conformal transform to ground state solitary waves. For (CM-DNLS) on the other hand, we will see below that the mechanism that prevents the existence of minimal mass blowup is due to the slow algebraic decay of ground states $\mathcal{R} \in H_+^1(\mathbb{R})$ with $|\mathcal{R}(x)| \sim \frac{1}{|x|}$ as $|x| \to +\infty$.

We begin with the L^2-tightness property for H^1-solutions on finite time intervals.

Lemma 3.1. *Let $I \subset \mathbb{R}$ be an interval of finite length $|I| < \infty$ and suppose that $u \in C(I, H^1(\mathbb{R}))$ solves (CM-DNLS). Then the family $\{u(t)\}_{t\in I}$ is tight in $L^2(\mathbb{R})$, that is, for every $\varepsilon > 0$ there exists $R = R(\varepsilon) > 0$ such that*

$$\int_{|x|\geq R} |u(t,x)|^2\,dx \leq \varepsilon \quad \textit{for all } t \in I.$$

Remarks.

(1) It should be noted that no assumption on the size of the solution $u(t)$, that is, we do *not* assume that $M(u_0)$ is sufficiently small.
(2) Notice that we do not assume that $u(t)$ belongs to the Hardy space $H^1_+(\mathbb{R})$, but in fact we allow for general H^1-valued solutions of (CM-DNLS).

Proof. The idea is to adapt an elegant argument in [6] developed for studying minimal mass blowup for L^2-critical NLS. Let $\psi \in C^\infty(\mathbb{R})$ be a smooth real-valued function with bounded derivative $\partial_x \psi \in L^\infty(\mathbb{R})$. For any $a \in \mathbb{R}$ and $u \in H^1(\mathbb{R})$, the non-negativity of the energy implies that

$$E(\mathrm{e}^{ia\psi} u) \geq 0.$$

Expanding the right-hand side, we find

$$\begin{aligned} E(\mathrm{e}^{ia\psi} u) &= \frac{1}{2} \int_{\mathbb{R}} |\partial_x(\mathrm{e}^{ia\psi} u) - i\Pi_+(|u|^2)\mathrm{e}^{ia\psi} u|^2 \, dx \\ &= \frac{1}{2} \int_{\mathbb{R}} |ia\partial_x \psi u + \partial_x u - i\Pi_+(|u|^2)u|^2 \, dx \\ &= \frac{a^2}{2} \int_{\mathbb{R}} |\partial_x \psi|^2 |\partial_x u|^2 \, dx + a\mathrm{Re}\, \langle i\partial_x \psi u, \partial_x u - i\Pi_+(|u|^2)u \rangle + E(u). \end{aligned}$$

Now we observe that

$$\mathrm{Re}\, \langle i\partial_x \psi u, \partial_x u \rangle = \int_{\mathbb{R}} \partial_x \psi \cdot \mathrm{Im}(\overline{u}\partial_x u) \, dx,$$

$$\mathrm{Re}\langle i\partial_x \psi u, -i\Pi_+(|u|^2)u \rangle = -\frac{1}{2}\mathrm{Re}\langle \partial_x \psi u, (1 + i\mathsf{H})(|u|^2)u \rangle = -\frac{1}{2} \int_{\mathbb{R}} \partial_x \psi |u|^4 \, dx,$$

using that $\Pi_+ = \frac{1}{2}(1 + iH)$ and $\mathrm{Re}\langle u, iH(|u|^2)u \rangle = 0$ since $H(|u|^2)$ is real-valued. Therefore, the quadratic expansion in a together with $E(\mathrm{e}^{ia\psi} u) \geq 0$ implies that

$$\left| \int_{\mathbb{R}} \partial_x \psi \left(\mathrm{Im}(\overline{u}\partial_x u) - \frac{1}{2}|u|^4 \right) dx \right| \leq \sqrt{2E_0} \left(\int_{\mathbb{R}} |\partial_x \psi|^2 |\partial_x u|^2 \, dx \right)^{1/2}, \tag{3.1}$$

with the energy $E_0 = E(u) \geq 0$.

Next, we apply (3.1) to obtain the claimed L^2-tightness bound. From (CM-DNLS) we deduce

$$\partial_t |u|^2 = -2\partial_x \left(\mathrm{Im}(\overline{u}\partial_x u) - \frac{1}{2}|u|^4 \right) \tag{3.2}$$

in view of $\mathrm{Re}(\overline{u}(2iD_+|u|^2)u) = \mathrm{Re}((\partial_x |u|^2)|u|^2) = \frac{1}{2}\partial_x |u|^4$. Now let χ be a smooth nonnegative function such that $\chi(x) \equiv 0$ for $|x| \leq 1/2$ and $\chi(x) \equiv 1$ for $|x| \geq 1$. For $R > 0$, we set $\chi_R(x) = \chi(x/R)$. Integrating by parts and using (3.2) and (3.1), we infer that

$$\left| \frac{d}{dt} \int_{\mathbb{R}} \chi_R |u|^2 \, dx \right| = 2 \left| \int_{\mathbb{R}} \partial_x \chi_R \left(\mathrm{Im}(\overline{u}\partial_x u) - \frac{1}{2}|u|^4 \right) dx \right| \tag{3.3}$$

$$\lesssim \sqrt{E_0} \left(\int_{\mathbb{R}} |\partial_x \chi_R|^2 |u|^2 \, dx \right)^{1/2} \lesssim \frac{\sqrt{E_0 M_0}}{R} \tag{3.4}$$

with the conserved L^2-mass $M_0 = \|u\|_{L^2}^2$. If we integrate this bound over the finite time interval $I \subset \mathbb{R}$ with some $t_0 \in I$ fixed, we finally obtain

$$\int_{\mathbb{R}} \chi_R(x)|u(t,x)|^2\,dx \le \int_{\mathbb{R}} \chi_R(x)|u(t_0,x)|^2\,dx + \frac{C|I|}{R},$$

for all $t \in I$. This readily implies the claimed tightness bound. □

Theorem 3.1 (No Minimal Mass Blowup). *Let $I \subset \mathbb{R}$ with $0 \in I$ and finite length $|I| < \infty$. Suppose $u \in C(I; H^1(\mathbb{R}))$ solves (CM-DNLS) with $M(u_0) = M(\mathcal{R}) = 2\pi$. Then it holds*

$$\sup_{t\in I} \|u(t)\|_{H^1} < +\infty.$$

Remark. (1) Note again that we allow for general H^1-valued solutions $u(t)$.

Proof. We argue by contradiction. Without loss of generality, we can assume that $I = [0,T)$ with some finite time $T \in (0,+\infty)$ and let $u \in C([0,T); H^1(\mathbb{R}))$ satisfy

$$\lim_{t\to T^-} \|u(t)\|_{H^1} = +\infty.$$

Step 1. We first show that $u(t)$ must have finite variance, that is, we have

$$\int_{\mathbb{R}} |x|^2|u(t,x)|^2\,dx < +\infty \quad \text{for } t \in [0,T). \tag{3.5}$$

Here we adapt a strategy developed for classifying minimal-mass finite-time blowup solutions for L^2-critical NLS; originally due to Merle in [25]. To prove the claim (3.5), we follow the arguments laid out in [6, 19].

Let $t_n \to T^-$ be a sequence of times. We define

$$\varepsilon_n := \frac{1}{\|\partial_x u(t_n)\|_{L^2}}\,, \quad v_n(x) := \varepsilon_n^{1/2} u(t_n, \varepsilon_n x)\,.$$

Applying the Minimal Mass Bubble Lemma B.1 from Appendix B to v_n, after passing to a subsequence if necessary, there exist sequences $x_n \in \mathbb{R}$ and $\lambda_n > 0$ such that $\lambda_n \to 0$ and

$$\lambda_n^{1/2} u(t_n, \lambda_n(x+x_n)) \to e^{i\theta}\mathcal{R}(x) \quad \text{strongly in } L^2(\mathbb{R})$$

for some $\theta \in [0,2\pi[$, with the ground state $\mathcal{R} \in H^1_+(\mathbb{R})$ minimizing the energy functional $E(u)$ on $H^1(\mathbb{R})$. Thus we obtain

$$|u(t_n,x)|^2\,dx - \|\mathcal{R}\|_{L^2}^2 \delta_{x=\lambda_n x_n} \rightharpoonup 0$$

in the weak sense of measures. By the L^2-tightness property in Lemma 3.1, we easily deduce that $\lambda_n|x_n| \le C$ with some constant $C > 0$. From this fact (and passing to a subsequence) and by translational invariance we can henceforth assume that $\lambda_n x_n \to 0$ holds.

Next, let $\psi \in C_0^\infty(\mathbb{R})$ be a non-negative function such that $\psi(x) \equiv |x|^2$ for $|x| < 1$ and $|\partial_x \psi(x)|^2 \leq C\psi(x)$ with some constant $C > 0$. For $R > 0$, we set $\psi_R(x) = R^2\psi(x/R)$ and we define

$$g_R(t) = \int_{\mathbb{R}} \psi_R(x)|u(t,x)|^2\, dx.$$

In analogy to (3.3) based on (3.1) we find

$$\left|\frac{d}{dt} g_R(t)\right| \lesssim \int_{\mathbb{R}} |\partial_x \psi_R|^2 |u|^2\, dx \lesssim \sqrt{g_R(t)} \tag{3.6}$$

where the last step used that $|\partial_x \psi_R|^2 \lesssim \psi_R$ by construction. By integrating this on $[t, t_n]$ and using that $g_R(t_n) \to 0$ as $n \to \infty$, we deduce

$$g_R(t) = \int_{\mathbb{R}} \psi_R(x)|u(t,x)|^2\, dx \lesssim (T-t)^2 \quad \text{for } t \in [0,T).$$

Passing to the limit $R \to +\infty$, this yields

$$\int_{\mathbb{R}} |x|^2 |u(t,x)|^2\, dx \lesssim (T-t)^2 \quad \text{for } t \in [0,T). \tag{3.7}$$

In particular, this implies that (3.5) holds, showing that $u(t)$ has finite variance.

Step 2. By Step 1, we have $u_0 = u(0) \in \Sigma = H^1(\mathbb{R}) \cap L^2(\mathbb{R}; |x|^2 dx)$ and thus we can apply the pseudo-conformal identity (see Lemma C.1) to conclude that

$$8t^2 E(\mathrm{e}^{i|x|^2/4t} u_0) = \int_{\mathbb{R}} |x|^2 |u(t,x)|^2\, dx \lesssim (T-t)^2. \tag{3.8}$$

If we pass to the limit $t \to T^-$, we obtain

$$E(\mathrm{e}^{i|x|^2/4T} u_0) = 0. \tag{3.9}$$

Since $\|\mathrm{e}^{i|x|^2/4T} u_0\|_{L^2}^2 = 2\pi$, the uniqueness result in Lemma 4.1 implies that

$$\mathrm{e}^{i|x|^2/4T} u_0(x) = \mathcal{R}(x) \tag{3.10}$$

up to translation, phase, and scaling. However, the fact that $\mathrm{e}^{i|x|^2/4T} u_0 \in \Sigma$ contradicts that $\mathcal{R}(x)$ has infinite variance, that is,

$$\int_{\mathbb{R}} |x|^2 |\mathcal{R}(x)|^2\, dx = +\infty.$$

This contradiction shows that any $u(t) \in H^1(\mathbb{R})$ solving (CM-DNLS) with $M(u(0)) = M(\mathcal{R}) = 2\pi$ cannot blowup in finite time. This completes the proof of Theorem 3.1. Therefore, the proof of Theorem 1.1 is also complete. □

4 | GROUND STATES AND TRAVELING SOLITARY WAVES

In this section, we show that all ground states (minimizers) for the energy functional $E(u)$ for (CM-DNLS) are given by the rational function

$$\mathcal{R}(x) = \frac{\sqrt{2}}{x+i} \in H^1_+(\mathbb{R}) \tag{4.1}$$

modulo translation, phase, and scaling. As a second main result of this section, we prove that all traveling solitary waves in $H^1_+(\mathbb{R})$ for (CM-DNLS) are given by Galilean boosts (with positive velocity) of $\mathcal{R}$ up to the symmetries just mentioned.

We start with the following key result, which shows uniqueness of non-trivial solutions of the (first-order) Euler-Lagrange equation for minimizers of $E(u)$.

Lemma 4.1 (Uniqueness of Ground States). *Suppose that $u \in H^1(\mathbb{R})$ with $u \not\equiv 0$ solves*

$$Du - \Pi_+(|u|^2)u = 0.$$

Then it holds

$$u(x) = \mathrm{e}^{i\theta}\lambda^{1/2}\mathcal{R}(\lambda x + y)$$

with some $\theta \in [0, 2\pi)$, $\lambda > 0$, $y \in \mathbb{R}$, and $\mathcal{R} \in H^1_+(\mathbb{R})$ is given by (4.1).

As a consequence, all minimizers $u \in H^1(\mathbb{R}) \setminus \{0\}$ for $E(u)$ are of the form $u = \mathcal{R}$ modulo phase, translation, and scaling.

Remarks.

(1) Note we that only assume that $u \in H^1(\mathbb{R})$ and we obtain *a posteriori* that $u \in H^1_+(\mathbb{R})$ due to its explicit form.
(2) As an interesting aside, we remark that the equation for $u \in H^1(\mathbb{R})$ above can be recast into the *nonlocal Liouville equation* on the real line:

$$|D|w = \mathrm{e}^{w} \quad \text{in } \mathbb{R}. \tag{4.2}$$

To prove this claim (neglecting any technicalities of function spaces for simplicity), we first apply the gauge transform introducing the function $v = \mathrm{e}^{-i/2\int_{-\infty}^x |u|^2} u$, which leads to the equation for v given by

$$\partial_x v + \frac{1}{2}\mathsf{H}(|v|^2)v = 0 \quad \text{in } \mathbb{R}, \tag{4.3}$$

where we recall that H denotes the Hilbert transform. It is easy to see non-trivial solutions v are (up to a complex phase factor) strictly positive $v > 0$. Finally, if we define $w : \mathbb{R} \to \mathbb{R}$ by setting $w = \log(v^2)$, we readily check that w solves (4.2). In [11, 12, 37] it is proven that all solutions $w \in L^1(\mathbb{R}; \frac{dx}{1+x^2})$ of (4.2) are explicitly given by

$$w(x) = \log\left(\frac{2\lambda}{1 + \lambda^2(x-y)}\right) \tag{4.4}$$

with some constants $\lambda > 0$ and $y \in \mathbb{R}$. From this uniqueness result for w, we could obtain the result of Lemma 4.1. However, we will below give a self-contained (and short) uniqueness proof based on Hardy space arguments and complex ODEs. This method also provides (yet) another proof of the stated uniqueness result for w solving (4.2). See also [2] for a recent uniqueness result for solutions w of the nonlocal Liouville equation $|D|w = Ke^w$ in $\mathbb{R}$ with the prescribed function $K : \mathbb{R} \to \mathbb{R}$.

Proof of Lemma 4.1. We introduce the function

$$w := \Pi_+(|u|^2) \in H^1_+(\mathbb{R}),$$

so that $u' = iuw$ and $|u|^2 = w + \overline{w}$. We obtain the complex ordinary differential equation

$$w' = \Pi_+(u'\overline{u} + \overline{u}'u) = \Pi_+(iw(w + \overline{w}) - i\overline{w}(\overline{w} + w)) = iw^2,$$

using that $\Pi_+(\overline{w}^2) = 0$ for $w \in H^1_+(\mathbb{R})$. Since $w \not\equiv 0$, we deduce

$$w(x) = \frac{i}{x - z}$$

for some constant $z \in \mathbb{C}$ with $\mathrm{Im}\, z < 0$ because $w \in L^2_+(\mathbb{R})$. Consequently,

$$u' = iuw = -\frac{u}{x - z}.$$

Thus we have $(x - z)u(x) = c$ with some constant $c \in \mathbb{C}$. To determine c, we notice

$$\frac{i}{x - z} = \Pi_+(|u|^2) = \Pi_+\left(\frac{|c|^2}{|x - z|^2}\right) = \frac{|c|^2}{(\overline{z} - z)(x - z)}.$$

This implies that $|c|^2 = -2\mathrm{Im}\, z = 2|\mathrm{Im}\, z|$. In summary, we have found that

$$u(x) = \mathrm{e}^{i\theta}\frac{\sqrt{2|\mathrm{Im}\, z|}}{x - z}$$

with some $\theta \in [0, 2\pi)$ and $z \in \mathbb{C}_-$. From this fact we readily deduce that $u(x) = \mathrm{e}^{i\theta}\lambda^{1/2}\mathcal{R}(\lambda x + y)$ with $\lambda = -(\mathrm{Im} z)^{-1} > 0$ and $y = -\mathrm{Re}\, z$. □

Our next goal is to classify all traveling solitary wave solutions for (CM-DNLS). By a **traveling solitary waves** (with finite energy), we mean solutions of the form

$$u(t, x) = \mathrm{e}^{i\omega t}\mathcal{R}_{v,\omega}(x - vt) \tag{4.5}$$

where $\omega \in \mathbb{R}$ is a frequency parameter and $v \in \mathbb{R}$ denotes the velocity. Here the non-trivial profile $\mathcal{R}_{v,\omega} \in H^1(\mathbb{R})$ is allowed to depend on ω and v. Note that a-priori we allow also for $\mathcal{R}_{v,\omega}$ in $H^1(\mathbb{R})$ and not just restricted to $H^1_+(\mathbb{R})$. We have the following complete classification result, which shows that all traveling solitary waves for (CM-DNLS) are generated by Galilean boosts, translations, scaling and phase transformations of the ground state $\mathcal{R}(x)$.

Proposition 4.1. *Let $v, \omega \in \mathbb{R}$ and suppose $u(t,x)$ is a traveling solitary wave for (CM-DNLS) with profile $\mathcal{R}_{v,\omega} \in H^1(\mathbb{R}) \setminus \{0\}$. Then it holds*

$$\omega = -\frac{v^2}{4} \quad \textit{and} \quad \mathcal{R}_{v,\omega}(x) = \mathrm{e}^{\frac{i}{2}vx}\mathrm{e}^{i\theta}\lambda^{1/2}\mathcal{R}(\lambda x + y)$$

with some $\theta \in [0, 2\pi)$, $\lambda > 0$, and $y \in \mathbb{R}$. Moreover, we have $\mathcal{R}_{v,\omega} \in H^1_+(\mathbb{R})$ if and only if $v \geq 0$ holds.

Proof. We divide the proof into the following steps.

Step 1. We easily check that $\mathcal{R}_{v,\omega} \in H^1(\mathbb{R})$ must solve

$$-\partial_{xx}\mathcal{R}_{v,\omega} + iv\partial_x\mathcal{R}_{v,\omega} - 2D_+(|\mathcal{R}_{v,\omega}|^2)\mathcal{R}_{v,\omega} = \omega\mathcal{R}_{v,\omega}. \tag{4.6}$$

The term $iv\partial_x\mathcal{R}_{v,\omega}$ can be removed by a Galilean boost transform. That is, we write

$$\mathcal{R}_{v,\omega}(x) = \mathrm{e}^{\frac{i}{2}vx}\mathcal{R}_{\tilde{\omega}}(x). \tag{4.7}$$

An elementary calculation yields that $\mathcal{R}_{\tilde{\omega}} \in H^1(\mathbb{R})$ satisfies

$$-\partial_{xx}\mathcal{R}_{\tilde{\omega}} - 2D_+(|\mathcal{R}_{\tilde{\omega}}|^2)\mathcal{R}_{\tilde{\omega}} = \tilde{\omega}\mathcal{R}_{\tilde{\omega}} \quad \text{with} \quad \tilde{\omega} = \omega + \frac{v^2}{4}. \tag{4.8}$$

Step 2. We claim that every solution $\mathcal{R}_{\tilde{\omega}} \in H^1(\mathbb{R})$ of (4.8) has zero energy:

$$E(\mathcal{R}_{\tilde{\omega}}) = 0. \tag{4.9}$$

Since $\mathcal{R}_{\tilde{\omega}} \not\equiv 0$, we see from Lemma 4.1 that

$$\mathcal{R}_{\tilde{\omega}} = \mathcal{R}$$

modulo symmetries. As a consequence, we obtain $\tilde{\omega} = 0$ and therefore $\omega = -\frac{v^2}{4}$ as claimed.

The proof of (4.9) follows from applying the gauge transform together with a Pohozaev-type argument. Since $D_+(|\mathcal{R}_{\tilde{\omega}}|^2) \in L^2(\mathbb{R})$, we notice that Equation (4.8) tells us that $\mathcal{R}_{\tilde{\omega}} \in H^2(\mathbb{R})$ holds. Next, we apply the gauge transform Φ discussed in Appendix C to $\mathcal{R}_{\tilde{\omega}}$. That is, we set

$$S(x) := \Phi(\mathcal{R}_{\tilde{\omega}})(x) = \mathrm{e}^{-\frac{i}{2}\int_{-\infty}^x |\mathcal{R}_{\tilde{\omega}}(y)|^2\,dy}\mathcal{R}_{\tilde{\omega}}(x). \tag{4.10}$$

We directly check that $S \in H^2(\mathbb{R})$ and $|S|^2 = |\mathcal{R}_{\tilde{\omega}}|^2$. Using that $|D| = \mathsf{H}\partial_x$ and $\Pi_+ = \frac{1}{2}(1 + i\mathsf{H})$, a calculation yields that

$$-\partial_{xx}S - (|D||S|^2)S + \frac{1}{4}|S|^4 S = \tilde{\omega}S. \tag{4.11}$$

If we integrate this equation against $\overline{S}$, we directly obtain

$$\int_{\mathbb{R}} |\partial_x S|^2 - \int_{\mathbb{R}} |S|^2(|D||S|^2) + \frac{1}{4}\int_{\mathbb{R}} |S|^6 = \tilde{\omega}\int_{\mathbb{R}} |S|^2. \tag{4.12}$$

Next, we integrate (4.11) against $x\partial_x \overline{S}$ over the compact interval $[-R, R]$. By taking the real part and taking the limit $R \to +\infty$, we find the identity

$$\frac{1}{2}\int_{\mathbb{R}} |\partial_x S|^2 - \frac{1}{24}\int_{\mathbb{R}} |S|^6 = -\frac{\tilde{\omega}}{2}\int_{\mathbb{R}} |S|^2. \tag{4.13}$$

For details of this step, we refer to Appendix C. The combination of (4.12) and (4.13) yields

$$\frac{1}{2}\int_{\mathbb{R}} |\partial_x S|^2 - \frac{1}{4}\int_{\mathbb{R}} |S|^2(|D||S|^2) + \frac{1}{24}\int_{\mathbb{R}} |S|^6 = 0. \tag{4.14}$$

Recall that the right side can be written as a complete square (see Appendix C)

$$\frac{1}{2}\int_{\mathbb{R}} |\partial_x S + \frac{1}{2}\mathsf{H}(|S|^2)S|^2\, dx = \frac{1}{2}\int_{\mathbb{R}} |\partial_x \mathcal{R}_{\tilde{\omega}} - \frac{i}{2}\Pi_+(|\mathcal{R}_{\tilde{\omega}}|^2)\mathcal{R}_{\tilde{\omega}}|^2\, dx,$$

where the last step follows by using (4.10). Thus we deduce that (4.9) holds, which completes the proof. □

4.1 | Proof of theorem 1.2

This claim immediately follows from Proposition 4.1 by taking $\eta = v/2$.

5 | ANALYSIS OF THE LAX OPERATOR

In this section, we will further study the Lax operator

$$L_u = D - T_u T_{\bar{u}}$$

introduced in Section 2 above, where $T_b(f) = \Pi_+(bf)$ denotes the Toeplitz operator on $L^2_+(\mathbb{R})$ with symbol b. In particular, we will derive important commutator identities and prove simplicity of eigenvalues of L_u for general potentials $u \in H^1_+(\mathbb{R})$ along with optimal bounds on the number of eigenvalues of L_u. Moreover, we will define the notion of **multi-soliton potentials** below.

5.1 | Simplicity of eigenvalues and sharp bounds for L_u

Given $u \in H^1_+(\mathbb{R})$[2], we recall that the operator L_u defined above is an unbounded self-adjoint operator on the Hardy space $L^2_+(\mathbb{R})$, with the operator domain $H^1_+(\mathbb{R})$. First we show simplicity of eigenvalues for L_u together with a bound on the number of eigenvalues.

Proposition 5.1. *If $L_u\psi = \lambda\psi$, then*

$$|\langle u, \psi\rangle|^2 = 2\pi\|\psi\|^2_{L^2}. \tag{5.1}$$

[2] Recall that the regularity assumption can be relaxed to $u \in L^2_+(\mathbb{R})$ via an approach using quadratic forms; see Appendix A below.

In particular, every eigenvalue of L_u is simple, and the number N of eigenvalues of L_u is finite with

$$N \leq \frac{\|u\|_{L^2}^2}{2\pi} .$$

In particular, the operator L_u has no point spectrum if $\|u\|_{L^2}^2 < 2\pi$.

Remark. In [36], an identity reminiscent to (5.1) was derived for the Lax operator of the Benjamin-Ono equation on the real line.

Proof. The key is to introduce the *Lax–Beurling semigroup of contractions* defined on $L^2_+(\mathbb{R})$ as

$$S(\eta)f(x) = \mathrm{e}^{i\eta x} f(x) , \ \eta \geq 0 .$$

and the corresponding *adjoint semigroup* defined as

$$S(\eta)^* f = \Pi_+\left(\mathrm{e}^{-ix\eta} f\right) , \ \eta \geq 0 .$$

Clearly, $S(\eta)$ and $S(\eta)^*$ act on the domain $H^1_+(\mathbb{R})$ of L_u. Also notice that, via the Fourier transform, we have

$$\widehat{S(\eta)^* f}(\xi) = \hat{f}(\xi + \eta) , \ \xi \geq 0, \eta \geq 0 .$$

To complete the proof of Proposition 5.1, we need the following identity.

Lemma 5.1. *For every $f \in H^1_+(\mathbb{R})$, we have the following limit in the L^2-norm,*

$$\lim_{\eta \to 0} \left[\frac{S(\eta)^*}{\eta}, L_u\right] f = f - \frac{1}{2\pi}\langle f, u\rangle u .$$

Proof of Lemma 5.1. Notice that, for $\xi > 0$,

$$\widehat{L_u f}(\xi) = \xi \hat{f}(\xi) - \frac{1}{4\pi^2} \int_0^\xi \hat{u}(\xi - \zeta)\left[\int_0^\infty \overline{\hat{u}(\tau)} \hat{f}(\zeta + \tau)\, d\tau\right] d\zeta .$$

Set

$$A(\eta) := \left[\frac{S(\eta)^*}{\eta}, L_u\right] , \ \xi \geq 0 .$$

Then

$$\widehat{A(\eta) f}(\xi) = \hat{f}(\xi + \eta) - \frac{1}{4\pi^2 \eta} \int_0^\eta \hat{u}(\xi + \eta - \zeta)\left[\int_0^\infty \overline{\hat{u}(\tau)} \hat{f}(\zeta + \tau)\, d\tau\right] d\zeta .$$

Passing to the limit in L^2, we get

$$\lim_{\eta \to 0} \widehat{A(\eta) f}(\xi) = \hat{f}(\xi) - \frac{1}{4\pi^2}\hat{u}(\xi) \int_0^\infty \overline{\hat{u}(\tau)} \hat{f}(\tau)\, d\tau ,$$

which yields Lemma 5.1. □

For future reference, we state the following lemma, which can be proved similarly to Lemma 5.1.

Lemma 5.2. *Let $a, b \in L^2_+(\mathbb{R})$. For every $f \in L^2_+(\mathbb{R})$, we have the following limit in the L^2-norm,*

$$\lim_{\eta\to 0}\left[\frac{S(\eta)^*}{\eta}, T_a T_{\overline{b}}\right] f = \frac{1}{2\pi}\langle f, b\rangle a\,.$$

Let us come back to the proof Proposition 5.1. If we apply Lemma 5.1 to $f = \psi$, we obtain

$$\lim_{\eta\to 0}\frac{(\lambda I - L_u)S(\eta)^*\psi}{\eta} = \psi - \frac{1}{2\pi}\langle \psi, u\rangle u\,.$$

Taking inner product of both sides with ψ, we obtain

$$0 = \|\psi\|^2_{L^2} - \frac{1}{2\pi}|\langle \psi, u\rangle|^2$$

which is (5.1). Now the next statements of Proposition 5.1 follow easily. Indeed, identity (5.1) implies that the kernel of the linear form $\langle ., u\rangle$ on any eigenspace is reduced to $\{0\}$, so that this eigenspace is one dimensional. Finally, if $\psi_1, \dots, \psi_N$ is an orthonormal system of eigenvectors, we have

$$\|u\|^2_{L^2} \geq \sum_{j=1}^{N} |\langle u, \psi_j\rangle|^2 = 2\pi N.$$

This proof of Proposition 5.1 is now complete. □

5.2 | Multi-soliton potentials

Next, we exhibit a class of potentials $u \in H^1_+(\mathbb{R})$ which are given by rational functions and which optimize the general bound for the number N of eigenvalues of L_u found in Proposition 5.1 above. To this end, we use $\sigma_{pp}(L_u)$ to denote the pure point spectrum of L_u and correspondingly we define

$$\mathcal{E}_{pp}(u) = \mathrm{span}\{\psi \in \ker(L_u - \lambda I) : \lambda \in \sigma_{pp}(L_u)\}$$

to be the space spanned by the eigenfunctions of L_u. We have the following spectral characterization result.

Proposition 5.2. *Let $u \in H^1_+(\mathbb{R})$ be given and let $N \geq 1$ be an integer. Then the following properties are equivalent and preserved by the flow of (CM-DNLS).*

(i) The space $\mathcal{E}_{pp}(u)$ has dimension N and is invariant under the adjoint semigroup $\{S(\eta)^*\}_{\eta\geq 0}$ and it holds that $u \in \mathcal{E}_{pp}(u)$.

(ii) There exist a polynomial $Q \in \mathbb{C}[x]$ of degree N, with all its zeros in $\mathbb{C}_-$, and a polynomial $P \in \mathbb{C}[x]$ of degree at most $N-1$ such that

$$u(x) = \frac{P(x)}{Q(x)} \quad and \quad P\overline{P} = i(Q'\overline{Q} - \overline{Q}'Q).$$

Before we prove this result, let us make some general comments as follows. We refer to the rational functions $u \in H^1_+(\mathbb{R})$ above as a **multi-soliton potential** or N**-soliton potential**. In Section 6 below, we will study the time evolution of multi-soliton potentials, which correspond to **multi-solitons**. In view of Proposition 5.1, we see that multi-solitons must have a quantized L^2-mass according to

$$\|u\|_{L^2}^2 = 2\pi N$$

with some integer $N \geq 1$. As a generic example for a multi-soliton potential, we can take the polynomial Q to have simple zeros, that is,

$$Q(x) = \prod_{j=1}^{N}(x - z_j),$$

where $z_1, \dots, z_N \in \mathbb{C}_-$ are pairwise distinct. Then the condition in (ii) above yields that

$$u(x) = \sum_{j=1}^{N} \frac{a_j}{x - z_j} \quad \text{with} \quad \sum_{j=1}^{N} \frac{a_j \overline{a}_k}{z_j - \overline{z}_k} = i \quad \text{with } k = 1, \dots, N.$$

Let us also remark that, for a given polynomial $Q \in \mathbb{C}[x]$ of degree $N \geq 1$, there exist only a finite number of polynomials $P \in \mathbb{C}[x]$ satisfying the constraint in (ii) up to a constant complex phase. We can find them as follows. Consider the polynomial $F := i(Q'\overline{Q} - \overline{Q}' Q) \in \mathbb{C}[x]$ of degree $2N - 2$. Suppose that

$$Q(x) = \prod_{j=1}^{N}(x - z_j)$$

with zeros $z_1, \dots, z_N \in \mathbb{C}_-$, which are not necessarily distinct. For $x \in \mathbb{R}$, we get

$$\frac{F(x)}{|Q(x)|^2} = \sum_{j=1}^{N} \frac{-2\mathrm{Im} z_j}{|x - z_j|^2} > 0\,.$$

In particular, the zeroes of F come in pairs as $(\alpha_j, \overline{\alpha}_j)$, $j = 1, \dots, N-1$, with $\alpha_j \notin \mathbb{R}$. Thus we can write

$$P(x) = c \prod_{j=1}^{N-1}(x - \alpha_j),$$

where the constant $c \in \mathbb{C}$ is adjusted so that $P\overline{P} = F$ holds. Of course, exchanging one α_j with $\overline{\alpha}_j$ leads to a different P and hence to a different function u.

Proof of Proposition 5.2. We first proof the equivalence of statements (i) and (ii). Finally, we address the preservation by the flow of (CM-DNLS).

Step 1: (ii) ⇒ (i). For $u(x) = \frac{P(x)}{Q(x)} \in H^1_+(\mathbb{R})$ as in (ii), we define

$$\theta(x) = \frac{\overline{Q}(x)}{Q(x)} \quad \text{and} \quad K_\theta := \frac{\mathbb{C}_{N-1}[x]}{Q(x)}\,,$$

so that we have the orthogonal decomposition

$$L^2_+(\mathbb{R}) = K_\theta \oplus \theta L^2_+(\mathbb{R}) \,.$$

We claim that K_θ is an invariant subspace of L_u. Indeed, if $f = A/Q \in K_\theta$, then

$$\begin{aligned} L_u(f) &= \frac{-iA'}{Q} + \frac{iQ'A}{Q^2} - \frac{P}{Q}\Pi_+\left(\frac{\overline{P}}{\overline{Q}}\frac{A}{Q}\right) \\ &= \frac{-iA'}{Q} + \Pi_+\left[\frac{P\overline{P}}{Q\overline{Q}}\frac{A}{Q} + i\frac{\overline{Q}'}{\overline{Q}}\frac{A}{Q}\right] - \frac{P}{Q}\Pi_+\left(\frac{\overline{P}}{\overline{Q}}\frac{A}{Q}\right) \\ &= \frac{-iA'}{Q} + i\Pi_+\left(\frac{\overline{Q}'}{\overline{Q}}\frac{A}{Q}\right) + \Pi_+\left(\frac{P}{Q}\Pi_-\left(\frac{\overline{P}A}{\overline{Q}Q}\right)\right). \end{aligned}$$

Observe that, if $R \in L^2(\mathbb{R})$ is a rational function, and if the denominator of R reads Q_+Q_-, where $Q_\pm$ is a polynomial with zeroes in $\mathbb{C}_\pm$, then $\Pi_+(R)$ is of the form P/Q_-, where P is a polynomial of degree less than the degree of Q_-. From this observation and the above identity, we conclude that $L_u(f) \in K_\theta$. Therefore L_u is a self-adjoint endomorphism on the finite dimensional space K_θ, which has dimension N. This implies that L_u has at least N eigenvalues. Since $u \in K_\theta$, it follows that u is a linear combination of an orthonormal basis of eigenfunctions $\psi_1, \dots, \psi_N \in K_\theta$. From Proposition 5.1 we conclude that

$$\|u\|^2_{L^2} = \sum_{k=1}^N |\langle u, \psi_k\rangle|^2 = 2\pi N.$$

By invoking Proposition 5.1 again, we deduce that L_u has exactly N eigenvalues. Thus we have shown $\mathcal{E}_{pp}(u) = K_\theta$ and therefore we conclude $\dim \mathcal{E}_{pp}(u) = N$ as well as $u \in \mathcal{E}_{pp}(u)$. Since the semigroup $\{S(\eta)\}_{\eta\geq 0}$ leaves the space $\theta L^2_+(\mathbb{R})$ invariant, we obtain that its adjoint semigroup $\{S(\eta)^*\}_{\eta\geq 0}$ leaves its orthogonal complement $\mathcal{E}_{pp}(u) = K_\theta$ invariant.

Step 2: (i) ⇒ (ii). Suppose that $\mathcal{E}_{pp}(u)$ has dimension $N \geq 1$ and is invariant under the adjoint semigroup $\{S(\eta)^*\}_{\eta\geq 0}$. Thus the orthogonal complement $(\mathcal{E}_{pp}(u))^\perp$ is invariant preserved by the action of the semigroup $S(\eta)$ with $\eta \geq 0$. By the Lax–Beurling theorem [23], we conclude

$$(\mathcal{E}_{pp}(u))^\perp = \theta L^2_+(\mathbb{R})$$

with some inner function θ defined on the upper complex halfplane $\mathbb{C}_+$. Furthermore, since $\mathcal{E}_{pp}(u)$ is N–dimensional, one can choose θ of the form

$$\theta(x) = \frac{\overline{Q}(x)}{Q(x)}\,, \; Q(x) = \prod_{j=1}^N (x - z_j)\,, \; \mathrm{Im} z_j < 0\,.$$

Consequently,

$$\mathcal{E}_{pp}(u) = (\theta L^2_+(\mathbb{R}))^\perp = K_\theta = \frac{\mathbb{C}_{N-1}[x]}{Q(x)}\,.$$

Since $u \in \mathcal{E}_{pp}(u)$, there exists $P \in \mathbb{C}_{N-1}[x]$ such that $u = P/Q$. Since L_u is self-adjoint, we have

$$L_u[(\mathcal{E}_{pp}(u))^\perp \cap H^1_+(\mathbb{R})] \subset (\mathcal{E}_{pp}(u))^\perp ,$$

alternatively $L_u(\theta H^1_+(\mathbb{R})) \subset \theta L^2_+(\mathbb{R})$. Let $h \in H^1_+(\mathbb{R})$. We have

$$L_u(\theta h) = \theta Dh + (D\theta)h - u\Pi_+(\overline{u}\theta h) = \theta Dh + (D\theta)h - |u|^2\theta h ,$$

because

$$\overline{u}\theta = \frac{\overline{P}}{\overline{Q}}\frac{\overline{Q}}{Q} = \frac{\overline{P}}{Q} \in L^2_+(\mathbb{R}) .$$

We infer $(D\theta)h - \theta|u|^2 h \in \theta L^2_+(\mathbb{R})$ for every $h \in H^1_+(\mathbb{R})$, or

$$\frac{D\theta}{\theta} - |u|^2 \in L^2_+(\mathbb{R}) .$$

Notice that, for every $x \in \mathbb{R}$,

$$\frac{D\theta}{\theta} = \sum_{j=1}^{N} \frac{2\mathrm{Im} z_j}{|x - z_j|^2}$$

hence $D\theta/\theta - |u|^2$ is real valued, therefore it belongs to $L^2_+(\mathbb{R}) \cap \overline{L^2_+(\mathbb{R})} = \{0\}$. Reformulating this identity in terms of P and Q, we obtain $P\overline{P} = i(Q'\overline{Q} - \overline{Q}'Q)$.

Step 3: Preservation by the Flow. In order to prove the last part of Proposition 5.2, we introduce the infinitesimal generator of the adjoint Lax–Beurling semi–group, namely the operator G such that

$$S(\eta)^* = \mathrm{e}^{-i\eta G} .$$

Notice that its operator domain is given by

$$\mathrm{dom}(G) = \{f \in L^2_+(\mathbb{R}) \,:\, \hat{f}_{|]0,+\infty[} \in H^1(]0, +\infty[)\}$$

and that

$$\widehat{(Gf)}(\xi) = i\frac{d\hat{f}}{d\xi} \quad \text{for} \quad \xi > 0.$$

In particular, the operator G acts on K_θ for every finite Blaschke product θ. We claim that properties (i) and (ii) are equivalent to the following statement:

(iii) The space $\mathcal{E}_{pp}(u)$ has dimension N, contains u, and moreover $u \in \mathrm{dom}(G)$ with $Gu \in \mathcal{E}_{pp}(u)$.

Indeed, as we just observed, if u is a N-soliton, then it satisfies (iii). Conversely, assume that $u \in H^1_+$ satisfies (iii). We appeal to a corollary of Lemma 5.1.

Lemma 5.3. *Let $f \in \mathrm{dom}(G) \cap H^1_+$ such that $L_u f \in \mathrm{dom}(G)$. Then $Gf \in H^1_+$ and*

$$GL_u f - L_u G f = if - i\frac{\langle f, u\rangle}{2\pi} u .$$

Proof. For every $h \in \mathrm{dom}(G)$, we have

$$Gh = i \lim_{\eta\to 0^+} \frac{S(\eta)^* h - h}{\eta} .$$

Rewriting Lemma 5.1 as

$$\lim_{\eta\to 0} \left[\frac{S(\eta)^* - I}{\eta}, L_u \right] f = f - \frac{1}{2\pi}\langle f, u\rangle u.$$

Using that L_u is a closed operator, the lemma follows. □

Applying Lemma 5.3 to $f = u$, we infer that $L_u(u) \in \mathrm{dom}(G)$ and that

$$GL_u(u) = L_u(Gu) + iu - i\frac{\|u\|_{L^2}^2}{2\pi} u .$$

In particular, $GL_u(u) \in \mathcal{E}_{pp}(u)$. Iterating this process, we conclude by an easy induction that, for every integer k, $L_u^k u \in \mathrm{dom}(G)$ and that $GL_u^k u \in \mathcal{E}_{pp}(u)$. Now recall from Proposition 5.1 that L_u has N simple eigenvalues on $\mathcal{E}_{pp}(u)$, and that the component of u on any eigenvector is different from 0. Consequently, u is a cyclic vector for L_u in $\mathcal{E}_{pp}(u)$, namely the N vectors $u, L_u, \dots , L_u^{N-1}u$ form a basis of $\mathcal{E}_{pp}(u)$. From this we infer that G acts on $\mathcal{E}_{pp}(u)$, and finally that $S(\eta)^*$ acts on $\mathcal{E}_{pp}(u)$, whence (i).

Let us prove that property (iii) is preserved by the flow of (CM-DNLS). Consider u_0 satisfying (iii), and denote by u the solution of (CM-DNLS) with $u(0) = u_0$, on its maximal time interval. Notice that we know from (ii) that u_0 belongs to every H^s. Therefore, by the well-posedness result Proposition 2.1, $u(t)$ belongs to every H^s, hence we do not have to worry about its regularity. We are going to use the Lax equation provided by Lemma 2.3. Denote by $U(t)$ the one-parameter family of unitary operators on $L^2_+(\mathbb{R})$ defined as

$$\frac{d}{dt}U(t) = B_{u(t)}U(t)\, ,\ U(0) = I\, .$$

Then Lemma 2.3 implies

$$L_{u(t)} = U(t)L_{u_0}U(t)^* \, . \tag{5.2}$$

Consequently, $\mathcal{E}_{pp}(u(t)) = U(t)[\mathcal{E}_{pp}(u_0)]$ has dimension N. Furthermore, in view of Lemma 2.4, the spectral measure of $L_{u(t)}$ associated to the vector $u(t)$ is the same as the spectral measure of L_{u_0} associated to the vector u_0. This implies that $\mathcal{E}_{pp}(u(t))$ contains $u(t)$. It remains to prove that $u(t) \in \mathrm{dom}(G)$ and that $Gu(t) \in \mathcal{E}_{pp}(u(t))$. Let us appeal to the reformulation of the dynamics as

$$\partial_t u = \tilde{B}_u u\, ,$$

where $\tilde{B}_u = B_u - iL_u^2$ according to (2.6). Given $\eta > 0$, define

$$v(t, \eta) := i\frac{S(\eta)^* u(t) - u(t)}{\eta}$$

and observe that

$$\partial_t v(t,\eta) = \tilde{B}_u v(t,\eta) + g(t,\eta)\,,\ g(t,\eta) := i\left[\frac{S(\eta)^*}{\eta}, \tilde{B}_{u(t)}\right] u(t)\,,$$

which, in view of (5.2), can be solved as

$$U(t)^* v(t,\eta) = \mathrm{e}^{-itL_{u_0}^2} v(0,\eta) + \int_0^t \mathrm{e}^{i(\tau-t)L_{u_0}^2} U(\tau)^* g(\tau,\eta)\, d\tau\,.$$

Using Lemma 5.2 and the expression of B_u, we obtain

Lemma 5.4. *If $u \in H^1_+(\mathbb{R})$, we have, for every $f \in L^2_+(\mathbb{R})$,*

$$\lim_{\eta\to 0} i\left[\frac{S(\eta)^*}{\eta}, B_u\right] f = \frac{1}{2\pi}(\langle f, L_u u\rangle u + \langle f, u\rangle L_u u)\,.$$

Combining Lemma 5.4 with Lemma 5.1, we infer, locally uniformly in t,

$$\lim_{\eta\to 0} g(t,\eta) = 2L_{u(t)} u(t)\,.$$

This shows that $v(t,\eta)$ has a limit $Gu(t)$ in L^2_+ as $\eta \to 0$, characterized by

$$\begin{aligned} U(t)^* Gu(t) &= \mathrm{e}^{-itL_{u_0}^2} Gu_0 + 2\int_0^t \mathrm{e}^{i(\tau-t)L_{u_0}^2} L_{u_0} U(\tau)^* u(\tau)\, d\tau \\ &= \mathrm{e}^{-itL_{u_0}^2} Gu_0 + 2t\, \mathrm{e}^{-itL_{u_0}^2} L_{u_0} u_0\,. \end{aligned}$$

Notice that, by (iii), the right hand side of the above equation belongs to $\mathcal{E}_{pp}(u_0)$. Consequently, $Gu(t) \in U(t)[\mathcal{E}_{pp}(u_0)] = \mathcal{E}_{pp}(u(t))$. This completes the proof. □

Remark. In fact, one can easily check that the operator L_u restricted to the invariant subspace $\theta L^2_+ = (K_\theta)^\perp$ has absolutely continuous simple spectrum with

$$L_u(\theta h) = \theta D h \quad \text{for all } h \in H^1_+(\mathbb{R}).$$

For any $u \in H^1_+(\mathbb{R})$, we notice that the operator L_u has the essential spectrum $\sigma_{ess}(L_u) = [0,\infty)$. In the case of multi-solitons, we find that 0 is always an embedded eigenvalue.

Proposition 5.3. *For any multi-soliton potential $u \in H^1_+(\mathbb{R})$, we have that*

$$L_u(1-\theta) = 0,$$

where $\theta(x) = \dfrac{\overline{Q}(x)}{Q(x)}$ with the notation from Proposition 5.2 (ii) above.

Proof. We observe that

$$L_u(1-\theta) = -D\theta - u\Pi_+(\overline{u}(1-\theta)) = -D\theta + u\Pi_+(\overline{u}\theta)\,.$$

Notice that the function

$$\overline{u}\theta = \frac{\overline{P}}{\overline{Q}}\frac{\overline{Q}}{Q} = \frac{\overline{P}}{Q}$$

belongs to $L^2_+(\mathbb{R})$. Hence we can deduce

$$L_u(1-\theta) = -D\theta + u\overline{u}\theta = 0,$$

because of the constraint in Proposition 5.2 (ii). □

6 | DYNAMICS OF MULTI-SOLITONS

This section is devoted to the study of multi-solitons, that is, solutions $u(t,x)$ with initial datum giving by a multi-soliton potential $u_0 \in H^1_+(\mathbb{R})$ (see Proposition 5.2 above). By means of an inverse spectral formula, we will be able to prove global-in-time existence for all multi-solitons. This is a large data result which is beyond the scope of a-priori bounds. Second, we prove that all multi-solitons with $N \geq 2$ exhibit an energy cascade (growth of Sobolev norms) as $t \to \pm\infty$.

6.1 | Preliminary discussion

Let us first consider the following *pole ansatz* of the form

$$u(t,x) = \sum_{j=1}^{N} \frac{a_j(t)}{x - z_j(t)} \in H^1_+(\mathbb{R}), \tag{6.1}$$

where $a_1(t), \dots, a_N(t) \in \mathbb{C} \setminus \{0\}$ and pairwise distinct poles $z_1(t), \dots, z_N(t)$ in the complex lower halfplane $\mathbb{C}_-$. If we plug this ansatz into (CM-DNLS), then a straightforward calculation shows that the self-consistency of (6.1) leads to the set of nonlinear constraints given by

$$\sum_{j=1}^{N} \frac{a_j(t)\overline{a}_k(t)}{z_j(t) - \overline{z}_k(t)} = i \quad \text{for} \quad k = 1, \dots, N. \tag{6.2}$$

Note that these conditions have already appeared in the discussion of multi-soliton potentials (see Proposition 5.2 above). Furthermore, the equations of motions which govern the parameters $\{a_j(t), z_j(t)\}_{j=1}^N$ are found to be

$$\dot{a}_k = 2i \sum_{\ell \neq k}^{N} \frac{a_\ell - a_k}{(z_k - z_\ell)^2} \quad \text{and} \quad a_k \dot{z}_k = -2i \sum_{\ell \neq k}^{N} \frac{a_\ell}{z_k - z_\ell} \tag{6.3}$$

with $k = 1, \dots, N$. A tedious calculation shows that the constraints (6.2) are indeed preserved by the time evolution determined by (6.3). Finally, we remark that the first-order system (6.3) can be used to derive that

$$\ddot{z}_k = \sum_{\ell \neq k}^{N} \frac{8}{(z_k - z_\ell)^3} \quad \text{for} \quad k = 1, \dots, N, \tag{6.4}$$

which is again confirmed by a lengthy calculation that we omit here. We note that (6.4) can be regarded as a complexified version[3] of the **Calogero–Moser (CM) system** on the real line, whose complete integrability was proven by J. Moser in [27] (see also [21, 28, 29]). Let us mention that the pole dynamics of rational solutions of completely integrable PDEs are governed by (complexified versions) of CM systems have also been found for the Benjamin-Ono, KdV and Half-Wave Maps equations in [3, 7, 9].

However, we emphasize that working with the pole ansatz in (6.1) leads to the following *caveats* that ultimately need to be addressed.

(1) *Collision of poles:* It may happen that $z_j(t) \to z_k(t)$ for some pair $j \neq k$ as $t \to T$ with some finite time $T > 0$. From (6.4) we expect that the solution $z_k(t)$ blows up in C^2 as $t \to T$. But the solution $u(t,x)$ itself may stay smooth as $t \to T$, whereas the pole ansatz (6.1) becomes invalid only. Explicit examples of pole collisions will be given in Subsection 6.4 below.
(2) *Showing that $z_k(t) \in \mathbb{C}_-$:* The major step in showing global-in-time existence for multi-solitons consists in proving that the poles $z_k(t)$ stay in the lower complex half-plane $\mathbb{C}_-$.

To systematically tackle the problems (1) and (2), we will derive an *inverse spectral formula* for multi-solitons, which entails the pole ansatz (6.1) as a special case. Furthermore, the dynamical evolution of multi-solitons $u(t,x)$ will be encoded by the *linear flow* of a suitable matrix $\mathsf{M}(t) \in \mathbb{C}^{N \times N}$ such that

$$\mathsf{M}(t) = 2\mathsf{V}t + \mathsf{W} \tag{6.5}$$

with some constant matrices V and W in $\mathbb{C}^{N\times N}$; see Proposition 6.2 and (6.12)–(6.13) below. Moreover, we remark that solving classical CM systems on the real line by means of linear flows of $N \times N$-matrices was successfully used in [28]. Finally, notice that similar inverse formulae for multi-soliton solutions were derived in the case of the Benjamin–Ono equation in [34] and in the case of the cubic Szegő equation in [17]. However, in these two examples, the global well-posedness result was established directly by other methods, while it seems to be the first time that such inverse formulae provide global existence.

6.2 | Inverse spectral formula and time evolution

Let

$$u(x) = \frac{P(x)}{Q(x)} \in H^1_+(\mathbb{R})$$

be a multi-soliton potential. We use the notation introduced in the proof of Proposition 5.2 with

$$\theta(x) = \frac{\overline{Q}(x)}{Q(x)} \quad \text{and} \quad K_\theta = \frac{\mathbb{C}_{N-1}[x]}{Q(x)}.$$

Recall that the Lax–Beurling semigroup $S(\eta)$ leaves the space θL^2_+ invariant, and hence its adjoint semigroup $S(\eta)^*$ leaves the subspace K_θ invariant. This observation leads to the following formula, where G is the operator introduced in subsection 5.2.

[3] That is, we formally generalize the positions $x_j \in \mathbb{R}$ to complex numbers $z_j \in \mathbb{C}_-$.

Proposition 6.1. *For every* $f \in K_\theta$,

$$f(x) = \frac{1}{2\pi i}\langle (G - xI)^{-1} f, 1-\theta\rangle \quad \textit{for} \quad \mathrm{Im}(x) > 0\,.$$

Proof. We start from the inversion Fourier formula for every element of $L^2_+(\mathbb{R})$,

$$f(x) = \frac{1}{2\pi}\int_0^\infty \mathrm{e}^{ix\xi}\hat f(\xi)\,d\xi\,.$$

We claim that if $f \in K_\theta$ we have

$$\hat f(\xi) = \langle S(\xi)^* f, 1-\theta\rangle\,,\ \xi > 0\,. \tag{6.6}$$

Indeed, by the Plancherel theorem,

$$\hat f(\xi) = \lim_{\varepsilon\to 0^+}\int_{\mathbb{R}} \mathrm{e}^{-ix\xi}\frac{f(x)}{1+i\varepsilon x}\,dx = \lim_{\varepsilon\to 0^+}\left\langle S(\xi)^* f, \frac{1}{1-i\varepsilon x}\right\rangle,$$

since $(1-i\varepsilon x)^{-1} \in L^2_+$. For the same reason $\theta(1-i\varepsilon x)^{-1} \in \theta L^2_+$ which is orthogonal to K_θ and in particular to $S(\xi)^* f$. Therefore

$$\hat f(\xi) = \lim_{\varepsilon\to 0^+}\left\langle S(\xi)^* f, \frac{1-\theta}{1-i\varepsilon x}\right\rangle,$$

which yields (6.6). It remains to plug (6.6) into the inversion Fourier formula, reminding that $S(\xi)^* = \mathrm{e}^{-i\xi G}$, and the proposition follows. □

Next, we are going to use Proposition 6.1 in the particular case $f = u$. The interesting feature is that the inner product in this formula takes place in K_θ, of which we can choose an orthonormal basis made of eigenvectors $(\psi_1, \dots, \psi_N)$ of L_u so that

$$L_u\psi_j = \lambda_j\psi_j \quad \text{for} \quad j = 1, \dots, N.$$

Here and throughout the following we label the eigenvalues such that

$$\lambda_1 = 0, \lambda_2, \dots, \lambda_N,$$

where we recall that 0 is always an eigenvalue of L_u by Proposition 5.3. In view of Proposition 5.1, we can choose the normalisation

$$\langle u, \psi_j\rangle = \sqrt{2\pi} \quad \text{for} \quad j = 1, \dots, N.$$

Note that, because of Proposition 5.3, it holds

$$\langle 1-\theta, \psi_j\rangle = 0 \quad \text{for} \quad j = 2, \dots, N.$$

Let us first discuss the case of $\langle 1-\theta, \psi_1\rangle$. Notice that every element f of K_θ is smooth on $]0, +\infty[$ with limits at 0^+. In particular, we can pass to the limit in formula (6.6) as $\xi \to 0^+$ to obtain

$$\hat f(0^+) = \langle f, 1-\theta\rangle\,.$$

On the other hand, the identity $L_u\psi_1 = 0$ reads

$$\xi\hat{\psi}_1(\xi) = \frac{1}{4\pi^2}\int_0^\xi \hat{u}(\xi-\zeta)\left(\int_0^\infty \overline{\hat{u}(\tau)}\hat{\psi}_1(\tau+\zeta)\,d\tau\right)d\zeta$$

so that

$$\langle \psi_1, 1-\theta\rangle = \hat{\psi}_1(0^+) = \frac{\hat{u}(0^+)}{2\pi}\langle u,\psi_1\rangle = \frac{\hat{u}(0^+)}{\sqrt{2\pi}}\,. \tag{6.7}$$

Coming back to the equation satisfied by u,

$$i\partial_t u + \partial_x^2 u + 2D_+(|u|^2)u = 0\,, \tag{6.8}$$

and taking the Fourier transform, we observe that

$$i\partial_t\hat{u}(t,\xi) - \xi^2\hat{u}(t,\xi) + 2\int_0^\infty \mathrm{e}^{-ix\xi}D_+(|u|^2)(t,x)u(t,x)\,dx = 0\,.$$

Passing to the limit as $\xi \to 0^+$ and observing that $D_+(|u|^2)$ and u both belong to L^2_+, we infer

$$\partial_t\hat{u}(t,0^+) = 0.$$

Hence $\hat{u}(0^+)$ is a conserved quantity and, consequently, the inner product product $\langle 1-\theta,\psi_1\rangle$ is conserved as well. Let us come back to the inverse formula

$$u(x) = \frac{1}{2i\pi}\langle (G-xI)^{-1}u, 1-\theta\rangle\,,\ \mathrm{Im}(x) > 0\,. \tag{6.9}$$

In the orthonormal basis $(\psi_1,\dots,\psi_N)$ of K_θ, we have just observed that the components of u and of $1-\theta$ are conserved quantities. Next, let us discuss the matrix of G in this basis, which we denote as

$$\mathsf{M} := (\langle G\psi_k,\psi_j\rangle)_{1\le j,k\le N}\,.$$

Of course, the matrix $\mathsf{M} = \mathsf{M}(t)$ will depend on t as well through the evolution of the eigenfunctions ψ_j as given by the Lax structure in order to keep $\langle u,\psi_j\rangle$ constant, that is, we have $\dot{\psi}_j = \tilde{B}_u\psi_j$. Indeed, we will later use this to derive explicitly formulas for M. For notational ease, we will sometimes omit the t-dependence of $\mathsf{M}(t)$.

We first consider the case $j \neq k$. For this We recall that every element of K_θ belongs to $\mathrm{dom}(G)$. Then, for $j \neq k$, we observe that

$$(\lambda_k - \lambda_j)\langle G\psi_k,\psi_j\rangle = \langle (GL_u - L_uG)\psi_k,\psi_j\rangle = i\langle\psi_k,\psi_j\rangle - i\frac{\langle\psi_k,u\rangle\langle u,\psi_j\rangle}{2\pi} = -i$$

so that

$$\langle G\psi_k,\psi_j\rangle = \frac{i}{\lambda_j-\lambda_k} \quad \text{if} \quad j\neq k. \tag{6.10}$$

Finally, let us discuss the diagonal elements $\langle G\psi_j, \psi_j\rangle$. Notice that their imaginary parts are easy to calculate, since

$$\mathrm{Im}\,\langle G\psi_j, \psi_j\rangle = \frac{1}{2\pi}\mathrm{Re}\left\langle \frac{d\hat{\psi}_j}{d\xi}, \hat{\psi}_j\right\rangle = -\frac{|\hat{\psi}_j(0^+)|^2}{4\pi} = -\frac{|\langle\psi_j, 1-\theta\rangle|^2}{4\pi}$$

which is 0 whenever $j \neq 1$. For $j = 1$, we use (6.7) to conclude

$$\mathrm{Im}\langle G\psi_1, \psi_1\rangle = -\frac{|\hat{u}(0^+)|^2}{8\pi^2} .$$

As for the real part of the diagonal elements, we are going to compute their time derivatives if u is a solution of (6.8). From the Lax pair formula, we may assume that

$$\dot{\psi}_j = B_u\psi_j\ ,\ B_u = T_u T_{\partial_x \overline{u}} - T_{\partial_x u} T_{\overline{u}} + i(T_u T_{\overline{u}})^2 .$$

Then

$$\frac{d}{dt}\langle G\psi_j, \psi_j\rangle = \langle [G, \tilde{B}_u]\psi_j, \psi_j\rangle = \langle [G, B_u]\psi_j, \psi_j\rangle .$$

Passing to the limit in Lemma 5.4, we have

$$[G, B_u]f = \frac{1}{2\pi}(\langle f, L_u u\rangle u + \langle f, u\rangle L_u u\ ).$$

Consequently, we get

$$\begin{aligned}\langle [G, B_u]\psi_j, \psi_j\rangle &= \frac{1}{2\pi}\langle L_u\psi_j, u\rangle\langle u, \psi_j\rangle + \frac{1}{2\pi}\langle \psi_j, u\rangle\langle L_u u, \psi_j\rangle \\ &= 2\lambda_j .\end{aligned}$$

Summing up, we have proved that

$$\frac{d}{dt}\langle G\psi_j, \psi_j\rangle = 2\lambda_j . \tag{6.11}$$

The inverse spectral formula therefore reads, setting

$$\gamma_j := \mathrm{Re}(\mathsf{M}_{jj})\ ,\ \sqrt{2\varrho}\,\mathrm{e}^{i\varphi} := \frac{\hat{u}(0^+)}{2\pi i}\ ,\ \varrho > 0.$$

The discussion above shows that the following results holds.

Proposition 6.2. *If $u(t) \in H^1_+(\mathbb{R})$ is a multi-soliton potential such that L_u has eigenvalues $\lambda_1 = 0, \lambda_2, \dots, \lambda_N$, then u can be recovered as*

$$u(t,x) = \sqrt{2\varrho}\mathrm{e}^{i\varphi}\langle(\mathrm{M}(t) - xI)^{-1}X, Y\rangle_{\mathbb{C}^N} \text{ for } \mathrm{Im}\, x > 0,$$

where

$$X := (1, \dots, 1)^T\ ,\ Y := (1, 0, \dots, 0)^T\ ,$$

$$\mathsf{M}_{jk} = \frac{i}{\lambda_j - \lambda_k}\ (1 \leq j \neq k \leq N), \quad \mathsf{M}_{jj} = \gamma_j - i\varrho\delta_{j1}\ (j = 1, \dots, N).$$

Furthermore, the following evolution laws hold:

$$\frac{d}{dt}\varphi = 0\,, \frac{d}{dt}\varrho = 0\,, \frac{d}{dt}\gamma_j = 2\lambda_j, \quad (j = 1, \dots, N).$$

6.3 | Global-in-time existence

In order to prove that N-soliton solutions $u(t,x)$ can be extended to all times $t \in \mathbb{R}$, we are going to study the eigenvalues of the matrix $\mathsf{M}(t) \in \mathbb{C}^{N\times N}$ introduced above. To this end, we note that the time evolution of $\mathsf{M}(t)$ stated in Proposition 6.2 can be written as

$$\mathsf{M}(t) = 2\mathsf{V}t + \mathsf{W} \tag{6.12}$$

with the constant complex $N \times N$-matrices $\mathsf{V} = (\mathsf{V}_{jk})_{1\le j,k\le N}$ and $\mathsf{W} = (\mathsf{W}_{jk})_{1\le j,k\le N}$ having the entries:

$$\mathsf{V}_{jk} = \lambda_j \delta_{jk}, \quad \mathsf{W}_{jk} = \begin{cases} \gamma_j - i\varrho\delta_{j1} & \text{if } j = k, \\ \frac{i}{\lambda_j - \lambda_k} & \text{if } j \neq k. \end{cases} \tag{6.13}$$

Recall that $\lambda_1 = 0, \lambda_2, \dots, \lambda_N \in \mathbb{R}$ are real and pairwise distinct and $\varrho > 0$ is a strictly positive real number, whereas $\gamma_1, \dots, \gamma_N \in \mathbb{R}$ are real numbers (which are not necessarily pairwise distinct).

To prove that multi-soliton solutions extend to all times $t \in \mathbb{R}$, we show that all the eigenvalues of $\mathsf{M}(t)$ are always in the lower complex plane $\mathbb{C}_- = \{z \in \mathbb{C} : \operatorname{Im} z < 0\}$.

Lemma 6.1. *For any $t \in \mathbb{R}$, all eigenvalues of the matrix $\mathsf{M}(t)$ have strictly negative imaginary parts, that is, it holds*

$$\sigma(\mathsf{M}(t)) \subset \mathbb{C}_- \quad \textit{for } t \in \mathbb{R}.$$

Remark. Below we will prove the remarkable fact that all the eigenvalues of $\mathsf{M}(t)$ except for one will asymptotically converge to the real axis as $t \to \pm\infty$. As a consequence, this implies that all N-solitons with $N \geq 2$ will have an algebraic growth of their Sobolev norms according to $\|u(t)\|_{H^s} \sim |t|^{2s}$ for any $s > 0$.

Proof. Let $t \in \mathbb{R}$ be given. For notational convenience we write $\mathsf{M} = \mathsf{M}(t)$ in what follows. We readily verify the identities

$$\frac{(\mathsf{M} - \mathsf{M}^*)_{jk}}{2i} = -2\varrho\delta_{1j}\delta_{1k}, \quad [\mathsf{M}, \mathsf{V}] = [\mathsf{W}, \mathsf{V}] = iI - i\langle \cdot, X\rangle X, \tag{6.14}$$

where $\langle \cdot, \cdot \rangle_{\mathbb{C}^N}$ denotes the standard inner product on $\mathbb{C}^N$ and $X = (1, \dots, 1)^T \in \mathbb{C}^N$. Because of the first identity above, any eigenvalue $z \in \sigma(\mathsf{M})$ must satisfy

$$\operatorname{Im} z \leq 0.$$

Let us prove that $z \notin \mathbb{R}$ holds. We argue by contradiction as follows. Suppose that $z \in \mathbb{R}$ and let $v = (v_1, \dots, v_N)^T \in \mathbb{C}^N \setminus \{0\}$ be a corresponding eigenvector, that is,

$$\mathsf{M}v = zv.$$

Since we assume that z is real, we can use the first identity in (6.14) to deduce that

$$0 = \frac{1}{2i}\langle(\mathsf{M} - \mathsf{M}^*)v, v\rangle_{\mathbb{C}^N} = -2\varrho|v_1|^2$$

and therefore $v_1 = 0$. Projecting the equation $\mathsf{M}v = zv$ onto the 1st mode and recalling that $\lambda_1 = 0$ by assumption, we infer

$$\sum_{j=2}^{N} \frac{v_j}{\lambda_j} = 0.$$

This can be written as

$$\langle w, X\rangle = 0 \quad \text{with} \quad w := \left(0, \frac{v_2}{\lambda_2}, \dots, \frac{v_N}{\lambda_N}\right)^T.$$

Noticing that $\mathsf{V}w = v$ and applying the second identity in (6.14), we find

$$\mathsf{V}(\mathsf{M} - zI)w = i(w - \langle w, X\rangle_{\mathbb{C}^N} X) = iw. \tag{6.15}$$

If we take inner product of the left side with w and using that $\mathsf{V} = \mathsf{V}^*$, we conclude

$$\begin{aligned}\langle w, \mathsf{V}(\mathsf{M} - zI)w\rangle_{\mathbb{C}^N} &= \langle \mathsf{V}w, \mathsf{M}w\rangle_{\mathbb{C}^N} - z\langle w, \mathsf{V}w\rangle_{\mathbb{C}^N} = \langle \mathsf{M}^*v, w\rangle_{\mathbb{C}^N} - z\langle w, v\rangle_{\mathbb{C}^N} \\ &= z\langle v, w\rangle_{\mathbb{C}^N} - z\langle w, v\rangle_{\mathbb{C}^N} = z\sum_{j=2}^{N}\frac{|v_j|^2}{\lambda_j} - z\sum_{j=2}^{N}\frac{|v_j|^2}{\lambda_j} = 0,\end{aligned}$$

where we used that $z \in \mathbb{R}$ and that $\mathsf{M}^*v = \mathsf{M}v = zv$ holds thanks to the form of M and $v_1 = 0$. Thus from (6.15) we deduce that $0 = i|w|^2$. This shows that $w = 0$ and consequently we find $v = 0$, which is a contradiction. Therefore $z \in \mathbb{R}$ cannot be an eigenvalue of M. ☐

As a direct consequence of Proposition 6.2 and Lemma 6.1, we deduce the following result.

Theorem 6.1 (Global-in-Time Existence). *Suppose $u_0 \in H^1_+(\mathbb{R})$ is an N-soliton potential with some $N \geq 1$. Then the corresponding solution $u(t)$ of (CM-DNLS) with $u(0) = u_0$ extends to all times $t \in \mathbb{R}$.*

Remark 6.1. The previous study shows that, to every N–soliton u, one can associate

$$\Lambda(u) := (\varphi, \varrho, \lambda_2, \dots, \lambda_N, \gamma_1, \dots, \gamma_N) \in \mathbb{T} \times (0, +\infty) \times \mathbb{R}^{2N-1}$$

with the condition that $\lambda_1 = 0, \lambda_2, \dots, \lambda_N$ are pairwise distinct, so that

$$\sqrt{2\varrho}\mathrm{e}^{i\varphi} = \frac{\hat{u}(0^+)}{2i\pi}$$

and $\lambda_1 = 0, \lambda_2, \dots, \lambda_N$ are the eigenvalues of L_u, while $\gamma_j = \langle G\psi_j, \psi_j\rangle - i\varrho\delta_{j1}$ if $L_u\psi_j = \lambda_j\psi_j$, $\|\psi_j\|_{L^2} = 1$.

We claim that the spectral mapping $u \mapsto \Lambda(u)$ is surjective. Indeed, assume we are given $(\varphi, \varrho, \lambda_2, \dots, \lambda_N, \gamma_1, \dots, \gamma_N) \in \mathbb{T} \times (0, +\infty) \times \mathbb{R}^{2N-1}$ with the condition that $\lambda_1 = 0, \lambda_2, \dots, \lambda_N$ are pairwise distinct, and consider the associated matrix M. According to Lemma 6.1, we already know that the eigenvalues of M belong to the lower half plane, so that we may define u as

$$u(x) = \sqrt{2\varrho}\, e^{i\varphi} \langle (\mathsf{M} - xI)^{-1} X, Y \rangle_{\mathbb{C}^N} \quad \text{for} \quad \operatorname{Im} x \geq 0.$$

We claim that u is a N–soliton with $Q(x) = \det(xI - \mathsf{M})$. Indeed, this is equivalent to the identity

$$|u(x)|^2 = \frac{1}{i} \left(\frac{\overline{Q}'(x)}{\overline{Q}(x)} - \frac{Q'(x)}{Q(x)} \right), \; x \in \mathbb{R}\,. \tag{6.16}$$

Using the expressions of $u(x)$ and $Q(x)$, (6.16) is equivalent to

$$|\langle Z(x), X \rangle_{\mathbb{C}^N}|^2 = \|Z(x)\|^2$$

where $Z(x) := (xI - \mathsf{M}^*)^{-1} Y$. The latter identity can be proved as follows. From $(xI - \mathsf{M}^*) Z(x) = Y$, we infer $\mathsf{V}(xI - \mathsf{M}^*) Z(x) = 0$. Taking the imaginary part of the inner product of both sides with $Z(x)$, we obtain

$$\begin{aligned} 0 &= \langle (\mathsf{V}(xI - \mathsf{M}^*) - (xI - \mathsf{M})\mathsf{V}) Z(x), Z(x) \rangle_{\mathbb{C}^N} \\ &= \langle [\mathsf{V}, (xI - \mathsf{M}^*)] Z(x), Z(x) \rangle_{\mathbb{C}^N} = i(|\langle Z(x), X \rangle_{\mathbb{C}^N}|^2 - \|Z(x)\|^2)\,. \end{aligned}$$

Furthermore, from Proposition 6.2, the value of $\hat{u}(0^+)$ can be obtained by identifying the coefficient of $1/x$ in the expansion of $u(x)$ as $x \to \infty$. In order to complete the proof of the surjectivity, we just have to check the following identities,

$$L_u \psi_j = \lambda_j \psi_j\,, \; \langle G\psi_j, \psi_j \rangle = (\gamma_j - i\varrho\delta - j1) \|\psi_j\|_{L^2}^2\,,$$

if we define ψ_j as $\psi_j(x) = \langle (\mathsf{M} - xI)^{-1} Y_j, Y \rangle_{\mathbb{C}^N}$, where Y_j denotes the column with 1 on the line j and 0 on the other lines. This can be done by direct calculations. For instance, since $\mathsf{V} Y_j = \lambda_j Y_j$ and $\mathsf{V} Y = 0$,

$$\begin{aligned} \lambda_j \psi_j(x) &= \langle (\mathsf{M} - xI)^{-1} \mathsf{V} Y_j, Y \rangle_{\mathbb{C}^N} = \langle (\mathsf{M} - xI)^{-1} [\mathsf{V}, \mathsf{M}] (\mathsf{M} - xI)^{-1} Y_j, Y \rangle_{\mathbb{C}^N} \\ &= D\psi_j(x) + i \langle (\mathsf{M} - xI)^{-1} Y_j, X \rangle_{\mathbb{C}^N} \langle (\mathsf{M} - xI)^{-1} X, Y \rangle_{\mathbb{C}^N} \end{aligned}$$

while

$$\begin{aligned} &\langle (\overline{\mathsf{M}} - xI)^{-1} X, Y \rangle_{\mathbb{C}^N} \langle (\mathsf{M} - xI)^{-1} Y_j, Y \rangle_{\mathbb{C}^N} = \langle (\mathsf{M}^* - xI)^{-1} Y Y^T (\mathsf{M} - xI)^{-1} Y_j, X \rangle_{\mathbb{C}^N} \\ &= (2i\varrho)^{-1} [\langle (\mathsf{M} - xI)^{-1} X, Y \rangle_{\mathbb{C}^N} - \langle (\mathsf{M}^* - xI)^{-1} Y_j, X \rangle_{\mathbb{C}^N}], \end{aligned}$$

so that $i\langle (\mathsf{M} - xI)^{-1} Y_j, X \rangle_{\mathbb{C}^N} \langle (\mathsf{M} - xI)^{-1} X, Y \rangle_{\mathbb{C}^N} = -u(x) \Pi_+(\overline{u}\psi_j)(x)$.

Below we will show that all multi-soliton solutions with $N \geq 2$ exhibit growth of Sobolev norms such that $(\phi, \varrho, \lambda_2, \dots, \lambda_N, \gamma_1, \dots, \gamma_N) \in \mathbb{T} \times (0, +\infty) \times \mathbb{R}^{2N-1}$ is given so that

$$\|u(t)\|_{H^s} \sim |t|^{2s} \quad \text{as } t \to \pm\infty \text{ for any } s > 0.$$

This demonstrates that global-in-time existence for such $u(t,x)$ cannot be inferred from a-priori bounds in H^1. Furthermore, it shows that the infinite hierarchy of conversation laws $\{I_k(u)\}_{k=0}^{\infty}$ generally fail to produce a-priori bounds on solutions of (CM-DNLS) in the large data regime with $\|u(0)\|_{L^2}^2 > 2\pi$.

6.4 | Explicit example: Two-soliton solutions

Before we study the case of N-solitons with arbitrary $N \geq 1$, it is instructive to first consider the case $N = 2$ in detail. In this case, the constraints in Proposition 5.2 can be solved explicitly. Suppose that

$$u(x) = \frac{P(x)}{Q(x)} \in H^1_+(\mathbb{R}) \tag{6.17}$$

is a two-soliton potential, that is, $Q \in \mathbb{C}[x]$ is a polynomial of degree 2 with zeros in $\mathbb{C}_-$ and $P \in \mathbb{C}[x]$ is a polynomial of degree at most 1 satisfying the condition stated in Proposition 5.2 above.

We start with generic case by assuming that $Q(x)$ has two different zeros $z_1 \neq z_2$ in $\mathbb{C}_-$. Then condition (ii) in Proposition 5.2 implies that the polynomial $P(x) = \text{const}$ is constant and we find that

$$u(x) = \frac{a_1}{x - x_1} + \frac{a_2}{x - z_2}, \quad \sum_{j=1}^{2} \frac{a_j \overline{a}_k}{z_j - \overline{z}_k} = i \quad \text{with } k = 1, 2.$$

The constraints on $a_1, a_2 \in \mathbb{C}$ with given $z_1, z_2 \in \mathbb{C}_-$ can be solved explicitly as follows. Writing for convenience $y_j = -\mathrm{Im} z_j > 0$, we find

$$\frac{|a_1|^2}{2y_1} - i\frac{a_2\overline{a}_1}{z_2 - \overline{z}_1} = 1\,, \quad -i\frac{a_1\overline{a}_2}{z_1 - \overline{z}_2} + \frac{|a_2|^2}{2y_2} = 1\,.$$

Let us set

$$\xi := \frac{|a_1|^2}{2y_1} = \frac{|a_2|^2}{2y_2}\,, \; \eta := -i\frac{a_2\overline{a}_1}{z_2 - \overline{z}_1} = -i\frac{a_1\overline{a}_2}{z_1 - \overline{z}_2} = 1 - \xi\,.$$

Then

$$a_j = \sqrt{2\xi y_j}\mathrm{e}^{i\theta_j}$$

and

$$1 - \xi = \eta = -2i\sqrt{y_1 y_2}\,\xi\,\frac{\mathrm{e}^{i(\theta_1 - \theta_2)}}{z_1 - \overline{z}_2}\,.$$

Now we discuss the two solutions, according to $\xi < 1$ or $\xi > 1$. If $\xi < 1$,

$$1 - \xi = \xi\frac{2\sqrt{y_1 y_2}}{|z_1 - \overline{z}_2|}\,,$$

hence

$$\xi = \frac{1}{1 + \frac{2\sqrt{y_1 y_2}}{|z_1 - \overline{z}_2|}}.$$

Thus we get

$$a_1 = i\left(\frac{2y_1}{1 + 2\frac{\sqrt{y_1 y_2}}{|z_1 - \overline{z}_2|}}\right)^{\frac{1}{2}} \frac{z_1 - \overline{z}_2}{|z_1 - \overline{z}_2|} \mathrm{e}^{i\theta}\,, \quad a_2 = \left(\frac{2y_2}{1 + 2\frac{\sqrt{y_1 y_2}}{|z_1 - \overline{z}_2|}}\right)^{\frac{1}{2}} \mathrm{e}^{i\theta}\,.$$

If $\xi > 1$,

$$\xi - 1 = \xi \frac{2\sqrt{y_1 y_2}}{|z_1 - \overline{z}_2|}\,,$$

hence

$$\xi = \frac{1}{1 - \frac{2\sqrt{y_1 y_2}}{|z_1 - \overline{z}_2|}}.$$

Therefore,

$$a_1 = -i\left(\frac{2y_1}{1 - 2\frac{\sqrt{y_1 y_2}}{|z_1 - \overline{z}_2|}}\right)^{\frac{1}{2}} \frac{z_1 - \overline{z}_2}{|z_1 - \overline{z}_2|} \mathrm{e}^{i\theta}\,, \quad a_2 = \left(\frac{2y_2}{1 - 2\frac{\sqrt{y_1 y_2}}{|z_1 - \overline{z}_2|}}\right)^{\frac{1}{2}} \mathrm{e}^{i\theta}\,.$$

Let us compute the remaining eigenvalue λ of L_u in both cases. It is enough which is the trace of L_u on $\mathcal{E}_{\mathrm{pp}} = K_\theta$ in view of Proposition 5.3. If $\{j, k\} = \{1, 2\}$, one checks that

$$L_u\left(\frac{1}{x - z_k}\right) = -i\frac{a_j}{a_k(z_k - z_j)}\frac{1}{x - z_k} + i\frac{a_j}{a_k(z_k - z_j)}\frac{1}{x - z_j}\,.$$

This implies

$$\lambda = -i\frac{a_2}{a_1(z_1 - z_2)} - i\frac{a_1}{a_2(z_2 - z_1)}.$$

After some calculations we obtain

$$\lambda = \frac{-(y_1 + y_2)}{\sqrt{y_1 y_2}|z_1 - \overline{z}_2|} < 0 \quad \text{and} \quad \lambda = \frac{(y_1 + y_2)}{\sqrt{y_1 y_2}|z_1 - \overline{z}_2|} > 0$$

in the first and in the second case, respectively.

Finally, we consider the non-generic case when $z_1 = z_2 = z$ and $Q(x) = (x - z)^2$. Then

$$F(x) := i(Q'(x)\overline{Q}(x) - Q(x)\overline{Q}'(x)) = 4y(x - z)(x - \overline{z}).$$

Writing $y = -\mathrm{Im}\, z > 0$ again, we have two possibilities:

$$P(x) = \sqrt{4y}(x - z)\,,\ u(x) = \frac{\sqrt{4y}}{x - z}$$

and

$$P(x) = \sqrt{4y}(x - \overline{z})\,,\ u(x) = \frac{\sqrt{4y}(x - \overline{z})}{(x - z)^2}\,.$$

Notice that the first case should not be confused with the case $N = 1$, where

$$Q(x) = x - z\,,\ u(x) = \frac{\sqrt{2y}}{x - z}\,.$$

In the first case, we have

$$L_u\left(\frac{1}{(x - z)^2}\right) = -\frac{1}{y}\frac{1}{(x - z)^2},$$

so that $\lambda = -y^{-1}$, which can obtained from the first formula by letting $z_1 \to z$ and $z_2 \to z$. Similarly, in the second case, one obtains that $\lambda = y^{-1}$.

As a next step, we study the time evolution of two-solitons. For $N = 2$, the matrix $\mathsf{M}(t)$ is given by

$$\mathsf{M}(t) = \begin{pmatrix} \gamma_1 - i\varrho & -i\lambda^{-1} \\ i\lambda^{-1} & \gamma_2 + 2\lambda t \end{pmatrix} \tag{6.18}$$

with some positive number $\varrho > 0$ and real numbers $\gamma_1, \gamma_2 \in \mathbb{R}$ and $\lambda \neq 0$ denotes the non-zero eigenvalue of L_u. From Proposition 6.2 we deduce

$$u(t,x) = \frac{\mathrm{e}^{i\varphi}\sqrt{2\varrho}(\gamma_2 + 2\lambda t + i\lambda^{-1} - x)}{x^2 - (\gamma_1 - i\varrho + \gamma_2 + 2\lambda t)x + (\gamma_1 - i\varrho)(\gamma_2 + 2\lambda t) - \lambda^{-2}} \tag{6.19}$$

with some constant $\varphi \in [0, 2\pi)$. The discriminant of the denominator of $u(t,x)$ is

$$\Delta(t) = (\gamma_1 - i\varrho - \gamma_2 - 2\lambda t)^2 + 4\lambda^{-2}. \tag{6.20}$$

Note that $\Delta(t) = 0$ if and only if $\gamma_1 = \gamma_2 + 2\lambda t$ and $\varrho|\lambda| = 2$. This corresponds to the degenerate two-solitons, the cases 1 and 2 occurring according to the sign of λ. Notice that, if $\varrho|\lambda| = 2$, the two-soliton solution of (6.8) will be degenerate at exactly one time $t \in \mathbb{R}$ characterized by $\gamma_1 = \gamma_2 + 2\lambda t$.

Finally, let us study the large-time behaviour of a two-soliton. Let $z_+(t)$ and $z_-(t)$ denote the poles of $u(t,x)$ at time t. We see that

$$z_\pm(t) = \frac{1}{2}\Big(\gamma_0 - i\varrho + \gamma_1 + 2\lambda t \pm \sqrt{(\gamma_0 - i\varrho - \gamma_1 - 2\lambda t)^2 + 4\lambda^{-2}}\Big). \tag{6.21}$$

As $t \to \pm\infty$, we obtain

$$z_+(t) \to \gamma_0 - i\varrho, \quad \mathrm{Re}\, z_-(t) = 2\lambda t + O(1), \quad \mathrm{Im}\, z_-(t) = \frac{-\varrho}{4\lambda^4 t^2} + O(t^{-3}). \tag{6.22}$$

The vanishing of $\operatorname{Im} z_-(t)$ as $t \to \pm\infty$ implies growth of the Sobolev norms for the two-soliton solution. More precisely, we claim that

$$\|u(t)\|_{H^s} \sim c_s |t|^{2s} \quad \text{as } t \to \pm\infty \text{ for any } s > 0. \tag{6.23}$$

To prove this, we note that $z_+(t) \neq z_-(t)$ for all $t \in \mathbb{R}$ (except for one time t at most) and we have that

$$u(t,x) = \frac{a_+(t)}{x - z_+(t)} + \frac{a_-(t)}{x - z_-(t)}, \quad a_\pm(t) = \sqrt{2\varrho} e^{i\varphi} \frac{\gamma_1 + 2\lambda t + i\lambda^{-1} - z_\pm(t)}{\sqrt{\Delta(t)}}.$$

From (6.22), we infer as $|t| \to \infty$ that

$$a_+(t) = O(1) \quad \text{and} \quad a_-(t) \sim \frac{1}{|t|}.$$

Since

$$\hat{u}(t,\xi) = -2\pi i \left(a_+(t) e^{-iz_+(t)\xi} + a_-(t) e^{-iz_-(t)\xi} \right) \quad \text{for} \quad \xi > 0,$$

we deduce the bound (6.23) from (6.22) and by direct calculation.

6.5 | Long-time asymptotics

We now study the long-time behavior for N-solitons with general $N \geq 2$. The key ingredient for the general understanding is the following result about the long-time asymptotics for the eigenvalues of the matrix $\mathsf{M}(t)$ in (6.12).

Lemma 6.2. *There exists $T_0 > 0$ sufficiently large such that all eigenvalues*

$$\{z_1(t), \dots, z_N(t)\} \subset \mathbb{C}_-$$

of $\mathsf{M}(t)$ are simple for $|t| \geq T_0$. As $|t| \to +\infty$, we have the asymptotic expansions:

$$\operatorname{Re} z_k(t) = 2\lambda_k t + \gamma_k + O(t^{-1}) \quad \textit{for } k = 1, \dots, N$$

$$\operatorname{Im} z_1(t) = -\varrho + O(t^{-1}), \quad \operatorname{Im} z_k(t) = -\frac{\varrho}{4\lambda_k^4 t^2} + O(t^{-3}) \quad \textit{for } k = 2, \dots, N.$$

Proof. We use standard eigenvalue perturbation theory for

$$\mathsf{A}(\varepsilon) = \mathsf{A} + \varepsilon \mathsf{B}$$

with a small parameter $|\varepsilon| \ll 1$. Here $\mathsf{A} = \mathsf{A}^* \in \mathbb{C}^{N \times N}$ is a Hermitian matrix and B denotes an arbitrary matrix in $\mathbb{C}^{N \times N}$.

Since $\mathsf{M}(t) = t(2\mathsf{V} + t^{-1}\mathsf{W})$ and by taking $\varepsilon = t^{-1}$, it suffices to study the eigenvalues of

$$\mathsf{A}(\varepsilon) = \mathsf{A} + \varepsilon \mathsf{B} \quad \text{with} \quad \mathsf{A} = 2\mathsf{V} \text{ and } \mathsf{B} = \mathsf{W},$$

where the constant matrices V and W are displayed in (6.13). Because $\mathsf{A} = 2\mathrm{diag}(\lambda_1, \dots, \lambda_N)$ has N simple eigenvalues, standard perturbation theory yields that $\mathsf{A}(\varepsilon)$ has N simple eigenvalue provided that $|\varepsilon| \ll 1$ is sufficiently small. Thus for $|t| \geq T_0$ with $T_0 > 0$ sufficiently large, we see that $\mathsf{M}(t) = t\mathsf{A}(t^{-1})$ has N simple eigenvalues $\{z_1(t), \dots, z_N(t)\}$ which all belong to $\mathbb{C}_-$ by Lemma 6.1.

The derivation of the claimed asymptotics for the eigenvalues requires an expansion up to order $\varepsilon^3 = t^{-3}$. Let $\{\mu_1(\varepsilon), \dots, \mu_N(\varepsilon)\}$ denote the eigenvalues of $\mathsf{A} + \varepsilon\mathsf{B}$, which are simple for $|\varepsilon| \ll 1$ sufficiently small. From [32, Section XII.1] we recall that

$$\mu_k(\varepsilon) = \mu_k + \mu_k^{(1)}\varepsilon + \mu_k^{(2)}\varepsilon^2 + \mu_k^{(3)}\varepsilon^3 + O(\varepsilon^4), \quad \text{with } \mu_k = 2\lambda_k\,.$$

The first-order coefficient is the well-known expression

$$\mu_k^{(1)} = \mathsf{B}_{kk} = \gamma_k - i\varrho\delta_{k1}.$$

Next, the second-order contribution is purely real and given by

$$\mu_k^{(2)} = -\sum_{j=1, j\neq k}^{N} \frac{1}{\mu_j - \mu_k}\mathsf{B}_{kj}\mathsf{B}_{jk} = -\frac{1}{2}\sum_{j=1, j\neq k}^{N} \frac{1}{(\mu_j - \mu_k)^3} \in \mathbb{R}.$$

Finally, the third-order term reads

$$\mu_k^{(3)} = \sum_{j\neq k, \ell\neq k}^{N} \frac{1}{(\mu_j - \mu_k)(\mu_\ell - \mu_k)}\mathsf{B}_{kj}\mathsf{B}_{j\ell}\mathsf{B}_{\ell k} - \sum_{j\neq k}^{N} \frac{1}{(\mu_j - \mu_k)^2}\mathsf{B}_{kj}\mathsf{B}_{jk}\mathsf{B}_{kk}.$$

For the proof of the lemma, it suffices to determine $\mathrm{Im}\,\mu_k^{(3)}$ for $k = 2, \dots, N$. Since $\mathsf{B}_{kk} \in \mathbb{R}$ for $k \geq 2$ and $\mathsf{B}_{kj}\mathsf{B}_{jk} \in \mathbb{R}$ if $j \neq k$, we see

$$\mathrm{Im} \sum_{j\neq k}^{N} \frac{1}{(\mu_j - \mu_k)^2}\mathsf{B}_{kj}\mathsf{B}_{jk}\mathsf{B}_{kk} = 0 \quad \text{for } k = 2, \dots, N.$$

As for the first sum in the expression for $\mu_k^{(3)}$, we notice the symmetry property

$$\mathsf{B}_{kj}\mathsf{B}_{j\ell}\mathsf{B}_{\ell k} = -\mathsf{B}_{\ell j}\mathsf{B}_{\ell j}\mathsf{B}_{jk} \quad \text{for } j \neq \ell.$$

Thus we only need to consider the diagonal case when $j = \ell$. This leads to

$$\mathrm{Im}\,\mu_k^{(3)} = \mathrm{Im} \sum_{j\neq k}^{N} \frac{1}{(\mu_j - \mu_k)^2}\mathsf{B}_{kj}\mathsf{B}_{jj}\mathsf{B}_{jk}.$$

Recalling that $\mathsf{B}_{kj}\mathsf{B}_{jk} \in \mathbb{R}$ for $j \neq k$ and $\mathsf{B}_{jj} \in \mathbb{R}$ if $j \geq 2$, we deduce

$$\mathrm{Im}\,\mu_k^{(3)} = \mathrm{Im}\frac{1}{(\mu_1 - \mu_k)^2}\mathsf{B}_{1k}\mathsf{B}_{11}\mathsf{B}_{k1} = -\frac{\varrho}{4\lambda_k^4} \quad \text{for } k = 2, \dots, N,$$

using that $\mu_k = 2\lambda_k$ and $\mu_1 = 2\lambda_1 = 0$. Since $\mathsf{M}(t) = t^{-1}\mathsf{A}(t^{-1})$, we obtain the claimed asymptotic formulae. □

As a consequence of the preceding lemma, we obtain the following result.

Proposition 6.3. *Let $u(t,x)$ be an N-soliton solution. Then there exists $T_0 > 0$ sufficiently large such that*

$$u(t,x) = \sum_{j=1}^{N} \frac{a_j(t)}{x - z_j(t)} \quad \textit{for } |t| \geq T_0,$$

where $\{z_1(t), \dots, z_N(t)\} \subset \mathbb{C}_-$ denote the simple eigenvalues of $\mathsf{M}(t)$ and coefficients $a_1(t), \dots, a_N(t) \in \mathbb{C} \setminus \{0\}$.

Proof. By Lemma 6.2, the eigenvalues $\{z_1(t), \dots, z_N(t)\} \subset \mathbb{C}_-$ of $\mathsf{M}(t)$ are simple whenever $|t| \geq T_0$, where $T_0 > 0$ is some sufficiently large constant.

Fix some time $t \in \mathbb{R}$ with $|t| \geq T_0$. For notational convenience, we will omit the dependence of $u(t,x)$ and $\mathsf{M}(t)$ for the rest of the proof. Since $u \in H^1_+(\mathbb{R})$ is a multi-soliton potential, we have

$$u(x) = \frac{P(x)}{Q(x)}$$

with some polynomial $Q \in \mathbb{C}[x]$ of degree N having all its zeros in $\mathbb{C}_-$ and some polynomial $P \in \mathbb{C}_{N-1}[x]$ satisfying the condition in Proposition 5.2. Recall that M denotes the matrix (with respect to a suitable orthonormalbasis) of the operator G acting on the invariant space $K_\theta = \frac{\mathbb{C}_{N-1}[x]}{Q(x)}$. From [34, Lemma 3.3] we observe that $Q(x)$ is the characteristic polynomial of G acting on K_θ. Hence we conclude

$$Q(x) = \det(xI - \mathsf{M}).$$

Since M has only simple eigenvalues, we find $Q(x) = \prod_{j=1}^{N}(x - z_j)$ with pairwise distinct zeros $z_j \in \mathbb{C}_-$. Since $P\overline{P} = i(Q'\overline{Q} - \overline{Q}'Q)$, we see that $P \in \mathbb{C}_{N-1}[x]$ has no common zeros with $Q(x)$. Thus, by partial fraction expansion, we conclude that

$$u(x) = \frac{P(x)}{Q(x)} = \sum_{j=1}^{N} \frac{a_j}{x - z_j}$$

with some $a_1, \dots, a_N \in \mathbb{C} \setminus \{0\}$. □

6.6 | Growth of Sobolev norms: Proof of theorem 1.3

Let $N \geq 2$ and suppose $u(t,x)$ is an N-soliton, which by Theorem 6.1 exists for all times $t \in \mathbb{R}$. We claim that

$$\|u(t)\|_{H^s} \sim_s t^{2s} \quad \text{as} |t| \to +\infty \tag{6.24}$$

for any $s > 0$.

For convenience, we discuss the limit $t \to +\infty$. (The case $t \to -\infty$ follows by the same reasoning.) We divide the proof into following steps.

Step 1. We first derive the asymptotics for $a_1(t), \dots, a_N(t)$. From Proposition 6.3 we deduce that

$$u(t,x) = \sum_{j=1}^{N} \frac{a_j(t)}{x - z_j(t)} \quad \text{for} \quad t \geq T_0 \tag{6.25}$$

with some sufficiently large constant $T_0 > 0$. Now we are in the position to use the system of differential equations in (6.3) for the time evolution of $a_1(t), \dots, a_N(t)$. We claim that

$$a_k(t) = \frac{\alpha}{\lambda_k^2 t} + O(t^{-2}) \quad \text{for } k = 2, \dots, N \tag{6.26}$$

with some non-zero constant $\alpha \in \mathbb{C} \setminus \{0\}$. Indeed, using that $\dot{z}_k = 2\lambda_k + o(1)$ as $t \to +\infty$ by Lemma 6.2, we deduce from (6.3) that

$$\vec{a}(t) = A(t)\vec{a}(t) + \vec{b}(t)$$

where $\vec{a}(t) = (a_2(t), \dots, a_N(t)) \in \mathbb{C}^{N-1}$ and $A(t) = (A(t))_{2 \leq k,\ell \leq N} \in \mathbb{C}^{N-1} \times \mathbb{C}^{N-1}$, $\vec{b}(t) = (b_2(t), \dots, b_N(t)) \in \mathbb{C}^{N-1}$ are given by

$$(A(t))_{k\ell} = \frac{-i}{(\lambda_k + o(1))(z_k - z_\ell)} = O(t^{-1}) \quad \text{for } k \neq \ell, \quad (A(t))_{kk} = 0,$$

$$b_k(t) = \frac{-ia_0(t)}{(\lambda_k + o(1))(z_k - z_1)} = O(t^{-1})a_1(t).$$

Here we also used that $|z_k(t) - z_\ell(t)| \sim t$ as $|t| \to \infty$ which follows from Lemma 6.2 together with the fact that all $\lambda_k \neq \lambda_\ell$ for $k \neq \ell$. Next, as a direct consequence of (6.3), we infer the conservation law

$$A = \sum_{j=1}^{N} a_j(t) = \frac{\hat{u}(t, 0^+)}{-2\pi i} \quad \text{for } t \geq T_0, \tag{6.27}$$

with some constant $A \in \mathbb{C}$, where the last equation follows from taking the Fourier transform of the right-hand side in (6.25). Since we must have $\hat{u}(t, 0^+) \neq 0$ by the discussion above, we conclude that $A \neq 0$ as well. Hence we deduce that

$$\vec{a}(t) = O(t^{-1})\vec{a}(t) + \vec{f}(t) \tag{6.28}$$

where $\vec{f}(t) = (f_2(t), \dots, f_N(t))$ is given by

$$f_k(t) = \frac{-iA}{(\lambda_k + o(1))(z_k(t) - z_1(t))} = \frac{-iA}{2\lambda_k^2 t} + O(t^{-2}),$$

thanks to the fact that $z_k(t) - z_1(t) = 2\lambda_k t + O(1)$ by Lemma 6.2 and $\lambda_1 = 0$. In view of (6.28), we conclude that (6.26) holds with the constant $\alpha = \frac{-iA}{2} \neq 0$.

Finally, by the conservation law (6.27) together with (6.26) we immediately find

$$\lim_{t \to +\infty} a_1(t) = A \neq 0. \tag{6.29}$$

Step 2. For $t \geq T_0$, we write

$$u(t,x) = \sum_{j=1}^{N} \phi_j(t,x) \quad \text{with} \quad \phi_j(t,x) = \frac{a_j(t)}{x - z_j(t)}. \tag{6.30}$$

We note that

$$\hat{\phi}_j(t,\xi) = -2\pi i a_j(t) \mathrm{e}^{-iz_j\xi} \mathbb{1}_{\xi \geq 0}.$$

By Plancherel, we find

$$\langle \phi_j(t), \phi_k(t) \rangle_{H^s} = -i a_j(t) \overline{a}_k(t) \int_0^{\infty} \mathrm{e}^{-i(z_j(t) - \overline{z}_k(t))\xi} \langle \xi \rangle^{2s} \, d\xi$$

To estimate this expression for $j \neq k$, we observe, by Lemma 6.2, that $\omega_{jk}(t) := z_j(t) - \overline{z}_k(t)$ satisfies $\operatorname{Im} \omega_{jk}(t) < 0$ as well as

$$\omega_{jk}(t) = 2(\lambda_j - \lambda_k)t + O(1) \sim t \quad \text{as} \quad t \to +\infty,$$

since $\lambda_j \neq \lambda_j$ when $j \neq k$. Integrating by parts sufficiently many times depending on $s > 0$, we deduce that

$$\left| \int_0^{\infty} \mathrm{e}^{-i\omega_{jk}(t)\xi} \langle \xi \rangle^{2s} \, d\xi \right| \lesssim_s \frac{1}{|\omega_{jk}(t)|} \sim \frac{1}{t} \quad \text{for } t \geq T_0,$$

provided that $j \neq k$. If recall the bounds for $a_1(t), \dots, a_N(t)$ derived in **Step 1** above, we can conclude

$$\left| \langle \phi_j(t), \phi_k(t) \rangle_{H^s} \right| \lesssim_s \frac{1}{t^2} \quad \text{for } t \geq T_0 \text{ and } j \neq k.$$

Next, we consider the case $j = k$. This yields

$$\begin{aligned} \langle \phi_j(t), \phi_j(t) \rangle_{H^s} &\sim |a_j(t)|^2 \int_0^{\infty} \mathrm{e}^{2\operatorname{Im}(z_j(t))\xi} (1 + |\xi|^{2s}) d\xi \\ &= |a_j(t)|^2 \left(\frac{1}{2|\operatorname{Im} z_j(t)|} + \frac{C_s}{2|\operatorname{Im} z_j(t)|^{1+2s}} \right) \end{aligned}$$

with the constant $C_s = \int_0^{\infty} \mathrm{e}^{-y} y^{2s} \, dy > 0$. By combining the estimates for the coefficients $\{a_1(t), \dots, a_N(t)\}$ from **Step 1** and the poles $\{z_1(t), \dots, z_N(t)\}$ from Lemma 6.2 we finally obtain

$$\langle \phi_j(t), \phi_j(t) \rangle_{H^s} \simeq_s \begin{cases} 1 & \text{for } j = 1, \\ t^{4s} & \text{for } j = 2, \dots, N, \end{cases}$$

for all times $t \geq T_0$. In summary, we conclude

$$t^{4s} + t^{-2} \lesssim_s \|u(t)\|_{H^s}^2 = \sum_{j=1}^{N} \langle \phi_j(t), \phi_j(t) \rangle + \sum_{j \neq k}^{N} \langle \phi_j(t), \phi_k(t) \rangle \lesssim_s t^{4s} + t^{-2}.$$

This proves (6.24) and completes the proof of Theorem 1.3.

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APPENDIX A: DEFINITION OF THE LAX OPERATOR FOR $u \in L^2_+(\mathbb{R})$

In view of the L^2-criticality of (CM-DNLS), it is worthwhile giving a definition of L_u via quadratic forms if we only assume that $u \in L^2_+(\mathbb{R})$ holds. We start with the following basic estimate.

Lemma A.1. *For every $u \in L^2_+(\mathbb{R})$ and $f \in H^{\frac{1}{2}}_+(\mathbb{R})$, we have $T_{\overline{u}} f \in L^2_+(\mathbb{R})$ with*

$$\|T_{\overline{u}} f\|^2_{L^2} \leq \frac{1}{2\pi} \|u\|^2_{L^2} \langle Df, f\rangle \, .$$

Proof. Applying the Fourier transformation, we have

$$\widehat{T_{\overline{u}} f}(\xi) = \int_0^{+\infty} \hat{f}(\xi+\eta)\overline{\hat{u}(\eta)} \, \frac{d\eta}{2\pi} \, .$$

Thus, by the Cauchy–Schwarz inequality,

$$\begin{aligned}
\int_0^{\infty} |\widehat{T_{\overline{u}} f}(\xi)|^2 \, d\xi &\leq \int_0^{+\infty} \int_0^{+\infty} |\hat{f}(\xi+\eta)|^2 \, \frac{d\xi \, d\eta}{2\pi} \int_0^{+\infty} |\hat{u}(\eta)|^2 \, \frac{d\eta}{2\pi} \\
&\leq \int_0^{+\infty} \zeta |\hat{f}(\zeta)|^2 \, \frac{d\zeta}{2\pi} \int_0^{+\infty} |\hat{u}(\eta)|^2 \, \frac{d\eta}{2\pi} \, ,
\end{aligned}$$

and the claim follows from Plancherel's theorem. □

Next, we define L_u for given $u \in L^2_+(\mathbb{R})$ via a densely defined quadratic form on $L^2_+(\mathbb{R})$ as follows. For $f, g \in H^{1/2}_+(\mathbb{R})$, we set

$$Q_u(f,g) = \langle Df, g\rangle - \langle T_{\overline{u}} f, T_{\overline{u}} g\rangle .$$

We claim that, for every $\varepsilon > 0$, there exists a constant $C_\varepsilon(u) > 0$ such that

$$\forall h \in H^{1/2}_+(\mathbb{R}) ,\ \|T_{\overline{u}} h\|_{L^2} \le \varepsilon \langle Dh, h\rangle^{1/2} + C_\varepsilon(u)\|h\|_{L^2} . \tag{A.1}$$

Indeed, for every $\lambda > 0$, set

$$u_{<\lambda} := \mathbf{1}_{[0,\lambda[}(D)u ,\ u_{\ge\lambda} := \mathbf{1}_{[\lambda,+\infty[}(D)u .$$

Then $\|u_{\ge\lambda}\|_{L^2} \to 0$ as $\lambda \to +\infty$, while

$$\|u_{<\lambda}\|_{L^\infty} \le \left(\frac{\lambda}{2\pi}\right)^{1/2} \|u\|_{L^2} .$$

Choose $\lambda = \lambda(\varepsilon, u)$ such that $(2\pi)^{-1/2}\|u_{\ge\lambda}\|_{L^2} \le \varepsilon$. Then, by Lemma A.1,

$$\|T_{\overline{u}} h\|_{L^2} \le \|T_{\overline{u}_{\ge\lambda}} h\|_{L^2} + \|T_{\overline{u}_{<\lambda}} h\|_{L^2} \le \varepsilon\langle Dh, h\rangle^{1/2} + \|u_{<\lambda}\|_{L^\infty}\|h\|_{L^2} ,$$

and (A.1) follows. Applying (A.1), we obtain

$$\langle Df, f\rangle \ge Q_u(f,f) \ge (1-2\varepsilon^2)\langle Df, f\rangle - 2C_\varepsilon(u)^2\|f\|^2_{L^2} . \tag{A.2}$$

Choosing ε small enough, we find a constant $K = K(u) > 0$ such that

$$\tilde{Q}_u(f,g) := Q_u(f,g) + K(u)\langle f, g\rangle$$

is an inner product on $H^{1/2}_+(\mathbb{R})$, defining a norm which is equivalent to the standard one. Then we just define

$$\mathrm{dom}(L_u) = \{f \in H^{1/2}_+(\mathbb{R}) \,:\, \exists C > 0 \text{ s. t. } |Q_u(f,g)| \le C\|g\|_{L^2} \text{ for } g \in H^{1/2}_+(\mathbb{R})\}$$

and

$$\langle L_u(f), g\rangle = Q_u(f,g) \quad \text{for } f \in \mathrm{dom}(L_u) \text{ and } g \in H^{1/2}_+(\mathbb{R}) .$$

Then the standard theory of quadratic forms (see [33]) implies that $\mathrm{dom}(L_u)$ is dense in $H^{1/2}_+(\mathbb{R})$, hence in $L^2_+(\mathbb{R})$, and that L_u is self-adjoint and bounded below. Furthermore, using the quadratic form Q_u, Lemma 5.1 and Proposition 5.1 extend easily to the case $u \in L^2_+(\mathbb{R})$.

APPENDIX B: VARIATIONAL PROPERTIES OF $E(u)$

We recall the energy functional

$$E(u) = \frac{1}{2}\int_{\mathbb{R}} |\partial_x u - i\Pi_+(|u|^2)u|^2\, dx,$$

where we allow for general $u \in H^1(\mathbb{R})$, which are not necessarily in the Hardy-Sobolev space $H^1_+(\mathbb{R})$. It is elementary to show that $E : H^1(\mathbb{R}) \to \mathbb{R}_{\geq 0}$ is *weakly lower semi-continuous*, that is, if $u_k \rightharpoonup u$ weakly in $H^1(\mathbb{R})$ then

$$\liminf_{n\to\infty} E(u_n) \geq E(u). \tag{B.1}$$

Lemma B.1 (Minimal Mass Bubble Lemma). *Suppose $(v_n)_{n\in\mathbb{N}} \subset H^1(\mathbb{R})$ is a sequence such that*

$$\sup_{n\geq 1} \|v_n\|_{L^2} < +\infty \quad \textit{and} \quad \|\partial_x v_n\|_{L^2} = \mu \quad \textit{for all } n \in \mathbb{N}.$$

with some constant $\mu > 0$. In addition, we assume that

$$\lim_{n\to\infty} E(v_n) = 0.$$

Then it holds that

$$\liminf_{n\to\infty} \|v_n\|^2_{L^2} \geq \|\mathcal{R}\|^2_{L^2} = 2\pi.$$

Here equality holds if and only if, after possibly passing to a subsequence,

$$v_n(x + x_n) \to \mathrm{e}^{i\theta}\lambda^{1/2}\mathcal{R}(\lambda x) \quad \textit{strongly in } L^2(\mathbb{R})$$

with some constants $\theta \in [0, 2\pi)$, $\lambda > 0$, and some sequence $x_n \in \mathbb{R}$.

Proof. We will give a proof that is based on a compactness lemma in [24]. Alternatively, we could use more refined analysis with a profile decomposition [15].

By rescaling $v_n \to \mu^{-1/2} v_n(\mu^{-1}\cdot)$, we can assume that $\|\partial_x v_n\|_{L^2} = 1$ for all $n \in \mathbb{N}$. From the triangle inequality and the form of $E(u)$, we find that

$$\|\Pi_+(|v_n|^2)v_n\|_{L^2} \geq \|\partial_x v_n\|_{L^2} - \|\partial_x v_n - i\Pi_+(|v_n|^2)v_n\|_{L^2} = 1 - \sqrt{2E(v_n)}.$$

Since $E(v_n) \to 0$ by assumption, we deduce that

$$1 \lesssim \|\Pi_+(|v_n|^2)v_n\|_{L^2} \lesssim \|\Pi_+(|v_n|^2)\|_{L^3}\|v_n\|_{L^6} \lesssim \||v_n|^2\|_{L^3}\|v_n\|_{L^6} \lesssim \|v_n\|^3_{L^6},$$

by Hölder's inequality and the classical fact that $\Pi_+ : L^3(\mathbb{R}) \to L^3(\mathbb{R})$ is bounded. Thus we have found that

$$\|v_n\|_{L^6} \geq C > 0$$

with some constant $C > 0$. On the other hand, by the fact $\sup_n \|v_n\|_{H^1} < +\infty$ and by Sobolev embeddings, we deduce that

$$\|v_n\|_{L^2} \leq C_1 \quad \text{and} \quad \|v_n\|_{L^8} \leq C_2$$

with some constants $C_1, C_2 > 0$. Thus, by applying the *pqr*-Lemma in [14], we deduce that there exist constants $\varepsilon > 0$ and $\delta > 0$ such that

$$\mu(\{x \in \mathbb{R} : |v_n(x)| > \varepsilon\}) \geq \delta \quad \text{for all } n \in \mathbb{N},$$

where μ denotes the Lebesgue measure on $\mathbb{R}$. Hence we can apply Lieb's compactness lemma in [24] to deduce that, after passing to a subsequence, there exists a sequence of translations $(y_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ such that

$$v_n(\cdot + y_n) \rightharpoonup v \quad \text{weakly in } H^1(\mathbb{R})$$

for some $v \in H^1(\mathbb{R})$ with $v \not\equiv 0$.

By translation invariance of the energy E, we can henceforth assume that $y_n = 0$ for all n. Furthermore, the weak lower semi–continuity of E implies that

$$0 = \lim_{n\to\infty} E(v_n) \geq E(v).$$

On the other hand, we have $E(v) \geq 0$ in general and hence we conclude $E(v) = 0$. By Lemma 4.1, the equality $E(v) = 0$ for $v \not\equiv 0$ holds if and only if

$$v(x) = \mathrm{e}^{i\theta}\lambda^{1/2}\mathcal{R}(\lambda x + y)$$

with some constants $\theta \in [0, 2\pi)$, $\lambda > 0$ and $y \in \mathbb{R}$. Since $v_n \rightharpoonup v$ in L^2, the weak lower semi–continuity of the L^2-norm implies that

$$\liminf_{n\to\infty} \|v_n\|_{L^2}^2 \geq \|v\|_{L^2}^2 = \|\mathcal{R}\|_{L^2}^2 = 2\pi.$$

Finally, we have equality (after passing to a subsequence if necessary) if and only if $v_n \to v$ strongly in $L^2(\mathbb{R})$, which completes the proof. □

APPENDIX C: USEFUL IDENTITIES AND GAUGE TRANSFORMATION

For sufficiently regular and decaying functions $v : \mathbb{R} \to \mathbb{C}$ and the Hilbert transform H, we have the following identities:

$$\mathrm{Re}\langle xv, \mathsf{H}(v)\rangle = \frac{1}{2\pi}\left|\int_{\mathbb{R}} v\, dx\right|^2, \tag{C.1}$$

$$\mathrm{Re}\,\langle \partial_x v, \mathsf{H}(|v|^2)v\rangle = -\frac{1}{2}\langle |v|^2, |D||v|^2\rangle, \tag{C.2}$$

$$\langle v\mathsf{H}(|v|^2), v\mathsf{H}(|v|^2)\rangle = \frac{1}{3}\int_{\mathbb{R}} |v|^6\, dx. \tag{C.3}$$

Let us check these identities. Using the Plancherel theorem, we have

$$\mathrm{Re}\langle xv, \mathsf{H}(v)\rangle = -\frac{1}{2\pi}\int_0^\infty \mathrm{Re}[\overline{\hat{v}}(\xi)\partial_\xi \hat{v}(\xi)]\, d\xi + \frac{1}{2\pi}\int_{-\infty}^0 \mathrm{Re}[\overline{\hat{v}}(\xi)\partial_\xi \hat{v}(\xi)]\, d\xi = \frac{1}{2\pi}|\hat{v}(0)|^2,$$

which is (C.1).

Since H preserves real valued functions, we have

$$\mathrm{Re}\,\langle \partial_x v, \mathsf{H}(|v|^2)v\rangle = \langle \mathrm{Re}[\overline{v}\partial_x v], \mathsf{H}(|v|^2\rangle = \frac{1}{2}\langle \partial_x(|v|^2), \mathsf{H}(|v|^2\rangle = -\frac{1}{2}\langle |v|^2, \mathsf{H}\partial_x(|v|^2\rangle,$$

which leads to (C.2) since $\mathsf{H}\partial_x = |D|$.

Finally, setting $\varrho := |v|^2$, we have $\varrho = \Pi_+\varrho + \overline{\Pi_+\varrho}$ and

$$\|v\|_{L^6}^6 = \int_{\mathbb{R}} \varrho^3 \, dx = \int_{\mathbb{R}} (\Pi_+\varrho + \overline{\Pi_+\varrho})^3 \, dx = 3 \int_{\mathbb{R}} [(\Pi_+\varrho)^2 \overline{\Pi_+\varrho} + \Pi_+\varrho(\overline{\Pi_+\varrho})^2] \, dx \,,$$

because $H^s_\pm$ is preserved by the product (if s is large enough) and the ranges of Π_+ and of Π_- are orthogonal. We obtain

$$\int_{\mathbb{R}} \varrho^3 \, dx = 3 \int_{\mathbb{R}} |\Pi_+\varrho|^2 \varrho \, dx = \frac{3}{4} \int_{\mathbb{R}} (\varrho^2 + (H\varrho)^2)\varrho \, dx \,,$$

whence

$$\int_{\mathbb{R}} \varrho^3 \, dx = 3 \int \varrho (H\varrho)^2 \, dx \,,$$

which is precisely (C.3).

C.1 | Derivation of Pohozaev identity

In this paragraph we provide the details for the proof of Proposition 4.1 when deriving the Pohozaev type identities. We integrate (4.11) against $x\partial_x \overline{S}$ over the compact interval $[-R, R]$. By taking the real part, we find

$$\mathrm{Re} \int_{-R}^{R} \left(-x\overline{S}' S'' - x\overline{S}' (|D||S|^2) S + \frac{1}{4} x\overline{S}' |S|^4 S \right) dx = \tilde{\omega} \mathrm{Re} \int_{-R}^{R} x\overline{S}' S \, dx.$$

For the first term on the left-hand side, we find

$$\mathrm{Re} \int_{-R}^{R} x\overline{S}' S'' \, dx = \frac{1}{2} \int_{-R}^{R} x\partial_x |S'|^2 \, dx = \frac{1}{2} x|S'(x)|^2 \Big|_{-R}^{R} - \frac{1}{2} \int_{-R}^{R} |S'(x)|^2 \, dx.$$

In view of $|D| = \mathsf{H}\partial_x$ and integrating by parts, we obtain

$$\begin{aligned} \mathrm{Re} \int_{-R}^{R} x\overline{S}' (|D||S|^2) S \, dx &= \frac{1}{2} \int_{-R}^{R} x\partial_x |S|^2 (\mathsf{H}\partial_x |S|^2) \, dx \\ &\to \frac{1}{2\pi} \left(\int_{-\infty}^{\infty} \partial_x |S|^2 \, dx \right)^2 = 0 \quad \text{as} \quad R \to +\infty, \end{aligned}$$

where we also used (C.1). Next, we notice

$$\mathrm{Re} \int_{-R}^{R} x\partial_x \overline{S} |S|^4 S \, dx = \frac{1}{6} x|S(x)|^6 \Big|_{-R}^{R} - \frac{1}{6} \int_{-R}^{R} |S|^6 \, dx$$

$$\tilde{\omega} \mathrm{Re} \int_{-R}^{R} x\partial_x \overline{S} S \, dx = \frac{\tilde{\omega}}{2} x|S(x)|^2 \Big|_{-R}^{R} - \frac{\tilde{\omega}}{2} \int_{-R}^{R} |S(x)|^2 \, dx.$$

Since $|S|^2 + |S'|^2 \in L^1(\mathbb{R})$, there exists a sequence $R_n \to +\infty$ such that $x(|S(x)|^2 + |S'(x)|^2) \to 0$ with $x = \pm R_n$ as $n \to \infty$ and we obtain

$$\frac{1}{2} \int_{\mathbb{R}} |\partial_x S|^2 \, dx - \frac{1}{24} \int_{\mathbb{R}} |S|^6 \, dx = -\frac{\tilde{\omega}}{2} \int_{\mathbb{R}} |S|^2 \, dx.$$

C.2 | Gauge transformation and pseudo-conformal law

Let $s \geq 0$ be given. We consider the nonlinear map

$$\Phi : H^s(\mathbb{R}) \to H^s(\mathbb{R}), \quad u(x) \mapsto u(x)\mathrm{e}^{-\frac{i}{2}\int_{-\infty}^{x} |u(y)|^2\, dy}. \tag{C.4}$$

Clearly, we have mass preservation property $M[\Phi(u)] = M(u)$ and it can be shown that $\Phi : H^s(\mathbb{R}) \to H^s(\mathbb{R})$ is a diffeomorphism. We refer to the map Φ as the **gauge transform**. We remark that the map Φ (with different numerical factors in the exponential term) also plays an important role for the derivative NLS (DNLS).

Suppose $u(t) \in H^s(\mathbb{R})$ (not necessarily restricted to $H^s_+(\mathbb{R})$) solves (CM-DNLS) on some time interval $I \subset \mathbb{R}$. Then $v(t) = \Phi(u(t)) \in H^s(\mathbb{R})$ is found to solve the derivative type NLS equation:

$$i\partial_t v = -\partial_{xx} v - (|D||v|^2)v + \frac{1}{4}|v|^4 v, \tag{C.5}$$

using that $\Pi_+ = \frac{1}{2}(1 + i\mathsf{H})$ and $|D| = \mathsf{H}\partial_x$. We readily check that (C.5) has the conserved energy

$$\widetilde{E}(v) = \frac{1}{2}\int_{\mathbb{R}} |\partial_x v|^2 - \frac{1}{4}\int_{\mathbb{R}} |v|^2(|D||v|^2) + \frac{1}{24}\int_{\mathbb{R}} |v|^6. \tag{C.6}$$

From identities (C.2) and (C.3), we have

$$\widetilde{E}(v) = \frac{1}{2}\int_{\mathbb{R}} |\partial_x v + \frac{1}{2}\mathsf{H}(|v|^2)v|^2 = \frac{1}{2}\int_{\mathbb{R}} |\partial_x u - i\Pi_+(|u|^2)u|^2 = E(u) \tag{C.7}$$

since $v = \Phi(u)$ and $\Pi_+ = \frac{1}{2}(1 + i\mathsf{H})$.

Now, let $\Sigma = \{u \in H^1(\mathbb{R}) : xu \in L^2(\mathbb{R})\}$ denote the space of solutions with finite variance and energy (not necessarily restricted to the Hardy space).

Lemma C.1 (Pseudo-Conformal Law). *Suppose that $u \in C([-T,T]; H^1(\mathbb{R}))$ solves (CM-DNLS) with $u(0) = u_0 \in \Sigma$. Then $u(t) \in \Sigma$ for all $t \in [-T,T]$ and we have*

$$\frac{d^2}{dt^2}\int_{\mathbb{R}} |x|^2 |u(t,x)|^2\, dx = 16E(u_0) \quad \textit{for } t \in [-T,T].$$

As a consequence, it holds that

$$8t^2 E(\mathrm{e}^{i|x|^2/4t}u_0) = \int_{\mathbb{R}} |x|^2 |u(t,x)|^2\, dx$$

for all $t \in [-T,T]$ with $t \neq 0$.

Remarks.

(1) Recall that the ground state $\mathcal{R}(x) = \frac{\sqrt{2}}{x+i} \in H^1_+(\mathbb{R})$ does *not* belong to Σ. In fact, it can be shown that all multi-solitons for (CM-DNLS) fail to have finite variance as well.
(2) In view of the non-negativity $E(u) \geq 0$, we see that the classical Zakharov–Glassey argument to prove existence for negative energy data (with finite variance) cannot be invoked for (CM-DNLS).

Proof. Let $v(t) = \Phi(u(t))$ for $t \in [-T, T]$. We readily check that $\Phi : \Sigma \to \Sigma$ holds and, in particular, we have $v_0 = v(0) \in \Sigma$. Recall that $v \in C([-T, T]; H^1(\mathbb{R}))$ solves (C.5). By following arguments for standard-type NLS, see, for example, [10][Section 7.6], we can deduce that $v(t) \in \Sigma$ for all $t \in [-T, T]$. Moreover, by calculations analogous to L^2-critical NLS (see [10] again), we find the variance-virial identities

$$\frac{d}{dt} \int_{\mathbb{R}} |x|^2 |v(t,x)|^2 \, dx = 4 \int_{\mathbb{R}} x \mathrm{Im}(\overline{v} \partial_x v) \, dx, \tag{C.8}$$

$$\frac{d^2}{dt^2} \int_{\mathbb{R}} |x|^2 |v(t,x)|^2 \, dx = 16 \widetilde{E}_0, \tag{C.9}$$

with $v_0 = \Phi(u_0)$ and $\widetilde{E}_0 = \widetilde{E}(v_0)$. By integration in $t \in [-T, T]$, we obtain that the variance $V(t) = \int_{\mathbb{R}} |x|^2 |v(t,x)|^2 \, dx$ is given by

$$V(t) = 8\widetilde{E}_0 t^2 + A_0 t + V_0 \tag{C.10}$$

with the constants $A_0 = 4 \int_{\mathbb{R}} x \mathrm{Im}(\overline{v}_0 \partial_x v_0) \, dx$ and $V_0 = V(0)$. For $t \in [-T, T]$ with $t \neq 0$, we observe

$$\begin{aligned} 8t^2 \widetilde{E}(\mathrm{e}^{i|x|^2/4t} v_0) &= 8t^2 \left(\frac{1}{2} \int_{\mathbb{R}} |\partial_x (\mathrm{e}^{i|x|^2/4t} v_0)|^2 - \frac{1}{4} \int_{\mathbb{R}} |v_0|^2 (|D||v_0|^2) + \frac{1}{24} \int_{\mathbb{R}} |v_0|^6 \right) \\ &= 8t^2 \widetilde{E}_0 + A_0 t + V_0 = V(t). \end{aligned}$$

Finally, we go back to the function $u = u(t,x)$. Here we note that $|u(t,x)| = |v(t,x)|$ and $\widetilde{E}(v_0) = E(u_0)$ since $v(t) = \Phi(u(t))$. This proves this first claim in Lemma C.1. For the second statement, we remark that Φ commutes with multiplication by $\mathrm{e}^{i|x|^2/4t}$ for $t \neq 0$, that is, we have $\Phi(\mathrm{e}^{i|x|^2/4t} u_0) = \mathrm{e}^{i|x|^2/4t} \Phi(u_0) = \mathrm{e}^{i|x|^2/4t} v_0$. Therefore $\widetilde{E}(\mathrm{e}^{i|x|^2|4t} v_0) = E(\mathrm{e}^{i|x|^2/4t} u_0)$. This completes the proof. □