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Tensor-spinor theory of gravitation in general even space-time dimensions

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ABSTRACT

We present a purely tensor-spinor theory of gravity in arbitrary even $D = 2n$ space-time dimensions. This is a generalization of the purely vector-spinor theory of gravitation by Bars and MacDowell (BM) in 4D to general even dimensions with the signature $(2n-1, 1)$. In the original BM-theory in $D = (3, 1)$, the conventional Einstein equation emerges from a theory based on the vector-spinor field ψ_μ from a lagrangian *free of* both the fundamental metric $g_{\mu\nu}$ and the vierbein $e_\mu{}^m$. We first improve the original BM-formulation by introducing a compensator χ, so that the resulting theory has manifest invariance under the nilpotent local fermionic symmetry: $\delta_\epsilon \psi = D_\mu \epsilon$ and $\delta_\epsilon \chi = -\epsilon$. We next generalize it to $D = (2n-1, 1)$, following the same principle based on a lagrangian *free of* fundamental metric or vielbein now with the field content $(\psi_{\mu_1 \cdots \mu_{n-1}}, \omega_\mu{}^{rs}, \chi_{\mu_1 \cdots \mu_{n-2}})$, where $\psi_{\mu_1 \cdots \mu_{n-1}}$ (or $\chi_{\mu_1 \cdots \mu_{n-2}}$) is a $(n-1)$ (or $(n-2)$) rank tensor-spinor. Our action is shown to produce the Ricci-flat Einstein equation in arbitrary $D = (2n-1, 1)$ space-time dimensions.

1. Introduction

In late 1970's, Bars and MacDowell (BM) presented an interesting re-formulation of the general relativity by A. Einstein, based on a purely vector-spinor field ψ_μ in four-dimensions (4D) [1]. Its action has only two fundamental fields: a vector-spinor ψ_μ and a Lorentz connection $\omega_\mu{}^{rs}$. The ψ_μ is a Majorana vector-spinor similar to the spin-3/2 field used in supergravity [2]. In other words, *neither* the fundamental metric $g_{\mu\nu}$ nor the vierbein $e_\mu{}^m$ is present in the basic lagrangian. Interestingly enough, upon using reasonable ansätze, the ψ_μ-field equation yields the conventional Einstein field equation, as desired. The basic philosophy behind the result in [1] is that a vector-spinor ψ_μ is *more fundamental* than the metric $g_{\mu\nu}$ or vierbein $e_\mu{}^m$, that controls the 'geometry of space-time'.

The action of BM-theory $I_4^{(0)} \equiv \int d^4x\, \mathcal{L}_4^{(0)}$ has the very simple lagrangian [1][2]

$$\mathcal{L}_4^{(0)} = -\tfrac{i}{4}\epsilon^{\mu\nu\rho\sigma}\left(\overline{\mathcal{R}}^{(0)}_{\mu\nu}\gamma_5\mathcal{R}^{(0)}_{\rho\sigma}\right) \overset{\nabla}{=} -\tfrac{i}{8}\epsilon^{\mu\nu\rho\sigma}\left(\overline{\psi}_\mu\gamma_5\gamma_{rs}\psi_\nu\right)R^{(0)rs}_{\rho\sigma} \quad , \tag{1.1}$$

where $\mathcal{R}_{\mu\nu}$ and $R_{\rho\sigma}{}^{rs}$ are the field-strengths of ψ_μ and $\omega_\mu{}^{rs}$:

$$\mathcal{R}^{(0)}_{\mu\nu} \equiv \left(\partial_\mu\psi_\nu - \tfrac{1}{4}\omega_\mu{}^{rs}\gamma_{rs}\psi_\nu\right) - (\mu \leftrightarrow \nu) \equiv D_\mu\psi_\nu - D_\nu\psi_\mu \quad , \tag{1.2a}$$

$$R_{\mu\nu}{}^{rs} \equiv \partial_\mu\omega_\nu{}^{rs} - \partial_\nu\omega_\mu{}^{rs} + \omega_\mu{}^{rt}\omega_{\nu t}{}^s - \omega_\nu{}^{rt}\omega_{\mu t}{}^s \quad . \tag{1.2b}$$

Most importantly, there is *neither* metric $g_{\mu\nu}$ *nor* vierbein $e_\mu{}^m$ introduced to define the lagrangian $\mathcal{L}_4^{(0)}$ [1].

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As *will be* explained in section 2, the field-equations of ψ_μ and $\omega_\mu{}^{rs}$ in (2.1) are satisfied by the peculiar ansätze (2.2), based on the covariantly-constant Majorana-spinor θ. Under these ansätze, the field equations of ψ_μ and $\omega_\mu{}^{rs}$ are reduced to the Ricci-flat equation $R_{\mu\nu}(e) \doteq 0$,[3] *i.e.*, the Einstein equation in vacuum with the torsion-free Lorentz connection in general relativity.

The original BM-formulation [1] seems to work *only in 4D*, and therefore our 4D space-time is uniquely singled out. This is because of the combination of the totally antisymmetric 4th rank constant tensor $\epsilon^{\mu\nu\rho\sigma}$, multiplied by the bilinear form of the 2nd-rank gravitino field-strength $\mathcal{R}^{(0)}_{\mu\nu}$. One possible generalization is to increase the rank of the vector-spinor ψ_μ to a tensor-spinor $\psi_{\mu_1\cdots\mu_n}$ $(n \ge 2)$. But we are discouraged to do so, due to the common problem of the consistency with higher-rank spinors [3][4]. Even for a vector-spinor ψ_μ, supergravity [2] theory is supposed to be the *only* consistent interacting system. It is nowadays a common notion [5][6] that unless we introduce infinite tower of massive higher spins as in (super)string theory [7], no higher-spin fields can interact consistently.

However, in our recent paper on supersymmetric BF-theories in diverse dimensions [8], we consider systems with higher-rank tensor-spinor $\psi_{\mu_1\cdots\mu_n}$ without inconsistency. Our system is also shown to embed integrable models in lower dimensions, such as super KP systems. This example indicates that such higher-rank tensor can exist consistently even with consistent non-trivial interactions.

In the original BM-theory [1], there is also a technical weak point. It is related to the invariance of the action $I_4^{(0)}$ under the 'gradient' symmetry: $\delta_\epsilon \psi_\mu = D_\mu \epsilon$. We will improve this point in the next section by introducing the Proca-Stëckelberg-type [9] compensator field χ, such that our improved action is invariant under the symmetry δ_ϵ. Since $\delta_\epsilon \psi_\mu = D_\mu \epsilon$ is regarded as nilpotent local fermionic symmetry [10][11][12][13][14], the original BM-theory [1] can be re-interpreted as nothing but an application of nilpotent local fermionic symmetry.

In our present paper, we will also accomplish the generalization of the original BM-theory [1] into general even dimensions $D = 2n$ with the signature $(2n-1, 1) = (\overbrace{+, +, \cdots, +}^{2n-1}, -)$, with the manifest action-invariance under the tensor-spinor symmetry: $\delta_\epsilon \psi_{\mu_1\cdots\mu_{n-1}} = (n-1) D_{[\mu_1} \epsilon_{\mu_2\cdots\mu_{n-1}]}$. Our lagrangian in $D = 2n$ has only three fields $\psi_{\mu_1\cdots\mu_{n-1}}$, $\omega_\mu{}^{rs}$ and $\chi_{\mu_1\cdots\mu_{n-2}}$, where the 1st and the last fields are Majorana-spinors, with a possible additional index for the **2** of $Sp(2)$, depending on $D = (2n-1, 1)$.

2. An improved action with invariance

In the BM-formulation with the lagrangian (1.1) [1], there are only two fundamental fields ψ_μ and $\omega_\mu{}^{rs}$, whose field equations are [1]

$$\frac{\delta \mathcal{L}_4^{(0)}}{\delta \overline{\psi}_\mu} = -\tfrac{i}{4} \epsilon^{\mu\nu\rho\sigma} (\gamma_5 \gamma_{rs} \psi_\nu) R_{\rho\sigma}{}^{rs} \doteq 0 \ , \tag{2.1a}$$

$$\frac{\delta \mathcal{L}_4^{(0)}}{\delta \omega_\mu{}^{rs}} = -\tfrac{i}{4} \epsilon^{\mu\nu\rho\sigma} (\overline{\psi}_\nu \gamma_5 \gamma_{rs} \mathcal{R}^{(0)}_{\rho\sigma}) \doteq 0 \ . \tag{2.1b}$$

Note that *neither* the metric $g_{\mu\nu}$ *nor* the vierbein $e_\mu{}^m$ is involved in these field equations.

These field equations are solved under the following ansätze [1][4]

$$\psi_\mu \ \overset{*}{=} \ (\gamma_\mu \theta) \equiv e_\mu{}^m (\gamma_m \theta) \ , \quad D_\mu \theta \ \overset{*}{=} 0 \ , \tag{2.2}$$

where $e_\mu{}^m$ is the conventional vierbein. The Majorana-spinor θ resembles the fermionic coordinates in superspace [15], but the difference is that it is a covariantly-constant spinor [1], as (2.2) shows. First of all, (2.1b) implies that

$$\begin{aligned} i\epsilon^{\mu\nu\rho\sigma} (\overline{\psi}_\nu \gamma_5 \gamma_{rs} \mathcal{R}^{(0)}_{\rho\sigma}) \ &\overset{*}{=} \ + i\epsilon^{\mu\nu\rho\sigma} (\overline{\theta} \gamma_5 \gamma_\nu \gamma_{rs} \gamma_t \theta) \, T_{\rho\sigma}{}^t \\ &\overset{*}{=} \ 2e(\overline{\theta}\theta) \left(T_{rs}{}^\mu + 2e_{[r}{}^\mu T_{s]} \right) + ie(\overline{\theta}\gamma_5\theta) \, \epsilon_{rs}{}^{tu} \left(T_{tu}{}^\mu + 2e_{[t}{}^\mu T_{u]} \right) \ \overset{*}{=} \ 0 \\ &\Longrightarrow \quad T_{rs}{}^\mu + 2e_{[r}{}^\mu T_{s]} \ \overset{*}{=} \ 0 \quad \Longrightarrow \quad T_{\mu\nu}{}^r \ \overset{*}{=} 0 \ , \end{aligned} \tag{2.3}$$

where $T_m \equiv T_{mr}{}^r$. The last equation in (2.3) is obtained by taking the trace of the 1st equation in (2.3) *via* $T_m \doteq 0$. We also regard two sectors $(\overline{\theta}\theta)$ and $i(\overline{\theta}\gamma_5\theta)$ as two independent sectors, but they are mutually consistent. This implies that $\omega_{mrs}(e) \equiv (1/2)(C_{mrs} - C_{msr} - C_{rsm})$ with $C_{\mu\nu}{}^r \equiv 2\partial_{[\mu} e_{\nu]}{}^r$. In our present paper, we do *not* regard the torsion-freedom $T_{\mu\nu}{}^r \ \overset{*}{=} 0$ as one of the ansätze, but it is *implied* by the ω-field equation under the other ansätze (2.2).

Second, the Riemann tensor in (2.1a) is now $R_{\mu\nu}{}^{rs}(\omega(e)) \equiv R_{\mu\nu}{}^{rs}(e)$, satisfying the Bianchi identity $R_{[mrs]}{}^t(e) \equiv 0$ and $R_{mnrs}(e) \equiv R_{rsmn}(e)$. Using this identity and the ansätze (2.2), we see that the Ricci-flatness (or Einstein equation in empty space-time):

$$i\epsilon_\mu{}^{\nu\rho\sigma} (\gamma_5 \gamma_{rs} \psi_\nu) R_{\rho\sigma}{}^{rs}(e) \ \overset{*}{=} \ + 4e(\gamma^\nu \theta) \left[R_{\mu\nu}(e) - \tfrac{1}{2} g_{\mu\nu} R(e) \right] \doteq 0 \ . \tag{2.4}$$

In the ansätze (2.2), the vierbein $e_\mu{}^m$ is introduced, *not* as the basic fundamental field in the theory, but is defined by the no-fermion background-state $|B\rangle$ and one-fermion state $|B, \alpha\rangle$ as $\langle B | \psi_\mu{}^\beta | B, \alpha \rangle = e_\mu{}^m (\gamma_m)_\alpha{}^\beta$ [1].

As has been mentioned, a weak-point in this original BM-theory [1] is the lack of action-invariance principle. For example, the lagrangian $\mathcal{L}_4^{(0)}$ in (1.1) is *not* invariant under the proper vector-spinor transformation $\delta_\epsilon \psi_\mu = D_\mu \epsilon$, because $\delta_\epsilon \mathcal{R}_{\mu\nu}{}^{(0)} = (1/4)(\gamma_{rs}\epsilon) R_{\mu\nu}{}^{rs} \neq 0$. To improve this point, we replace the *bare* ψ_μ-field in (1.1) by the field-strength of the Proca-Stückelberg field χ [9], following tensor-hierarchy formulations [16][17] or our previous work on nilpotent local fermionic symmetry [10][11][12][13][14]. To be more specific, we introduce a Proca-Stückelberg compensator field χ [9] in its modified field strength

$$\mathcal{P}_\mu \equiv D_\mu \chi + \psi_\mu \ , \tag{2.5}$$

[3] The symbol $\doteq$ stands for a field equation.

[4] We use the symbol $\overset{*}{=}$ for an ansatz, in order to reproduce conventional general relativity.

which is invariant: $\delta_\epsilon \mathcal{P}_\mu = 0$ under the infinitesimal local fermionic symmetry

$$\delta_\epsilon \psi_\mu = D_\mu \epsilon \ , \quad \delta_\epsilon \chi = -\epsilon \ . \tag{2.6}$$

The $\mathcal{P}_\mu$ and the *modified* $\mathcal{R}_{\mu\nu}$ satisfy their Bianchi identities

$$D_{[\mu}\mathcal{P}_{\nu]} \equiv \frac{1}{2}\mathcal{R}_{\mu\nu} \ , \quad \mathcal{R}_{\mu\nu} \equiv \mathcal{R}^{(0)}_{\mu\nu} + \tfrac{1}{4}R_{\mu\nu}{}^{rs}(\gamma_{rs}\chi) \ , \tag{2.7a}$$

$$D_{[\mu}\mathcal{R}_{\nu\rho]} \equiv \frac{1}{4}R_{[\mu\nu|}{}^{rs}(\gamma_{rs}\mathcal{P}_{|\rho]}) \ . \tag{2.7b}$$

In other words, we can simply replace ψ_μ everywhere in the original BM-theory by $\mathcal{P}_\mu$ to make the improved action $I_4 \equiv \int d^4x\, \mathcal{L}_4$ invariant under (2.6). Namely, the lagrangian (1.1) is improved to

$$\mathcal{L}_4 = \tfrac{i}{2}\epsilon^{\mu\nu\rho\sigma}\left(\overline{\mathcal{R}}_{\mu\nu}\gamma_5\mathcal{R}_{\rho\sigma}\right) \overset{\nabla}{=} \tfrac{i}{2}\epsilon^{\mu\nu\rho\sigma}\left(\overline{\mathcal{P}}_\mu\gamma_5\gamma_{rs}\mathcal{P}_\nu\right)R_{\rho\sigma}{}^{rs}(\omega) \ . \tag{2.8}$$

As usual [9], the compensator χ is gauged away by the use of (2.6), and there is *no* essential effect in physical degrees of freedom. For example, the new χ-field equation is a *sufficient condition* of the ψ-field equation. However, the important mission of χ is to make the modified action $I_4 \equiv \int d^4x\, \mathcal{L}_4$ manifestly invariant under δ_ϵ.

Note that the previous procedure of getting the Ricci-flatness equation $R_{mn}(e) \overset{*}{=} 0$ is *not* affected by the modification of ψ_μ to $\mathcal{P}_\mu$, because we can always require $\chi \overset{*}{=} 0$ as an additional ansatz to (2.2).

Accordingly, the original ansätze (2.2) is modified to

$$\mathcal{P}_\mu \overset{*}{=} (\gamma_\mu\theta) \equiv e_\mu{}^m(\gamma_m\theta) \ , \quad D_\mu\theta \overset{*}{=} 0 \ , \quad \chi \overset{*}{=} 0 \ . \tag{2.9}$$

The BM-theory [1] has *neither* metric *nor* vierbein, but it is induced out of the vector-spinor ψ_μ or $\mathcal{P}_\mu$ by the relationship [1]

$$(\overline{\mathcal{P}}_\mu\mathcal{P}_\nu) = -(\overline{\theta}\theta)\, g_{\mu\nu}\big|_{\text{classical}} \overset{*}{=} -g_{\mu\nu} \tag{2.10}$$

under the ansatz $\mathcal{P}_\mu \overset{*}{=} e_\mu{}^m(\gamma_m\theta)$.

The fermionic generators Q_α for our *local* transformation δ_ϵ in (2.6) satisfy the *nilpotent* anti-commutator:

$$\{Q_\alpha,\ Q_\beta\} = 0 \ . \tag{2.11}$$

In other words, our modified BM-theory of [1] is noting but an example of many other previously-established nilpotent local fermionic theories [10][11][12][13][14]. In our generalization to $D = (2n-1, 1)$, we will use the modified field-strength $\mathcal{P}_{[n-1]}$ of the higher-rank generalization of the compensator $\chi_{[n-2]}$.

3. The Lagrangian in general $D=(2n-1,1)$

As has been stated, the field content of our system is $(\psi_{[n-1]},\ \omega_\mu{}^{rs},\ \chi_{[n-2]})$.[5] Accordingly, we have to define the generalized γ_5-tensor γ_{2n+1}, as described in the Appendix. Our action is $I_{2n} \equiv \int d^{2n}x\, \mathcal{L}_{2n}$ with

$$\mathcal{L}_{2n} = -\frac{i^{n+1}c_n}{(n!)^2}\,\epsilon^{[n][n]'}\left(\overline{\mathcal{R}}_{[n]}\gamma_{2n+1}\mathcal{R}_{[n]'}\right) \tag{3.1a}$$

$$= +\frac{i^{n+1}c_n}{8[\,(n-1)!\,]^2}\,\epsilon^{[n-1][n-1]'\mu\nu}\left(\overline{\mathcal{P}}_{[n-1]}\gamma_{2n+1}\gamma_{rs}\mathcal{P}_{[n-1]'}\right)R_{\mu\nu}{}^{rs} \ , \tag{3.1b}$$

where the constant c_n depends on $D = (2n-1, 1)$ as

$$c_n = \begin{cases} +1 & (n = 4\ (mod\ 4)) \ , \\ -1 & (n \neq 4\ (mod\ 4)) \ , \end{cases} \tag{3.2}$$

associated with the flipping property explained in Appendix. The $\epsilon^{[2n]}$ is the 2n-th rank totally antisymmetric constant tensor in $D = (2n-1, 1)$, while the field-strengths of $\psi_{[n-1]}$ and $\chi_{[n-2]}$ are defined by

$$\mathcal{R}_{\mu_1\mu_2\cdots\mu_n} \equiv +nD_{[\mu_1}\psi_{\mu_2\cdots\mu_n]} + \tfrac{1}{8}n(n-1)R_{[\mu_1\mu_2|}{}^{rs}(\gamma_{rs}\chi_{|\mu_3\cdots\mu_n]}) \ . \tag{3.3a}$$

$$\mathcal{P}_{\mu_1\cdots\mu_{n-1}} \equiv +(n-1)D_{[\mu_1}\chi_{\mu_2\cdots\mu_{n-1}]} + m\,\psi_{\mu_1\cdots\mu_{n-1}} \ , \tag{3.3b}$$

satisfying their Bianchi identities

$$D_{[\mu_1}\mathcal{R}_{\mu_2\cdots\mu_{n+1}]} \equiv +\tfrac{1}{8}nR_{[\mu_1\mu_2|}{}^{rs}(\gamma_{rs}\mathcal{P}_{|\mu_3\cdots\mu_{n+1}]}) \ , \tag{3.4a}$$

$$D_{[\mu_1}\mathcal{P}_{\mu_2\cdots\mu_n]} \equiv +\frac{1}{n}\mathcal{R}_{\mu_1\cdots\mu_n} \ . \tag{3.4b}$$

[5] The symbol $[n]$ stands for totally antisymmetric n space-time indices: $[n] \equiv [\mu_1\cdots\mu_n]$, in order to save space.

Eqs. (3.3) and (3.4) with generalized Chern-Simons terms follow the same pattern as the general tensor-hierarchy formulations [16][17], even though the latter have been developed only for *bosonic* fields.

As in general tensor-hierarchy formulation [16][17], the ψ and χ-fields have their proper gauge-symmetries. In particular, $\chi_{[n-2]}$ itself is also a tensor with its own gauge symmetry:

$$\delta_\epsilon \psi_{\mu_1\cdots\mu_{n-1}} = (n-1) D_{[\mu_1}\epsilon_{\mu_2\cdots\mu_{n-1}]} \tag{3.5a}$$

$$\delta_\epsilon \chi_{\mu_1\cdots\mu_{n-2}} = -\epsilon_{\mu_1\cdots\mu_{n-2}} \ , \tag{3.5b}$$

$$\delta_\eta \psi_{\mu_1\cdots\mu_{n-1}} = -\tfrac{1}{8}(n-1)(n-2) R_{[\mu_1\mu_2|}{}^{rs}(\gamma_{rs}\eta_{|\mu_3\cdots\mu_{n-1}]}) \tag{3.6a}$$

$$\delta_\eta \chi_{\mu_1\cdots\mu_{n-2}} = +(n-2) D_{[\mu_1}\eta_{\mu_2\cdots\mu_{n-2}]} \ . \tag{3.6b}$$

In the previous case of 4D with $n=2$, the χ-field was *not* a gauge field lacking its proper gauge symmetry. Relevantly, the field strengths $\mathcal{R}$ and $\mathcal{P}$ are invariant under δ_ϵ and δ_η:

$$\delta_\epsilon \mathcal{R}_{[n]} = 0 \ , \quad \delta_\epsilon \mathcal{P}_{[n-1]} = 0 \ , \tag{3.7a}$$

$$\delta_\eta \mathcal{R}_{[n]} = 0 \ , \quad \delta_\eta \mathcal{P}_{[n-1]} = 0 \ . \tag{3.7b}$$

The field equations for ψ, ω and χ are

$$\frac{\delta \mathcal{L}_{2n}}{\delta \overline{\psi}_{[n-1]}} = +\frac{i^{n+1}c_n}{4[(n-1)!]^2}\epsilon^{[n-1][n-1]'\mu\nu}(\gamma_{2n+1}\gamma_{rs}\mathcal{P}_{[n-1]'})\, R_{\mu\nu}{}^{rs} \doteq 0 \ , \tag{3.8a}$$

$$\frac{\delta \mathcal{L}_{2n}}{\delta \omega_\mu{}^{rs}} = -\frac{i^{n+1}c_n}{2n[(n-1)!]^2}\epsilon^{\mu[n-1][n]}(\overline{\mathcal{P}}_{[n-1]}\gamma_{2n+1}\gamma_{rs}\mathcal{R}_{[n]}) \doteq 0 \ , \tag{3.8b}$$

$$\frac{\delta \mathcal{L}_{2n}}{\delta \chi_{[n-2]}} = -(n-1)\, D_\rho \left(\frac{\delta \mathcal{L}_{2n}}{\delta \overline{\psi}_{\rho[n-2]}} \right) \doteq 0 \ . \tag{3.8c}$$

4. Our action yields Einstein equations in $D=(2n-1,1)$

The ansätze we need for the action I_{2n} to yield the conventional Einstein equation (Ricci-flat equation in vacuum) are the generalization of (2.9):

$$\mathcal{P}_{\mu_1\cdots\mu_{n-1}} \stackrel{*}{=} (\gamma_{\mu_1\cdots\mu_{n-1}}\theta) \equiv e_{\mu_1}{}^{r_1}\cdots e_{\mu_n}{}^{r_n}(\gamma_{r_1\cdots r_n}\theta) \ , \tag{4.1a}$$

$$D_\mu\theta \stackrel{*}{=} 0 \ , \quad \chi_{[n-2]} \stackrel{*}{=} 0 \ . \tag{4.1b}$$

The satisfaction of ω-field equation (3.8b) works in a fashion parallel to the $n=2$ case. First, (4.1b) and (3.3a) imply that

$$\mathcal{R}_{\mu_1\cdots\mu_n} \stackrel{*}{=} nD_{[\mu_1}\left[e_{\mu_2}{}^{t_2} e_{\mu_3}{}^{t_3}\cdots e_{\mu_n]}{}^{t_n}(\gamma_{t_2\cdots t_n}\theta) \right] \stackrel{*}{=} \tfrac{1}{2}n(n-1)T_{[\mu_1\mu_2|}{}^t(\gamma_{t|\mu_3\cdots\mu_n]}\theta) \ , \tag{4.2}$$

so that our ω-field equation (3.8b) is equivalent to

$$(\overline{\theta}\gamma^{m[n-2]uv}\gamma_{rs}\gamma_{w[n-2]}\theta)\, T_{uv}{}^w \stackrel{*}{=} 0 \ . \tag{4.3}$$

We used here the Hodge-duality (A.2) in Appendix for the γ_{2n+1}. Note also that once we have introduced the vielbein $e_\mu{}^m$ in (4.1), we can freely change the world-indices $\mu, \nu, \cdots$ into local Lorentz-indices $m, n, \cdots$.

Second, our next objective is to show that the equation $T_{rs}{}^t \stackrel{*}{=} 0$ comes out as the necessary condition of the ω-field equation under the ansätze. To this end, we note that among many possible γ-matrix terms, only

$$(\overline{\theta}\gamma^{[p]}\theta) \neq 0 \quad (p = 0,\, 3,\, 4,\, 7,\, 8,\, \cdots) \ , \tag{4.4}$$

survive, while all others are zero [18]. The lemma considerably simplifies (4.3). First of all the number of γ-matrices sandwiched by the θ's in (4.3) are at most six, while we have to consider only $(\overline{\theta}\gamma^{[0]}\theta)$, $(\overline{\theta}\gamma^{[3]}\theta)$ and $(\overline{\theta}\gamma^{[4]}\theta)$. Moreover, $(\overline{\theta}\gamma^{[3]}\theta)$ is impossible, due to the even number of γ-matrices sandwiched.

Since we can regard the two groups of terms: $(\overline{\theta}\gamma^{[0]}\theta)$ and $(\overline{\theta}\gamma^{[4]}\theta)$ as independent, if the former groups imply that $T_{rs}{}^t \stackrel{*}{=} 0$, then the latter group of terms vanish, because they are all proportional to $T_{rs}{}^t$.

We concentrate on the $(\overline{\theta}\gamma^{[0]}\theta)$–terms. We recast the γ-matrix part of (4.3) into

$$\gamma^{m[n-2]uv}\gamma_{rs}\gamma_{w[n-2]} \tag{4.5a}$$

$$= (-1)^n\gamma^{m[n-2]uv}\gamma_{rs}\gamma_{[n-2]}\gamma_w - (-1)^n(n-2)\gamma^{m[n-3]wuv}\gamma_{rs}\gamma_{[n-3]}\gamma_w \tag{4.5b}$$

$$\begin{aligned} &= +(-1)^n\gamma^{m[n-2]uv}\gamma_{[n-2]}\gamma_{rs}\gamma_w + 4(-1)^n(n-2)\gamma^m{}_{[s|}{}^{[n-3]uv}\gamma_{|r][n-3]}\gamma_w \\ &\quad - (-1)^n(n-2)\gamma^{m[n-3]wuv}\gamma_{[n-3]}\gamma_{rs}\gamma_w - (-1)^n 4(n-2)(n-3)\gamma^m{}_{[s|}{}^{[n-4]wuv}\gamma_{|r][n-4]}\gamma_w \ . \end{aligned} \tag{4.5c}$$

From (4.5a) to (4.5b), we have separated γ_w from $\gamma_{[n-2]w}$ into $\gamma_{[n-2]}\gamma_w$. from (4.5b) to (4.5c), we commuted γ_{rs} with or $\gamma_{[n-3]}$.

At this stage, it is easy to see that only the 1st-term in (4.5c) contributes to $\gamma^{[0]}$, while all the remaining three terms do *not* produce the $\gamma^{[0]}$-term. In fact, the 2nd-term of (4.5c) has the anti-symmetrized indices $[rs]$ and $[muv]$, so it has the two irreducible terms:

$$(\xi\,\gamma_{rs}{}^{muv} + \eta\,\delta_{[r}{}^{[m}\gamma_{s]}{}^{uv]})\,\gamma_w \ , \tag{4.6}$$

where the constants ξ and η depend on the dimensionality $D=(2n-1,1)$. For example, there is *no* structure like $\delta_{[r}{}^{[m}\delta_{s]}{}^{u}\gamma^{v]}$, because there arises *no* term like $\gamma^{[1]}$ out of $\gamma^{m}{}_{[s|}{}^{[n-3]uv}\gamma_{|r][n-3]}$. Hence, there is *no* way to produce a $\gamma^{[0]}$ after multiplying the parentheses in (4.6) with γ_w.

Similarly, the 3rd term in (4.5c) is proportional to $\gamma^{mwuv}\gamma_{rs}\gamma_w$, which can produce only $\gamma^{[6]}$, $\gamma^{[4]}$ and $\gamma^{[2]}$ but *no* $\gamma^{[0]}$-term. Finally, the 4th term in (4.5c) is also similar, because it has the irreducible terms

$$\xi'\,\gamma_{rs}{}^{mwuv} + \eta'\,\delta_{[r}{}^{[m}\gamma_{s]}{}^{wuv]} \tag{4.7}$$

without any $\gamma^{[0]}$-term.

Eventually, the ω-field equation is reduced to (4.5c) only with its 1st term, which is easy to evaluate:

$$\begin{aligned}
&(\overline{\theta}\gamma^{m[n-2]uv}\gamma_{rs}\gamma_{w[n-2]}\theta)\,T_{uv}{}^{w}\big|_{\gamma^{(0)}} \\
&\quad=(-1)^{n}(\overline{\theta}\gamma^{m[n-2]uv}\gamma_{[n-2]}\gamma_{rs}\gamma_{w}\theta)\,T_{uv}{}^{w}\big|_{\gamma^{(0)}} \\
&\quad=(-1)^{n+(n-2)(n-3)/2}\,\tfrac{(2n-3)!}{(n-1)!}\,(\overline{\theta}\gamma^{muv}\gamma_{rs}\gamma_{w}\theta)\,T_{uv}{}^{w}\big|_{\gamma^{(0)}} \\
&\quad=(-1)^{n+(n-2)(n-3)/2}\,\tfrac{2(2n-3)!}{(n-1)!}\,(\overline{\theta}\theta)\,\big(T_{rs}{}^{m}+2_{[r}{}^{m}T_{s]}\big) \ \stackrel{*}{=}0 \ \Longrightarrow \ T_{\mu\nu}{}^{r} \ \stackrel{*}{=}0 \ .
\end{aligned} \tag{4.8}$$

All terms left over contain $T_{rs}{}^{t}$ linearly, so they vanish as *necessary condition* of the solution $T_{rs}{}^{t} \stackrel{*}{=} 0$. Most importantly, the torsion-freedom $T_{rs}{}^{t} \stackrel{*}{=} 0$ is the *unique* result of our ω-field equation, but it is *not* required from outside, unlike our other ansätze. No other solution with $T_{mn}{}^{r}\neq 0$ to our ω-field equation (3.8b) is possible under (4.1).

The satisfaction of the χ-field equation (3.8c) is the *necessary* condition of the ψ-field equation (3.8a) with ansätze (4.1), because the former is given by a divergence of the latter equation (3.8a). So, we skip its confirmation.

The satisfaction of the ψ-field equation is highly non-trivial. First of all, since we have confirmed $T_{rs}{}^{t} \stackrel{*}{=} 0$, we can restrict $\omega_{\mu}{}^{rs}$ to be $\omega_{\mu}{}^{rs}(e)$. Accordingly, we can use the Bianchi identity $R_{[mrs]}{}^{t}(e)\equiv 0$, $R_{mrst}(e)\equiv R_{stmr}(e)$ and $R_{[rs]}(e)\equiv 0$.

After substituting the ansatz (4.2) for $\mathcal{R}$ and $\omega_{\mu}{}^{rs} \stackrel{*}{=} \omega_{\mu}{}^{rs}(e)$ into (3.8a), the original ψ-field equation is equivalent to

$$\epsilon^{[n-1][n-1]'\mu\nu}(\gamma_{2n+1}\gamma_{rs}\gamma_{[n-1]'}\theta)R_{\mu\nu}{}^{rs} \ \stackrel{*}{=}0 \ \Longrightarrow \ (\gamma_{rs}\gamma^{m_1m_2\cdots m_{n-1}tu}\theta)R_{tu}{}^{rs} \ \stackrel{*}{=}0 \ . \tag{4.9}$$

If we multiply (4.9) by the gamma-matrix $\gamma^{m_2\cdots m_{n-1}}$ from the left, we get

$$(\gamma_{[n-2]}\gamma_{rs}\gamma^{m[n-2]tu}\theta)R_{tu}{}^{rs} \ \stackrel{*}{=}0 \ , \tag{4.10}$$

as the necessary condition of our ψ-field equation upon our ansätze (4.1). Note that the index m_1 in (4.9) is still a free index after the multiplication, as symbolized by m in (4.10). There are only five free-indices $r,\ s,\ m,\ t,\ u$ on the γ-matrices in (4.10) that give only three irreducible structures:

$$\gamma_{[n-2]}\gamma_{rs}\gamma^{m[n-2]tu} = \xi''\gamma_{rs}{}^{mtu} + \eta''\delta_{[r}{}^{[m}\gamma_{s]}{}^{tu]} + \zeta''\delta_{[r}{}^{[m}\delta_{s]}{}^{t}\gamma^{u]} \ . \tag{4.11}$$

As before, the constants ξ'', η'' and ζ'' depend on n of $D=(2n-1,1)$. Using (4.11) in (4.10), we get

$$\begin{aligned}
0 \ &\stackrel{*}{=} \Big(\cancel{\xi''\gamma_{rs}{}^{mtu}} + \cancel{\eta''\delta_{[r}{}^{[m}\gamma_{s]}{}^{tu]}} + \zeta''\delta_{[r}{}^{[m}\delta_{s]}{}^{t}\gamma^{u]}\Big)\,\theta\,R_{tu}{}^{rs}(e) \\
&= -\tfrac{2}{3}\zeta''(\gamma^{t}\theta)\left[R_{mt}(e) - \frac{1}{2}\eta_{mt}R(e)\right] = -\tfrac{2}{3}\zeta''(\gamma^{t}\theta)\,G_{mt}(e) \ \Longrightarrow \ (\gamma^{s}\theta)\,G_{rs}(e) \ \stackrel{*}{=}0 \ ,
\end{aligned} \tag{4.12}$$

where $G_{rs}(e)\equiv R_{rs}(e)-(1/2)\eta_{rs}R(e)$ is the Einstein tensor. The terms with η'' do *not* contribute, because of the Bianchi identities $R_{[rst]}{}^{u}(e)\equiv 0$ and $R_{[rs]}(e)\equiv 0$.

In the last step in (4.12), we implicitly assumed that $\zeta''\neq 0$. This can be explicitly confirmed for general $n\geq 2$, by multiplying (4.11) by $\delta_{[m}{}^{r}\delta_{t}{}^{s}\gamma_{u]}$. To be more specific, we get

$$\zeta'' = (-1)^{(n-1)(n-2)/2}\,\tfrac{3\cdot(2n-2)!}{(n-1)!(n-1)} \qquad (n\geq 2) \ , \tag{4.13}$$

and therefore $\zeta''\neq 0$ for $n\geq 2$.

The proof of $G_{rs}(e) \stackrel{*}{=} 0$ from (4.12) is as follows: We multiply (4.12) by $(\overline{\theta}\gamma_t)$ to get

$$0 \ \stackrel{*}{=} (\overline{\theta}\gamma_t)(\gamma^{s}\theta)\,G_{rs}(e) = (\overline{\theta}\gamma_t\gamma^{s}\theta)\,G_{rs}(e) = (\overline{\theta}\theta)\,G_{rt}(e) \ \Longrightarrow \ G_{rs} \ \stackrel{*}{=}0 \quad (Q.E.D.) \tag{4.14}$$

The above method is simple, but started with the multiplication of our ψ-field equation (3.1a) by $\gamma^{m_2\cdots m_{n-1}}$ from the left. So, the final condition (4.12) may be too strong. There is a direct way of proving the satisfaction of our ψ-field equation.

$$\epsilon^{\mu_1\cdots\mu_{n-1}[n-1]\rho\sigma}(\gamma_{2n+1}\gamma_{rs}\mathcal{P}_{[n-1]})\,R_{\rho\sigma}{}^{rs} \ \stackrel{*}{=} \epsilon^{\mu_1\cdots\mu_{n-1}[n-1]\rho\sigma}(\gamma_{2n+1}\gamma_{rs}\gamma_{[n-1]}\theta)\,R_{\rho\sigma}{}^{rs}(e) \ \stackrel{*}{=}0 \tag{4.15a}$$

$$\begin{aligned}
&\Longrightarrow +(\gamma^{m_1\cdots m_{n-1}tu}\gamma_{rs}\theta)R_{tu}{}^{rs}(e) \\
&\qquad -2\left[2(n-1)\,\delta_s{}^{[m_1}\gamma_r{}^{m_1\cdots m_{n-1}]tu} + 4\delta_r{}^{u}\gamma^{m_1\cdots m_{n-1}t}{}_{s}\right]\theta\,R_{tu}{}^{rs}(e)
\end{aligned} \tag{4.15b}$$

$$\begin{aligned}
&= +4(\gamma^{m_1\cdots m_{n-1}tu}[D_r(e),\ D_s(e)]\theta) \\
&\qquad -2\left[\cancel{2(n-1)\,\delta_s{}^{[m_1}\gamma_r{}^{m_1\cdots m_{n-1}]tu}} + \cancel{4(-1)^n\delta_r{}^{u}\gamma_s{}^{m_1\cdots m_{n-1}t}}\right]\theta\,R_{tu}{}^{rs}(e) \ \stackrel{*}{=}0 \ .
\end{aligned} \tag{4.15c}$$

From (4.15a) to (4.15b), we commuted γ_{rs} with $\gamma^{[n-1]}$. From (4.15b) to (4.15c), we have used $[D_r(e),\, D_s(e)] = (1/4)R_{rs}{}^{tu}(e)\,\gamma_{tu}$. Fortunately, each of the terms in (4.15c) vanishes, because of the ansatz $D_m(e)\theta \overset{*}{=} 0$, and the Bianchi identities $R_{[rtu]}{}^s(e) \equiv 0$ and $R_{[ur]}(e) \equiv 0$. This shows that $R_{rs}(e) \overset{*}{=} 0$ is the sufficient condition of our ψ-field equation under the ansätze (4.1).

Some readers may question, where the condition $R_{rs}(e) \overset{*}{=} 0$ emerges explicitly in the method above. Actually, the ansatz $D_\mu(e)\theta \overset{*}{=} 0$ in (4.2) implies $R_{rs}(e) \overset{*}{=} 0$, as the *necessary* condition of $D_\mu\theta \overset{*}{=} 0$. Consider the commutator

$$[D_\mu(e),\, D_\nu(e)]\theta = \tfrac{1}{4}(\gamma_{rs}\theta)\, R_{\mu\nu}{}^{rs} \overset{*}{=} 0 \quad\Longrightarrow\quad (\gamma_{st}\theta)\, R_{mr}{}^{st}(e) \overset{*}{=} 0 \ . \tag{4.16}$$

Similar to (4.12) through (4.14), we multiply (4.16) by $(\overline{\theta}\gamma_u{}^r)$ to get

$$(\overline{\theta}\gamma_u{}^r\gamma_{st}\theta)R_{mr}{}^{st}(e) = -2(\overline{\theta}\theta)R_{mu}(e) \overset{*}{=} 0 \implies R_{st}(e) \overset{*}{=} 0 \ . \tag{4.17}$$

Here again, we used $(\overline{\theta}\gamma_{[2]}\theta) \overset{*}{=} 0$ and $R_{m[rst]} \overset{\bullet}{=} 0$.

We can re-confirm our result from a different viewpoint. We can answer the question whether $R_{mn}(e) \overset{*}{=} 0$ is the *only* non-trivial solution other than $R_{\mu\nu}{}^{rs}(e) \overset{\bullet}{=} 0$ under (4.1). A *short* answer is in the affirmative, because we have confirmed that Einstein space-time is the *necessary* condition of our ψ-field equation (3.8a) under (4.1), *except for* the flat space-time $R_{\mu\nu}{}^{rs}(e) \overset{*}{=} 0$.

A *long* answer is as follows: The question is whether the condition

$$[D_\rho(e),\, D_\sigma(e)]\theta = \tfrac{1}{4}R_{\rho\sigma}{}^{rs}(e)(\gamma_{rs}\theta) \overset{*}{=} 0 \tag{4.18}$$

allows non-trivial space-time other than the *flat* one: $R_{\mu\nu}{}^{rs}(e) \overset{*}{=} 0$. Mathematically, this question is very difficult to answer, because we have to categorize all possible Riemann or Weyl curvature tensors in $D = (2n-1, 1)$ [19].

However, we can rely on a much simpler and empirical method in physics. Consider a possible Kaluza-Klein type simple dimensional-reduction [20] of (4.18) in $D = (2n-1, 1)$ into $D = (3, 1)$. From our experience with surviving supersymmetries in $D = (3, 1)$ [21], we know that for a $D = (3, 1)$ space-time with a *covariantly-constant spinor* ϵ, there is *non-trivial* curved space-time background. Since the $D = (3, 1)$ part of the original Riemann tensor in (4.18) is non-zero, the original Riemann tensor in $D = (2n-1, 1)$ has corresponding *non-vanishing* components. In other words, we know empirically that the equation such as (4.18) has solutions with *curved* space-times in $D = (2n-1, 1)$ with $R_{\mu\nu}{}^{rs}(e) \neq 0$, but $R_{\mu\nu}(e) \overset{*}{=} 0$ as desired.

We have thus confirmed that an Einstein space-time $G_{rs}(e) \overset{*}{=} 0$ in vacuum (or Ricc-flat space-time) is the *necessary* condition of our ψ-field equation (3.8a) under the ansätze (4.1), other than the flat space-time $R_{\mu\nu}{}^{rs}(e) \overset{*}{=} 0$. We can conclude that our ψ-field equation (3.8a) yields uniquely the Einstein space-time with $G_{rs}(e) \overset{*}{=} 0$ under our ansätze (4.1) in $D = (2n-1, 1)$, other than the trivial flat space-time $R_{rs}{}^{tu}(e) \overset{*}{=} 0$. It is *not* far-fetched to conclude that $(n-1)$-th rank tensor-spinor $\psi_{[n-1]}$ is more fundamental than the geometrical metrics or vielbeins in general $D = (2n-1, 1)$.

Some readers may wonder, what is the relationship corresponding to (2.10), *i.e.,* how to define the metric $g_{\mu\nu}$ in terms of tensor-spinor $\psi_{\mu_1\cdots\mu_{n-1}}$. To this end, consider the $n = 3,\ D = (5, 1)$ case. For example, relating $(\overline{\mathcal{P}}_{\mu\rho}\mathcal{P}_{\nu\sigma})$ to $g_{\mu\nu}$ by[6]

$$g^{\rho\sigma}(\overline{\mathcal{P}}_{\mu\rho}\mathcal{P}_{\nu\sigma}) \ \overset{*}{=}\ -(\overline{\theta}\gamma_{\mu\rho}\gamma_\nu{}^\rho\theta) = 5(\overline{\theta}\theta)\, g_{\mu\nu} \tag{4.19}$$

does *not* work, because the most left-side needs the inverse metric $g^{\rho\sigma}$, and therefore, it does *not* define the metric in terms of $\psi_{\mu\nu}$ in a *closed form*. However, we can still solve (4.19) *perturbatively* for the metric as in quantum gravity. Defining $g_{\mu\nu} \equiv \eta_{\mu\nu} + h_{\mu\nu}$ and $g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} + h^{\mu\rho}h_\rho{}^\nu + \mathcal{O}(h^3)$ into (4.19), we can get the *perturbative* solution for $h_{\mu\nu}$:

$$h_{\mu\nu}(\overline{\theta}\theta) \ \overset{*}{=}\ + \tfrac{1}{4}X_{\mu\rho,\,\nu}{}^\rho - \tfrac{1}{40}\eta_{\mu\nu}X_{\rho\sigma}{}^{\rho\sigma} \ \ , \tag{4.20}$$

where $X_{\mu\nu,\rho\sigma}$ is defined by

$$X_{\mu\nu,\rho\sigma} \equiv (\overline{\mathcal{P}}_{\mu\nu}\mathcal{P}_{\rho\sigma}) - 2\eta_{\rho[\mu}\eta_{\nu]\sigma}(\overline{\theta}\theta) \ . \tag{4.21}$$

In other words, we can express $h_{\mu\nu}$ in terms of $\mathcal{P}_{\mu\nu}$ at least perturbatively. Even though this is *not a closed form*, still our fundamental field $\psi_{\mu\nu}$ (or $\mathcal{P}_{\mu\nu}$) *defines* the metric $g_{\mu\nu}$. This is *not* limited to the special $n = 3$ case, but we can get similar perturbative expressions also for any arbitrary $n = 2,\ 3,\ 4,\ \cdots$.

5. Concluding remarks

In this paper, we have accomplished two major objectives: First, we introduced the compensator field χ in $D = (3, 1)$, so that the total action of the original BM-theory [1] becomes invariant under the nilpotent local fermionic symmetry $\delta_\epsilon\psi_\mu = D_\mu\epsilon$ and $\delta_\epsilon\chi = -\epsilon$. In other words, the original BM-theory [1] is re-interpreted as another example of local nilpotent fermionic symmetry [10][11][13][12][14].

The 2nd one is the generalization of the original theory [1] with the compensator χ in $D = (3, 1)$ to general space-time $D = (2n-1, 1)$. For this generalization, both the recent developments in tensor-hierarchy formulation [16][17] and nilpotent local fermionic symmetry formulations [10][11][13][13][12][14] played significant roles. The tensor-hierarchy formulation [16][17] is usually for bosonic fields, but now it is applied to the tensor-spinor $\psi_{[n-1]}$ combined with the compensator spinor $\chi_{[n-2]}$.

In the past, the importance of nilpotent local fermionic symmetry *never* drew enough attention. The reason is that the vanishing poor-looking anti-commutator such as (2.11) does *not* seem to produce any significant interactions, at least, compared with supersymmetry or

[6] There is *no* term like $(\overline{\theta}\gamma^{[4]}\theta)$ because of the reason already explained with (4.5).

supergravity [22]. We keep showing that that is *not* the case in our past papers [10][11][12][13][14]. In the present paper, we have shown yet another important application of nilpotent local fermionic symmetry in terms of BM-theory [1] generalized to $D = (2n-1, 1)$ *without* basic vielbeins.

We stress that the success of our formulation in this paper is based on the peculiar combination of both nilpotent fermionic symmetry formulations [10][11][12][13][14] and tensor-hierarchy formulations [16][17]. Our postulate for the gamma-matrix analysis (4.5) through (4.7) and (4.11) also played a technically significant role. Without any of these formulations and techniques, our theory would *not* be successful in such a straightforward way as in the present paper.

The original BM-theory was presented in late 1970's [1]. There are four main reasons for the delayed development afterwards. First, the tensor-hierarchy formulation [16][17] is the foundation of our present theory, in particular, for the invariance of our action I_{2n} under two distinct symmetries δ_ϵ and δ_η in (3.5) for $\psi_{[n-1]}$ and $\chi_{[n-2]}$. In the original BM-formulation [1], the invariance of $I_4^{(0)}$ was obscure. Second, the categorization of all fermions in general even dimensional space-time is possible, thanks to the systematic analyses in [23][24]. This is merely a technical point, but it still plays an important practical role in our formulation. Third, the non-trivial and crucial γ-matrix algebra, such as (4.5), (4.6), (4.7) and (4.11) in general $D = (2n-1, 1)$ was hard to handle. Fourth, the method (4.17) is very decisive for our analysis, which does *not* seem well-known for general $D = (2n-1, 1)$ dimensions, even though a similar method works in Euclidian space as (4.18).

We have *not* introduced a separate action for 'matter' fields in our formulation. However, our higher-dimensional Einstein field equation $\widehat{G}_{\hat{\mu}\hat{\nu}} \stackrel{*}{=} 0$[7] in $D = (2n-1, 1)$ creates Yang-Mills field equations, or σ-model type scalar-field equations, out of simple dimensional reductions [20]. Consider a simple dimensional-reduction [20] from $D = (2n-1, 1)$ into $D = (3, 1)$ with the metric-tensor reduction

$$(\widehat{g}_{\hat{\mu}\hat{\nu}}) = \begin{pmatrix} g_{\mu\nu} - 4A_\mu{}^\gamma A_\nu{}^\gamma & -2A_{\mu\beta} \\ -2A_{\nu\alpha} & \delta_{\alpha\beta} \end{pmatrix} \quad . \tag{5.1}$$

For simplicity, we truncated scalar fields from the extra dimensions. The indices $\mu, \nu, \cdots = 0, 1, 2, 3$ are for $D = (3, 1)$, while $\alpha, \beta, \cdots = 1, \cdots, 2n-4$ are for the extra $D = (2n-4, 0)$ compact dimensions. Correspondingly, the $D = (3.1)$-components of the Einstein equation yields

$$\widehat{G}_{\mu\nu}(\widehat{e}) = +G_{\mu\nu}(e) + 2F_\mu{}^{\rho\alpha} F_{\nu\rho}{}^\alpha - \tfrac{1}{2} g_{\mu\nu} (F_{\rho\sigma}{}^\alpha)^2 \doteq 0 \ . \tag{5.2}$$

This is nothing but the Einstein equation with the vector-fields $A_\mu{}^\alpha$ in $D = (3, 1)$. Even though this mechanism covers only bosonic fields, such as the vector fields and σ-model scalar fields, we expect that fermionic fields may well be produced by a certain lagrangians that contain *neither metric nor vielbeins* in the future.

An important aspect of our formulation is that the conventional metric and vielbein emerge out of higher-rank tensor-spinor $\psi_{[n-1]}$ in $D = (2n-1, 1)$. This is because of the ansätze (4.1) absorbing higher-rank tensor-indices. The introduction of such higher-rank tensor spinors has been regularly avoided in the past, because of possible inconsistent interactions [3]. However, the combination of the recent formulations of nilpotent local fermionic symmetries [10][11][12][13][14] and tensor-hierarchy formulation [16][17] has made it possible to consider such a sophisticated theory of pure tensor-spinors as the more fundamental physical quantities than geometrical metrics and vielbeins in general $D = (2n-1, 1)$ dimensions.

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Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Appendix A. Spinors in D=(2n−1,1)

We give here the general properties of spinors in $D = (2n-1, 1)$. First, the ϵ-tensor has the properties $\epsilon^{012\cdots(2n-1)} = +1$, and

$$\tfrac{1}{r!} \epsilon_{m_1 \cdots m_{2n-r}[r]} \, \epsilon^{[r] s_1 \cdots s_{2n-r}} = -(-1)^r [\,(2n-r)!\,] \, \delta_{[m_1}{}^{s_1} \cdots \delta_{m_{2n-r}]}{}^{s_{2n-r}} \quad . \tag{A.1}$$

Accordingly, the generalized γ_5-matrix is defined by

$$\gamma_{2n+1} \equiv - \frac{i^{n+1}}{(2n)!} \epsilon^{[2n]} \gamma_{\,[2n]} \quad . \tag{A.2}$$

The factor of the power of i is fixed by the normalization $(\gamma_{2n+1})^2 = +I$.

These properties are common to any $D = (2n-1, 1)$. However, the detailed structure of spinors depends on n. According to [23][24], fermionic structures repeat every eight space-time dimensions. We need the Majorana spinor θ with the normalization $(\overline{\theta}\theta) = +1$ in even space-time dimensions. To this end, we have to require that such a bilinear without any gamma-matrix sandwiched is non-vanishing. To this end, we categorize all even space-time dimensions higher than three into four cases: $D = (3+8p, 1)$, $D = (5+8p, 1)$, $D = (7+8p, 1)$ and $D = (9+8p, 1)$ $\;(p = 0, 1, 2, \cdots)$, in turn:

(i) D=(3+8p,1): In this case, spinors can be *Majorana*-spinors [23][24], because the condition $s = 3+8p, \ t = 1$ *i.e.,* $s-t = 8p+2$ in the notation of [24]. The flipping and hermitian-conjugation properties of two Majorana spinors are

[7] Only for the discussion of dimensional-reductions, we use the 'Hat'-symbol for fields and indices in $D = (2n-1, 1)$, while no hat-symbols for fields and indices in $D = (3, 1)$ as in [20].

$$(\overline{\psi}\gamma^{[m]}\chi) = -(-1)^{(m-1)(m-2)/2}(\overline{\chi}\gamma^{[m]}\psi) = \begin{cases} +(\overline{\chi}\psi) & (m=0) \ , \quad \text{(a)} \\ -(\overline{\chi}\gamma^{\mu}\psi) & (m=1) \ , \quad \text{(b)} \\ -(\overline{\chi}\gamma^{\mu\nu}\psi) & (m=2) \ , \quad \text{(c)} \end{cases} \tag{A.3}$$

$$(\overline{\psi}\gamma^{[m]}\chi)^{\dagger} = +(\overline{\psi}\gamma^{[m]}\chi) \ . \tag{A.3d}$$

In (A.3a) through (A.3c), only $m = 1, \ 2$ and 3 cases are given as examples. Eq. (A.3a) is desirable to normalize $(\overline{\theta}\theta) = +1$, while (A.3b) implies $c_{2+4p} = -1$. Eq. (A.3) also means that $\overline{\gamma^{rs}\psi} = -(\overline{\psi}\gamma^{rs})$, that is used in the confirmation of the equivalence between two expressions in (3.1). Our lagrangian (3.1) is real and does *not* vanish with the imaginary unit 'i' in front, because of (A.3c) and (A.3d).

(ii) $\boldsymbol{D=(5+8p,1)}$: In this case, since $s = 5 + 8p, \ t = 1, \ s - t = 8p + 4$ as in [23][24], spinors can be *symplectic Majorana*-spinors [23][24], carrying the indices $A, \ B, \ \cdots = 1, \ 2$ for the **2** of $Sp(1)$:

$$(\overline{\psi}^{A}\gamma^{[m]}\chi_{A}) = -(-1)^{(m-1)(m-2)/2}(\overline{\chi}^{A}\gamma^{[m]}\psi_{A}) = \begin{cases} +(\overline{\chi}^{A}\psi_{A}) & (m=0) \ , \quad \text{(a)} \\ -(\overline{\chi}^{A}\gamma^{\mu}\psi_{A}) & (m=1) \ , \quad \text{(b)} \\ -(\overline{\chi}^{A}\gamma^{\mu\nu}\psi_{A}) & (m=2) \ , \quad \text{(c)} \end{cases} \tag{A.4}$$

$$(\overline{\psi}^{A}\gamma^{[m]}\chi_{A})^{\dagger} = +(\overline{\psi}^{A}\gamma^{[m]}\chi_{A}) \ , \tag{A.4d}$$

where the contractions of the indices $A, \ B, \ \cdots$ are by the $Sp(1)$ metric $\epsilon^{AB} = -\epsilon^{BA}$ with $\epsilon^{12} = +1$, like $(\overline{\psi}^{A}\gamma^{[m]}\chi_{A}) \equiv \epsilon^{AB}(\overline{\psi}_{B}\gamma^{[m]}\chi_{A})$. Accordingly, we have $(\overline{\theta}^{A}\theta_{A}) = +1$, and $c_{3+8p} = -1$. Relevantly, $\overline{\gamma^{rs}\psi^{A}} = -(\overline{\psi}^{A}\gamma^{rs})$. Sometimes, we make the **2**-indices implicit, like $(\overline{\theta}^{A}\theta_{A}) \equiv (\overline{\theta}\theta) \ \overset{*}{=} 1$. Our lagrangian (3.1) is real and does *not* vanish, due to (A.4c) and (A.4d).

(iii) $\boldsymbol{D=(7+8p,1)}$: In this case, spinors can be *pseudo-Majorana*-spinors, because $s - t = 8p + 6$ [23][24]:

$$(\overline{\psi}\gamma^{[m]}\chi) = -(-1)^{m(m-1)/2}(\overline{\chi}\gamma^{[m]}\psi) = \begin{cases} -(\overline{\chi}\psi) & (m=0) \ , \quad \text{(a)} \\ -(\overline{\chi}\gamma^{\mu}\psi) & (m=1) \ . \quad \text{(b)} \\ +(\overline{\chi}\gamma^{\mu\nu}\psi) & (m=2) \ , \quad \text{(c)} \end{cases} \tag{A.5}$$

$$(\overline{\psi}\gamma^{[m]}\chi)^{\dagger} = -(-1)^{m}(\overline{\psi}\gamma^{[m]}\chi) \ . \tag{A.5d}$$

Due to (A.5a), we have to introduce an **2**-index of $Sp(1)$, so that $(\overline{\theta}^{A}\theta_{A}) \neq 0$, even though $(\overline{\theta}\theta) = 0$. Accordingly, $c_{4+4p} = +1$. Also, due to (A.5d), we have to introduce the imaginary unit. Eventually, the original condition $(\overline{\theta}\theta) = +1$ is modified to $i(\overline{\theta}^{A}\theta_{A}) = +1$. Eq. (A.5c) also implies that $\overline{\gamma^{rs}\psi^{A}} = -(\overline{\psi}^{A}\gamma^{rs})$. Our lagrangian (3.1) is real and does *not* vanish, due to $(\overline{\psi}^{A}\gamma_{\ 9+8p}\chi_{A}) = +(\overline{\chi}^{A}\gamma_{\ 9+8p}\psi_{A})$ because of the **2**-index.

(iv) $\boldsymbol{D=(9+8p,1)}$: In this case, spinors can be *Majorana*-spinors, because $s - t = 8p + 8$ [23][24]. The flipping and hermitian-conjugation properties are

$$(\overline{\psi}\gamma^{[m]}\chi) = -(-1)^{(m-1)(m-2)/2}(\overline{\chi}\gamma^{[m]}\psi) = \begin{cases} +(\overline{\chi}\psi) & (m=0) \ , \quad \text{(a)} \\ -(\overline{\chi}\gamma^{\mu}\psi) & (m=1) \ . \quad \text{(b)} \\ -(\overline{\chi}\gamma^{\mu\nu}\psi) & (m=2) \ , \quad \text{(c)} \end{cases} \tag{A.6}$$

$$(\overline{\psi}\gamma^{[m]}\chi)^{\dagger} = +(\overline{\psi}\gamma^{[m]}\chi) \ . \tag{A.6d}$$

Due to (A.6a), we can define $(\overline{\theta}\theta) = +1$. Our lagrangian (3.1) is real and does *not* vanish, due to (A.6c) and (A.6d).

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