Conformal Symmetries of Some Space Times in f(R) Theories of Gravity

By

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Conformal Symmetries of Some Space Times in f(R) Theories of Gravity

By

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Supervised by

Dr. Muhammad Ramzan

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<u>Student's Declaration</u>

I, **Fiaz Hussain** S/O **Saleh Muhammad**, Ph.D. Scholar at Department of Mathematics, The Islamia University of Bahawalpur, hereby declare that the research work titled **"Conformal Symmetries of Some Space Times in f(R) Theories of Gravity"** is carried out by me. I also certify that nothing has been incorporated in this research work without acknowledgement.

Fiaz Hussain

S/O

Saleh Muhammad

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It is hereby to certify that work presented by **Fiaz Hussain** in the thesis titled **"Conformal Symmetries of Some Space Times in f(R) Theories of Gravity"** is based on the results of research study conducted by the candidate under our supervision. No portion of this work has been formerly offered for higher degree in this university or any other institute of learning and to the best of the author's knowledge, no material has been used in this thesis which is not his own work, except where due acknowledgements has been made. He has fulfilled all the requirements and is qualified to submit this thesis in the partial fulfilment for the degree of the Doctor of Philosophy in Mathematics in Faculty of Science, The Islamia University of Bahawalpur.

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Dedicated to

My Venerable Teachers, Kind Parents and All Members of My Family

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Dated: **Fiaz Hussain**

Abstract

In this thesis, a study of conformal symmetries of some space times in the f(R) theories of gravity have been presented. The study includes static spherically symmetric, static plane symmetric, static cylindrically symmetric, Bianchi type I, II, III, V, Kantowski Sachs, Spatially homogeneous rotating space-times, a class of pp waves and non-static plane symmetric space-times. Initially, we have found some solutions of Einstein field equations (EFEs) using different fluid matters in the f(R) theories of gravity and then we have found conformal vector fields (CVFs) of the obtained solutions by means of direct integration technique. In the static spherically symmetric space-times, six cases have been discussed out of which there exists only one case for which CVFs become homothetic vector fields (HVFs) while in the rest of the cases CVFs become Killing vector fields (KVFs). In the static plane symmetric space-times, again there exist six cases. Out of these six cases, the space-times in five cases become conformally flat therefore admit fifteen independent CVFs while in the sixth case CVFs become KVFs. In the static cylindrically symmetric space-times, the dimension of CVFs turns out to be 4, 5, 6 and 15. In the Bianchi type I space-times, there exist fourteen cases. Out of these fourteen cases, the space-times in nine cases admit 4 and 5 HVFs while in five cases, space-times admit 6 and 15 CVFs. In Bianchi type II space-times, there exist seven cases while studying each case we found that in four cases, space-times admit proper HVFs while in rest of the three cases, CVFs become KVFs. In the Bianchi type V space-times, again there exist seven cases. From these seven cases, the space-times in six cases admit three KVFs while in the seventh case, the space-times become conformally flat therefore admit fifteen independent CVFs. In the Kantowski Sachs and Bianchi type III space-times, there exist eight cases. Studying each case in detail, we found that in six cases, space-times admit four and six KVFs while in two cases, the space-times admit proper CVFs of dimension six. In spatially homogeneous rotating space-times, there exist six cases. Within these six cases, there exist three cases in which space-times admit three and four KVFs. In two cases, the space-times admit four HVFs while in the remaining case, the space-time admits fifteen independent CVFs. In the pp-wave space-times, there exist ten cases. Studying each case, we found that in eight cases, space-times admit proper HVFs while in two cases, space-times admit proper CVFs. In non-static plane symmetric space-times, there exist, seven

cases. From these seven cases, there exist two cases in which CVFs become KVFs. In three cases, space-times admit five HVFs while in the remaining two cases, the space-times admit six and fifteen independent CVFs.

List of Publications

[1] **F. Hussain,** G. Shabbir, F. M. Mahomed and M. Ramzan, Conformal vector fields in proper non-static plane symmetric space-times in f(R) theory gravity, International Journal of Geometric Methods in Modern Physics, **17** (2020) 2050077. **ISI Impact Factor (1.022)**

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[3] **F. Hussain**, G. Shabbir and M. Ramzan, Classification of static cylindrically symmetric space-times in f(R) theory of gravity by conformal motions with perfect fluid matter, Arabian Journal of Mathematics, **8** (2019) 115. **ISI Impact Factor (0.00)**

[4] **F. Hussain,** G. Shabbir, S. Jamal and M. Ramzan, Some Bianchi type II space-times and their conformal vector fields in f(R) theory of gravity, Modern Physics Letter A, **34** (2019) 1950320. **ISI Impact Factor (1.367)**

[5] G. Shabbir, **F. Hussain**, M. Ramzan and A. H. Bokhari, A note on classification of spatially homogeneous rotating space-times in f(R) theory of gravity according to their proper conformal vector fields, International Journal of Geometric Methods in Modern Physics, **16** (2019) 1950111. **ISI Impact Factor (1.022)**

[6] G. Shabbir, **F. Hussain**, A. H. Kara and M. Ramzan, A note on some perfect fluid Kantowski Sachs and Bianchi type III spacetimes and their conformal vector fields in f(R) theory of gravity, Modern Physics Letters A **34** (2019) 1950079. **ISI Impact Factor (1.367)**

[7] G. Shabbir, **F. Hussain,** F. M. Mahomed and M. Ramzan, Dust static plane symmetric solutions and their conformal vector fields in f(R) theory of gravity, Modern Physics Letters A **33** (2018) 1850222. **ISI Impact Factor (1.367)**

[8] G. Shabbir, M. Ramzan, **F. Hussain** and S. Jamal, Classification of static spherically symmetric space-times in f(R) theory of gravity according to their conformal vector fields, International Journal of Geometric Methods in Modern Physics, **15** (2018) 1850193. **ISI Impact Factor (1.022)**

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Chapter 1

Preliminaries and Literature Review in f(R) Theory of Gravity

1.1 Introduction

A comprehensive view of the universe first started with Sir Isaac Newton's theory of gravitation, nearly three hundred years ago. This theory permitted scientists to describe the movement of universal bodies on earth with certain assumptions. Initially, Einstein theory was in two major forms namely special relativity (SR) and general relativity (GR). In SR, the space-time structure was assumed to be flat and there was no discussion related to the effect of gravity on the space-time structure. Within its limits, the SR had proven itself a satisfactory theory but in the years following 1905, Einstein became convinced that gravitation should be expressed in terms of curvature. Consequently in 1915, Einstein offered the theory of GR whose structure was space, time and gravitation. This theory describes the gravitation due to the existence of matter and energy. As in comparison with the SR, the space-time is not necessarily flat in GR. Einstein theory of GR is based upon the well-known EFEs which will be defined in the upcoming sections.

In this chapter, we will give some definitions and terminologies which will formulate a basis to understand the research work. A comprehensive review of literature on the theory of f(R) will be given. Rest of this chapter is planned as follows: In section (1.2) to (1.12), several basic definitions will be given. These definitions include manifold, tangent space, tensors, covariant derivative, Christofell symbols, Riemann curvature tensor, Ricci tensor, Ricci scalar, EFEs, energy momentum tensor (EMT), space-time, Lie derivative and space-time symmetries respectively. Review of some important literature related to the f(R) gravity will be given in section (1.13). Material related to the definitions mentioned above is taken from (Oneill, 1983, Wald, 1984 and Hall, 2004).

1.2 Manifold

In common words, manifold is defined as a space of points that is locally flat and globally seems to be a curve. Mathematically, an n-dimensional manifold M is defined as an extension of ordinary space. It fulfills the topological axioms (Qadir and Saifullah, 2006):

(i) It is separable.

(ii) It is connected.

(iii) It is Hausdorff.

(iv) There exists a homomorphism from its open cover to set of n tupples.

1.3 Tangent Space

Before defining a tangent space, first we have to define a tangent vector (TV). A tangent vector T at point $p \in M$, where M denotes the manifold is basically a map $T : \eta \to R$, η being family of real valued C^{∞} functions from M into R which fulfills linearity and Leibnitz rule i.e

(i) $T(as + bq) = aT(s) + bT(q)$ for all $s, q \in \eta$ and $a, b \in R.$

(ii) $T(sq) = T(s)q(p) + s(p)T(q)$ for all $s, q \in \eta.$

Assortment of all TVs at each point $p \in M$, is represented by $T_p(M)$ and is called tangent space (Nail, 1983 and Wald, 1984). A visible picture of such concepts is shown below:

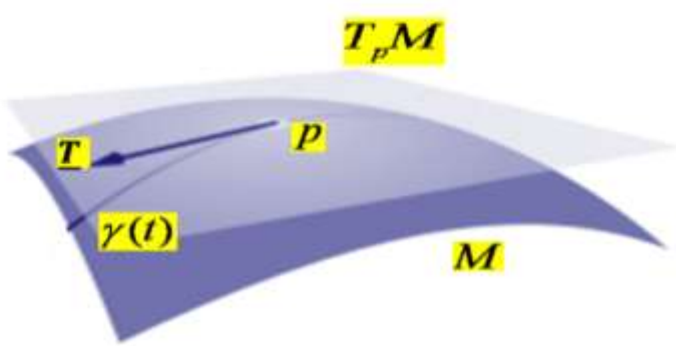


Figure 1.1

In figure 1.1, $p \in M$ is a point in the manifold M and $\gamma(t)$ is a curve in M.

1.4 Tensors

The quantities which remain unchanged when subjected to some transformation are called tensors. Mathematically, a tensor T of type (k,l) over $V_p(M)$ where $V_p(M)$ is representing finite dimensional vector space is a multilinear map which acts on vectors and yield a number. Mathematical expression of such a map is given below:

$$T : \underbrace{V_p(M)\times V_p(M)\times ...\times V_p(M)}_{k} \times \underbrace{V_p^*M \times V_p^*M ...\times V_p^*M}_{l} \tag{1.4.1}$$

where k, l and V_p^*M symbolizes the ordinary vectors or tangent vectors, dual vectors and dual vector space respectively. Now, we introduce a simple but important operation on tensors which is called a contraction. A contraction is a map on a M which reduces the rank by two. Symbolically, it is represented as $C : \Im(k,l) \rightarrow \Im(k-1, l-1)$. Moreover, components of a tensor satisfying

$$E'^{ab}(x') = E^{ij}(x)\partial x'^{\,a}_{,i} \partial x'^{\,b}_{,j}, \tag{1.4.2}$$

is called contravariant tensor with comma showing partial derivative. A covariant tensor is defined as:

$$E'_{ab}(x') = E_{ij}(x)\partial x'^{i}_{,a}\, \partial x'^{j}_{,a}. \tag{1.4.3}$$

Similarly, a mixed tensor is defined as

$$E'^{a}_{b}(x') = \frac{\partial x'^{a}}{\partial x^{i}} \frac{\partial x^{j}}{\partial x'^{b}} E^{i}_{j}(x). \tag{1.4.4}$$

A tensor T_{ab} is called symmetric if $T_{ab} = T_{ba}$ and anti-symmetric if $T_{ab} = -T_{ba}$. Similarly, if T_{ab} is a tensor of type $(0,2)$, then

$$T_{ab} = \frac{1}{2}(T_{ab} + T_{ba}) + \frac{1}{2}(T_{ab} - T_{ba}). \qquad (1.4.5)$$

We have decomposed T_{ab} into symmetric and skew-symmetric parts i.e. $T_{(ab)} = \frac{1}{2}(T_{ab} + T_{ba})$ and $T_{[ab]} = \frac{1}{2}(T_{ab} - T_{ba})$ respectively (Wald, 1984). One of the most significant form of tensors is **metric tensor**. To define a metric tensor, we start with the notion of metric which is considered as infinitely small squared distance associated with an "infinitely small displacement". The basic idea of infinitely small displacement is precisely captured from the concept of tangent vector. A metric g should fulfil the properties:

(i) It is a linear map whose domain contains the product of tangent space with tangent space onto real number i.e. $g : T_p(M) \times T_p(M) \rightarrow R.$

(ii) It is symmetric i.e. $g(v_1, v_2) = g(v_2, v_1) \;\; \forall \;\; v_1, v_2 \in T_p(M).$

(iii) It is non degenerate i.e. determinant of g_{ab} is non zero.

In the coordinate basis, we may write g_{ab} as

$$g = \sum_{a,b} g_{ab} dx^a \otimes dx^b. \qquad (1.4.6)$$

One may replace the notation of g into ds^2 so that, we have

$$ds^2 = \sum_{a,b} g_{ab} dx^a dx^b. \qquad (1.4.7)$$

One can define the inverse of g as g^{ab}. Thus, by definition $g^{ab} g_{bc} = \delta^a_c$, where δ^a_c denotes the Kronecker delta and is defined as

$$\delta_c^a = \begin{cases} 1 & \text{if } a = c \\ 0 & \text{if } a \neq c \end{cases}. \tag{1.4.8}$$

It is well-known that the tensor under discussion is used for lowering or raising the indices.

1.5 Covariant Derivative

The Covariant derivative is basically a map say $H:(k,l) \to (k,l+1)$, where (k,l) is representing the type of any smooth tensor field into another type $(k,l+1)$. It is represented by the symbol ∇ or the semicolon (;). It is considered as a map which fulfills the linearity property and obeys Leibnitz rule. It also fulfills the commutative property of contraction. In addition, this map is torsion free and when acts on some scalar field it represents partial derivative instead of covariant derivative. This sort of the derivative have a deep relationship with the differential geometry which is further have a link for the core study of fibre bundles. It is also used to elaborate several mathematical representations of theoretical physics including GR. The advantage of this form of derivative over the usual differentiation is that it helps to calculate those changes which happen to occur on the curved space-times. Other forms of the derivative include Lie and usual differentiation. These types of the derivative don't have the property to map a tensor quantity into the tensor quantity whereas covariant derivative have this property which made this sort of dervitive significant in the field of tensor analysis.

1.6 Christofell Symbol

We know that the metric tensor is covariantly constant i.e. (Wald, 1984)

$$\nabla_c g_{ab} = 0, \tag{1.6.1}$$

where ∇_c denotes the covariant derivative of the metric tensor g_{ab}. Expanding the above equation (1.6.1) give

$$g_{ab,c} - \Gamma^d_{ac} g_{bd} - \Gamma^d_{bc} g_{ad} = 0. \tag{1.6.2}$$

$$g_{ab,c} = \Gamma^d_{ac} g_{bd} + \Gamma^d_{bc} g_{ad}. \tag{1.6.3}$$

Interchanging c with b in equation (1.6.3), we have

$$g_{cb,a} = \Gamma^d_{ca} g_{bd} + \Gamma^d_{ba} g_{cd}. \tag{1.6.4}$$

Similarly, interchanging b with a in equation (1.6.4), we have

$$g_{ca,b} = \Gamma^d_{cb} g_{ad} + \Gamma^d_{ab} g_{cd}. \tag{1.6.5}$$

Adding equations (1.6.3) and (1.6.4), we get

$$g_{ab,c} + g_{cb,\mathrm{a}} = \Gamma^d_{ac} g_{bd} + \Gamma^d_{bc} g_{ad} + \Gamma^d_{ca} g_{bd} + \Gamma^d_{ba} g_{cd}. \tag{1.6.6}$$

Subtracting equation (1.6.5) from equation (1.6.6), we have

$$g_{ab,c} + g_{cb,a} - g_{ca,b} = \Gamma^d_{ac} g_{bd} + \Gamma^d_{bc} g_{ad} + \Gamma^d_{ca} g_{bd} + \Gamma^d_{ba} g_{cd} - \Gamma^d_{cb} g_{ad} - \Gamma^d_{ab} g_{cd}. \tag{1.6.7}$$

Using the fact that $\Gamma^d_{ac} = \Gamma^d_{ca}$ and $\Gamma^d_{ba} = \Gamma^d_{ab}$ in the above equation (1.6.7), we have

$$\Gamma^d_{ac} = \frac{1}{2} g^{bd} \left[g_{ab,c} + g_{cb,a} - g_{ca,b} \right]. \tag{1.6.8}$$

It is important to note that Christofell symbols are not a tensor quantity. These symbols are important objects as they arise in describing the effects of parallel transport of vectors. These are also used in the calculations of covariant derivative, Riemann curvature tensor and in the geodesic equation. Further, Christofell symbols act as a dominant tool to investigate the geometry of manifold. For instance, a manifold with all the vanishing components is the indication that the assocated space-time geometry is flat. This property further shows that the components of the Ricci tensor also going to vanish and hence one seeks for the vaccum solutions of the EFEs. These symbols have a majour role to calculate the components of the curvature tensor.

1.7 Riemann Curvature Tensor and its Derived Forms

The Riemann Curvature tensor in terms of Christofell symbols is defined by

$$R^a_{bcd} = \Gamma^a_{bd,c} - \Gamma^a_{bc,d} + \Gamma^a_{ce}\Gamma^e_{bd} - \Gamma^a_{de}\Gamma^e_{bc}. \tag{1.7.1}$$

If we contract the above expression, we get the Ricci tensor $R_{fh} = R^e_{feh}$ and the trace of Ricci tensor yields Ricci scalar $R = g^{ef} R_{fe}$ (Wald, 1984). In the later terminology, Ricci scalar R has a significant role as it acts as the key factor while introducing f(R) theory of gravity. On the other hand, dealing with the curvature tensor yields curvature of a manifold in the subject of differential geometry. Curvature tensor has a majour impact in the development of GR as the core of this theory seems to coincide with the well-known field equations. One of the bright ingradiant of such equations is the Ricci tensor which is obtained by taking the trace of curvature tensor. Furthermore by taking the trace of Ricci tensor one gets the scalar curvature. In the next section, mathematical form of the EFEs is given.

1.8 Einstein Field Equations in General Relativity

The EFEs in GR are defined by (Stephani et al., 2003)

$$G_{ab} = R_{ab} - \frac{1}{2} R g_{ab} + \Lambda g_{ab} = k T_{ab}, \tag{1.8.1}$$

where the quantities are defined below:

$G_{ab} =$ Eienstien tensor,

$R_{ab} =$ Ricci tensor,

$R =$ Ricci scalar,

$g_{ab} =$ Metric tensor,

$\Lambda =$ Cosmological constant,

$k = \frac{8\pi G}{c^4} = \text{Coupling constant},$

G being gravitational constant,

c is the speed of light,

$T_{ab} = \text{Energy momentum tensor}.$

Actually, the EFEs provide a link between geometry and physics of the space-time as the left hand side of the above field equations denotes the geometry while the right hand side denotes the physics of the space-time. In the above equation (1.8.1), T_{ab} is the source to provide the gravitational contribution. The case when T_{ab} vanishes, one is confined to get the vacuum solutions to these equations as the door which provide physics of space-time structure closes. The coming slots of thesis define the EMT.

1.9 Energy Momentum Tensor

The right hand side of equations (1.8.1) is known to be the energy momentum tensor (EMT). EMT has a major role in GR as it provides the link to discuss the physics related to the considered space-time structure. In addition, when one needs to explain the gravitational field in EFEs, one must look to use EMT. Other well-known terminologies related to the physics such as mass which is defined as amount of matter in a certain object and energy which is capability of doing work are also directs to define EMT. There are different types of EMT but we are focusing only on two types which we have used in our research work. The first one is T_{ab} for the perfect fluids which is:

$$T_{ab} = (\rho + p)s_a s_b + p g_{ab}, \qquad (1.9.1)$$

where ρ is the matter density, p is the pressure and s_a is the 4-velocity vector. A perfect fluid can be characterized as the following properties:

(i) It does not have shear stress.

(ii) It does not have anisotropic pressure.

(iii) It does not have heat conductivity.

(iv) It does not have viscosity.

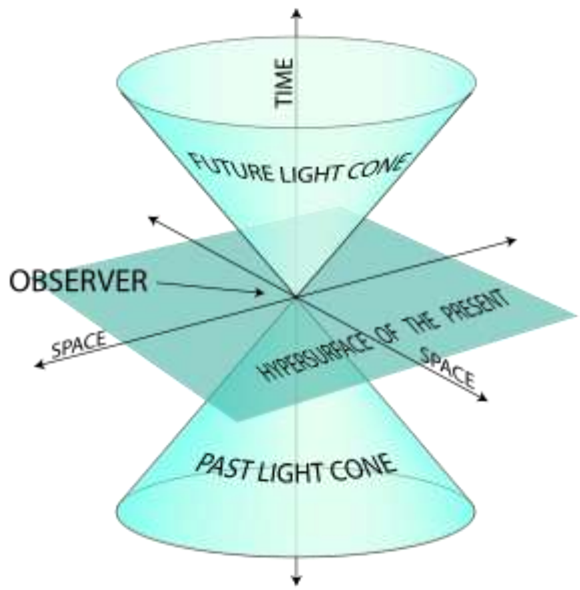
It is completely characterized by its rest frame isotropic pressure and mass density. The second form of EMT T_{ab} for the dust matter is of the form

$$T_{ab} = \rho s_a s_b. \quad (1.9.2)$$

Clearly, one can define energy momentum tensor for dust matter by putting the pressure term p equal to zero in equation (1.9.1).

1.10 Space-time

A space-time is defined as a combination (M, g), where M represents the manifold defined earlier in section (1.2) and g is the Lorentzian metric on M (Wald, 1984). A space-time comprises of elements which are known as events. It is further classified as stationary, static and non-static. The term stationary space-time is referred to those having time-like KVF. A combination of stationary space-time along with the property that it holds a KVF orthogonal to the hypersurface are termed as static space-time. The space-time other than the stationary and static are said to be non-static. Therefore, we can define a non-static space-time as the space-time which do not admit time-like KVF (Stephani et al., 2003). There are also some other space-times like Minkowski or flat which have vanishing Riemann tensor. Following diagram shows the structure of space-time. It is usually referred to as light cone:



Light cone. **Figure 1.2**

1.11 Lie Derivative

The Lie derivative L of the metric tensor g_{ab} along the vector field X is basically a directional derivative and is defined as: (Hall, 2004)

$$L_X g_{ab} = \frac{\partial g_{ab}}{\partial x^c} X^c + g_{ac} \frac{\partial X^c}{\partial x^b} + g_{bc} \frac{\partial X^c}{\partial x^a}, \tag{1.11.1}$$

Another way to define Lie derivative is (Hall, 2004, Hawking, 1973)

$$L_X T = \lim_{t \to 0} \left\{ \frac{\phi_t^* T - T}{t} \right\}, \tag{1.11.2}$$

where ϕ_t is the one parameter local diffeomorphism and $\phi_t^* T$ denotes the pullback map. If X_i and X_j are two vector fields, then the Lie bracket is defined as:

$$\left[X_i, X_j \right] = X_i(X_j) - X_j(X_i). \tag{1.11.3}$$

If we concentrate within mathematical physics and look into the structure of GR, we see that the Lie derivative is an important terminology. Especially in the field of symmetries of gravitational fields. Moreover, it is proved to be helpful while one needs to calculate the changes that happen to occur in curved geometry.

1.12 Space-time Symmetries

In the theory of GR, the space-time symmetries act as the key probes in solving many mathematical problems. The EFEs defined in equation (1.8.1) have no general solution. One is not able to solve these equations unless one fixes the space-time geometry. In addition, if a space-time admits some sort of symmetry then there exists a possibility of finding the exact solutions to some extent. Space-time symmetries are not only used in finding the solutions, they are also helpful to classify the already existing solutions with respect to the symmetry which the space-time admits. The transformations that force the physical quantities to leave invariant are proved to be extremely important in modern theoretical physics. The existence of symmetries in space-time provide the conservation laws. In GR, different forms of symmetries include:

(i) Killing symmetry.

(ii) Homothetic symmetry.

(iii) Conformal symmetry.

(iv) Affine symmetry.

(v) Projective symmetry.

(vi) Curvature collineation.

(vii) Ricci collineation.

(viii) Matter collineation, etc.

Precise definitions of above mentioned symmetries can be found in (Hall, 2004). Among these symmetries conformal symmetry is very important due to its important applications in the modern theoretical physics. From the geometrical point of view, Maxwell's laws of electromagnetic theory and light cone structures remain invariant under conformal transformations. Moreover, for massless particles along the null geodesics, conformal symmetry produces a constant of the motion. A number of applications of conformal KVFs exist in the theory of irreversible processes and many more can be seen in (Khan et al., 2015) and the references therein. Conformal symmetry is also used in quantum electrodynamics as the classical field equations for massless electrodynamics which are invariant under a much larger group of space-time transformations known as conformal group (Baker and Johnson, 1979). In astrophysics, conformal symmetry is used to study compact stars. Compact stars are used to study certain properties of gravitational fields. Conformal symmetry is also used to study wormhole solutions and Brane world gravstars. Moreover, the role which conformal KVFs play at the kinematic and dynamic level is documented in (Martin et al., 1986). In view of the important applications of CVFs, our focus in this thesis will be on searching the CVFs. A vector field X is said to be conformal vector field, if (Hall, 2004)

$$L_X g_{ab} = 2\psi g_{ab}, \tag{1.12.1}$$

where ψ is smooth real valued function on M. There is a close relationship between the vector field X and the conformal factor ψ as it is a well-known fact that

$$X = \begin{cases} \text{HVF}, & \text{if} \quad \psi = \text{constant}. \\ \text{Proper HVF}, & \text{if} \quad \psi = \text{constant} \neq 0. \\ \text{KVF}, & \text{if} \quad \psi = 0. \\ \text{Proper CVF}, & \text{Otherwise}. \end{cases}$$

For a space-time, the maximum dimension of CVFs is fifteen and it is obtained when the space-time is conformally flat. If the space-time is non conformally flat, then maximum dimension of CVFs is seven (Hall, 2004).

1.13 Literature Review in f(R) Theory of Gravity

GR is a sophisticated theory of gravitation and was given by Albert Einstein in 1915. The mathematical form of this theory was described in the form of EFEs which are defined in equation (1.8.1). These equations were derived by a technique known as Einstein Hilbert action in linear function of scalar curvature R (Stephani et al., 2003). After four years of its birth, people started thinking that what will be the form of field equations, if the scalar curvature R in the action become some function of R. The idea was first presented by Weyl in 1919 and then supplemented by Eddington in 1923. They argued that inclusion of higher order invariants in the action can produce interesting results (Weyl, 1919 and Eddington, 1923). In 1929, an American astronomer Edwin Hubble was performing experiment where he observed that the distance of galaxies at the edges of the universe from the observer is increasing with the passage of time. From this he concluded that our universe is expanding with a fixed rate (Hubble, 1929). Due to this expansion, space-time geometry was needed to be updated. As the left hand side of equations (1.8.1) reflects the space-time geometry which is also called the curvature part, therefore the change in the curvature part of EFEs seems useful for the better explanation of phenology of expanding behavior of universe. This was possible only if one makes the modification in the action of GR. Later on, investigation of (Utiyama and De Witt, 1962), put forward the idea to modify the action of GR by adding the curvature invariants. Eventually, in 1970, A. H. Buchdahl made the modification in action of GR and replaced Ricci scalar R with a general function f(R) in the Einstein Hilbert action. The refined action $S\left[g\right]$ is given below (Nojiri and Odintsov, 2003)

$$S[g]=\frac{1}{2k}\int f\left(R\right)\sqrt{-g}d^4x, \qquad (1.13.1)$$

where k is the coupling constant, $g=\left|g_{\mu\nu}\right|$ and f(R) is representing the function of the scalar curvature or Ricci scalar R. The action defined in equation (1.13.1) serves as a basic tool to formulate equations of motions in the f(R) theory of gravity. In fact, the variation of action (1.13.1) yields following equations in f(R) gravity (Nojiri and Odintsov, 2003)

$$F\left(R\right)R_{ab}-\frac{1}{2}f\left(R\right)g_{ab}-\nabla_a\nabla_bF\left(R\right)+g_{ab}\Box F\left(R\right)=kT_{ab}, \qquad (1.13.2)$$

where $F(R)=\frac{d}{dR}f(R)$, $\Box\equiv\nabla^a\nabla_a$ in which ∇_a is the covariant derivative, T_{ab} is the standard matter energy-momentum tensor. In contrast with the GR, the equations defined in equation (1.13.2) have order four, therefore create a possibility of obtaining more solutions than GR whose equations of motion are of order two. The replacement $f(R)=R$ led towards the equations of motion in GR. There is a huge amount of works on the solutions of equations (1.13.2). A brief description is given here. In the early 2006 and 2007, some work on spherically symmetric space-times was carried by (Multamaki and Vilja, 2006). The same authors extended the work from vacuum to non-vacuum solutions and sought solutions using source of EMT as perfect fluid. More work related to solutions of EFEs in f(R) gravity have been done in (Multamaki and Vilja, 2007), (Capozziello et al., 2007), (Hollenstein and Lobo, 2008), (Capozziello et al., 2008), (Azadi et al., 2008), (Carames and Bezarra de Mello, 2009), (Sharif and Shamir, 2009), (Sharif and Shamir, 2010), (Sharif and Shamir, 2010), (Reboucas and Santos, 2010), (Shamir, 2010), (Hendi and Momeni, 2011), (Sharif and Kausar, 2011), (Sebasiani and Zerbini, 2011), (Sharif and Kausar, 2011), (Shojai and Shojai, 2012), (Capozziello et al., 2012), (Hindi et al., 2012), (Gutierrez-Pineres, 2012), (Yavari, 2013), (Sharif and Zahra, 2013), (Shamir and Raza, 2014), (Shamir and Raza, 2014), (Arbuzova et al., 2014), (Hendi, 2014,) (Ohta et al., 2015), (Amirabi et al., 2016), (Shamir, 2016), (Elmardi et al., 2016), (Tripathy and Mishra, 2016), (Gao and Shen, 2016), (Banik et al., 2017), (Mahraj et al., 2017) and (Nashed and Capozziello, 2019).

Our purpose in this thesis is to discuss CVFs of distinguish class of space-times in the f(R) theory of gravity. In the second chapter, we will try to find CVFs of some static space-times in the f(R) theory of gravity. These include space-times, static spherically symmetric, static cylindrically symmetric and static plane symmetric. In the third chapter, we will find proper CVFs of spatially homogenous rotating space-times and well-known class of plane fronted gravitational waves (GWs) also called pp-waves space-times. In the fourth chapter, a comprehensive study of some Bianchi models will be presented. In particular, conformal symmetries of Bianchi type I, II, V, III and Kantowski Sachs space-times will be discussed in detail. In chapter five, conformal symmetries of proper non static plane symmetric space-times will be discussed. In chapter six, conclusion of overall analysis will be given. At the end of this thesis, bibliography of our work will be presented.

Chapter 2

Conformal Symmetry of Some Static Space-Times in f(R) Theory of Gravity

2.1 Introduction

In this chapter, we will study CVFs of some static space-times in the f(R) theory of gravity. The methodology which is adopted here is twofold. In the initial step, we have tried to formulate the space-times in the theory of f(R). Further, we have used these space-times to construct CVFs. The procedure of finding the CVFs is direct integration. Three space-times have been considered in this chapter to find CVFs. These space-times are static spherically symmetric (SSS), static plane symmetric (SPS) and static cylindrically symmetric (SCS). The whole chapter is divided into five sections. In the section (2.2), we don't explore the space-times. Instead of this, we have used the results of two papers (Capozziello et al., 2012 and Amirabi et al., 2016) to find CVFs. SS space-time is of incredible passion because of it having various significant physical and hypothetical aspects. This space-time is considered as one of the initial solution of the EFEs whose significant example include the Schwarzschild solution. Looking at the end of physical scenario, the Schwarzschild solution is utilized to clarify the gravitational field outside to static round stars. SS space-times are likewise used to talk about the nearby planetary group tests and can be considered as key fixings to figure related physical amounts like weight, thickness and gravitational fields. Spherical symmetric space-times are important to study from several point of view. Especially the theory of black holes entirely depends on the spherical geometry of space-times. The SSS stars are used to compute pressure, density and gravitational fields and to treat the solar system tests. On the other hand exact spherical symmetric solutions yield a deeper approach to look into the Steller models and related red-shifts. Keeping the important applications of spherical symmetric space-times in mind, it seems interesting to study this space-time from the symmetry perspective. The upcoming three sections are specified for detailed study of CVFs for spme well-known space-times whereas in the last section (2.5), a brief summary of the obtained results is given.

2.2 Conformal Vector Fields of Static Spherically Symmetric Space-times in the f(R) Theory of Gravity

Consider SSS space-times in the coordinates (t, r, θ, ϕ) (given by $\left(x^0, x^1, x^2, x^3\right)$ respectively) with line element (Stephani et al., 2003)

$$ds^2 = -A(r)dt^2 + \frac{dr^2}{A(r)} + r^2 d\Omega^2, \qquad (2.2.1)$$

where $d\Omega^2 = \left[d\theta^2 + \sin^2\theta d\phi^2\right]$. The minimal number of isometries which the above space-times (2.2.1) admit are (Stephani et al., 2003)

$$\frac{\partial}{\partial t},\ \frac{\partial}{\partial \phi},\ \cot\theta\cos\phi\frac{\partial}{\partial \phi} + \sin\phi\frac{\partial}{\partial \theta},\ \cos\phi\frac{\partial}{\partial \theta} - \cot\theta\sin\phi\frac{\partial}{\partial \phi}. \qquad (2.2.2)$$

Expanding equation (1.12.1) and using equation (2.2.1), we have

$$A'X^1 + 2AX^0_{,0} = 2A\psi, \qquad (2.2.3)$$

$$A^2X^0_{,1} - X^1_{,0} = 0, \qquad (2.2.4)$$

$$AX^0_{,2} - r^2X^2_{,0} = 0, \qquad (2.2.5)$$

$$AX^0_{,3} - r^2\sin^2\theta X^3_{,0} = 0, \qquad (2.2.6)$$

$$-A'X^1 + 2AX^1_{,1} = 2A\psi, \qquad (2.2.7)$$

$$X^1_{,2} + Ar^2X^2_{,1} = 0, \qquad (2.2.8)$$

$$X^1_{,3} + Ar^2\sin^2\theta X^3_{,1} = 0, \qquad (2.2.9)$$

$$X^1 + rX^2_{,2} = r\psi, \qquad (2.2.10)$$

$$X^2_{,3} + \sin^2\theta X^3_{,2} = 0, \qquad (2.2.11)$$

$$X^1 + r\cot\theta X^2 + rX^3_{,3} = r\psi, \qquad (2.2.12)$$

where prime signifies $\frac{d}{dr}$. Multiplying equation (2.2.7) with $\frac{1}{\sqrt{A}}$ and then integrating over r, one has $X^1 = \sqrt{A}\int \frac{\psi}{\sqrt{A}}dr + \sqrt{A}B^1(t,\theta,\phi)$, where $B^1(t,\theta,\phi)$ is a function of integration (FOI). Using the value of X^1 in equations (2.2.4), (2.2.8) and (2.2.9) after some algebraic calculations, we have

$$\left.\begin{aligned} X^0 &= B_t^1(t,\theta,\phi)\int A^{\frac{-3}{2}}dr + B^2(t,\theta,\phi), \\ X^2 &= -B_\theta^1(t,\theta,\phi)\int \frac{dr}{r^2\sqrt{A}} + B^3(t,\theta,\phi) \\ X^3 &= -\operatorname{cosec}^2\theta B_\phi^1(t,\theta,\phi)\int \frac{dr}{r^2\sqrt{A}} + B^4(t,\theta,\phi) \end{aligned}\right\}, \tag{2.2.13}$$

where $B^i(t,\theta,\phi)$, $i = 2,3,4$ are also FOIs. For the sake of obtaining CVF X, we search the functions $B^i(t,\theta,\phi)$, $i = 1,2,3,4$ and the conformal factor ψ. As already mentioned that we are going to use the results of the two papers (Capozziello et al., 2012 and Amirabi. et al., 2016) to find CVFs. The procedure will be straight forward, we will use the values of metric component $A = A(r)$ which are given in the papers (Capozziello et al., 2012 and Amirabi et al., 2016) in the above system of equations (2.2.3) to (2.2.12) and then we will find the components of CVFs. The process of integration is lengthy but straightforward. Omitting the details of calculations which are performed in this process we arrive at the following cases:

Case (i)

In this case, we have the space-time (Capozziello et al., 2012)

$$ds^2 = -\left[1 - \frac{\Lambda}{3}r^2 - \frac{A_1}{2r}\right]dt^2 + \left[1 - \frac{\Lambda}{3}r^2 - \frac{A_1}{2r}\right]^{-1}dr^2 + r^2 d\Omega^2, \tag{2.2.14}$$

where $A_1 \in \Re \setminus \{0\}$ and Λ is cosmological constant. Next, we substitute the value of $A(r)$ in the above ten equations from (2.2.3) to (2.2.12) and after some lengthy calculations, we found that $\psi = 0$ which implies CVFs are KVFs and are given in equation (2.2.2). The above space-times are known as static Schwarzschild–de Sitter solutions.

Case (ii)

Here, we have Schwarzschild–de/anti-de Sitter black hole space-time (Amirabi et al., 2016)

$$ds^2 = -\left[1 - \frac{2M}{r} + c_2 r^2\right] dt^2 + \left[1 - \frac{2M}{r} + c_2 r^2\right]^{-1} dr^2 + r^2 d\Omega^2, \qquad (2.2.15)$$

where M represents the Arnowitt–Deser–Misner (ADM) mass and $c_2 \in R$. In this case, again we are using the value of $A(r)$ in the above ten equations from (2.2.3) to (2.2.12) and after some lengthy calculations, we found that $\psi = 0$ which implies CVFs are KVFs and are shown by the equation (2.2.2).

Case (iii)

Here the space-time (Amirabi et al., 2016) is,

$$ds^2 = -\left[\frac{-\Lambda}{3} r^2 + \frac{1}{2} + \frac{1}{3\alpha r}\right] dt^2 + \left[\frac{-\Lambda}{3} r^2 + \frac{1}{2} + \frac{1}{3\alpha r}\right]^{-1} dr^2 + r^2 d\Omega^2, \qquad (2.2.16)$$

where $\alpha \in R \setminus \{0\}$ and Λ is cosmological constant. The space-time (2.2.16) do not admit CVFs as $\psi = 0$, here CVFs are KVFs which are given in equation (2.2.2).

Case (iv)

The space-time in this case is (Amirabi et al., 2016)

$$ds^2 = -\left[\frac{1}{2} + \frac{2\eta^2 + 1}{3\alpha r}\right] dt^2 + \left[\frac{1}{2} + \frac{2\eta^2 + 1}{3\alpha r}\right]^{-1} dr^2 + r^2 d\Omega^2, \qquad (2.2.17)$$

where η is the global monopole charge. Here, using the same technique which we have used in the previous cases, we again found that $\psi = 0, \Longrightarrow$ CVFs become KVFs which are already mentioned in equation (2.2.2).

Case (v)

Here, we have the space-time (Amirabi et al., 2016)

$$ds^2 = -\left[\frac{1}{2}\right]dt^2 + \left[\frac{1}{2}\right]^{-1} dr^2 + r^2 d\Omega^2, \qquad (2.2.18)$$

Substituting the value of $A(r)$ in the above ten equations from (2.2.3) to (2.2.12) and after some lengthy calculations, we found that $\psi = c_1$ which implies CVFs become HVFs. Here, the proper HVF after subtracting KVFs is

$$X = (t, r, 0, 0). \qquad (2.2.19)$$

Case (vi)

Here, considering the space-time of the form (Capozziello et al., 2012)

$$ds^2 = -\left[\frac{1}{2} - \frac{1}{r}\right]dt^2 + \left[\frac{1}{2} - \frac{1}{r}\right]^{-1} dr^2 + r^2 d\Omega^2, \qquad (2.2.20)$$

which is asymptotically flat and will be physically consistent for $r > 0$ as value of Ricci scalar is negative. Again in this case, we found that $\psi = 0,$ which implies that CVFs are the KVFs which are given in equation (2.2.2).

2.3 Conformal Vector Fields of Dust Static Plane Symmetric Space-times in the f(R) Theory of Gravity

The line element of SPS space-times in the usual coordinates (t, x, y, z) (given by (x^0, x^1, x^2, x^3) respectively) is given by (Stephani et al., 2003)

$$ds^2 = -Adt^2 + dx^2 + B[dy^2 + dz^2], \qquad (2.3.1)$$

where $A = A(x)$ and $B = B(x)$ are nowhere zero functions of x only. The minimal set of isometries which the space-times (2.3.1) admit are (Stephani et al., 2003)

$$\partial_t, \ \partial_y, \ \partial_z, \ \ y\partial_z - z\partial_y. \qquad (2.3.2)$$

The value of Ricci scalar R for the above space-times (2.3.1) turn out to be

$$R = 2\left[\frac{4B''}{B} - \frac{B'^2}{B^2} + \frac{2A''}{A} + \frac{2A'B'}{AB} - \frac{A'^2}{A^2}\right], \qquad (2.3.3)$$

where prime appearing in the above equation signifies $\frac{d}{dx}$. Expanding equation (1.12.1) and using equation (2.3.1), we have

$$A'X^1 + 2AX^0_{,0} = 2A\psi, \qquad (2.3.4)$$

$$AX^0_{,1} - X^1_{,0} = 0, \qquad (2.3.5)$$

$$AX^0_{,2} - BX^2_{,0} = 0, \qquad (2.3.6)$$

$$AX^0_{,3} - BX^3_{,0} = 0, \qquad (2.3.7)$$

$$X^1_{,1} = \psi, \qquad (2.3.8)$$

$$X^1_{,2} + BX^2_{,1} = 0, \qquad (2.3.9)$$

$$X^1_{,3} + BX^3_{,1} = 0, \qquad (2.3.10)$$

$$B'X^1 + 2BX^2_{,2} = 2B\psi, \qquad (2.3.11)$$

$$X^2_{,3} + X^3_{,2} = 0, \qquad (2.3.12)$$

$$B'X^1 + 2BX^3_{,3} = 2B\psi. \qquad (2.3.13)$$

From equations (2.3.9), (2.3.10) and (2.3.12), we have $X^1 = \int Q^1(t,x,y)dy + Q^2(t,x,z)$, where $Q^1(t,x,y)$ and $Q^2(t,x,z)$ are FOI. Now, utilizing the value of X^1 in equations (2.3.5), (2.3.9) and (2.3.12), we have

$$X^0 = \iint \frac{Q^1_t(t,x,y)}{A} dx\,dy + \int \frac{Q^2_t(t,x,z)}{A} dx + Q^5(t,y,z),$$

$$X^2 = -\int \frac{Q^1(t,x,y)}{B}\,dx + Q^3(t,y,z),$$

$$X^3 = -\int Q_z^3(t,y,z)\,dy + Q^4(t,x,z), \tag{2.3.14}$$

where $Q^3(t,y,z)$, $Q^4(t,x,z)$ and $Q^5(t,x,z)$ are also FOI. As we are in search of CVFs in f(R) theory of gravity therefore, we need to deduce the solutions in this theory. The theory of f(R) is based on the set of field equations defined in equation (1.13.2). For finding the solutions of equations (1.13.2), we will use equation (2.3.1) in the set of equations (1.13.2) and will look for the metric coeffiecnts. Further, we are using dust matter as a source of EMT which is defined by

$$T_{ab} = \rho u_a u_b, \tag{2.3.15}$$

where ρ denotes the matter density and u_a is the four velocity vector. Using equations (2.3.1), (2.3.15) and (1.13.2), after some algebraic manipulations, we have

$$\frac{A''}{2A} - \frac{A'^2}{4A^2} + \frac{A'B'}{4AB} - \frac{B''}{2B} + \frac{A'F'}{2AF} - \frac{F''}{F} - \frac{B'F'}{2BF} = \frac{k\rho}{FA}. \tag{2.3.16}$$

$$\frac{B'^2}{2B^2} - \frac{B''}{B} + \frac{A'B'}{2AB} - \frac{F''}{F} + \frac{A'F'}{2AF} = \frac{k\rho}{FA}. \tag{2.3.17}$$

A comparison of equations (2.3.16) and (2.3.17) provide the following equation:

$$\frac{A'B'}{2AB} + \frac{A'^2}{2A^2} - \frac{A''}{A} + \frac{B'^2}{B^2} - \frac{B''}{B} + \frac{B'F'}{BF} - \frac{2F''}{F} = 0. \tag{2.3.18}$$

Looking at equation (2.3.18), we see that it contain three unknowns viz. A, B and F. Choosing different $f(R)$ models, a number of solutions have been obtained by (Shamir, 2016). In this study, we are interested in discovering analytic solutions of equation (2.3.18) by imposing some restrictions on derivative of $f(R)$. These restrictions are:

(a) $A = \text{constant}$, $B = B(x)$, $F' \neq 0$, $F'' = 0$ and $B'^2F - BB''F + BB'F' = 0$.

(b) $A = \text{constant}$, $B = B(x)$, $F' = 0$, $F'' = 0$ and $B'^2 - BB'' = 0$.

(c) $B = \text{constant}$, $A = A(x)$, $F' \neq 0$, $F'' = 0$ and $A'^2 - 2AA'' = 0$.

(d) $B = \text{constant}$, $A = A(x)$, $F' = 0$, $F'' = 0$ and $A'^2 - 2AA'' = 0$.

(e) $A = A(x)$, $B = B(x)$, $A = B$, $F' = 0$, $F'' = 0$ and $A'^2 - AA'' = 0$.

(f) $A = A(x)$, $B = B(x)$, $A = B$, $F' \neq 0$, $F'' = 0$ and $A'^2 F - AA''F + AA'F' = 0$. Solutions of equation (2.3.18) using the above restrictions turns out to be

(i) $A = \text{constant}$, $B = e^{\alpha x^2 + \beta x}$, $f(R) = \left(c_1 x + c_2\right) R + c_3$ and $R = 2(3x^2 + 2)\alpha^2 + 6\alpha\beta x + \frac{3}{2}\beta^2$, where $\alpha, \beta, c_1, c_2, c_3 \in \Re \setminus \{0\}$.

(ii) $A = \text{constant}$, $B = e^{c_4 x + c_5}$, $f(R) = c_6 R + c_7$ and $R = \frac{3}{2} c_4^2$, where $c_4, c_5, c_6, c_7 \in \Re \setminus \{0\}$.

(iii) $B = \text{constant}$, $A = (c_1 x + c_2)^2$, $f(R) = (d_1 x + d_2) R + d_3$ and $R = 0$, where $c_1, c_2, d_1, d_2, d_3 \in \Re \setminus \{0\}$.

(iv) $B = \text{constant}$, $A = (c_1 x + c_2)^2$, $f(R) = d_1 R + d_2$ and $R = 0$, where $c_1, c_2, d_1, d_2 \in \Re \setminus \{0\}$.

(v) $A = B = e^{d_1 x + d_2}$, $f(R) = a_1 R + a_2$ and $R = 3d_1^2$, where $a_1, a_2, d_1, d_2 \in \Re \setminus \{0\}$.

(vi) $A = B = e^{(a_1 x + a_2)^{3/2}}$, $f(R) = (a_1 x + a_2) R + a_3$ and $R = 9a_1^2 \left[\frac{3(a_1 x + a_2)}{4} + \frac{1}{4\sqrt{(a_1 x + a_2)}} \right]$, where $a_1, a_2, a_3 \in \Re \setminus \{0\}$.

Next, we will make use of above information in the equations (2.3.4) to (2.3.13) to sorted out CVFs.

Case (i)

Constraints of this case are $A = \text{constant}$, $B = e^{\alpha x^2 + \beta x}$, $f(R) = \left(c_1 x + c_2\right) R + c_3$ and $R = 2(3x^2 + 2)\alpha^2 + 6\alpha\beta x + \frac{3}{2}\beta^2$, where $\alpha, \beta, c_1, c_2, c_3 \in \Re \setminus \{0\}$. The space-time (2.3.1) performing rescaling in the coordinate t take the form

$$ds^2 = -dt^2 + dx^2 + e^{\alpha x^2 + \beta x}[dy^2 + dz^2]. \qquad (2.3.19)$$

Now, we are interested in finding CVFs of space-time (2.3.19) with the help of equations (2.3.4) to (2.3.13). Skipping the process of integration, we arrive at $\psi = 0,$ which shows that no proper CVFs exist. CVFs here are KVFs which are given in equation (2.3.2).

Case (ii)

The values of metric coeffiecnts and related function $f(R)$ in the present case are $A = \text{constant},$ $B = e^{c_4 x + c_5},$ $f(R) = c_6 R + c_7$ and $R = \frac{3}{2} c_4^2,$ where $c_4, c_5, c_6, c_7 \in \Re \backslash \{0\}.$ Note that for $c_5 = 1$ and $c_6 = 0,$ solution corresponds to GR. The space-times (2.3.1) after an appropriate rescaling of t has the form:

$$ds^2 = -dt^2 + dx^2 + e^{c_4 x}[dy^2 + dz^2]. \qquad (2.3.20)$$

Again solving equations (2.3.4) to (2.3.13) with the help of space-time (2.3.20), we get fifteen linearly independent CVFs which are:

$$X^0 = \left(\frac{c_4^2 y^2 + c_4^2 z^2 + 4e^{-c_4 x}}{2c_4^2}\right)\left[-a_5 e^{\frac{c_4 \omega_1}{2}} + a_6 e^{-\frac{c_4 \omega_2}{2}}\right] + \left[a_7 e^{\frac{c_4 \omega_1}{2}} + a_8 e^{-\frac{c_4 \omega_2}{2}}\right] z$$
$$+ \left[a_9 e^{\frac{c_4 \omega_1}{2}} + a_{10} e^{-\frac{c_4 \omega_2}{2}}\right] y + a_1 e^{\frac{c_4 \omega_1}{2}} - a_2 e^{-\frac{c_4 \omega_2}{2}} + a_3,$$

$$X^1 = \left(\frac{-c_4^2 y^2 - c_4^2 z^2 + 4e^{-c_4 x}}{2c_4^2}\right)\left[a_5 e^{\frac{c_4 \omega_1}{2}} + a_6 e^{-\frac{c_4 \omega_2}{2}}\right] + \left[a_7 e^{\frac{c_4 \omega_1}{2}} - a_8 e^{-\frac{c_4 \omega_2}{2}}\right] z$$
$$+ \left[a_9 e^{\frac{c_4 \omega_1}{2}} - a_{10} e^{-\frac{c_4 \omega_2}{2}}\right] y + a_1 e^{\frac{c_4 \omega_1}{2}} + a_2 e^{-\frac{c_4 \omega_2}{2}} - \frac{2}{c_4} a_{11} z + \frac{2}{c_4} a_{12} y - \frac{2}{c_4} a_{13},$$

$$X^2 = \left(\frac{-c_4^2 y^2 + c_4^2 z^2 + 4e^{-c_4 x}}{2c_4^2}\right) a_{12} + \frac{2}{c_4}\left[a_9 e^{\frac{c_4 \omega_2}{2}} - a_{10} e^{-\frac{c_4 \omega_1}{2}}\right] + a_{11} yz + a_4 z + a_{13} y$$
$$- \frac{2}{c_4}\left[a_5 e^{\frac{c_4 \omega_2}{2}} + a_6 e^{-\frac{c_4 \omega_1}{2}}\right] y + a_{14},$$

$$X^3=\left(\frac{-c_4^2y^2+c_4^2z^2-4e^{-c_4x}}{2c_4^2}\right)a_{11}-\frac{2}{c_4}\left[a_5e^{\frac{c_4\omega_2}{2}}+a_6e^{-\frac{c_4\omega_1}{2}}\right]z-(a_{12}z+a_4)y$$
$$+\frac{2}{c_4}\left[a_7e^{\frac{c_4\omega_2}{2}}-a_8e^{-\frac{c_4\omega_1}{2}}\right]+a_{13}z+a_{15}, \qquad (2.3.21)$$

where $\omega_1=(t+x)$, $\omega_2=(t-x)$ and $a_i\in\Re$ with $i=1,2,3,...,15.$ Conformal factor in this case is

$$\psi=\left(\frac{c_4^2y^2+c_4^2z^2+4e^{-c_4x}}{4c_4}\right)\left[-a_5e^{\frac{c_4\omega_1}{2}}-a_6e^{-\frac{c_4\omega_2}{2}}\right]+\frac{c_4}{2}\left[a_7e^{\frac{c_4\omega_1}{2}}-a_8e^{-\frac{c_4\omega_2}{2}}\right]z$$
$$+\frac{c_4}{2}\left[a_9e^{\frac{c_4\omega_1}{2}}-a_{10}e^{-\frac{c_4\omega_2}{2}}\right]y+\frac{c_4}{2}\left[a_1e^{\frac{c_4\omega_1}{2}}+a_2e^{-\frac{c_4\omega_2}{2}}\right].$$

Case (iii)

Here, we have $B=$ constant, $A=(c_1x+c_2)^2$, $f(R)=(d_1x+d_2)R+d_3$ and $R=0$, where $c_1,c_2,d_1,d_2,d_3\in\Re\setminus\{0\}$. The space-times (2.3.1) after a suitable rescaling of y and z turn to be

$$ds^2=-(c_1x+c_2)^2dt^2+dx^2+dy^2+dz^2. \qquad (2.3.22)$$

Again the above space-times (2.3.22) is conformally flat, therefore admit fifteen independent CVFs which are:

$$X^0=\frac{1}{x}\left[\begin{array}{l}\left(\frac{x^2+y^2+z^2}{2}\right)\left(c_3e^t+c_4e^{-t}\right)+z\left(c_5e^t+c_6e^{-t}\right)+\\ y\left(c_7e^t+c_8e^{-t}\right)-c_9e^t+c_{10}e^{-t}\end{array}\right]+c_{11},$$

$$X^1=-\left[\begin{array}{l}\left(\frac{y^2+z^2-x^2}{2}\right)\left(c_3e^t-c_4e^{-t}\right)+z\left(c_5e^t-c_6e^{-t}\right)+\\ y\left(c_7e^t-c_8e^{-t}\right)-c_{12}xz+c_{13}xy\end{array}\right]$$
$$+c_9e^t+c_{10}e^{-t}+c_{14}x,$$

$$X^2=\left(\frac{x^2-y^2+z^2}{2}\right)c_{13}+c_{12}yz+c_{15}z+c_{14}y+xy\left(c_3e^t-c_4e^{-t}\right)$$
$$+x\left(c_7e^t-c_8e^{-t}\right)+c_{16},$$

$$X^3 = \left(\frac{z^2 - x^2 - y^2}{2}\right)c_{12} - c_{13}yz - c_{15}y + c_{14}z + xz\left(c_3 e^t - c_4 e^{-t}\right) + x\left(c_5 e^t - c_6 e^{-t}\right) + c_{17}, \tag{2.3.23}$$

where $c_i \in \Re$ with $i = 3, 4, 5, ..., 17.$ Note that for the sack of simplicity, the above components of CVFs have been found by assuming $c_1 = 1$ and $c_2 = 0.$ Conformal factor in this case is $\psi = x\left(c_3 e^t - c_4 e^{-t}\right) + c_{12}z - c_{13}y + c_{14}.$

Case (iv)

The values of space-time components along with the supplementary function $f(R)$ in this case are $B = \text{constant},$ $A = (c_1 x + c_2)^2,$ $f(R) = d_1 R + d_2$ and $R = 0,$ where $c_1, c_2, d_1, d_2 \in \Re \setminus \{0\}.$ It is to be noted that the space-time formulated in this case turn out to be same as we obtain in the previous case having equation (2.3.22). The only difference is found in the value of $f(R).$ CVFs for this case are given in equation (2.3.23). As the space-times in this case coincides with the space-times (2.3.22), therefore CVFs for this case are given by the equation (2.3.23).

Case (v)

In this case, we have $A = B = e^{d_1 x + d_2},$ $f(R) = a_1 R + a_2$ and $R = 3d_1^2,$ where $a_1, a_2, d_1, d_2 \in \Re \setminus \{0\}.$ The space-times (2.3.1) after an appropriate frame has the form

$$ds^2 = dx^2 + e^{d_1 x}[-dt^2 + dy^2 + dz^2]. \tag{2.3.24}$$

The above space-times (2.3.24) resembles with the space-like version of FLRW for $k = 0$ and is conformally flat therefore, CVFs turn out to be:

$$X^0=\left(\frac{(t^2+y^2+z^2)}{2}+\frac{2e^{-d_1x}}{d_1^2}\right)c_1-\frac{2}{d_1}\left(c_2t+c_3\right)e^{\frac{-d_1x}{2}}+\left(c_4t+c_5\right)z+$$
$$\left(-c_6t+c_7\right)y+c_8t+c_9,$$
$$X^1=\left(\frac{(t^2-y^2-z^2)e^{\frac{d_1x}{2}}}{2}+\frac{2e^{\frac{-d_1x}{2}}}{d_1^2}\right)c_2+\frac{2}{d_1}\left(c_6y-c_4z-c_1t-c_8\right)-$$
$$e^{\frac{d_1x}{2}}\left(c_{10}z+c_{11}y-c_3t-c_{12}\right),$$
$$X^2=\left(\frac{(z^2-y^2-t^2)}{2}+\frac{2e^{-d_1x}}{d_1^2}\right)c_6-\frac{2}{d_1}\left(c_2y+c_{11}\right)e^{\frac{-d_1x}{2}}+\left(c_4y+c_{13}\right)z+$$
$$\left(c_1t+c_8\right)y+c_7t+c_{14},$$
$$X^3=\left(\frac{(t^2-y^2+z^2)}{2}-\frac{2e^{-d_1x}}{d_1^2}\right)c_4-\frac{2}{d_1}\left(c_2z+c_{10}\right)e^{\frac{-d_1x}{2}}+\left(c_1t+c_8\right)z-$$
$$\left(c_6z+c_{13}\right)y+c_5t+c_{15}, \qquad (2.3.25)$$

Conformal factor in this case is

$$\psi=\left(\frac{-e^{\frac{-d_1x}{2}}}{d_1}+\frac{d_1(t^2-y^2-z^2)e^{\frac{d_1x}{2}}}{4}\right)c_2-\frac{d_1}{2}e^{\frac{d_1x}{2}}\left(c_{10}z+c_{11}y-c_3t-c_{12}\right),$$

where $c_i\in\Re$ with $i=1,2,3,\ldots,15.$

Case (vi)

Here, we have the values $A=B=e^{(a_1x+a_2)^{3/2}}$, $f(R)=(a_1x+a_2)R+a_3$ and

$R=9a_1^2\left[\frac{3(a_1x+a_2)}{4}+\frac{1}{4\sqrt{(a_1x+a_2)}}\right]$, where $a_1,a_2,a_3\in\Re\setminus\{0\}.$ The space-time (2.3.1) become

$$ds^2=dx^2+e^{(a_1x+a_2)^{3/2}}\left[-dt^2+dy^2+dz^2\right]. \qquad (2.3.26)$$

Again the above space-time (2.3.26) is conformally flat therefore admits fifteen independent CVFs which are

$$X^0 = \left(\frac{(t^2 + y^2 + z^2)}{2} + \int \left(e^{-D} \int e^{-D} dx \right) dx \right) c_1 + (c_6 t + c_7) \int e^{-D} dx + (c_2 t + c_3) z$$
$$- (c_4 t - c_5) y + c_8 t + c_9,$$

$$X^1 = e^D \begin{bmatrix} \left(\frac{(t^2 - y^2 - z^2)}{2} + \int \left(e^{-D} \int e^{-D} dx \right) dx \right) c_6 + c_2 \left(\int e^{-D} dx \right) z - c_4 \left(\int e^{-D} dx \right) y \\ -c_{11} y - c_{10} z + (c_1 t + c_8) \int e^{-D} dx + c_7 t + c_{12} \end{bmatrix},$$

$$X^2 = \left(\frac{(z^2 - y^2 - t^2)}{2} + \int \left(e^{-D} \int e^{-D} dx \right) dx \right) c_4 + (c_6 y + c_{11}) \left(\int e^{-D} dx \right) + (c_2 y + c_{13}) z$$
$$+ (c_1 t + c_8) y + c_5 t + c_{14},$$

$$X^3 = \left(\frac{(z^2 - y^2 + t^2)}{2} - \int \left(e^{-D} \int e^{-D} dx \right) dx \right) c_2 + (c_6 z + c_{10}) \left(\int e^{-D} dx \right) - (c_4 z + c_{13}) y \qquad (2.3.27)$$
$$+ (c_1 t + c_8) z + c_3 t + c_{15},$$

where $c_i \in \Re$ with $i = 1, 2, 3, \ldots, 15$ and $D = \frac{1}{2}(a_1 x + a_2)^{3/2}$. Conformal factor in this case is

$$\psi = D' e^D \begin{bmatrix} \left(\frac{(t^2 - y^2 - z^2)}{2} + \int \left(e^{-D} \int e^{-D} dx \right) dx \right) c_6 + c_2 \left(\int e^{-D} dx \right) z - c_4 \left(\int e^{-D} dx \right) y \\ -c_{11} y - c_{10} z + (c_1 t + c_8) \int e^{-D} dx + c_7 t + c_{12} \end{bmatrix}$$
$$+ c_2 z - c_4 y + c_1 t + c_6 \int e^{-D} dx + c_8.$$

2.4 Conformal Vector Fields of Static Cylindrically Symmetric space-times in the f(R) Theory of Gravity

A space-times with cylindrical symmetric static geometry is defined by (Stephani et al., 2003)

$$ds^2 = -e^{\nu(r)} dt^2 + dr^2 + e^{\lambda(r)} d\theta^2 + e^{\mu(r)} dz^2, \qquad (2.4.1)$$

where $\nu = \nu(r)$, $\lambda = \lambda(r)$ and $\mu = \mu(r)$ are non-zero functions of r only. Note that the space-times given in the above equation (2.4.1) have a general form because a SPS space-times could be obtained by setting $\lambda(r) = \mu(r)$. The linearly independent KVFs which the space-times (2.4.1) admits are (Stephani et al., 2003)

$$\frac{\partial}{\partial t},\ \frac{\partial}{\partial \theta},\ \frac{\partial}{\partial z}. \tag{2.4.2}$$

The Ricci tensors associated with the space-times (2.4.1) are of the form (Sharif, 2004)

$$R_{00} = \frac{1}{4}e^{\nu}\left[2\nu'' + \nu'^2 + \nu'\lambda' + \nu'\mu'\right],$$

$$R_{11} = -\frac{1}{4}\left[2\nu'' + 2\lambda'' + 2\mu'' + \nu'^2 + \lambda'^2 + \mu'^2\right],$$

$$R_{22} = -\frac{1}{4}e^{\lambda}\left[2\lambda'' + \lambda'^2 + \lambda'\mu' + \nu'\lambda'\right],$$

$$R_{33} = -\frac{1}{4}e^{\mu}\left[2\mu'' + \mu'^2 + \lambda'\mu' + \nu'\mu'\right], \tag{2.4.3}$$

where the notation prime is the derivative with respect to r. Expanding equation (1.12.1) and using equation (2.4.1), we obtain the conformal equations:

$$\nu' X^1 + 2X^0_{,0} = 2\psi, \tag{2.4.4}$$

$$e^{\nu} X^0_{,1} - X^1_{,0} = 0, \tag{2.4.5}$$

$$e^{\nu} X^0_{,2} - e^{\lambda} X^2_{,0} = 0, \tag{2.4.6}$$

$$e^{\nu} X^0_{,3} - e^{\mu} X^3_{,0} = 0, \tag{2.4.7}$$

$$X^1_{,1} = \psi, \tag{2.4.8}$$

$$X^1_{,2} + e^{\lambda} X^2_{,1} = 0, \tag{2.4.9}$$

$$X^1_{,3} + e^{\mu} X^3_{,1} = 0, \tag{2.4.10}$$

$$\lambda' X^1 + 2X^2_{,2} = 2\psi, \tag{2.4.11}$$

$$e^{\lambda} X^2_{,3} + e^{\mu} X^3_{,2} = 0, \tag{2.4.12}$$

$$\mu' X^1 + 2X^3_{,3} = 2\psi. \tag{2.4.13}$$

Solving equations (2.4.6), (2.4.7) and (2.4.12) simultaneously, we have $X^0=\int E^1(t,r,\theta)\,d\theta+E^2(t,r,z)$, where $E^1(t,r,\theta)$ and $E^2(t,r,z)$ are FOI. Now, using value of X^0 in equations (2.4.5), (2.4.6) and (2.4.7), we have

$$X^1=\int\Big(e^{\nu}\int E^1_r(t,r,\theta)d\theta\Big)dt+e^{\nu}\int E^2_r(t,r,z)dt+E^5(r,\theta,z),$$

$$X^2=e^{\nu-\lambda}\int E^1(t,r,\theta)\,dt+E^3(r,\theta,z),$$

$$X^3=e^{\nu-\mu}\int E^2_z(t,r,z)\,dt+E^4(r,\theta,z), \qquad (2.4.14)$$

where $E^i(r,\theta,z)$ with $i=3,4,5$ are FOI. Up to now, we have found components of CVFs in terms of the unknown functions of integration and metric components. In order to find CVF X in the theory under consideration, first we look for the solutions of EFEs defined in equation (1.13.2). Here, these solutions have been obtained by taking the matter part as perfect fluid

$$T_{ab}=(\rho+p)s_a s_b+pg_{ab}, \qquad (2.4.15)$$

where the symbols ρ, p and s_a are specified for the quantities density, pressure and four velocity vector respectively. Here, we define our velocity vector as $s_a=-e^{\frac{\nu(r)}{2}}\delta^0_a$. Surviving components of EMT defined for the space-times (2.4.1) are

$$T_{00}=\rho e^{\nu},\ T_{11}=p,\ T_{22}=pe^{\lambda},\ T_{33}=pe^{\mu}. \qquad (2.4.16)$$

Using equations (2.4.1), (2.4.3) and (2.4.16) in (1.13.2) after some algebraic manipulations, we have

$$F''-\frac{F'}{2}\nu'+\frac{F}{4}\left[2\lambda''+2\mu''+\mu'^2+\lambda'^2-\nu'\lambda'-\nu'\mu'\right]+k(\rho+p)=0. \qquad (2.4.17)$$

$$\frac{F'}{2}(\lambda'-\nu')+\frac{F}{4}\left[2\lambda''-2\nu''+\lambda'^2-\nu'^2+\lambda'\mu'-\nu'\mu'\right]+k(\rho+p)=0. \qquad (2.4.18)$$

$$\frac{F'}{2}(\mu'-\nu')+\frac{F}{4}\left[2\mu''-2\nu''+\mu'^2-\nu'^2+\lambda'\mu'-\nu'\lambda'\right]+k(\rho+p)=0. \qquad (2.4.19)$$

The above equations (2.4.17) to (2.4.19) are non-linear, therefore to make the computational process easy, we are classifying the space-times (2.4.1) by imposing some restrictions on the metric coefficients along with the condition given in $F' = 0$, (Shamir, 2016). The classification has following cases:

(i) $\nu = \nu(r)$, $\lambda = \lambda(r)$ and $\mu = \text{constant}$.

(ii) $\nu = \nu(r)$, $\mu = \mu(r)$ and $\lambda = \text{constant}$.

(iii) $\lambda = \lambda(r)$, $\mu = \mu(r)$ and $\nu = \text{constant}$.

(iv) $\nu = \nu(r)$ and $\lambda(r) = \mu(r)$.

(v) $\lambda = \lambda(r)$ and $\nu(r) = \mu(r)$.

(vi) $\mu = \mu(r)$ and $\nu(r) = \lambda(r)$.

(vii) $\nu = \nu(r)$ and $\mu = \lambda = \text{constant}$.

(viii) $\lambda = \lambda(r)$ and $\nu = \mu = \text{constant}$.

(ix) $\mu = \mu(r)$ and $\nu = \lambda = \text{constant}$.

(x) $\nu(r) = \lambda(r)$ and $\mu = \text{constant}$.

(xi) $\nu(r) = \mu(r)$ and $\lambda = \text{constant}$.

(xii) $\lambda(r) = \mu(r)$ and $\nu = \text{constant}$.

(xiii) $\nu = \lambda = \mu \neq \text{constant}$.

(xiv) $\nu = \lambda = \mu = \text{constant}$.

Case (i)

Constraints of this case are $\nu = \nu(r)$, $\lambda = \lambda(r)$ and $\mu = \text{constant}$. Under this assumption, equations (2.4.17) to (2.4.19) give

$$2\lambda'' - 2\nu'' - \nu'\lambda' - \nu'^2 + \lambda'^2 = 0. \tag{2.4.20}$$

Here, we assume solution of the form

$$\nu = n\lambda, \tag{2.4.21}$$

where $n \in \Re \setminus \{0,1\}$. Using equation (2.4.21) in equation (2.4.20), we have $\nu = \ln\left(\frac{4}{c_1 r + c_2}\right)^4$ and $\lambda = \ln\left(\frac{c_1 r + c_2}{4}\right)^4$. The space-times (2.4.1) with the suitable rescaling in the coordinate z has the shape given below:

$$ds^2 = -\left(\frac{4}{c_1 r + c_2}\right)^4 dt^2 + dr^2 + \left(\frac{c_1 r + c_2}{4}\right)^4 d\theta^2 + dz^2, \tag{2.4.22}$$

where $c_1, c_2 \in \Re$. Now, solving equations (2.4.4) to (2.4.13) with the help of space-time (2.4.22), we found that $\psi = c_3$, which implies that no proper CVFs exist. Here, CVFs coincide with the HVFs shown below:

$$X^0 = 3c_3 t + c_4,\ \ X^1 = \left(\frac{c_1 r + c_2}{c_1}\right)c_3,\ X^2 = -c_3\theta + c_5,\ \ X^3 = c_3 z + c_6, \tag{2.4.23}$$

where $c_i \in \Re,\ i = 3,4,5,6.$ The above space-times (2.4.22) admit four CVFs including minimal set of isometries listed in equation (2.4.2). The remaining is proper HVF given by

$$X^0 = 3c_3 t,\ \ X^1 = \left(\frac{c_1 r + c_2}{c_1}\right)c_3,\ \ X^2 = -c_3\theta,\ \ X^3 = c_3 z. \tag{2.4.24}$$

Generators of conformal algebra which are labelled by ξ_i in this case are

$$\xi_1 = 3t\frac{\partial}{\partial t} + \left(\frac{c_1 r + c_2}{c_1}\right)\frac{\partial}{\partial r} - \theta\frac{\partial}{\partial \theta} + z\frac{\partial}{\partial z},$$

$$\xi_2 = \frac{\partial}{\partial t}, \qquad \xi_3 = \frac{\partial}{\partial \theta}, \qquad \xi_4 = \frac{\partial}{\partial z}.$$

These generators form a closed form conformal algebra whose non-zero commutation relations are:

$$[\xi_1,\xi_2]=-3\xi_2,\ [\xi_2,\xi_1]=3\xi_2,$$

$$[\xi_1,\xi_3]=\xi_3,\ [\xi_3,\xi_1]=-\xi_3,$$

$$[\xi_1,\xi_4]=-\xi_4,\ [\xi_4,\xi_1]=\xi_4.$$

Case (ii)

Here, we have $\nu=\nu(r),\ \mu=\mu(r)$ and $\lambda=\text{constant}$. Using equation (2.4.17) to (2.4.19), we have $\nu=\ln\left(\frac{c_1r+c_2}{2}\right)^2$ and $\mu=\ln\left(\frac{c_3r+c_4}{2}\right)^2$. The space-times (2.4.1) after an appropriate rescaling of θ take the form

$$ds^2=-\left(\frac{c_1r+c_2}{2}\right)^2dt^2+dr^2+d\theta^2+\left(\frac{c_3r+c_4}{2}\right)^2dz^2, \tag{2.4.25}$$

where $c_i\in\Re,$ with $i=1,2,3,4.$ Now, solving equations (2.4.4) to (2.4.13) with the help of the space-time (2.4.25), we found that $\psi=0,$ indicating that the CVFs are the KVFs which are given in equation (2.4.2). These KVFs form a closed Lie algebra whose non-zero brackets are:

$$[\xi_1,\xi_2]=\xi_3,\ [\xi_2,\xi_3]=\xi_1,\ [\xi_1,\xi_3]=\xi_2,$$

$$[\xi_2,\xi_1]=-\xi_3,\ [\xi_3,\xi_2]=-\xi_1,\ [\xi_3,\xi_1]=-\xi_2.$$

Case (iii)

Constraints for this case are $\lambda=\lambda(r),\ \mu=\mu(r)$ and $\nu=\text{constant}.$ Using equations (2.4.17) to (2.4.19), we have $\mu=\ln\left(\frac{c_1r+c_2}{2}\right)^2$ and $\lambda=\ln\left(\frac{c_3r+c_4}{2}\right)^2$. The space-times (2.4.1) by a suitable rescaling in the coordinate t has the shape:

$$ds^2=-dt^2+dr^2+\left(\frac{c_3r+c_4}{2}\right)^2d\theta^2+\left(\frac{c_1r+c_2}{2}\right)^2dz^2, \tag{2.4.26}$$

where $c_i \in \Re,\ i=1,2,3,4.$ Again, we found that $\psi = 0,$ leading to the KVFs of equation (2.4.2). These KVFs also form a closed Lie algebra with the following brackets:

$$[\xi_1,\xi_2]=\xi_3,\ [\xi_2,\xi_3]=\xi_1,$$

$$[\xi_1,\xi_3]=\xi_2, [\xi_2,\xi_1]=-\xi_3,$$

$$[\xi_3,\xi_2]=-\xi_1,\ [\xi_3,\xi_1]=-\xi_2.$$

Case (iv)

Here, we have $\nu=\nu(r)$ and $\lambda(r)=\mu(r).$ Again, equations (2.4.17) to (2.4.19) yields

$$2\lambda''-\nu'\lambda'+2\nu''+\nu'^2=0. \qquad (2.4.27)$$

The solution of above equation (2.4.27) turns out to be $\nu=\ln(c_1r+c_2)^6$ and $\lambda=\ln(c_1r+c_2)^3.$ The space-times (2.3.1) take the form

$$ds^2=-(c_1r+c_2)^6dt^2+dr^2+(c_1r+c_2)^3\left[d\theta^2+dz^2\right], \qquad (2.4.28)$$

where $c_i \in \Re,$ with $i=1,2.$ Components of CVFs turn out to be

$$X^0=4c_3t+c_4,\ \ X^1=2c_3\left(\frac{c_1r+c_2}{c_1}\right),$$

$$X^2=-c_3\theta-c_5z+c_6,\ \ X^3=-c_3z+c_5\theta+c_7, \qquad (2.4.29)$$

where $c_i \in \Re,\ i=3,4,5,6,7.$ Checking the consistency of equations (2.4.4) to (2.4.13), conformal factor turns out to be $\psi=2c_3.$ The five CVFs given in equation (2.4.29) are further classified as four KVFs and one proper HVF. The proper HVF after subtracting KVF from (2.4.29) is

$$X^0=4c_3t,\ \ X^1=2c_3\left(\frac{c_1r+c_2}{c_1}\right),\ X^2=-c_3\theta,\ \ X^3=-c_3z. \qquad (2.4.30)$$

The generators of conformal algebra in this case are

$$\xi_1 = 4t\frac{\partial}{\partial t} + 2\left(\frac{c_1 r + c_2}{c_1}\right)\frac{\partial}{\partial r} - \theta\frac{\partial}{\partial \theta} - z\frac{\partial}{\partial z},$$

$$\xi_2 = \theta\frac{\partial}{\partial z} - z\frac{\partial}{\partial \theta},$$

$$\xi_3 = \frac{\partial}{\partial t}, \ \ \xi_4 = \frac{\partial}{\partial \theta}, \ \ \xi_5 = \frac{\partial}{\partial z}.$$

These generators constitute a closed form conformal algebra whose non-zero commutation relations satisfy:

$$[\xi_1, \xi_4] = \frac{-1}{4}\xi_4, \ \ [\xi_4, \xi_1] = \frac{1}{4}\xi_4,$$

$$[\xi_1, \xi_5] = \frac{-1}{4}\xi_5, \ \ [\xi_5, \xi_1] = \frac{1}{4}\xi_5,$$

$$[\xi_2, \xi_4] = -\xi_5, \ \ [X_4, X_2] = X_5,$$

$$[\xi_2, \xi_5] = \xi_4, \ \ [\xi_5, \xi_2] = -\xi_4,$$

$$[\xi_1, \xi_3] = -\xi_3, \ [\xi_3, \xi_1] = \xi_3.$$

Case (v)

In this case, solution of equations (2.4.17) to (2.4.19) is $\mu = \nu = \ln\left(\frac{c_1 r + c_2}{2}\right)^2$ and $\lambda = \ln\left(\frac{c_1 r + c_2}{2}\right)^4$, where $c_1, c_2 \in \Re$. The space-times (2.4.1) take the form

$$ds^2 = -\left(\frac{c_1 r + c_2}{2}\right)^2 dt^2 + dr^2 + \left(\frac{c_1 r + c_2}{2}\right)^4 d\theta^2 + \left(\frac{c_1 r + c_2}{2}\right)^2 dz^2, \quad (2.4.31)$$

This is the space-time which admits proper CVFs. The components of CVFs are:

$$X^0 = c_5 z + c_6,$$

$$X^1 = 2c_3\theta\left(\frac{c_1 r + c_2}{c_1}\right) - 16c_4\left(\frac{c_1 r + c_2}{c_1^5}\right),$$

$$X^2 = \left[\frac{16 - c_1^2\theta^2\left(c_1 r + c_2\right)^2}{c_1^2\left(c_1 r + c_2\right)^2}\right]c_3 + \frac{16c_4\theta}{c_1^4} + c_7,$$

$$X^3 = c_5 t + c_8, \qquad (2.4.32)$$

where $c_i \in \Re$ with $i = 3,4,5,6,7,8.$ Conformal factor in this case is $\psi = 2c_3\theta - \frac{16c_4}{c_1^4}$. This is the case in which the space-times (2.4.31) admit proper CVFs. After eliminating HVFs from equation (2.4.32), the proper CVF is

$$X^0 = 0,\ \ X^1 = 2c_3\theta\left(\frac{c_1 r + c_2}{c_1}\right),\ \ X^2 = \left[\frac{16 - c_1^2\theta^2\left(c_1 r + c_2\right)^2}{c_1^2\left(c_1 r + c_2\right)^2}\right]c_3,\ \ X^3 = 0. \qquad (2.4.33)$$

The above components of CVFs given in equation (2.4.32) has the generators:

$$\xi_1 = z\frac{\partial}{\partial t} + t\frac{\partial}{\partial z}, \qquad \xi_2 = 2\theta\left(\frac{c_1 r + c_2}{c_1}\right)\frac{\partial}{\partial r} + \left[\frac{16 - c_1^2\theta^2\left(c_1 r + c_2\right)^2}{c_1^2\left(c_1 r + c_2\right)^2}\right]\frac{\partial}{\partial \theta}, \qquad \xi_3 = \frac{\partial}{\partial t}, \qquad \xi_4 = \frac{\partial}{\partial \theta},$$

$\xi_5 = -16\left(\frac{c_1 r + c_2}{c_1^5}\right)\frac{\partial}{\partial r} - \frac{16\theta}{c_1^4}\frac{\partial}{\partial \theta}, \quad \xi_6 = \frac{\partial}{\partial z}$. These generators form a closed form conformal algebra whose non-zero commutation relations satisfy:

$$[\xi_1, \xi_2] = \frac{-16}{c_1^4}\xi_1,\ \ [\xi_1, \xi_5] = 2\xi_2,\ \ [\xi_2, \xi_5] = \frac{-16}{c_1^4}\xi_5,$$

$$[\xi_3, \xi_4] = 4\xi_6,\ \ [\xi_3, \xi_6] = 4\xi_4.$$

Case (vi)

Here, the values of metric components as a result of solution of equations (2.4.17) to (2.4.19) turn out to be $\lambda = \nu = \ln(c_1 r + c_2)$ and $\mu = \ln(c_1 r + c_2)^3$, where $c_1, c_2 \in \Re.$ The space-times (2.4.1) become

$$ds^2 = -(c_1 r + c_2)dt^2 + dr^2 + (c_1 r + c_2)d\theta^2 + (c_1 r + c_2)^3 dz^2, \qquad (2.4.34)$$

Now, solving equations (2.3.4) to (2.3.13) with the help of the space-time (2.3.34), we found that $\psi = -2c_3,$ which implies that no proper CVFs exist. Here, the CVFs are HVFs which are

$$X^0 = -c_3 t + c_4\theta + c_5, \;\; X^1 = -2c_3\left(\frac{c_1 r + c_2}{c_1}\right),$$

$$X^2 = -c_3\theta + c_4 t + c_6, \;\; X^3 = c_3 z + c_7, \qquad (2.4.35)$$

where $c_i \in \Re,\; i = 3,4,5,6,7.$ The above space-times (2.4.34) admit five CVFs including four isometries and one proper HVF. The proper HVF after subtracting KVFs from (2.4.35) is

$$X^0 = -c_3 t, \;\; X^1 = -2c_3\left(\frac{c_1 r + c_2}{c_1}\right), \; X^2 = -c_3\theta, \;\; X^3 = c_3 z. \qquad (2.4.36)$$

The generators of conformal algebra are

$\xi_1 = \theta\frac{\partial}{\partial t} + t\frac{\partial}{\partial\theta},\;\; \xi_2 = -t\frac{\partial}{\partial t} - 2\left(\frac{c_1 r + c_2}{c_1}\right)\frac{\partial}{\partial r} - \theta\frac{\partial}{\partial\theta} + z\frac{\partial}{\partial z},\;\; \xi_3 = \frac{\partial}{\partial t},\;\; \xi_4 = \frac{\partial}{\partial\theta},\;\; \xi_5 = \frac{\partial}{\partial z}.$ These generators form a closed form conformal algebra whose non-zero commutation relations satisfy:

$$[\xi_1, \xi_2] = -\xi_2,\; [\xi_1, \xi_4] = \xi_4,\; [\xi_1, \xi_5] = \xi_5,\; [\xi_3, \xi_4] = \xi_5,\; [\xi_3, \xi_5] = \xi_4.$$

Case (vii)

Here, the assumptions are $\nu=\nu(r)$ and the remaining two metric coeffiecnts μ and λ both are equal to some constant. The solution of equations (2.4.17) to (2.4.19) turns out to be $\nu=\ln\left(c_1 r+c_2\right)^2$, where $c_1, c_2 \in \Re (c_1 \neq 0)$. The space-times (2.4.1) with an appropriate rescaling of θ and z become

$$ds^2=-\left(c_1 r+c_2\right)^2 dt^2+dr^2+d\theta^2+dz^2. \qquad (2.4.37)$$

For simplicity, one can choose $c_1=1$ and $c_2=0$ so that the above space-times (2.4.37) being conformally flat admits fifteen independent CVFs which are

$$X^0=\frac{1}{r}\left[\begin{array}{l}\left(\dfrac{r^2+\theta^2+z^2}{2}\right)\left(c_3 e^t+c_4 e^{-t}\right)+z\left(c_5 e^t+c_{10} e^{-t}\right)+\theta\left(c_7 e^t+c_8 e^{-t}\right) \\ -c_9 e^t+c_{10} e^{-t}\end{array}\right]+c_{11},$$

$$X^1=-\left[\begin{array}{l}\left(\dfrac{\theta^2+z^2-r^2}{2}\right)\left(c_3 e^t-c_4 e^{-t}\right)+z\left(c_5 e^t-c_{10} e^{-t}\right)+\theta\left(c_7 e^t-c_8 e^{-t}\right) \\ -c_{12} r z+c_{13} r\theta\end{array}\right]+c_{14} r$$
$$+c_9 e^t+c_{10} e^{-t},$$

$$X^2=\left(\frac{r^2-\theta^2+z^2}{2}\right)c_{13}+c_{12}\theta z+c_{15} z+c_{14}\theta+r\theta\left(c_3 e^t-c_4 e^{-t}\right)+r\left(c_7 e^t-c_8 e^{-t}\right)$$
$$+c_{16},$$

$$X^3=\left(\frac{z^2-r^2-\theta^2}{2}\right)c_{12}-c_{13}\theta z-c_{15}\theta+c_{14} z+rz\left(c_3 e^t-c_4 e^{-t}\right)+ \qquad (2.4.38)$$
$$r\left(c_5 e^t-c_6 e^{-t}\right)+c_{17},$$

where $c_i \in \Re$ with $i = 3,4,5,...,17.$ Conformal factor in this case is $\psi = r\left(c_3 e^t - c_4 e^{-t}\right) + c_{12} z - c_{13}\theta + c_{14}$. Generators of conformal algebra are in this case are

$$\xi_1 = e^t\left(\frac{r^2+\theta^2+z^2}{2r}\right)\frac{\partial}{\partial t} - e^t\left(\frac{\theta^2+z^2-r^2}{2}\right)\frac{\partial}{\partial r} + e^t\theta r\frac{\partial}{\partial\theta} + e^t zr\frac{\partial}{\partial z},\quad \xi_2 = e^{-t}\left(\frac{\partial}{\partial r} + \frac{1}{r}\frac{\partial}{\partial t}\right),$$

$$\xi_3 = e^{-t}\left(\frac{r^2+\theta^2+z^2}{2r}\right)\frac{\partial}{\partial t} + e^{-t}\left(\frac{\theta^2+z^2-r^2}{2}\right)\frac{\partial}{\partial r} - e^{-t}\theta r\frac{\partial}{\partial\theta} - e^{-t} zr\frac{\partial}{\partial z},\quad \xi_4 = e^t\left(\frac{\partial}{\partial r} - \frac{1}{r}\frac{\partial}{\partial t}\right),$$

$$\xi_5 = \left(\frac{r^2-\theta^2+z^2}{2}\right)\frac{\partial}{\partial\theta} - \theta z\frac{\partial}{\partial z} - r\theta\frac{\partial}{\partial r},\quad \xi_6 = \left(\frac{z^2-r^2-\theta^2}{2}\right)\frac{\partial}{\partial z} + rz\frac{\partial}{\partial r} + \theta z\frac{\partial}{\partial\theta},\quad X_7 = \frac{\partial}{\partial t},$$

$$\xi_8 = e^t\left(\frac{z}{r}\frac{\partial}{\partial t} + P_1\right),\quad \xi_9 = e^{-t}\left(\frac{z}{r}\frac{\partial}{\partial t} + P_2\right),\quad \xi_{10} = e^t\left(\frac{\theta}{r}\frac{\partial}{\partial t} + P_3\right),\quad \xi_{11} = e^{-t}\left(\frac{\theta}{r}\frac{\partial}{\partial t} + \theta\frac{\partial}{\partial r} - r\frac{\partial}{\partial\theta}\right),$$

$$\xi_{12} = r\frac{\partial}{\partial r} + \theta\frac{\partial}{\partial\theta} + z\frac{\partial}{\partial z},\quad \xi_{13} = z\frac{\partial}{\partial y} - \theta\frac{\partial}{\partial z},\quad \xi_{14} = \frac{\partial}{\partial\theta}, \xi_{15} = \frac{\partial}{\partial z},\text{ where } P_1 = -z\frac{\partial}{\partial r} + r\frac{\partial}{\partial z},$$

$P_2 = z\frac{\partial}{\partial r} - r\frac{\partial}{\partial z}$ and $P_3 = -\theta\frac{\partial}{\partial r} + r\frac{\partial}{\partial\theta}$. One can find the Lie Algebra using the Lie bracket given in equation (1.11.3).

Case (viii)

Here, we have $\lambda = \lambda(r)$ and $\mu = \nu = \text{constant}.$ The solution of equations (2.4.17) to (2.4.19) turn out to be $\lambda = \ln\left(c_1 r + c_2\right)^2$, where $c_i \in \Re$ with $i = 1,2$ and $c_1 \neq 0.$ The space-times (2.4.1) after an appropriate rescaling of the coordinates t and z reduce to be

$$ds^2 = -dt^2 + dr^2 + \left(c_1 r + c_2\right)^2 d\theta^2 + dz^2. \tag{2.4.39}$$

Again, for calculation purpose one can choose $c_1 = 1$ and $c_2 = 0,$ the space-times (2.4.39) is conformally flat, therefore admit fifteen independent CVFs which are

$$X^0 = c_6\left(\frac{t^2+r^2+z^2}{2}\right) + c_3 tz + rt\omega_1 - r\omega_2 + c_7 t + c_8 z + c_{15},$$

$$X^1=\left(\frac{t^2+r^2-z^2}{2}\right)\omega_1-z\omega_3+c_3rz+c_6tr-c_{13}\cos\theta+c_{14}\sin\theta$$
$$-t\omega_2+c_7r,$$

$$X^2=\left(\frac{t^2-r^2-z^2}{2r}\right)\omega_4-\frac{z\omega_5}{r}+\frac{t\omega_6}{r}+\frac{\omega_7}{r}+c_{17},$$

$$X^3=c_3\left(\frac{t^2-r^2+z^2}{2}\right)+rz\omega_1+r\omega_3+c_6tz+c_8t+c_7z+c_{16}, \qquad (2.4.40)$$

where $c_i\in\Re$, with $i=3,4,5,...,17$, $\omega_1=c_4\sin\theta+c_5\cos\theta$, $\omega_2=c_{11}\cos\theta-c_{12}\sin\theta$, $\omega_3=c_9\sin\theta+c_{10}\cos\theta$, $\omega_4=c_4\cos\theta-c_5\sin\theta$, $\omega_5=c_9\cos\theta-c_{10}\sin\theta$, $\omega_6=c_{11}\sin\theta+c_{12}\cos\theta$, $\omega_7=c_{13}\sin\theta+c_{14}\cos\theta$. Conformal factor in this case is $\psi=c_3z+r\left(c_4\sin\theta+c_5\cos\theta\right)+c_6t+c_7$. Generators of conformal algebra are

$$\xi_1=tz\frac{\partial}{\partial t}+rz\frac{\partial}{\partial r}+\left(\frac{t^2+z^2-r^2}{2}\right)\frac{\partial}{\partial z},\qquad \xi_2=tr\frac{\partial}{\partial r}+tz\frac{\partial}{\partial z}+\left(\frac{t^2+z^2+r^2}{2}\right)\frac{\partial}{\partial t},\qquad \xi_3=z\frac{\partial}{\partial t}+t\frac{\partial}{\partial z},$$

$$\xi_4=t\frac{\partial}{\partial t}+r\frac{\partial}{\partial r}+z\frac{\partial}{\partial z},\quad \xi_5=-\cos\theta\frac{\partial}{\partial r}+\frac{\sin\theta}{r}\frac{\partial}{\partial\theta},\quad \xi_6=\frac{\partial}{\partial\theta},\quad \xi_7=\sin\theta\frac{\partial}{\partial r}+\frac{\cos\theta}{r}\frac{\partial}{\partial\theta},$$

$$\xi_8=tr\sin\theta\frac{\partial}{\partial t}+\left(\frac{t^2-z^2+r^2}{2}\right)\sin\theta\frac{\partial}{\partial r}+\left(\frac{t^2-z^2-r^2}{2r}\right)\cos\theta\frac{\partial}{\partial\theta}+rz\sin\theta\frac{\partial}{\partial z},\qquad \xi_9=\frac{\partial}{\partial t},$$

$$\xi_{10}=tr\cos\theta\frac{\partial}{\partial t}+\left(\frac{t^2-z^2+r^2}{2}\right)\cos\theta\frac{\partial}{\partial r}+\left(\frac{r^2+z^2-t^2}{2r}\right)\sin\theta\frac{\partial}{\partial\theta}+rz\cos\theta\frac{\partial}{\partial z},\qquad \xi_{11}=\frac{\partial}{\partial z},$$

$$\xi_{12}=-z\sin\theta\frac{\partial}{\partial r}+r\sin\theta\frac{\partial}{\partial z}-\frac{z}{r}\cos\theta\frac{\partial}{\partial\theta},\qquad \xi_{13}=-z\cos\theta\frac{\partial}{\partial r}+r\cos\theta\frac{\partial}{\partial z}+\frac{z}{r}\sin\theta\frac{\partial}{\partial\theta},$$

$$\xi_{14}=-r\cos\theta\frac{\partial}{\partial t}-t\cos\theta\frac{\partial}{\partial r}+\frac{t}{r}\sin\theta\frac{\partial}{\partial\theta},\qquad \xi_{15}=-r\sin\theta\frac{\partial}{\partial t}+t\sin\theta\frac{\partial}{\partial r}+\frac{t}{r}\cos\theta\frac{\partial}{\partial\theta}.$$

Adopting the same procedure one can find the Lie Algebra using the Lie bracket given in equation (1.11.3).

Case (ix)

Here, the constraints over metric coeffiecnts are $\mu = \mu(r)$ and $\nu = \lambda = \text{constant}$. The solution of equations (2.4.17) to (2.4.19) turn out to be $\mu = \ln\left(c_1 r + c_2\right)^2$, where $c_i \in \Re$ with $i = 1, 2$ and $c_1 \neq 0$. The space-times (2.4.1) with a suitable rescaling of t and θ takes the shape

$$ds^2 = -dt^2 + dr^2 + d\theta^2 + \left(c_1 r + c_2\right)^2 dz^2. \qquad (2.4.41)$$

Again, for calculation purpose one can choose $c_1 = 1$ and $c_2 = 0$, the space-times (2.4.41) is conformally flat, therefore admit fifteen independent CVFs which are

$$\begin{aligned}
X^0 &= c_4\left(\frac{t^2 + r^2 + z^2}{2}\right) - c_3 t\theta + tr\left[c_6 \cos z - c_7 \sin z\right] + \\
&\quad r\left[c_{10} \sin z - c_{14} \cos z\right] + c_5 t + c_{13}\theta + c_{15}, \\
X^1 &= \left(\frac{t^2 + r^2 - \theta^2}{2}\right)\left[c_6 \cos z - c_7 \sin z\right] - \theta\left[c_8 \cos z - c_{11} \sin z\right] \\
&\quad - c_3 r\theta + c_4 tr + t\left[c_{10} \sin z - c_{14} \cos z\right] + c_5 r - c_9 \cos z + c_{12} \sin z, \\
X^2 &= c_3\left(\frac{r^2 - t^2 - \theta^2}{2}\right) + c_4 t\theta + r\theta\left[c_6 \cos z - c_7 \sin z\right] + \\
&\quad r\left[c_8 \cos z - c_{11} \sin z\right] + c_{13} t + c_5\theta + c_{16}, \\
X^3 &= \left(\frac{r^2 - t^2 + \theta^2}{r}\right)\left[c_6 \sin z + c_7 \cos z\right] + \frac{\theta}{r}\left[c_8 \sin z + c_{11} \cos z\right] + \\
&\quad \frac{t}{r}\left[c_{10} \cos z + c_{14} \sin z\right] + \frac{1}{r}\left[c_9 \sin z + c_{12} \cos z\right] + c_{17},
\end{aligned} \qquad (2.4.42)$$

where $c_i \in \Re$ with $i = 3, 4, 5, \ldots, 17.$ Conformal factor in this case is $\psi = -c_3\theta + r\left(c_6 \cos z + c_7 \sin z\right) + c_4 t + c_5$. Generators of conformal algebra in this case are

$\xi_1 = -t\theta\frac{\partial}{\partial t} - r\theta\frac{\partial}{\partial r} + \left(\frac{r^2 - t^2 - \theta^2}{2}\right)\frac{\partial}{\partial \theta}$, $\quad \xi_2 = \frac{\partial}{\partial \theta}$, $\quad \xi_3 = tr\frac{\partial}{\partial r} + t\theta\frac{\partial}{\partial \theta} + \left(\frac{t^2 + \theta^2 + r^2}{2}\right)\frac{\partial}{\partial t}$,

$\xi_4 = -\cos z\frac{\partial}{\partial r} + \frac{\sin z}{r}\frac{\partial}{\partial z}$, $\quad \xi_5 = \sin z\frac{\partial}{\partial r} + \frac{\cos z}{r}\frac{\partial}{\partial z}$, $\quad \xi_6 = -r\cos z\frac{\partial}{\partial t} - t\cos z\frac{\partial}{\partial r} + \frac{t}{r}\sin z\frac{\partial}{\partial z}$,

$\xi_7 = \frac{\partial}{\partial t}$, $\quad \xi_8 = \theta\sin z\frac{\partial}{\partial r} - r\sin z\frac{\partial}{\partial \theta} + \frac{\theta}{r}\cos z\frac{\partial}{\partial z}$, $\quad \xi_9 = r\sin z\frac{\partial}{\partial t} + t\sin z\frac{\partial}{\partial r} + \frac{t}{r}\cos z\frac{\partial}{\partial z}$,

$\xi_{10} = -\theta\sin z\frac{\partial}{\partial r} + r\cos z\frac{\partial}{\partial \theta} + \frac{\theta}{r}\sin z\frac{\partial}{\partial z}$, $\quad \xi_{11} = t\frac{\partial}{\partial t} + r\frac{\partial}{\partial r} + \theta\frac{\partial}{\partial \theta}$, $\quad \xi_{12} = \theta\frac{\partial}{\partial t} + t\frac{\partial}{\partial \theta}$,

$\xi_{13} = -tr\sin z\frac{\partial}{\partial t} + \sin z\left(\frac{\theta^2 - t^2 - r^2}{2}\right)\frac{\partial}{\partial r} + \cos z\left(\frac{r^2 + \theta^2 - t^2}{r}\right)\frac{\partial}{\partial z} - r\theta\sin z\frac{\partial}{\partial \theta}$, $\quad \xi_{14} = \frac{\partial}{\partial z}$,

$\xi_{15} = tr\cos z\frac{\partial}{\partial t} + \cos z\left(\frac{t^2 - \theta^2 + r^2}{2}\right)\frac{\partial}{\partial r} + r\theta\cos z\frac{\partial}{\partial \theta} + \sin z\left(\frac{r^2 - t^2 - \theta^2}{r}\right)\frac{\partial}{\partial z}$. Adopting the same procedure one can find the Lie Algebra using the Lie bracket given in equation (1.11.3).

Case (x)

Here, we have $\nu(r) = \lambda(r)$ and $\mu = \text{constant}$. Equations (2.4.17) to (2.4.19) implies $\nu'^2 = 0$, therefore $\nu = \alpha$, where $\alpha \in \Re^+$. The space-times (2.4.1) with the rescaling of z turn to be

$$ds^2 = dr^2 + dz^2 + \alpha[-dt^2 + d\theta^2]. \qquad (2.4.43)$$

Solving equations (2.4.4) to (2.4.13) for the space-times (2.4.43), we obtain fifteen independent CVFs which are

$$X^0 = \left(\frac{\alpha t^2 + r^2 + \alpha\theta^2 + z^2}{2\alpha}\right)c_1 + c_3 t\theta + c_2 tr - c_4 tz + c_5 t + \frac{1}{\alpha}c_6 r + c_7\theta + c_8 z + c_9,$$

$$X^1 = \left(\frac{\alpha t^2 + r^2 - \alpha\theta^2 - z^2}{2}\right)c_2 + c_1 tr + c_3 r\theta - c_4 rz + c_6 t + c_5 r + c_{10}\theta + c_{11}z + c_{12},$$

$$X^2 = \left(\frac{\alpha t^2 - r^2 + \alpha\theta^2 - z^2}{2\alpha}\right)c_3 + c_1 t\theta + c_2 r\theta - c_4\theta z + c_7 t - \frac{1}{\alpha}c_{10}r + c_5\theta + c_{13}z + c_{14},$$

$$X^3 = \left(\frac{r^2 - \alpha t^2 + \alpha\theta^2 - z^2}{2}\right)c_4 + c_1 tz + c_2 rz + c_3\theta z + \alpha c_8 t - c_{11}r - \alpha c_{13}\theta + c_5 z + c_{15}, \qquad (2.4.44)$$

where $c_i \in \Re$ with $i = 1, 2, 3, \ldots, 15$. Conformal factor in this case is $\psi = c_1 t + c_2 r + c_3 \theta - c_4 z + c_5$. The generators of conformal algebra in this case are

$$X_1 = \left(\frac{\alpha t^2 + r^2 + \alpha\theta^2 + z^2}{2\alpha} \right) \frac{\partial}{\partial t} + rt \frac{\partial}{\partial r} + t\theta \frac{\partial}{\partial \theta} + tz \frac{\partial}{\partial z}, \quad X_2 = t \frac{\partial}{\partial t} + r \frac{\partial}{\partial r} + \theta \frac{\partial}{\partial \theta} + z \frac{\partial}{\partial z},$$

$$X_3 = \left(\frac{\alpha t^2 + r^2 - \alpha\theta^2 - z^2}{2} \right) \frac{\partial}{\partial r} + rt \frac{\partial}{\partial t} + r\theta \frac{\partial}{\partial \theta} + rz \frac{\partial}{\partial z}, \quad X_4 = \frac{r}{\alpha} \frac{\partial}{\partial t} + t \frac{\partial}{\partial r}, \quad X_5 = \frac{\partial}{\partial t},$$

$$X_6 = \left(\frac{\alpha t^2 - r^2 + \alpha\theta^2 - z^2}{2\alpha} \right) \frac{\partial}{\partial \theta} + r\theta \frac{\partial}{\partial r} + t\theta \frac{\partial}{\partial t} + \theta z \frac{\partial}{\partial z}, \quad X_7 = t \frac{\partial}{\partial \theta} + \theta \frac{\partial}{\partial t}, \quad X_8 = \frac{\partial}{\partial r},$$

$$X_9 = \left(\frac{r^2 - \alpha t^2 + \alpha\theta^2 - z^2}{2} \right) \frac{\partial}{\partial z} - rz \frac{\partial}{\partial r} - tz \frac{\partial}{\partial z} - tz \frac{\partial}{\partial t}, \quad X_{10} = z \frac{\partial}{\partial t} + \alpha t \frac{\partial}{\partial z}, \quad X_{11} = \frac{\partial}{\partial \theta},$$

$$X_{12} = \theta \frac{\partial}{\partial r} - \frac{r}{\alpha} \frac{\partial}{\partial \theta}, \quad X_{13} = z \frac{\partial}{\partial r} - r \frac{\partial}{\partial z}, \quad X_{14} = z \frac{\partial}{\partial \theta} - \alpha\,\theta \frac{\partial}{\partial z}, \quad X_{15} = \frac{\partial}{\partial z}.$$

Conformal algebra may be discussed by using the above generators.

Case (xi)

Here, we take $\nu(r) = \mu(r)$ and $\lambda =$ constant. Again, solution of equations (2.4.17) to (2.4.19), give $\nu'^2 = 0$, therefore $\nu = \alpha$, where $\alpha \in \Re^+$. The space-times (2.4.1) after suitable rescaling of θ take the form

$$ds^2 = dr^2 + d\theta^2 + \alpha[-dt^2 + dz^2]. \qquad (2.4.45)$$

The above space-times (2.4.45) being conformally flat again admit fifteen independent CVFs which are

$$X^0 = \left(\frac{\alpha t^2 + r^2 + \theta^2 + \alpha z^2}{2\alpha^2} \right) c_4 + \frac{c_1 t\theta}{\alpha} + \frac{c_5 tr}{\alpha} + \frac{c_2 tz}{\alpha} + \frac{c_6 t}{\alpha} + \frac{c_{13} r}{\alpha} + \frac{c_7 \theta}{\alpha} + \frac{c_{10} z}{\alpha} + c_{14},$$

$$X^1 = \left(\frac{\alpha t^2 + r^2 - \theta^2 - \alpha z^2}{2\alpha} \right) c_5 + \frac{c_4 tr}{\alpha} + \frac{c_1 r\theta}{\alpha} + \frac{c_2 rz}{\alpha} + \frac{c_6 r}{\alpha} + c_{13} t - c_8 \theta - c_{11} z + c_{15},$$

$$X^2=\left(\frac{\alpha t^2-r^2+\theta^2-\alpha z^2}{2\alpha}\right)c_1+\frac{c_4 t\theta}{\alpha}+\frac{c_5 r\theta}{\alpha}+\frac{c_2\theta z}{\alpha}+\frac{c_6\theta}{\alpha}+c_7 t+c_8 r-c_3 z+c_9,$$

$$X^3=\left(\frac{\alpha t^2+\alpha z^2-r^2-\theta^2}{2\alpha^2}\right)c_2+\frac{c_4 tz}{\alpha}+\frac{c_5 rz}{\alpha}+\frac{c_1\theta z}{\alpha}+\frac{c_{10}t}{\alpha}+\frac{c_{11}r}{\alpha}+\frac{c_3\theta}{\alpha}+\frac{c_6 z}{\alpha}+c_{12}, \quad (2.4.46)$$

where $c_i\in\Re$ with $i=1,2,3,...,15$. Conformal factor in this case is $\psi=c_4 t+c_5 r+c_1\theta-c_2 z+c_6$. The generators of conformal algebra in this case are

$$X_1=\left(\frac{\alpha t^2+r^2+\theta^2+\alpha z^2}{2\alpha^2}\right)\frac{\partial}{\partial t}+\frac{rt}{\alpha}\frac{\partial}{\partial r}+\frac{t\theta}{\alpha}\frac{\partial}{\partial\theta}+\frac{tz}{\alpha}\frac{\partial}{\partial z},\quad X_2=\frac{t}{\alpha}\frac{\partial}{\partial t}+\frac{r}{\alpha}\frac{\partial}{\partial r}+\frac{\theta}{\alpha}\frac{\partial}{\partial\theta}+\frac{z}{\alpha}\frac{\partial}{\partial z},$$

$$X_3=\left(\frac{\alpha t^2+r^2-\theta^2-\alpha z^2}{2\alpha}\right)\frac{\partial}{\partial r}+\frac{rt}{\alpha}\frac{\partial}{\partial t}+\frac{r\theta}{\alpha}\frac{\partial}{\partial\theta}+\frac{rz}{\alpha}\frac{\partial}{\partial z},\quad X_4=\frac{r}{\alpha}\frac{\partial}{\partial t}+t\frac{\partial}{\partial r},\quad X_5=\frac{\partial}{\partial t},$$

$$X_6=\left(\frac{\alpha t^2-r^2+\theta^2-\alpha z^2}{2\alpha}\right)\frac{\partial}{\partial\theta}+\frac{r\theta}{\alpha}\frac{\partial}{\partial r}+\frac{t\theta}{\alpha}\frac{\partial}{\partial t}+\frac{\theta z}{\alpha}\frac{\partial}{\partial z},\quad X_7=t\frac{\partial}{\partial\theta}+\frac{\theta}{\alpha}\frac{\partial}{\partial t},\quad X_8=\frac{\partial}{\partial r},$$

$$X_9=\left(\frac{\alpha t^2+\alpha z^2-\theta^2-r^2}{2\alpha^2}\right)\frac{\partial}{\partial z}+\frac{rz}{\alpha}\frac{\partial}{\partial r}+\frac{t\theta}{\alpha}\frac{\partial}{\partial\theta}+\frac{tz}{\alpha}\frac{\partial}{\partial t},\quad X_{10}=\frac{z}{\alpha}\frac{\partial}{\partial t}+\frac{t}{\alpha}\frac{\partial}{\partial z},\quad X_{11}=\frac{\partial}{\partial\theta},$$

$$X_{12}=r\frac{\partial}{\partial\theta}-\theta\frac{\partial}{\partial r},\quad X_{13}=\frac{r}{\alpha}\frac{\partial}{\partial z}-z\frac{\partial}{\partial r},\quad X_{14}=\frac{\theta}{\alpha}\frac{\partial}{\partial z}-z\frac{\partial}{\partial\theta},\quad X_{15}=\frac{\partial}{\partial z}.$$

Conformal Algebra may also be discussed using the above generators.

Case (xii)

Here, we have $\lambda(r)=\mu(r)$ and $\nu=$ constant. The solution of equations (2.4.17) to (2.4.19), give $\lambda(r)=\mu(r)=e^{a_1 r+a_2}$ where $a_1,\ a_2\in\Re$. The space-times (2.4.1) with an appropriate form

$$ds^2=-dt^2+dr^2+e^{a_1 r}[d\theta^2+dz^2]. \quad (2.4.47)$$

CVFs in this case are:

$$X^0=\left(\frac{a_1^2\theta^2+a_1^2z^2+4e^{-a_1r}}{2a_1^2}\right)\left[-c_5e^{\frac{a_1w_1}{2}}+c_6e^{-\frac{a_1w_2}{2}}\right]+\left[c_7e^{\frac{a_1w_1}{2}}+c_8e^{-\frac{a_1w_2}{2}}\right]z$$
$$+\left[c_9e^{\frac{a_1w_1}{2}}+c_{10}e^{-\frac{a_1w_2}{2}}\right]\theta+c_1e^{\frac{a_1w_1}{2}}-c_2e^{-\frac{a_1w_2}{2}}+c_3,$$

$$X^1=\left(\frac{-a_1^2\theta^2-a_1^2z^2+4e^{-a_1r}}{2a_1^2}\right)\left[c_5e^{\frac{a_1w_1}{2}}+c_6e^{-\frac{a_1w_2}{2}}\right]+\left[c_7e^{\frac{a_1w_1}{2}}-c_8e^{-\frac{a_1w_2}{2}}\right]z$$
$$+\left[c_9e^{\frac{a_1w_1}{2}}-c_{10}e^{-\frac{a_1w_2}{2}}\right]\theta+c_1e^{\frac{a_1w_1}{2}}+c_2e^{-\frac{a_1w_2}{2}}-\frac{2}{a_1}c_{11}z+\frac{2}{a_1}c_{12}\theta-\frac{2}{a_1}c_{13},$$

$$X^2=\left(\frac{-a_1^2\theta^2+a_1^2z^2+4e^{-a_1r}}{2a_1^2}\right)c_{12}+\frac{2}{a_1}\left[c_9e^{\frac{a_1w_2}{2}}-c_{10}e^{-\frac{a_1w_1}{2}}\right]+c_{11}\theta z-$$
$$c_4z+c_{13}\theta-\frac{2}{a_1}\left[c_5e^{\frac{a_1w_2}{2}}+c_6e^{-\frac{a_1w_1}{2}}\right]\theta+c_{14},$$

$$X^3=\left(\frac{-a_1^2\theta^2+a_1^2z^2-4e^{-a_1r}}{2a_1^2}\right)c_{11}-\frac{2}{a_1}\left[c_5e^{\frac{a_1w_2}{2}}+c_6e^{-\frac{a_1w_1}{2}}\right]z-c_{12}\theta z+$$
$$c_4\theta+c_{13}z+\frac{2}{a_1}\left[c_7e^{\frac{a_1w_2}{2}}-c_8e^{-\frac{a_1w_1}{2}}\right]+c_{15}, \tag{2.4.48}$$

where $w_1=(t+r),\ w_2=(t-r)$ and $c_i\in\Re$ and $i=1,2,3,...,15.$ Conformal factor in this case is

$$\psi=\left(\frac{a_1^2\theta^2+a_1^2z^2+4e^{-a_1r}}{4a_1}\right)\left[-c_5e^{\frac{a_1w_1}{2}}-c_6e^{-\frac{a_1w_2}{2}}\right]+\frac{a_1}{2}\left[c_7e^{\frac{a_1w_1}{2}}-c_8e^{-\frac{a_1w_2}{2}}\right]z$$
$$+\frac{a_1}{2}\left[c_9e^{\frac{a_1w_1}{2}}-c_{10}e^{-\frac{a_1w_2}{2}}\right]\theta+\frac{a_1}{2}\left[c_1e^{\frac{a_1w_1}{2}}+c_2e^{-\frac{a_1w_2}{2}}\right].$$

Tracking the lines discussed previously, one find the Lie Algebra of obtained vector fields.

Case (xiii)

In this case $\nu=\lambda=\mu\neq$ constant. Again Equations (2.4.17) to (2.4.19) implies $\nu''=0,$ therefore $\nu=(d_1r+d_2)$ where $d_1,d_2\in\Re(d_1\neq 0).$ The space-times (2.4.1) in an appropriate frame take the form

$$ds^2=dr^2+e^{d_1r}\left[-dt^2+d\theta^2+dz^2\right]. \tag{2.4.49}$$

The above space-time (2.3.49) is a space-like version of FLRW for $k=0,$ therefore admits fifteen independent CVFs which are

$$\begin{aligned}
X^0&=\left(\frac{(t^2+\theta^2+z^2)}{2}+\frac{2e^{-d_1r}}{d_1^2}\right)c_1-\frac{2}{d_1}\left(c_2t+c_3\right)e^{\frac{-d_1r}{2}}+\left(c_4t+c_5\right)z+\\
&\quad\left(-c_6t+c_7\right)\theta+c_8t+c_9,\\
X^1&=\left[\frac{(t^2-\theta^2-z^2)e^{\frac{d_1r}{2}}}{2}+\frac{2e^{\frac{-d_1r}{2}}}{d_1^2}\right]c_2+\frac{2}{d_1}\left(c_6\theta-c_4z-c_1t-c_8\right)-\\
&\quad e^{\frac{d_1r}{2}}\left(c_{10}z+c_{11}\theta-c_3t-c_{12}\right),\\
X^2&=\left(\frac{(z^2-\theta^2-t^2)}{2}+\frac{2e^{-d_1r}}{d_1^2}\right)c_6-\frac{2}{d_1}\left(c_2\theta+c_{11}\right)e^{\frac{-d_1r}{2}}+\left(c_4\theta+c_{13}\right)z+\\
&\quad\left(c_1t+c_8\right)\theta+c_7t+c_{14},\\
X^3&=\left[\frac{(t^2-\theta^2+z^2)}{2}-\frac{2e^{-d_1r}}{d_1^2}\right]c_4-\frac{2}{d_1}\left(c_2z+c_{10}\right)e^{\frac{-d_1r}{2}}+\left(c_1t+c_8\right)z-\\
&\quad\left(c_6z+c_{13}\right)\theta+c_5t+c_{15},
\end{aligned} \tag{2.4.50}$$

Conformal factor in this case is

$$\psi=\left(\frac{-e^{\frac{-d_1r}{2}}}{d_1}+\frac{d_1(t^2-\theta^2-z^2)e^{\frac{d_1r}{2}}}{4}\right)c_2-\frac{d_1}{2}e^{\frac{d_1r}{2}}\left(c_{10}z+c_{11}\theta-c_3t-c_{12}\right),$$

where $c_i \in \Re$ with $i = 1, 2, 3, ..., 15.$ The generators of conformal algebra in this case are

$$X_1 = \left(\frac{(t^2 + \theta^2 + z^2)}{2} + \frac{2e^{-d_1 r}}{d_1^2} \right) \frac{\partial}{\partial t} - \frac{2t}{d_1} \frac{\partial}{\partial r} + t\theta \frac{\partial}{\partial \theta} + tz \frac{\partial}{\partial z}, \qquad X_2 = t\frac{\partial}{\partial t} - \frac{2}{d_1}\frac{\partial}{\partial r} + \theta \frac{\partial}{\partial \theta} + z\frac{\partial}{\partial z},$$

$$X_3 = \left(\frac{(t^2 - \theta^2 - z^2)e^{\frac{d_1 r}{2}}}{2} + \frac{2e^{\frac{-d_1 r}{2}}}{d_1^2} \right) \frac{\partial}{\partial r} - \frac{2te^{\frac{-d_1 r}{2}}}{d_1} \frac{\partial}{\partial t} - \frac{2\theta e^{\frac{-d_1 r}{2}}}{d_1} \frac{\partial}{\partial \theta} - \frac{2ze^{\frac{-d_1 r}{2}}}{d_1} \frac{\partial}{\partial z}, \qquad X_4 = \frac{\partial}{\partial t},$$

$$X_5 = \left(\frac{(t^2 - \theta^2 + z^2)}{2} - \frac{2e^{-d_1 r}}{d_1^2} \right) \frac{\partial}{\partial z} - \frac{2z}{d_1} \frac{\partial}{\partial r} + tz \frac{\partial}{\partial t} + \theta z \frac{\partial}{\partial \theta}, \qquad X_6 = t\frac{\partial}{\partial \theta} + \theta \frac{\partial}{\partial t}, \qquad X_7 = \frac{\partial}{\partial \theta},$$

$$X_8 = \left(\frac{(z^2 - \theta^2 - t^2)}{2} + \frac{2e^{-d_1 r}}{d_1^2} \right) \frac{\partial}{\partial \theta} + \frac{2\theta}{d_1} \frac{\partial}{\partial r} - z\theta \frac{\partial}{\partial z} - t\theta \frac{\partial}{\partial t}, \qquad X_9 = z\frac{\partial}{\partial t} + t\frac{\partial}{\partial z}, \qquad X_{10} = \frac{\partial}{\partial z},$$

$$X_{11} = -\theta e^{\frac{d_1 r}{2}} \frac{\partial}{\partial r} + \frac{-2}{d_1} e^{\frac{-d_1 r}{2}} \frac{\partial}{\partial \theta}, \; X_{12} = -ze^{\frac{d_1 r}{2}} \frac{\partial}{\partial r} + \frac{-2}{d_1} e^{\frac{-d_1 r}{2}} \frac{\partial}{\partial z}, \qquad X_{13} = z\frac{\partial}{\partial \theta} - \theta \frac{\partial}{\partial z},$$

$X_{14} = te^{\frac{d_1 r}{2}} \frac{\partial}{\partial r} + \frac{-2e^{\frac{-d_1 r}{2}}}{d_1} \frac{\partial}{\partial t}, \qquad X_{15} = e^{\frac{k_1 r}{2}} \frac{\partial}{\partial r}.$ Additionally, one can find the conformal algebra by using the above generators.

2.5 Summary

In this chapter, we have studied CVFs of SSS, SPS and SCS space-times in the f(R) theory of gravity. The method to find CVFs for the stated space-times is direct integration. The results of this study are:

(a) In the SSS space-times, we have used the results of two papers (Capozziello et al., 2012 and Amirabi et al., 2016) to find CVFs. Six cases were discussed. The findings of the study of static spherically symmetric space-times are as under:

(a-1) The space-times in the cases (i), (ii), (iii), (iv) and (vi) do not admit proper CVFs. CVFs for these cases become KVFs which are shown in equation (2.2.2).

(a-2) The space-time in the case (v) admits proper HVFs which are presented in equation (2.2.19).

(b) In the section (2.3), we have investigated CVFs of dust SPS space-times in the f(R) gravity setup. Our drive was two folded: firstly, we found some solutions of EFFs in the f(R) theory of gravity. Secondly, we obtained the CVFs of the resulting solutions. To explore the solutions, we have used dust matter as a source of EMT. It is important to mention that these solutions are deduced by imposing some restrictions on the derivative of f(R) i.e. F(R). In general, six cases (i) to (vi) were discussed which produce the following results:

(b-1) In the case (i), CVFs become KVFs. The KVFs for this case are given in equation (2.3.2) and the space-time in this case is takes the form given by the equation (2.3.19).

(b-2) The space-times in the cases (ii) to (vi) are conformally flat and clearly admit fifteen independent CVFs, The space-times are given in equations (2.3.20), (2.3.22), (2.3.24) and (2.3.26) and the expressions for CVFs are shown by the equations (2.3.21), (2.3.23), (2.3.25) and (2.3.27). Note that in the cases (iii) and (iv), we obtain exactly the same space-times which is given in equation (2.3.22) with the only difference in the value of $f(R)$. In both the cases, CVFs are given in equation (2.3.23).

(c) In the third section, we have classified general form of SCS space-times in the view of f(R) theory of gravity by their CVFs. The SCS solutions of EFEs belongs to the general class of space-time in the sense that these are further linked to generate SPS solutions which happen to lie in the framework of SCS space-times under some particular circumstances. These space-times have a productive applications in the field of black holes thermodynamics, electric and magnetic strings. The models having cylindrical symmetry are also used to discuss the interface between matter and GWs. Due to having cylindrical symmetry the waves associated with them are termed as cylindrical GWs. Going through the vast amount of applications are information possessed by the SCS space-times, a step by step analysis have been made in the theory of f(R) gravity to classify the space-time under consideration. Additionally, the source for providing the gravitational contribution is assumed to be perfect fluid. In this classification, there exist thirteen cases which on further study provide the following results:

(c-1) The space-times in the cases (i), (iv) and (vi), admit proper HVFs rather than proper CVFs. The space-times admitting such vector fields are shown in equations (2.4.22), (2.4.28) and (2.4.34). The proper HVFs for these cases are given in equations (2.4.24), (2.4.30) and (2.4.36).

(c-2) In the cases (ii) and (iii), the space-times admit three linearly independent KVFs. The space-times admitting such vector fields are (2.4.25) and (2.4.26) and the KVFs for these cases are given in equation (2.4.2).

(c-3) The space-time given by the case (v) admits proper CVFs. This is the space-time (2.4.31) and the proper CVFs is given in equation (2.4.33).

(c-4) The space-times studied in the cases (vii) to (xiii) become conformally flat, therefore admit fifteen independent CVFs. These are the space-times (2.4.37), (2.4.39), (2.4.41), (2.4.43), (2.4.45), (2.4.47) and (2.4.49). The CVFs for these cases are expressed by the equations (2.4.38), (2.4.40), (2.4.42), (2.4.44), (2.4.46), (2.4.48) and (2.4.50).

Chapter 3

Conformal Vector Fields of Kantowski Sachs and Some Bianchi Type Models in f(R) Theory of Gravity

3.1 Introduction

In this chapter, a study of CVFs of some Bianchi type models in the f(R) theory of gravity has been presented. Bianchi models have been remained a topic of special interest by theoretical physists as these are spatially homogeneous and are often used in the study of anisotropic cosmological models. These models have the capability to seek in to the internal structure of our universe. They have appeared as a power full tool in the theory of GR and belongs to well-known class of EFEs. One of the prominent examples of such solutions are the class of Bianchi models which are nine in number. In this chapter, we study Bianchi type I, II, V, Kantowski Sachs and Bianchi type III space-times. The following lines presents the layout of this chapter:

In section (3.2), CVFs of Bianchi type I space-times have been presented. From this study, we found that the CVFs are of dimension four, five, six and fifteen. In section (3.3), we have studied CVFs of Bianchi type II space-times and reach at the conclusion that the Bianchi type II space-times admit CVFs of dimension three, four and five only. In section (3.4), CVFs of Bianchi type V space-times have been presented. The study consists of seven cases. From these seven cases, we found that the CVFs in six cases reduce to isometries, while in the seventh case, the space-times become conformally flat, therefore admit fifteen independent CVFs. In section (3.5), a study of CVFs of Kantowski Sachs and Bianchi type III space-times in the f(R) theory of gravity according to their proper CVFs has been presented. In this study, we found that the Kantowski Sachs and Bianchi type III space-times admit CVFs of dimension four and six respectively.

3.2 Conformal Vector Fields of Bianchi type I Space-Times in f(R) Gravity

The line element of a Bianchi type I space-times in the usual coordinates (t, x, y, z) (given by (x^0, x^1, x^2, x^3) respectively) is (Stephani et al., 2003)

$$ds^2 = -dt^2 + A^2dx^2 + B^2dy^2 + C^2dz^2, \qquad (3.2.1)$$

where $A = A(t)$, $B = B(t)$ and $C = C(t)$ are nowhere zero functions of t. From the above space-times (3.2.1), we see that in built KVFs are (Stephani et al., 2003)

$$\partial_x,\ \partial_y,\ \partial_z. \tag{3.2.2}$$

The value of scalar curvature R for the space-times (3.2.1) is

$$R = 2\left[\frac{\ddot{A}}{A}+\frac{\ddot{B}}{B}+\frac{\ddot{C}}{C}+\frac{\dot{A}\dot{B}}{AB}+\frac{\dot{B}\dot{C}}{BC}+\frac{\dot{C}\dot{A}}{CA}\right], \tag{3.2.3}$$

where dot signifies the derivative with respect to t. To obtain CVFs in the f(R) theory of gravity considering Bianchi type I space-times, we start with the standard EFEs in vacuum (Nojiri and Odintsov, 2003)

$$F(R)R_{ab} - \frac{1}{2} f(R)g_{ab} - \nabla_a\nabla_b F(R) + g_{ab}\Box F(R) = 0, \tag{3.2.4}$$

where $f(R)$ is the function of Ricci scalar R, $F(R) = \frac{d}{dR} f(R)$ and $\Box \equiv \nabla^a\nabla_a$ in which ∇_a is the covariant derivative. Using equation (3.2.1) in equation (3.2.4) one has (Sharif and Shamir, 2009)

$$\frac{\ddot{A}}{A}-\frac{\ddot{B}}{B}+\frac{\dot{C}}{C}\left(\frac{\dot{A}}{A}-\frac{\dot{B}}{B}\right)+\frac{\dot{F}}{F}\left(\frac{\dot{A}}{A}-\frac{\dot{B}}{B}\right)=0. \tag{3.2.5}$$

$$\frac{\ddot{B}}{B}-\frac{\ddot{C}}{C}+\frac{\dot{A}}{A}\left(\frac{\dot{B}}{B}-\frac{\dot{C}}{C}\right)+\frac{\dot{F}}{F}\left(\frac{\dot{B}}{B}-\frac{\dot{C}}{C}\right)=0. \tag{3.2.6}$$

$$\frac{\ddot{A}}{A}-\frac{\ddot{C}}{C}+\frac{\dot{B}}{B}\left(\frac{\dot{A}}{A}-\frac{\dot{C}}{C}\right)+\frac{\dot{F}}{F}\left(\frac{\dot{A}}{A}-\frac{\dot{C}}{C}\right)=0. \tag{3.2.7}$$

The goal now is to look for the solutions of equations (3.2.5) to (3.2.7). Here, we are using the approach adopted by (Nojiri and Odintsov, 2003) and assume $F(R)$ to be of the form

$$F(R) = f_0 R^m, \tag{3.2.8}$$

where $f_0, m \in \Re$. Motivation behind considering $F(R)$ given in equation (3.2.8) is that it has proven a viable f(R) model which is compatible with cosmological observations and has passed through solar system tests. It can also mimic dark energy hypothesis when one deal with past and current expansion of the universe. In spite of these, it has been widely studied in finding well known exact solutions and is also used in the study of stability analysis of f(R) models (Ntahompagaze et al., 2018). Using equation (3.2.8) in equations (3.2.5) to (3.2.7), we have

$$\frac{\ddot{A}}{A}-\frac{\ddot{B}}{B}+\frac{\dot{C}}{C}\left(\frac{\dot{A}}{A}-\frac{\dot{B}}{B}\right)+m\frac{\dot{R}}{R}\left(\frac{\dot{A}}{A}-\frac{\dot{B}}{B}\right)=0. \tag{3.2.9}$$

$$\frac{\ddot{B}}{B}-\frac{\ddot{C}}{C}+\frac{\dot{A}}{A}\left(\frac{\dot{B}}{B}-\frac{\dot{C}}{C}\right)+m\frac{\dot{R}}{R}\left(\frac{\dot{B}}{B}-\frac{\dot{C}}{C}\right)=0. \tag{3.2.10}$$

$$\frac{\ddot{A}}{A}-\frac{\ddot{C}}{C}+\frac{\dot{B}}{B}\left(\frac{\dot{A}}{A}-\frac{\dot{C}}{C}\right)+m\frac{\dot{R}}{R}\left(\frac{\dot{A}}{A}-\frac{\dot{C}}{C}\right)=0. \tag{3.2.11}$$

Now, we find the solutions of equations (3.2.9) to (3.2.11) using the following approach:

(i) $A=A(t),\ B=B(t)$ and $C=\text{constant}.$

(ii) $A=A(t),\ C=C(t)$ and $B=\text{constant}.$

(iii) $B=B(t),\ C=C(t)$ and $A=\text{constant}.$

(iv) $A=A(t)$ and $B(t)=C(t).$

(v) $B=B(t)$ and $A(t)=C(t).$

(vi) $C=C(t)$ and $A(t)=B(t).$

(vii) $A=\text{constant}$ and $B(t)=C(t).$

(viii) $B=\text{constant}$ and $A(t)=C(t).$

(ix) $C=\text{constant}$ and $A(t)=B(t).$

(x) $A=A(t)$ and $B=C=\text{constant}.$

(xi) $B=B(t)$ and $A=C=\text{constant}.$

(xii) $C=C(t)$ and $A=B=$ constant.

(xiii) $A=B=C\neq$ constant.

(xiv) $A=B=C=$ constant.

Here, we will explain the procedure to find the solution of equations (3.2.9) to (3.2.11) in only one case which is (i). Substituting $C=\text{constant}$ in equation (3.2.9) to (3.2.11) results in three equations having four unknowns $A,\ B,\ R$ and m therefore, we need some extra conditions to solve the above equation. Here, we assume that $A=B^n$ and $\ddot{B}=0,$ where $n\in R\setminus\{0,1\}.$ After some lengthy calculations, we found that $B(t)=(c_1t+c_2),\ A(t)=(c_1t+c_2)^{-1},\ R=2\left(\frac{c_1}{c_1t+c_2}\right)^2,$ $f(R)=2f_0R^{1/2}+d_1$ and $m=-\frac{1}{2},$ where $c_1,c_2,f_0,d_1\in\Re\left(c_1,f_0\neq 0\right).$ Using the same procedure with different conditions, one can find the remaining cases. There exist the following solutions of equations (3.2.9) to (3.2.11):

(i) $A(t)=(c_1t+c_2)^{-1},\quad B(t)=(c_1t+c_2),\quad R=2\left(\frac{c_1}{c_1t+c_2}\right)^2,\quad m=-\frac{1}{2}$ and $C=$ constant, where $c_1,c_2\in\Re.$

(ii) $A(t)=(c_1t+c_2)^{-1},\quad C(t)=(c_1t+c_2),\quad R=2\left(\frac{c_1}{c_1t+c_2}\right)^2,\quad m=-\frac{1}{2}$ and $B=$ constant, where $c_1,c_2\in\Re.$

(iii) $B(t)=(c_1t+c_2)^{-1},\quad C(t)=(c_1t+c_2),\quad R=2\left(\frac{c_1}{c_1t+c_2}\right)^2,\quad m=-\frac{1}{2}.$ and $A=$ constant, where $c_1,c_2\in\Re.$

(iv) $A(t)=(c_1t+c_2)^{-2},\quad R=6\left(\frac{c_1}{c_1t+c_2}\right)^2,\quad m=-\frac{1}{2}$ and $B(t)=C(t)=(c_1t+c_2),$ where $c_1,c_2\in\Re.$

(v) $B(t)=(c_1t+c_2)^{-2}$, $R=6\left(\frac{c_1}{c_1t+c_2}\right)^2$, $m=-\frac{1}{2}$ and $A(t)=C(t)=(c_1t+c_2)$, where $c_1,c_2\in\Re$.

(vi) $C(t)=(c_1t+c_2)^{-2}$, $R=6\left(\frac{c_1}{c_1t+c_2}\right)^2$, $m=-\frac{1}{2}$ and $B(t)=A(t)=(c_1t+c_2)$, where $c_1,c_2\in\Re$.

(vii) $A=$constant, $R=\frac{7}{2t^2}$, $m=-1$ and $B(t)=C(t)=t^{\frac{-1}{2}}$.

(viii) $B=$constant, $R=\frac{7}{2t^2}$, $m=-1$ and $A(t)=C(t)=t^{\frac{-1}{2}}$.

(ix) $C=$constant, $R=\frac{7}{2t^2}$, $m=-1$ and $A(t)=B(t)=t^{\frac{-1}{2}}$.

(x) $A(t)=t^{-1}$, $R=\frac{4}{t^2}$, $m=-1$ and $B=C=$constant.

(xi) $B(t)=t^{-1}$, $R=\frac{4}{t^2}$, $m=-1$ and $A=C=$constant.

(xii) $C(t)=t^{-1}$, $R=\frac{4}{t^2}$, $m=-1$ and $A=B=$constant.

(xiii) $A(t)=B(t)=C(t)=e^{k_1t}$ and $R=6\left(\frac{\ddot{A}}{A}+\frac{\dot{A}^2}{A^2}\right)$, where $k_1\in\Re\setminus\{0\}$.

(xiv) $A=B=C=$constant.

We will consider each case in turn:

Case (i)

The information regarding over here is $A(t)=(c_1t+c_2)^{-1}$, $B(t)=(c_1t+c_2)$, $R=2\left(\frac{c_1}{c_1t+c_2}\right)^2$, $m=-\frac{1}{2}$ and $C=$constant, where $c_1,c_2\in\Re$. The space-times (3.2.1) after suitable rescaling of z take the form

$$ds^2=-dt^2+(c_1t+c_2)^{-2}dx^2+(c_1t+c_2)^2dy^2+dz^2. \qquad (3.2.12)$$

Now, we are interested to find CVFs of the above space-time (3.2.12). The traditional expansion of equation (1.12.1) and using equation (3.2.12), yields the following ten equations:

$$X^0_{,0}=\psi, \qquad (3.2.13)$$

$$X^1_{,0}-(c_1t+c_2)^2X^0_{,1}=0, \qquad (3.2.14)$$

$$X^0_{,2}-(c_1t+c_2)^2X^2_{,0}=0, \qquad (3.2.15)$$

$$X^0_{,3}-X^3_{,0}=0, \qquad (3.2.16)$$

$$-c_1(c_1t+c_2)^{-1}X^0+X^1_{,1}=\psi, \qquad (3.2.17)$$

$$X^1_{,2}+(c_1t+c_2)^4X^2_{,1}=0, \qquad (3.2.18)$$

$$X^1_{,3}+(c_1t+c_2)^2X^3_{,1}=0, \qquad (3.2.19)$$

$$c_1(c_1t+c_2)^{-1}X^0+X^2_{,2}=\psi, \qquad (3.2.20)$$

$$X^3_{,2}+(c_1t+c_2)^2X^2_{,3}=0, \qquad (3.2.21)$$

$$X^3_{,3}=\psi. \qquad (3.2.22)$$

Solving equations (3.2.13) to (3.2.16), we have the following information:

$$X^0 = \int \psi dt + E^1, \ \ X^1 = \frac{(c_1 t + c_2)^3}{3c_1} E^1_x + E^2,$$

$$X^2 = \frac{-1}{c_1(c_1 t + c_2)} E^1_y + E^3, \ \ X^3 = tE^1_z + E^4, \qquad (3.2.23)$$

where $E^i = E^i(x, y, z)$ with $i = 1, 2, 3, 4$ are FOI. For approaching the required CVF X, we need to search the functions $E^i(x, y, z)$ and the function ψ. Whose final form will help to categorize the related vector fields. Ignoring the process of straightforward integration, we reach at the following components of CVFs

$$X^0 = \left(\frac{c_1 t + c_2}{c_1}\right) k_2, \ \ X^1 = 2k_2 x + k_9, \ \ X^2 = k_4, \ \ X^3 = k_2 z + k_3, \qquad (3.2.24)$$

with conformal factor $\psi = k_2$, where $k_2, k_3, k_4, k_9 \in \Re$. It follows that the above space-time (3.2.12) do not admit proper CVFs. In this case CVFs are HVFs due to the constant conformal factor. If we look at the dimension of CVFs, we find that it is four, three of which are isometries and one is proper homothetic which becomes

$$\left(\frac{c_1 t + c_2}{c_1}\right)\frac{\partial}{\partial t} + 2x\frac{\partial}{\partial x} + z\frac{\partial}{\partial z}. \qquad (3.2.25)$$

Case (ii)

Here the ingredients of are $A(t) = (c_1 t + c_2)^{-1}$, $C(t) = (c_1 t + c_2)$, $R = 2\left(\frac{c_1}{c_1 t + c_2}\right)^2$, $m = -\frac{1}{2}$ and $B =$constant, where $c_1, c_2 \in \Re (c_1 \neq 0)$. The space-times (3.2.1) after appropriate rescaling of y turn to the following form

$$ds^2 = -dt^2 + (c_1 t + c_2)^{-2} dx^2 + dy^2 + (c_1 t + c_2)^2 dz^2. \qquad (3.2.26)$$

Following the lines of previous case (i), we found that $\psi = k_1$, which is the indication of non existence of proper CVFs. Here, the CVFs are HVFs again which are

$$X^0 = \left(\frac{c_1 t + c_2}{c_1}\right) k_1, \;\; X^1 = 2k_1 x + k_2, \;\; X^2 = k_1 y + k_3, \;\; X^3 = k_4, \qquad (3.2.27)$$

where $k_1, k_2, k_3, k_4 \in \Re$. Clearly, the above space-time (3.2.26) admit four CVFs consisting of three KVFs and one proper HVFs. The proper HVF except KVFs is

$$\left(\frac{c_1 t + c_2}{c_1}\right)\frac{\partial}{\partial t} + 2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}. \qquad (3.2.28)$$

Case (iii)

The values of metric components along with the necessary information possessed over here are $B(t) = (c_1 t + c_2)^{-1}$, $C(t) = (c_1 t + c_2)$, $R = 2\left(\frac{c_1}{c_1 t + c_2}\right)^2$, $m = -\frac{1}{2}$. and $A = \text{constant}$, where $c_1, c_2 \in \Re (c_1 \neq 0)$. The space-times (3.2.1) after suitable rescaling of x provide the way to write the it in the form

$$ds^2 = -dt^2 + dx^2 + (c_1 t + c_2)^{-2} dy^2 + (c_1 t + c_2)^2 dz^2. \qquad (3.2.29)$$

Again, in this case CVFs become the HVFs, which are

$$X^0 = \left(\frac{c_1 t + c_2}{c_1}\right) k_1, \;\; X^1 = k_1 x + k_2, \;\; X^2 = 2k_1 y + k_3, \;\; X^3 = k_4, \qquad (3.2.30)$$

where $k_1, k_2, k_3, k_4 \in \Re$. It is clear that the space-time (3.2.29) admits four CVFs which are not proper. CVFs reduced to proper HVFs given by

$$\left(\frac{c_1 t + c_2}{c_1}\right)\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y}. \qquad (3.2.31)$$

Case (iv)

Here, we have $A(t)=(c_1t+c_2)^{-2}$, $R=6\left(\frac{c_1}{c_1t+c_2}\right)^2$, $m=-\frac{1}{2}$ and $B(t)=C(t)=(c_1t+c_2)$, where $c_1,c_2\in\Re(c_1\neq 0)$. The space-times (3.2.1) take the form

$$ds^2=-dt^2+(c_1t+c_2)^{-4}dx^2+(c_1t+c_2)^2[dy^2+dz^2]. \qquad (3.2.32)$$

The function f(R) for the above space-time (3.2.32) is $f(R)=2f_0R^{1/2}+d_2$, where $f_0,d_2\in\Re$. Adopting the similar procedure as we did in the previous cases, it can be shown that the CVFs in the form of components are:

$$X^0=\left(k_8x+k_9\right)\left(c_1t+c_2\right),$$

$$X^1=\left[\frac{\left(c_1t+c_2\right)^6}{6c_1}+\frac{3c_1x^2}{2}\right]k_8+3k_9c_1x+k_{10},$$

$$X^2=-k_6z+k_3,\ \ X^3=k_6y+k_7, \qquad (3.2.33)$$

with conformal factor $\psi=c_1\left(k_8x+k_9\right)$, where $k_3,k_6,k_7,k_8,k_9,k_{10}\in\Re$. The above space-time (3.2.32) admit six CVFs in which four are KVFs. From these four KVFs three are given in equation (3.2.2) and fourth is $y\frac{\partial}{\partial z}-z\frac{\partial}{\partial y}$. One is proper HVF which is $t\frac{\partial}{\partial t}+3x\frac{\partial}{\partial x}$ and rest is proper CVF. The proper CVF has the following form

$$X^0=k_8x\left(c_1t+c_2\right)+k_{11},\ \ X^1=\left[\frac{\left(c_1t+c_2\right)^6}{6c_1}+\frac{3c_1x^2}{2}\right]k_8,\ \ X^2=0,\ \ X^3=0, \qquad (3.2.34)$$

where $k_{11}=c_2k_9$.

Case (v)

Here the information is $B(t)=(c_1t+c_2)^{-2}$, $R=6\left(\dfrac{c_1}{c_1t+c_2}\right)^2$, $m=-\dfrac{1}{2}$ and $A(t)=C(t)=(c_1t+c_2)$, where $c_1,c_2\in\Re(c_1\neq 0)$. The space-times (3.2.1) take the form

$$ds^2=-dt^2+(c_1t+c_2)^{-4}dy^2+(c_1t+c_2)^2[dx^2+dz^2]. \qquad (3.2.35)$$

The procedure to find the CVFs is direct integration technique, so the CVFs in this case are found to be

$$X^0=\left(k_8y+k_9\right)\left(c_1t+c_2\right),\ X^1=-k_6z+k_{10},\ X^3=k_6x+k_7,$$

$$X^2=\left[\frac{\left(c_1t+c_2\right)^6}{6c_1}+\frac{3c_1y^2}{2}\right]k_8+3k_9c_1y+k_3, \qquad (3.2.36)$$

with conformal factor $\psi=c_1\left(k_8y+k_9\right)$, where $k_3,k_6,k_7,k_8,k_9,k_{10}\in\Re$. The above space-time (3.2.35) admit six CVFs in which four are KVFs. From these four KVFs three are given in equation (3.2.2) and fourth is $x\dfrac{\partial}{\partial z}-z\dfrac{\partial}{\partial x}$. One is proper HVF which is $t\dfrac{\partial}{\partial t}+3y\dfrac{\partial}{\partial y}$ and one is proper CVF. The proper CVF excluding HVF from equation (3.2.36) is

$$X^0=k_8y\left(c_1t+c_2\right)+k_{11},\ X^1=0,\ X^2=\left[\frac{\left(c_1t+c_2\right)^6}{6c_1}+\frac{3c_1y^2}{2}\right]k_8,\ X^3=0, \qquad (3.2.37)$$

where $k_{11}=c_2k_9$.

Case (vi)

Here, we have the following information $C(t)=(c_1t+c_2)^{-2}$, $R=6\left(\dfrac{c_1}{c_1t+c_2}\right)^2$, $m=-\dfrac{1}{2}$ and $B(t)=A(t)=(c_1t+c_2)$, where $c_1,c_2\in\Re(c_1\neq 0)$. The space-times (3.2.1) reduced to be

$$ds^2 = -dt^2 + (c_1 t + c_2)^{-4} dz^2 + (c_1 t + c_2)^2 [dx^2 + dy^2]. \qquad (3.2.38)$$

The CVFs in this case are

$$X^0 = (k_8 z + k_9)(c_1 t + c_2), \;\; X^1 = -k_6 y + k_{10}, \;\; X^2 = k_6 x + k_3,$$

$$X^3 = \left[\frac{(c_1 t + c_2)^6}{6c_1} + \frac{3c_1 y^2}{2}\right] k_8 + 3k_9 c_1 z + k_7, \qquad (3.2.39)$$

with conformal factor $\psi = c_1(k_8 z + k_9)$, where $k_3, k_6, k_7, k_8, k_9, k_{10} \in \Re.$ The above space-time (3.2.35) admit six CVFs in which four are KVFs. From these four KVFs three are given in equation (3.2.2) and fourth is $x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}.$ One is proper HVF which is $t\frac{\partial}{\partial t} + 3z\frac{\partial}{\partial z}$ and one is proper CVF. The proper CVF neglecting HVF from equation (3.2.39) is

$$X^0 = k_8 z(c_1 t + c_2) + k_{11}, \;\; X^1 = 0, \;\; X^2 = 0, \;\; X^3 = \left[\frac{(c_1 t + c_2)^6}{6c_1} + \frac{3c_1 y^2}{2}\right] k_8. \qquad (3.2.40)$$

where $k_{11} = c_2 k_9.$

Case (vii)

The information possessed by this case are $A = \text{constant},\; R = \frac{7}{2t^2},\; m = -1$ and $B(t) = C(t) = t^{\frac{-1}{2}}.$ The space-times (3.2.1) after an appropriate frame takes the following mathematical shape

$$ds^2 = -dt^2 + dx^2 + t^{-1}[dy^2 + dz^2]. \qquad (3.2.41)$$

The f(R) for the above space-time (3.2.41) is $f(R) = f_0 \ln R + d_3,$ where $f_0, d_3 \in \Re.$ The CVFs in this case are

$$X^0 = k_2 t, \;\; X^1 = k_2 x + k_4, \;\; X^2 = \frac{3}{2}k_2 y - k_7 z + k_6, \;\; X^3 = \frac{3}{2}k_2 z + k_7 y + k_8, \qquad (3.2.42)$$

with conformal factor $\psi = k_2$, where $k_2, k_4, k_6, k_7, k_8 \in \Re.$ In this case, the CVFs are HVFs. The dimension of CVFs is five in which four are KVFs, three are given in equation (3.2.1) and fourth is $y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}.$ One is proper HVF. The compact form of the proper HVF is given below

$$X^0 = k_2 t, \ \ X^1 = k_2 x, \ \ X^2 = \frac{3}{2}k_2 y, \ \ X^3 = \frac{3}{2}k_2 z. \tag{3.2.43}$$

Case (viii)

Here, we have $B =$constant, $R = \frac{7}{2t^2}$, $m = -1$ and $A(t) = C(t) = t^{\frac{-1}{2}}$. The space-times (3.2.1) after an appropriate rescaling of y take the form

$$ds^2 = -dt^2 + dy^2 + t^{-1}[dx^2 + dz^2]. \tag{3.2.44}$$

Here, $\psi = k_1$, which means that no proper CVFs exist. Here, the CVFs become HVFs which are

$$X^0 = k_1 t, \ \ X^1 = \frac{3}{2}k_1 x - k_2 z + k_3, \ \ X^2 = k_1 y + k_4, \ \ X^3 = \frac{3}{2}k_1 z + k_2 x + k_5, \tag{3.2.45}$$

where $k_1, k_2, k_3, k_4, k_5 \in \Re.$ It follows that the above space-time (3.2.44) admit five CVFs in which four are KVFs, three are given in equation (3.2.1) and fourth is $x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x}.$ One is proper HVF. The proper HVF after subtracting KVFs is of the form

$$X^0 = k_1 t, \ \ X^1 = \frac{3}{2}k_1 x, \ \ X^2 = k_1 y, \ \ X^3 = \frac{3}{2}k_1 z. \tag{3.2.46}$$

Case (ix)

Here, we are with the data $C =$constant, $R = \frac{7}{2t^2}$, $m = -1$ and $A(t) = B(t) = t^{\frac{-1}{2}}$. The space-times (3.2.1) after an appropriate rescaling of z become

$$ds^2 = -dt^2 + dx^2 + t^{-1}[dx^2 + dy^2]. \tag{3.2.47}$$

Adopting similar procedure as we did in the previous cases, we come to know that $\psi = k_1,$ which means that no proper CVFs exist. Here, the CVFs become HVFs which are

$$X^0 = k_1 t,\ \ X^1 = \frac{3}{2}k_1 x - k_2 y + k_3,\ \ X^2 = \frac{3}{2}k_1 y + k_2 x + k_4,\ \ X^3 = k_1 z + k_5, \qquad (3.2.48)$$

where $k_1, k_2, k_3, k_4, k_5 \in \Re.$ It follows that the above space-time (3.2.47) admit five CVFs in which four are KVFs, three are given in equation (3.2.1) and fourth is $x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}.$ One is proper HVF. The proper HVF after subtracting KVFs from equation (3.2.48) is

$$X^0 = k_1 t,\ \ X^1 = \frac{3}{2}k_1 x,\ \ X^2 = \frac{3}{2}k_1 y,\ \ X^3 = k_1 z. \qquad (3.2.49)$$

Case (x)

Keeping the restrictions which are $A(t) = t^{-1},\ R = \frac{4}{t^2},\ m = -1$ and $B = C =$ constant. The space-times (3.2.1) now has the form after observing the rescaling

$$ds^2 = -dt^2 + t^{-2}dx^2 + dy^2 + dz^2. \qquad (3.2.50)$$

f(R) for the above space-time (3.2.50) is $f(R) = f_0 \ln R + d_4,$ where $f_0, d_4 = \Re.$ Again, in this case CVFs become HVFs which are

$$X^0 = k_5 t,\ \ X^1 = 2k_5 x + k_1,\ \ X^2 = k_5 y - k_2 z + k_3,\ \ X^3 = k_5 z + k_2 y + k_4, \qquad (3.2.51)$$

where $\psi = k_5$ and $k_1, k_2, k_3, k_4, k_5 \in \Re.$ From equation (3.2.51), we see that the above space-time (3.2.50) admit five CVFs in which four are KVFs, three are given in equation (3.2.2) and fourth is $y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}.$ One is proper HVF. The proper HVF after eliminating KVFs from equation (3.2.51) is

$$X^0 = k_5 t,\ \ X^1 = 2k_5 x,\ \ X^2 = k_5 y,\ \ X^3 = k_5 z. \qquad (3.2.52)$$

Case (xi)

This case has the information, $B(t)=t^{-1}$, $R=\frac{4}{t^2}$, $m=-1$ and $A=C=$constant. The space-times (3.2.1) after appropriate rescaling of x and z become

$$ds^2=-dt^2+dx^2+t^{-2}dy^2+dz^2. \qquad (3.2.53)$$

Succeeding on the same lines, we found that the CVFs become HVFs which are

$$X^0=k_1t,\ \ X^1=k_1y-k_2z+k_3,\ \ X^2=2k_1x+k_4,\ \ X^3=k_1z+k_2x+k_5, \qquad (3.2.54)$$

where $\psi=k_1$ and $k_1,k_2,k_3,k_4,k_5\in\Re$. From equation (3.2.54), we see that the above space-time (3.2.53) admit five CVFs in which four are KVFs, three are given in equation (3.2.2) and fourth is $x\frac{\partial}{\partial z}-z\frac{\partial}{\partial x}$. One is proper HVF. The proper HVF after eliminating KVFs from equation (3.2.54) is

$$X^0=k_1t,\ \ X^2=2k_1x,\ \ X^1=k_1y,\ \ X^3=k_1z. \qquad (3.2.55)$$

Case (xii)

Here, we have $C(t)=t^{-1}$, $R=\frac{4}{t^2}$, $m=-1$ and $A=B=$constant. Under these constraints the space-times (3.2.1) after keeping in mind the process of rescaling become

$$ds^2=-dt^2+dx^2+dy^2+t^{-2}dz^2. \qquad (3.2.56)$$

Again, in this case the CVFs become HVFs which are

$$X^0=k_1t,\ \ X^1=k_1x-k_2y+k_3,\ \ X^2=k_1y+k_2x+k_4,\ \ X^3=2k_1z+k_5, \qquad (3.2.57)$$

where $\psi=k_1$ and $k_1,k_2,k_3,k_4,k_5\in\Re$. From equation (3.2.57), we see that the above space-time (3.2.56) admit five CVFs in which four are KVFs, three are given in equation (3.2.2) and fourth is $x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x}$. One is proper HVF. The proper HVF after eliminating KVFs from equation (3.2.57) is

$$X^0=k_1t,\;\; X^1=k_1x,\;\; X^2=k_1y,\;\; X^3=2k_1z. \tag{3.2.58}$$

Case (xiii)

Here all the metric components are equal to the function e^{k_1t} and $R=6\left(\frac{\ddot{A}}{A}+\frac{\dot{A}^2}{A^2}\right)$, where k_1 is non zero real number. The space-time (3.2.1) becomes

$$ds^2=-dt^2+e^{k_1t}[dx^2+dy^2+dz^2]. \tag{3.2.59}$$

The above space-time (3.2.59) is conformally flat which is well known FRW model for $k=0$ and CVFs in this case are:

$$\begin{aligned}
X^0 &= -\left(\frac{e^{-k_1t}}{2k_1^3}+\frac{(x^2+y^2+z^2)e^{k_1t}}{2k_1}\right)c_4-\frac{1}{k_1}\left(c_6x+c_1y-c_2z+c_7\right)\\
&\quad +e^{k_1t}\left(c_{14}x+c_8y+c_{11}z+c_{15}\right),\\
X^1 &= \left(\frac{e^{-2k_1t}}{2k_1^2}+\frac{(x^2-y^2-z^2)}{2}\right)c_6+\frac{e^{-k_1t}}{k_1^2}c_4x-\frac{e^{-k_1t}}{k_1}c_{14}+c_1xy\\
&\quad -c_2xz+c_7x-c_9y-c_{12}z+c_{16},\\
X^2 &= \left(\frac{e^{-2k_1t}}{2k_1^2}+\frac{(y^2-x^2-z^2)}{2}\right)c_1+\frac{e^{-k_1t}}{k_1^2}c_4y-\frac{e^{-k_1t}}{k_1}c_8+c_6xy\\
&\quad -c_2yz+c_9x+c_7y-c_3z+c_{10},\\
X^3 &= \left(\frac{-e^{-2k_1t}}{2k_1^2}+\frac{(x^2+y^2-z^2)}{2}\right)c_2+\frac{e^{-k_1t}}{k_1^2}c_4z-\frac{e^{-k_1t}}{k_1}c_{11}+c_6xz\\
&\quad +c_1yz+c_{12}x+c_3y+c_7z+c_{13},
\end{aligned} \tag{3.2.60}$$

with conformal factor

$$\psi(t,x,y,z)=\left(\frac{e^{-k_1t}}{2k_1^2}-\frac{(x^2+y^2+z^2)e^{k_1t}}{2}\right)c_4+k_1e^{k_1t}\left(c_{14}x+c_8y+c_{11}z+c_{15}\right), \tag{3.2.61}$$

where $c_i\in\Re$ with $i=1,2,3,4,6,7...,16.$

Case (xiv)

The restrictions of this case after utilizing in the original space-times lead to the following form

$$ds^2=-dt^2+dx^2+dy^2+dz^2, \tag{3.2.62}$$

which is Minkowski space-time. The above space-times (3.2.62) admit fifteen CVFs which are:

$$X^0=\left(\frac{t^2+x^2+y^2+z^2}{2}\right)c_4+c_1ty+c_5tx-c_2tz+c_6t+c_{13}x+c_7y+c_{10}z+c_{14},$$

$$X^1=\left(\frac{t^2+x^2-y^2-z^2}{2}\right)c_5+c_4tx+c_1xy-c_2xz+c_{13}t+c_6x-c_8y-c_{11}z+c_{15},$$

$$X^2=\left(\frac{t^2+y^2-x^2-z^2}{2}\right)c_1+c_4ty+c_5xy-c_2yz+c_7t+c_8x+c_6y-c_3z+c_9,$$

$$X^3=\left(\frac{x^2+y^2-t^2-z^2}{2}\right)c_2+c_4tz+c_5xz+c_1yz+c_{10}t+c_{11}x+c_3y+c_6z+c_{12}, \tag{3.2.63}$$

with conformal factor

$$\psi(t,x,y,z)=c_4t+c_5x+c_1y-c_2z+c_6, \tag{3.2.64}$$

where $c_i\in\Re$ with $i=1,2,3,...,15.$

It is essential to mention here that in order to explore the solutions, we have used the form of f(R) i.e. $F(R)=f_0R^m$, where $f_0,m\in\Re.$ Starting from this general form of the function f(R) and adopting the approach (a1) to (a14), we have analytically found two particular types of f(R) models corresponding to the values $m=-\frac{1}{2}$ and $m=-1.$ These are $f(R)=2f_0\sqrt{R}+d_2$ and

$f(R) = f_0 \ln\left(R\right) + d_4,$ where $f_0, d_2, d_4 \in \Re.$ The model with $\ln\left(R\right)$ term has been found to be significantly feasible at cosmological scale as it has well qualified solar system tests and is free from the instability problem (Capozziello et al., 2016 and Paul, 2009). On the other hand, the form with $f(R) = 2 f_0 \sqrt{R} + d_2$ contain positive power of the scalar curvature and is treated as viable regarding the inflationary era.

3.3 Conformal Vector Fields of Bianchi type II Space-Times in f(R) Gravity

Bianchi type II space-times belongs to the class of Bianchi models. This model has enormous applications in discussing phenomenon of universe at the large scale structure. The aim over here is to consider this model for looking CVFs in the setup of f(R) gravity. Here, we take model of Bianchi type II space-times in coordinates (t, x, y, z) consuming the line element (Hickman and Yazdan, 2017, Camci and Sahin, 2006 and Shabbir and Khan, 2010)

$$ds^2 = -dt^2 + A(t)dx^2 + B(t)dy^2 + \left[B(t)x^2 + C(t)\right]dz^2 - 2xB(t)\,dy\,dz, \qquad (3.3.1)$$

where $A = A(t),\ B = B(t)$ and $C = C(t)$ are nowhere zero functions of t. The in built isometries admitted by the space-times (3.3.1) are (Shabbir and Khan, 2010)

$$\frac{\partial}{\partial y},\ \frac{\partial}{\partial x} + z\frac{\partial}{\partial y},\ \frac{\partial}{\partial z}. \qquad (3.3.2)$$

The scalar curvature R associated with the space-times (3.3.1) has the value

$$R = \frac{-1}{2}\left[\frac{2\ddot{A}}{A} + \frac{2\ddot{B}}{B} + \frac{2\ddot{C}}{C} - \frac{\dot{A}^2}{A^2} + \frac{\dot{A}\dot{B}}{AB} - \frac{\dot{B}^2}{B^2} + \frac{\dot{B}\dot{C}}{BC} - \frac{\dot{C}^2}{C^2} + \frac{\dot{C}\dot{A}}{CA} - \frac{B}{AC}\right], \qquad (3.3.3)$$

where overhead dot symbolizes $\frac{d}{dt}$. Using equation (3.3.1) in conformal motion equation (1.12.1), we obtain

$$X^0_{,0} = \psi, \qquad (3.3.4)$$

$$AX^1_{,0} - X^0_{,1} = 0, \qquad (3.3.5)$$

$$BX^2_{,0} - xBX^3_{,0} - X^0_{,2} = 0, \tag{3.3.6}$$

$$xBX^2_{,0} - \left(Bx^2 + C\right)X^3_{,0} + X^0_{,3} = 0, \tag{3.3.7}$$

$$\dot{A}X^0 + 2AX^1_{,1} = 2A\psi, \tag{3.3.8}$$

$$BX^2_{,1} - xBX^3_{,1} + AX^1_{,2} = 0, \tag{3.3.9}$$

$$xBX^2_{,1} - \left(Bx^2 + C\right)X^3_{,1} - AX^1_{,3} = 0, \tag{3.3.10}$$

$$\dot{B}X^0 + 2BX^2_{,2} - 2xBX^3_{,2} = 2B\psi, \tag{3.3.11}$$

$$x\dot{B}X^0 + BX^1 + xBX^2_{,2} - \left(Bx^2 + C\right)X^3_{,2} - BX^2_{,3} + xBX^3_{,3} = 2xB\psi, \tag{3.3.12}$$

$$\left(\dot{B}x^2 + \dot{C}\right)X^0 + 2xB(X^1 - X^2_{,3}) + 2\left(Bx^2 + C\right)X^3_{,3} = 2\left(Bx^2 + C\right)\psi. \tag{3.3.13}$$

From equations (3.3.4), (3.3.5), (3.3.6) and (3.3.7), we have

$$X^0 = \int \psi dt + D^1, \quad X^1 = \int\left(\frac{1}{A}\int \psi_x dt\right)dt + D^1_x \int \frac{dt}{A} + D^2,$$

$$X^2 = \int\left(\frac{1}{B}\int \psi_y dt\right)dt + D^1_y \int \frac{dt}{B} + x^2\int\left(\frac{1}{C}\int \psi_y dt\right)dt + x^2 D^1_y \int \frac{dt}{C}$$
$$+ x\int\left(\frac{1}{C}\int \psi_z dt\right)dt + xD^1_z \int \frac{dt}{C} + D^4,$$

$$X^3 = \int\left(\frac{1}{C}\int \psi_z dt\right)dt + D^1_z \int \frac{dt}{C} + x\int\left(\frac{1}{C}\int \psi_y dt\right)dt + xD^1_y \int \frac{dt}{C} + D^3, \tag{3.3.14}$$

where $D^i = D^i\left(x, y, z\right)$ with $i = 1,2,3,4$ are FOIs. As the study is dedicated purely for seeking CVFs in the theory under discussion, hence, we are making use of equations (3.3.1) in equation (3.2.4) to get

$$\left[\frac{\dot{A}}{2A} + \frac{\dot{B}}{2B} + \frac{\dot{C}}{2C}\right]\dot{F} + \left[\frac{\dot{A}\dot{C}}{4AC} + \frac{\dot{A}\dot{B}}{4AB} + \frac{\dot{B}\dot{C}}{4BC} - \frac{B}{4AC}\right]F + \frac{f}{2} - \frac{RF}{2} = 0. \tag{3.3.15}$$

$$\left[\frac{\dot{B}}{2B}+\frac{\dot{C}}{2C}\right]\dot{F}+\left[\frac{\ddot{B}}{2B}+\frac{\ddot{C}}{2C}-\frac{\dot{C}^2}{4C^2}+\frac{\dot{B}\dot{C}}{4BC}-\frac{\dot{B}^2}{4B^2}+\frac{B}{4AC}\right]F+\frac{f}{2}-\frac{RF}{2}+\ddot{F}=0. \qquad (3.3.16)$$

$$\left[\frac{\dot{A}}{2A}+\frac{\dot{C}}{2C}\right]\dot{F}+\left[\frac{\ddot{A}}{2A}+\frac{\ddot{C}}{2C}-\frac{\dot{A}^2}{4A^2}-\frac{\dot{C}^2}{4C^2}+\frac{\dot{A}\dot{C}}{4AC}-\frac{3B}{4AC}\right]F+\frac{f}{2}-\frac{RF}{2}+\ddot{F}=0. \qquad (3.3.17)$$

$$\frac{F}{(Bx^2+C)}\left[\begin{array}{l} Bx^2\left(\frac{\ddot{A}}{2A}+\frac{\ddot{C}}{2C}-\frac{\dot{A}^2}{4A^2}-\frac{\dot{C}^2}{4C^2}+\frac{\dot{A}\dot{C}}{4AC}-\frac{3B}{4AC}\right)+ \\ C\left(\frac{\ddot{A}}{2A}+\frac{\ddot{B}}{2B}-\frac{\dot{A}^2}{4A^2}-\frac{\dot{B}^2}{4B^2}+\frac{\dot{A}\dot{B}}{4AB}+\frac{B}{4AC}\right) \end{array}\right]+\frac{f}{2}-\frac{RF}{2}+\ddot{F}$$
$$+\left[\frac{\dot{A}}{2A}+\frac{\dot{B}}{2B}+\frac{\dot{C}}{2C}+\frac{2\dot{B}x^2}{C}-\frac{\dot{B}x^2+\dot{C}}{2(Bx^2+C)}\right]\dot{F}=0. \qquad (3.3.18)$$

As the above equations (3.3.15) to (3.3.18) are highly nonlinear and are difficult to solve, therefore, we use the technique of (Ram and Singh, 1993) and assume the following adhoc relation:

$$\frac{\ddot{A}}{2A}+\frac{\ddot{C}}{2C}-\frac{\dot{A}^2}{4A^2}-\frac{\dot{C}^2}{4C^2}+\frac{\dot{A}\dot{C}}{4AC}-\frac{3B}{4AC}=\frac{\ddot{A}}{2A}+\frac{\ddot{B}}{2B}-\frac{\dot{A}^2}{4A^2}-\frac{\dot{B}^2}{4B^2}+\frac{\dot{A}\dot{B}}{4AB}+\frac{B}{4AC}. \qquad (3.3.19)$$

Using equation (3.3.19) in equation (3.3.18) and then subtracting the resulting equation from (3.3.17), we have $\left(B\dot{C}-\dot{B}C\right)\dot{F}=0$ which implies:

(a) $\dot{F}\neq 0$ and $B\dot{C}-\dot{B}C=0.$

(b) $\dot{F}=0$ and $B\dot{C}-\dot{B}C=0.$

(c) $\dot{F}=0$ and $B\dot{C}-\dot{B}C\neq 0.$

When, $\dot{F}\neq 0$ and $B\dot{C}-\dot{B}C=0 \Rightarrow B=\alpha C,$ where $\alpha\in\Re\setminus\{0\}$. Using $B=\alpha C$ in the previous equation (22), we get $\alpha=0$, which give contradiction. Similarly, possibility (b) also give contradiction. Now, when $\dot{F}=0$ and $B\dot{C}-\dot{B}C\neq 0,$ then the equations (18) to (21) takes the form

$$\left[\frac{\dot{A}\dot{C}}{4AC}+\frac{\dot{A}\dot{B}}{4AB}+\frac{\dot{B}\dot{C}}{4BC}-\frac{B}{4AC}\right]F+\frac{f}{2}-\frac{RF}{2}=0. \qquad (3.3.20)$$

$$\left[\frac{\ddot{B}}{2B}+\frac{\ddot{C}}{2C}-\frac{\dot{C}^2}{4C^2}+\frac{\dot{B}\dot{C}}{4BC}-\frac{\dot{B}^2}{4B^2}+\frac{B}{4AC}\right]F+\frac{f}{2}-\frac{RF}{2}=0. \qquad (3.3.21)$$

$$\left[\frac{\ddot{A}}{2A}+\frac{\ddot{C}}{2C}-\frac{\dot{A}^2}{4A^2}+\frac{\dot{A}\dot{C}}{4AC}-\frac{\dot{C}^2}{4C^2}-\frac{3B}{4AC}\right]F+\frac{f}{2}-\frac{RF}{2}=0. \qquad (3.3.22)$$

Here, we solve equations (3.3.19) to (3.3.22). Forgoing the details, we found following seven cases in which the solutions of the above equations (3.3.19) to (3.3.22) have been investigated. These cases are

(i) $A=t^3,\ B=t^{-2},\ C=t^{-3}$ and $R=\frac{-8}{t^2}$.

(ii) $A=(k_1t+k_2)^3,\ B=(k_1t+k_2),\ C=\frac{-2}{k_1^2}$ and $R=\frac{-11k_1^2}{4(k_1t+k_2)^2}$, where $k_1,k_2\in\Re(k_1\neq 0)$.

(iii) $A=\frac{-1}{k_1^2},\ B=(k_1t+k_2)^{-1},\ C=(k_1t+k_2)$ and $R=\frac{-k_1^2}{(k_1t+k_2)^2}$, where $k_1,k_2\in\Re(k_1\neq 0)$.

(iv) $A=B=\left(2t^2+k_1t+k_2\right)^{1/2},\quad C=\left(2t^2+k_1t+k_2\right)$ and $R=\frac{5(4t+k_1)^2-60\left(2t^2+k_1t+k_2\right)k_1^2}{8\left(2t^2+k_1t+k_2\right)^2}$,

where $k_1,k_2\in\Re(k_1\neq 0)$.

(v) $A=C=(k_1t+k_2)^{3/2},\ B=\left(k_1t+k_2\right)$ and $R=\frac{4-11k_1^2}{8(k_1t+k_2)^2}$, where $k_1=\pm 2,\ k_2\in\Re$.

(vi) $A=C=k_1,\ B=-2k_1^2t^{-2}$ and $R=\frac{-5}{t^2}$, where $k_1\in\Re\setminus\{0\}$.

(vii) $A=B=\left(k_2\cos\sqrt{\frac{2}{k_1}}t+k_3\sin\sqrt{\frac{2}{k_1}}t\right),\quad C=k_1$ and $R=\frac{1}{2}\left[\frac{k_1\dot{A}^2+A^2-4k_1A\ddot{A}}{k_1A^2}\right]$, where

$k_1,k_2,k_3\in\Re(k_1\neq 0)$.

We will discuss each case one by one. Infect, we will substitute the values of metric components in the above equation (3.3.14) and try to find the unknown FOIs i.e. $D^i\left(x,y,z\right)$ with

$i=1,2,3,4.$ When these FOI are determine then we reach at the required result. In the upcoming lines we will adopt the technique discussed above and find CVFs.

Case (i)

Here, we have the information $A=t^3,\ B=t^{-2},\ C=t^{-3}$ and $R=\frac{-8}{t^2}.$ The space-time (3.3.1) takes the form

$$ds^2=-dt^2+t^3dx^2+t^{-2}dy^2+\left[t^{-2}x^2+t^{-3}\right]dz^2-2xt^{-2}dy\,dz. \qquad (3.3.23)$$

Now, we find CVFs of the space-time (3.3.23) using equations (3.3.4) to (3.3.13). Excluding the calculations, one finds that $\psi=\frac{c_1}{2},$ where $c_1\in\Re$ which means that the CVFs become HVFs which are

$$\frac{t}{2}\frac{\partial}{\partial t}-\frac{x}{4}\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+\frac{5z}{4}\frac{\partial}{\partial z},\ \frac{\partial}{\partial x}+z\frac{\partial}{\partial y},\ \frac{\partial}{\partial y},\ \ \frac{\partial}{\partial z}. \qquad (3.3.24)$$

The above space-time (3.3.23) admits four CVFs in which three are KVFs which are given in equation (3.3.2) and one is proper HVF. The proper HVF after subtracting KVFs from (3.3.24) is

$$\frac{t}{2}\frac{\partial}{\partial t}-\frac{x}{4}\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+\frac{5z}{4}\frac{\partial}{\partial z}. \qquad (3.3.25)$$

Case (ii)

Here, we have $A=(k_1t+k_2)^3,\ \ B=(k_1t+k_2),\ \ C=\frac{-2}{k_1^2}$ and $R=\frac{-11k_1^2}{4(k_1t+k_2)^2},$ where $k_1,k_2\in\Re(k_1\neq 0).$ The space-time (3.3.1) turn to be in the following form:

$$ds^2=-dt^2+(k_1t+k_2)^3dx^2+(k_1t+k_2)\left[dy^2+x^2dz^2-2xdydz\right]-\frac{2}{k_1^2}dz^2. \qquad (3.3.26)$$

Again solving equations (3.3.4) to (3.3.13) with the help of equation (3.3.26), one finds that $\psi=2c_1,$ which means that the CVFs become HVFs which are

$$X^0 = 2c_1\left(\frac{k_1 t + k_2}{k_1}\right),\ X^1 = -c_1 x + c_2, \quad X^2 = c_1 y + c_2 z + c_3,\ \ X^3 = 2c_1 z + c_4 \Bigg\}, \tag{3.3.27}$$

where $c_1, c_2, c_3, c_4 \in \Re \setminus \{0\}$. The above space-time (3.3.26) admits four CVFs in which three are KVFs which are given in equation (3.3.2) and one is proper HVF. The proper HVF after subtracting KVFs from (3.3.27) is

$$X^0 = 2c_1\left(\frac{k_1 t + k_2}{k_1}\right),\ X^1 = -c_1 x,\ X^2 = c_1 y,\ \ X^3 = 2c_1 z. \tag{3.3.28}$$

Case (iii)

The values of metric components along with the scalar curvature possessed by this case are $A = \frac{-1}{k_1^2}$, $B = (k_1 t + k_2)^{-1}$, $C = (k_1 t + k_2)$ and $R = \frac{-k_1^2}{(k_1 t + k_2)^2}$, where $k_1, k_2 \in \Re (k_1 \neq 0)$. The space-time (3.3.1) takes the form

$$ds^2 = -dt^2 - \frac{1}{k_1^2} dx^2 + (k_1 t + k_2)^{-1}\left[dy^2 + x^2 dz^2 - 2x dy dz\right] + (k_1 t + k_2) dz^2. \tag{3.3.29}$$

Using the direct integration technique, we come to know that $\psi = \frac{2}{3}c_1$, implies CVFs become HVFs which are

$$X^0 = \frac{2}{3}\left(\frac{k_1 t + k_2}{k_1}\right)c_1,\ X^1 = \frac{2}{3}c_1 x + c_2, \quad X^2 = c_1 y + c_2 z + c_3,\ \ X^3 = \frac{1}{3}c_1 z + c_4 \Bigg\}, \tag{3.3.30}$$

where $c_i \in \Re,\ i = 1,2,3,4.$ Clearly the four CVFs shown in (3.3.30) are decomposed as three KVFs which are given in equation (3.3.2) and the remaining one is proper HVF. The proper HVF after subtracting KVFs from (3.3.30) is

$$X^0 = \frac{2}{3}\left(\frac{k_1 t + k_2}{k_1}\right)c_1,\ X^1 = \frac{2}{3}c_1 x, X^2 = c_1 y,\ \ X^3 = \frac{1}{3}c_1 z. \tag{3.3.31}$$

Case (iv)

The information possessed by this case is $A = B = \left(2t^2 + k_1 t + k_2\right)^{1/2}$, $C = \left(2t^2 + k_1 t + k_2\right)$ and $R = \dfrac{5(4t + k_1)^2 - 60\left(2t^2 + k_1 t + k_2\right)k_1^2}{8\left(2t^2 + k_1 t + k_2\right)^2}$, where $k_1, k_2 \in \Re (k_1 \neq 0).$ The space-time (3.3.1) become

$$ds^2 = -dt^2 + \left(2t^2 + k_1 t + k_2\right)^{1/2}\left[dx^2 + dy^2 + x^2 dz^2 - 2xdydz\right] + \left(2t^2 + k_1 t + k_2\right)dz^2. \quad (3.3.32)$$

Solving equations (3.3.4) to (3.3.13) with the help of space-time (3.3.32) implies $\psi = 0$, which directs that the space-time admit KVFs which could be seen in equaion (3.3.2).

Case (v)

The constraints here are $A = C = \left(k_1 t + k_2\right)^{3/2}$, $B = \left(k_1 t + k_2\right)$ and $R = \dfrac{4 - 11k_1^2}{8(k_1 t + k_2)^2}$, where $k_1 = \pm 2,\ k_2 \in \Re.$. The space-time (2) takes the form

$$ds^2 = -dt^2 + \left(k_1 t + k_2\right)^{3/2}\left[dx^2 + dz^2\right] + \left(k_1 t + k_2\right)\left[dy^2 + x^2 dz^2 - 2xdydz\right]. \quad (3.3.33)$$

Again solving equations (3.3.4) to (3.3.13) with the help of space-time (3.3.33) implies that $\psi = 0$, therefore CVFs become KVFs which are

$$z\frac{\partial}{\partial x} + \left(\frac{x^2 - z^2}{2}\right)\frac{\partial}{\partial y} - x\frac{\partial}{\partial z},\ \frac{\partial}{\partial x} + z\frac{\partial}{\partial y},\ \frac{\partial}{\partial y},\ \frac{\partial}{\partial z}. \quad (3.3.34)$$

From the above four KVFs, three are given in equation (3). Here, one thing which is necessary to note that the space-time (3.3.33) admit extra KVF $z\frac{\partial}{\partial x} + \left(\frac{x^2 - z^2}{2}\right)\frac{\partial}{\partial y} - x\frac{\partial}{\partial z}$ which is different from the minimal set of isometries admitted by the space-times (3.3.2) giving extra conservation law.

Case (vi)

Here, we have $A = C = k_1,\ B = -2k_1^2 t^{-2}$ and $R = \frac{-5}{t^2}$, where $k_1 \in \Re \setminus \{0\}$. The space-time (3.3.1) in this case become

$$ds^2 = -dt^2 + k_1\left[dx^2 + dz^2\right] - 2k_1^2 t^{-2}\left[dy^2 + x^2 dz^2 + 2xk_1 dy\, dz\right]. \tag{3.3.35}$$

Solving equations (3.3.4) to (3.3.13) using the space-time (3.3.35) and avoiding from the lengthy calculations, we find that $\psi = \frac{1}{4}c_1$, implies CVFs become HVFs which are

$$\left.\begin{aligned} &X^0 = \frac{1}{4}c_1 t,\ X^1 = \frac{1}{4}c_1 x - c_2 z + c_3, \\ &X^2 = \frac{1}{4}c_1 y - \left(\frac{x^2 - z^2}{2}\right)c_2 + c_3 z + c_4,\ \ X^3 = \frac{1}{4}c_1 z + c_2 x + c_5 \end{aligned}\right\}, \tag{3.3.36}$$

where $c_1, c_2, c_3, c_4, c_5 \in \Re \setminus \{0\}$. The above space-time (3.3.35) admit five CVFs. From these five CVFs, four are KVFs and one is proper HVF. The proper HVF after subtracting KVFs from (3.3.36) is

$$X^0 = \frac{1}{4}c_1 t,\ X^1 = \frac{1}{4}c_1 x, X^2 = \frac{1}{4}c_1 y,\ \ X^3 = \frac{1}{4}c_1 z. \tag{3.3.37}$$

Case (vii)

The ingredients possessed over here are $A = B = \left(k_2 \cos\sqrt{\frac{2}{k_1}}t + k_3 \sin\sqrt{\frac{2}{k_1}}t\right)$, $C = k_1$ and $R = \frac{1}{2}\left[\frac{k_1 \dot{A}^2 + A^2 - 4k_1 A\ddot{A}}{k_1 A^2}\right]$, where $k_1, k_2, k_3 \in \Re (k_1 \neq 0)$. The space-time (3.3.1) in this case takes the form

$$ds^2 = -dt^2 + \left(k_2 \cos\sqrt{\frac{2}{k_1}}t + k_3 \sin\sqrt{\frac{2}{k_1}}t\right)\left[dx^2 + dy^2 + x^2 dz^2 - 2xdydz\right] + k_1 dz^2. \tag{3.3.38}$$

Solving equations (3.3.4) to (3.3.13) using the space-time (3.3.38) and without inducting from the lengthy calculations one finds that $\psi = 0$, indicating that the CVFs become KVFs which are

$$X^0 = 0,\ X^1 = c_1 z + c_2, X^2 = \left(\frac{k_1 x^2 - z^2}{2}\right) c_1 + c_2 z + c_3, X^3 = -k_1 c_1 x + c_4, \qquad (3.3.39)$$

where $c_1, c_2, c_3, c_4 \in \Re \setminus \{0\}$. From the above four KVFs, three are given in equation (3.3.2) and the remaining one KVF is $z\frac{\partial}{\partial x} + \left(\frac{k_1 x^2 - z^2}{2}\right)\frac{\partial}{\partial y} - k_1 x \frac{\partial}{\partial z}$.

3.4 Conformal Vector Fields of Bianchi type V Space-Times in f(R) Gravity

The line element representing geometry of Bianchi type V space-time in coordinates (t, x, y, z) is (Stephani et al., 2003)

$$ds^2 = -dt^2 + A^2 dx^2 + e^{2px}\left[B^2 dy^2 + C^2 dz^2\right], \qquad (3.4.1)$$

where $A = A(t)$, $B = B(t)$ and $C = C(t)$ are no-where zero functions of t only and $p \in \Re$. The least number of isometries associated with the above space-times (3.4.1) are (Shabbir et al., 2018)

$$\frac{\partial}{\partial y},\ \frac{\partial}{\partial z},\ \frac{\partial}{\partial x} - p\,y\frac{\partial}{\partial y} - p\,z\frac{\partial}{\partial z}. \qquad (3.4.2)$$

Ricci scalar R for the space-time (3.4.1) is

$$R = -2\left[\frac{\ddot{A}}{A} + \frac{\dot{A}\dot{B}}{AB} + \frac{\ddot{B}}{B} + \frac{\dot{B}\dot{C}}{BC} + \frac{\ddot{C}}{C} + \frac{\dot{C}\dot{A}}{CA} - \frac{3p^2}{A^2}\right], \qquad (3.4.3)$$

where dot represents $\frac{d}{dt}$. Using equation (3.4.1) in equation (1.12.1), one arrives at

$$X^0_{,0} = \psi, \qquad (3.4.4)$$

$$X^0_{,1} - A^2 X^1_{,0} = 0, \qquad (3.4.5)$$

$$X^0_{,2} - B^2 e^{2px} X^2_{,0} = 0, \tag{3.4.6}$$

$$X^0_{,3} - C^2 e^{2px} X^3_{,0} = 0, \tag{3.4.7}$$

$$\dot{A} X^0 + A X^1_{,1} = A\psi, \tag{3.4.8}$$

$$A^2 X^1_{,2} + B^2 e^{2px} X^2_{,1} = 0, \tag{3.4.9}$$

$$A^2 X^1_{,3} + C^2 e^{2px} X^3_{,1} = 0, \tag{3.4.10}$$

$$\dot{B} X^0 + pBX^1 + BX^2_{,2} = B\psi, \tag{3.4.11}$$

$$B^2 X^2_{,3} + C^2 X^3_{,2} = 0, \tag{3.4.12}$$

$$\dot{C} X^0 + pCX^1 + CX^3_{,3} = C\psi. \tag{3.4.13}$$

From equation (3.4.4), we have $X^0 = \int \psi dt + S^1$, where S^1 is a FOI depending on the coordinates (x, y, z). Now, by utilizing the value of X^0 in equations (3.4.5), (3.4.6) and (3.4.7), we have

$$X^0 = \int \psi dt + S^1, \quad X^1 = \int \left(\frac{1}{A^2} \int \psi_x dt \right) dt + S^1_x \int \frac{dt}{A^2} + S^2,$$

$$X^2 = e^{-2px} \int \left(\frac{1}{B^2} \int \psi_y dt \right) dt + e^{-2px} S^1_y \int \frac{dt}{B^2} + S^3,$$

$$X^3 = e^{-2px} \int \left(\frac{1}{C^2} \int \psi_z dt \right) dt + e^{-2px} S^1_z \int \frac{dt}{C^2} + S^4, \tag{3.4.14}$$

where $S^i = S^i(x, y, z)$, with $i = 2,3,4$ are FOI. To find the CVFs for the space-times under consideration in the f(R) theory of gravity, we must use equations (3.4.1) and (1.13.2). We are using the source of matter a perfect fluid to explore the solutions of equation (1.13.2). The formula for the perfect fluid source is

$$T_{ab} = (\rho + p) u_a u_b + p g_{ab}, \tag{3.4.15}$$

where ρ and p are defined in section (1.9). Here, u_a showing the four velocity vector defined as $u_a = -\delta_a^0$. Now, using equations (3.4.1), (1.13.2) and (3.4.15), one has (Sharif and Shamir, 2010)

$$\frac{\ddot{F}}{F} - \frac{\dot{A}\dot{F}}{AF} - \frac{\dot{C}\dot{A}}{CA} - \frac{\dot{A}\dot{B}}{AB} + \frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} + \frac{2p^2}{A^2} = \frac{k}{F}(\rho + p). \quad (3.4.16)$$

$$\frac{\ddot{F}}{F} - \frac{\dot{B}\dot{F}}{BF} - \frac{\dot{B}\dot{C}}{BC} - \frac{\dot{A}\dot{B}}{AB} + \frac{\ddot{A}}{A} + \frac{\ddot{C}}{C} + \frac{2p^2}{A^2} = \frac{k}{F}(\rho + p). \quad (3.4.17)$$

$$\frac{\ddot{F}}{F} - \frac{\dot{C}\dot{F}}{CF} - \frac{\dot{A}\dot{C}}{AC} - \frac{\dot{B}\dot{C}}{BC} + \frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{2p^2}{A^2} = \frac{k}{F}(\rho + p). \quad (3.4.18)$$

$$2\frac{\dot{A}}{A} - \frac{\dot{B}}{B} - \frac{\dot{C}}{C} = 0. \quad (3.4.19)$$

The above equations, (3.4.16) to (3.4.18) after some algebraic manipulations leads to the following equation (3.4.20)

$$\frac{\dot{F}}{F}\left(\frac{\dot{B}}{B} - \frac{\dot{A}}{A}\right) + \frac{\dot{C}}{C}\left(\frac{\dot{B}}{B} - \frac{\dot{A}}{A}\right) + \frac{\ddot{B}}{B} - \frac{\ddot{A}}{A} = 0. \quad (3.4.20)$$

Now, our purpose is to obtain exact solutions of equations (3.4.19) and (3.4.20). Both these equations are difficult to solve as these having non-linearity. In addition to this, these equations contains four unknowns where as we have two equations, therefore one must impose extra conditions to solve them. Here, we are using the following approach to find the solutions of equations (3.4.19) and (3.4.20).

(a) $A = A(t),\ B = B(t),\ C = \text{constant},\ \ddot{A} = 0,\ B = A^2,\ \dot{F} \neq 0$ and $A\dot{F} + 2\dot{A}F = 0.$

(b) $A = A(t),\ B = B(t),\ C = \text{constant},\ B = A^2,\ \dot{F} = 0$ and $A\ddot{A} + 2\dot{A}^2 = 0.$

(c) $A = A(t),\ C = C(t),\ B = \text{constant},\ \ddot{A} = 0,\ C = A^2,\ \dot{F} \neq 0$ and $C\dot{F} + \dot{C}F = 0.$

(d) $A = A(t),\ C = C(t),\ B = \text{constant},\ C = A^2,\ \dot{F} = 0$ and $\dot{A}\dot{C} + C\ddot{A} = 0.$

(e) $C = C(t),\ B = B(t),\ A = \text{constant},\ \ddot{B} = 0,\ C = B^{-1},\ \dot{F} \neq 0$ and $C\dot{F} + \dot{C}F = 0.$

(f) $C=C(t),\ \ B=B(t),\ \ A=\text{constant},\ \ \dot{F}=0,\ \ C=B^{-1}$ and $B\ddot{B}-\dot{B}^2=0.$

(g) $A=A(t),\ \ B=B(t),\ \ C=C(t)$ and $A=B=C.$

The above equations (3.4.19) and (3.4.20) admit following solutions by using the above approach:

(a) $A=(c_1t+c_2),$ $B=(c_1t+c_2)^2,$ $C=\text{constant},$ $R=2\left[\dfrac{3q^2-4c_1^2}{(c_1t+c_2)^2}\right]$ and

$f(R)=\dfrac{c_3}{2(3q^2+4c_1^2)}R^2+c_4,$ where $c_1,c_2,c_3,c_4\in\Re\setminus\{0\}.$

(b) $A=(c_1t+c_2)^{1/3},$ $B=(c_1t+c_2)^{2/3},$ $C=\text{constant},$ $R=2\left[\dfrac{2c_1^2}{9(c_1t+c_2)^2}+\dfrac{3q^2}{(c_1t+c_2)^{2/3}}\right]$ and

$f(R)=c_3R+c_4,$ where $c_1,c_2,c_3,c_4\in\Re\setminus\{0\}.$

(c) $A=(c_1t+c_2),$ $C=(c_1t+c_2)^2,$ $B=\text{constant},$ $R=2\left[\dfrac{3q^2-4c_1^2}{(c_1t+c_2)^2}\right]$ and

$f(R)=\dfrac{c_3}{2(3q^2+4c_1^2)}R^2+c_4,$ where $c_1,c_2,c_3,c_4\in\Re\setminus\{0\}.$

(d) $A=(c_1t+c_2)^{1/3},$ $C=(c_1t+c_2)^{2/3},$ $B=\text{constant},$ $R=2\left[\dfrac{2c_1^2}{9(c_1t+c_2)^2}+\dfrac{3q^2}{(c_1t+c_2)^{2/3}}\right]$ and

$f(R)=c_3R+c_4,$ where $c_1,c_2,c_3,c_4\in\Re\setminus\{0\}.$

(e) $B=(c_1t+c_2),$ $C=(c_1t+c_2)^{-1},$ $A=\text{constant},$ $R=2\left[\dfrac{3q^2(c_1t+c_2)^2-c_1^2}{(c_1t+c_2)^2}\right]$ and

$f(R)=c_1\sqrt{\dfrac{2}{6q^2-R}}R+c_3,$ where $c_1,c_2,c_3\in\Re\setminus\{0\}.$

(f) $B=e^{c_1t+c_2},$ $C=e^{-(c_1t+c_2)},$ $A=\text{constant},$ $R=2\left[3q^2-c_1^2\right]$ and $f(R)=c_3R+c_4,$ where $c_1,c_2,c_3,c_4\in\Re\setminus\{0\}.$

(g) $A=B=C=\text{constant}$, $R=6q^2$ and $f(R)=\left[F_1+F_2\right]R+c_1$, where $F_1=c_2\cos\sqrt{\frac{R}{3}}t+c_3\sin\sqrt{\frac{R}{3}}t$, $F_2=\frac{3}{3D^2+R}k\left(\rho+p\right)$, in which $D=\frac{d}{dt}$ and $c_1,c_2,c_3\in\Re\setminus\{0\}$.

CVFs for each of the above cases are given below:

Case (i)

In this case, $A=(c_1t+c_2)$, $B=(c_1t+c_2)^2$, $C=\text{constant}$, $R=2\left[\frac{3q^2-4c_1^2}{(c_1t+c_2)^2}\right]$ and $f(R)=\frac{c_3}{2(3q^2+4c_1^2)}R^2+c_4$, where $c_1,c_2,c_3,c_4\in\Re\setminus\{0\}$. Under these constraints, the space-time (3.4.1) takes the form:

$$ds^2=-dt^2+(c_1t+c_2)^2dx^2+e^{2qx}\left[(c_1t+c_2)^4dy^2+dz^2\right]. \tag{3.4.21}$$

Equations (3.4.4) to (3.4.13) along with the (3.4.21) lead to vanishing ψ which directs towards the KVFs expressed in equation (3.4.2).

Case (ii)

Here, $A=(c_1t+c_2)^{1/3}$, $B=(c_1t+c_2)^{2/3}$, $C=\text{constant}$, $R=2\left[\frac{2c_1^2}{9(c_1t+c_2)^2}+\frac{3q^2}{(c_1t+c_2)^{2/3}}\right]$ and $f(R)=c_3R+c_4$, where $c_1,c_2,c_3,c_4\in\Re\setminus\{0\}$. The space-time (3.4.1) after applying an appropriate rescaling of the coordinate z has the form:

$$ds^2=-dt^2+(c_1t+c_2)^{2/3}dx^2+e^{2qx}\left[(c_1t+c_2)^{4/3}dy^2+dz^2\right]. \tag{3.4.22}$$

Now, we are interested in finding CVFs of the above space-time (3.4.22). Performing direct integration techniques and skipping lengthy calculations, one has $\psi=0$, which means that CVFs become KVFs which are given in equation (3.4.2).

Case (iii)

In this case, $A=(c_1t+c_2)$, $C=(c_1t+c_2)^2$, $B=\text{constant}$, $R=2\left[\frac{3q^2-4c_1^2}{(c_1t+c_2)^2}\right]$ and $f(R)=\frac{c_3}{2(3q^2+4c_1^2)}R^2+c_4$, where $c_1,c_2,c_3,c_4\in\Re\setminus\{0\}$. The space-time (3.4.1) takes the following shape after assuming the rescaling in the coordinate y

$$ds^2=-dt^2+(c_1t+c_2)^2dx^2+e^{2qx}\left[dy^2+(c_1t+c_2)^4dz^2\right]. \tag{3.4.23}$$

Using the same technique, we come to know that $\psi=0$, implies CVFs become KVFs which are given in equation (3.4.2).

Case (iv)

In this case, $A=(c_1t+c_2)^{1/3}$, $C=(c_1t+c_2)^{2/3}$, $B=\text{constant}$, $R=2\left[\frac{2c_1^2}{9(c_1t+c_2)^2}+\frac{3q^2}{(c_1t+c_2)^{2/3}}\right]$ and $f(R)=c_3R+c_4$, where $c_1,c_2,c_3,c_4\in\Re\setminus\{0\}$. The space-time (3.4.1) comes into the form

$$ds^2=-dt^2+(c_1t+c_2)^{2/3}dx^2+e^{2qx}\left[dy^2+(c_1t+c_2)^{4/3}dz^2\right]. \tag{3.4.24}$$

CVFs in this case are also become KVFs and are given in equation (3.4.2).

Case (v)

Now, we have $B=(c_1t+c_2)$, $C=(c_1t+c_2)^{-1}$, $A=\text{constant}$, $R=2\left[\frac{3q^2(c_1t+c_2)^2-c_1^2}{(c_1t+c_2)^2}\right]$ and $f(R)=c_1\sqrt{\frac{2}{6q^2-R}}R+c_3$, where $c_1,c_2,c_3\in\Re\setminus\{0\}$. Now, in this case the space-times (3.4.1) after suitable rescaling of x has the form:

$$ds^2=-dt^2+dx^2+e^{2qx}\left[(c_1t+c_2)^2dy^2+(c_1t+c_2)^{-2}dz^2\right]. \tag{3.4.25}$$

Solving equations (3.4.4) to (3.4.13) using the space-time (3.4.25) and skipping lengthy and tedious calculations one finds that $\psi = 0$. Obviously, CVFs become KVFs given by equation (3.4.2).

Case (vi)

Here, we have $B = e^{c_1 t + c_2}$, $C = e^{-(c_1 t + c_2)}$, $A =$ constant, $R = 2\left[3q^2 - c_1^2\right]$ and $f(R) = c_3 R + c_4$, where $c_1, c_2, c_3, c_4 \in \Re \setminus \{0\}$. The space-time (3.4.1) after suitable rescaling of x has the form:

$$ds^2 = -dt^2 + dx^2 + e^{2qx}\left[e^{2c_1 t} dy^2 + e^{-2c_1 t} dz^2\right]. \qquad (3.4.26)$$

Again solving equations (3.4.4) to (3.4.13) with the help of the space-time (3.4.26) and avoiding lengthy and tedious calculations one finds that $\psi = 0$, which means that CVFs become KVFs which are given in equation (3.4.2).

Case (vii)

Here, we have $A = B = C =$ constant, $R = 6q^2$ and $f(R) = \left[F_1 + F_2\right] R + c_1$, where $F_1 = c_2 \cos\sqrt{\frac{R}{3}} t + c_3 \sin\sqrt{\frac{R}{3}} t$, $F_2 = \frac{3}{3D^2 + R} k(\rho + p)$, in which $D = \frac{d}{dt}$ and $c_1, c_2, c_3 \in \Re \setminus \{0\}$. The space-time (3.4.1) after an appropriate rescaling takes the form:

$$ds^2 = -dt^2 + dx^2 + e^{2qx}[dy^2 + dz^2]. \qquad (3.4.27)$$

The above space-time (3.4.26) is conformally flat, therefore admits fifteen independent CVFs which are:

$$X^0 = e^{qx}\left[\left(\frac{y^2 + z^2}{2} + \frac{e^{-2qx}}{2q^2}\right)\left(c_4 e^{qt} + c_5 e^{-qt}\right) + y\left(c_7 e^{qt} + c_8 e^{-qt}\right) + z\left(c_{10} e^{qt} + c_{11} e^{-qt}\right)\right]$$
$$+ e^{qx}\left(c_{14} e^{qt} - c_{15} e^{-qt}\right) + c_{13},$$

$$X^1 = e^{qx}\left[\left(\frac{y^2+z^2}{2}-\frac{e^{-2qx}}{2q^2}\right)\left(c_4e^{qt}-c_5e^{-qt}\right)+y\left(c_7e^{qt}-c_8e^{-qt}\right)+z\left(c_{10}e^{qt}-c_{11}e^{-qt}\right)\right]$$
$$-\frac{1}{q}\left(c_1y-c_2z+c_6\right)+e^{qx}\left(c_{14}e^{qt}+c_{15}e^{-qt}\right),$$

$$X^2 = \left(\frac{y^2-z^2}{2}-\frac{e^{-2qx}}{2q^2}\right)c_1+\frac{e^{-qx}}{q}\left[\left(c_4e^{qt}-c_5e^{-qt}\right)y+\left(c_7e^{qt}-c_8e^{-qt}\right)\right]-c_2yz-c_3z$$
$$+c_6y+c_9,$$

$$X^3 = \left(\frac{y^2-z^2}{2}+\frac{e^{-qx}}{2q^2}\right)c_2+\frac{e^{-qx}}{q}\left[\left(c_4e^{qt}-c_5e^{-qt}\right)z+\left(c_{10}e^{qt}-c_{11}e^{-qt}\right)\right]+c_1yz+c_3y \qquad (3.4.28)$$
$$+c_6z+c_{12},$$

with conformal factor

$$\psi = qe^{qx}\left[\left(\frac{y^2+z^2}{2}+\frac{e^{-2qx}}{2q^2}\right)\left(c_4e^{qt}-c_5e^{-qt}\right)+y\left(c_7e^{qt}-c_8e^{-qt}\right)+z\left(c_{10}e^{qt}-c_{11}e^{-qt}\right)\right]$$
$$+qe^{qx}\left(c_{14}e^{qt}+c_{15}e^{-qt}\right),$$

where $c_i \in \Re$ with $i = 1, 2, 3, \ldots, 15.$

3.5 Conformal Vector Fields of Kantowski Sachs and Bianchi type III Space-Times in f(R) Gravity

The line element representing Kantowski-Sachs and Bianchi type III space-times has the form (Stephani et al., 2003)

$$ds^2 = -dt^2 + Adr^2 + B\left[d\theta^2 + f(\theta)^2 d\phi^2\right], \qquad (3.5.1)$$

where $A = A(t)$ and $B = B(t)$ are nowhere zero functions of t only. For $f(\theta) = \mathrm{Sin}\,\theta,$ the above space-times (3.5.1) become Kantowski-Sachs space-times and for $f(\theta) = \mathrm{Sinh}\,\theta,$ the above space-times become Bianchi type III space-times. The above space-time (3.5.1) admit four linearly independent KVFs which are (Stephani et al., 2003)

$$\frac{\partial}{\partial r},\ \frac{\partial}{\partial \phi},\ \cos\phi\frac{\partial}{\partial \theta}-\frac{f'}{f}\sin\phi\frac{\partial}{\partial \phi},\ \sin\phi\frac{\partial}{\partial \theta}+\frac{f'}{f}\cos\phi\frac{\partial}{\partial \phi}, \tag{3.5.2}$$

where prime over f shows the derivative with respect to θ. Ricci scalar R for the space-time (3.5.1) is

$$R=\frac{1}{2}\left[\frac{\dot{A}^2}{A^2}-\frac{2\dot{A}\dot{B}}{AB}+\frac{\dot{B}^2}{B^2}-\frac{2\ddot{A}}{A}-\frac{4\ddot{B}}{B}+\frac{4\alpha}{B}\right], \tag{3.5.3}$$

where $\alpha=\frac{f''(\theta)}{f(\theta)}$ and dot denotes $\frac{d}{dt}$. Using equation (3.5.1) in equation (1.12.1), one arrives at

$$X^0_{,0}=\psi, \tag{3.5.4}$$

$$X^0_{,1}-AX^1_{,0}=0, \tag{3.5.5}$$

$$X^0_{,2}-BX^2_{,0}=0, \tag{3.5.6}$$

$$X^0_{,3}-Bf^2(\theta)X^3_{,0}=0, \tag{3.5.7}$$

$$\dot{A}X^0+2AX^1_{,1}=2A\psi, \tag{3.5.8}$$

$$AX^1_{,2}+BX^2_{,1}=0, \tag{3.5.9}$$

$$AX^1_{,3}+Bf^2(\theta)X^3_{,1}=0, \tag{3.5.10}$$

$$\dot{B}X^0+2BX^2_{,2}=2B\psi, \tag{3.5.11}$$

$$X^2_{,3}+f^2(\theta)X^3_{,2}=0, \tag{3.5.12}$$

$$\dot{B}X^0+2B\frac{f'}{f}X^2+2BX^3_{,3}=2B\psi. \tag{3.5.13}$$

Solving equation (3.5.4), we have $X^0=\int\psi dt+S^1(r,\theta,\phi),$ where $S^1(r,\theta,\phi)$ is a FOI. Now, by utilizing the value of X^0 in equations (3.5.5), (3.5.6) and (3.5.7), we get

$$\left.\begin{aligned}
&X^0 = \int \psi dt + S^1(r,\theta,\phi), X^1 = \int\left(\frac{1}{A}\int \psi_r dt\right)dt + S_r^1(r,\theta,\phi)\int \frac{dt}{A} + S^2(r,\theta,\phi),\\
&X^2 = \int\left(\frac{1}{B}\int \psi_\theta dt\right)dt + S_\theta^1(r,\theta,\phi)\int \frac{dt}{B} + S^3(r,\theta,\phi),\\
&X^3 = \int\left(\frac{1}{Bf^2(\theta)}\int \psi_\phi\, dt\right)dt + S_\phi^1(r,\theta,\phi)\int \frac{dt}{Bf^2(\theta)} + S^4(r,\theta,\phi),
\end{aligned}\right\} \quad (3.5.14)$$

where $S^i(r,\theta,\phi)$ with $i=2,3,4$ are functions of integration. As, we are in search of CVF X for the space-times under consideration in f(R) theory of gravity therefore, first we need to explore solutions in the said theory. The EMT for the perfect fluid is defined by

$$T_{ab} = (\rho + p)u_a u_b + p g_{ab}, \quad (3.5.15)$$

where ρ and p already defined in section (1.9) with u_a being the four velocity vector defined as $u_a = -\delta_a^0$. By utilizing equation (3.5.1) and (3.5.15) in equation (1.13.2) after some algebraic manipulations, one has

$$\ddot{F} + \left(\frac{\dot{A}}{2A} + \frac{2\dot{B}}{B}\right)\dot{F} + \left(\frac{\dot{A}\dot{B}}{2AB} + \frac{\ddot{B}}{B} - \frac{2\alpha}{B} - R\right)F + f = k(\rho - p). \quad (3.5.16)$$

$$\ddot{F} + \left(\frac{\dot{A}}{A} + \frac{3\dot{B}}{2B}\right)\dot{F} + \left(\frac{3\dot{A}\dot{B}}{4AB} + \frac{\ddot{B}}{2B} + \frac{\ddot{A}}{2A} - \frac{\dot{A}^2}{4A^2} - \frac{\alpha}{B} - R\right)F + f = k(\rho - p). \quad (3.5.17)$$

Subtracting equation (3.5.16) from equation (3.5.17), gives

$$\left(\frac{\dot{A}}{A} - \frac{\dot{B}}{B}\right)\dot{F} + \left(\frac{\ddot{A}}{A} - \frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{2AB} - \frac{\dot{A}^2}{2A^2} + \frac{2\alpha}{B}\right)F = 0. \quad (3.5.18)$$

Here, we solution equation (3.5.18). Skipping the detail, solution of equation (3.5.18) is found in the form of eight cases which are:

(i) $A = (c_1 t + c_2)$, $B = (c_1 t + c_2)^{1/2}$, $R = \frac{1}{2}\left[\frac{4\alpha}{\sqrt{(c_1 t + c_2)}} + \frac{5c_1^2}{4(c_1 t + c_2)^2}\right]$ and

$f(R) = e^{\left(-\frac{8\alpha}{3c_1^2}(c_1 t + c_2)^{3/2} + c_3\right)} R + c_4$, where $c_1, c_2, c_3, c_4 \in \Re \setminus \{0\}$.

(ii) $A=4(c_1t+c_2)^{-2}$, $B=\frac{1}{4}(c_1t+c_2)^2$, $R=2\left[\frac{4\alpha-c_1^2}{(c_1t+c_2)^2}\right]$ and $f(R)=\left[c_3(c_1t+c_2)^{2\alpha/c_1^2}\right]R+c_4$, where $c_1,c_2,c_3,c_4\in\Re\setminus\{0\}$.

(iii) $A=(2c_1t+2c_2)^{1/2}$, $B=(2c_1t+2c_2)^{3/4}$, $R=\frac{1}{32}\left[\frac{16(2c_1t+2c_2)^{5/4}\alpha+21c_1^2}{(c_1t+c_2)^2}\right]$ and $f(R)=\mathrm{e}^{\left(\frac{8\alpha}{5c_1^2}(2c_1t+2c_2)^{5/4}+c_3\right)}R+c_4$, where $c_1,c_2,c_3,c_4\in\Re\setminus\{0\}$.

(iv) $A=\left(\frac{c_1t+c_2}{3}\right)^3$, $B=\left(\frac{c_1t+c_2}{3}\right)^{-1/2}$, $R=\frac{1}{24}\left[16\sqrt{3c_1t+3c_2}-\frac{33c_1^2}{(c_1t+c_2)^2}\right]$ and $f(R)=\mathrm{e}^{\left(\frac{-8\alpha}{35\sqrt{3}c_1^2}(c_1t+c_2)^{5/2}+c_3\right)}R+c_4$, where $c_1,c_2,c_3,c_4\in\Re\setminus\{0\}$.

(v) $A=\text{constant}$, $B=(\alpha t^2+c_1t+c_2)$, $R=\frac{1}{2}\left[\frac{c_1^2-4\alpha c_2}{(\alpha t^2+c_1t+c_2)^2}\right]$ and $f(R)=c_3R+c_4$, where $c_1,c_2,c_3,c_4\in\Re\setminus\{0\}$.

(vi) $A=(c_1t+c_2)^2$, $B=\text{constant}$, $R=2\alpha$ and $f(R)=\mathrm{e}^{\left[\frac{-\alpha}{c_1}\left(c_1\frac{t^2}{2}+c_2t\right)+c_3\right]}R+c_4$, where $c_1,c_2,c_3,c_4\in\Re\setminus\{0\}$.

(vii) $A=t^4$, $B=t^{-1}$, $R=\frac{1}{2}\left[\frac{4\alpha t^3-7}{t^2}\right]$ and $f(R)=c_1\,\mathrm{e}^{\frac{-2\alpha t^3}{15}}R+c_2$, where $c_1,c_2\in\Re\setminus\{0\}$.

(viii) $A=\text{constant}$, $B=t^2$, $R=2\left[\frac{\alpha-1}{t^2}\right]$ and $f(R)=c_1t^{\alpha-1}R+c_2$, where $c_1,c_2\in\Re\setminus\{0\}$.

Case (i)

The information associated over here is $A=(c_1t+c_2)$, $B=(c_1t+c_2)^{1/2}$, $R=\frac{1}{2}\left[\frac{4\alpha}{\sqrt{(c_1t+c_2)}}+\frac{5c_1^2}{4(c_1t+c_2)^2}\right]$ and $f(R)=\mathrm{e}^{\left(-\frac{8\alpha}{3c_1^2}(c_1t+c_2)^{3/2}+c_3\right)}R+c_4$, where $c_1,c_2,c_3,c_4\in\Re\setminus\{0\}$.

The space-time (3.5.1) takes the form:

$$ds^2=-dt^2+(c_1t+c_2)dr^2+(c_1t+c_2)^{1/2}\left[d\theta^2+f(\theta)^2d\phi^2\right]. \qquad (3.5.19)$$

Now, we find CVFs of the space-time (3.5.19) using equations (3.5.4) to (3.5.13). Omitting the process of calculations, we reach at $\psi=0$, representing the KVFs of equation (3.5.2).

Case (ii)

Here, $A=4(c_1t+c_2)^{-2}$, $B=\frac{1}{4}(c_1t+c_2)^2$, $R=2\left[\frac{4\alpha-c_1^2}{(c_1t+c_2)^2}\right]$ and $f(R)=\left[c_3(c_1t+c_2)^{2\alpha/c_1^2}\right]R+c_4$,

where $c_1,c_2,c_3,c_4\in\Re\setminus\{0\}$. The space-time (3.5.1) in this case has the form:

$$ds^2=-dt^2+4(c_1t+c_2)^{-2}dr^2+\frac{1}{4}(c_1t+c_2)^2\left[d\theta^2+f(\theta)^2d\phi^2\right]. \qquad (3.5.20)$$

Again solving equations (3.5.4) to (3.5.13) taking space-times (3.5.20) into account, one has following components of CVFs:

$$\left.\begin{aligned}&X^0=(c_1t+c_2)(c_5r+c_6),\ X^1=2c_1\left[\left(\frac{16c_1^2r^2+(c_1t+c_2)^4}{32c_1^2}\right)c_5+c_6r\right]+c_7,\\&X^2=c_8\cos\phi+c_9\sin\phi,\ \ X^3=\frac{f'}{f}(-c_8\sin\phi+c_9\cos\phi)+c_{10}\end{aligned}\right\}, \qquad (3.5.21)$$

with conformal factor $\psi=c_1(c_5r+c_6)$, where $c_i\in\Re$ with $i=5,6,7,8,9,10$. The above space-times (3.5.21) admit six CVFs out of which four are KVFs which are given in equation (3.5.2), one is HVF which is $(c_1t+c_2)\frac{\partial}{\partial t}+2c_1r\frac{\partial}{\partial r}$, and one is proper conformal vector field which is

$$(c_1t+c_2)r\frac{\partial}{\partial t}+2c_1\left(\frac{16c_1^2r^2+(c_1t+c_2)^4}{32c_1^2}\right)\frac{\partial}{\partial r}. \qquad (3.5.22)$$

Case (iii)

In this case, $A=(2c_1t+2c_2)^{1/2}$, $B=(2c_1t+2c_2)^{3/4}$, $R=\frac{1}{32}\left[\frac{16(2c_1t+2c_2)^{5/4}\alpha+21c_1^2}{(c_1t+c_2)^2}\right]$ and $f(R)=\mathrm{e}^{\left(\frac{8\alpha}{5c_1^2}(2c_1t+2c_2)^{5/4}+c_3\right)}R+c_4$, where $c_1,c_2,c_3,c_4\in\Re\setminus\{0\}$. The space-time (3.5.1) takes the form:

$$ds^2=-dt^2+(2c_1t+2c_2)^{1/2}dr^2+(2c_1t+2c_2)^{3/4}\left[d\theta^2+f(\theta)^2d\phi^2\right]. \qquad (3.5.23)$$

By the same procedure of solving the system of conformal equations from (3.5.4) to (3.5.13) as examined previous case, we come to know that $\psi=0$, $\Rightarrow$ CVFs become KVFs which are given in equation (3.5.2).

Case (iv)

Now, if $A=\left(\frac{c_1t+c_2}{3}\right)^3$, $B=\left(\frac{c_1t+c_2}{3}\right)^{-1/2}$, $R=\frac{1}{24}\left[16\sqrt{3c_1t+3c_2}-\frac{33c_1^2}{(c_1t+c_2)^2}\right]$ and $f(R)=\mathrm{e}^{\left(\frac{-8\alpha}{35\sqrt{3}c_1^2}(c_1t+c_2)^{5/2}+c_3\right)}R+c_4$, where $c_1,c_2,c_3,c_4\in\Re\setminus\{0\}$, the space-time (3.5.1) becomes:

$$ds^2=-dt^2+\left(\frac{c_1t+c_2}{3}\right)^3dr^2+\left(\frac{c_1t+c_2}{3}\right)^{-1/2}\left[d\theta^2+f(\theta)^2d\phi^2\right]. \qquad (3.5.24)$$

Solving equations (3.5.4) to (3.5.13) with the help of space-time (3.5.24) implies that CVFs in this case also become KVFs which are expressed by the equation (3.5.2).

Case (v)

If $A = \text{constant}$, $B = (\alpha t^2 + c_1 t + c_2)$, $R = \frac{1}{2}\left[\frac{c_1^2 - 4\alpha c_2}{(\alpha t^2 + c_1 t + c_2)^2}\right]$ and $f(R) = c_3 R + c_4$, where $c_1, c_2, c_3, c_4 \in \Re \setminus \{0\}$. Now, in this case the space-times (3.5.1) after suitable rescaling of r has the form

$$ds^2 = -dt^2 + dr^2 + (\alpha t^2 + c_1 t + c_2)\left[d\theta^2 + f(\theta)^2 d\phi^2\right]. \qquad (3.5.25)$$

Solving equations (3.5.4) to (3.5.13) using the space-times (3.5.25) one finds $\psi = 0$, which means that no proper CVFs exist. CVFs in this case are also KVFs which are linearly independent shown by equation (3.5.2).

Case (vi)

Here, we have $A = (c_1 t + c_2)^2$, $B = \text{constant}$, $R = 2\alpha$ and $f(R) = \mathrm{e}^{\left[\frac{-\alpha}{c_1}\left(c_1 \frac{t^2}{2} + c_2 t\right) + c_3\right]} R + c_4$, where $c_1, c_2, c_3, c_4 \in \Re \setminus \{0\}$. The space-time (3.5.1) after suitable rescaling of θ and ϕ takes the form:

$$ds^2 = -dt^2 + (c_1 t + c_2)^2 dr^2 + \left[d\theta^2 + f(\theta)^2 d\phi^2\right]. \qquad (3.5.26)$$

Solving equations (3.5.4) to (3.5.13) with the help of the space-times (3.5.26) and avoiding from the lengthy and tedious calculations one finds that $\psi = 0$, giving KVFs which are

$$\left.\begin{array}{l} X^0 = c_5 e^{c_1 r} + c_6 e^{-c_1 r},\ X^1 = \dfrac{-1}{(c_1 t + c_2)}\left[c_5 e^{c_1 r} - c_6 e^{-c_1 r}\right] + c_7, \\ X^2 = c_8 \cos\phi + c_9 \sin\phi, X^3 = \dfrac{f'}{f}(-c_8 \sin\phi + c_9 \cos\phi) + c_{10} \end{array}\right\}, \qquad (3.5.27)$$

where $c_5, c_6, c_7, c_8, c_9, c_{10} \in \Re.$. Here, clearly there are six KVFs admitted by the space-time (3.5.26) out of which four are trivial while other two KVFs are $e^{c_1 r}\frac{\partial}{\partial t} - \left(\frac{e^{c_1 r}}{c_1 t + c_2}\right)\frac{\partial}{\partial r}$ and $e^{-c_1 r}\frac{\partial}{\partial t} + \left(\frac{e^{-c_1 r}}{c_1 t + c_2}\right)\frac{\partial}{\partial r}$.

Case (vii)

Here, we have $A=t^4$, $B=t^{-1}$, $R=\frac{1}{2}\left[\frac{4\alpha t^3-7}{t^2}\right]$ and $f(R)=c_1\,\mathrm{e}^{\frac{-2\alpha t^3}{15}}R+c_2$, where $c_1,c_2\in\Re\setminus\{0\}$. The space-time (3.5.1) takes the form:

$$ds^2=-dt^2+t^4dr^2+t^{-1}\left[d\theta^2+f(\theta)^2d\phi^2\right]. \qquad (3.5.28)$$

Solving equations (3.5.4) to (3.5.13) using the space-times (3.5.28), again we obtain $\psi=0$, which means that no proper CVFs exist. CVFs in this case are basic KVFs.

Case (viii)

In this case, $A=\text{constant}$, $B=t^2$, $R=2\left[\frac{\alpha-1}{t^2}\right]$ and $f(R)=c_1t^{\alpha-1}R+c_2$, where $c_1,c_2\in\Re\setminus\{0\}$. hence, space-time (3.5.1) after an appropriate rescaling of r takes the form:

$$ds^2=-dt^2+dr^2+t^2\left[d\theta^2+f(\theta)^2d\phi^2\right]. \qquad (3.5.29)$$

The above space-times (3.5.29) admit six linearly independent CVFs which are

$$\left.\begin{aligned} &X^0=(c_3r+c_4)t,\, X^1=\left(\frac{t^2+r^2}{2}\right)c_3+c_4r+c_5,\\ &X^2=c_6\cos\phi+c_7\sin\phi,\ X^3=\frac{f'}{f}(-c_6\sin\phi+c_7\cos\phi)+c_8 \end{aligned}\right\}, \qquad (3.5.30)$$

with conformal factor $\psi=(c_3r+c_4)$, where $c_i\in\Re$ with $i=3,4,5,6,7,8$. It is clear from equation (3.5.30) that the above space-time (3.5.29) admits four KVFs which are given in equation (3.5.2), one is HVF which is $t\frac{\partial}{\partial t}+r\frac{\partial}{\partial r}$ and one is proper CVF which is

$$rt\frac{\partial}{\partial t}+\left(\frac{t^2+r^2}{2}\right)\frac{\partial}{\partial r}. \qquad (3.5.31)$$

Hence, the space-time in this case admits proper CVF.

3.6 Summary

The clear picture of obtained results by the above study is given in the following four tables. The first column of the tables is representing case number, second column is showing the equation number of space-time in the respective case while the third and fourth columns are showing equations number of CVFs and dimension of CVFs respectively.

Table 3.1
Results of Bianchi type I space-times

Case No.	Space-time	Equations showing CVFs	Dimension of CVFs
(i)	Eq. (3.2.12)	Eq. (3.2.24)	4
(ii)	Eq. (3.2.26)	Eq. (3.2.27)	4
(iii)	Eq. (3.2.29)	Eq. (3.2.30)	4
(iv)	Eq. (3.2.32)	Eq. (3.2.33)	6
(v)	Eq. (3.2.35)	Eq. (3.2.36)	6
(vi)	Eq. (3.2.38)	Eq. (3.2.39)	6
(vii)	Eq. (3.2.41)	Eq. (3.2.42)	5
(viii)	Eq. (3.2.44)	Eq. (3.2.45)	5
(ix)	Eq. (3.2.47)	Eq. (3.2.48)	5
(x)	Eq. (3.2.50)	Eq. (3.2.51)	5
(xi)	Eq. (3.2.53)	Eq. (3.2.54)	5
(xii)	Eq. (3.2.56)	Eq. (3.2.57)	5
(xiii)	Eq. (3.2.59)	Eq. (3.2.60)	15
(xiv)	Eq. (3.2.62)	Eq. (3.2.63)	15

Table 3.2

Results of Bianchi type II space-times

Case No.	Space-time	Equations showing CVFs	Dimension of CVFs
(i)	Eq. (3.3.23)	Eq. (3.3.24)	4
(ii)	Eq. (3.3.26)	Eq. (3.3.27)	4
(iii)	Eq. (3.3.29)	Eq. (3.3.30)	4
(iv)	Eq. (3.3.32)	Eq. (3.3.2)	3
(v)	Eq. (3.3.33)	Eq. (3.3.34)	4
(vi)	Eq. (3.3.35)	Eq. (3.3.36)	5
(vii)	Eq. (3.3.38)	Eq. (3.3.39)	4

Table 3.03

Results of Bianchi type V space-times

Case No.	Space-time	Equations showing CVFs	Dimension of CVFs
(i)	Eq. (3.4.21)	Eq. (3.4.2)	3
(ii)	Eq. (3.4.22)	Eq. (3.4.2)	3
(iii)	Eq. (3.4.23)	Eq. (3.4.2)	3
(iv)	Eq. (3.4.24)	Eq. (3.4.2)	3
(v)	Eq. (3.4.25)	Eq. (3.4.2)	3
(vi)	Eq. (3.4.26)	Eq. (3.4.2)	3
(vii)	Eq. (3.4.27)	Eq. (3.4.28)	15

Table 3.04

Results of Kantowski Sachs and Bianchi type III space-times

Case No.	Space-time	Equations showing CVFs	Dimension of CVFs
(i)	Eq. (3.5.19)	Eq. (3.5.2)	4
(ii)	Eq. (3.5.20)	Eq. (3.5.21)	6
(iii)	Eq. (3.5.23)	Eq. (3.5.2)	4
(iv)	Eq. (3.5.24)	Eq. (3.5.2)	4
(v)	Eq. (3.5.25)	Eq. (3.5.2)	4
(vi)	Eq. (3.5.26)	Eq. (3.5.27)	6
(vii)	Eq. (3.5.28)	Eq. (3.5.2)	4
(viii)	Eq. (3.5.29)	Eq. (3.5.30)	6

It is important to see over here that in the section (3.4) and (3.5), the techniques generates possible forms of the function of scalar curvature. These forms contain both linear as well as nonlinear functions of the function f(R). We do not fixes these forms rather we have analytically calculated them. As a result we also obtain various shapes of the metric potentials which one can further used to calculate the physical quantities pressure and density for each of the space-time. In particular, forms of the functions having linearity in the scalar curvature provide a way to switch off to the background geometry. For instance, functions like $f(R) = c_1 R + c_2$, where c_1 and c_2 are constants. Clearly, for the vanishing c_2 with value of c_1 to be unity one can recover the results of GR. On the other aspect, assuming nonlinear modes of the function f(R) have several advantages over the linear ones. Going at the level of aspects related to cosmology, nonlinear modes of functions help to discuss the glitches of interior space maintenance, flourishing aspects termed as expanding universe with dark matter etc.

Chapter 4

Conformal Vector Fields of Spatially Homogeneous Rotating Space-Times and PP-Waves Space-Times in f(R) Theory of Gravity

4.1 Introduction

In this chapter, we have found CVFs of spatially homogeneous rotating space-times and pp-waves space-times in the f(R) theory of gravity. Both the space-times retain their own importance in the theory of GR. Rotating solutions of EFEs provide a way for a better understanding of real physical universe. There is a long route of solutions with rotating geometry. Initially, such solutions to the EFEs was sorted out by Gamow. In the subsequent work followed by the Gamow some solutions with rotating geometry was found by Gödel. Later on this idea was further extended the Reboucus who found exact rotating solution of EFEs by making the assumption of perfect fluid and electromagnetic field as a source of curvature (Shabbir, 2019). Similarly, pp-waves space-times admit a very special class of solutions which is known as plane fronted GWs with parallel propagations in the f(R) theory of gravity. PP-waves are GWs introduced by Ehlers and Kund in 1962. In the study of high-energy phenomena and neutron stars, GWs has put a great contribution. PP waves are infect falls in the category of GWs and have a marginal literature. In particular, the concept of kinetic energy of freely falling bodies and the phenomenon of memory effect are well addressed by such waves. A complete work connected with the center of mass density of the GWs exists in a literature. On the other aspect, GWs have a capability to judge any gravitational theory. Therefore, it is quite necessary to study such waves. To make the study more fruitful and easy, we have solved the problem by making classification. For searching proper CVFs, plane waves are further classified in ten cases. The breakup of this chapter is, first to explore some solutions in the theory of f(R) for both the considered space-times and then pursue for the CVFs. This procedure is given in sections (4.2) and (4.3). In the last section of this chapter, a brief summary of obtained results will be presented.

4.2 Conformal Vector Fields of Spatially Homogeneous Rotating Space-Times in the f(R) Theory of Gravity

Consider the spatially homogenous rotating space-times in the usual coordinates (t, r, ϕ, z) with the line element (Stephani et al., 2003)

$$ds^2 = -dt^2 + dr^2 + A(r)d\phi^2 + dz^2 - 2B(r)dtd\phi, \qquad (4.2.1)$$

where $A = A(r)$ and $B = B(r)$ are nowhere zero functions of r only. The minimal isometries for the above space-times (4.2.1) are (Hall, 2004)

$$\frac{\partial}{\partial t}, \ \frac{\partial}{\partial \phi}, \ \frac{\partial}{\partial z}. \qquad (4.2.2)$$

Scalar curvature R associated with the space-times (4.2.1) has the value

$$R = \frac{1}{2(B^2 + A)^2}\begin{bmatrix} 4B^3B'' - B^2B'^2 + 4ABB'' + 3AB'^2 \\ +2B^2A'' - 4BB'A' + 2AA'' - A'^2 \end{bmatrix}, \qquad (4.2.3)$$

where prime is equivalent to $\frac{d}{dr}$. Using equation (4.2.1) in equation (1.12.1), we have ten non-linear conformal equations

$$X^0_{,0} + BX^2_{,0} = \psi, \qquad (4.2.4)$$

$$X^0_{,1} + BX^2_{,1} - X^1_{,0} = 0, \qquad (4.2.5)$$

$$B'X^1 + X^0_{,2} + BX^2_{,2} - AX^2_{,0} + BX^0_{,0} = 2B\psi, \qquad (4.2.6)$$

$$X^0_{,3} + BX^2_{,3} - X^3_{,0} = 0, \qquad (4.2.7)$$

$$X^1_{,1} = \psi, \qquad (4.2.8)$$

$$X^1_{,2} + AX^2_{,1} - BX^0_{,1} = 0, \qquad (4.2.9)$$

$$X^1_{,3} + X^3_{,1} = 0, \qquad (4.2.10)$$

$$A'X^1+2AX^2_{,2}-2BX^0_{,2}=2A\psi, \quad (4.2.11)$$

$$AX^2_{,3}-BX^0_{,3}+X^3_{,2}=0, \quad (4.2.12)$$

$$X^3_{,3}=\psi. \quad (4.2.13)$$

From equations (4.2.7), (4.2.8), (4.2.12) and (4.2.13), we have

$$X^0=\frac{1}{(A+B^2)}\left[A\left(\iint\psi_t dz^2+zE_t^2(t,r,\phi)\right)+B\left(\iint\psi_\phi dz^2+zE_\phi^2(t,r,\phi)\right)\right]+E^3(t,r,\phi),$$

$$X^1=\int\psi dr+E^1(t,\phi,z),$$

$$X^2=\frac{1}{B}\begin{bmatrix}-\frac{1}{(A+B^2)}\left\{A\left(\iint\psi_t dz^2+zE_t^2(t,r,\phi)\right)+B\left(\iint\psi_\phi dz^2+zE_\phi^2(t,r,\phi)\right)\right\}\\ +\iint\psi_t dz^2+zE_t^2(t,r,\phi)\end{bmatrix}$$
$$+E^4(t,r,\phi),$$

$$X^3=\int\psi dz+E^2(t,r,\phi), \quad (4.2.14)$$

where $E^1(t,\phi,z)$, $E^i(t,r,\phi)$ with $i=2,3,4$ are functions of integration. The next procedure is to find these values of integration constants via considering the metrics in the theory under consideration which is f(R) theory. For this, we use equation (4.2.1) in equation (1.13.2) assuming $T_{ab}=0,$ we have

$$F''+\frac{F}{4(A+B^2)^2}\begin{bmatrix}4B^3B''-3B^2B'^2+4ABB''+AB'^2\\ +2B^2A''-4BB'A'+2AA''-A'^2\end{bmatrix}+\left[\frac{AB'+2BA'}{4AB}\right]F'$$
$$-\frac{f}{2}+\frac{RF}{2}=0. \quad (4.2.15)$$

$$F''+\frac{F}{4(A+B^2)^2}\begin{bmatrix}B^3B''-\frac{B^2B'^2}{2}-\frac{A^2B''}{B}+\frac{3AB'^2}{2}\\ +B^2A''-\frac{3BB'A'}{2}+AA''+\frac{AA'B'}{2B}-\frac{A'^2}{2}\end{bmatrix}-\frac{B'F'}{4B}-\frac{f}{2}+\frac{RF}{2}$$
$$-\left[\frac{AB'+2BA'}{4AB}\right]F'=0. \quad (4.2.16)$$

$$\left[\frac{AB'+2BA'}{4AB}\right]F'-\frac{B'^2F}{4(A+B^2)}+\frac{f}{2}-\frac{RF}{2}=0. \qquad (4.2.17)$$

$$F''+\frac{F}{4(A+B^2)^2}\left[\begin{array}{l}4B^3B''-3B^2B'^2-\frac{2B^4A''}{A}+\frac{2B^3A'B'}{A}+4ABB''\\ +AB'^2-2B^2A''-2BB'A'+\frac{B^2A'^2}{A}\end{array}\right]-\frac{A'F'}{2A}-\frac{f}{2}$$
$$+\frac{RF}{2}-\left[\frac{AB'+2BA'}{4AB}\right]F'=0. \qquad (4.2.18)$$

$$F''+\frac{F}{4(A+B^2)^2}\left[\begin{array}{l}4B^3B''-B^2B'^2+4ABB''+3AB'^2\\ +2B^2A''-4BB'A'+2AA''-A'^2\end{array}\right]-\frac{f}{2}+\frac{RF}{2}$$
$$-\left[\frac{AB'+2BA'}{4AB}\right]F'=0. \qquad (4.2.19)$$

Using equation (4.2.17) in equations (4.2.15), (4.2.16), (4.2.18) and (4.2.19), after some simplifications, we have

$$\frac{A'F'}{AF}+\frac{1}{4(A+B^2)^2}\left[\begin{array}{l}3B^2A''+\frac{BB'A'}{2}+AA''-\frac{A'^2}{2}+\frac{2B^4A''}{A}-\frac{2B^3A'B'}{A}-3B^3B''\\ +\frac{AA'B'}{2B}+\frac{3B^2B'^2}{2}-\frac{A^2B''}{B}-\frac{B^2A'^2}{A}-4ABB''-\frac{AB'^2}{2}\end{array}\right]=0. \qquad (4.2.20)$$

The above equation (4.2.20) is obviously hard to solve due to nonlinear terms in the unknowns A, B and F. In such a situation, different stretages may be used to tackle this problem. For example, one can specify the function $f(R)$ and then look for the metric components. It would be better to classify above equation by imposing certain restrictions on the unknowns involved in the above equation (4.2.20). Here are some solutions of above equation (4.2.20):

(i) $A=r^{8/9}$, $B=r^{4/9}$, $R=-\frac{44}{81r^2}$ and $f(R)=c_1R+c_2$, where $c_1,c_2\in\Re(c_1\neq 0)$.

(ii) $A=-\frac{1}{2}e^{2ar}$, $B=e^{ar}$, $R=2a^2$ and $f(R)=c_1Re^{\frac{-3r}{8}\sqrt{\frac{R}{2}}}+c_2$, where $a,c_1,c_2\in\Re(a\neq 0)$.

(iii) $A=r^2-r^4,\quad B=r^2,\quad R=-2$ and $f(R)=c_1 R\,\mathrm{e}^{\left(\frac{r^4+15r^2}{16R}\right)}(1+r^2R)^{-15/64}+c_2,$ where $c_1,c_2\in\Re(c_1\neq 0).$

(iv) $A=\left(c_2 r+c_3\right),\quad B=c_1,\quad R=\dfrac{-c_2^2}{2(c_2 r+c_3+c_1^2)}$ and $f(R)=c_4 R\left(\dfrac{-c_2^2}{2R}\right)^{1/8}\mathrm{e}^{R\left(\frac{c_1}{2c_2}\right)^2}+c_5,$ where $c_1,c_2,c_3,c_4,c_5\in\Re(c_2\neq 0).$

(v) $A=B=(c_1 r+c_2),\qquad R=-\dfrac{1}{2}\left[\dfrac{(c_1^2r^2+2c_1c_2r+c_1r+c_2^2+c_1^2+c_2)c_1^2}{(c_1^2r^2+2c_1c_2r+c_1r+c_2^2+c_2)^2}\right]$ and $f(R)=c_3 e^{\frac{1}{8}\int\frac{(2+A)A'}{(1+A)^2}dr}R+c_4,$ where $c_1,c_2,c_3,c_4\in\Re(c_1\neq 0).$

(vi) $A=B=\text{constant},\ R=0$ and $f(R)=(c_1 r+c_2)R+c_3,$ where $c_1,c_2,c_3\in\Re(c_1\neq 0).$

We will discuss each case one by one.

Case (i)

This case has the constraints along with the function $f(R),\ A=r^{8/9},\ B=r^{4/9},\ R=-\dfrac{44}{81r^2}$ and $f(R)=c_1R+c_2,$ where $c_1,c_2\in\Re(c_1\neq 0).$ The space-time (4.2.1) takes the form:

$$ds^2=-dt^2+dr^2+r^{8/9}d\phi^2+dz^2-2r^{4/9}dtd\phi, \tag{4.2.21}$$

Now, we find CVFs of the space-time (4.2.21) using equations (4.2.4) to (4.2.13). After some lengthy calculations, we find that $\psi=c_3,$ which means that no proper CVFs exist. Here, CVFs become HVFs which are

$$\left.\begin{array}{l} X^0=c_3t+c_4,\ X^1=c_3r,\\ X^2=\dfrac{5\phi}{9}c_3+c_5,\ X^3=c_3z+c_6 \end{array}\right\}, \tag{4.2.22}$$

where $c_3,c_4,c_5,c_6\in\Re\backslash\{0\}.$ The above system of CVFs contain three KVFs which are given in equation (4.2.2) and one is proper HVF. Proper HVF without considering KVFs is

$$X^0 = c_3 t,\ X^1 = c_3 r,\ X^2 = \frac{5\phi}{9} c_3,\ X^3 = c_3 z. \qquad (4.2.23)$$

Case (ii)

Here, we have $A = -\frac{1}{2} e^{2ar}$, $B = e^{ar}$, $R = 2a^2$ and $f(R) = c_1 R e^{\frac{-3r}{8}\sqrt{\frac{R}{2}}} + c_2$, where $a, c_1, c_2 \in \Re (a \neq 0).$ Under the above restrictions, we obtain

$$ds^2 = -dt^2 + dr^2 - \frac{1}{2} e^{2ar} d\phi^2 + dz^2 - 2e^{ar} dt d\phi. \qquad (4.2.24)$$

Again solving equations (4.1.4) to (4.1.13) with the help of equation (4.1.24), one finds that $\psi = 0$, which means that the CVFs are isometries which are given below:

$$\left.\begin{aligned} &X^0 = \frac{2}{a} c_3 e^{-ar} + c_4,\ X^1 = c_3\phi + c_5,\ X^3 = c_7 \\ &X^2 = -c_3\left[\frac{a}{2}\phi^2 + \frac{2}{a} e^{-ar} \sinh ar\right] - c_5 a\phi + c_6 \end{aligned}\right\}, \qquad (4.2.25)$$

where $c_3, c_4, c_5, c_6, c_7 \in \Re \setminus \{0\}$. Form the information (4.2.25), we see that there are five KVFs admitted by the above space-time (4.2.24) in which three are given in equation (4.2.2) and the remaining two KVFs are $\frac{2}{a} e^{-ar} \frac{\partial}{\partial t} + \phi \frac{\partial}{\partial r} - \left(\frac{a}{2}\phi^2 + \frac{2}{a} e^{-ar} \sinh ar\right)\frac{\partial}{\partial \phi}$ and $\frac{\partial}{\partial r} - a\phi \frac{\partial}{\partial \phi}$. Equation (4.2.24) usually known as stationary Gödel space-time (Shabbir et al., 2011).

Case (iii)

In this case, $A = r^2 - r^4$, $B = r^2$, $R = -2$ and $f(R) = c_1 R \mathrm{e}^{\left(\frac{r^4 + 15r^2}{16R}\right)} (1 + r^2 R)^{-15/64} + c_2$, where $c_1, c_2 \in \Re (c_1 \neq 0)$. It is worth mentioning here that for $A = r^2 - r^4$ and $B = r^2$, the space-time (4.2.1) become Som-Raychaudhuri space-time (Shabbir et al., 2011)

$$ds^2 = -dt^2 + dr^2 + r^2(1 - r^2) d\phi^2 + dz^2 - 2r^2 dt d\phi. \qquad (4.2.26)$$

Using the previously described method, we come to the information that $\psi = 0,$ implies CVFs become KVFs which are:

$$\left.\begin{aligned} &X^0 = r\left[c_3 \sin\phi - c_4 \cos\phi\right] + c_5,\ X^1 = c_3 \cos\phi + c_4 \sin\phi, \\ &X^2 = -\frac{1}{r}\left[c_3 \sin\phi - c_4 \cos\phi\right] + c_6,\ X^3 = c_7, \end{aligned}\right\} \tag{4.2.27}$$

where $c_i \in \Re$ with $i = 3, 4, 5, 6, 7.$ The above system given in equation (4.2.27) consists of five KVFs. From these five KVFs three are given in equation (4.2.2) and the remaining two KVFs are $r\sin\phi\frac{\partial}{\partial t} + \cos\phi\frac{\partial}{\partial r} - \frac{1}{r}\sin\phi\frac{\partial}{\partial\phi}$ and $-r\cos\phi\frac{\partial}{\partial t} + \sin\phi\frac{\partial}{\partial r} + \frac{1}{r}\cos\phi\frac{\partial}{\partial\phi}.$

Case (iv)

Here, we have $A = \left(c_2 r + c_3\right),$ $B = c_1,$ $R = \dfrac{-c_2^2}{2(c_2 r + c_3 + c_1^2)}$ and $f(R) = c_4 R\left(\dfrac{-c_2^2}{2R}\right)^{1/8} e^{R\left(\frac{c_1}{2c_2}\right)^2} + c_5,$

where $c_i \in \Re$ with $i = 1, 2, 3, 4, 5.$ In an appropriate frame, we have

$$ds^2 = -dt^2 + dr^2 + \left(c_2 r + c_3\right) d\phi^2 + dz^2 - 2dtd\phi. \tag{4.2.28}$$

Solving equations (4.2.4) to (4.2.13) with the help of space-time (4.2.28) implies that $\psi = c_6,$ therefore CVFs in this case become HVFs which are

$$\left.\begin{aligned} &X^0 = \left(\frac{\phi}{2} + t\right)c_6 + c_7 z + c_8,\ X^1 = \left(\frac{c_2 r + c_3 + 1}{c_2}\right)c_6, \\ &X^2 = \frac{\phi}{2}c_6 + c_9,\ X^3 = c_6 z + (t + \phi)c_7 + c_{10}, \end{aligned}\right\} \tag{4.2.29}$$

where $c_i \in \Re$ with $i = 6, 7, 8, 9, 10.$ Information given in the system (4.2.29) is a combination of four isometries and one proper HVF. The following equation represents proper HVF which is obtained by exclusion of isometries from (4.2.29)

$$X^0 = \left(\frac{\phi}{2} + t\right)c_6,\ X^1 = \left(\frac{c_2 r + c_3 + 1}{c_2}\right)c_6,\ X^2 = \frac{\phi}{2}c_6,\ X^3 = c_6 z. \tag{4.2.30}$$

From the remaining four KVFs one is $z\frac{\partial}{\partial t}+(t+\phi)\frac{\partial}{\partial z}$ and other three are given in equation (4.2.2).

Case (v)

Here, we have $A=B=(c_1r+c_2),$ $R=-\frac{1}{2}\left[\frac{(c_1^2r^2+2c_1c_2r+c_1r+c_2^2+c_1^2+c_2)c_1^2}{(c_1^2r^2+2c_1c_2r+c_1r+c_2^2+c_2)^2}\right]$ and $f(R)=c_3e^{\frac{1}{8}\int\frac{(2+A)A'}{(1+A)^2}dr}R+c_4,$ where $c_i\in\Re$ with $i=1,2,3,4.$ The new shape of space-time (4.2.1) in this case becomes

$$ds^2=-dt^2+dr^2+(c_1r+c_2)d\phi^2+dz^2-2(c_1r+c_2)dtd\phi. \qquad (4.2.31)$$

Solving equations (4.2.4) to (4.2.13) using the space-time (4.2.31) and avoiding from the lengthy calculations one finds that $\psi=0,$ which means that no proper CVFs exist. CVFs in this case are basic KVFs as represented by equation (4.2.2).

Case (vi)

The information $A=B=\text{constant},$ $R=0$ and $f(R)=(c_1r+c_2)R+c_3,$ where $c_1,c_2,c_3\in\Re(c_1\neq 0).$ led towards the following equation in an appropriate frame

$$ds^2=-dt^2+dr^2+d\phi^2+dz^2-2dtd\phi. \qquad (4.2.32)$$

CVFs in this case are

$$X^0=\left(\frac{3t^2+r^2+\phi^2+z^2-2\phi t}{4}\right)c_6+\left(\frac{-t^2+r^2+\phi^2+z^2+2\phi t}{4}\right)c_7+\left(\frac{\phi+t}{2}\right)c_8$$
$$+\left(\frac{\phi-t}{2}\right)c_{14}+\left(\frac{z}{2}\right)c_9+\left(\frac{z}{2}\right)c_{11}-\left(\frac{r}{2}\right)c_{13}+\left(\frac{r}{2}\right)c_{16}+c_4tz+c_5rt+c_{15},$$

$$X^1=\left(\frac{t^2+r^2-\phi^2-z^2+2\phi t}{2}\right)c_5+c_4rz+c_6rt+c_7\phi r+c_8r-c_{10}z-c_{13}\phi+c_{16}t+c_{17},$$

$$X^2=\left(\frac{t^2-r^2+3\phi^2-z^2+2\phi t}{4}\right)c_7+\left(\frac{-t^2+r^2+\phi^2+z^2+2\phi t}{4}\right)c_6+\left(\frac{3\phi+t}{2}\right)c_8$$
$$+\left(\frac{\phi+t}{2}\right)c_{14}+\left(\frac{z}{2}\right)c_9-\left(\frac{z}{2}\right)c_{11}+\left(\frac{r}{2}\right)c_{13}+\left(\frac{r}{2}\right)c_{16}+c_4\phi z+c_5\phi r+c_{18},$$

$$X^3=\left(\frac{t^2-r^2-\phi^2+z^2+2\phi t}{2}\right)c_4+c_5rz+c_6tz+c_7\phi z+c_8z+c_{10}r+c_{11}\phi+c_9t+c_{12}, \qquad (4.2.33)$$

where $c_i\in\Re$ for $i=4,5,6,...,18$. Conformal factor in this case is $\psi=c_4z+c_5r+c_6t+c_7\phi+c_8$. One can find proper CVFs by ignoring HVFs from equation (4.2.33) to get

$$\left.\begin{aligned}
X^0&=\left(\frac{3t^2+r^2+\phi^2+z^2-2\phi t}{4}\right)c_6+\left(\frac{-t^2+r^2+\phi^2+z^2+2\phi t}{4}\right)c_7+c_4tz+c_5rt\\
X^1&=\left(\frac{t^2+r^2-\phi^2-z^2+2\phi t}{2}\right)c_5+c_4rz+c_6rt+c_7\phi r\\
X^2&=\left(\frac{t^2-r^2+3\phi^2-z^2+2\phi t}{4}\right)c_7+\left(\frac{-t^2+r^2+\phi^2+z^2+2\phi t}{4}\right)c_6+c_4\phi z+c_5\phi r\\
X^3&=\left(\frac{t^2-r^2-\phi^2+z^2+2\phi t}{2}\right)c_4+c_5rz+c_6tz+c_7\phi z
\end{aligned}\right\}. \qquad (4.2.34)$$

From the remaining eleven CVFs one is proper HVF which is $\left(\frac{\phi+t}{2}\right)\frac{\partial}{\partial t}+r\frac{\partial}{\partial r}+\left(\frac{3\phi+t}{2}\right)\frac{\partial}{\partial\phi}+z\frac{\partial}{\partial z}$ and other ten KVFs are $\frac{\partial}{\partial t}$, $\frac{\partial}{\partial r}$, $\frac{\partial}{\partial\phi}$, $\frac{\partial}{\partial z}$, $r\frac{\partial}{\partial z}-z\frac{\partial}{\partial r}$, $\frac{z}{2}\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial\phi}\right)+\phi\frac{\partial}{\partial z}$, $\frac{z}{2}\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial\phi}\right)$, $-\frac{r}{2}\frac{\partial}{\partial t}-\phi\frac{\partial}{\partial r}+\frac{r}{2}\frac{\partial}{\partial\phi}$, $\frac{r}{2}\frac{\partial}{\partial t}+t\frac{\partial}{\partial r}+\frac{r}{2}\frac{\partial}{\partial\phi}$, $\left(\frac{\phi-t}{2}\right)\frac{\partial}{\partial t}+\left(\frac{\phi+t}{2}\right)\frac{\partial}{\partial\phi}$.

4.3 Conformal Vector Fields of PP-Wave Space-Times in the f(R) Theory of Gravity

Consider a pp-wave space-times in the harmonic coordinates $(u, x, y, v) = (x^0, x^1, x^2, x^3)$ with the line element (Ehlers and Kundt, 1962)

$$ds^2 = 2Hdu^2 + dx^2 + dy^2 + 2\,du\,dv, \qquad (4.3.1)$$

where H is depending on $u,$ $x,$ and $y.$ The space-times (4.3.1) is unique in the sense as it contain only one KVF $\frac{\partial}{\partial v}.$ Expanding the conformal equation (1.12.1) and using equation (4.3.1) yield

$$H_{,u}X^0 + H_{,x}X^1 + H_{,y}X^2 + 2HX^0_{,0} + X^3_{,0} = 2H\psi, \qquad (4.3.2)$$

$$2HX^0_{,1} + X^3_{,1} + X^1_{,0} = 0, \qquad (4.3.3)$$

$$2HX^0_{,2} + X^3_{,2} + X^2_{,0} = 0, \qquad (4.3.4)$$

$$2HX^0_{,3} + X^3_{,3} + X^0_{,0} = 2\psi, \qquad (4.3.5)$$

$$X^1_{,1} = \psi, \qquad (4.3.6)$$

$$X^1_{,2} + X^2_{,1} = 0, \qquad (4.3.7)$$

$$X^1_{,3} + X^0_{,1} = 0, \qquad (4.3.8)$$

$$X^2_{,2} = \psi, \qquad (4.3.9)$$

$$X^2_{,3} + X^0_{,2} = 0, \qquad (4.3.10)$$

$$X^0_{,3} = 0. \qquad (4.3.11)$$

Initial system obtained by solving the equations (4.3.5), (4.3.8), (4.3.10) and (4.3.11), has the form:

$$\left.\begin{aligned} X^0 &= Z^1, \\ X^1 &= -vZ^1_x + Z^3, \\ X^2 &= -vZ^1_y + Z^2, \\ X^3 &= 2\int \Omega dv - vZ^1_u + Z^4, \end{aligned}\right\} \tag{4.3.12}$$

where Z^i with $i = 1, 2, 3, 4$ are functions of $u,$ x and $y.$ It is worth mentioning that at the first step of this study, we would like to construct some pp-wave solutions of EFEs in the f(R) theory of gravity and then we will find the proper CVFs of the obtained solutions. For doing this, we use equation (4.3.1) in equation (1.13.2) assuming $T_{ab} = 0,$ we have

$$\left[H_{,xx} + H_{,yy}\right]F - 2\left[F_{,xx} + F_{,yy}\right]H + F_{,uu} + H_{,x}F_{,x} + H_{,y}F_{,y} + Hf = 0. \tag{4.3.13}$$

$$2F_{,yy} = f,\ 2F_{,xx} = f,\ \ F_{,ux} = 0,\ \ F_{,uy} = 0,\ \ F_{,xy} = 0. \tag{4.3.14}$$

The above equations (4.3.13) and (4.3.14) contain two unknowns namely F and H which need to be determined. Indeed, upon integration of $F_{,ux} = 0,$ with respect to x and u give

$$F = \int A^1(u, y)du + A^2(x, y), \tag{4.3.15}$$

where $A^1(u, y)$ and $A^2(x, y)$ are functions of integration. Now, making use of equation (4.3.15) in $F_{,uy} = 0$ and $F_{,xy} = 0$ leads to $A^1(u, y) = Q(u)$ and $A^2(x, y) = \int A^4(x)dx + A^5(y),$ where $Q(u),$ $A^4(x)$ and $A^5(y)$ are functions of integration. Hence, equation (4.3.15) becomes

$$F = \int Q(u)du + \int A^4(x)dx + A^5(y). \tag{4.3.16}$$

Now, using equation (4.3.16) in $2F_{,yy} = f$ and $2F_{,xx} = f$ implies $A^5_{,yy}(y) = A^4_{x}(x),$ which on differentiating with respect to $x,$ give $A^4_{,xx}(x) = 0$ and hence $A^4(x) = c_1 x + c_2,$ where $c_1, c_2 \in R (c_1 \neq 0).$ Similarly, the value of $A^5(y)$ turns out to be $A^5(y) = c_1 \dfrac{y^2}{2} + c_3 y + c_4,$ where

$c_3, c_4 \in R.$ Substituting the values of $A^4(x)$ and $A^5(y)$. in equation (4.3.16), give $F = c_1\left(\frac{x^2+y^2}{2}\right) + c_2 x + c_3 y + \int Q(u)du + c_4.$ Using the value of F in equation (4.3.13), we get

$$\left[H_{,xx} + H_{,yy}\right]F - 2c_1 H + c_1\left[xH_{,x} + yH_{,y}\right] + c_2 H_{,x} + c_3 H_{,y} + Q_{,u}(u) = 0, \qquad (4.3.17)$$

Now, our purpose is to obtain the solution of equation (4.3.17), which by substituting $c_2 = c_3 = Q_{,u}(u) = 0,$ admits a plane wave solution of the form (Aichelburg, 1970)

$$H = J(u)\left(\frac{x^2-y^2}{2}\right) + K(u)xy, \qquad (4.3.18)$$

where $J(u)$ and $K(u)$ are known as the two polarization states of the plane wave depending on u. From the physical point of view, plane waves have a role in the advancement of electrodynamics starting from the time of earliest radio transmissions via modern communication system. Moreover, the solution (4.3.18) is the mathematical form of generalized plane wave and require further insight. For better understanding the space-time structure of the plane wave, it would be interesting to find the nature of plane waves. The plane wave solution given in equation (4.3.18) becomes linearly polarized if we take $J(u) = \text{constant}$ and $K(u) = 0.$ Similarly, it becomes screw symmetric if H is only a function of x and y. It is important to mention here that there are numerous special possible choices of the polarization states $J(u)$ and $K(u)$ whose Killing vector fields have already been discussed in (Sipple and Goenner, 1986). Further, a special choice of taking $J(u) = u^{-2}$ and $K(u) = 0,$ or $K(u) = u^{-2}$ and $J(u) = 0$ or $J(u) = K(u) = u^{-2}$ in equation (4.3.18) yields an extra Killing vector field, so extra conservation law (Jamal and Shabbir, 2016). In this study, we will look for proper CVFs of the space-times (4.3.1) by taking equation (4.3.18) into account. Further, it is necessary to mention here that if the function H in equation (4.3.18) satisfy the condition $uH_u + 2H = 0,$ then $J(u) = K(u) = u^{-2}.$ Similarly, $uH_u + xH_x + yH_y + 2H = 0$ implies that $J(u) = K(u) = u^{-4}$ (Kuhnel and Rademacher, 2004). For better understanding the geometry of pp waves, we will classify the above equation (4.3.18) by putting some other restrictions on $J(u)$ and $K(u)$. This classification involves the following cases:

(i) $H = u^{-2}\left(\frac{x^2 - y^2}{2}\right) + axy,$ where $a \in \Re \setminus \{0\}$.

(ii) $H = a\left(\frac{x^2 - y^2}{2}\right) + u^{-2}xy,$ where $a \in \Re \setminus \{0\}$.

(iii) $H = u^{-2}\left(\frac{x^2 - y^2}{2}\right)$.

(iv) $H = u^{-2}xy$. (v) $H = u^{-2}\left[\frac{x^2 - y^2}{2} + xy\right]$.

(vi) $H = a\left[\frac{x^2 - y^2}{2} + xy\right],$ where $a \in \Re \setminus \{0\}$.

(vii) $H = a\left(\frac{x^2 - y^2}{2}\right),$ where $a \in \Re \setminus \{0\}$.

(viii) $H = axy,$ where $a \in \Re \setminus \{0\}$.

(ix) $H = au^{-4}\left(\frac{x^2 - y^2}{2}\right),$ where $a \in \Re \setminus \{0\}$.

(x) $H = au^{-4}xy,$ where $a \in \Re \setminus \{0\}$.

In the following lines, we will use the values of H from the above cases into equation (4.3.1) to formulate the space-times and then try to investigate the CVFs for each of the above case.

Case (i)

In this case, we have $H = u^{-2}\left(\frac{x^2 - y^2}{2}\right) + axy,$ where $a \in \Re \setminus \{0\}$. The space-times (4.3.1) take the form

$$ds^2 = \left[u^{-2}\left(x^2 - y^2\right) + 2axy\right]du^2 + dx^2 + dy^2 + 2\,du\,dv. \qquad (4.3.19)$$

Now, we find proper CVFs of the space-times (4.3.19) using equations (4.3.2) to (4.3.11). If one proceeds further after some calculations one finds that $\psi = c_6,$ which means that no proper CVFs exist. The CVFs become HVFs which are

$$X^0 = 0, \;\; X^1 = c_6 x, \;\; X^2 = c_6 y, \;\; X^3 = 2c_6 v + c_7, \qquad (4.3.20)$$

where $c_6, c_7 \in \Re$. The components of CVFs given in equation (4.3.20) are combination of one isometry and other is proper HVF. The proper HVF after subtracting KVF from (4.3.20) is

$$X = \left(0,\, x,\, y,\, 2v\right). \qquad (4.3.21)$$

Case (ii)

Here, we have $H = a\left(\frac{x^2 - y^2}{2}\right) + u^{-2}xy,$ where $a \in \Re \setminus \{0\}$. The space-times (4.3.1) take the form

$$ds^2 = \left[a\left(x^2 - y^2\right) + 2u^{-2}xy\right]du^2 + dx^2 + dy^2 + 2\,du\,dv. \qquad (4.3.22)$$

Again solving equations (4.3.2) to (4.3.11) with the help of space-time (4.3.22), one finds that $\psi = c_6,$ which implies that no proper CVFs exist. Here, the CVFs become HVFs which are given in equation (4.3.20). The Proper HVF for this case is exactly the same as given in equation (4.3.21).

Case (iii)

With $H = u^{-2}\left(\frac{x^2 - y^2}{2}\right)$, the space-times (4.3.1) has the form

$$ds^2 = u^{-2}\left(x^2 - y^2\right)du^2 + dx^2 + dy^2 + 2\,du\,dv. \qquad (4.3.23)$$

The space-time (4.3.23) and equations (4.3.2) to (4.3.11) implies that $\psi = c_7$, which means that no proper CVFs exist. The CVFs again become HVFs which are

$$X^0 = c_8 u,\;\; X^1 = c_7 x + c_9 u^{\frac{\alpha}{2}} + c_{10} u^{\frac{-\beta}{2}},\;\; X^2 = c_7 y + \sqrt{u}\left[c_{11}\sin\lambda + c_{12}\cos\lambda\right],$$

$$X^3 = 2c_7 v - c_8 v + \frac{c_{12} y}{2\sqrt{u}}\left[\sqrt{3}\sin\lambda - \cos\lambda\right] - \frac{c_{11} y}{2\sqrt{u}}\left[\sin\lambda + \sqrt{3}\cos\lambda\right] - \frac{\gamma x}{2} + c_{13}, \qquad (4.3.24)$$

where $c_i \in \Re$ with $i = 7,8,9,10,11,12,13,$ $\alpha = \left(1+\sqrt{5}\right)$, $\beta = \left(-1+\sqrt{5}\right)$, $\gamma = c_9 \alpha u^{\frac{\beta}{2}} + c_{10}\beta u^{\frac{-\alpha}{2}}$ and $\lambda = \frac{\sqrt{3}\ln u}{2}$. The space-time (4.3.23) admits seven CVFs in which six are KVFs which are

$$\frac{\partial}{\partial v}, \qquad u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v}, \qquad \sqrt{u}\sin\lambda\frac{\partial}{\partial y} - \frac{y}{2\sqrt{u}}\left[\sin\lambda + \sqrt{3}\cos\lambda\right]\frac{\partial}{\partial v}, \qquad u^{\frac{\alpha}{2}}\frac{\partial}{\partial x} - \frac{\alpha x}{2}u^{\frac{\beta}{2}}\frac{\partial}{\partial v}$$

$\sqrt{u}\cos\lambda\frac{\partial}{\partial y} + \frac{y}{2\sqrt{u}}\left[\sqrt{3}\sin\lambda - \cos\lambda\right]\frac{\partial}{\partial v}$, $u^{\frac{-\beta}{2}}\frac{\partial}{\partial x} - \frac{\beta x}{2}u^{\frac{-\alpha}{2}}\frac{\partial}{\partial v}$, and one is proper HVF. The proper HVF after subtracting KVFs from (4.3.24) is given in equation (4.3.21).

Case (iv)

Here, we have $H = u^{-2}xy$ and the space-times (4.3.1) take the form

$$ds^2 = 2u^{-2}xy\,du^2 + dx^2 + dy^2 + 2\,du\,dv. \qquad (4.3.25)$$

In this case, we also have $\psi = c_7$ which means that no proper CVFs exist, the CVFs become HVFs which are:

$$X^0 = c_6 u,\ X^1 = c_7 x - \sqrt{u}\left[c_8 \sin\lambda + c_9 \cos\lambda\right] + c_{10} u^{\frac{\alpha}{2}} + c_{11} u^{\frac{-\beta}{2}},$$

$$X^2 = c_7 y + \sqrt{u}\left[c_8 \sin\lambda + c_9 \cos\lambda\right] + c_{10} u^{\frac{\alpha}{2}} + c_{11} u^{\frac{-\beta}{2}}, \tag{4.3.26}$$

$$X^3 = 2c_7 v - c_6 v + \frac{c_8}{2\sqrt{u}}\left[x\sin\lambda - y\sin\lambda + \sqrt{3}\left(x\cos\lambda - y\cos\lambda\right)\right] + c_{12},$$
$$-\frac{c_9}{2\sqrt{u}}\left[\sqrt{3}\left(x\sin\lambda - y\sin\lambda\right) - x\cos\lambda + y\cos\lambda\right] - \frac{c_{10}\alpha\left(x+y\right)}{2}u^{\frac{\beta}{2}} + \frac{c_{11}\beta\left(x+y\right)}{2}u^{\frac{-\alpha}{2}},$$

where $\alpha = \left(1+\sqrt{5}\right)$, $\beta = \left(-1+\sqrt{5}\right)$, $\lambda = \frac{\sqrt{3}\ln u}{2}$ and $c_i \in \Re$ with $i = 6,7,8,9,10,11,12.$ The set of isometries over hare are

$$\sqrt{u}\sin\lambda\left(\frac{\partial}{\partial y} - \frac{\partial}{\partial x}\right) + \frac{1}{2\sqrt{u}}\left[x\sin\lambda - y\sin\lambda + \sqrt{3}\left(x\cos\lambda - y\cos\lambda\right)\right]\frac{\partial}{\partial v},\quad u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v},\quad \frac{\partial}{\partial v},$$

$$\sqrt{u}\cos\lambda\left(\frac{\partial}{\partial y} - \frac{\partial}{\partial x}\right) - \frac{1}{2\sqrt{u}}\left[\sqrt{3}\left(x\sin\lambda - y\sin\lambda\right) - x\cos\lambda + y\cos\lambda\right]\frac{\partial}{\partial v},$$

$u^{\frac{\alpha}{2}}\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) - \frac{\alpha(x+y)}{2}u^{\frac{\beta}{2}}\frac{\partial}{\partial v},\quad u^{\frac{-\beta}{2}}\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) + \frac{\beta(x+y)}{2}u^{\frac{-\alpha}{2}}\frac{\partial}{\partial v}$. The remaining one is proper HVF. The proper HVF after subtracting KVFs from (4.3.26) is given in equation (4.3.21).

Case (v)

In this case, we have $H = u^{-2}\left[\frac{x^2 - y^2}{2} + xy\right]$ and the space-times (4.3.1) take the form

$$ds^2 = u^{-2}\left[x^2 - y^2 + 2xy\right]du^2 + dx^2 + dy^2 + 2\,du\,dv. \tag{4.3.27}$$

Solving equations (4.3.2) to (4.3.11) with the help of the space-time (4.3.27) yields $\psi = c_6$, which indicates that the CVFs become HVFs which are

$$X^0 = c_5 u,\ X^1 = c_6 x + \eta_1\left[c_7 u^{\frac{1+\alpha}{2}} + c_8 u^{\frac{1-\alpha}{2}}\right] + \eta_2\sqrt{u}\left[c_9 \sin\left(\frac{\beta\ln u}{2}\right) + c_{10}\cos\left(\frac{\beta\ln u}{2}\right)\right],$$

$$X^2 = c_6 y + c_7 u^{\frac{1+\alpha}{2}} + c_8 u^{\frac{1-\alpha}{2}} + \sqrt{u}\left[c_9 \sin\left(\frac{\beta \ln u}{2}\right) + c_{10} \cos\left(\frac{\beta \ln u}{2}\right)\right], \qquad (4.3.28)$$

$$X^3 = 2c_6 v - c_5 v - \frac{\gamma_1}{2}\left[c_7(1+\alpha)u^{\frac{-1+\alpha}{2}} - c_8(-1+\alpha)u^{\frac{-1-\alpha}{2}}\right] - \frac{c_9\gamma_2}{2\sqrt{u}}\left[\beta\cos\left(\frac{\beta \ln u}{2}\right) + \sin\left(\frac{\beta \ln u}{2}\right)\right]$$
$$+ \frac{c_{10}\gamma_2}{2\sqrt{u}}\left[\beta\sin\left(\frac{\beta \ln u}{2}\right) - \cos\left(\frac{\beta \ln u}{2}\right)\right] + c_{11},$$

where $\alpha = \sqrt{1+4\sqrt{2}}$, $\beta = \sqrt{-1+4\sqrt{2}}$, $\gamma_1 = x + y + \sqrt{2}x$, $\gamma_2 = x + y - \sqrt{2}x$, $\eta_1 = 1+\sqrt{2}$, $\eta_2 = 1-\sqrt{2}$ and $c_i \in \Re$ with $i = 5,6,7,8,9,10,11$. Again the the set of isometries are

$$\eta_1 u^{\frac{1-\alpha}{2}}\left[\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right] + \frac{\gamma_1}{2}(-1+\alpha)u^{\frac{-1-\alpha}{2}}\frac{\partial}{\partial v},\ \eta_1 u^{\frac{1+\alpha}{2}}\left[\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right] - \frac{\gamma_1}{2}(1+\alpha)u^{\frac{-1+\alpha}{2}}\frac{\partial}{\partial v},$$

$$\eta_2\sqrt{u}\sin\left(\frac{\beta \ln u}{2}\right)\left[\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right] - \frac{\gamma_2}{2\sqrt{u}}\left[\beta\cos\left(\frac{\beta \ln u}{2}\right) + \sin\left(\frac{\beta \ln u}{2}\right)\right]\frac{\partial}{\partial v},\ u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v},$$

$$\eta_2\sqrt{u}\cos\left(\frac{\beta \ln u}{2}\right)\left[\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right] + \frac{\gamma_2}{2\sqrt{u}}\left[\beta\sin\left(\frac{\beta \ln u}{2}\right) - \cos\left(\frac{\beta \ln u}{2}\right)\right]\frac{\partial}{\partial v},\ \frac{\partial}{\partial v}.$$

The remaining one is proper HVF. The proper HVF after subtracting KVFs from (4.3.28) is same as given in equation (4.3.21).

Case (vi)

Here, we have a sub class of screw symmetric pp-waves and H is of the form $H = a\left[\frac{x^2 - y^2}{2} + xy\right]$, where $a \in \Re \setminus \{0\}$. The space-times (4.3.1) after an appropriate rescaling of u take the form

$$ds^2 = \left[x^2 - y^2 + 2xy\right]du^2 + dx^2 + dy^2 + 2\,du\,dv. \qquad (4.3.29)$$

Again in this case $\psi = c_7$, $\Rightarrow$ that the CVFs become HVFs which are

$$X^0 = c_6,\ X^1 = c_7 x + \beta\left[c_8 \sin\left(2^{1/4}u\right) - c_9 \cos\left(2^{1/4}u\right)\right] + \alpha\left[c_{10} e^{\left(-2^{1/4}\right)u} + c_{11} e^{\left(2^{1/4}\right)u}\right],$$

$$X^2 = c_7 y - \left[c_8 \sin\left(2^{1/4}u\right) - c_9 \cos\left(2^{1/4}u\right)\right] + c_{10} e^{\left(-2^{1/4}\right)u} + c_{11} e^{\left(2^{1/4}\right)u},$$

$$\begin{aligned} X^3 &= 2c_7 v + \frac{2^{-1/4}}{\alpha}\left[\sqrt{2}\,(3x+y) + 4x + 2y\right]\left[c_{10} e^{\left(-2^{1/4}\right)u} - c_{11} e^{\left(2^{1/4}\right)u}\right] \\ &\quad + \frac{2^{-1/4}}{\alpha}\left[\sqrt{2}\,(x-y) - 2y\right]\left[-c_8 \cos\left(2^{1/4}u\right) - c_9 \sin\left(2^{1/4}u\right)\right] + c_{12}, \end{aligned} \tag{4.3.30}$$

where $\alpha = \left(1+\sqrt{2}\right)$, $\beta = \left(-1+\sqrt{2}\right)$ and $c_i \in \Re$ with $i = 6,7,8,9,10,11,12$. The space-time (4.3.29) admits seven CVFs in which six are KVFs which are

$$\alpha \sin\left(2^{-1/4}u\right)\frac{\partial}{\partial x} - \sin\left(2^{1/4}u\right)\frac{\partial}{\partial y} - \frac{2^{-1/4}}{\alpha}\left[\sqrt{2}\,(x-y) - 2y\right]\cos\left(2^{1/4}u\right)\frac{\partial}{\partial v},$$

$$-\beta \cos\left(2^{1/4}u\right)\frac{\partial}{\partial x} + \cos\left(2^{1/4}u\right)\frac{\partial}{\partial y} - \frac{2^{-1/4}}{\alpha}\left[\sqrt{2}\,(x-y) - 2y\right]\sin\left(2^{1/4}u\right)\frac{\partial}{\partial v},$$

$$e^{\left(-2^{1/4}\right)u}\left[\alpha\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right] + \frac{2^{-1/4}}{\alpha}\left[\sqrt{2}\,(3x+y) + 4x + 2y\right]e^{\left(-2^{1/4}\right)u}\frac{\partial}{\partial v},\quad \frac{\partial}{\partial u},$$

$$e^{\left(2^{1/4}\right)u}\left[\alpha\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right] - \frac{2^{-1/4}}{\alpha}\left[\sqrt{2}\,(3x+y) + 4x + 2y\right]e^{\left(2^{1/4}\right)u}\frac{\partial}{\partial v},\quad \frac{\partial}{\partial v}.$$

The remaining one is proper HVF. The proper HVF is given in equation (4.3.21).

Case (vii)

The plane wave linearly polarized is represented by $H = a\left(\frac{x^2 - y^2}{2}\right)$, where $a \in \Re \setminus \{0\}$. The space-times (4.3.1) after an appropriate rescaling of u take the form

$$ds^2 = \left(x^2 - y^2\right) du^2 + dx^2 + dy^2 + 2\,du\,dv. \qquad (4.3.31)$$

Here, again we get $\psi = c_7$, showing that the CVFs become HVFs which are

$$X^0 = c_6,\;\; X^1 = c_7 x + c_{10} e^{-u} + c_{11} e^{u},\;\; X^2 = c_7 y + c_8 \sin u + c_9 \cos u,$$

$$X^3 = 2c_7 v - y\left[c_8 \cos u - c_9 \sin u\right] + x\left[c_{10} e^{-u} - c_{11} e^{u}\right] + c_{12}, \qquad (4.3.32)$$

where $c_i \in \Re$ with $i = 6,7,8,9,10,11,12.$ The space-time (4.3.31) admits seven CVFs in which six are KVFs which are $\frac{\partial}{\partial u}$, $\sin u \frac{\partial}{\partial y} - y \cos u \frac{\partial}{\partial v}$, $\cos u \frac{\partial}{\partial y} + y \sin u \frac{\partial}{\partial v}$, $e^{-u}\left[\frac{\partial}{\partial x} + x \frac{\partial}{\partial v}\right]$, $e^{u}\left[\frac{\partial}{\partial x} - x \frac{\partial}{\partial v}\right]$, $\frac{\partial}{\partial v}$ and the remaining one is proper HVF represented by the equation (4.3.21).

Case (viii)

Here, we have another form of screw symmetric pp-wave and $H = axy$, where $a \in \Re \setminus \{0\}$. The space-times (4.3.1) after an appropriate rescaling of u take the form

$$ds^2 = 2xy\,du^2 + dx^2 + dy^2 + 2\,du\,dv. \qquad (4.3.33)$$

For this case, we have the conformal factor $\psi = c_7$ which is the indication that the CVFs are infect HVFs:

$$X^0 = c_6,\; X^1 = c_7 x - \left[c_{10} \sin u + c_{11} \cos u\right] + c_8 e^{u} + c_9 e^{-u},$$

$$X^2 = c_7 y + c_{10} \sin u + c_{11} \cos u + c_8 e^{u} + c_9 e^{-u},$$

$$X^3 = 2c_7 v + \left(x - y\right)\left[c_{10} \cos u - c_{11} \sin u\right] - \left(x + y\right)\left[c_8 e^{u} - c_9 e^{-u}\right] + c_{12}, \qquad (4.3.34)$$

where $c_i \in \Re$ with $i = 6,7,8,9,10,11,12.$ From equation (4.3.34), we see that there are six isometries which are $\sin u\left[\frac{\partial}{\partial y} - \frac{\partial}{\partial x}\right] + \cos u \left(x - y\right)\frac{\partial}{\partial v}$, $e^{u}\left[\frac{\partial}{\partial x} + \frac{\partial}{\partial y} - \left(x + y\right)\frac{\partial}{\partial v}\right]$,

$\cos u\left[\frac{\partial}{\partial y}-\frac{\partial}{\partial x}\right]-\sin u\left(x-y\right)\frac{\partial}{\partial v},\;\; e^{-u}\left[\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\left(x+y\right)\frac{\partial}{\partial v}\right],\;\; \frac{\partial}{\partial u},\;\; \frac{\partial}{\partial v}$ and one proper HVF. The proper HVF is given in equation (4.3.21).

Case (ix)

Here, we have $H=au^{-4}\left(\frac{x^2-y^2}{2}\right)$, where $a\in\Re\setminus\{0\}$. The space-times (4.3.1) after an appropriate rescaling of u take the form:

$$ds^2=u^{-4}\left(x^2-y^2\right)du^2+dx^2+dy^2+2\,du\,dv. \qquad (4.3.35)$$

The CVFs in this case are:

$$X^0=c_6u^2,$$

$$X^1=\left(c_6u+c_7\right)x+u\left[c_{10}P+c_{11}Q\right],$$

$$X^2=\left(c_6u+c_7\right)y+u\left[c_8\phi+c_9\omega\right],$$

$$\begin{aligned}X^3=&-c_6\left(\frac{x^2+y^2}{2}\right)+2c_7v-\frac{c_8y}{u}\left[u\phi-\omega\right]-\frac{c_9y}{u}\left[u\omega+\phi\right]\\&-\frac{c_{10}x}{u}\left[uP-Q\right]+\frac{c_{11}x}{u}\left[-uQ+P\right]+c_{12},\end{aligned} \qquad (4.3.36)$$

where $P=sinhu^{-1}$, $Q=\cos hu^{-1}$, $\phi=\sin u^{-1}$, $\omega=\cos u^{-1}$, $\psi=\left(c_6u+c_7\right)$ and $c_i\in\Re$ with $i=6,7,8,9,10,11,12$. Components of CVFs represented by the system (4.3.36) are composed of six KVFs, one proper HVF and one proper which is

$$X^0=c_6u^2,\;\; X^1=c_6ux,\;\; X^2=c_6uy,\;\; X^3=-c_6\left(\frac{x^2+y^2}{2}\right). \qquad (4.3.37)$$

Case (x)

In this case, we have $H = au^{-4}xy,$ where $a \in \Re \setminus \{0\}$. The space-times (4.3.1) after suitable rescaling of u take the form:

$$ds^2 = 2u^{-4}xy\,du^2 + dx^2 + dy^2 + 2\,du\,dv. \tag{4.3.38}$$

Here, the CVFs are:

$$\begin{aligned}
X^0 &= c_6 u^2, \\
X^1 &= (c_6 u + c_7)x - u[c_8\omega + c_9\phi] + u[c_{10}Q + c_{11}P], \\
X^2 &= (c_6 u + c_7)y + u[c_8\omega + c_9\phi] + u[c_{10}Q + c_{11}P], \\
X^3 &= -c_6\left(\frac{x^2+y^2}{2}\right) + 2c_7 v + \frac{c_8(x-y)}{u}[u\omega + \phi] \\
&\quad + \frac{c_9(x-y)}{u}[u\phi - \omega] - \frac{c_{10}(x+y)}{u}[uQ - P] \\
&\quad - \frac{c_{11}(x+y)}{u}[uP - Q] + c_{12},
\end{aligned} \tag{4.3.39}$$

where $P = sinhu^{-1}$, $Q = \cos hu^{-1}$, $\phi = \sin u^{-1}$, $\omega = \cos u^{-1}$, $\psi = (c_6 u + c_7)$ and $c_i \in \Re$ with $i = 6, 7, 8, 9, 10, 11, 12.$ The proper CVF here turn out to be

$$X^0 = c_6 u^2,\ X^1 = c_6 ux,\ X^2 = c_6 uy,\ X^3 = -c_6\left(\frac{x^2+y^2}{2}\right). \tag{4.3.40}$$

4.4 Summary

The plan of the present chapter was: firstly, we have found some rotating and pp-wave solutions of EFEs in the f(R) theory of gravity, secondly, we have found CVFs of obtained solutions. From this study, the following results are obtained:

(a) For the section (4.2), we have the following results:

(a-1) The space-times in the cases (i) and (iv) admit proper HVFs of dimension four and five. These are the space-times (4.2.21) and (4.2.28). The proper HVFs for these cases are given in equations (4.2.23) and (4.2.30).

(a-2) For the cases (ii), (iii) and (v), the CVFs become KVFs of dimension three and five. The space-times for these cases are given in equations (4.2.24), (4.2.26) and (4.2.31). KVFs are given in equations (4.2.2), (4.2.25) and (4.2.27).

(a-3) The space-time in the case (vi) is conformally flat, therefore admit fifteen independent CVFs. This is the space-time (4.2.32) and the proper CVFs are given in equation (4.2.34).

(b) In the second section, a classification of pp-waves space-times according to their proper CVFs is presented. Parallel propagating waves (pp-waves) also known as plane fronted GWs has remained a topic of special interest in the last couple of years and in the modern theoretical physics because of the new theoretical ideas like kinetic energy of the free particle, center of mass density of gravitational wave and the memory effect (Maluf et al., 2019). Study of conservation laws in the background of well-known class of plane GWs is important. In this regard, most basic symmetry is Killing symmetry which give rise to certain conservation laws. (Sipple and Goenner, 1986) classified pp-waves according to KVFs and thus developed a variety of conservation laws. Here, we have studied a more general class of symmetries than Killing and homothetic symmetry which is conformal symmetry of pp-wave space-times in theory of f(R). The pp-wave space-times admit a very special class of solutions known as plane fronted GWs which is given in equation (4.3.16). Further, we have found proper CVFs of this special type of solution by classifying it into ten cases. The results of this classification are:

(b-1) The space-times in the cases (i) to (viii) admit proper HVFs of dimension two and seven. These are the space-times (4.3.19), (4.3.22), (4.3.23), (4.3.25), (4.3.27), (4.3.29), (4.3.31) and (4.3.33). Proper HVFs for these cases is same which is given in equation (4.3.21). Note that our result is the verification of the corollary of (Kuhnel and Rademacher, 2004) which states that for any pp-waves, if $xH_{,x} + yH_{,y} = 2H,$ then it admits proper HVF. Moreover, the space-times in the cases (iii), (iv) and (v) admit an additional KVF $u\partial_u - v\partial_v$ which is also known as boost vector field or boost rotation in the literature. This boost vector field appears due to $uH_{,u} + 2H = 0,$ which verify the corollary of the same paper (Kuhnel and Rademacher, 2004).

(b-2) The space-times in the cases (ix) and (x) admit proper CVFs. These are the spacetimes (4.3.35) and (4.3.38). The proper CVFs for these cases are given in equations (4.3.37) and (4.3.40). Our results are the verification of corollary of (Kuhnel and Rademacher, 2004) which states that on any pp-wave space-time if $uH_{,u} + xH_{,x} + yH_{,y} + 2H = 0,$ then the space-time admit proper CVF.

Chapter 5

Conformal Vector Fields of Proper Non-Static Plane Symmetric Space-Times in f(R) Theory of Gravity

5.1 Introduction

In this chapter, firstly we will look for the proper non static plane symmetric space-times in the theory of f(R). The term proper non static is referred to those which do not admit time like KVF. The methodology which we will adopt to find such space-times is purely algebraic. These space-times have been further investigated to get CVFs in the theory under consideration. This chapter is composed of three sections. The second section is specified for investigation of solutions in the theory of f(R) along with the CVFs which the space-time admits. The third section contains a summary of overall analysis.

5.2 Conformal Vector Fields of Proper Non-Static Plane Symmetric Space-Times

The line element of proper non-static plane symmetric space-times is represented as (Stephani et al., 2003)

$$ds^2 = -P^2(t,x)dt^2 + dx^2 + Q^2(t,x)[dy^2 + dz^2], \qquad (5.2.1)$$

where $P = P(t,x)$ and $Q = Q(t,x)$ are nowhere zero functions of t and x. From equation (5.2.1), we see that there do not exists time like KVF. The minimal set of isometries for the space-times (5.2.1) are (Stephani et al., 2003)

$$\partial_y,\ \partial_z,\quad y\partial_z - z\partial_y. \qquad (5.2.2)$$

For the above space-times scalar curvature R read as

$$R = 2\left[\frac{2Q''}{Q} + \frac{P''}{P} + \frac{Q'^2}{Q^2} + \frac{2P'Q'}{PQ} - \frac{2\ddot{Q}}{QP^2} - \frac{\dot{Q}^2}{P^2Q^2} + \frac{2\dot{P}\dot{Q}}{QP^3}\right], \qquad (5.2.3)$$

where the prime is representing $\frac{d}{dr}$ whereas dot is specified as $\frac{d}{dt}$. A vector field E is said to be CVF, if

$$L_E S_{ab} \equiv S_{ab,c}E^c + S_{bc}E^c_{,a} + S_{ac}E^c_{,b} = 2\phi S_{ab}, \tag{5.2.4}$$

where ϕ, L, S_{ab} and comma (,) represents the conformal function, the Lie derivative, metric tensor and partial derivative respectively. Writing equation (5.2.4) explicitly and using equation (5.2.1), we get

$$\dot{P}E^0 + P'E^1 + PE^0_{,0} = P\phi, \tag{5.2.5}$$

$$E^1_{,0} - P^2E^0_{,1} = 0, \tag{5.2.6}$$

$$Q^2E^2_{,0} - P^2E^0_{,2} = 0, \tag{5.2.7}$$

$$Q^2E^3_{,0} - P^2E^0_{,3} = 0, \tag{5.2.8}$$

$$E^1_{,1} = \phi, \tag{5.2.9}$$

$$E^1_{,2} + Q^2E^2_{,1} = 0, \tag{5.2.10}$$

$$E^1_{,3} + Q^2E^3_{,1} = 0, \tag{5.2.11}$$

$$\dot{Q}E^0 + Q'E^1 + QE^2_{,2} = Q\phi, \tag{5.2.12}$$

$$E^2_{,3} + E^3_{,2} = 0, \tag{5.2.13}$$

$$\dot{Q}E^0 + Q'E^1 + QE^3_{,3} = Q\phi. \tag{5.2.14}$$

Performing simple algebraic manipulations on equations (5.2.6), (5.2.10), (5.2.11) and (5.2.13), we obtain

$$E^0 = \int\left[\frac{1}{P^2}\int N^1_t dy\right]dx + \int \frac{N^2_t}{P^2}dx + N^5,$$

$$E^1 = \int N^1 dy + N^2,$$

$$E^2 = -\int \frac{N^1}{Q^2} dx + N^3,$$

$$E^3 = -\int N_z^3 \, dy + N^4. \qquad (5.2.15)$$

In the above system (5.2.15), N^1, N^2, N^3, N^4 and N^5 are defined below:

(a) $N^1 = N^1(t, x, y).$

(b) $N^2 = N^2(t, x, z).$

(c) $N^3 = N^3(t, y, z).$

(d) $N^4 = N^4(t, x, z).$

(e) $N^5 = N^5(t, y, z).$

The final form of CVFs would be obtained if one find the values of above unknown functions. Here, we are interested to find CVFs in the f(R) gravity, whose field equations are (Nojiri and Odintsov, 2003)

$$F(R)R_{ab} - \frac{1}{2} f(R)S_{ab} - \nabla_a \nabla_b F(R) + S_{ab} \Box F(R) = kT_{ab}, \qquad (5.2.16)$$

where $F(R) \equiv \frac{d}{dR} f(R)$, T_{ab} is the EMT, k denotes the coupling constant and $\Box \equiv \nabla^e \nabla_e$ is the de-Alembert's operator in which ∇ denotes the covariant derivative. The above equation (5.2.16) after rearranging the terms and by taking $T_{ab} = 0$, takes the form (Andra et al., 2019)

$$G_{ab} = \frac{1}{F(R)} \left[\left(\frac{f(R) - RF(R)}{2} \right) S_{ab} + \nabla_a \nabla_b F(R) - S_{ab} \nabla^e \nabla_e F(R) \right], \qquad (5.2.17)$$

where G_{ab} denotes the Einstein tensor. Equation (5.2.17) is important as it clearly shows the relation between geometry and the gravitational field given by the curvature and this is one of the main reasons that led to interpret the dark side of the gravitational contribution (Katsuragawa et al., 2019). Using equation (5.2.1) in equation (5.2.17), we have

$$\frac{F''}{F}+\frac{2Q'F'}{QF}-\frac{2\dot{Q}\dot{F}}{QFP^2}-\frac{\dot{Q}^2}{P^2Q^2}+\frac{2Q''}{Q}+\frac{Q'^2}{Q^2}+\frac{f}{2F}-\frac{R}{2}=0. \tag{5.2.18}$$

$$\frac{\dot{P}\dot{F}}{FP^3}-\frac{\ddot{F}}{FP^2}-\frac{2\dot{Q}\dot{F}}{QFP^2}+\frac{2Q'F'}{QF}-\frac{2\ddot{Q}}{QP^2}-\frac{\dot{Q}^2}{P^2Q^2}+\frac{2\dot{Q}\dot{P}}{QP^3}-\frac{Q'^2}{Q^2}+\frac{2P'Q'}{PQ}+\frac{f}{2F}-\frac{R}{2}=0. \tag{5.2.19}$$

$$\frac{F''}{F}-\frac{\ddot{F}}{FP^2}+\frac{\dot{P}\dot{F}}{FP^3}-\frac{\dot{Q}\dot{F}}{QFP^2}+\frac{P'F'}{PF}+\frac{Q'F'}{QF}-\frac{\ddot{Q}}{QP^2}+\frac{\dot{Q}\dot{P}}{QP^3}+\frac{P''}{P}+\frac{Q''}{Q}+\frac{P'Q'}{PQ}$$
$$+\frac{f}{2F}-\frac{R}{2}=0. \tag{5.2.20}$$

$$\frac{\dot{F}'}{F}-\frac{P'\dot{F}}{PF}+\frac{2\dot{Q}'}{Q}-\frac{2\dot{Q}P'}{QP}=0. \tag{5.2.21}$$

It should be noted that the above equations (5.2.18) to (5.2.21) involve the metric coefficients and the function f(R) along with their derivatives, which makes these equations difficult to solve. To overcome this problem, one must look for the numerical solutions or impose some sorts of restrictions on the metric coefficients to obtain analytic solutions. Another approach which may be applied is perturbation approach is to find the solutions of above equations. But it requires complicated calculations. Further, finding solutions become easier if first we perform some algebraic manipulations on equations (5.2.18) to (5.2.20). As a first step in this direction, we are subtracting equations (5.2.19) and (5.2.20) from equation (5.2.18) results in two equations which on subtraction yields

$$\frac{F''}{F}+\frac{Q''}{Q}+\frac{Q'^2}{Q^2}+\frac{\ddot{Q}}{QP^2}-\frac{\dot{Q}\dot{P}}{QP^3}-\frac{P'Q'}{PQ}-\frac{Q'F'}{QF}+\frac{\dot{Q}\dot{F}}{QFP^2}+\frac{\dot{Q}^2}{Q^2P^2}+\frac{P'F'}{PF}+\frac{P''}{P}=0. \tag{5.2.22}$$

As already mentioned that we are in search of those solutions for which the space-times (5.2.2) become proper non static. Here, we are omitting the details and only giving solutions of equations (5.2.21) and (5.2.22) in the form of following cases:

(i) $P=\text{constant}$, $Q=(2a_1t+2a_2)^{1/2}$, $R=\dfrac{2a_1^2}{(2a_1t+2a_2)^2}$ and $f(R)=(a_3x+a_4)R+a_5$, where $a_i\in\Re$ with $i=1,2,3,4,5(a_1\neq 0)$.

(ii) $P=\left[H^1(t)\dfrac{(2a_1x+2a_2)^{3/2}}{3a_1}+H^2(t)\right]$, $Q=(2a_1x+2a_2)^{3/2}$, $R=\dfrac{a_1^2u}{2}$, $f(R)=a_3R+a_4$, where

$$u=\left[\frac{32H^1(t)(a_1x+a_2)^2-3a_1H^2(t)\sqrt{2a_1x+2a_2}}{(a_1x+a_2)^2\sqrt{2a_1x+2a_2}\left\{H^1(t)(2a_1x+2a_2)^{3/2}+3a_1H^2(t)\right\}}\right],$$ $H^1(t)$, $H^2(t)$ are functions of integration and $a_i\in\Re$ with $i=1,2,3,4(a_1\neq 0)$.

(iii) $P=H^1(t)x+H^2(t)$, $Q=\text{constant}$, $R=0$, $f(R)=a_1R+a_2$, where $H^1(t)$, $H^2(t)$ are functions of integration and $a_1,a_2\in\Re(a_1\neq 0)$.

(iv) $P=t^{2n-1}$, $Q=t^n$, $R=2n^2t^{4n}$ and $f(R)=(a_1x+a_2)R+a_3$, where $n,a_1,a_2,a_3\in\Re(n\neq 0,1)$.

(v) $P=Q=(a_1t+a_2)$, $R=\dfrac{2a_1^2}{(a_1t+a_2)^4}$ and $f(R)=(a_3x+a_4)R+a_5$, where $a_1,a_2,a_3,a_4,a_5\in\Re(a_1\neq 0)$.

(vi) $P=Q=(a_1tx+a_2x+a_3t+a_4)$, $R=2\left[\dfrac{3Q^2(a_1t+a_2)^2+(a_1x+a_3)^2}{Q^4}\right]$ and $f(R)=a_5R+a_6$, where $a_i\in\Re$, $i=1,2,3,4,5,6$ and $\left(a_1=\dfrac{a_2a_3}{a_4},a_2,a_3,a_4\neq 0\right)$.

(vii) $P=\text{constant}$, $Q=(a_1t+a_2)$, $R=\dfrac{-2a_1^2}{(a_1t+a_2)^2}$ and $f(R)=\dfrac{a_3R}{(a_1t+a_2)}+a_4$, where $a_1,a_2,a_3,a_4\in\Re(a_1\neq 0)$.

Now, we will find proper CVFs for each of the above cases by putting the values of metric components in equation (5.2.1) and then solving the system of equations (5.2.5) to (5.2.14) applying direct integration approach. The procedure is somewhat lengthy and laborious therefore we are omitting the details and discussing each case briefly in the sequel.

Case (i)

The constraints with function $f(R)$ in this case are $P = \text{constant}$, $Q = (2a_1 t + 2a_2)^{1/2}$, $R = \frac{2a_1^2}{(2a_1 t + 2a_2)^2}$ and $f(R) = (a_3 x + a_4) R + a_5$, where $a_i \in \Re$ with $i = 1,2,3,4,5 (a_1 \neq 0)$. The space-times (5.2.1) after an appropriate rescaling of t take the form

$$ds^2 = -dt^2 + dx^2 + (2a_1 t + 2a_2)\left[dy^2 + dz^2\right]. \qquad (5.2.23)$$

Adopting the procedure discussed above, we found that $\phi = 2c_1$ which indicates that CVFs are HVFs which are

$$E^0 = 2c_1\left(\frac{a_1 t + a_2}{a_1}\right),\ E^1 = 2c_1 x + c_4,\ E^2 = c_1 y - c_2 z + c_3,\ E^3 = c_1 z + c_2 y + c_5, \qquad (5.2.24)$$

where $c_1, c_2, c_3, c_4, c_5 \in \Re \setminus \{0\}$. The above space-time (5.2.24) admit five CVFs in which four are KVFs. From these four KVFs, three are given in equation (5.2.2) while fourth KVF is ∂_x. Remaining fifth is proper HVF. The proper HVF after eliminating KVFs from (5.2.24) is

$$E^0 = 2c_1\left(\frac{a_1 t + a_2}{a_1}\right),\ E^1 = 2c_1 x,\ E^2 = c_1 y,\ E^3 = c_1 z. \qquad (5.2.25)$$

Case (ii)

Here, we have $P = \left[H^1(t)\frac{(2a_1 x + 2a_2)^{3/2}}{3a_1} + H^2(t)\right]$, $Q = (2a_1 x + 2a_2)^{3/2}$, $R = \frac{a_1^2 u}{2}$, $f(R) = a_3 R + a_4$, where $u = \left[\frac{32H^1(t)(a_1 x + a_2)^2 - 3a_1 H^2(t)\sqrt{2a_1 x + 2a_2}}{(a_1 x + a_2)^2\sqrt{2a_1 x + 2a_2}\left\{H^1(t)(2a_1 x + 2a_2)^{3/2} + 3a_1 H^2(t)\right\}}\right]$,

$H^1(t),\ H^2(t)$ are FOI and $a_1, a_2, a_3, a_4 \in \Re(a_1 \neq 0).$ The space-times (5.2.1) in this case become

$$ds^2 = -\left[H^1(t)\frac{(2a_1x+2a_2)^{3/2}}{3a_1} + H^2(t)\right]^2 dt^2 + dx^2 + (2a_1x+2a_2)^3\left[dy^2+dz^2\right]. \qquad (5.2.26)$$

It is important to mention here that there exist the following two possibilities:

(c) $H^1(t) \neq H^2(t).$

(d) $H^1(t) = H^2(t).$

(c) When $H^1(t) \neq H^2(t),$ then we found that $\phi = 0$ which implies that the CVFs are KVFs which are given in equation (5.2.2). (d) When $H^1(t) = H^2(t)$ then the above space-time (5.2.26) after appropriate rescaling become static, therefore we do not consider this case further as we are interested in only those cases where the above space-times (5.2.1) become proper non static.

Case (iii)

Here is $P = H^1(t)x + H^2(t),$ $Q =$ constant, $R = 0,$ $f(R) = a_1R + a_2,$ where $H^1(t),$ $H^2(t)$ being functions depending on t and $a_1, a_2 \in \Re(a_1 \neq 0).$ The space-times (5.2.1) after appropriate rescaling of y and z become

$$ds^2 = -\left[H^1(t)x + H^2(t)\right]^2 dt^2 + dx^2 + dy^2 + dz^2. \qquad (5.2.27)$$

The above space-time (5.2.27) is conformally flat, therefore admits fifteen CVFs which are:

$$E^0 = c_3\left(\frac{(t+x)^2 + y^2 + z^2 - 2(t+x)}{2(t+x)}\right)e^t - c_2\left(\frac{(t+x)^2 + y^2 + z^2 + 2(t+x)}{2(t+x)}\right)e^{-t} - c_{13}$$
$$+\frac{1}{(t+x)}\left[\left(c_6e^t - c_7e^{-t}\right)z + \left(c_9e^t - c_{10}e^{-t}\right)y - c_{11}e^{-t} - c_{12}e^t - c_1z - c_4y - c_5\right],$$

$$E^1 = c_2\left(\frac{(t+x)^2 - y^2 - z^2 + 2(t+x+1)}{2}\right)e^{-t} + c_3\left(\frac{(t+x)^2 - y^2 - z^2 - 2(t+x-1)}{2}\right)e^{t}$$
$$+(t+x)\left[c_1 z + c_4 y + c_5\right] - z\left(c_6 e^{t} + c_7 e^{-t}\right) - y\left(c_9 e^{t} + c_{10} e^{-t}\right) - c_{11}e^{-t} + c_{12}e^{t} + c_{13},$$

$$E^2 = -c_4\left(\frac{(t+x)^2 - y^2 + z^2}{2}\right) + (t+x+1)\left[c_2 y + c_{10}\right]e^{-t} + (t+x-1)\left[c_3 y + c_9\right]e^{t} + c_1 yz$$
$$+c_5 y - c_8 z + c_{14},$$

$$E^3 = -c_1\left(\frac{(t+x)^2 + y^2 - z^2}{2}\right) + (t+x+1)\left[c_2 z + c_7\right]e^{-t} + (t+x-1)\left[c_3 z + c_6\right]e^{t} + c_4 yz \qquad (5.2.28)$$
$$+c_5 z + c_8 y + c_{15},$$

where $c_i \in \Re$ with $i = 1,2,3,...,15.$ Here, $H^1(t) = \text{constant}$ and $H^2(t) = t.$ Conformal factor turns out to be $\phi = c_1 z + c_2(t+x+1)e^{-t} + c_3(t+x-1)e^{t} + c_4 y + c_5.$

Case (iv)

Values of metric components are $P = t^{2n-1},$ $Q = t^{n},$ $R = 2n^2 t^{4n}$ and $f(R) = \left(a_1 x + a_2\right)R + a_3,$ where $n, a_1, a_2, a_3 \in \Re (n \neq 0,1).$ The space-times (5.2.1) take the form

$$ds^{2} = -t^{4n-2}dt^2 + dx^2 + t^{2n}\left[dy^2 + dz^2\right]. \qquad (5.2.29)$$

Here, system of equations (5.2.5) to (5.2.14) for the space-time (5.2.29) yields $\phi = 2c_1$ which indicates that the CVFs become HVFs which are

$$E^0 = \frac{c_1 t}{n},\ E^1 = 2c_1 x + c_3,\ E^2 = c_1 y - c_2 z + c_4,\ E^3 = c_1 z + c_2 y + c_5, \qquad (5.2.30)$$

where $c_1, c_2, c_3, c_4, c_5 \in \Re \setminus \{0\}.$ Again, we obtain five CVFs including four KVFs and one proper HVF which is

$$E^0 = \frac{c_1 t}{n},\ E^1 = 2c_1 x,\ E^2 = c_1 y,\ E^3 = c_1 z. \qquad (5.2.31)$$

Case (v)

Here, we have $P=Q=\left(a_1t+a_2\right)$, $R=\dfrac{2a_1^2}{\left(a_1t+a_2\right)^4}$ and $f(R)=\left(a_3x+a_4\right)R+a_5$, where $a_1,a_2,a_3,a_4,a_5\in\Re(a_1\neq 0)$. The space-times (5.2.1) become

$$ds^2=dx^2+\left(a_1t+a_2\right)^2\left[-dt^2+dy^2+dz^2\right]. \qquad (5.2.32)$$

Ignoring the details, we come to know that $\phi=2c_1$ which shows that the CVFs become HVFs which are

$$E^0=c_1\left(\frac{a_1t+a_2}{a_1}\right),\ E^1=2c_1x+c_5,\ E^2=c_1y-c_2z+c_3,\ E^3=c_1z+c_2y+c_4, \qquad (5.2.33)$$

where $c_1,c_2,c_3,c_4,c_5\in\Re\setminus\{0\}$. One can find proper HVF by excluding KVFs from (5.2.33) to get

$$E^0=c_1\left(\frac{a_1t+a_2}{a_1}\right),\ E^1=2c_1x,\ E^2=c_1y,\ E^3=c_1z. \qquad (5.2.34)$$

Case (vi)

The constraints here are $P=\text{constant}$, $Q=\left(a_1t+a_2\right)$, $R=\dfrac{-2a_1^2}{\left(a_1t+a_2\right)^2}$ and $f(R)=\dfrac{a_3R}{\left(a_1t+a_2\right)}+a_4$, where $a_1,a_2,a_3,a_4\in\Re(a_1\neq 0)$. The space-times (5.2.1) after suitable rescaling of t become

$$ds^2=-dt^2+dx^2+\left(a_1t+a_2\right)^2\left[dy^2+dz^2\right]. \qquad (5.2.35)$$

CVFs here are:

$$E^0=\left(2c_1x+c_2\right)\left(\frac{a_1t+a_2}{a_1}\right),\ E^2=-c_3z+c_5,$$

$$E^1=\left[\left(\frac{a_1t+a_2}{a_1}\right)^2+x^2\right]c_1+c_2x+c_4,\ E^3=c_3y+c_6, \qquad (5.2.36)$$

where $c_1, c_2, c_3, c_4, c_5, c_6 \in \Re \setminus \{0\}$. The proper CVF after eliminating HVF from equation (5.2.36) is

$$E^0 = 2c_1\left(\frac{a_1 t + a_2}{a_1}\right)x, \;\; E^1 = \left[\left(\frac{a_1 t + a_2}{a_1}\right)^2 + x^2\right]c_1, \;\; E^2 = 0, \;\; E^3 = 0. \qquad (5.2.37)$$

Conformal factor in this case is $\phi = (2c_1 x + c_2)$.

Case (vii)

Here, $P = Q = (a_1 tx + a_2 x + a_3 t + a_4)$, $R = 2\left[\frac{3Q^2 (a_1 t + a_2)^2 + (a_1 x + a_3)^2}{Q^4}\right]$ and $f(R) = a_5 R + a_6$,

where $a_i \in \Re$, $i = 1, 2, 3, 4, 5, 6$ and $\left(a_1 = \frac{a_2 a_3}{a_4}, a_2, a_3, a_4 \neq 0,\right)$. The space-times (5.2.1) become

$$ds^2 = dx^2 + (a_1 tx + a_2 x + a_3 t + a_4)^2\left[-dt^2 + dy^2 + dz^2\right]. \qquad (5.2.38)$$

Calculations show that the conformal factor turn out to be zero and we obtain the KVFs as shown in equation (5.2.2).

5.3 Summary

In the first section of this chapter, we have found some proper non static solutions of EFEs in the f(R) theory of gravity while in the second section, we have found CVFs of the obtained space-times. During the process of finding solutions of EFEs, seven cases arose. It turns out that the dimension of CVFs for the proper non static plane symmetric space-times is three, five, six and fifteen. The clear picture of results is given in the following table:

Table 5.1

Case No	Metric components	Equations showing CVFs	Conformal Factor	Description and dimension
(i)	$P=\text{constant}$, $Q=\left(2a_1t+2a_2\right)^{1/2}$.	Equation (5.2.24).	$\phi=c_1$.	HVFs and 5
(ii)	$P=\left[H^1(t)\dfrac{\left(2a_1x+2a_2\right)^{3/2}}{3a_1}+H^2(t)\right]$, $Q=\left(2a_1x+2a_2\right)^{3/2}$.	Equation (5.2.2).	$\phi=0$.	KVFs and 3
(iii)	$P=Q=\left(x+t\right)$.	Equation (5.2.28).	$\phi=c_2(t+x+1)e^{-t}+c_3(t+x-1)e^{t}+c_4y+c_1z++c_5$.	CVFs and 15
(iv)	$P=t^{2n-1},\ Q=t^{n}$.	Equation (5.2.30).	$\phi=c_1$.	HVFs and 5
(v)	$P=Q=\left(a_1t+a_2\right)$.	Equation (5.2.33).	$\phi=c_1$.	HVFs and 5
(vi)	$P=\text{constant}$, $Q=\left(a_1t+a_2\right)$.	Equation (5.2.36).	$\phi=\left(2c_1x+c_2\right)$.	CVFs and 6
(vii)	$P=Q=\left(a_1tx+a_2x+a_3t+a_4\right)$.	Equation (5.2.2).	$\phi=0$.	KVFs and 3

From the above table, we see that

(a) The space-times in the cases (i), (iv) and (v) admits five CVFs. Out of these five CVFs, four are KVFs and one is proper HVF. The space-times for these cases are given in equation (5.2.23), (5.2.29) and (5.2.32).

(b) In the cases (ii) and (vii), CVFs are the KVFs. The space-times for these cases are given in equations (5.2.26) and (5.2.38). KVFs provide laws of conservation for example ∂_t corresponds to energy conservation, ∂_y, ∂_z represents spatial translational giving well defined conservation of linear momentum and $y\partial_z - z\partial_y$ is the rotation in the pair y and z giving conservation of angular momentum (Jamal and Shabbir, 2018).

(c) The space-time in the cases (iii) and (vi) admits proper CVFs. In the case (iii), the space-time is conformally flat, therefore admits fifteen independent CVFs. The space-time for this case is given in equation (5.2.28). In the case (vi), the space-time admits six CVFs out of which four are KVFs, one is proper HVF and one is proper CVF. CVFs for this case are given in equation (5.2.36) and the space-time is given in equation (5.2.35).

Chapter 6

6.1 Conclusion

In this thesis, we have discussed conformal symmetries of some space-times in the f(R) theories of gravity. The study includes static spherically symmetric, static plane symmetric, static cylindrically symmetric, Bianchi type I, II, III, V, Kantowski Sachs, Spatially homogeneous rotating space-times, pp-waves and non-static plane symmetric space-times. Initially, we have found some solutions of EFEs using different fluid matters in the f(R) theories of gravity by means of some algebraic techniques. After finding these solutions, we have found CVFs using direct integration technique. Studying all the considered space-times in details, we determined that there exist sixteen conformally flat cases and admit fifteen independent CVFs. The space-times for these cases are given in equations (2.3.20), (2.3.22), (2.3.24), (2.3.26), (2.4.37), (2.4.39), (2.4.41), (2.4.43), (2.4.45), (2.4.47), (2.4.49), (3.2.59), (3.2.62), (3.4.27), (4.2.32), (5.2.27) and the equations showing the expressions of CVFs are (2.3.21), (2.3.23), (2.3.25), (2.3.27), (2.4.38), (2.4.40), (2.4.42), (2.4.44), (2.4.46), (2.4.48), (2.4.50), (3.2.60), (3.2.63), (3.4.28), (4.2.33) and (5.2.28). In nine cases, the space-times admit proper CVFs. The space-times admitting proper CVFs are shown by the equations (2.4.31), (3.2.32), (3.2.35), (3.2.38), (3.5.20), (3.5.29), (4.3.35), (4.3.38) and (5.2.35). The forms of proper CVFs for the above ten space-times are given by the equations (2.4.33), (3.2.34), (3.2.37), (3.2.40), (3.5.22), (3.5.31), (4.3.37), (4.3.40) and (5.2.37). In thirty cases, space-times admit proper HVFs. The space-times admitting proper HVFs are represented by the equations (2.2.18), (2.4.22), (2.4.28), (2.4.34), (3.2.12), (3.2.26), (3.2.29), (3.2.41), (3.2.44), (3.2.47), (3.2.50), (3.2.53), (3.2.56), (3.3.23), (3.3.26), (3.3.29), (3.3.35), (4.2.21), (4.2.28), (4.3.19), (4.3.22), (4.3.23), (4.3.25), (4.3.27), (4.3.29), (4.3.31), (4.3.33), (5.2.23), (5.2.29) and (5.2.32). The forms of proper HVFs are shown by the equations (2.2.19), (2.4.24), (2.4.30), (2.4.36), (3.2.25), (3.2.28), (3.2.31), (3.2.43), (3.2.46), (3.2.49), (3.2.52), (3.2.55), (3.2.58), (3.3.25), (3.3.28), (3.3.31), (3.3.37), (4.2.23), (4.2.30), (4.3.21), (5.2.25), (5.2.31) and (5.2.34). Note that the proper HVFs corresponding to the space-times (4.3.19), (4.3.22), (4.3.23), (4.3.25), (4.3.27), (4.3.29), (4.3.31) and (4.3.33) are the same and are shown in the equation (4.3.21). In rest of the cases, CVFs become KVFs. As a result of the above study, we have obtained three types of vector fields namely proper CVFs, proper HVFs and KVFs. From the physical point of view, HVFs form homothetic algebra which coincides with the Lie and Noether point symmetries of wave and Klein-Gordon equations. Analysis of such equations is important in the problem of stability. Further, HVFs play a significant role in the dynamics of cosmological models and have capability to model the universe which enables one to find new facts related to singularities in general relativity. On the other hand, the KVFs give laws of conservation. For example $r\partial_z - z\partial_r$ represents rotational invariance in the coordinates $r,\ z$ and the conservation law is angular momentum. Similarly, ∂_t shows that the total energy of a system is conserved. Other Killing vector fields like ∂_ϕ and ∂_z denote the translational invariance in ϕ and z respectively and the conservation law is linear momentum. The existence of conformal symmetries predicate something about the inner structure of a space-time. As discussed earlier that the wave and Klein-Gordon equations are also related to the conformal algebra of pseudo-Riemannian spaces. The generators of the conformal algebra are used to classify the potentials of wave and Klein-Gordon equations. CVFs are important objects for studying the geometry of several kinds of manifolds as well as acts as a key probes in the characterization of several kinds of important spaces, like Euclidean space, Euclidean sphere and the complex projective space. There is a close relationship between the potential functions of CVFs and Obata's differential equation. One can obtain energy $e(X)$ of a smooth CVF X on a Riemannian manifold M using the relation given by (Deshmukh, 2017).

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