OPEN ACCESS
IOP Publishing

Classical and Quantum Gravity

Class. Quantum Grav. **40** (2023) 025006 (21pp)

https://doi.org/10.1088/1361-6382/acaa83

Parallel spinors, pp-waves, and gravitational perturbations

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Received 1 August 2022; revised 22 November 2022
Accepted for publication 9 December 2022
Published 23 December 2022

Abstract

We prove that any real, vacuum gravitational perturbation of a four-dimensional vacuum pp-wave space-time can be locally expressed, modulo gauge transformations, as the real part of a Hertz/Debye potential, where the scalar potential satisfies the wave equation. We discuss relations with complex perturbations, complex space-times, non-linear structures, and real spaces with split (ultra-hyperbolic/Kleinian) signature. Motivated by generalized notions of parallel spinors, we also discuss generalizations of the result to other space-times.

Keywords: pp-waves, gravitational perturbations, potentials, parallel spinors

(Some figures may appear in colour only in the online journal)

1. Introduction

Pp-wave space-times (plane-fronted waves with parallel propagation) are exact solutions to the Einstein equations modelling gravitational radiation, and defined by the existence of a (non-trivial) parallel null vector field[1]. These space-times are interesting both physically and mathematically for many reasons: they are relevant for gravitational wave physics; they satisfy, in appropriate cases, a linear superposition principle; they represent a universal limit for general relativity in that, as shown by Penrose [1], every Lorentzian space-time looks like a plane wave near a null geodesic; all their curvature invariants vanish (which is relevant e.g. for string theory); etc. In addition, closer to our motivation, they represent the simplest case of

[1] A special case corresponds to *plane waves*, which have an isometry group that is at least five-dimensional, while generic pp-waves have in general only one Killing vector.

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a four-dimensional Lorentzian geometry that admits a parallel spinor field [2]. In this work we study vacuum gravitational perturbations of pp-waves in four dimensions, and the problem of representations of solutions to linearized gravity in terms of the so-called Hertz/Debye potentials.

The general question we want to address is: can any real vacuum gravitational perturbation be represented, modulo gauge, in terms of a Hertz/Debye potential? This is conjectured to be true, locally, for perturbations of all algebraically special vacuum spaces, cf the introduction in [3]; but, as far as we know, the problem has only been completely solved for the case of Minkowski space-time [4, 5], [6, section 5.7.] Our main result is given in sections 3.1 and 4.

Although in this work we study the special case of pp-waves, the techniques we use also apply to perturbations of the above more general class of solutions. This is because our procedure is based on exploiting the existence of special geometric structures called α- and β-surfaces, or simply *twistor surfaces*, that are present, in particular, for any algebraically special Einstein space-time. Pp-waves have the advantage that, while having a very simple curvature structure that facilitates computations, the conceptual difficulties one has to deal with in the other more complicated cases are already present in this class. We illustrate this point by studying the more general case of a 'half-Káhler' vacuum space-time (see sections 2.2 and 4).

Furthermore, the fact that our method is based on twistor surfaces allows us to give, in the pp-wave case, a precise description of the close connection that exists between the Hertz/Debye representation of real linear gravitational perturbations and the fully non-linear geometry of a complex analogue of a pp-wave: a complex four-manifold admitting a parallel spinor field. Notably, the situation can also be understood in terms of *real* geometry, but for a metric with split (also called ultra-hyperbolic, Kleinian, or neutral) signature.

Finally, parallel spinors constitute the major motivation in this work, since as detailed in section 2.2 below, a pp-wave is the simplest case of a general scheme in which special geometries (including e.g. black hole space-times) are characterized in terms of 'generalized parallel spinors'. Our approach exploits a simple link between generalized parallel spinors and complex geometry, and it has direct connections to the twistor programme and the heavenly formalisms of Penrose, Newman and Plebański; see section 2.

1.1. Summary

In section 2 we give an elementary review of spinors in 4d; present our motivation relating parallel spinors and complex geometry; and deduce the structure of a four-geometry that admits a parallel spinor in Lorentz signature and also for complex metrics. Our main result is presented in section 3 where we study gravitational perturbations. In section 4 we study a generalization of this result, to the case of a 'half-Káhler' vacuum space-time. Some final remarks are given in section 5. We include appendix with additional details of calculations. We follow the notation and conventions of Penrose and Rindler [6, 7]; in particular, we use abstract indices.

2. Parallel spinors, real and complex space-times

2.1. Preliminaries

Given a 4d complex vector space with a metric g_{ab} and an orientation, the orthogonal group is

$$\mathrm{SO}(4,\mathbb{C}) = (\mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C}))/\mathbb{Z}_2. \tag{2.1}$$

The relation between the two sides of (2.1) is understood by fixing an isomorphism σ between $\mathbb{C}^4$ and $\mathbb{C}^2\otimes\mathbb{C}^2$, i.e. one writes a column vector v in $\mathbb{C}^4$ as a matrix $\sigma(v)$ in $\mathbb{C}^2\otimes\mathbb{C}^2$. Then (2.1) means that, for any orthogonal transformation $\Lambda\in \mathrm{SO}(4,\mathbb{C})$, there are elements L and R in $\mathrm{SL}(2,\mathbb{C})$ such that $\sigma(\Lambda v) = L\sigma(v)R^{\mathrm{t}}$.

Elements in each copy of $\mathbb{C}^2$ are called spinors. Since each $\mathbb{C}^2$ has an independent action of $\mathrm{SL}(2,\mathbb{C})$, there are two different kinds of spinors. We say that the two kinds have opposite 'chirality'. In abstract indices, these are distinguished by primed and unprimed indices, e.g. $\psi^{A'}$ and φ^A, and the isomorphism σ is $v^a\to\sigma(v)^{AA'}\equiv v^{AA'}$. We usually omit σ, so that we identify $v^a\equiv v^{AA'}$. This way we have the usual identification of indices $a=AA'$, $b=BB'$, etc which we follow in this work. From the relation $g(v,v)=2\det\sigma(v)$ one deduces that the metric is $g_{ab}=\epsilon_{AB}\epsilon_{A'B'}$, where ϵ_{AB} is the natural volume element of $\mathbb{C}^2$.

Without any reality conditions, spinors of opposite chirality are independent. Real forms of (2.1) corresponding to different metric signatures are recovered by using different reality structures. These structures can in turn be understood as operations on spinors, that we call 'spinor conjugations', and they may or may not lead to relations between chiralities.

For Lorentzian reality conditions, spinor conjugation interchanges chirality, so the action of the two factors in the RHS of (2.1) is not independent, and one recovers the Lorentz group $\mathrm{SO}(1,3)=\mathrm{SL}(2,\mathbb{C})/\mathbb{Z}_2$. We denote Lorentzian spinor conjugation with an overbar, e.g. $\varphi^A\to\bar\varphi^{A'}$. A spinor φ^A and its complex conjugate $\bar\varphi^{A'}$ produce a real null vector $N^a=\varphi^A\bar\varphi^{A'}$. Given a basis of $\mathbb{C}^2$, $\{o^A,\iota^A\}$, one can consider the complex conjugate basis $\{\bar o^{A'},\bar\iota^{A'}\}$ and construct four linearly independent null vectors as

$$\ell^a=o^A\bar o^{A'},\qquad n^a=\iota^A\bar\iota^{A'},\qquad m^a=o^A\bar\iota^{A'},\qquad \bar m^a=\iota^A\bar o^{A'}. \tag{2.2}$$

If the basis $\{o^A,\iota^A\}$ is normalized by $\epsilon_{AB}o^A\iota^B=1$, then the vectors (2.2) satisfy the usual conditions for a null tetrad: $g_{ab}\ell^a n^b=1=-g_{ab}m^a\bar m^b$, and the rest vanishes.

For Euclidean (/Riemannian) reality conditions, spinor conjugation $\dagger$ preserves chirality, but a spinor φ^A and its complex conjugate $\varphi^{\dagger A}$ are linearly independent: if φ^A has components (a,b) relative to some basis, then $\varphi^{\dagger A}$ has components $(-\bar b,\bar a)$. Since $\dagger$ is anti-linear and it holds $\dagger^2=-1$, this is really a quaternionic structure. The Euclidean form of (2.1) is $\mathrm{SO}(4,\mathbb{R})=(\mathrm{SU}(2)\times\mathrm{SU}(2))/\mathbb{Z}_2$, and chiralities are independent. Given a spinor o^A, one has a spin basis $\{o^A,o^{\dagger A}\}$, but unlike Lorentz signature, this does not give a basis for the opposite chirality.

Finally, the restriction to real elements in $\mathrm{SL}(2,\mathbb{C})$ corresponds to a metric with split signature. The isomorphism (2.1) becomes $\mathrm{SO}(2,2)=(\mathrm{SL}(2,\mathbb{R})\times\mathrm{SL}(2,\mathbb{R}))/\mathbb{Z}_2$, spinors are real and chiralities are independent.

Over an open neighbourhood on a smooth manifold M equipped with a metric g_{ab}, one constructs the primed and unprimed spinor bundles $\mathbb{S}'$, $\mathbb{S}$, and the considerations above apply pointwise on each fibre. A spinor field is a (local) section of $\mathbb{S}$ or $\mathbb{S}'$ (or tensor products of them). If x^a are local coordinates on M, we use the identification of indices $a=AA'$, etc to write e.g. $\mathrm{d}x^a\equiv\mathrm{d}x^{AA'}$. Similarly, the Levi-Civita connection is $\nabla_a=\nabla_{AA'}$. If (M,g_{ab}) is real, the operator $\nabla_{AA'}$ is also real.

2.2. *Motivation: parallel spinors*

A parallel (or covariantly constant) spinor is a spinor field o^A that satisfies

$$\nabla_{AA'}o^B=0. \tag{2.3}$$

The existence of a non-trivial solution to (2.3) imposes strong restrictions on the geometry. Specific restrictions depend on the metric signature, see [2].

Table 1. Different notions of parallel spinors give different special four-geometries. $\Theta_{AA'}$ is the 'GHP connection', and $\mathcal{C}_{AA'}$ is a conformally invariant version of it. Apart from $\nabla_{AA'}o^B = 0$, the other equations are non-linear, since the connections depend on o^A. The terminology in the Lorentzian case is perhaps not standard, although similar names have been used by Flaherty [14]. In the split case, the conditions can be related to *para-complex* geometry.

Condition	Riemann signature	Lorentz signature	Split signature
$\nabla_{AA'}o^B = 0$	hyper-Kähler	Pp-wave	Null-Kähler
$\Theta_{AA'}o^B = 0$	Kähler	'half-Kähler'	No name
$\mathcal{C}_{AA'}o^B = 0$	Hermitian	'half-Hermitian'	No name

In Lorentz signature, complex conjugation of (2.3) gives a parallel spinor with opposite chirality, $\nabla_{AA'}\bar{o}^{B'} = 0$. The real null vector $\ell^b = o^B\bar{o}^{B'}$ is therefore covariantly constant, so the geometry is a pp-wave. In this work we are interested in this case, see section 2.3. In Riemann signature, the complex conjugate of (2.3) is $\nabla_{AA'}o^{\dagger B} = 0$. One then has a parallel spin frame, so the manifold must be hyper-Kähler. We will not focus on this case. In split signature, a (real) solution to (2.3) is equivalent to a null Kähler structure, see [8, 9].

Our interest in parallel spinors actually arises from 'generalized' versions of them, where one considers connections more general than the Levi-Civita connection. Such generalizations are important both in physics and in mathematics. For example, these objects appear in supergravity in relation to the existence of supersymmetries; and they are also relevant in certain areas of pure geometry, for instance concerning different definitions of 'mass'. See e.g. [10].[2]

But our major motivation is the connection that generalized parallel spinors turn out to have with complex geometry. For example, a Kähler manifold can be characterized by the existence of a parallel (pure) *projective* spinor, see [11]. In four dimensions (where all spinors are pure), this can be expressed in terms of the Riemannian version of a connection that is well-known in general relativity, the so-called 'GHP' connection $\Theta_{AA'}$. (See [6, section 4.12] for the meaning of this symbol and the associated formalism.) Interestingly enough, a Hermitian manifold can be similarly defined via parallel spinors, using a generalization of $\Theta_{AA'}$, that we may call 'complex-conformal connection' or 'conformally invariant GHP connection', and we denote by $\mathcal{C}_{AA'}$, cf [12, 13]. We summarize the situation in table 1.

The operators $\Theta_{AA'}$ and $\mathcal{C}_{AA'}$ are well-defined in any signature[3]. Our interest in the (generalized) parallel spinor equations presented in table 1 is that they imply the existence of twistor surfaces, which are the basic object that give integration procedures. The kind of algebraic and differential manipulations that one has to follow in these procedures is essentially the same in all cases, which is why we find the parallel spinors viewpoint attractive: it is both conceptually (geometrically) meaningful and computationally practical. In this paper we are interested in the simplest case, equation (2.3), and its applications to the linearized gravity problem in general relativity. We will also discuss the 'half-Káhler' case, see section 4. For the treatment of $\mathcal{C}_{AA'}o^B = 0$ in conformal geometry, see [13] (perturbations are not treated in this reference).

2.3. Lorentz signature: pp-wave space-times

We define a pp-wave space-time as a four-dimensional Lorentzian manifold (M, g_{ab}) that admits a non-trivial parallel real null vector N^a, $N_aN^a = 0$, $\nabla_aN^b = 0$, and such that the Ricci

[2] We are interested in parallel *Weyl* spinors, while in supergravity and related areas one considers *Dirac* spinors.

[3] One needs a *pair* of spinors in the construction of $\Theta_{AA'}$, $\mathcal{C}_{AA'}$. In the Riemannian case a single spinor is enough since its complex conjugate gives the other. In the other cases the extra spinor can be chosen at will.

tensor is $R_{ab} \propto N_a N_b$. As shown in [15], any such geometry admits a parallel spinor, that in the rest of this work we denote by o^A. The associated parallel null vector is denoted by $\ell^a = o^A \bar{o}^{A'}$.

The following result is just the standard characterization of pp-waves in terms of Brinkmann coordinates, and it is well-known:

Proposition 2.1. *Let (M, g_{ab}) be a Lorentzian space-time admitting a non-trivial parallel spinor field o^A, equation (2.3). Then there exist (locally) a coordinate system $(u, v, \zeta, \bar{\zeta})$ and a real scalar field $H = H(v, \zeta, \bar{\zeta})$ such that the metric is*

$$g = 2(\mathrm{d}u\mathrm{d}v - \mathrm{d}\zeta\mathrm{d}\bar{\zeta}) + H\mathrm{d}v^2. \tag{2.4}$$

The Ricci scalar vanishes, and the rest of the curvature is given by

$$\Phi_{ABA'B'} = \frac{1}{2} H_{\zeta\bar{\zeta}} o_A o_B \bar{o}_{A'} \bar{o}_{B'}, \tag{2.5}$$

$$\Psi_{ABCD} = \frac{1}{2} H_{\bar{\zeta}\bar{\zeta}} o_A o_B o_C o_D \tag{2.6}$$

where $H_{\zeta\bar{\zeta}} = \partial_\zeta \partial_{\bar{\zeta}} H$ and $H_{\bar{\zeta}\bar{\zeta}} = \partial_{\bar{\zeta}} \partial_{\bar{\zeta}} H$.

It is instructive to look at the derivation of this result from the perspective of twistor surfaces; we will do this in the rest of the section 2.3.

Consider the two-dimensional complex distribution in $TM \otimes \mathbb{C}$ given by $D = \{o^A \beta^{A'} \mid \beta^{A'} \in \mathbb{S}'\}$. The condition for this to be involutive (i.e. $[D, D] \subset D$) is the shear-free equation $o^A o^B \nabla_{AA'} o_B = 0$ (cf [7, section 7.3]), which is certainly satisfied if (2.3) holds. This implies that there exist complex two-surfaces in the complexified space-time $\mathbb{C}M$, called β-surfaces, such that their tangent bundle is D. Analogously, the distribution $\bar{D} = \{\bar{o}^{A'} \alpha^A \mid \alpha^A \in \mathbb{S}\}$ is involutive, and is the tangent bundle to a different kind of complex two-surfaces in $\mathbb{C}M$, called α-surfaces. Let us focus on the former. The β-surfaces are labelled by two complex coordinates (v, ζ) that are constant on them, namely $o^A \nabla_{AA'} v = 0$, $o^A \nabla_{AA'} \zeta = 0$ (see [7, lemma (7.3.15)]). From these two equations we deduce that there are two spinor fields, say $v_{A'}, \bar{\iota}_{A'}$, such that $\nabla_{AA'} v = o_A v_{A'}$, $\nabla_{AA'} \zeta = o_A \bar{\iota}_{A'}$. Since $\ell_a = o_A \bar{o}_{A'}$ is covariantly constant, it is in particular closed, so we can take $v_{A'} = \bar{o}_{A'}$. So (v, ζ) are defined by

$$\mathrm{d}v = o_A \bar{o}_{A'} \mathrm{d}x^{AA'}, \qquad \mathrm{d}\zeta = o_A \bar{\iota}_{A'} \mathrm{d}x^{AA'}. \tag{2.7}$$

We see that v is real, whereas ζ is complex. They are functionally independent, which means that $\bar{o}_{A'} \bar{\iota}^{A'} = N$ for some scalar field $N \neq 0$.

From the condition $\mathrm{d}^2\zeta = 0$ we deduce that $o^A \nabla_{AA'} \bar{\iota}_{B'} = 0$. Therefore, $o^A \nabla_{AA'} N = 0$, which implies that N is a holomorphic function of v, ζ (i.e. $\mathrm{d}N$ is a linear combination of $\mathrm{d}v$ and $\mathrm{d}\zeta$), so it can be set to 1 by a coordinate transformation $\zeta \to \zeta'(v, \zeta)$. We drop the prime and denote again by ζ the new coordinate, with $\bar{o}_{A'} \bar{\iota}^{A'} = 1$.

Notice that $v, \bar{\zeta}$ satisfy $\bar{o}^{A'} \nabla_{AA'} v = 0$, $\bar{o}^{A'} \nabla_{AA'} \bar{\zeta} = 0$, so these scalars are constant on α-surfaces. This is a generic feature of Lorentz signature: spinor complex conjugation interchanges α- and β-surfaces.

Using some of the previous identities, one can show that the vector fields $o^A \bar{o}^{A'}$ and $o^A \bar{\iota}^{A'}$ commute, so we have two functionally independent scalar fields u, w defined by $\partial_u = o^A \bar{o}^{A'} \partial_{AA'}$, $\partial_w = o^A \bar{\iota}^{A'} \partial_{AA'}$. We see that u is real and w is complex. These are coordinates along the β-surfaces. The coordinate w is however not functionally independent of $(v, \bar{\zeta})$, since a short calculation gives $\partial_{\bar{\zeta}} = -o^A \bar{\iota}^{A'} \partial_{AA'}$. Summarizing, we have

$$\partial_u = o^A \bar{o}^{A'} \partial_{AA'}, \qquad \partial_{\bar{\zeta}} = -o^A \bar{\iota}^{A'} \partial_{AA'}. \tag{2.8}$$

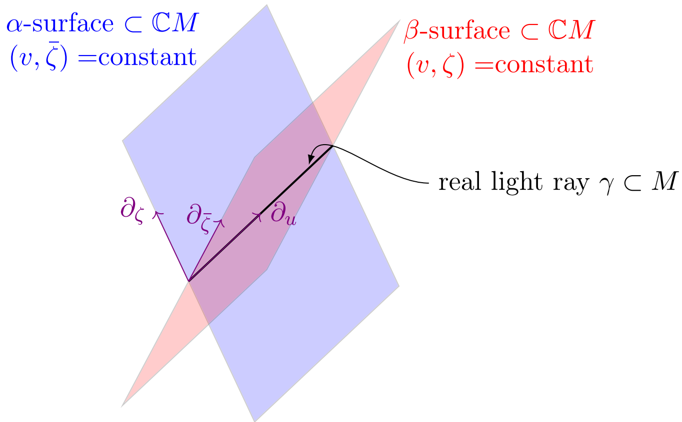

Figure 1. An α-surface, a β-surface, and a real (Lorentzian) space-time intersect in a real light ray γ, that has tangent vector $\ell^a = o^A\bar{o}^{A'} = \partial_u^a$. The coordinate system defined by these twistor surfaces coincides, in the pp-wave case, with Brinkmann coordinates.

So α- and β-surfaces give a coordinate system $(u,v,\zeta,\bar{\zeta})$ for M, that we illustrate in figure 1. For pp-waves these are simply Brinkmann coordinates, so the interpretation is known: the integral curves of $\ell^a = o^A\bar{o}^{A'}$ are the rays of the wave and u is an affine parameter along them, the hypersurfaces $v =$ constant are 'wave surfaces', and the real and imaginary parts of ζ are coordinates transverse to the direction of propagation of the wave.

With the above information, the structure of the metric can be deduced from the expression $g_{ab} = \epsilon_{AB}\bar{\epsilon}_{A'B'}$, by replacing $\epsilon_{AB} = o_A\iota_B - \iota_A o_B$, its complex conjugate, and the definition of the coordinates (2.7). When doing this computation, one finds that the only piece of information missing at this point is an expression for the 1-form $\iota_A\bar{\iota}_{A'}\mathrm{d}x^{AA'}$. This can be obtained as follows. For any function f, we have $\mathrm{d}f = (\nabla_{AA'}f)\mathrm{d}x^{AA'}$. Using the identities $\nabla_{AA'} = \delta_A^B\bar{\delta}_{A'}^{B'}\nabla_{BB'}$ and $\delta_A^B = o_A\iota^B - \iota_A o^B$, together with definitions (2.7) and (2.8), and putting $f = u$, we get

$$\iota_A\bar{\iota}_{A'}\mathrm{d}x^{AA'} = \mathrm{d}u - \left(\iota^A\bar{\iota}^{A'}\nabla_{AA'}u\right)\mathrm{d}v.$$

The expression (2.4) for the metric then follows straightforwardly, by defining the real scalar field

$$H := -2\iota^A\bar{\iota}^{A'}\nabla_{AA'}u. \tag{2.9}$$

This function H represents the wave profile, and the case $H = 0$ reduces to Minkowski space-time.

We also notice that in the coordinate system $(u,v,\zeta,\bar{\zeta})$, the wave operator acting on an arbitrary scalar field φ is

$$\Box\varphi = 2(\varphi_{uv} - \varphi_{\zeta\bar{\zeta}}) - H\varphi_{uu}, \tag{2.10}$$

where $\varphi_{uv} = \partial_u\partial_v\varphi$, etc.

There is a natural spin frame: the parallel spinor o_A, and the spinor ι_A used in (2.7). The associated connection 1-form has only one non-trivial component:

$$\nabla_{AA'}\iota_B = -\kappa' o_A o_B \bar{o}_{A'}, \tag{2.11}$$

where $\kappa' := -\iota^A\iota^B\bar{\iota}^{B'}\nabla_{BB'}\iota_A$. In order to show this, notice first that, from (2.7) and $\mathrm{d}^2\bar{\zeta} = 0$, it follows that $\bar{o}^{A'}\nabla_{AA'}\iota_B = 0$. In addition, from (2.7) one deduces that $o_A = -\bar{o}^{A'}\nabla_{AA'}\zeta$ and

$\iota_A = \bar{\iota}^{A'}\nabla_{AA'}\bar{\zeta}$. These identities can then be used to show that $o^A\nabla_{AA'}\iota_B = 0$, so (2.11) follows. In terms of H, the expression for κ' is

$$\kappa' = \frac{1}{2}H_\zeta. \tag{2.12}$$

This can be shown by using (2.9), which gives $o^A\nabla_{AA'}H = 2\iota^B\bar{\iota}^{B'}\nabla_{BB'}\bar{\iota}_{A'}$.

For the curvature, equation (2.3) implies $[\nabla_a,\nabla_b]o^D = 0$, so it follows that $\Lambda = 0$, $\Phi_{ABA'B'} = \Phi_{22}o_Ao_B\bar{o}_{A'}\bar{o}_{B'}$, and $\Psi_{ABCD} = \Psi_4 o_Ao_Bo_Co_D$, where Φ_{22} and Ψ_4 are defined by contractions with $\iota^A,\bar{\iota}^{A'}$ on the left-hand sides. Expressions for Φ_{22} and Ψ_4 in terms of H can be deduced, for example, by using the Newman–Penrose equations and the fact that all spin coefficients except κ' vanish: equations $(4.11.12)(a')$ and $(4.11.12)(b')$ in [6] give $\Phi_{22} = -\delta\kappa'$ and $\Psi_4 = -\delta'\kappa'$ (in Newman–Penrose notation). Using then $\delta = -\partial_{\bar{\zeta}}$, $\delta' = -\partial_\zeta$ (which follow from (2.8)) and (2.12), one obtains the expressions (2.5) and (2.6).

2.4. Complex space-times

We now consider a complex space-time with a parallel spinor. Here, we are referring to a genuinely complex four-manifold, not to a complexified pp-wave space-time. See [7, section 6.9] for the distinction between 'complex' and 'complexified' space-time. In the complexified case we still have two parallel spinors of opposite chirality (o^A and $\bar{o}^{A'}$ become independent but they are both parallel), whereas in the genuinely complex case we only retain one parallel spinor.

The situation is very similar to the case of split signature. This is because in that case, spinors are real and the two chiralities are independent. The following result is for complex space-times, but if one replaces 'complex' by 'real' everywhere then exactly the same holds true for split signature metrics, as was already shown in [8, 9]:

Proposition 2.2 (see [8, 9]). *Let $(\mathbb{C}M, g^{\mathbb{C}}_{ab})$ be a complex space-time with a non-trivial parallel spinor field o^A. Then there exist, locally, a complex coordinate system (u,v,ζ,w) and a complex scalar field Θ [4] such that the metric is*

$$g^{\mathbb{C}} = 2(\mathrm{d}u\mathrm{d}v + \mathrm{d}\zeta\mathrm{d}w) - 2\Theta_{ww}\mathrm{d}v^2 + 4\Theta_{wu}\mathrm{d}v\mathrm{d}\zeta - 2\Theta_{uu}\mathrm{d}\zeta^2 \tag{2.13}$$

where $\Theta_{ww} = \partial_w\partial_w\Theta$, etc. The Ricci scalar vanishes, and the rest of the curvature is given by

$$\Phi_{ABA'B'} = o_Ao_B\tilde{\nabla}_{A'}\tilde{\nabla}_{B'}f, \tag{2.14}$$

$$\Psi_{ABCD} = -\frac{1}{2}o_Ao_Bo_Co_D\Box f, \tag{2.15}$$

$$\tilde{\Psi}_{A'B'C'D'} = -\tilde{\nabla}_{A'}\tilde{\nabla}_{B'}\tilde{\nabla}_{C'}\tilde{\nabla}_{D'}\Theta, \tag{2.16}$$

where $\tilde{\nabla}_{A'} := o^A\nabla_{AA'}$, $\Box$ is the wave operator associated to (2.13), and

$$f = \Theta_{uv} + \Theta_{\zeta w} + \Theta_{uu}\Theta_{ww} - \Theta_{uw}^2. \tag{2.17}$$

In coordinates, the wave operator $\Box$ acting on an arbitrary scalar field φ is

$$\Box\varphi = 2(\varphi_{uv} + \varphi_{w\zeta} + \Theta_{uu}\varphi_{ww} + \Theta_{ww}\varphi_{uu} - 2\Theta_{uw}\varphi_{uw}). \tag{2.18}$$

From proposition 2.2 we see that the vacuum Einstein equations are now more complicated than in the pp-wave case: the Ricci-flat condition is equivalent to $\tilde{\nabla}_{A'}\tilde{\nabla}_{B'}f = 0$, which in

[4] The symbol Θ here should not be confused with the GHP connection $\Theta_{AA'}$ used in other parts of this paper.

the coordinate system of the proposition reads $f_{uu} = f_{uw} = f_{ww} = 0$. The solution to this is $f = p(v,\zeta)u + q(v,\zeta)w + r(v,\zeta)$, where p,q,r are arbitrary functions of (v,ζ), so in terms of the 'potential' Θ, the Einstein equations are

$$\Theta_{uv} + \Theta_{\zeta w} + \Theta_{uu}\Theta_{ww} - \Theta_{uw}^2 = p(v,\zeta)u + q(v,\zeta)w + r(v,\zeta), \tag{2.19}$$

see [8]. This is a very special case of the hyper-heavenly equation of Plebański and Robinson [16]. The non-trivial right hand side in (2.19) (i.e. $f \neq 0$) complicates the analysis of the integrability properties of this equation. The special case $f \equiv 0$ is Plebański's second heavenly equation, and notice from (2.15) that this case gives a self-dual (half-flat) space, which is an integrable system by virtue of the non-linear graviton twistor construction of Penrose.

It is useful to briefly discuss the structures involved in the derivation of the result of proposition 2.2. As in section 2.3, the condition $\nabla_{AA'}o^B = 0$ implies that the distribution $D = \{o^A\beta^{A'} \mid \beta^{A'} \in \mathbb{S}'\}$ is involutive, and this gives origin to β-surfaces in $\mathbb{C}M$, which are labelled by two complex coordinates v,ζ defined by $o^A\nabla_{AA'}v = 0 = o^A\nabla_{AA'}\zeta$. Unlike the Lorentzian case, there are no α-surfaces now. In addition, both coordinates v,ζ are now complex. There are two independent spinor fields $v_{A'}, \zeta_{A'}$, with $v_{A'}\zeta^{A'} = N \neq 0$, such that

$$\mathrm{d}v = o_A v_{A'}\mathrm{d}x^{AA'}, \qquad \mathrm{d}\zeta = o_A\zeta_{A'}\mathrm{d}x^{AA'}. \tag{2.20}$$

From the conditions $\mathrm{d}^2 v = 0 = \mathrm{d}^2\zeta$, it follows that $\tilde{\nabla}_{A'}v_{B'} = \tilde{\nabla}_{A'}\zeta_{B'} = 0$, which give $\tilde{\nabla}_{A'}N = 0$, so we can set $N \equiv 1$ by a coordinate transformation.

Using the above information, a short calculation shows that the vector fields $o^A v^{A'}$ and $o^A\zeta^{A'}$ commute, so we have two functionally independent complex scalar fields u, w defined by

$$o^A v^{A'}\partial_{AA'} = \partial_u, \qquad o^A\zeta^{A'}\partial_{AA'} = \partial_w. \tag{2.21}$$

They correspond to complex coordinates along the β-surfaces. Thus, we see again that twistor surfaces produce a natural coordinate system (v,ζ,u,w) for $\mathbb{C}M$: these are the coordinates used in proposition 2.2, and they generalize the Brinkmann coordinates of the pp-wave case of proposition 2.1.

The structure (2.13) of the metric can be deduced in a similar way to what we did in section 2.3: there is a flat (complex) metric η_{ab} and a symmetric spinor field $H_{A'B'}$ such that

$$g_{ab} = \eta_{ab} + o_A o_B H_{A'B'}. \tag{2.22}$$

The components of $H_{A'B'}$ generalize the pp-wave profile function H (2.9). In addition, a short calculation shows that $H_{A'B'}$ satisfies $\tilde{\nabla}^{A'}H_{A'B'} = 0$, so there exists a scalar field Θ (see Remark 3.2 below) such that

$$H_{A'B'} = -2\tilde{\nabla}_{A'}\tilde{\nabla}_{B'}\Theta. \tag{2.23}$$

The equations (2.22) and (2.23) give a coordinate-free expression for (2.13). The Einstein equation (2.19) is, in coordinate-free terms:

$$\Box\Theta - 2\left(\tilde{\nabla}_{A'}\tilde{\nabla}_{B'}\Theta\right)\left(\tilde{\nabla}^{A'}\tilde{\nabla}^{B'}\Theta\right) = f, \qquad \tilde{\nabla}_{A'}\tilde{\nabla}_{B'}f = 0. \tag{2.24}$$

3. Perturbations

We will now study real gravitational perturbations of a real, Lorentzian, vacuum pp-wave space-time, and connections with complex space-times. We recall that the structure of the background pp-wave space-time is described in proposition 2.1: one has Brinkmann coordinates $(u,v,\zeta,\bar{\zeta})$ defined by α- and β-surfaces, the spinor field o^A is parallel, the spinor ι^A is defined in equation (2.7), and all the information of the geometry is encoded in the real scalar field H. In addition, the vacuum condition for the background implies that $H_{\zeta\bar{\zeta}} = 0$.

3.1. Main result

Theorem 3.1. *Let (M, g_{ab}) be a vacuum pp-wave space-time, equation (2.4) with $H_{\zeta\bar{\zeta}} = 0$. For any real metric perturbation h_{ab} satisfying the linearized Einstein vacuum equations, there exist, locally, a real vector field V_a and a complex scalar field Φ, such that h_{ab} can be written as*

$$h_{ab} = 2\mathrm{Re}\left(h^{\mathrm{H}}_{ab}\right) + \nabla_a V_b + \nabla_b V_a \tag{3.1}$$

where $2\mathrm{Re}(h^{\mathrm{H}}_{ab}) = h^{\mathrm{H}}_{ab} + \overline{h^{\mathrm{H}}_{ab}}$, and h^{H}_{ab} is given by

$$h^{\mathrm{H}}_{ab} = o_A o_B \tilde{\nabla}_{A'} \tilde{\nabla}_{B'} \Phi = \Phi_{\bar{\zeta}\bar{\zeta}} \ell_a \ell_b + 2\Phi_{\bar{\zeta}u} \ell_{(a} m_{b)} + \Phi_{uu} m_a m_b \tag{3.2}$$

with $\Phi_{\bar{\zeta}u} = \partial_{\bar{\zeta}} \partial_u \Phi$, etc, and $\tilde{\nabla}_{A'} = o^A \nabla_{AA'}$. The scalar field Φ satisfies the wave equation

$$\Box \Phi = 0, \tag{3.3}$$

where $\Box$ is the wave operator associated to g_{ab}, equation (2.10).

We prove this result in section 3.2 below. Note that in tensor terms, the tensor field (3.2) can also be written as

$$h^{\mathrm{H}}_{ab} = \nabla_c \nabla_d [\mathcal{H}_{(a}{}^{cd}{}_{b)} \Phi], \qquad \mathcal{H}_{abcd} = 4 \ell_{[a} m_{b]} \ell_{[c} m_{d]}. \tag{3.4}$$

This is the usual expression for a Hertz/Debye potential in perturbation theory, particularized to the special background of a pp-wave. (The superscript 'H' is from 'Hertz'.)

Combining the result of theorem 3.1 with the discussion of section 2.4, we see some sort of correspondence between a *real linear* problem and a *complex non-linear* one: modulo gauge, the linearized gravity problem for real pp-wave space-times would seem to be a 'linear version' (see below) of the structure of a complex space with a parallel spinor. Furthermore, as observed in section 2.4, the non-linear structures can actually be understood in a *real* context, by going to a real space with a split signature metric.

Note that this correspondence can be established only *after* one proves theorem 3.1: we want to show that *there exists* a scalar potential for the gravitational perturbation, while in a linear version of (2.22)–(2.24) one is already assuming that a potential exists. Actually, a closer look at the linear version of (2.22)–(2.24) reveals that the situation is subtle:

- By 'linear version' we mean that, in the complex metric (2.22) and (2.23) and in the complex Einstein equations (2.19), one formally replaces Θ by $\Theta + \varepsilon\Phi$ (where ε is a parameter), and one keeps only linear terms in ε. Then the perturbation to the background complex metric is exactly (3.2), and the scalar field Φ would seem to satisfy the wave equation $\Box\Phi = 0$. In addition, using the general expression [6, equation (5.7.15)] for the perturbed Weyl spinor, it is not difficult to show that for the complex perturbation (3.2) one has

$$\dot{\Psi}_{ABCD}[h^{\mathrm{H}}] = \frac{1}{8} o_A o_B o_C o_D \Box\Box\Phi, \qquad \dot{\tilde{\Psi}}_{A'B'C'D'}[h^{\mathrm{H}}] = \frac{1}{2} \tilde{\nabla}_{A'} \tilde{\nabla}_{B'} \tilde{\nabla}_{C'} \tilde{\nabla}_{D'} \Phi, \tag{3.5}$$

  which resemble (2.15), (2.16).
- However, the linear version of the complex Einstein equations is the fourth order equation $\tilde{\nabla}_{A'} \tilde{\nabla}_{B'} \Box\Phi = 0$, see (2.14). Analogously to the discussion around equation (2.19), this implies that $\Box\Phi = F$, where F is a function such that $F_{uu} = F_{uw} = F_{ww} = 0$, so one does

not really get the homogeneous wave equation for Φ. We will encounter a similar issue in our proof of theorem 3.1 below, where we will show that one can get rid of inhomogeneous terms by considering gauge transformations.

- Even if one manages to get the homogeneous wave equation, the background real and complex geometries, equations (2.4) and (2.13) respectively, are different, which means that the wave equations, while *formally* equal, are different in practice. Explicitly, the wave operators of the real and complex geometries are given by equations (2.10) and (2.18).

3.2. Proof of theorem 3.1

3.2.1. Preliminaries. We consider a smooth mono-parametric family of real space-times $(M, g_{ab}(\varepsilon))$, where $g_{ab} \equiv g_{ab}(0)$ is the background space-time and is assumed to satisfy the vacuum Einstein equations $R_{ab} = 0$. The background Levi-Civita connection is denoted by ∇_a, and the linearization of the metric is $h_{ab} = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}|_{\varepsilon=0}[g_{ab}(\varepsilon)]$. The linearizations of the Ricci tensor and of the curvature scalar are linear operators acting on h_{ab}. They will be denoted by $\dot{R}_{ab}[h]$ and $\dot{R}[h]$ respectively, and explicit expressions for them are (see e.g. [17])

$$\dot{R}_{ab}[h] = -\frac{1}{2}\Box h_{ab} - \frac{1}{2}\nabla_a\nabla_b(g^{cd}h_{cd}) + \frac{1}{2}\nabla^c\nabla_a h_{bc} + \frac{1}{2}\nabla^c\nabla_b h_{ac}, \tag{3.6}$$

$$\dot{R}[h] = \nabla^a\nabla^b h_{ab} - \Box(g^{ab}h_{ab}). \tag{3.7}$$

Calculations are greatly simplified by using spinors. We emphasize that we do not perturb spinors themselves, we just use the spinor structure of the background space-time. Since all perturbations are tensor fields, they can be written in spinor language, using the usual dictionary between tensor indices and pairs of spinor indices (see section 2.1). For example, using the background Levi-Civita connection $\nabla_a = \nabla_{AA'}$, for the perturbed Ricci tensor we can write $\dot{R}_{ab}[h] \equiv \dot{R}_{ABA'B'}[h]$, with

$$\dot{R}_{ABA'B'}[h] = -\frac{1}{2}\Big[\Box h_{ABA'B'} + \nabla_{AA'}\nabla_{BB'}(g^{cd}h_{cd}) - \nabla^{CC'}\nabla_{AA'}h_{BB'CC'} - \nabla^{CC'}\nabla_{BB'}h_{AA'CC'}\Big]. \tag{3.8}$$

Notice that this does not mean that we are linearizing a spinor field. In this work, the meaning of 'perturbed field' is the ordinary one in perturbation theory in general relativity (see e.g. [17]).

Let $\{o^A, \iota^A\}$ be a spin frame for the background space-time, $\epsilon_{AB}o^A\iota^B = 1$. For the calculations in section 3.2.1, it is useful to define the operators

$$\tilde{\nabla}_{A'} := o^A\nabla_{AA'}, \quad \nabla_{A'} := \iota^A\nabla_{AA'}, \quad \bar{\nabla}_A := \bar{o}^{A'}\nabla_{AA'}, \quad \nabla_A := \bar{\iota}^{A'}\nabla_{AA'}. \tag{3.9}$$

For the particular case of a pp-wave background, from the discussion of section 2.3 we have $\tilde{\nabla}_{A'}o^B = \nabla_{A'}o^B = 0$, $\tilde{\nabla}_{A'}\iota^B = 0$ and $\nabla_{A'}\iota^B = -\kappa'\bar{o}_{A'}o^B$.

3.2.2. The radiation gauge. As is well-known, diffeomorphism invariance in general relativity implies that in linearized gravity, any metric perturbation h_{ab} is physically equivalent to $h_{ab} + K[\xi]_{ab}$, where $K[\xi]_{ab} = \nabla_a\xi_b + \nabla_b\xi_a$ and ξ_a is arbitrary. For a vacuum background, it identically holds $\dot{R}_{ab}[K[\xi]] \equiv 0$ for any ξ_a. For a pp-wave, in appendix 'The gauge operator' we give explicit expressions for the components of $K[\xi]_{ab}$.

For a background space-time possessing a null vector ℓ^a associated to a geodesic shear-free congruence (which is certainly the case for the pp-waves studied in this work), one can impose (see [18]) the so-called *radiation gauge*:

$$\ell^a h_{ab} = 0, \qquad g^{ab} h_{ab} = 0. \tag{3.10}$$

A short calculation then shows that in terms of a null tetrad $\{\ell_a, n_a, m_a, \bar{m}_a\}$, it holds

$$h_{ab} = h_{nn}\ell_a\ell_b - 2h_{n\bar{m}}\ell_{(a}m_{b)} - 2h_{nm}\ell_{(a}\bar{m}_{b)} + h_{\bar{m}\bar{m}}m_am_b + h_{mm}\bar{m}_a\bar{m}_b \tag{3.11}$$

where $h_{nn} = n^a n^b h_{ab}$, etc. Replacing the expression (2.2) for the null vectors, in spinor language we get

$$h_{ab} = o_A o_B \mathring{X}_{A'B'} + \bar{o}_{A'}\bar{o}_{B'}\bar{\mathring{X}}_{AB},$$

where

$$\mathring{X}_{A'B'} = \frac{1}{2}h_{nn}\bar{o}_{A'}\bar{o}_{B'} - 2h_{n\bar{m}}\bar{o}_{(A'}\bar{\iota}_{B')} + h_{\bar{m}\bar{m}}\bar{\iota}_{A'}\bar{\iota}_{B'}.$$

Now, let ψ be an arbitrary real scalar field. Then we have, trivially,

$$\begin{aligned} h_{ab} &= o_A o_B \mathring{X}_{A'B'} + \bar{o}_{A'}\bar{o}_{B'}\bar{\mathring{X}}_{AB} + i\psi\, o_A o_B \bar{o}_{A'}\bar{o}_{B'} - i\psi\, o_A o_B \bar{o}_{A'}\bar{o}_{B'} \\ &= o_A o_B \underbrace{(\mathring{X}_{A'B'} + i\psi\,\bar{o}_{A'}\bar{o}_{B'})}_{=:X_{A'B'}} + \bar{o}_{A'}\bar{o}_{B'}\underbrace{(\bar{\mathring{X}}_{AB} - i\psi\, o_A o_B)}_{=\bar{X}_{AB}}, \end{aligned} \tag{3.12}$$

so the tensor field (3.11) is

$$h_{ab} = \gamma_{ab} + \bar{\gamma}_{ab}, \qquad \gamma_{ab} = o_A o_B X_{A'B'}. \tag{3.13}$$

The reason for including the arbitrary scalar field ψ will become clear in the section 3.2.3. The relation between the components of $X_{A'B'}$ and the components in (3.11) is

$$X_{0'0'} = h_{\bar{m}\bar{m}}, \qquad X_{0'1'} = h_{n\bar{m}}, \qquad X_{1'1'} = \frac{1}{2}h_{nn} + i\psi. \tag{3.14}$$

It is important to note that the conditions (3.10) do not exhaust the gauge freedom. In appendix 'Residual radiation gauge freedom' we analyse the residual gauge transformations under which (3.10) is preserved. This plays an important role below.

3.2.3. Potentials. We now assume that we are given a real metric perturbation h_{ab} in radiation gauge, equation (3.13), that satisfies the linearized Einstein vacuum equations:

$$\dot{R}_{ABA'B'}[h] = \dot{R}_{ABA'B'}[\gamma] + \dot{R}_{ABA'B'}[\bar{\gamma}] = 0. \tag{3.15}$$

Notice that this equation does not imply that $\dot{R}_{ABA'B'}[\gamma]$ vanishes. Using (3.8), (3.9) and (3.13), after some calculations we find the following expressions for the linearized Ricci tensor and Ricci scalar of the tensor field $\gamma_{ab} = o_A o_B X_{A'B'}$:

$$\dot{R}[\gamma] = \tilde{\nabla}^{A'}\tilde{\nabla}^{B'}X_{A'B'}, \tag{3.16a}$$

$$\dot{R}_{ABA'B'}[\gamma] = -\frac{1}{2}\left[o_A o_B \Box X_{A'B'} + o_B\tilde{\nabla}^{A'}\nabla_{AA'}X_{B'C'} + o_A\tilde{\nabla}^{A'}\nabla_{BB'}X_{A'C'}\right]. \tag{3.16b}$$

For the complex conjugate $\bar{\gamma}_{ab} = \bar{o}_{A'}\bar{o}_{B'}\bar{X}_{AB}$, the corresponding formulas are obtained by simply taking the complex conjugate of the above. Note that, regardless of (3.15), it follows immediately that

$$o^A o^B \dot{R}_{ABA'B'}[\gamma] \equiv 0, \qquad \bar{o}^{A'}\bar{o}^{B'}\dot{R}_{ABA'B'}[\bar{\gamma}] \equiv 0. \tag{3.17}$$

From (3.16a) and its complex conjugate, we find

$$\begin{aligned} \dot{R}[h] &= \tilde{\nabla}^{A'}\tilde{\nabla}^{B'}X_{A'B'} + \bar{\nabla}^{A}\bar{\nabla}^{B}\bar{X}_{AB} \\ &= \left(\tilde{\nabla}^{A'}\tilde{\nabla}^{B'}\mathring{X}_{A'B'} + i\partial_u^2\psi\right) + \left(\bar{\nabla}^{A}\bar{\nabla}^{B}\bar{\mathring{X}}_{AB} - i\partial_u^2\psi\right) = 0, \end{aligned} \tag{3.18}$$

where in the second line we used the definition of $X_{A'B'}$ given in equation (3.12). We now see the reason for including the arbitrary scalar field ψ: since it is free, we can choose it so as to satisfy

$$\partial_u^2\psi = i\tilde{\nabla}^{A'}\tilde{\nabla}^{B'}\mathring{X}_{A'B'}, \tag{3.19}$$

or more explicitly:

$$\partial_u^2\psi = i\left[\partial_{\bar{\zeta}}^2 h_{\bar{m}\bar{m}} + 2\partial_u\partial_{\bar{\zeta}}h_{n\bar{m}} + \frac{1}{2}\partial_u^2 h_{nn}\right]. \tag{3.20}$$

(One still has the freedom $\psi \to \psi + \chi$ with $\partial_u^2\chi = 0$, but we will not need this.) The choice (3.20) of ψ has the consequence that

$$\tilde{\nabla}^{A'}\tilde{\nabla}^{B'}X_{A'B'} = 0, \tag{3.21}$$

which implies that there exists, locally, a spinor field $Y_{A'}$ such that

$$X_{A'B'} = \tilde{\nabla}_{(A'}Y_{B')}. \tag{3.22}$$

Remark 3.2. The argument for deducing (3.22) from (3.21) is essentially a variant of the argument given by Penrose in [5, section 4]. It is always true locally, and it can be extended globally to a region that has vanishing first and second homotopy groups. As explained by Penrose, this topological restriction accounts for the impossibility of finding *global* potentials in certain cases, such as for Coulomb fields.

In coordinate terms, the above means that, by virtue of the vanishing of the perturbed Ricci scalar of h_{ab}, and by choosing ψ in the form (3.20), one can locally find two fields $Y_{0'}$, $Y_{1'}$ (which are the components of a spinor field $Y_{A'}$ in the spin frame $\{\bar{o}_{A'}, \bar{\iota}_{A'}\}$) such that

$$\partial_u Y_{0'} = h_{\bar{m}\bar{m}}, \tag{3.23a}$$

$$\partial_u Y_{1'} - \partial_{\bar{\zeta}}Y_{0'} = 2h_{n\bar{m}}, \tag{3.23b}$$

$$\partial_{\bar{\zeta}}Y_{1'} = -\left(\frac{1}{2}h_{nn} + i\psi\right). \tag{3.23c}$$

From these equations we see that there is some freedom in $Y_{0'}$, $Y_{1'}$: one can check that the equations are invariant under $Y_{0'} \to Y_{0'} + \tau_{0'}$, $Y_{1'} \to Y_{1'} + \tau_{1'}$, where $\tau_{0'} = p(v,\zeta)\bar{\zeta} + q_{0'}(v,\zeta)$ and $\tau_{1'} = p(v,\zeta)u + q_{1'}(v,\zeta)$, with $p, q_{0'}, q_{1'}$ arbitrary functions of v, ζ. Alternatively, this is seen from integrating equations (3.23), which gives

$$Y_{0'} = \int \mathrm{d}u h_{\bar{m}\bar{m}} + p(v,\zeta)\bar{\zeta} + q_{0'}(v,\zeta), \tag{3.24a}$$

$$Y_{1'} = -\int \mathrm{d}\bar{\zeta}\left(\frac{1}{2}h_{nn} + i\psi\right) + p(v,\zeta)u + q_{1'}(v,\zeta). \tag{3.24b}$$

From a coordinate-free perspective, the freedom in $p, q_{0'}, q_{1'}$ corresponds to the fact that equation (3.22) is invariant under $Y_{A'} \to Y_{A'} + \tau_{A'}$, where $\tau_{A'}$ is any solution to $\tilde{\nabla}_{(A'}\tau_{B')} = 0$. We will not need to use this freedom.

In view of (3.22), the original real metric perturbation is

$$h_{ab} = \gamma_{ab} + \bar{\gamma}_{ab}, \qquad \gamma_{ab} = o_A o_B \tilde{\nabla}_{(A'}Y_{B')}. \tag{3.25}$$

The linearized Ricci operator for tensor fields of the form $\gamma_{ab} = o_A o_B\tilde{\nabla}_{(A'}Y_{B')}$ is, of course, a special case of (3.16*b*). After some calculations, we find

$$-2\dot{R}_{ABA'B'}[\gamma] = o_A o_B\left[2\tilde{\nabla}_{A'}\tilde{\nabla}_{B'}\nabla^{C'}Y_{C'} - \nabla_{(A'}\tilde{\nabla}_{B')}\tilde{\nabla}^{C'}Y_{C'}\right] - o_{(A}\iota_{B)}\tilde{\nabla}_{A'}\tilde{\nabla}_{B'}\tilde{\nabla}^{C'}Y_{C'}. \tag{3.26}$$

Summarizing, so far we have only imposed the vanishing of the perturbed Ricci scalar, and we used this to deduce the structure (3.25) of the metric perturbation. Using (3.26) and its complex conjugate, the rest of the Einstein equations (3.15) is

$$o_A o_B \left[2\tilde{\nabla}_{A'}\tilde{\nabla}_{B'}\nabla^{C'}Y_{C'} - \nabla_{(A'}\tilde{\nabla}_{B')}\tilde{\nabla}^{C'}Y_{C'}\right] - o_{(A}\iota_{B)}\tilde{\nabla}_{A'}\tilde{\nabla}_{B'}\tilde{\nabla}^{C'}Y_{C'} + \text{c.c} = 0 \quad (3.27)$$

where 'c.c' stands for complex conjugate. This equation is automatically satisfied *if* $\nabla^{C'}Y_{C'}$ and $\tilde{\nabla}^{C'}Y_{C'}$ vanish. In this case the result of theorem 3.1 would follow immediately: the equation $\tilde{\nabla}^{C'}Y_{C'} = 0$ would imply that $Y_{C'} = \tilde{\nabla}_{C'}\Phi$ for some (locally defined) complex scalar field Φ, and $\nabla^{C'}Y_{C'} = 0$ would give the wave equation $\Box\Phi = 0$. However, the converse of the above statement is not necessarily true: the equation $\dot{R}_{ABA'B'}[h] = 0$ does not imply that $\nabla^{C'}Y_{C'}$ and $\tilde{\nabla}^{C'}Y_{C'}$ vanish.

The non-vanishing of $\nabla^{C'}Y_{C'}$ and $\tilde{\nabla}^{C'}Y_{C'}$ makes the completion of the proof of theorem 3.1 more difficult. What we will show is that these fields can be set to zero by a gauge transformation. To this end, recall that we mentioned in section 3.2.2 that we still have the freedom to perform residual gauge transformations, i.e. transformations of the form

$$h_{ab} \to h'_{ab} = h_{ab} - K[\xi]_{ab} \quad (3.28)$$

where $K[\xi]_{ab} = \nabla_a\xi_b + \nabla_b\xi_a$ satisfies $\ell^a K[\xi]_{ab} = 0 = g^{ab}K[\xi]_{ab}$. We analyse this residual freedom in appendix 'Residual radiation gauge freedom', where we show that there exists a spinor field $g_{A'}$ such that $K[\xi]_{ab}$ can be written as in equation (A.11). The gauge-transformed metric is then

$$h'_{ab} = \gamma'_{ab} + \bar{\gamma}'_{ab}, \quad (3.29)$$

where

$$\gamma'_{ab} = o_A o_B \tilde{\nabla}_{(A'}Z_{B')}, \qquad Z_{B'} = Y_{B'} - g_{B'}. \quad (3.30)$$

Since (3.28)–(3.29) is a gauge transformation, we have $\dot{R}_{ABA'B'}[h] = \dot{R}_{ABA'B'}[h']$, thus, the Einstein equations (3.27) are equivalently

$$o_A o_B \left[2\tilde{\nabla}_{A'}\tilde{\nabla}_{B'}\nabla^{C'}Z_{C'} - \nabla_{(A'}\tilde{\nabla}_{B')}\tilde{\nabla}^{C'}Z_{C'}\right] - o_{(A}\iota_{B)}\tilde{\nabla}_{A'}\tilde{\nabla}_{B'}\tilde{\nabla}^{C'}Z_{C'} + \text{c.c} = 0. \quad (3.31)$$

Proposition 3.3. *The gauge transformation (3.28) and (3.29) can be chosen such that the spinor field $Z_{A'}$ satisfies the neutrino equation*

$$\nabla^{AA'}Z_{A'} = 0. \quad (3.32)$$

We defer the proof of this proposition to appendix 'Proof of proposition 3.3'. Now, any solution to (3.32) can be written (locally) as $Z_{A'} = \tilde{\nabla}_{A'}\Phi$ for some complex scalar field that satisfies the wave equation. To see this, first contract (3.32) with o_A, which gives $\tilde{\nabla}^{A'}Z_{A'} = 0$. This implies that there is, locally, a complex scalar field Φ such that $Z_{A'} = \tilde{\nabla}_{A'}\Phi$ (see remark 3.2). Contracting now (3.32) with ι_A, we get $\nabla^{A'}\tilde{\nabla}_{A'}\Phi = 0$, which is the same as $\Box\Phi = 0$.

Summarizing, the original real metric perturbation is $h_{ab} = h'_{ab} + K[\xi]_{ab}$, where $h'_{ab} = \gamma'_{ab} + \bar{\gamma}'_{ab}$, γ'_{ab} is given by

$$\gamma'_{ab} = o_A o_B \tilde{\nabla}_{A'}\tilde{\nabla}_{B'}\Phi, \quad (3.33)$$

and Φ satisfies the wave equation $\Box\Phi = 0$ on the background pp-wave space-time. This concludes the proof of theorem 3.1.

Remark 3.4.

(a) The perturbation $h'_{ab} = \gamma'_{ab} + \bar{\gamma}'_{ab}$ is both in radiation gauge and in Lorenz gauge: one can check that $\ell^a h'_{ab} = 0 = g^{ab}h'_{ab}$ as well as $\nabla^a h'_{ab} = 0$.

(b) The residual radiation gauge freedom is essential for the proof of theorem 3.1. Note that this must also be used if one wants to apply the same method to even the simplest case of Minkowski space-time, which can be obtained by simply setting $H=0$ in our formulas[5].
(c) As a by-product of the above construction, we obtained a method to generate solutions to the linearized Einstein vacuum equations from solutions to the neutrino equation.
(d) If the potential Φ is independent of u, i.e. $\Phi_u = 0$, then the perturbation actually gives a solution to the full (non-linear) Einstein equations. This can be seen from equations (3.2), (3.3), (2.5), (2.10): Φ is just a perturbation to the background wave profile H.

4. Generalization to a 'half-Káhler' vacuum space-time

In section 4 we briefly show how the ideas of the previous sections can be carried over to a more general (real, Lorentzian) space-time: the 'half-Káhler' case of table 1. This is defined by the condition that there is a parallel *projective* spinor, that is, a 'spinor field up to scale' that is parallel. We are not aware of a description of this space-time (or its complex generalization) analogous to the one given in propositions 2.1, 2.2. In the Euclidean case, the manifold must be Kähler, so the Lorentzian version might be of interest on its own right[6]. In addition, a *real* version of the complex result of proposition 2.2 for this case would correspond again to a split signature metric. Here we restrict ourselves merely to the description of gravitational perturbations.

A convenient way of expressing the existence of a spinor up to scale that is parallel is to use GHP language (see [6, section 4.12]). As is known, the use of spinors/vectors up to scale in relativity brings about the notion of 'GHP weight'. Let $\{o^A, \iota^A\}$ be two spinor fields in a Lorentzian space-time (M, g_{ab}), with $o_A\iota^A = 1$. A (scalar/tensor/spinor) field η is said to have GHP weight $\{p,q\}$ if, under the rescaling $o^A \to \lambda o^A$, $\iota^A \to \lambda^{-1}\iota^A$ (with λ a complex scalar different from zero), it transforms as $\eta \to \lambda^p \bar{\lambda}^q \eta$. A derivative operator that is covariant under this transformation is the GHP connection $\Theta_a = \nabla_a + p\omega_a + q\bar{\omega}_a$, where $\omega_a := \iota_B \nabla_a o^B$. The existence of a parallel projective spinor can then be expressed as the condition $\Theta_{AA'}o^B = 0$.

If o^A satisfies $\Theta_{AA'}o^B = 0$, then a few calculations using $[\Theta_a, \Theta_b]o^C = 0$ show that the space-time must be of Petrov type II. If in addition, we impose the vacuum condition $\Phi_{ABA'B'} = 0 = \Lambda$, then $\Psi_2 = 0$, $\nabla_{A(A'}\omega^A_{B')} = 0$ and $\nabla_{A'(A}\omega^{A'}_{B)} = -\Psi_3 o_A o_B$. The GHP connection is then self-dual and algebraically special. Notice that $\Psi_2 = 0$ excludes the type D case, so in particular, black holes are not included in this vacuum class.

The vector field $\ell^b = o^B \bar{o}^{B'}$ satisfies $\Theta_a \ell^b = 0$, so it is tangent to a null congruence that is both geodesic and shear-free. The radiation gauge for gravitational perturbations can then be imposed [18]. The discussion from now on is analogous to what we did in sections 3.2.2 and 3.2.3, the only extra point to keep in mind is that all fields now carry GHP weights. If h_{ab} is a perturbation in radiation gauge, we can write it as in (3.12), that is $h_{ab} = \gamma_{ab} + \bar{\gamma}_{ab}$, with $\gamma_{ab} = o_A o_B X_{A'B'}$. The spinor field $X_{A'B'}$ has weights $\{-2,0\}$. The linearized Einstein equations are then $\dot{R}_{ab}[h] = \dot{R}_{ab}[\gamma] + \dot{R}_{ab}[\bar{\gamma}] = 0$, and a calculation shows that

$$-2\dot{R}_{ab}[\gamma] = o_A o_B[\Box^{\Theta} X_{A'B'} + 2\tilde{\Theta}^{C'}\Theta_{(A'}X_{B')C'}] - 2o_{(A}\iota_{B)}\tilde{\Theta}^{C'}\tilde{\Theta}_{(A'}X_{B')C'} - \frac{1}{2}g_{ab}\dot{R}[\gamma], \quad (4.1)$$

[5] Note that the formulations of the Minkowski problem in [4, 5], [6, section 5.7] are very different to our method and only apply when the background is flat.

[6] Perhaps a closer Lorentzian analogue to a Kähler manifold would be a space-time where we have both $\Theta_{AA'}o^B = 0$ and $\Theta_{AA'}\iota^B = 0$. This may be seen as a complexified version of Kähler geometry. Black holes (in particular) are conformal to this space-time.

where

$$\dot{R}[\gamma] = \tilde{\Theta}^{A'}\tilde{\Theta}^{B'}X_{A'B'}, \tag{4.2}$$

and we defined

$$\square^{\Theta} := g^{ab}\Theta_a\Theta_b, \qquad \tilde{\Theta}_{A'} := o^A\Theta_{AA'}, \qquad \Theta_{A'} := \iota^A\Theta_{AA'}. \tag{4.3}$$

Choosing the free scalar field ψ in (3.12) so that $\tilde{\Theta}^{A'}\tilde{\Theta}^{B'}X_{A'B'} = 0$, we deduce that there is a spinor field $Y_{A'}$, with weights $\{-3,0\}$, such that $X_{A'B'} = \tilde{\Theta}_{(A'}Y_{B')}$. Thus, after some calculations, the linearized Einstein equations become (compare to (3.27))

$$o_Ao_B\left[2\tilde{\Theta}_{A'}\tilde{\Theta}_{B'}\Theta^{C'}Y_{C'} - \Theta_{(A'}\tilde{\Theta}_{B')}\tilde{\Theta}^{C'}Y_{C'}\right] - o_{(A}\iota_{B)}\tilde{\Theta}_{A'}\tilde{\Theta}_{B'}\tilde{\Theta}^{C'}Y_{C'} + \text{c.c} = 0. \tag{4.4}$$

For a residual radiation gauge transformation $h_{ab} \to h'_{ab} = h_{ab} - K[\xi]_{ab}$, (where $K[\xi]_{ab} = \nabla_a\xi_b + \nabla_b\xi_a$ satisfies (A.5)) there is a spinor $g_{A'}$ such that $K[\xi]_{ab} = o_Ao_B\tilde{\Theta}_{(A'}g_{B')} + \text{c.c}$. So the Einstein equations for h'_{ab} are the same as (4.4) with $Y_{A'}$ replaced by $Z_{A'} = Y_{A'} - g_{A'}$. The analysis of the residual gauge is analogous to the pp-wave case discussed in appendix 'Residual radiation gauge freedom', where instead of coordinate derivatives we use GHP operators. For example, instead of (A.7), we find $\text{þ}\xi_\ell = 0$, $\text{þ}^2\xi_m = \text{þ}^2\xi_{\bar{m}} = \text{þ}^2\xi_n = 0$, $\eth\eth'\xi_\ell = 0$. Choosing then the gauge such that $\tilde{\Theta}^{A'}g_{A'} = \tilde{\Theta}^{A'}Y_{A'}$, $\Theta^{A'}g_{A'} = \Theta^{A'}Y_{A'}$, we obtain $\tilde{\Theta}^{A'}Z_{A'} = 0$ and $\Theta^{A'}Z_{A'} = 0$, or equivalently a weighted neutrino equation

$$\Theta^{AA'}Z_{A'} = 0. \tag{4.5}$$

From $\tilde{\Theta}^{A'}Z_{A'} = 0$ we deduce that there is, locally, a complex scalar field Φ, with weights $\{-4,0\}$, such that $Z_{A'} = \tilde{\Theta}_{A'}\Phi$, and from $\Theta^{A'}Z_{A'} = 0$ we deduce that $\square^{\Theta}\Phi = 0$.

In summary, we see that any real gravitational perturbation to a 'half-Kähler' vacuum space-time, once written in radiation gauge and assuming that it satisfies the linearized Einstein vacuum equations, can be locally expressed as

$$h_{ab} = o_Ao_B\tilde{\Theta}_{A'}\tilde{\Theta}_{B'}\Phi + \text{c.c} + 2\nabla_{(a}\xi_{b)}, \tag{4.6}$$

where Φ satisfies the GHP weighted wave equation

$$\square^{\Theta}\Phi = 0. \tag{4.7}$$

This generalizes the pp-wave result, theorem 3.1. Notice that, analogously to the pp-wave case, the Hertz potential $h'_{ab} = o_Ao_B\tilde{\Theta}_{A'}\tilde{\Theta}_{B'}\Phi + \text{c.c}$ is both in radiation gauge and in Lorenz gauge (i.e. $\nabla^a h'_{ab} = 0$).

5. Conclusions

In this work we have shown that any real, linear gravitational perturbation of a (real, Lorentzian) vacuum pp-wave space-time can be locally expressed, modulo gauge transformations, as the real part of a Hertz/Debye potential, where the scalar Debye potential satisfies the wave equation of the background pp-wave solution. This is believed to hold for more general backgrounds as well (replacing the wave equation by, e.g. the Teukolsky equation), but to our knowledge, the result has been completely proven only for perturbations of Minkowski [4, 5], [6, section 5.7]. We stress that our result is local, cf remark 3.2 and also [19].

We also showed the connections between the Hertz/Debye representation for perturbations of pp-waves and the non-linear structure of a complex space-time with a parallel spinor. This illustrates the formal relation between this representation and a particular case of the hyper-heavenly construction of Plebański and Robinson [16]. In addition, we argued that a *linear* problem in a real space with a Lorentzian metric is related to a *non-linear* problem also in a real space but with a split signature metric. This is interesting in view of modern developments where physics in split signature is relevant, especially in the context of scattering amplitudes and connections to gravitation, see [20, 21].

Our approach relied on using special complex two-surfaces in the complexified space-time, called α- and β-surfaces, which are the basic object of twistor theory, see [7]. These surfaces are present, in particular, for (complexifications of) any algebraically special, vacuum, real, Lorentzian space-time. Thus, the method employed in this work can also be applied to the analysis of linearized gravity on more general backgrounds. We illustrated this by generalizing our result to perturbations of a 'half-Káhler' vacuum space-time. Explicit computations in more general backgrounds are more involved due to the complicated structure of the curvature. The interpretation of the coordinates defined by twistor surfaces is also more difficult than in the pp-wave case (where these coordinates are simply Brinkmann coordinates).

From a physical point of view, our motivation came from perturbation theory in general relativity and its applications to gravitational wave physics, concerning the Hertz/Debye potential representation of perturbations, and gauge issues. While the currently most interesting space-times for gravitational wave physics are more general than pp-waves, representing e.g. single or binary black holes, the case of pp-waves already presents conceptual difficulties similar to those that appear in the other more general cases. This can be seen from our study of perturbations to 'half-Káhler' space-times. The application of these ideas to the general class of Petrov type II vacuum solutions (including type D and the Kerr solution) is left for future work [22]. It would also be interesting to compare our results with those of [23].

From a geometric perspective, our motivation originated in the relations that (generalized) parallel spinors have with complex geometry, as discussed in section 2.2. We focused on perturbations to the simplest case of a parallel spinor in Lorentz signature, a pp-wave metric. In Euclidean signature this corresponds to hyper-Kähler manifolds, and linear perturbations in this context have been studied e.g. in [24, 25]. Natural generalizations are the other cases described in table 1. We also studied the Lorentzian 'half-Káhler' case, which in a Riemannian setting would correspond to perturbations of Kähler manifolds. A natural next step would be the study of perturbations to the 'half-Hermitian' case $\mathcal{C}_{AA'}o^B = 0$.

Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

Acknowledgment

I am very grateful to Steffen Aksteiner, Lars Andersson and Bernard Whiting for many discussions on the topics of this work, and for ongoing collaboration on generalizations of the results presented in this manuscript. I also thank Abraham Harte for interesting discussions about plane waves, and Carlos Shahbazi for helpful discussions about parallel spinors in supergravity. The author is supported by the Alexander von Humboldt Foundation.

Appendix. Gauge issues

Throughout this appendix we assume a (real, Lorentzian) vacuum pp-wave background, with the special spin frame $\{o^A, \iota^A\}$, and its complex conjugate $\{\bar{o}^{A'}, \bar{\iota}^{A'}\}$, introduced in section 2.3.

The associated null tetrad $\{\ell^a, n^a, m^a, \bar{m}^a\}$ is defined as in equation (2.2). In terms of Brinkmann coordinates (see equation (2.4)), we have

$$\ell^a\partial_a = \partial_u, \qquad m^a\partial_a = -\partial_{\bar{\zeta}}, \qquad \bar{m}^a\partial_a = -\partial_\zeta, \qquad n^a\partial_a = \partial_v - \frac{1}{2}H\partial_u. \tag{A.1}$$

The connection coefficients are given by

$$\nabla_a\ell^b = 0, \quad \nabla_a m^b = -\bar{\kappa}'\ell_a\ell^b, \quad \nabla_a\bar{m}^b = -\kappa'\ell_a\ell^b, \quad \nabla_a n^b = -\ell_a(\kappa' m^b + \bar{\kappa}'\bar{m}^b). \tag{A.2}$$

The gauge operator

For an arbitrary covector field ξ_a, we define the 'gauge operator' (or Killing operator) K by

$$K[\xi]_{ab} = \nabla_a\xi_b + \nabla_b\xi_a. \tag{A.3}$$

In terms of a null tetrad, this can be written as follows:

$$\begin{aligned} K[\xi]_{ab} = {} & K_{nn}\ell_a\ell_b - 2K_{n\bar{m}}\ell_{(a}m_{b)} - 2K_{nm}\ell_{(a}\bar{m}_{b)} + K_{\bar{m}\bar{m}}m_a m_b + K_{mm}\bar{m}_a\bar{m}_b \\ & + K_{\ell\ell}n_a n_b - 2K_{\ell\bar{m}}n_{(a}m_{b)} - 2K_{\ell m}n_{(a}\bar{m}_{b)} + 2K_{\ell n}n_{(a}\ell_{b)} - 2K_{m\bar{m}}m_{(a}\bar{m}_{b)}. \end{aligned}$$

For a pp-wave, using (A.1) and (A.2) we find:

$$K_{\ell\ell} = 2\partial_u\xi_\ell, \quad K_{m\bar{m}} = -\left(\partial_{\bar{\zeta}}\xi_{\bar{m}} + \partial_\zeta\xi_m\right) \tag{A.4a}$$

$$K_{\ell\bar{m}} = \partial_u\xi_{\bar{m}} - \partial_\zeta\xi_\ell, \quad K_{nm} = \left(\partial_v - \frac{1}{2}H\partial_u\right)\xi_m + \bar{\kappa}'\xi_\ell - \partial_{\bar{\zeta}}\xi_n, \tag{A.4b}$$

$$K_{\ell m} = \partial_u\xi_m - \partial_{\bar{\zeta}}\xi_\ell, \quad K_{n\bar{m}} = \left(\partial_v - \frac{1}{2}H\partial_u\right)\xi_{\bar{m}} + \kappa'\xi_\ell - \partial_\zeta\xi_n, \tag{A.4c}$$

$$K_{\bar{m}\bar{m}} = -2\partial_\zeta\xi_{\bar{m}}, \quad K_{\ell n} = \partial_u\xi_n + \left(\partial_v - \frac{1}{2}H\partial_u\right)\xi_\ell, \tag{A.4d}$$

$$K_{mm} = -2\partial_{\bar{\zeta}}\xi_m, \quad K_{nn} = 2\left(\partial_v - \frac{1}{2}H\partial_u\right)\xi_n + 2\kappa'\xi_m + 2\bar{\kappa}'\xi_{\bar{m}}, \tag{A.4e}$$

where $K_{\ell\ell} = \ell^a\ell^b K_{ab}$, $\xi_m = m^a\xi_a$, etc.

Residual radiation gauge freedom. The radiation gauge (3.10) is preserved by transformations in which the new gauge vector ξ_a satisfies

$$\ell^a K[\xi]_{ab} = 0, \qquad g^{ab}K[\xi]_{ab} = 0. \tag{A.5}$$

Equivalently: $K_{\ell\ell} = K_{\ell m} = K_{\ell\bar{m}} = K_{\ell n} = K_{m\bar{m}} = 0$. Using identities (A.4), this is

$$\partial_u\xi_\ell = 0, \tag{A.6a}$$

$$\partial_u\xi_{\bar{m}} - \partial_\zeta\xi_\ell = 0, \tag{A.6b}$$

$$\partial_u\xi_m - \partial_{\bar{\zeta}}\xi_\ell = 0, \tag{A.6c}$$

$$\partial_u\xi_n + \partial_v\xi_\ell = 0, \tag{A.6d}$$

$$\partial_{\bar{\zeta}}\xi_{\bar{m}} + \partial_\zeta\xi_m = 0. \tag{A.6e}$$

From here we deduce

$$\partial_u^2\xi_{\bar{m}} = \partial_u^2\xi_m = \partial_u^2\xi_n = 0, \qquad \partial_\zeta\partial_{\bar{\zeta}}\xi_\ell = 0. \tag{A.7}$$

Notice that in view of (A.4), and given that ∂_u is a Killing vector of the background space-time, from (A.7) it follows that $\partial_u^2 K_{\mathbf{ab}} = 0$ for all $\mathbf{a},\mathbf{b} = u, v, \zeta, \bar{\zeta}$.

So we get the following general form for the components of the (real) gauge vector ξ_a:

$$\xi_\ell = f_1(v,\zeta) + f_2(v,\bar{\zeta}), \tag{A.8a}$$

$$\xi_{\bar{m}} = [\partial_\zeta f_1(v,\zeta)]u + f_3(v,\zeta,\bar{\zeta}), \tag{A.8b}$$

$$\xi_n = -[\partial_v f_1(v,\zeta) + \partial_v f_2(v,\bar{\zeta})]u + f_4(v,\zeta,\bar{\zeta}), \tag{A.8c}$$

for some functions $f_1(v,\zeta)$, $f_2(v,\bar{\zeta})$, $f_3(v,\zeta,\bar{\zeta})$ and $f_4(v,\zeta,\bar{\zeta})$. Apart from reality conditions for ξ_a, any restrictions on these functions will be differential.

Given that we here impose the gauge operator (A.3) to satisfy (A.5), the same reasoning that we used in section 3.2.2 to deduce (3.12) now gives

$$K[\xi]_{ab} = o_A o_B G_{A'B'} + \bar{o}_{A'}\bar{o}_{B'}\bar{G}_{AB}, \tag{A.9}$$

where

$$G_{A'B'} = \left(\frac{1}{2}K_{nn} + i\eta\right)\bar{o}_{A'}\bar{o}_{B'} - 2K_{n\bar{m}}\bar{o}_{(A'}\bar{\iota}_{B')} + K_{\bar{m}\bar{m}}\bar{\iota}_{A'}\bar{\iota}_{B'} \tag{A.10}$$

and we have included an arbitrary scalar field η. Explicit expressions for K_{nn}, $K_{n\bar{m}}$, $K_{\bar{m}\bar{m}}$ in terms of ξ_a are given in (A.4). We can now express $G_{A'B'}$ in terms of a 1-index spinor by doing the same trick that we did in section 3.2.3. The linearized Ricci scalar of (A.9) is $\dot{R}[K[\xi]] = \tilde{\nabla}^{A'}\tilde{\nabla}^{B'}G_{A'B'} + \bar{\nabla}^A\bar{\nabla}^B\bar{G}_{AB}$ (which vanishes identically since the background is vacuum), where

$$\tilde{\nabla}^{A'}\tilde{\nabla}^{B'}G_{A'B'} = \partial_{\bar{\zeta}}^2 K_{\bar{m}\bar{m}} + 2\partial_u\partial_{\bar{\zeta}}K_{n\bar{m}} + \frac{1}{2}\partial_u^2 K_{nn} + i\partial_u^2\eta.$$

A short calculation shows that $\partial_u^2 K_{nn} = 0 = \partial_u\partial_{\bar{\zeta}}K_{n\bar{m}}$, while $\partial_{\bar{\zeta}}^2 K_{\bar{m}\bar{m}}$ is independent of u. Therefore, if we choose the arbitrary scalar η in (A.10) in the form $\eta = \frac{i}{2}[\partial_{\bar{\zeta}}^2 K_{\bar{m}\bar{m}}]u^2$, then $\tilde{\nabla}^{A'}\tilde{\nabla}^{B'}G_{A'B'} = 0$, so there is a spinor field $g_{A'}$ such that $G_{A'B'} = \tilde{\nabla}_{(A'}g_{B')}$, and

$$K[\xi]_{ab} = o_A o_B \tilde{\nabla}_{(A'}g_{B')} + \bar{o}_{A'}\bar{o}_{B'}\bar{\nabla}_{(A}\bar{g}_{B)}. \tag{A.11}$$

The relation between $g_{A'}$ and ξ_a is given by

$$\partial_u g_{0'} = K_{\bar{m}\bar{m}}, \tag{A.12a}$$

$$\partial_u g_{1'} - \partial_{\bar{\zeta}} g_{0'} = 2K_{n\bar{m}}, \tag{A.12b}$$

$$\partial_{\bar{\zeta}} g_{1'} = -\left(\frac{1}{2}K_{nn} + i\eta\right), \tag{A.12c}$$

where in the right-hand sides one replaces the expressions (A.4). We deduce from here that

$$\partial_u^3 g_{0'} = 0, \qquad \partial_{\bar{\zeta}}\partial_u^2 g_{0'} = 0, \qquad \partial_u^3 g_{1'} = 0. \tag{A.13}$$

Thus, the general structure of $g_{0'}$, $g_{1'}$ is

$$g_{0'} = A_{0'}(v,\zeta)u^2 + B_{0'}(v,\zeta,\bar{\zeta})u + C_{0'}(v,\zeta,\bar{\zeta}), \tag{A.14a}$$

$$g_{1'} = A_{1'}(v,\zeta,\bar{\zeta})u^2 + B_{1'}(v,\zeta,\bar{\zeta})u + C_{1'}(v,\zeta,\bar{\zeta}), \tag{A.14b}$$

for some functions $A_{0'}, \ldots, C_{1'}$ where the arguments are as specified in the previous equations. Using (A.12) and (A.4), one can relate these functions to the ones appearing in (A.8); this way we see, for example, that they do not identically vanish. For instance, we get

$A_{0'}(v,\zeta) = -\partial_\zeta^2 f_1(v,\zeta)$. However, in general the explicit expressions do not seem to be particularly enlightening.

Proof of proposition 3.3

We have the identity $\nabla^{AA'}Z_{A'} = o^A\nabla^{A'}Z_{A'} - \iota^A\tilde{\nabla}^{A'}Z_{A'}$, so $\nabla^{AA'}Z_{A'} = 0$ iff $\nabla^{A'}Z_{A'} = 0$ and $\tilde{\nabla}^{A'}Z_{A'} = 0$. Since $Z_{A'} = Y_{A'} - g_{A'}$, we have

$$\tilde{\nabla}^{A'}Z_{A'} = \tilde{\nabla}^{A'}Y_{A'} - \tilde{\nabla}^{A'}g_{A'}, \tag{A.15a}$$

$$\nabla^{A'}Z_{A'} = \nabla^{A'}Y_{A'} - \nabla^{A'}g_{A'}, \tag{A.15b}$$

so we want to show that, as long as the Einstein equations are satisfied, we can choose the gauge transformation such that the associated spinor field $g_{A'}$ satisfies $\tilde{\nabla}^{A'}g_{A'} = \tilde{\nabla}^{A'}Y_{A'}$ and $\nabla^{A'}g_{A'} = \nabla^{A'}Y_{A'}$.

The first observation is that any requirement for the function $\tilde{\nabla}^{A'}g_{A'}$ restricts $g_{A'}$ only up to the addition of terms of the form $\tilde{\nabla}_{A'}S$. In other words, we can write $g_{A'} = V_{A'} + 2\tilde{\nabla}_{A'}S$ where $V_{A'}$ and S are independent, then (A.15) become

$$\tilde{\nabla}^{A'}Z_{A'} = \tilde{\nabla}^{A'}Y_{A'} - \tilde{\nabla}^{A'}V_{A'}, \tag{A.16a}$$

$$\nabla^{A'}Z_{A'} = \nabla^{A'}Y_{A'} - \nabla^{A'}V_{A'} - \Box S \tag{A.16b}$$

(where we used the identity $\Box S = 2\nabla^{A'}\tilde{\nabla}_{A'}S$), and we want to show that $V_{A'}$ and S can be chosen such that $\tilde{\nabla}^{A'}Z_{A'} = 0 = \nabla^{A'}Z_{A'}$. Restrictions on $V_{A'}$ and S arise from the fact that they come from a gauge transformation: the general form of the components of $g_{A'}$ was obtained in (A.14). See (A.20) below.

In order to obtain expressions for the fields $\tilde{\nabla}^{A'}Y_{A'}$, $\nabla^{A'}Y_{A'}$, we use the linearized Einstein equations (3.27). The non-trivial, independent components are:

$$\partial_u^2\left(\tilde{\nabla}^{A'}Y_{A'}\right) = 0, \tag{A.17a}$$

$$2\partial_u^2\left(\nabla^{A'}Y_{A'}\right) + \partial_u\partial_\zeta\left(\tilde{\nabla}^{A'}Y_{A'}\right) = 0, \tag{A.17b}$$

$$\partial_u\partial_{\bar\zeta}\left(\tilde{\nabla}^{A'}Y_{A'}\right) + \partial_u\partial_\zeta\left(\bar{\nabla}^{A}\bar{Y}_{A}\right) = 0, \tag{A.17c}$$

$$2\partial_u\partial_{\bar\zeta}\left(\nabla^{A'}Y_{A'}\right) + \frac{1}{2}\left(\partial_u\partial_v + \partial_\zeta\partial_{\bar\zeta}\right)\left(\tilde{\nabla}^{A'}Y_{A'}\right) - \frac{1}{2}\partial_\zeta^2\left(\bar{\nabla}^{A}\bar{Y}_{A}\right) = 0, \tag{A.17d}$$

$$\partial_{\bar\zeta}^2\left(\tilde{\nabla}^{A'}Y_{A'}\right) - 4\partial_u\partial_\zeta\left(\nabla^{A}\bar{Y}_{A}\right) - 2\left(\partial_u\partial_v + \partial_\zeta\partial_{\bar\zeta}\right)\left(\bar{\nabla}^{A}\bar{Y}_{A}\right) = 0, \tag{A.17e}$$

$$2\partial_{\bar\zeta}^2\left(\nabla^{A'}Y_{A'}\right) + \partial_{\bar\zeta}\left(\partial_v - \frac{1}{2}H\partial_u\right)\left(\tilde{\nabla}^{A'}Y_{A'}\right) + 2\partial_\zeta^2\left(\nabla^{A}\bar{Y}_{A}\right)$$
$$+ \partial_\zeta\left(\partial_v - \frac{1}{2}H\partial_u\right)\left(\bar{\nabla}^{A}\bar{Y}_{A}\right) = 0. \tag{A.17f}$$

Equations (A.17a) and (A.17b) correspond to $\bar{o}^{A'}\bar{o}^{B'}\dot{R}_{ABA'B'}[h] = 0$, (A.17$c$) and (A.17$d$) correspond to $\bar{o}^{A'}\bar{\iota}^{B'}\dot{R}_{ABA'B'}[h] = 0$, and (A.17$e$) and (A.17$f$) correspond to $\bar{\iota}^{A'}\bar{\iota}^{B'}\dot{R}_{ABA'B'}[h] = 0$. Notice that (A.17$a$) and (A.17$b$) are the only equations that involve only $\tilde{\nabla}^{A'}Y_{A'}$ and $\nabla^{A'}Y_{A'}$ (not their complex conjugates); this is because of the identities (3.17). We will then use (A.17a) and (A.17b) to deduce the structure of $\tilde{\nabla}^{A'}Y_{A'}$, $\nabla^{A'}Y_{A'}$.

From (A.17a) we deduce that

$$\tilde{\nabla}^{A'}Y_{A'} = a(v,\zeta,\bar\zeta)u + b(v,\zeta,\bar\zeta) \tag{A.18}$$

for some functions a, b. These functions can be written in terms of the metric perturbation h_{ab} (up to an arbitrary function of v, ζ, that we can set to zero), by noticing that $\tilde{\nabla}^{A'} Y_{A'} = -(\partial_u Y_{1'} + \partial_{\bar{\zeta}} Y_{0'})$ and using equations (3.24*a*) and (3.24*b*). Taking a u-derivative in (A.17*b*) and using (A.17*a*), we see that $\partial_u^3(\nabla^{A'} Y_{A'}) = 0$, so $\nabla^{A'} Y_{A'}$ is quadratic in u. Using also (A.18), it follows that

$$\nabla^{A'} Y_{A'} = \left[-\frac{1}{4} a_\zeta(v, \zeta, \bar{\zeta})\right] u^2 + c(v, \zeta, \bar{\zeta}) u + d(v, \zeta, \bar{\zeta}) \tag{A.19}$$

for some functions c, d. The rest of the equations in (A.17) involve also the complex conjugate fields, and they give additional restrictions on the functions that appear in the right hand sides of (A.18) and (A.19).

Notice that, since the fields $\tilde{\nabla}^{A'} g_{A'}$, $\nabla^{A'} g_{A'}$ come from a gauge transformation, the same equations (A.17) hold for them. In other words, any restrictions on $\tilde{\nabla}^{A'} Y_{A'}$, $\nabla^{A'} Y_{A'}$ coming from (A.17) are also satisfied by $\tilde{\nabla}^{A'} g_{A'}$, $\nabla^{A'} g_{A'}$. But $\tilde{\nabla}^{A'} g_{A'}$, $\nabla^{A'} g_{A'}$ must also fulfil restrictions that come from the gauge condition. These additional restrictions were analysed in appendix 'Residual radiation gauge freedom', where the general expressions (A.14) were found. In our current context, we have $g_{A'} = V_{A'} + 2\tilde{\nabla}_{A'} S$, or in components $g_{0'} = V_{0'} + 2\partial_u S$, $g_{1'} = V_{1'} - 2\partial_{\bar{\zeta}} S$. The restrictions (A.13) together with the fact that $V_{A'}$ and S are independent imply that $\partial_u^3 V_{0'} = \partial_{\bar{\zeta}} \partial_u^2 V_{0'} = \partial_u^3 V_{1'} = 0$ and $\partial_u^4 S = \partial_u^3 \partial_{\bar{\zeta}} S = 0$. So we have the following form:

$$V_{0'} = \alpha_{0'}(v, \zeta) u^2 + \beta_{0'}(v, \zeta, \bar{\zeta}) u + \gamma_{0'}(v, \zeta, \bar{\zeta}), \tag{A.20a}$$

$$V_{1'} = \alpha_{1'}(v, \zeta, \bar{\zeta}) u^2 + \beta_{1'}(v, \zeta, \bar{\zeta}) u + \gamma_{1'}(v, \zeta, \bar{\zeta}), \tag{A.20b}$$

$$S = S_3(v, \zeta) u^3 + S_2(v, \zeta, \bar{\zeta}) u^2 + S_1(v, \zeta, \bar{\zeta}) u + S_0(v, \zeta, \bar{\zeta}). \tag{A.20c}$$

After some tedious calculations, this gives (using in particular the expression (2.10) for $\Box$):

$$\tilde{\nabla}^{A'} g_{A'} = -(2\alpha_{1'} + \partial_{\bar{\zeta}} \beta_{0'}) u - (\beta_{1'} + \partial_{\bar{\zeta}} \gamma_{0'}), \tag{A.21}$$

$$\begin{aligned} \nabla^{A'} g_{A'} = & \left[\partial_v \alpha_{0'} + \partial_\zeta \alpha_{1'} + 6\partial_v S_3 - 2\partial_\zeta \partial_{\bar{\zeta}} S_2\right] u^2 \\ & + \left[\partial_v \beta_{0'} - H\alpha_{0'} + \partial_\zeta \beta_{1'} + 4\partial_v S_2 - 2\partial_\zeta \partial_{\bar{\zeta}} S_1 - 6HS_3\right] u \\ & + \left[\partial_v \gamma_{0'} - \frac{1}{2} H\beta_{0'} + \partial_\zeta \gamma_{1'} + 2\partial_v S_1 - 2\partial_\zeta \partial_{\bar{\zeta}} S_0 - 2HS_2\right] \end{aligned} \tag{A.22}$$

Comparing these expressions to (A.18) and (A.19), we see that we can choose the free functions in (A.20) so that $\tilde{\nabla}^{A'} g_{A'} = \tilde{\nabla}^{A'} Y_{A'}$ and $\nabla^{A'} g_{A'} = \nabla^{A'} Y_{A'}$, which is what we wanted to prove.

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