

# Asymptotic flatness at null infinity in five dimensions

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## Abstract

We discuss the asymptotic structure of null infinity in five dimensional space-time. Since it is known that the conformal infinity is not useful for odd higher dimensions, we shall employ the coordinate based method like the Bondi coordinate firstly introduced in four dimensions. Then we define the null infinity and identify the asymptotic symmetry. We also derive the Bondi mass expression and show its conservation law.

## 1 Introduction

Inspired by superstring theory, fundamental studies of higher dimensional space-time is important. One of open issue is the asymptotic structure at null infinity. We often introduce the conformal infinity to discuss the asymptotic structure at infinities in four dimensions [1, 2]. Therein the space-time is compactified by conformal transformation and embedded into another space-time. For example, Minkowski space-time is conformally embedded into Einstein static universe. Conformally embedded space-time have two different infinities, i.e., spatial infinity and null infinity. In higher dimensional space-times, the asymptotic structure at spatial infinity can be well-defined. The asymptotic symmetry is identified with Poincare group and conserved quantities associated with the symmetry are constructed [3, 4].

On the other hand, the asymptotic structure at null infinity is not understood completely in higher dimensional space-times [5–7]. Indeed, the definition of null infinity is given only in even dimensions [5, 7]. The difficulty in the definition of null infinity compared with spatial infinity is due to the presence of the gravitational waves at null infinity. At spatial infinity, since there are no gravitational waves, the asymptotic structure is "stationary" and the total mass and total angular momentum are conserved, while the asymptotic structure at null infinity might be disturbed by gravitational waves. Hence, we need a stable definition of null infinity against gravitational waves. We can give such a definition in *even* dimensions if we use conformal embedding method, but cannot in odd dimensional space-times since we cannot show the smoothness of Einstein equations at null infinity. This non-smoothness would be related with the facts that in conformal embedding method we introduce the conformal factor  $\Omega \sim 1/r$ , and the behavior of gravitational waves near null infinity are order of  $\mathcal{O}(\Omega^{(d-2)/2})$  in  $d$  dimensional space-times. The problem comes from the half-integer power of  $\Omega$ .

In this paper, we define null infinity in five dimensional space-time. We do not use conformal embedding method, but the Bondi coordinate to define null infinity, which was introduced firstly by Bondi and Sachs in four dimensions [8–11]. The rest of this paper is organized as follows. In section 2, we introduce the Bondi coordinate in five dimensions. In section 3, we define the asymptotic flatness at null infinity in five dimensions and show the robustness of the definition against gravitational waves by solving Einstein equations. In this section, we define the Bondi mass, and obtain the Bondi mass loss law in five dimensions. In section 4, we discuss the asymptotic symmetry at null infinity associated with the asymptotic flatness defined in section 3. We show that there are no supertranslations in five dimensions unlike in four dimensions. Finally, in section 5, we give a summary.

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## 2 Bondi coordinate in five dimensions

We consider five dimensional space-time. We introduce the Bondi coordinate to define asymptotic flatness at null infinity. Suppose there is a function  $u(x^a)$  which satisfies the equation

$$u_{,a}u_{,b}g^{ab} = 0. \quad (1)$$

Let  $\theta, \phi, \psi$  be angular coordinates, which are constant along gradient  $u$ . Period of these each coordinates are taken to be  $\pi, 2\pi, 2\pi$ , respectively. For convenience, we introduce the notation  $(\theta, \phi, \psi) = x^A$ , where capital Latin indices run from 2 to 4. Now we define the function  $r$  by the equation

$$r^6 \sin^2 \theta \cos^2 \theta = \det(g_{AB}). \quad (2)$$

Using these coordinates (we call them the Bondi coordinates)

$$x^0 = u, x^1 = r, x^2 = \theta, x^3 = \phi, x^4 = \psi, \quad (3)$$

we can write down metric such that

$$ds^2 = -(Ve^B/r^2)du^2 - 2e^B dudr + r^2 h_{AB}(dx^A + U^A du)(dx^B + U^B du), \quad (4)$$

where

$$h_{AB} = \begin{pmatrix} e^{C_1} & \sin \theta \sinh D_1 & \cos \theta \sinh D_2 \\ \sin \theta \sinh D_1 & e^{C_2} \sin^2 \theta & \sin \theta \cos \theta \sinh D_3 \\ \cos \theta \sinh D_2 & \sin \theta \cos \theta \sinh D_3 & e^{C_3} \cos^2 \theta \end{pmatrix}. \quad (5)$$

$V, B, h_{AB}, U^A, C_1, C_2, C_3, D_1, D_2$  and  $D_3$  are function of  $u, r$  and  $x^A$ . In this coordinate, null infinity is represented by  $r \rightarrow \infty$ .

## 3 Boundary conditions

The boundary conditions in the Bondi coordinates

$$C_1(u, r, x^A) = \frac{C_{11}(u, x^A)}{r^{3/2}} + O(1/r^2) \quad (6)$$

$$C_2(u, r, x^A) = \frac{C_{21}(u, x^A)}{r^{3/2}} + O(1/r^2) \quad (7)$$

$$C_3(u, r, x^A) = \frac{C_{31}(u, x^A)}{r^{3/2}} + O(1/r^2) \quad (8)$$

$$D_1(u, r, x^A) = \frac{D_{11}(u, x^A)}{r^{3/2}} + O(1/r^2) \quad (9)$$

$$D_2(u, r, x^A) = \frac{D_{21}(u, x^A)}{r^{3/2}} + O(1/r^2) \quad (10)$$

$$D_3(u, r, x^A) = \frac{D_{31}(u, x^A)}{r^{3/2}} + O(1/r^2) \quad (11)$$

$$V = r^2 + V_1(u, x^A)r^{1/2} - M(u, x^A) + O(1/r^{1/2}) \quad (12)$$

$$B = \frac{B_1(u, x^A)}{r^3} + O(1/r^{7/2}) \quad (13)$$

$$U^A = \frac{U_{A1}(u, x^A)}{r^{5/2}} + O(1/r^3) \quad (14)$$

$$\mathbb{C} = \frac{\mathbb{C}_1(u, x^A)}{r^{3/2}} + O(1/r^2) \quad (15)$$

define asymptotic flatness at null infinity. These boundary conditions are determined from Einstein equations  $R_{ab} = 0$ . We define  $M$  in  $V$  as Bondi mass and from Einstein equation  $R_{uu} = 0$ , we get

$\partial_u M(u, x^A)$  as

$$\frac{\partial M}{\partial u} = -\frac{1}{3} \left( (C_{11,u})^2 + C_{11,u} C_{21,u} + (C_{21,u})^2 + (D_{11,u})^2 + (D_{21,u})^2 + (D_{31,u})^2 \right). \quad (16)$$

Eq. (16) represents mass loss rate by gravitational waves, and total mass always decreases as in four dimension. Then  $M(u, x^A)$  describes the mass in  $u = \text{constant}$  surfaces.

## 4 Asymptotic symmetry at null infinity

Asymptotic symmetry should be defined as transformations preserving the boundary conditions (12), (13), (14) and (15). By infinitesimal transformation  $\xi$ , metric is transformed as

$$\delta g_{ab} = 2\nabla_{(a}\xi_{b)}. \quad (17)$$

To preserve the boundary conditions given in the previous section, the variation of metric,  $\delta g_{ab}$ , should satisfy

$$\delta g_{rr} = 0, \delta g_{rA} = 0, g^{AB}\delta g_{AB} = 0, \quad (18)$$

$$\delta g_{uu} = O(r^{-3/2}), \delta g_{ur} = O(r^{-3}), \delta g_{uA} = O(r^{-1/2}), \delta g_{AB} = O(r^{1/2}). \quad (19)$$

From Eq. (18), the components of infinitesimal transformation  $\xi$  must take the following forms:

$$\xi_r = f(u, x^A)e^B, \quad (20)$$

$$\xi_B g^{AB} = f^A(u, x^A) - fU^A + \int_r^\infty dr' e^B f_{,B} g^{AB}, \quad (21)$$

$$\xi_u = -\frac{re^B}{3} (-\xi_{A,B} + \xi_C \Gamma_{AB}^C + \xi_r \Gamma_{AB}^r) g^{AB}, \quad (22)$$

where  $\mathcal{D}_A$  is the covariant derivative with respect to the standard metric  $h_{AB}^{(0)}$  on 3 sphere. The infinitesimal transformation  $\xi$  have four free parameters  $f, f^A$ .

Then, the other components of metric variation become

$$\delta g_{uu} = \frac{2r}{3} \mathcal{D}_A f^A_{,u} + \frac{2}{3} (3f + \mathcal{D}^2 f)_{,u} + \frac{2}{r^{1/2}} h_{AB}^{(0)} f^A_{,u} U_{B1} + O(r^{-3/2}), \quad (23)$$

$$\delta g_{ur} = \frac{1}{3} (\mathcal{D}_A f^A + 3f_{,u}) + \frac{1}{5r^{5/2}} h^{(1)AB} \mathcal{D}_A \mathcal{D}_B f + O(r^{-3}), \quad (24)$$

$$\delta g_{uA} = r^2 h_{AB}^{(0)} f^B_{,u} + \frac{r}{3} \mathcal{D}_A (3f_{,u} + \mathcal{D}_B f^B) + r^{1/2} h_{AB}^{(1)} f^B_{,u} + \frac{1}{3} \mathcal{D}_A (\mathcal{D}^2 f + 3f) + O(r^{-1/2}), \quad (25)$$

$$\delta g_{AB} = \frac{2r^2}{3} (-\mathcal{D}_C f^C h_{AB}^{(0)} + 3\mathcal{D}_{(A} f_{B)}) + \frac{2r}{3} (-\mathcal{D}^2 f h_{AB}^{(0)} + 3\mathcal{D}_A \mathcal{D}_B f) + T_{AB}(u, x^A) r^{1/2} + O(r^{-1/2}), \quad (26)$$

where  $X_{(AB)} := (1/2)(X_{AB} + X_{BA})$  for some tensor  $X_{AB}$ ,  $T_{AB}$  is some traceless tensor with respect to  $h_{AB}^{(0)}$  and we expand the metric  $h_{AB}$  as

$$h_{AB} = h_{AB}^{(0)} + \frac{1}{r^{3/2}} h_{AB}^{(1)} + O(r^{-2}), \quad (27)$$

and  $h^{(1)AB} = h^{(0)AC} h^{(0)BD} h_{CD}^{(1)}$ . In the Bondi coordinate,  $h_{AB}^{(0)}$  and  $h_{AB}^{(1)}$  are

$$h_{AB}^{(0)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sin^2 \theta & 0 \\ 0 & 0 & \cos^2 \theta \end{pmatrix} \quad (28)$$

and

$$h_{AB}^{(1)} = \begin{pmatrix} C_{11} & D_{11} \sin \theta & D_{21} \cos \theta \\ D_{11} \sin \theta & C_{21} \sin^2 \theta & D_{31} \sin \theta \cos \theta \\ D_{21} \cos \theta & D_{31} \sin \theta \cos \theta & -(C_{11} + C_{21}) \cos^2 \theta \end{pmatrix}. \quad (29)$$

To satisfy the condition (19), we find that  $f^A$  and  $f$  should satisfy following equations:

$$f^A{}_{,u} = 0, \quad (30)$$

$$\mathcal{D}_A f_B + \mathcal{D}_B f_A = -2 \frac{\partial f}{\partial u} h_{AB}^{(0)}, \quad (31)$$

$$\mathcal{D}_A \mathcal{D}_B f = \frac{1}{3} \mathcal{D}^2 f h_{AB}^{(0)}. \quad (32)$$

The part of  $f$  not proportional to  $u$  generates translation group, and since in general the equation (32) has five solutions, this part have only five degrees of freedom, there is no supertranslation freedom unlike in four dimension.  $f^A$  generate Lorentz transformation group, so asymptotic symmetry at null infinity in five dimensional space-time is Poincare group which is semi-direct of five dimensional transformation group and Lorentz group.

## 5 Summary

In this paper, we define asymptotic flatness at null infinity in five dimensional space-time by using the Bondi coordinates. In conformal embedding method, we cannot show the smoothness of asymptotic structure at null infinity because the gravitational waves behave  $\Omega^{3/2}$  near null infinity and in the coordinate using  $\Omega \sim 1/r$ , the regularity of gravitational fields at null infinity is not guaranteed in general in five dimensions. On the other hand, in the Bondi coordinates, we can show the robustness of the asymptotic structure at null infinity which is defined by boundary conditions of Eqs. (12), (13), (14) and (15). Solving Einstein equations under these boundary conditions, we find that total mass always decreases by gravitational waves as in four dimensions. In addition, we show that the asymptotic symmetry at null infinity would be Poincare group in five dimensions.

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