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Future complete S^1 symmetric solutions of the Einstein Maxwell Higgs system

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1. Introduction

We have obtained in previous works [6, 7, 3] a proof of the non linear stability (i.e. future completeness for small initial data) of Einsteinian vacuum spacetimes with compact space sections and a S^1 spatial isometry group, in the case where the space is an S^1 bundle over a surface Σ of genus greater than one. We intend in this paper to extend this result to the Einstein Maxwell Higgs system. Our previous result is proved by using the fact that the 4 dimensional vacuum Einstein equations with a one parameter isometry group are essentially (i.e. up to a harmonic one form on Σ which we chose to be zero) equivalent to an Einstein - wave map system on the quotient 3 dimensional manifold, the Einstein equations being equivalent to a time dependent elliptic system on Σ coupled in the case of genus(Σ) > 1 with ordinary differential equations governing the evolution of conformal classes of the time dependent metric of Σ. The appearance of a harmonic map generalizing the Ernst equation in the stationary Einstein Maxwell system has been found by Kinnersley and Mazur (see [11]) who studied the associated invariance group. The reduction of the Einstein Maxwell system, with a spatial S^1 isometry group to a coupled Einstein wave map system on three dimensional space has been proved by Moncrief [12] in the case $\Sigma = S^2$, and for a class of Einstein Maxwell Higgs equations in the case $\Sigma = T^2$. Though the notations and formalism used in these reductions were different from the one with which we obtained our global existence proof, it was likely, that the proof of future completeness obtained for the vacuum unpolarized Einstein equations in [3] with Σ of genus greater than one will extend to the Einstein Maxwell Higgs system.

It is known since the works of Kaluza and Klein that one can recover the Einstein Maxwell equations by considering a Lorentzian metric ${}^{(5)}g$ on a five dimensional manifold V_5 with a spacelike 1 - parameter isometry group G_1, writing the vacuum Einstein equations for $(V_5, {}^{(5)}g)$ and discarding a wave type equation for a scalar field taken to be constant. If one considers the full equations (see [10]) one finds them equivalent to the Einstein Maxwell equations with an additional scalar field ϕ which interacts with both gravitation and electromagnetism. If the gravitational and electromagnetic fields as well as the additional scalar field are themselves invariant under another spatial isometry group, then the 5 dimensional metric ${}^{(5)}g$ is invariant under a 2 - parameter abelian group G_2 endowing V_5 with a G_2 fiber bundle structure with base a manifold V_3. In this article we consider more generally a d dimensional manifold V_d with a Lorentzian metric ${}^{(d)}g$ which admits an m dimensional isometry group which endows it with a fiber bundle structure with base an $n = d - m$ dimensional manifold V_n. We write the general Kaluza Klein formulas expressing the Ricci tensor of ${}^{(d)}g$ in terms of geometric elements defined on V_n, in the case of an abelian isometry group. We show that in the case $n = 3$

the vacuum Einstein equations in dimension d, $Ricci(^{(d)}g) = 0$, are (modulo the choice zero for $d-m$ harmonic one forms) equivalent to a coupled Einstein wave map system on V_3. It gives in particular by taking $d = 5$ the Einstein wave map system which is equivalent to the 4 dimensional Einstein equations with Maxwell - Higgs sources and S^1 isometry group. We finally sketch the extension to that system of the proof used for the non linear stability theorem [3] of the vacuum unpolarized 4 dimensional Einstein equations with S^1 isometry group.

2. General Kaluza Klein equations

If the Lorentzian metric $^{(d)}g$ on the d manifold V_d admits a $d-n$ dimensional isometry group G_{d-n} endowing V_d with a fiber bundle structure with base the n manifold V_n this metric $^{(d)}g$ can be written as follows in a local trivialization of the bundle over an open set of V_n:

$$^{(d)}g \equiv \underline{g} + \xi_{mn}(\theta^m + a^m)(\theta^n + a^n), \tag{2.1}$$

where $\underline{g}$ is a Lorentzian metric on V_n, θ^m a basis of invariant 1 forms on G_{d-n}, $a \equiv (a^m)$ the representation V_n of a G_{d-n} connection on V_d, i.e. a 1 form with values in $\mathcal{G}$, the Lie algebra of G_{d-n}, we denote by $F \equiv (F^m)$ its curvature 2 form. In local coordinates on V_n we have:

$$\underline{g} \equiv \underline{g}_{\alpha\beta} dx^\alpha dx^\beta, a^m \equiv a^m_\alpha dx^\alpha. \tag{2.2}$$

We suppose that, as in the example given above, G_{d-n} is an **abelian group** then:

$$F = da, \text{i.e.} \quad F^m_{\alpha\beta} = \partial_\alpha a^m_\beta - \partial_\beta a^m_\alpha \tag{2.3}$$

and the gauge covariant derivatives on sections of vector bundles, associated with the principal fiber bundle with group G_{d-n} over $(V_n, \underline{g})$, coincide with the usual covariant derivatives in the metric $\underline{g}$

We recall the Kaluza-Klein form of the Ricci tensor of $^{(d)}g$ (see [5] II , V 13) and use it to write the vacuum Einstein equations on $(V_d, ^{(d)}g)$.

We set:

$$(\det\xi)^{\frac{1}{2}} =: e^X, \text{hence} \quad \xi^{mn}\partial_\alpha \xi_{mn} = 2\partial_\alpha X, \tag{2.4}$$

then the general Kaluza Klein formulas give the following system, all quantities being fields on V_n with greek indices raised with $\underline{g}$ and latin indices $m, n, ...$ with ξ :

$$^{(d)}R_{\alpha\beta} \equiv \underline{R}_{\alpha\beta} - \underline{\nabla}_\alpha \partial_\beta X + \frac{1}{4}\partial_\alpha \xi_{mn}\partial_\beta \xi^{mn} + \frac{1}{2}F_{m,\alpha}{}^\lambda \underline{F}^m_{\beta\lambda} = 0 \tag{2.5}$$

$$^{(d)}R_{\alpha m} \equiv -\frac{e^{-X}}{2}\underline{\nabla}_\lambda[\underline{F}_{m\alpha}{}^\lambda e^X] = 0 \tag{2.6}$$

$$^{(d)}R_{mn} \equiv -\frac{1}{2}[\underline{\nabla}^\alpha \partial_\alpha \xi_{mn} + \underline{\partial}^\alpha X \partial_\alpha \xi_{mn}] + \frac{1}{2}\xi^{pq}\partial_\alpha \xi_{mp}\underline{\partial}^\alpha \xi_{nq} - \frac{1}{4}F_{m,\alpha\beta}\underline{F}^{\alpha\beta}_n = 0. \tag{2.7}$$

We see that the equations $^{(d)}R_{mn} = 0$ are a quasidiagonal, semilinear system of wave equations for the field $\xi \equiv (\ \xi_{mn})$, they are first order in the derivatives of the other unknowns $\underline{g}$ and a.

The equations $^{(d)}R_{\alpha m} = 0$ look like Maxwell type equations for F, with a weight e^X.

The equations $^{(d)}R_{\alpha\beta} = 0$ look like Einstein equations with source the Maxwell and scalar fields, except for the appearance of second derivatives of X. The combination of these equations with the system $^{(d)}R_{mn} = 0$ takes the form of usual Einstein equations with sources by introduction of a metric $\tilde{g}$ conformal to $\underline{g}$. We will perform the explicit computation in the case of interest to us, $n = 3$, which permit the introduction of twist potentials to solve the Maxwell type equations.

3. Case n=3

3.1. Twist potentials, definition

The equations 2.6 are equivalent to

$$dE_m = 0\text{with} \quad E_m =: \underline{*}[F_m e^X] \tag{3.1}$$

with $\underline{*}$ the adjoint operator on forms in the metric $\underline{g}$. When $n = 3$ the adjoint E_m of the 2 form F_m is a 1 form. The general solution of 2.6 is then:

$$E_m = d\omega_m + H_m \tag{3.2}$$

with ω_m an arbitrary scalar function on V_3, called a twist potential, and H_m a representant of a 1 cohomology class. We consider the class of solutions such that $H_m = 0$.

3.2. Associated tensors

The inversion of the defining equations for E_m gives

$$F_m = e^{-X}\underline{*}E_m = e^{-X}\underline{*}(d\omega_m) \tag{3.3}$$

that is with $\underline{\eta}$ the volume form of $\underline{g}$:

$$F_{m,\alpha\beta} = \underline{\eta}_{\alpha\beta\lambda} e^{-X} \underline{g}^{\lambda\mu}\partial_\lambda\omega_m \text{ hence } \quad F^m_{\alpha\beta} = \underline{\eta}_{\alpha\beta\lambda} e^{-X} \underline{g}^{\lambda\mu}\xi^{mn}\partial_\lambda\omega_n. \tag{3.4}$$

Lemma 3.1 *The following identities hold (underlining means that greek indices are raised with $\underline{g}^{\alpha\beta}$) :*

$$\underline{F}_{m,\alpha}{}^{\lambda} F_{n,\beta\lambda} \equiv e^{-2X}[\underline{g}_{\alpha\beta}\partial_\lambda\omega_m\underline{\partial}^\lambda\omega_n - \partial_\alpha\omega_m\partial_\beta\omega_n] \tag{3.5}$$

$$\underline{F}_m{}^{\lambda\mu} F^m_{\lambda\mu} \equiv 2e^{-2X}\xi^{mn}\partial_\lambda\omega_m\underline{\partial}^\lambda\omega_n. \tag{3.6}$$

Proof. The first identity is obtained for instance by a computation in orthonormal frame, the second results by contraction on the 3 manifold V_3. ■

3.3. Equations

The 2 forms F^m being the differentials of the 1 forms a^m are necessarily closed, hence the functions ω_m must satisfy the equations

$$d[\underline{*}e^{-X}\xi^{mn}d\omega_m] = 0 \tag{3.7}$$

equivalently the functions ω_m must satisfy the following semilinear wave equations on $(V_3, \underline{g})$:

$$\underline{\nabla}^\alpha[e^{-X}\xi^{mn}\partial_\alpha\omega_m] = 0. \tag{3.8}$$

4. Conformal metric

We introduce on V_3 the conformal metric

$$\tilde{g}_{\alpha\beta} =: e^{2X}\underline{g}_{\alpha\beta}, \text{hence} \quad \underline{g}^{\alpha\beta} = e^{2X}\tilde{g}^{\alpha\beta}. \tag{4.1}$$

A simple calculation shows that the divergence of covariant vectors v are linked by the identity

$$\underline{\nabla}_\alpha\underline{v}^\alpha \equiv e^{2X}(\tilde{\nabla}_\alpha\tilde{v}^\alpha - \tilde{v}^\alpha\partial_\alpha X), \quad \underline{v}^\alpha = e^{2X}\tilde{v}^\alpha. \tag{4.2}$$

In particular the wave operators in the metrics $\underline{g}$ and $\tilde{g}$ are linked by the identity

$$\underline{\nabla}_\alpha\underline{\partial}^\alpha \equiv e^{2X}\tilde{\nabla}_\alpha\tilde{\partial}^\alpha - \underline{\partial}^\alpha X\partial_\alpha. \tag{4.3}$$

4.1. Scalar equations
We deduce from 2.7 that

$$ {}^{(d)}R_{mn} \equiv -\frac{e^{2X}}{2}[\tilde{\nabla}^\alpha \partial_\alpha \xi_{mn} + \xi^{pq}\partial_\alpha \xi_{mp}\tilde{\partial}^\alpha \xi_{nq}] - \frac{1}{4}F_{m,\alpha\beta}\underline{F}_n^{\alpha\beta}] = 0. \tag{4.4} $$

where $F_{m,\alpha\beta}\underline{F}_n^{\alpha\beta}$ is given by 3.5.

These equations and $X = \log(\det \xi)^{1/2}$ imply by a simple computation that:

$$ \xi^{mn(d)}R_{mn} \equiv -e^{2X}\tilde{\nabla}_\lambda \tilde{\partial}^\lambda X + \frac{1}{4}F_{m,\alpha\beta}\underline{F}^{m,\alpha\beta} = 0, \tag{4.5} $$

that is, using 3.6:

$$ \tilde{\nabla}_\lambda \tilde{\partial}^\lambda X - \frac{1}{2}e^{-2X}\xi^{mn}\partial_\lambda \omega_m \tilde{\partial}^\lambda \omega_n = 0. \tag{4.6} $$

For simplicity of proofs we will sometimes introduce the scalar function $X = \log(\det \xi)^{1/2}$ as an auxiliary unknown. The equations 4.4 and 4.6 will imply that

$$ X = \frac{1}{2} logdet\xi \tag{4.7} $$

if this property and its first time derivative are satisfied initially.

4.2. Twist potentials equations
We have the identities:

$$ \underline{\nabla}^\alpha[e^{-X}\xi^{mn}\partial_\alpha \omega_m] \equiv e^{2X}\{\tilde{\nabla}^\alpha[e^{-X}\xi^{mn}\partial_\alpha \omega_m] - \tilde{\partial}^\alpha X(e^{-X}\xi^{mn}\partial_\alpha \omega_m)]\} \tag{4.8} $$

$$ \equiv e^{3X}\tilde{\nabla}^\alpha[e^{-2X}\xi^{mn}\partial_\alpha \omega_m] \equiv e^X \xi^{mn}\tilde{\nabla}^\alpha[\partial_\alpha \omega_m)] + e^X[\tilde{\partial}^\alpha \xi^{mn} - 2\xi^{mn}\tilde{\partial}^\alpha X]\partial_\alpha \omega_m \tag{4.9} $$

The equations 3.8 for the twist potentials are therefore:

$$ \tilde{\nabla}^\alpha \partial_\alpha \omega_p + \xi_{np}[\tilde{\partial}^\alpha \xi^{mn} - 2\xi^{mn}\tilde{\partial}^\alpha X]\partial_\alpha \omega_m = 0 \tag{4.10} $$

that is

$$ \tilde{\nabla}^\alpha \partial_\alpha \omega_p - \xi^{mn}(\tilde{\partial}^\alpha \xi_{np} + 2\tilde{\partial}^\alpha X)\partial_\alpha \omega_m = 0 \tag{4.11} $$

4.3. Einstein equations on V_3
On the 3 dimensional manifold V_3 it holds that (see [5] I p351):

$$ \underline{R}_{\alpha\beta} \equiv \tilde{R}_{\alpha\beta} + \underline{\nabla}_\alpha \partial_\beta X - \partial_\alpha X \partial_\beta X + \underline{g}_{\alpha\beta}(\underline{\nabla}^\lambda \partial_\lambda X + \underline{\partial}^\lambda X \partial_\lambda X) \tag{4.12} $$

We deduce from 2.5 and 4.12 that the term $\underline{\nabla}_\alpha \partial_\beta X$ disappears from ${}^{(d)}R_{\alpha\beta}$. We find that:

$$ {}^{(d)}R_{\alpha\beta} \equiv \tilde{R}_{\alpha\beta} + \frac{1}{4}\partial_\alpha \xi_{mn}\partial_\beta \xi^{mn} - \frac{1}{2}\underline{F}_{m,\alpha}{}^\lambda F^m_{\beta\lambda} - \partial_\alpha X \partial_\beta X + \tilde{g}_{\alpha\beta}\tilde{\nabla}^\lambda \partial_\lambda X \tag{4.13} $$

and we deduce from 4.5 that the combination ${}^{(d)}R_{\alpha\beta} + e^{-2X}\tilde{g}_{\alpha\beta}\xi^{mn(d)}R_{mn} = 0$ reduces to the following Einstein equations on $(V_3, \tilde{g})$:

$$ \tilde{R}_{\alpha\beta} = \rho_{\alpha\beta} \tag{4.14} $$

with, using 3.5:

$$ \rho_{\alpha\beta} \equiv -\frac{1}{4}\partial_\alpha \xi_{mn}\partial_\beta \xi^{mn} + \partial_\alpha X \partial_\beta X + \frac{e^{-2X}}{2}\xi^{mn}\partial_\alpha \omega_m \partial_\beta \omega_n. \tag{4.15} $$

Theorem 4.1 *The equations 4.14 are the Einstein equations on $(V_3,\tilde{g})$ with source a mapping $u:(V_3,\tilde{g})\to(R^D,G)$ by $(x^\alpha)\mapsto(X,\xi_{mn},\omega_m)$, where G is the riemannian metric:*

$$G\equiv(dX)^2+\frac{1}{4}\xi^{mp}\xi^{nq}d\xi_{mn}d\xi_{pq}+\frac{1}{2}e^{-2X}\xi^{mn}d\omega_m d\omega_n. \quad (4.16)$$

PROOF. The stress energy tensor of the mapping $u=(X,\xi,\omega)$ is by definition

$$T_{\alpha\beta}=:\partial_\alpha u.\partial_\beta u-\frac{1}{2}\tilde{g}_{\alpha\beta}\partial_\lambda u.\tilde{\partial}^\lambda u \quad (4.17)$$

where a dot denotes a scalar product in the metric G. The corresponding right hand side for $\tilde{R}_{\alpha\beta}$ is

$$\rho_{\alpha\beta}\equiv\partial_\alpha u.\partial_\beta u \quad (4.18)$$

The identity of this $\rho_{\alpha\beta}$ with 4.15 results from the definition of G. ■

5. Einstein wave map system

The Bianchi identities satisfied by the Ricci tensor of $\tilde{g}$ together with the equations 4.14 imply that the stress energy tensor $T_{\alpha\beta}$ of the mapping u satisfies the conservation law $\tilde{\nabla}_\alpha\tilde{T}^{\alpha\beta}=0$. For any map between pseudo riemannian manifolds it holds that

$$\tilde{\nabla}_\alpha\tilde{T}^{\alpha\beta}\equiv\hat{\nabla}^\alpha\partial_\alpha u.\tilde{\partial}^\beta u. \quad (5.1)$$

where $\hat{\nabla}^\alpha\partial_\alpha$ is the wave map operator. It is straightforward to check that for the considered $\rho_{\alpha\beta}$, as foreseen due to the consistency of the original vacuum equations, the vector $\tilde{\nabla}_\alpha\tilde{T}^{\alpha\beta}$ takes the form of the scalar product of $\tilde{\partial}^\beta u$ with the semilinear wave operators found before for X, ξ and ω.

We have proved the following theorem, where "essentially" means "with the choice $H_m=0$ of harmonic 1 forms appearing in 3.2":

Theorem 5.1 *The class of d dimensional vacuum Einstein equations Ricci$({}^{(d)}g)=0$ with a $d-3$ dimensional spacelike abelian isometry group are essentially equivalent to Einstein equations for a lorentzian metric $\tilde{g}$ on V_3 with source a wave map $u:(V_3,\tilde{g})\to(R^D,G)$, that is to the system*

$$\tilde{R}_{\alpha\beta}=\partial_\alpha u.\partial_\beta u,\quad \hat{\nabla}_\alpha\tilde{\partial}^\alpha u=0, \quad (5.2)$$

with a dot the scalar product in G and $\hat{\nabla}$ the covariant derivative[1] *associated with u.*

6. Einstein Maxwell Higgs equations

We return to the case $d=5$, Einstein-Maxwell-scalar equations, to give the physical interpretation of the metric ξ wich is now on R^2. In the Kaluza - Klein formulation the spacetime is a Lorentzian manifold $(V_5,{}^{(5)}g)$ with a S^1 fiber bundle structure induced by a spacelike Killing vector field $\partial/\partial x^4$ and with base a 4 manifold V_4. The metric can be written:

$${}^{(5)}g=:{}^{(4)}g+e^{2\phi}(dx^4+B)^2. \quad (6.1)$$

with ${}^{(4)}g$ a Lorentzian metric, B a locally defined 1 form and ϕ a scalar field all defined on V_4, the vacuum Einstein equations for $(V_5,{}^{(5)}g)$ imply the Einstein Maxwell equations on $(V_4,{}^{(4)}g)$ with electromagnetic potential B coupled with the scalar field ϕ. Suppose that the gravitational

[1] That is $\hat{\nabla}_\alpha\partial_\beta u^A\equiv\partial_\alpha\partial_\beta u^A-\tilde{\Gamma}^\lambda_{\alpha\beta}\partial_\lambda u^A+\Gamma^A_{BC}\partial_\alpha u^B\partial_\beta u^C$, with $\tilde{\Gamma}$ and Γ connection coefficients respectively of the metrics $\tilde{g}$ and G.

and electromagnetic fields as well as the additional scalar field are invariant under another 1 - parameter group S^1 endowing V_4 with a S^1 fiber bundle structure, with Killing vector field $\partial/\partial x^3$ and base a 3 dimensional manifold V_3 with local coordinates x^α, $\alpha = 0, 1, 2$. This hypothesis is equivalent to the independence of ϕ and $B \equiv \underline{B} + B_3 dx^3$ on x^3, with $\underline{B} \equiv B_\alpha dx^\alpha$, together with the possibility of writing ${}^{(4)}g$ under the following form, all quantities defined on the quotient manifold V_3,

$$ {}^{(4)}g \equiv \underline{g} + e^{2\gamma}(dx^3 + A)^2, \quad \underline{g} \equiv \underline{g}_{\alpha\beta} dx^\alpha dx^\beta, \quad A =: A_\alpha dx^\alpha. \tag{6.2} $$

The hypotheses are equivalent to saying that the metric ${}^{(5)}g$ admits a 2 parameter abelian group G_2, as shown explicitly by the following lemma.

Lemma 6.1 *The metric*

$$ {}^{(5)}g =: \underline{g} + e^{2\gamma}(dx^3 + A)^2 + e^{2\phi}(dx^4 + \underline{B} + B_3 dx^3)^2 \tag{6.3} $$

reads as a metric on V_5 with an abelian G_2 isometry group, namely, setting $a =: a_\alpha dx^\alpha$, $b = b_\alpha dx^\alpha$:

$$ {}^{(5)}g \equiv \underline{g} + \xi_{33}(dx^3 + a)^2 + 2\xi_{34}(dx^3 + a)(dx^4 + b) + \xi_{44}(dx^4 + b)^2 \tag{6.4} $$

with

$$ e^{2\phi} = \xi_{44}, \quad B_3 = \frac{\xi_{34}}{\xi_{44}}, \quad e^{2\gamma} = \frac{\det \xi}{\xi_{44}} \equiv \xi_{33} - \frac{\xi_{34})^2}{\xi_{44}} \tag{6.5} $$

and

$$ A = a, \quad \underline{B} = b + \frac{\xi_{34}}{\xi_{44}} a \tag{6.6} $$

Proof. Straightforward calculation. ■

In the case where $B_3 = 0$, that is when the electromagnetic potential vector is orthogonal to the Killing vector $\partial/\partial x^3$, the field ξ reduces to:

$$ \xi_{44} = e^{2\phi}, \quad \xi_{34} = 0, \quad \xi_{33} = e^{2\gamma}, \quad X = \gamma + \phi. \tag{6.7} $$

and the metric G is given by[2]:

$$ G \equiv (dX)^2 + (d\phi)^2 + (d\gamma)^2 + \frac{1}{2}e^{-2(X+\phi)}(d\omega_4)^2 + \frac{1}{2}e^{-2(X+\gamma)}(d\omega_3)^2. \tag{6.8} $$

7. Cauchy problem

7.1. Wave map equation

The wave map equation is a semilinear wave equation on V_3 when the metric $\tilde{g}$ is known.

7.2. Einstein equations

Einstein equations on a 3 manifold are essentially[3] non dynamical, since the Riemann tensor is in that case equivalent to the Ricci tensor. To solve the Cauchy problem for the Einstein equations 5.2 one takes as usual $V_3 = \Sigma \times R$ and consider a 2+1 splitting of the metric $\tilde{g}$, with N and ν its lapse and shift, g a $t-$ dependent riemanian metric on Σ :

$$ \tilde{g} \equiv -N^2 dt^2 + g_{ab}(dx^a + \nu^a dt)(dx^b + \nu^b dt). $$

[2] In the case of vacuum 4 dimensional Einstein equations with 1 spacelike isometry group $\phi = \omega_4 = 0$ the metric G reduces to the metric of the Poincaré plane used in previous articles.

[3] We will see that they are non dynamical if and only if the surface Σ is topologically a sphere.

We denote by k the extrinsic curvature of a surface $\Sigma_t = \Sigma \times \{t\}$ embedded in $(V_3, \tilde{g})$, by τ its mean extrinsic curvature. Then, with h a traceless tensor:

$$k_{ab} \equiv -\frac{1}{2N}\bar{\partial}_0 g_{ab} \equiv h_{ab} + \frac{1}{2}g_{ab}\tau, \quad \tau = g^{ab}k_{ab}.$$

7.2.1. Einstein constraints. Part of the Einstein equations are the constraints on each Σ_t, that is:

$$R(g) - k.k + \tau^2 = |u'|^2 + g^{ab}\partial_a u.\partial_b u, \quad u' =: N^{-1}\partial_0 u, \tag{7.1}$$

$$\nabla_b k_a^b - \partial_a \tau = u'.\partial_a u. \tag{7.2}$$

To solve these constraints one uses the conformal method, that is we set:

$$g_{ab} = e^{2\lambda}\sigma_{ab} \tag{7.3}$$

and we take as a gauge condition that τ is a given function of t alone (CMC gauge).

The constraints become on each Σ_t, with covariant derivative, laplacian and norms in the metric σ_t :

- The equation linear in h, given σ, u, $\dot{u}$

$$D_b h_a^b = L_a \equiv -D_a u.\dot{u}, \quad \dot{u} =: e^{2\lambda}N^{-1}\partial_0 u$$

  One solves this system by setting $h = q + r$ with h a TT (Transverse, Traceless) tensor, that is a solution of the homogeneous system

$$D_b q_a^b = 0, \quad q_a^a = 0, \tag{7.4}$$

  and r a conformal Lie derivative of a vector field Y, solution then of an elliptic system.
- The nonlinear elliptic equation for λ

$$\Delta_\sigma \lambda = f(x, \lambda) \equiv \frac{\tau^2}{4}e^{2\lambda} - \frac{1}{2}p_2 e^{-2\lambda} + \frac{1}{2}p_3,$$

$$p_2 \equiv |\ \dot{u}\ |^2 + |\ h\ |^2, \quad p_3 \equiv R(\sigma) - |Du|^2$$

7.2.2. Equations for lapse and shift. The Einstein equation $\tilde{R}_{00} = \rho_{00}$ gives the following elliptic equation for N :

$$\Delta_\sigma N - aN = -e^{2\lambda}\partial_t \tau \tag{7.5}$$

$$a \equiv e^{-2\lambda}(|\ h\ |_g^2 + |\ \dot{u}\ |_g^2) + e^{2\lambda}\frac{\tau^2}{2} \tag{7.6}$$

A linear differential equation for ν, depending on λ, h, N and linearly on $\partial_t \sigma$, results form the expression of k. This differential equation admits solutions only if its non homogeneous term is L^2 orthogonal to conformal Killing vector fields.

When σ, q and the mapping u are known on V_3 the equations for r, λ, N are an elliptic system on each Σ_t.

7.3. Hyperbolic-elliptic system, given σ and q
When the family of metrics σ_t and TT tensor q_t are given, the wave map u and the metric coefficients N, λ, ν in CMC gauge satisfy a coupled hyperbolic - elliptic system with auxiliary unknown r. The Bianchi identities show that the space part of $Ricci(\tilde{g}) - \rho$ is a TT tensor on Σ_t when this system is satisfied.

7.4. Teichmuller parameters (Σ compact)
From now on we consider only the case of a **compact** Σ.

When Σ is diffeomorphic to S^2 all the possible metrics σ_t are conformal (up to diffeomorphisms) to the canonical metric, and there does not exist on (Σ, σ) any non identically zero TT tensor hence $q \equiv 0$, and $Ricci(\tilde{g}) - \rho = 0$ as soon as constraints and lapse equations are satisfied. We can take as gauge condition, in addition to CMC, $\partial_t \sigma = 0$. There are integrability conditions for the shift equation (see [8]).

When Σ is a 2 torus, i.e. Genus$(\Sigma) = 0$, any metric σ on Σ admits two linearly independent Killing vector fields, and two linearly independent TT tensors. A metric on Σ is conformal to a metric with zero scalar curvature, one can choose σ such that $R(\sigma) = 0$. We do not treat this case.

When G =:Genus$(\Sigma) > 1$, a metric on Σ is conformal to a metric with constant negative scalar curvature, one can choose σ such that $R(\sigma) = -1$. The space of classes of conformally equivalent metrics on Σ, called Teichmuller space T_{eich}, can be identified with M_{-1}/D_0, quotient of the space of metrics with $R(\sigma) = -1$ by the group of diffeomorphisms homotopic to the identity. $M_{-1} \rightarrow T_{eich}$ is a trivial fiber bundle whose base can be endowed with the structure of the manifold R^{6G-6}. As a gauge condition we impose to the metric σ_t to be in some chosen cross section $Q \rightarrow \psi(Q)$ of this fiber bundle. Let $Q^I, I = 1, ..., 6G-6$ be coordinates in T_{eich}, then $\partial\psi/\partial Q^I$ is a known tangent vector to M_{-1} at $\psi(Q)$, that is a traceless symmetric 2-tensor field on Σ, sum of a TT tensor field $X_I(Q)$ and Lie derivative of a vector field on $(\Sigma, \psi(Q))$. Solvability condition for the shift equation determines dQ/dt in terms of q_t and conversely. One obtains an ordinary differential system for the evolution of Q by the L^2 orthogonality of $\tilde{R}_{ab} - \rho_{ab}$ with the 6G-6 dimensional vector space of TT tensors over Σ.

7.5. Conclusion
We have to solve the coupled system:

1. Elliptic equations on (Σ, σ_t) with $\sigma_t = \psi(Q(t))$, with coefficients depending on u.
2. ODE on R for $Q(t)$ with coefficients depending on u and elliptic unknowns.
3. Wave map system on $(\Sigma \times R, \tilde{g})$, $\tilde{g}$ determined by σ_t and elliptic unknowns.

Theorem 7.1 *(local existence.)The Cauchy data on the compact orientable smooth surface Σ_{t_0}, genus$(\Sigma) > 1$ are:*

1. A C^∞ metric σ_0 and TT tensor q_0.

2. Wave map initial data: $u_0 = u(t_0, .) \in H_2$, $\dot{u}_0 = e^{2\lambda} u'(t_0, .) \in H_1$.

The 2+1 Einstein wave map system has a solution taking these Cauchy data and such that $u \in C^0([t_0, T), H_2) \cap C^1([t_0, T), H_1)$, $\lambda, N, \nu \in C^0([t_0, T), W_3^p) \cap C^1([t_0, T), W_2^p)$, 1¡p¡2, N¿0, if T-$t_0$ is small enough.

There is a corresponding Einteinian spacetime $(V_d, {}^{(d)} g)$ if the initial data satisfy the Chern integrability condition for the construction of A and B.

8. Scheme for global existence

We have chosen to work in CMC gauge: τ is a time parameter which, in our conventions taken from MTW inceases from $-\infty$ to 0 if the spacetime expands from a big-bang singularity to a moment of maximum expansion. To have notations more familiar to the analyst we choose as time parameter $t = -\tau^{-1}$, then t increases from $t_0 > 0$ to infinity when Σ_t expands from $\tau_0 < 0$ to zero. In the case where $R(g) < 0$ the constraint 7.1 shows that this moment of maximum expansion cannot be attained. Existence on (t_0, ∞) will result from a priori bounds of the norms appearing in local existence theorem. These a priori bounds result from energy estimates for the wave map and elliptic estimates, in particular for the confromal factor λ which satisfies a non linear equation for which estimation depends crucially on the negative sign of $R(\sigma)$, i.e. by the Gauss Bonnet theorem the fact that Genus$(\Sigma) > 1$. The estimates involve also the uniform equivalence of σ_t with σ_0. This requires decay in t of the "total energy". This decay is a consequence of the expansion of the metric $g(t, .)$ of Σ_t, obtained when Genus$(\Sigma) > 1$, but its proof requires the introduction of corrected energies (as already in [7]). The proofs are essentially the same as in the vacuum case of [3], at least for the Einstein - Maxwell Higgs system with $B_3 = 0$ where the target metric takes a simple form. We sketch below the main steps that we use.

9. First energy estimate

One defines the first energy not only as the energy of the wave map but by, with $|\cdot|_g$ the point wise norm in the metrics g and G,

$$E(t) \equiv \int_{\Sigma_t} (|\partial u|_g^2 + |u'|^2 + \frac{1}{2}|h|_g^2)\mu_g$$

The Hamiltonian constraint

$$R(g) + \frac{\tau^2}{2} = |\partial u|_g^2 + |u'|^2 + \frac{1}{2}|h|_g^2$$

together with the Gauss Bonnet formula (χ the Euler Poincaré constant)

$$\int_{\Sigma_t} R(g)\mu_g = 4\pi\chi \tag{9.1}$$

give, without using wave map equation, that:

$$\frac{dE(t)}{dt} = \frac{1}{2}\tau \int_{\Sigma_t} N(|u'|^2 + \frac{1}{2}|h|_g^2)\mu_g \leq 0. \tag{9.2}$$

We deduce from this equality that $E(t)$ is a non increasing function of t since $\tau < 0$, but we do not obtain its decay due to the absence of $|Du|^2$ in the right hand side.

10. First elliptic estimates

The definition of $E(t)$ implies

$$||h||_{L^2(\sigma)} \leq e^{2\lambda_M} E(t)$$

while under hypothesis $R(\sigma) = -1$ the maximum principle applied to ... implies

$$e^{2\lambda} \geq 2\tau^{-2} \tag{10.1}$$

and, with the parameter choice $\tau = -\frac{1}{t}$, it implies, applied to ...

$$0 < N \leq 2. \tag{10.2}$$

Further elliptic estimates require bounds of $\partial u.\partial u$ and $\partial u.u'$ in $W_1^p(\sigma)$, $1 < p < 2$, which are obtained in terms of the second energy of the wave map.

11. Second energy
We denote by $\hat{\nabla}$ and $\hat{\partial}_0$ covariant derivatives for mappings (Σ, g) or (R, dt^2) into (R^D, G). Set:

$$E^{(1)}(t) \equiv \int_{\Sigma_t} (|\ \hat{\Delta}_g u\ |^2 + |\ \hat{\nabla} u'\ |^2)\mu_g \tag{11.1}$$

$$\varepsilon^2 =: E(t), \quad \varepsilon_1^2 =: \tau^{-2} E_1(t). \tag{11.2}$$

One finds after long computations using elliptic estimates applied to the constraints and lapse equations that:

$$1 \leq \frac{1}{\sqrt{2}}|\tau|e^{\lambda} \leq 1 + C_{E,\sigma}(\varepsilon + \varepsilon_1), \tag{11.3}$$

$$0 \leq 2 - N \leq C_{E,\sigma}(\varepsilon^2 + \varepsilon\varepsilon_1) \tag{11.4}$$

where we denote by $C_{\sigma,E}$ numbers depending only on a priori bound of ε and ε_1, and on the domain of σ in $Teich$ supposed to be compact.

Using these bounds and the wave map $(\Sigma \times R, \tilde{g}) \to (R^D, G)$ which reads in our notations:

$$-N^{-1}\hat{\partial}_0 \partial_0 u + g^{ab}\hat{\nabla}_a(N\partial_b u) + Nu' = 0. \tag{11.5}$$

we find that:

$$\frac{dE^{(1)}}{dt} - 2\tau E^{(1)} = \tau \int_{\Sigma_t} N\ |\ Du'\ |^2\ \mu_g + Z \leq Z \tag{11.6}$$

with:

$$|Z| \leq C_{\sigma,E}(\varepsilon + \varepsilon_1)^3. \tag{11.7}$$

$$|Z| \leq |\tau|^3 C_{\sigma,E}(\varepsilon + \varepsilon_1)^3. \tag{11.8}$$

The inequalities 11.6 and 9.2, are not sufficient to prove the bound of $\varepsilon + \varepsilon_1$ and the fact that σ_t projects on a fixed compact subset of T_{eich}, the proof of this last property requires decay of $\varepsilon + \varepsilon_1$.

12. Corrected energies
To obtain the decay property one introduces corrected energies and exploit the negative (non definite) terms in the energies inequalities:

$$E_\alpha(t) = E(t) - \alpha\tau \int_{\Sigma_t} (u - \bar{u}).u'\mu_g \tag{12.1}$$

$$\bar{u} = \frac{1}{Vol_{\sigma_t}} \int_{\Sigma_t} u\mu_\sigma,$$

$$E_\alpha^{(1)}(t) = E^{(1)}(t) + \alpha\tau \int_{\Sigma_t} \hat{\Delta}_g u.u'\mu_g \tag{12.2}$$

The use of elliptic estimates leads to

$$\frac{dE_\alpha}{dt} - k\tau E_\alpha \leq |\tau| C_{\sigma,E}(\varepsilon + \varepsilon_1)^3, \tag{12.3}$$

$$\frac{dE_\alpha^{(1)}}{dt} - (2 + k)\tau E_\alpha^{(1)} + |\tau|^3 C_{\sigma,E}(\varepsilon + \varepsilon_1)^3. \tag{12.4}$$

We denote by Λ_σ the first positive eigenvalue of $-\Delta_\sigma$ and we prove that $E_\alpha + \tau^{-2}E_\alpha^{(1)}$ is equivalent to the total energy $\varepsilon^2 + \varepsilon_1^2$ under the following conditions:

- $$\alpha = \frac{1}{4}, \quad k = 1 \quad \text{if} \quad \Lambda_\sigma > \frac{1}{8} \tag{12.5}$$

- $$\alpha < \frac{4}{8 + \Lambda_\sigma^{-1}}, \quad 0 < k < 1, \quad \text{if} \quad \Lambda_\sigma \leq \frac{1}{8}.$$

All these differential inequalities imply the inequalities:

$$(\varepsilon^2 + \varepsilon_1^2)(t) \leq t^{-k} M_1 (\varepsilon^2 + \varepsilon_1^2)(t_0). \tag{12.6}$$

13. Future complete existence (non linear stability)

Theorem 13.1 *Let $(\sigma_0, q_0) \in C^\infty(\Sigma_0)$ and $(u_0, \bar{u}_0) \in H_2(\Sigma_0, \sigma_0) \times H_1(\Sigma_0, \sigma_0)$ be initial data on the compact manifold Σ_0, Genus(Σ_0) > 1, satisfying the Chern integrability condition. There exists a number $\eta > 0$ such that, if $E_{tot}(t_0) < \eta$, the 4 dimensional Einstein Maxwell Higgs system with S^1 isometry group and electromagnetic field orthogonal to the Killing field have a solution on $\Sigma \times S^1 \times [t_0, \infty)$, $t = -\tau^{-1}$, with initial values determined by $\sigma_0, q_0, u_0, \dot{u}_0$. This space time is globally hyperbolic, future timelike and null complete.*

Proof. Using the differential equation satisfied by Q and the decay of the total energy proved using its a priori bound one obtains the inequality:

$$|Q(t) - Q(t_0)| \leq M_2 (\varepsilon^2 + \varepsilon_1^2)(t_0). \tag{13.1}$$

This inequality together with 12.6 give a bound of the total energy and of Q by a bootstrap argument if the initial total energy small enough. One deduces the existence of the solution on $\Sigma \times [t_0, \infty)$, and the existence for an infinite proper time along the lines $\{x\} \times R$ after estimating the usual H_2 norms in terms of the geometrically defined second energy. This estimate depends, as in the proof given in [3], on the fact that the Riemann curvature of the target metric is negative. The special form of this metric plays also a role in the estimate of the second corrected energy.

The global hyperbolicity and completeness is a particular case of a theorem proved in [4]. ■

We have not checked the corresponding properties for the general G given by 4.16, but we conjecture that the proof goes through in that general case.

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