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Bimodule structure of the mixed tensor product over $\mathcal{U}_q s\ell(2|1)$ and quantum walled Brauer algebra

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Abstract

We study a mixed tensor product $\mathbf{3}^{\otimes m} \otimes \overline{\mathbf{3}}^{\otimes n}$ of the three-dimensional fundamental representations of the Hopf algebra $\mathcal{U}_q s\ell(2|1)$, whenever q is not a root of unity. Formulas for the decomposition of tensor products of any simple and projective $\mathcal{U}_q s\ell(2|1)$-module with the generating modules $\mathbf{3}$ and $\overline{\mathbf{3}}$ are obtained. The centralizer of $\mathcal{U}_q s\ell(2|1)$ on the mixed tensor product is calculated. It is shown to be the quotient $\mathsf{X}_{m,n}$ of the quantum walled Brauer algebra $\mathsf{qwB}_{m,n}$. The structure of projective modules over $\mathsf{X}_{m,n}$ is written down explicitly. It is known that the walled Brauer algebras form an infinite tower. We have calculated the corresponding restriction functors on simple and projective modules over $\mathsf{X}_{m,n}$. This result forms a crucial step in decomposition of the mixed tensor product as a bimodule over $\mathsf{X}_{m,n} \boxtimes \mathcal{U}_q s\ell(2|1)$. We give an explicit bimodule structure for all m, n.

1. Introduction

Over the course of the last twenty years Logarithmic conformal field theory (LCFT) has established itself as an area of extensive interaction between models of statistical physics such as percolation, the sand pile model, dense polymers as well as other models with nonlocal observables on the one hand, and modern topics in mathematics such as Nichols algebras, quantum

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groups, braided categories, VOA theory and diagram algebras on the other. One of the most developed approaches [1–3] to constructing LCFT is based on the intersection of screening operator kernels. In this approach one chooses a lattice vertex operator algebra (VOA) and fixes a set of fields v_i, which correspond to representations of VOA and are called screening currents. The zero modes of these currents $s_i = \oint v_i$ are called screenings. Under certain integer valuedness conditions on scalar products of the screening currents momenta, the screenings form a finite-dimensional Nichols algebra (see examples in [4,5]). Under these conditions the intersection of the screening kernels is a vacuum module of a rational LCFT: $\mathrm{Vac} = \bigcap_i \mathrm{Ker}\, s_i$. In this case, LCFT is a representation space of the rational W-algebra $\mathcal{W} = \mathrm{Vac}$.

The algebra $\mathcal{W}$ has only a finite number of irreducible representations. The set of simple and projective $\mathcal{W}$-modules is closed under fusion and the characters of the $\mathcal{W}$-irreducible modules generate a finite-dimensional representation of the modular group.

Another source of LCFT is given by various lattice models [6–12]. CFT appears naturally as a scaling limit of lattice models in the critical point, see e.g. [13]. Then, a mathematically rigorous program on algebraic construction of the scaling limits was initiated in [14]. If one considers nonlocal observables (for example, the cluster probability in percolation theory [15–18]) in the lattice model, then in the scaling limit an LCFT is in general expected to appear, and in several models [19,20,11,21] its appearance is shown explicitly.

The standard approach to studying the lattice models is the transfer-matrix method [22,23]. In this approach a connection with a spin chain is established by the Hamiltonian limit. Another feature of lattice models with nonlocal observables is that in the Hamiltonian limit there exist nontrivial Jordan blocks in the Hamiltonian [24–28] (see discussion on the Jordan blocks problem in the algebraic Bethe ansatz approach in [29]). From the side of LCFT the existence of nontrivial Jordan blocks in the Hamiltonian is expressed in the fact that the conformal dimension operator L_0 becomes non-diagonalizable and conformal blocks admit logarithmic terms.

In both approaches a quantum group plays a crucial role [30,31,8,12]. In the first case quantum group appears as a double bosonization of the algebra generated by screenings [32]. In the second case the spin-chain can be constructed as tensor product of fundamental representations of the quantum group.

For the simplest case of $(1, p)$ LCFT models [33–36], the corresponding spin-chain T_N is a tensor product of two-dimensional representations of the quantum group $\mathcal{U}_q s\ell(2)$, [37], and the T_N is called the Heisenberg spin-chain.

An interesting generalization of the Heisenberg spin-chain is a spin-chain based on the algebra $\mathcal{U}_q s\ell(M|N)$ [38]. Such spin-chains describe interaction between spin and other degrees of freedom. For instance, $\mathcal{U}_q s\ell(2|1)$-spin-chain is related with (integrable) t–J model which includes the interaction between spin and charge degrees of freedom, [39–41].

On the side of LCFT, models related with the quantum group $\mathcal{U}_q s\ell(2|1)$ are constructed in [32] by the approach based on intersection of kernels of the screening operators. Rational W-algebras $\mathcal{W}$ containing as a subalgebra $\hat{sl}(2)_k$ at a rational level k naturally occur in these models. At the same time models over $\hat{sl}(2)_k$ are not rational. In this case $\hat{sl}(2)_k$ is an analog of the Virasoro algebra in $(1, p)$-models. More on LCFT with $\hat{sl}(2)_k$ see in [42–44].

In order to investigate how LCFT with quantum group $\mathcal{U}_q s\ell(2|1)$ appears in the scaling limit of the spin-chain, it is natural to follow the approach proposed in [37]. In the present paper we study $\mathcal{U}_q s\ell(2|1)$ mixed tensor product which is the space of states for the spin-chains with $\mathcal{U}_q s\ell(2|1)$ symmetry. But the case when q is a root of unity $q = e^{i\pi/p}$, is more complicated and we leave it for a separate work. Therefore in the present paper we consider only the algebra with a generic value of the parameter q.

It is useful to make an analogy with the Heisenberg $\mathcal{U}_q s\ell(2)$-spin-chain with generic q. Its centralizer $\mathcal{C}(\mathcal{U}_q s\ell(2))$ on the chain is the Temperley–Lieb algebra $\mathcal{C}(\mathcal{U}_q s\ell(2)) = TL_N$ with the same value of the parameter q. Thus, the spin-chain space of states can be expressed as a bimodule $T_N = \bigoplus_i V_i \boxtimes M_i$, where V_i and M_i are some simple $\mathcal{U}_q s\ell(2)$- and TL_N-modules. When $N \to \infty$, the algebra TL_N conjecturally converges to the Virasoro algebra. When q is a root of unity the centralizer of the Temperley–Lieb algebra TL_N is the Lusztig limit $\mathcal{LU}_q s\ell(2)$ of $\mathcal{U}_q s\ell(2)$. In this case the bimodule decomposition of the spin chain contains non-semisimple summands [12].

When q is a root of unity, the algebra $\mathcal{LU}_q s\ell(2)$ contains the restricted quantum group $\overline{\mathcal{U}}_q s\ell(2)$ as a subalgebra, see details in [45]. In [37] it is shown that the centralizer of $\overline{\mathcal{U}}_q s\ell(2)$ on the spin-chain T_N is the algebra $\mathcal{W}_N$, which contains the algebra TL_N. In the limit $N \to \infty$ the algebra $\mathcal{W}_N$ gives the triplet algebra $\mathcal{W}$ built by the lattice VOA construction.

In case of $\mathcal{U}_q s\ell(2|1)$ we take its (mutually dual) fundamental representations which are three-dimensional and denote them by $\mathbf{3}$ and $\overline{\mathbf{3}}$. We study the mixed tensor product

$$\mathcal{T}_{m,n} = \mathbf{3}^{\otimes m} \otimes \overline{\mathbf{3}}^{\otimes n}. \tag{1.1}$$

The tensor product $\mathcal{T}_{m,n}$ is the space of states of different integrable spin-chains with $\mathcal{U}_q s\ell(2|1)$ symmetric hamiltonians, examples of which are considered in [46,7,47,48]. We let $\mathsf{X}_{m,n}$ denote the centralizer of $\mathcal{U}_q s\ell(2|1)$ on $\mathcal{T}_{m,n}$, $\mathcal{C}(\mathcal{U}_q s\ell(2|1)) = \mathsf{X}_{m,n}$. It is shown in [49–52] that $\mathsf{X}_{m,n}$ is isomorphic to some quotient of the quantum walled Brauer algebra $\mathsf{qwB}_{m,n}$. In this paper we do not give an explicit description of $\mathsf{X}_{m,n}$ itself, but describe simple and projective modules over $\mathsf{X}_{m,n}$. We find the decomposition of the chain $\mathcal{T}_{m,n}$ as a bimodule over $\mathcal{U}_q s\ell(2|1)$ and $\mathsf{X}_{m,n}$. Even for generic values of q, the bimodule is not semisimple. We give the bimodule in an explicit form in Theorem 5.3.

The quantum walled Brauer algebra $\mathsf{qwB}_{m,n}$ was introduced in [53–55]. The two-parametric algebra $\mathsf{qwB}_{m,n}$ was introduced in [56] and the structure of the simple modules was described implicitly. Modules over $\mathsf{qwB}_{m,n}$ and its classical analogue $\mathsf{wB}_{m,n}$ were investigated in [57,58, 61–63]. Some useful facts about $\mathsf{wB}_{m,n}$-projective modules can be obtained from [76].

For arbitrary values M, N the algebra $\mathcal{U}_q s\ell(M|N)$ on the appropriate mixed tensor product (which is the tensor product of its fundamental representations) is centralized by some quotient of $\mathsf{qwB}_{m,n}$, see also [52,64]. If $N = 0$ the bimodule is semisimple. We study the simplest non-semisimple case $N = 1$.

The outline of the article is as follows. In Sec. 2 we define the algebra $\mathcal{U}_q s\ell(2|1)$ and classify its finite-dimensional simple and projective modules. In Sec. 3 we describe the mixed tensor product and introduce the centralizer $\mathsf{X}_{m,n}$. First, we prove the formulas for the tensor products of modules needed to the mixed tensor product decomposition. Next, we show that the centralizer is a quotient of the algebra $\mathsf{qwB}_{m,n}$. In Sec. 4 we describe simple and projective modules over $\mathsf{X}_{m,n}$ and the restriction functors on them. In the last Sec. 5 we describe the bimodule structure and give a sketch of a proof for the bimodule decomposition formula.

2. The Hopf algebra $\mathcal{U}_q s\ell(2|1)$

2.1. Definition of $\mathcal{U}_q s\ell(2|1)$

Quantum analogues of superalgebras $s\ell(2|1)$ and $g\ell(2|1)$ was studied intensively in [65–68]. We describe the Hopf algebra $\mathcal{U}_q s\ell(2|1)$ by a system of generators and relations. In this section and in the entire paper we assume that the parameter q is not a root of unity. We choose the

generators adapted to the Hopf subalgebra structure $\mathcal{U}_q s\ell(2|1) \supset \mathcal{U}_q g\ell(2) \supset \mathcal{U}_q s\ell(2)$ (such that the embeddings act by identification of corresponding generators); we extensively use these subalgebras while working with $\mathcal{U}_q s\ell(2|1)$ modules in the sequel. The Hopf subalgebra $\mathcal{U}_q s\ell(2)$ in $\mathcal{U}_q s\ell(2|1)$ is generated as an associative algebra by E, K, and F with the relations

$$KF = q^{-2}FK, \quad EF - FE = \frac{K - K^{-1}}{q - q^{-1}}, \quad KE = q^2 EK. \tag{2.1}$$

The larger algebra $\mathcal{U}_q g\ell(2)$ contains an additional generator k satisfying the relations

$$kF = qFk, \quad kE = q^{-1}Ek, \quad kK = Kk. \tag{2.2}$$

We call the generators E, F, K and k bosonic. There are two additional generators B and C, which extend $\mathcal{U}_q g\ell(2)$ to $\mathcal{U}_q s\ell(2|1)$, and which we call fermionic, or simply fermions. The relations that involve the fermions B and C are

$$\begin{gathered} kB = -Bk, \quad KB = qBK, \quad kC = -Ck, \quad KC = q^{-1}CK, \\ B^2 = 0, \quad BC - CB = \frac{k - k^{-1}}{q - q^{-1}}, \quad C^2 = 0, \\ FC - CF = 0, \qquad BE - EB = 0, \\ FFB - [2]FBF + BFF = 0, \quad EEC - [2]ECE + CEE = 0, \end{gathered} \tag{2.3}$$

where we use q-integers defined as

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}}.$$

The Hopf-algebra structure of $\mathcal{U}_q s\ell(2|1)$ (the coproduct, the antipode, and the counit) is given by

$$\begin{aligned} &\Delta(F) = F \otimes 1 + K^{-1} \otimes F, \qquad \Delta(E) = E \otimes K + \mathbf{1} \otimes E, \\ &\Delta(B) = B \otimes 1 + k^{-1} \otimes B, \qquad \Delta(C) = C \otimes k + \mathbf{1} \otimes C, \end{aligned} \tag{2.4}$$

$$S(B) = -kB, \quad S(F) = -KF, \quad S(C) = -Ck^{-1}, \quad S(E) = -EK^{-1}, \tag{2.5}$$

$$\epsilon(B) = 0, \quad \epsilon(F) = 0, \quad \epsilon(C) = 0, \quad \epsilon(E) = 0, \tag{2.6}$$

with k and K being group-like.

2.2. *Simple* $\mathcal{U}_q s\ell(2|1)$ *modules*

Here and below we use the same notations as in [70]. We consider a subcategory of $\mathcal{U}_q s\ell(2|1)$-modules with k eigenvalues of the form q^{-n} for $n \in \mathbb{Z}$. The subcategory is closed under tensor products.

Each simple finite dimensional $\mathcal{U}_q s\ell(2|1)$ module contains a highest weight vector satisfying the conditions:

$$E|\alpha, s, \beta, r\rangle_0^{\leftarrow} = 0, \qquad C|\alpha, s, \beta, r\rangle_0^{\leftarrow} = 0, \tag{2.7}$$

where the labeling corresponds to the eigenvalues of K and k:

$$\begin{aligned} &K|\alpha, s, \beta, r\rangle_0^{\leftarrow} = \alpha q^{s-1}|\alpha, s, \beta, r\rangle_0^{\leftarrow}, \\ &k|\alpha, s, \beta, r\rangle_0^{\leftarrow} = \beta q^{-r}|\alpha, s, \beta, r\rangle_0^{\leftarrow}, \end{aligned} \qquad \alpha, \beta = \pm 1, \quad s, r \in \mathbb{Z}, \quad s \geqslant 1. \tag{2.8}$$

We choose such a parametrization because the dimension and structure of the modules are independent of α and β. We note that there are four nonisomorphic modules in each dimension, which differ in the sign of the Cartan eigenvalues.

The set of the simple finite-dimensional $\mathcal{U}_q s\ell(2|1)$-modules is

$$\mathcal{Z}_{s,r}^{\alpha,\beta}, \qquad \alpha,\beta=\pm 1, \quad s,r\in\mathbb{Z}, \quad s\geqslant 1. \tag{2.9}$$

The structure of the modules will be explained in more detail in the next subsection. For a pictorial explanation of the notations for vectors in the module see Fig. 1.

The modules have dimensions

$$\dim \mathcal{Z}_{s,r}^{\alpha,\beta}=\begin{cases} 2s-1, & r=0,\\ 2s+1, & r=s,\\ 4s, & r\neq 0,s. \end{cases} \tag{2.10}$$

The modules with $r=0$ and $r=s$ are atypical, and others are typical. In [65] it was shown that every finite-dimensional irreducible module over the general linear Lie superalgebra $g\ell(n|1)$ can be deformed into an irreducible module over $\mathcal{U}_q g\ell(n|1)$. Notations "typical" and "atypical" for modules in the present work are inherited from the theory of Lie superalgebras (see, for example [69]).

The trivial module corresponds to $\mathcal{Z}_{1,0}^{+1,+1}$, see (2.9). We note that there are four one-dimensional modules and only one of them is trivial.

2.2.1. $\mathcal{U}_q s\ell(2|1)$-action on simple modules

Each $\mathcal{U}_q s\ell(2|1)$-module decomposes into a direct sum of simple $\mathcal{U}_q g\ell(2)$-modules $\mathcal{X}_{s,r}^{\alpha,\beta}$, where $\alpha,\ \beta=\pm 1$, $s\geqslant 1$, $r\in\mathbb{Z}$. Their dimensions are $\dim \mathcal{X}_{s,r}^{\alpha,\beta}=s$. Eigenvalues of generators K and k on the highest weight vector in the module $\mathcal{X}_{s,r}^{\alpha,\beta}$ are αq^{s-1} and βq^{-r} correspondingly.

We describe (following [70]) the action of $\mathcal{U}_q s\ell(2|1)$ on its simple modules explicitly, using the basis adapted to the decomposition into $\mathcal{U}_q g\ell(2)$-modules. The examples of each type of modules are shown in Fig. 1.

Atypical modules with $r=0$, $\mathcal{Z}_{s,0}^{\alpha,\beta}$: As $\mathcal{U}_q g\ell(2)$-modules, these modules decompose as

$$\mathcal{Z}_{s,0}^{\alpha,\beta}=\mathcal{X}_{s,0}^{\alpha,\beta}\oplus\mathcal{X}_{s-1,-1}^{\alpha,-\beta}, \tag{2.11}$$

and we choose a basis in $\mathcal{Z}_{s,0}^{\alpha,\beta}$ in accordance with this decomposition, as

$$\left(|\alpha,s,\beta,0\rangle_n^{\leftarrow}\in\mathcal{X}_{s,0}^{\alpha,\beta}\right)_{0\leqslant n\leqslant s-1}, \qquad \left(|\alpha,s,\beta,0\rangle_m^{\rightarrow}\in\mathcal{X}_{s-1,-1}^{\alpha,\beta}\right)_{0\leqslant m\leqslant s-2}.$$

The fermionic generators relate these two types of vectors as

$$B|\alpha,s,\beta,0\rangle_n^{\leftarrow}=-[n]|\alpha,s,\beta,0\rangle_{n-1}^{\rightarrow}, \qquad C|\alpha,s,\beta,0\rangle_m^{\rightarrow}=\beta|\alpha,s,\beta,0\rangle_{m+1}^{\leftarrow}.$$

Atypical modules with $s=r$, $\mathcal{Z}_{s,s}^{\alpha,\beta}$: The modules decompose as

$$\mathcal{Z}_{s,s}^{\alpha,\beta}=\mathcal{X}_{s,s}^{\alpha,\beta}\oplus\mathcal{X}_{s+1,s}^{\alpha,-\beta}, \tag{2.12}$$

and we choose a basis in $\mathcal{Z}_{s,s}^{\alpha,\beta}$ accordingly, as

$$\left(|\alpha,s,\beta,s\rangle_n^{\leftarrow}\in\mathcal{X}_{s,s}^{\alpha,\beta}\right)_{0\leqslant n\leqslant s-1}, \qquad \left(|\alpha,s,\beta,s\rangle_m^{\rightarrow}\in\mathcal{X}_{s+1,s}^{\alpha,-\beta}\right)_{0\leqslant m\leqslant s}.$$

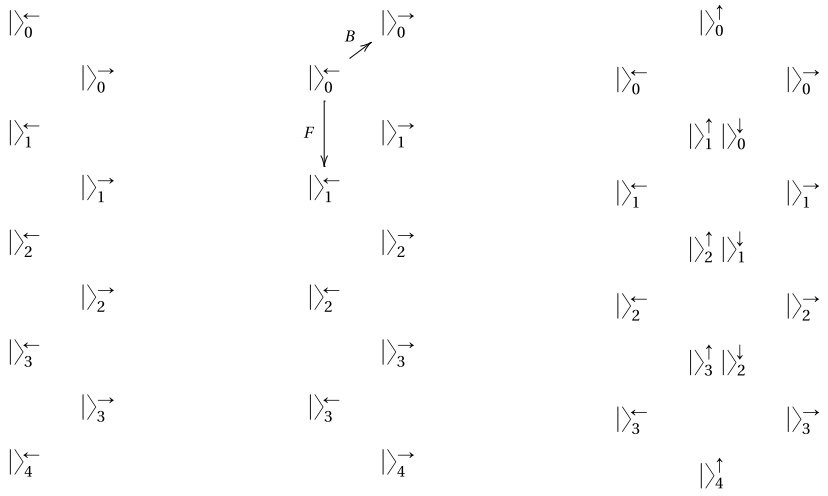


Fig. 1. Left: An atypical module $\mathcal{Z}^{\alpha,\beta}_{s,0}$ (with $s=5$). Each vertical column is a $\mathcal{U}_q g\ell(2)$ module in (2.11). The top state is $|\alpha,s;\beta,0\rangle^{\leftarrow}_0$ and the bottom, $|\alpha,s;\beta,0\rangle^{\leftarrow}_{s-1}$. MIDDLE: An atypical module $\mathcal{Z}^{\alpha,\beta}_{s,s}$ (with $s=4$). Each vertical column is a $\mathcal{U}_q g\ell(2)$ module in (2.12). The top state is $|\alpha,s;\beta,s\rangle^{\rightarrow}_0$ and the bottom, $|\alpha,s;\beta,s\rangle^{\rightarrow}_s$. Right: The typical module $\mathcal{Z}^{\alpha,\beta}_{s,r}$ ($s=4$). Each column is a $\mathcal{U}_q g\ell(2)$-module in (2.13). The directions in which the generators map are common for all modules.

The fermions act between these two sets of basis vectors as

$$B|\alpha,s,\beta,s\rangle^{\leftarrow}_n=[s-n]|\alpha,s,\beta,s\rangle^{\rightarrow}_n,\qquad C|\alpha,s,\beta,s\rangle^{\rightarrow}_m=\beta|\alpha,s,\beta,s\rangle^{\leftarrow}_m.$$

Typical modules ($r\neq 0, s$): The modules decompose as

$$\mathcal{Z}^{\alpha,\beta}_{s,r}=\mathcal{X}^{\alpha,\beta}_{s,r}\oplus\mathcal{X}^{\alpha,-\beta}_{s+1,r}\oplus\mathcal{X}^{\alpha,-\beta}_{s-1,r-1}\oplus\mathcal{X}^{\alpha,\beta}_{s,r-1},\tag{2.13}$$

and we choose the basis in $\mathcal{Z}^{\alpha,\beta}_{s,r}$ as

$$\big(|\alpha,s,\beta,r\rangle^{\leftarrow}_j\big)_{0\leqslant j\leqslant s-1},\ \big(|\alpha,s,\beta,r\rangle^{\uparrow}_m\big)_{0\leqslant m\leqslant s},$$
$$\big(|\alpha,s,\beta,r\rangle^{\downarrow}_n\big)_{0\leqslant n\leqslant s-2},\ \big(|\alpha,s,\beta,r\rangle^{\rightarrow}_j\big)_{0\leqslant j\leqslant s-1}.$$

The fermions act on these vectors as

$$B|\alpha,s,\beta,r\rangle^{\leftarrow}_j=\frac{[j]}{[s]}|\alpha,s,\beta,r\rangle^{\downarrow}_{j-1}+\beta\frac{[r][s-j]}{[s]}|\alpha,s,\beta,r\rangle^{\uparrow}_j,$$
$$B|\alpha,s,\beta,r\rangle^{\uparrow}_m=[m]|\alpha,s,\beta,r\rangle^{\rightarrow}_{m-1},\qquad C|\alpha,s,\beta,r\rangle^{\uparrow}_m=|\alpha,s,\beta,r\rangle^{\leftarrow}_m,$$
$$B|\alpha,s,\beta,r\rangle^{\downarrow}_n=\beta[r][n+1-s]|\alpha,s,\beta,r\rangle^{\rightarrow}_n,\quad C|\alpha,s,\beta,r\rangle^{\downarrow}_n=\beta[r-s]|\alpha,s,\beta,r\rangle^{\leftarrow}_{n+1},$$
$$C|\alpha,s,\beta,r\rangle^{\rightarrow}_j=\frac{1}{[s]}|\alpha,s,\beta,r\rangle^{\downarrow}_j+\beta\frac{[s-r]}{[s]}|\alpha,s,\beta,r\rangle^{\uparrow}_{j+1}.$$

In this paper we are mainly interested in two three-dimensional simple $\mathcal{U}_q s\ell(2|1)$-modules $\mathbf{3}=\mathcal{Z}^{1,-1}_{1,1}$ and $\overline{\mathbf{3}}=\mathcal{Z}^{1,1}_{2,0}$. They have the structure

$$\mathbf{3} = \begin{matrix} |\rangle_0^{\rightarrow}, \\ |\rangle_0^{\leftarrow} \\ |\rangle_1^{\rightarrow} \end{matrix} \qquad \overline{\mathbf{3}} = \begin{matrix} |\rangle_0^{\leftarrow} \\ |\rangle_0^{\rightarrow} \\ |\rangle_1^{\leftarrow} \end{matrix} \tag{2.14}$$

2.3. $\mathbf{Ext^1}$ *spaces for atypical modules*

For two modules $\mathcal{Z}_1$ and $\mathcal{Z}_2$, we define $\mathrm{Ext}^1(\mathcal{Z}_2, \mathcal{Z}_1)$ as a linear space with basis identified with nontrivial short exact sequences

$$0 \to \mathcal{Z}_1 \to \mathcal{Z}_1 \oplus \mathcal{Z}_2 \to \mathcal{Z}_2 \to 0$$

modulo a certain equivalence relation [71].

The groups Ext^1 vanish for the typical $\mathcal{U}_q s\ell(2|1)$ modules. For the atypical modules, the $\mathrm{Ext}^1(\mathcal{Z}_1, \mathcal{Z}_2)$ group is at most 1-dimensional. Whenever $\mathrm{Ext}^1(\mathcal{Z}_1, \mathcal{Z}_2)$ is nontrivial, we describe the algebra action in terms of generators: the action of a $\mathcal{U}_q s\ell(2|1)$-generator A on $\mathcal{Z}_1 \oplus \mathcal{Z}_2$ is given by

$$\rho_A = \rho_A^{(0)} + \xi_A,$$

where $\rho_A^{(0)}$ is the direct sum of actions of $\mathcal{U}_q s\ell(2|1)$-generators on the simple modules and $\xi_A = \xi_A^{\mathcal{Z}_1, \mathcal{Z}_2} : \mathcal{Z}_1 \to \mathcal{Z}_2$ are linear maps.

We list the ξ_A maps in terms of the bases introduced above. The formulas can be somewhat uniformized by adopting the following convention for the 1-dimensional modules $\mathcal{Z}_{1,0}^{\alpha,-\beta}$: we denote this module also by $\mathcal{Z}_{0,0}^{\alpha,\beta}$, with a basis vector $|\alpha, 0, \beta, 0\rangle_0^{\rightarrow} = |\alpha, 1, -\beta, 0\rangle_0^{\leftarrow}$ (and, formally, with $|\alpha, 0, \beta, 0\rangle_m^{\leftarrow} = 0$, $m \neq 0$). We then have

$$\begin{aligned}
&\mathrm{Ext}^1(\mathcal{Z}_{s,0}^{\alpha,\beta}, \mathcal{Z}_{s+1,0}^{\alpha,-\beta}) = \{b_{s+1}\}, && \xi_B : |\alpha, s, \beta, 0\rangle_m^{\leftarrow} \mapsto -[s-m]|\alpha, s+1, -\beta, 0\rangle_m^{\leftarrow}, \\
&\quad (s \geqslant 1) && \xi_B : |\alpha, s, \beta, 0\rangle_m^{\rightarrow} \mapsto [s-m-1]|\alpha, s+1, -\beta, 0\rangle_m^{\rightarrow}, \\
&\mathrm{Ext}^1(\mathcal{Z}_{s,0}^{\alpha,\beta}, \mathcal{Z}_{s-1,0}^{\alpha,-\beta}) = \{c_{s-1}\}, && \xi_C : |\alpha, s, \beta, 0\rangle_m^{\leftarrow} \mapsto |\alpha, s-1, -\beta, 0\rangle_m^{\leftarrow}, \\
&\quad (s \geqslant 2) && \xi_C : |\alpha, s, \beta, 0\rangle_m^{\rightarrow} \mapsto |\alpha, s-1, -\beta, 0\rangle_m^{\rightarrow}, \\
&\mathrm{Ext}^1(\mathcal{Z}_{s,s}^{\alpha,\beta}, \mathcal{Z}_{s-1,s-1}^{\alpha,-\beta}) = \{\bar{b}_{s-1}\}, && \xi_B : |\alpha, s, \beta, s\rangle_m^{\leftarrow} \mapsto -[m]|\alpha, s-1, -\beta, s-1\rangle_{m-1}^{\leftarrow}, \\
&\quad (s \geqslant 1) && \xi_B : |\alpha, s, \beta, s\rangle_m^{\rightarrow} \mapsto [m]|\alpha, s-1, -\beta, s-1\rangle_{m-1}^{\rightarrow}, \\
&\mathrm{Ext}^1(\mathcal{Z}_{s,s}^{\alpha,\beta}, \mathcal{Z}_{s+1,s+1}^{\alpha,-\beta}) = \{\bar{c}_{s+1}\}, && \xi_C : |\alpha, s, \beta, s\rangle_m^{\leftarrow} \mapsto |\alpha, s+1, -\beta, s+1\rangle_{m+1}^{\leftarrow}, \\
&\quad (s \geqslant 0) && \xi_C : |\alpha, s, \beta, s\rangle_m^{\rightarrow} \mapsto |\alpha, s+1, -\beta, s+1\rangle_{m+1}^{\rightarrow}.
\end{aligned}$$

2.4. *Projective* $\mathcal{U}_q s\ell(2|1)$*-modules*

There are two types of projective $\mathcal{U}_q s\ell(2|1)$-modules.

2.4.1. *Simple projective modules*

All simple typical modules described in 2.2.1 are projective.

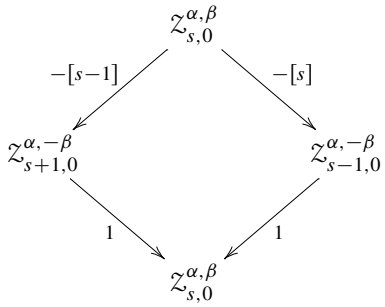
2.4.2. Projective covers of atypical modules

We use the notation $\mathcal{R}^{\alpha,\beta}_{s,0}$ and $\mathcal{R}^{\alpha,\beta}_{s,s}$ for projective covers of $\mathcal{Z}^{\alpha,\beta}_{s,0}$ and $\mathcal{Z}^{\alpha,\beta}_{s,s}$ (where, as before, $\alpha, \beta = \pm 1$ and $s \geqslant 1$). We describe the projective covers in terms of Loewy graphs (see definition in [77]). The reconstruction of the $\mathcal{U}_q s\ell(2|1)$-action on a projective module from its Loewy graph is described in detail in [70, Sec. 6]. The action $\rho_A(v)$ of a generator A on a vector v has three parts:

$$\rho_A(v) = \rho^{(0)}_A(v) + c(v)\xi_A(v) + \eta_A(v),$$

where $\rho^{(0)}_A(v)$ is the action of A in the irreducible subquotient, ξ_A is determined in 2.3, and for the map η_A we give explicit formulas after each Loewy graph (whenever η_A is nonzero). Here $c(v)$ are some coefficients depending on a pair of simple subquotients in the projective module in question. We write them on edges in Loewy graphs (see [70] for a detailed explanation).

It is convenient to distinguish between two series and two exceptional cases of projective covers. The first series is $\mathcal{R}^{\alpha,\beta}_{s,0}$, $s \geqslant 2$, with the Loewy graph



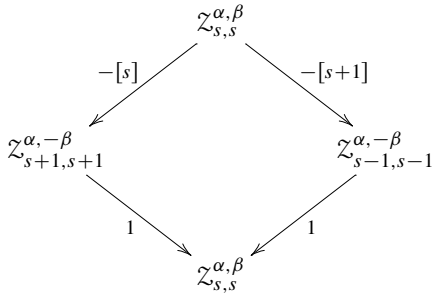
(2.15)

where

$$\eta_B : |\alpha, s, \beta, 0\rangle^{\leftarrow\,\triangle}_{n} \mapsto -\beta[n]|\alpha, s, \beta, 0\rangle^{\rightarrow}_{n-1\,\triangledown}.$$

Here $v^{\triangle}$ denotes the vector v from the top subquotient, and $v_{\triangledown}$ denotes vector v from the bottom subquotient.

The second series is $\mathcal{R}^{\alpha,\beta}_{s,s}$, $s \geqslant 2$, with the Loewy graph

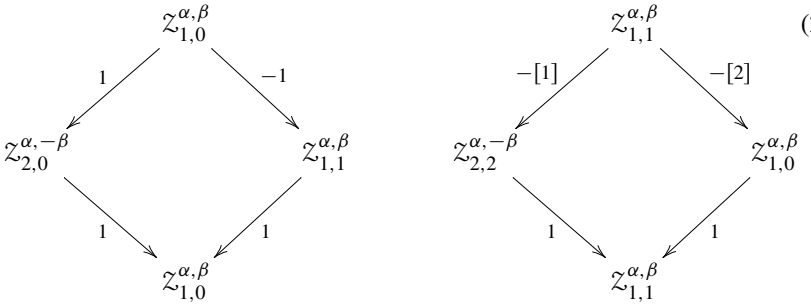


(2.16)

and with

$$\eta_C : |\alpha, s, \beta, s\rangle^{\rightarrow\,\triangle}_{n} \mapsto |\alpha, s, \beta, s\rangle^{\leftarrow}_{n\,\triangledown}.$$

The two exceptional cases are $\mathcal{R}^{\alpha,\beta}_{1,0}$ and $\mathcal{R}^{\alpha,\beta}_{1,1}$, with the respective Loewy graphs



(2.17)

These modules have dimensions

$$\begin{aligned}
&\dim \mathcal{R}^{\alpha,\beta}_{s,0} = 8s-4, \quad s>1,\\
&\dim \mathcal{R}^{\alpha,\beta}_{s,s} = 8s+4, \quad s\geqslant 1,\\
&\dim \mathcal{R}^{\alpha,\beta}_{1,0} = 8.
\end{aligned}$$

3. The mixed tensor product

We study the mixed tensor product ("spin-chain") (1.1), where $\mathbf{3}=\mathcal{Z}^{1,-1}_{1,1}$ and $\overline{\mathbf{3}}=\mathcal{Z}^{1,1}_{2,0}$ are the two three-dimensional simple $\mathcal{U}_q s\ell(2|1)$-modules, see (2.14). We are interested in decomposing $\mathcal{T}_{m,n}$ as a bimodule over $\mathcal{U}_q s\ell(2|1)$ and its centralizer $\mathsf{X}_{m,n}$. As a necessary first step, we decompose tensor products of relevant $\mathcal{U}_q s\ell(2|1)$-modules with the fundamental modules $\mathcal{Z}^{\alpha,\beta}_{1,1}$ and $\mathcal{Z}^{\alpha,\beta}_{2,0}$.

Theorem 3.1. *Tensor products $\mathcal{Z}\otimes\mathcal{Z}^{\alpha,\beta}_{1,1}$, where $\mathcal{Z}$ ranges the atypical and typical simple modules and their projective covers, decompose as follows:*

$$\begin{aligned}
&\mathcal{Z}^{\alpha_1,\beta_1}_{s,0}\otimes\mathcal{Z}^{\alpha_2,\beta_2}_{1,1} = \mathcal{Z}^{\alpha_{12},-\beta_{12}}_{s-1,0}+\mathcal{Z}^{\alpha_{12},\beta_{12}}_{s,1}, \quad s\geqslant 2,\\
&\mathcal{Z}^{\alpha_1,\beta_1}_{s,s}\otimes\mathcal{Z}^{\alpha_2,\beta_2}_{1,1} = \mathcal{Z}^{\alpha_{12},-\beta_{12}}_{s+1,s+1}+\mathcal{Z}^{\alpha_{12},\beta_{12}}_{s,s+1}, \quad s\geqslant 1,\\
&\mathcal{Z}^{\alpha_1,\beta_1}_{s,r}\otimes\mathcal{Z}^{\alpha_2,\beta_2}_{1,1} = \left\{\begin{array}{ll}
\mathcal{R}^{\alpha_{12},-\beta_{12}}_{s+1,0}+\mathcal{Z}^{\alpha_{12},-\beta_{12}}_{s-1,-1}, & r=-1,\\
\mathcal{R}^{\alpha_{12},-\beta_{12}}_{s-1,s-1}+\mathcal{Z}^{\alpha_{12},-\beta_{12}}_{s+1,s}, & r=s-1,\\
\mathcal{Z}^{\alpha_{12},\beta_{12}}_{s,r+1}+\mathcal{Z}^{\alpha_{12},-\beta_{12}}_{s+1,r+1}+\mathcal{Z}^{\alpha_{12},-\beta_{12}}_{s-1,r} & \textit{otherwise},
\end{array}\right\} \qquad s\geqslant 2
\end{aligned}$$

and

$$\begin{aligned}
&\mathcal{R}^{\alpha_1,\beta_1}_{s,0}\otimes\mathcal{Z}^{\alpha_2,\beta_2}_{1,1} = \mathcal{R}^{\alpha_{12},-\beta_{12}}_{s-1,0}+2\mathcal{Z}^{\alpha_{12},\beta_{12}}_{s,1}+\mathcal{Z}^{\alpha_{12},-\beta_{12}}_{s-1,1}+\mathcal{Z}^{\alpha_{12},-\beta_{12}}_{s+1,1}, \quad s\geqslant 3,\\
&\mathcal{R}^{\alpha_1,\beta_1}_{s,s}\otimes\mathcal{Z}^{\alpha_2,\beta_2}_{1,1} = \mathcal{R}^{\alpha_{12},-\beta_{12}}_{s+1,s+1}+2\mathcal{Z}^{\alpha_{12},\beta_{12}}_{s,s+1}+\mathcal{Z}^{\alpha_{12},-\beta_{12}}_{s-1,s}+\mathcal{Z}^{\alpha_{12},-\beta_{12}}_{s+1,s+2}, \quad s\geqslant 2,
\end{aligned}$$

where we write $\alpha_{12}=\alpha_1\alpha_2$ and $\beta_{12}=\beta_1\beta_2$.

The exceptional cases are listed below:

$$\begin{aligned}
&\mathcal{Z}^{\alpha_1,\beta_1}_{1,0}\otimes\mathcal{Z}^{\alpha_2,\beta_2}_{1,1} = \mathcal{Z}^{\alpha_{12},\beta_{12}}_{1,1},\\
&\mathcal{Z}^{\alpha_1,\beta_1}_{1,-1}\otimes\mathcal{Z}^{\alpha_2,\beta_2}_{1,1} = \mathcal{R}^{\alpha_{12},-\beta_{12}}_{2,0},
\end{aligned}$$

$$\mathcal{Z}^{\alpha_1,\beta_1}_{1,r}\otimes\mathcal{Z}^{\alpha_2,\beta_2}_{1,1}=\mathcal{Z}^{\alpha_{12},\beta_{12}}_{1,r+1}+\mathcal{Z}^{\alpha_{12},-\beta_{12}}_{2,r+1},\quad r\neq -1,0,1,$$
$$\mathcal{R}^{\alpha_1,\beta_1}_{2,0}\otimes\mathcal{Z}^{\alpha_2,\beta_2}_{1,1}=\mathcal{R}^{\alpha_{12},-\beta_{12}}_{1,0}+2\mathcal{Z}^{\alpha_{12},\beta_{12}}_{2,1}+\mathcal{Z}^{\alpha_{12},-\beta_{12}}_{3,1},$$
$$\mathcal{R}^{\alpha_1,\beta_1}_{1,0}\otimes\mathcal{Z}^{\alpha_2,\beta_2}_{1,1}=\mathcal{R}^{\alpha_{12},\beta_{12}}_{1,1}+\mathcal{Z}^{\alpha_{12},\beta_{12}}_{1,2}+\mathcal{Z}^{\alpha_{12},-\beta_{12}}_{2,1},$$
$$\mathcal{R}^{\alpha_1,\beta_1}_{1,1}\otimes\mathcal{Z}^{\alpha_2,\beta_2}_{1,1}=\mathcal{R}^{\alpha_{12},-\beta_{12}}_{2,2}+2\mathcal{Z}^{\alpha_{12},\beta_{12}}_{1,2}+\mathcal{Z}^{\alpha_{12},-\beta_{12}}_{2,3}.$$

The tensor products $\mathcal{Z}\otimes\mathcal{Z}^{\alpha,\beta}_{2,0}$ *decompose as:*

$$\left.\begin{aligned}
&\mathcal{Z}^{\alpha_1,\beta_1}_{s,0}\otimes\mathcal{Z}^{\alpha_2,\beta_2}_{2,0}=\mathcal{Z}^{\alpha_{12},\beta_{12}}_{s+1,0}+\mathcal{Z}^{\alpha_{12},\beta_{12}}_{s-1,-1},\\
&\mathcal{Z}^{\alpha_1,\beta_1}_{s,s}\otimes\mathcal{Z}^{\alpha_2,\beta_2}_{2,0}=\mathcal{Z}^{\alpha_{12},\beta_{12}}_{s-1,s-1}+\mathcal{Z}^{\alpha_{12},\beta_{12}}_{s+1,s},\\
&\mathcal{Z}^{\alpha_1,\beta_1}_{s,r}\otimes\mathcal{Z}^{\alpha_2,\beta_2}_{2,0}=\begin{cases}\mathcal{R}^{\alpha_{12},-\beta_{12}}_{s,0}+\mathcal{Z}^{\alpha_{12},\beta_{12}}_{s+1,1}, & r=1,\\ \mathcal{R}^{\alpha_{12},-\beta_{12}}_{s,s}+\mathcal{Z}^{\alpha_{12},\beta_{12}}_{s-1,s}, & r=s+1,\\ \mathcal{Z}^{\alpha_{12},\beta_{12}}_{s+1,r}+\mathcal{Z}^{\alpha_{12},-\beta_{12}}_{s,r-1}+\mathcal{Z}^{\alpha_{12},\beta_{12}}_{s-1,r-1}, & \textit{otherwise},\end{cases}
\end{aligned}\right\}\quad s\geqslant 2$$

and

$$\mathcal{R}^{\alpha_1,\beta_1}_{s,0}\otimes\mathcal{Z}^{\alpha_2,\beta_2}_{2,0}=\mathcal{R}^{\alpha_{12},\beta_{12}}_{s+1,0}+2\mathcal{Z}^{\alpha_{12},\beta_{12}}_{s-1,-1}+\mathcal{Z}^{\alpha_{12},-\beta_{12}}_{s,-1}+\mathcal{Z}^{\alpha_{12},-\beta_{12}}_{s-2,-1},\quad s\geqslant 3,$$
$$\mathcal{R}^{\alpha_1,\beta_1}_{s,s}\otimes\mathcal{Z}^{\alpha_2,\beta_2}_{2,0}=\mathcal{R}^{\alpha_{12},\beta_{12}}_{s-1,s-1}+2\mathcal{Z}^{\alpha_{12},\beta_{12}}_{s+1,s}+\mathcal{Z}^{\alpha_{12},-\beta_{12}}_{s+2,s+1}+\mathcal{Z}^{\alpha_{12},-\beta_{12}}_{s,s-1},\quad s\geqslant 2.$$

The exceptional cases are:

$$\mathcal{Z}^{\alpha_1,\beta_1}_{1,0}\otimes\mathcal{Z}^{\alpha_2,\beta_2}_{2,0}=\mathcal{Z}^{\alpha_{12},\beta_{12}}_{2,0},$$
$$\mathcal{Z}^{\alpha_1,\beta_1}_{1,1}\otimes\mathcal{Z}^{\alpha_2,\beta_2}_{2,0}=\mathcal{Z}^{\alpha_{12},-\beta_{12}}_{1,0}+\mathcal{Z}^{\alpha_{12},\beta_{12}}_{2,1},$$
$$\mathcal{Z}^{\alpha_1,\beta_1}_{1,2}\otimes\mathcal{Z}^{\alpha_2,\beta_2}_{2,0}=\mathcal{R}^{\alpha_{12},-\beta_{12}}_{1,1},$$
$$\mathcal{Z}^{\alpha_1,\beta_1}_{1,r}\otimes\mathcal{Z}^{\alpha_2,\beta_2}_{2,0}=\mathcal{Z}^{\alpha_{12},-\beta_{12}}_{1,r-1}+\mathcal{Z}^{\alpha_{12},\beta_{12}}_{2,r},\quad r\neq 0,1,2,$$
$$\mathcal{R}^{\alpha_1,\beta_1}_{2,0}\otimes\mathcal{Z}^{\alpha_2,\beta_2}_{2,0}=\mathcal{R}^{\alpha_{12},\beta_{12}}_{3,0}+2\mathcal{Z}^{\alpha_{12},\beta_{12}}_{1,-1}+\mathcal{Z}^{\alpha_{12},-\beta_{12}}_{2,-1},$$
$$\mathcal{R}^{\alpha_1,\beta_1}_{1,0}\otimes\mathcal{Z}^{\alpha_2,\beta_2}_{2,0}=\mathcal{R}^{\alpha_{12},\beta_{12}}_{2,0}+\mathcal{Z}^{\alpha_{12},-\beta_{12}}_{1,-1}+\mathcal{Z}^{\alpha_{12},\beta_{12}}_{2,1},$$
$$\mathcal{R}^{\alpha_1,\beta_1}_{1,1}\otimes\mathcal{Z}^{\alpha_2,\beta_2}_{2,0}=\mathcal{R}^{\alpha_{12},-\beta_{12}}_{1,0}+2\mathcal{Z}^{\alpha_{12},\beta_{12}}_{2,1}+\mathcal{Z}^{\alpha_{12},-\beta_{12}}_{3,2}.$$

3.1.1. It follows, in particular, that the set of simple modules and their projective covers is closed under tensor product decompositions.

Proof. We discuss two cases: $\mathcal{Z}^{\alpha_1,\beta_1}_{s,s}\otimes\mathcal{Z}^{\alpha_2,\beta_2}_{1,1}$ and $\mathcal{Z}^{\alpha_1,\beta_1}_{s,s-1}\otimes\mathcal{Z}^{\alpha_2,\beta_2}_{1,1}$. Other cases are similar.

We consider the $\mathcal{U}_q s\ell(2|1)$-modules in the left-hand side of the tensor product as $\mathcal{U}_q g\ell(2)$-modules (as explained in 2.2.1) and calculate their tensor product using the results in [45]. For the tensor product $\mathcal{Z}^{\alpha_1,\beta_1}_{s,s}\otimes\mathcal{Z}^{\alpha_2,\beta_2}_{1,1}$, we have

$$\begin{aligned}\mathcal{Z}^{\alpha_1,\beta_1}_{s,s}\otimes\mathcal{Z}^{\alpha_2,\beta_2}_{1,1}&=\left(\mathcal{X}^{\alpha_1,\beta_1}_{s,s}\oplus\mathcal{X}^{\alpha_1,-\beta_1}_{s+1,s}\right)\otimes\left(\mathcal{X}^{\alpha_2,\beta_2}_{1,1}\oplus\mathcal{X}^{\alpha_2,-\beta_2}_{2,1}\right)\\&=\mathcal{X}^{\alpha_{12}\beta_{12}}_{s,s+1}\oplus\mathcal{X}^{\alpha_{12},-\beta_{12}}_{s+1,s+1}\oplus\mathcal{X}^{\alpha_{12},-\beta_{12}}_{s-1,s}\oplus\mathcal{X}^{\alpha_{12},-\beta_{12}}_{s+1,s+1}\oplus\mathcal{X}^{\alpha_{12},\beta_{12}}_{s+2,s+1}\oplus\mathcal{X}^{\alpha_{12},\beta_{12}}_{s,s}.\end{aligned}\tag{3.1}$$

Decomposition (3.1) contains six $\mathcal{U}_q g\ell(2)$-modules. Taking into account that a typical module contains four $\mathcal{U}_q g\ell(2)$-summands and an atypical one contains two, the module in (3.1) can be the direct sum of either three atypical $\mathcal{U}_q s\ell(2|1)$-modules or one typical and one atypical

module. Explicitly writing the decompositions of possible $\mathcal{U}_q s\ell(2|1)$-modules shows that there exists only one $\mathcal{U}_q s\ell(2|1)$-module that has the decomposition (3.1). The second and the fifth summands can be combined into $\mathcal{Z}^{\alpha_{12},-\beta_{12}}_{s+1,s+1}$ and the other four summands give $\mathcal{Z}^{\alpha_{12},\beta_{12}}_{s,s+1}$. Thus, we have

$$\mathcal{Z}^{\alpha_1,\beta_1}_{s,s} \otimes \mathcal{Z}^{\alpha_2,\beta_2}_{1,1} = \mathcal{Z}^{\alpha_{12},-\beta_{12}}_{s+1,s+1} \oplus \mathcal{Z}^{\alpha_{12},\beta_{12}}_{s,s+1}.$$

We next consider the product $\mathcal{Z}^{\alpha_1,\beta_1}_{s,s-1} \otimes \mathcal{Z}^{\alpha_2,\beta_2}_{1,1}$. The $\mathcal{U}_q g\ell(2)$-decomposition is

$$\begin{aligned}\mathcal{Z}^{\alpha_1,\beta_1}_{s,s-1} \otimes \mathcal{Z}^{\alpha_2,\beta_2}_{1,1} &= \left(\mathcal{X}^{\alpha,\beta}_{s,s-1} \oplus \mathcal{X}^{\alpha_1,-\beta_1}_{s+1,s-1} \oplus \mathcal{X}^{\alpha_1,-\beta_1}_{s-1,s-2} \oplus \mathcal{X}^{\alpha_1,\beta_1}_{s,s-2}\right) \\ &\quad \otimes \left(\mathcal{X}^{\alpha_2,\beta_2}_{1,1} \oplus \mathcal{X}^{\alpha_2,-\beta_2}_{2,1}\right) \\ &= \mathcal{X}^{\alpha_{12},\beta_{12}}_{s,s} \oplus \mathcal{X}^{\alpha_{12},-\beta_{12}}_{s+1,s} \oplus \mathcal{X}^{\alpha_{12},-\beta_{12}}_{s-1,s-1} \oplus \mathcal{X}^{\alpha_{12},\beta_{12}}_{s,s-1} \oplus \mathcal{X}^{\alpha_{12},-\beta_{12}}_{s+1,s} \oplus \mathcal{X}^{\alpha_{12},\beta_{12}}_{s+2,s} \\ &\quad \oplus \mathcal{X}^{\alpha_{12},\beta_{12}}_{s,s-1} \oplus \mathcal{X}^{\alpha_{12},-\beta_{12}}_{s+1,s-1} \oplus \mathcal{X}^{\alpha,-\beta}_{s-1,s-1} \oplus \mathcal{X}^{\alpha_{12},\beta_{12}}_{s,s-1} \oplus \mathcal{X}^{\alpha_{12},\beta_{12}}_{s-2,s-2} \\ &\quad \oplus \mathcal{X}^{\alpha_{12},-\beta_{12}}_{s-1,s-2}.\end{aligned} \tag{3.2}$$

Because $\mathcal{Z}^{\alpha_1,\beta_1}_{s,s-1}$ is a projective simple module (see 2.4.1), the decomposition of $\mathcal{Z}^{\alpha_1,\beta_1}_{s,s-1} \otimes \mathcal{Z}^{\alpha_2,\beta_2}_{1,1}$ involves only projective modules, which, as we recall from 2.4.2, consist of all typical simple modules and the $\mathcal{R}^{\alpha,\beta}_{s,r}$. There are several $\mathcal{U}_q s\ell(2|1)$-modules that have the $\mathcal{U}_q g\ell(2)$-decomposition (3.2), but only one of them is projective.[1] Thus, we have

$$\mathcal{Z}^{\alpha_1,\beta_1}_{s,s-1} \otimes \mathcal{Z}^{\alpha_2,\beta_2}_{1,1} = \mathcal{R}^{\alpha_{12},-\beta_{12}}_{s-1,s-1} \oplus \mathcal{Z}^{\alpha_{12},-\beta_{12}}_{s+1,s}.$$

The cases $\mathcal{R}^{\alpha_1,\beta_1}_{s,0} \otimes \mathcal{Z}^{\alpha_2,\beta_2}_{1,1}$ and $\mathcal{R}^{\alpha_1,\beta_1}_{s,s} \otimes \mathcal{Z}^{\alpha_2,\beta_2}_{1,1}$ are worked out similarly. We consider $\mathcal{U}_q g\ell(2)$-decompositions of both tensorands and calculate tensor products of $\mathcal{U}_q g\ell(2)$-modules. This gives a long direct sum of simple and projective $\mathcal{U}_q g\ell(2)$-modules that each time are combined uniquely into a sum of projective $\mathcal{U}_q s\ell(2|1)$-modules. □

Remark 3.1.2. Decomposition of all tensor products of finite dimensional $s\ell(2|1)$-representations into their indecomposable building blocks was found in [72].

3.1.3. We calculate decomposition of $\mathcal{T}_{m,n}$ iteratively using Theorem 3.1. The multiplicities of $\mathcal{U}_q s\ell(2|1)$-indecomposable modules are dimensions of $\mathsf{X}_{m,n}$-modules, which we discuss below.

3.2. The centralizer of $\mathcal{U}_q s\ell(2|1)$ on the mixed tensor product

We fix bases in the **3** and $\overline{\mathbf{3}}$ modules in accordance with 2.2.1 and introduce a shorthand notation for them:

$$\begin{aligned} f_1 &= |1,1;-1,1\rangle^{\leftarrow}_0, & f_2 &= |1,1;-1,1\rangle^{\rightarrow}_0, & f_3 &= |1,1;-1,1\rangle^{\rightarrow}_1, \\ v_1 &= |1,2;1,0\rangle^{\rightarrow}_0, & v_2 &= |1,2;1,0\rangle^{\leftarrow}_1, & v_3 &= |1,2;1,0\rangle^{\leftarrow}_0. \end{aligned}$$

In the tensor products of two $\mathcal{U}_q s\ell(2|1)$ modules, we then have the operators

[1] For example, the direct sum of simple $\mathcal{U}_q s\ell(2|1)$-modules $2\mathcal{Z}^{\alpha_{12},-\beta_{12}}_{s-1,s-1} \oplus \mathcal{Z}^{\alpha_{12},\beta_{12}}_{s,s} \oplus \mathcal{Z}^{\alpha_{12},\beta_{12}}_{s-2,s-2} \oplus \mathcal{Z}^{\alpha_{12},-\beta_{12}}_{s+1,s}$ is compatible with the $\mathcal{U}_q g\ell(2)$-decomposition (3.2), but is not a projective $\mathcal{U}_q s\ell(2|1)$-module.

$$g : \mathbf{3}\otimes\mathbf{3} \mapsto \mathbf{3}\otimes\mathbf{3}, \qquad \mathcal{E} : \mathbf{3}\otimes\overline{\mathbf{3}} \mapsto \mathbf{3}\otimes\overline{\mathbf{3}}, \qquad h : \overline{\mathbf{3}}\otimes\overline{\mathbf{3}} \mapsto \overline{\mathbf{3}}\otimes\overline{\mathbf{3}},$$

that commute with $\mathcal{U}_q s\ell(2|1)$ and are explicitly given by

$$g : \begin{pmatrix} f_1\otimes f_1 & f_1\otimes f_2 & f_1\otimes f_3 \\ f_2\otimes f_1 & f_2\otimes f_2 & f_2\otimes f_3 \\ f_3\otimes f_1 & f_3\otimes f_2 & f_3\otimes f_3 \end{pmatrix} \mapsto$$
$$\begin{pmatrix} q^{-2} f_1\otimes f_1 & -q^{-1} f_2\otimes f_1 & -q^{-1} f_3\otimes f_1 \\ (q^{-2}-1) f_2\otimes f_1 - q^{-1} f_1\otimes f_2 & -f_2\otimes f_2 & -q^{-1} f_3\otimes f_2 \\ (q^{-2}-1) f_3\otimes f_1 - q^{-1} f_1\otimes f_3 & (q^{-2}-1) f_3\otimes f_2 - q^{-1} f_2\otimes f_3 & -f_3\otimes f_3 \end{pmatrix},$$
$$\mathcal{E} : \begin{pmatrix} f_1\otimes v_1 & f_1\otimes v_2 & f_1\otimes v_3 \\ f_2\otimes v_1 & f_2\otimes v_2 & f_2\otimes v_3 \\ f_3\otimes v_1 & f_3\otimes v_2 & f_3\otimes v_3 \end{pmatrix} \mapsto$$
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -q & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot (q^2 f_1\otimes v_1 + q f_2\otimes v_2 - f_3\otimes v_3),$$

and

$$h : \begin{pmatrix} v_1\otimes v_1 & v_1\otimes v_2 & v_1\otimes v_3 \\ v_2\otimes v_1 & v_2\otimes v_2 & v_2\otimes v_3 \\ v_3\otimes v_1 & v_3\otimes v_2 & v_3\otimes v_3 \end{pmatrix} \mapsto$$
$$\begin{pmatrix} q^{-2} v_1\otimes v_1 & (q^{-2}-1) v_1\otimes v_2 - q^{-1} v_2\otimes v_1 & (q^{-2}-1) v_1\otimes v_3 - q^{-1} v_3\otimes v_1 \\ -q^{-1} v_1\otimes v_2 & -v_2\otimes v_2 & (q^{-2}-1) v_2\otimes v_3 - q^{-1} v_3\otimes v_2 \\ -q^{-1} v_1\otimes v_3 & -q^{-1} v_2\otimes v_3 & -v_3\otimes v_3 \end{pmatrix}.$$

On $\mathcal{T}_{m,n}$, we define the operators

$$g_j = \underbrace{\mathbf{1}\otimes\ldots\otimes\mathbf{1}}_{m-j-1} \otimes g \otimes \underbrace{\mathbf{1}\otimes\ldots\otimes\mathbf{1}}_{n+j-1},$$
$$h_i = \underbrace{\mathbf{1}\otimes\ldots\otimes\mathbf{1}}_{m+i-1} \otimes h \otimes \underbrace{\mathbf{1}\otimes\ldots\otimes\mathbf{1}}_{n-i-1},$$
$$\mathcal{E} = \underbrace{\mathbf{1}\otimes\ldots\otimes\mathbf{1}}_{m-1} \otimes \mathcal{E} \otimes \underbrace{\mathbf{1}\otimes\ldots\otimes\mathbf{1}}_{n-1}.$$

These are the generators of a quantum walled Brauer algebra, which we discuss in the next subsection.

3.3. The quantum walled Brauer algebra

3.3.1. The algebra $\mathsf{qwB}_{m,n}$ is the associative unital algebra generated by g_i, $\mathcal{E}$, h_j, where $1 \leqslant i \leqslant m-1$ and $1 \leqslant j \leqslant n-1$, with relations (see [53–55])

$$g_i h_j = h_j g_i,$$
$$(g_i-\gamma)(g_i-\delta) = 0, \qquad (h_j-\gamma)(h_j-\delta) = 0,$$
$$g_i g_j = g_j g_i, \quad |i-j|>1, \qquad h_i h_j = h_j h_i, \quad |i-j|>1,$$
$$g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}, \qquad h_j h_{j+1} h_j = h_{j+1} h_j h_{j+1},$$

$$\begin{aligned} \mathcal{E}\mathcal{E} &= \frac{\theta+1}{\gamma+\delta}\mathcal{E}, \\ \mathcal{E}g_1\mathcal{E} = \mathcal{E}, &\qquad \mathcal{E}h_1\mathcal{E} = \mathcal{E}, \\ \mathcal{E}g_i = g_i\mathcal{E}, \quad 2 \leqslant i \leqslant m-1, &\qquad \mathcal{E}h_j = h_j\mathcal{E}, \quad 2 \leqslant j \leqslant n-1, \\ \mathcal{E}g_1h_1^{-1}\mathcal{E}(g_1-h_1) = 0, &\qquad (g_1-h_1)\mathcal{E}g_1h_1^{-1}\mathcal{E} = 0. \end{aligned}$$

These relations involve complex parameters γ, δ, and θ, and we sometimes use the notation $\mathsf{qw}\mathcal{B}_{m,n}(\gamma,\delta,\theta)$ for the algebra, although one parameter can be eliminated from the relations by renormalizing the generators. We write the relations in the present form for more convenient comparison with different choices in literature.

Remark 3.3.2. The algebra $\mathsf{qw}\mathcal{B}_{m,n}$ has a presentation by tangle diagrams, see [63].

Remark 3.3.3. In [52,57,58] the one-parameter walled Brauer algebra is discussed. It can be considered as a classical limit of quantum walled Brauer algebra $\mathsf{qw}\mathcal{B}_{m,n}$. To get this limit from the algebra with relations 3.3.1 we can do the following. By renormalization of the generators, parameter γ can be set to $\gamma = -1$. We introduce a complex parameter r:

$$\theta = -\delta^r$$

so that the relation reads $\mathcal{E}\mathcal{E} = -\frac{\delta^r-1}{\delta-1}\mathcal{E}$. Then we consider the limit $\delta \to 1$. The dependent on parameters algebra relations become

$$\begin{aligned} &g_i^2 = h_i^2 = 1, \\ &\mathcal{E}\mathcal{E} = -r\mathcal{E}. \end{aligned}$$

Such an algebra is called the (classical) walled Brauer algebra with (only one) parameter r. We use the notation $\mathsf{w}\mathcal{B}_{m,n}(-r)$ for it.

Theorem 3.3.4. *The generators defined in 3.2 satisfy the* $\mathsf{qw}\mathcal{B}_{m,n}$ *relations with the parameters*

$$\begin{aligned} &\gamma = -1, \\ &\delta = q^{-2}, \\ &\theta = -q^{-2}. \end{aligned} \tag{3.3}$$

Remark 3.3.5. By choice of normalization in matrices, the parameters γ and δ can be changed, however the relation

$$\theta = \frac{\delta}{\gamma} \tag{3.4}$$

remains invariant. This relation means that we consider a degenerate case in which the algebra becomes non-semisimple as we discuss below.

Corollary 3.3.6. *The endomorphism algebra of* $\mathcal{U}_q s\ell(2|1)$*-module* $\mathcal{T}_{m,n}$ *is isomorphic to the quotient of the algebra* $\mathsf{qw}\mathcal{B}_{m,n}$ *with special parameters* (3.3).

One can consider an algebra $\mathcal{U}_q s\ell(M|N)$ for arbitrary positive integers M and N. Let V and $V^\star$ be fundamental representation of $\mathcal{U}_q s\ell(M|N)$ and its dual. We let $\mathsf{X}_{m,n}^{M,N}$ denote the algebra of endomorphisms of $\mathcal{U}_q s\ell(M|N)$ on mixed tensor product $V^{\star\otimes m} \otimes V^{\otimes n}$. As was shown in [73] (see also [50,51,49,52]) there is a surjective homomorphism

$$\Psi_{m,n}^{M,N} : \mathsf{qwB}_{m,n}(\gamma = -1, \delta = q^{-2}, \theta = -q^{-2(M-N)}) \to \mathsf{X}_{m,n}^{M,N}. \tag{3.5}$$

Here the parameter q is the same as in the algebra $\mathcal{U}_q s\ell(M|N)$. In the classical limit we conclude that the algebra of endomorphisms of $s\ell(M|N)$ on mixed tensor product of its fundamental representations is a quotient of the algebra $\mathsf{wB}_{m,n}(r)$ with $r = N - M$. This is consistent with the results of [49,52] because classical algebras $\mathsf{wB}_{m,n}(r)$ and $\mathsf{wB}_{m,n}(-r)$ are isomorphic to each other. Indeed, the isomorphism is given by the formulas $g_i' = -g_i$, $h_j' = -h_j$ and $\mathcal{E}' = -\mathcal{E}$.

We note that for $N = 0$ the algebra $\mathsf{X}_{m,n}^{M,0}$ is semisimple and $\ker \Psi_{m,n}^{M,0}$ contains the whole radical of $\mathsf{qwB}_{m,n}$, see [54].

At the end of this section we formulate two statements important for the sequel.

Conjecture 3.4. *Representation categories of the algebra* $\mathsf{qwB}_{m,n}$ *with generic values of parameter* $\frac{\delta}{\gamma}$ *and of the (classical) walled Brauer algebra are equivalent as abelian categories.*

The walled Brauer algebra has quasi-hereditary structure, see [58]. According to our first conjecture (Conjecture 3.4) we suppose $\mathsf{qwB}_{m,n}$ with generic values of the parameter $\frac{\delta}{\gamma}$ to be also quasi-hereditary.

In the following sections we consider only the case $M = 2$, $N = 1$ and use the notation $\mathsf{X}_{m,n}$ for $\mathsf{X}_{m,n}^{2,1}$. The second important statement is (see also [74])

Conjecture 3.5. *The algebra* $\mathsf{X}_{m,n}$ *is quasi-hereditary.*[2]

4. Modules over $\mathsf{qwB}_{m,n}$ and $\mathsf{X}_{m,n}$

In this section we give the description of the $\mathsf{X}_{m,n}$-modules. We rely on the quasi-hereditary hypothesis 3.5 and borrow definitions from [58–60]. From these papers we use the notions of Specht module, simple head, projective cover and multiplicity.

In this section we describe Specht and simple modules for $\mathsf{qwB}_{m,n}$ and simple and projective modules for algebra $\mathsf{X}_{m,n}$.

4.1. $\mathsf{qwB}_{m,n}$ *Specht modules*

4.1.1. A finite integer sequence $\mu = (\mu_1, \mu_2, \dots \mu_r)$ is called a partition, if $\mu_1 \geqslant \mu_2 \geqslant \dots \mu_r > 0$.

A bipartition is a pair of partitions $\lambda = (\lambda^L, \lambda^R)$. Let Λ be the set of all bipartitions. For each integer $0 \leqslant f \leqslant \min(m, n)$, we set

$$\Lambda_{m,n}(f) = \{\lambda \in \Lambda \mid m - |\lambda^L| = n - |\lambda^R| = f\}, \tag{4.1}$$

where $|\lambda|$ is the sum of elements of a partition, and

$$\Lambda_{m,n} = \bigcup_{f=0}^{\min(m,n)} \Lambda_{m,n}(f). \tag{4.2}$$

[2] The conjecture about quasi-hereditary structure in the general case $\mathsf{X}_{m,n}^{M,N}$ can apparently be formulated but is beyond the scope of this paper.

The set $\Lambda_{m,n}$ is in bijective correspondence with the set of $\mathsf{qw}\mathcal{B}_{m,n}$ Specht modules [56]. We let $S(\lambda)$ denote the $\mathsf{qw}\mathcal{B}_{m,n}$-Specht module corresponding to the bipartition λ.

The following Theorem is given in [52].

Theorem 4.1.2. *For generic values of the* $\mathsf{qw}\mathcal{B}_{m,n}$ *parameters, each Specht module is simple, and the sets of Specht and simple modules coincide.*

4.2. Modules over $\mathsf{qw}\mathcal{B}_{m,n}$ *with special parameters*

We now consider the category of $\mathsf{qw}\mathcal{B}_{m,n}$ modules with the parameters related as in (3.4). The algebra is then nonsemisimple, and some of the Specht modules $S(\lambda)$ become reducible.

Let $D(\lambda)$ and $K(\lambda)$ be the simple head and the projective cover for $S(\lambda)$. Below we also use the notation $D\big[\lambda^L,\lambda^R\big]$ and $K\big[\lambda^L,\lambda^R\big]$ for $D((\lambda^L,\lambda^R))$ and $K((\lambda^L,\lambda^R))$ respectively.

In [57, Theorem 2.7] the full classification of simple modules over the walled Brauer algebra is given. Thus, assuming Conjecture 3.4 (but see also [56, Theorem 8.1]) we have the following

Lemma 4.2.1. *If* $\mathcal{E}\mathcal{E}\neq 0$*, the modules* $D(\lambda)$, $\lambda\in\Lambda_{m,n}$ *give a complete set of simple modules for the algebra* $\mathsf{qw}\mathcal{B}_{m,n}$.

The decomposition multiplicities $d_{\lambda,\mu}=\big[S(\mu):D(\lambda)\big]$ for the $S(\lambda)$-modules in terms of their simple subquotients are determined in [58]. Because of the quasi-hereditary structure of $\mathsf{qw}\mathcal{B}_{m,n}$ each projective module $K(\lambda)$ has a filtration by Specht modules. Let $\tilde{d}_{\lambda\mu}=\big[K(\lambda):S(\mu)\big]$ be the multiplicity of a given Specht module $S(\lambda)$ in the filtration; then, by the Brauer–Humphreys reciprocity (see [58] and references therein)

$$\tilde{d}_{\lambda\mu}=d_{\lambda\mu}. \tag{4.3}$$

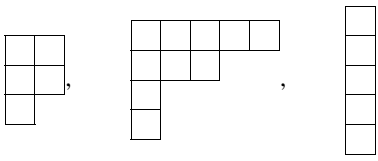
We use this statement to construct projective modules for $\mathsf{X}_{m,n}$ in the next subsection.

4.3. Modules in the decomposition of the mixed tensor product

As a $\mathsf{X}_{m,n}\boxtimes\mathcal{U}_q s\ell(2|1)$-bimodule, the mixed tensor product $\mathcal{T}_{m,n}$ decomposes into a direct sum of indecomposable bimodules.

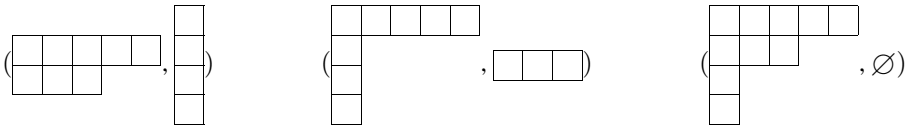
Definition 4.3.1. For non-negative integers p,q, a partition μ is called a (p,q)-hook partition if it doesn't contain a box in the $(p+1,q+1)$-position, i.e. $\mu_{p+1}<q+1$.

Some examples of $(2,1)$-hook partitions are



Definition 4.3.2 (*see [75]*). For non-negative integers p,q a bipartition $\lambda=(\lambda^L,\lambda^R)$ is called a (p,q)-cross bipartition if there exist non-negative integers p_1,p_2,q_1,q_2 such that λ^L is a (p_1,q_1)-hook partition, λ^R is a (p_2,q_2)-hook partition and $p_1+p_2\leqslant p$, $q_1+q_2\leqslant q$.

Some examples of $(2,1)$-cross bipartitions are



Let $\mathcal{C}\mathrm{r}_{m,n}$ be the subset of all $(2,1)$-cross bipartitions in $\Lambda_{m,n}$. Assuming Conjecture 3.4 and applying the statements from [52], [75] for $M=2$, $N=1$, we have

Proposition 4.3.3. *If $\lambda \in \mathcal{C}r_{m,n}$ then $\ker \Psi^{2,1}_{m,n}$ acts as zero on $D(\lambda)$. The modules $D(\lambda)$, $\lambda \in \mathcal{C}r_{m,n}$ give a complete set of simple $\mathsf{X}_{m,n}$-modules.*

Proposition 4.3.4. *Each $\mathsf{X}_{m,n}$-simple module $D(\lambda)$, $\lambda \in \mathcal{C}r_{m,n}$ occurs as a subquotient in the bimodule decomposition of $\mathcal{T}_{m,n}$.*

In the following we use notation $a = |m-n|$. For bipartitions from $\mathcal{C}\mathrm{r}_{m,n}$ we introduce the notation

for $m \geqslant n$:

$$
\begin{aligned}
&A^a_s = ((a,1^s),(s)), \quad a>0, \quad 0 \leqslant s \leqslant n,\\
&B^a_s = ((a,s),(1^s)), \quad a>0, \quad 1 \leqslant s \leqslant \min(a,n),\\
&C^a_s = ((s+1,a+1),(1^{s+2})), \quad a \leqslant s \leqslant n-2, \quad a \geqslant 0,
\end{aligned}
$$

for $m \leqslant n$:

$$
\begin{aligned}
&\hat{A}^a_s = ((s),(a,1^s)), \quad a>0, \quad 0 \leqslant s \leqslant m,\\
&A^0_0 = \hat{A}^0_0 = (\varnothing,\varnothing),\\
&\hat{B}^a_s = ((1^s),(a,s)), \quad a>0, \quad 1 \leqslant s \leqslant \min(a,m),\\
&\hat{C}^a_s = ((1^{s+2}),(s+1,a+1)), \quad a \leqslant s \leqslant m-2, \quad a \geqslant 0.
\end{aligned}
$$

We note that $B^a_1 = A^a_1$ and $\hat{C}^0_0 = C^0_0$.

For given m, n we define a subset $\mathcal{A}\mathrm{t}_{m,n}$ of bipartitions in $\mathcal{C}\mathrm{r}_{m,n}$ as

$$
\mathcal{A}\mathrm{t}_{m,n} = \begin{cases}
\{A^a_s | 0 \leqslant s \leqslant n\} \bigcup \{B^a_s | 2 \leqslant s \leqslant \min(a,n)\} \bigcup \{C^a_s | a \leqslant s \leqslant n-2\}, \\
\quad m > n, \\
\{A^0_0\} \bigcup \{C^0_s | 0 \leqslant s \leqslant n-2\} \bigcup \{\hat{C}^0_s | 1 \leqslant s \leqslant n-2\}, \\
\quad m = n, \\
\{\hat{A}^a_s | 0 \leqslant s \leqslant m\} \bigcup \{\hat{B}^a_s | 2 \leqslant s \leqslant \min(a,m)\} \bigcup \{\hat{C}^a_s | a \leqslant s \leqslant m-2\}, \\
\quad m < n.
\end{cases} \tag{4.4}
$$

We call these bipartitions atypical. If $\lambda \in \mathcal{A}\mathrm{t}_{m,n}$ we call corresponding modules $S(\lambda)$ and $D(\lambda)$ atypical also.

We define the operation $\hat{G}$ from the set of $\mathsf{qw}\mathcal{B}_{m,n}$-modules to the set of $\mathsf{qw}\mathcal{B}_{n,m}$-modules. The operation $\hat{G}$ acts on the simple $\mathsf{qw}\mathcal{B}_{m,n}$-module by the formula

$$
\hat{G}\Big(D\big[\lambda^L,\lambda^R\big]\Big) = D\big[\lambda^R,\lambda^L\big], \tag{4.5}
$$

i.e. it changes left and right partitions in a bipartition. We note that $\hat{G}A_s^a = \hat{A}_s^a$, and similarly for B_s^a, C_s^a. When applied to projective modules, the operation $\hat{G}$ acts on each simple subquotient by the formula (4.5) and does not change the structure of the Loewy graph. It is obvious that

$$K\left[\lambda^R, \lambda^L\right] = \hat{G}\Big(K\left[\lambda^L, \lambda^R\right]\Big). \tag{4.6}$$

The action of the algebra $\mathsf{X}_{m,n}$ on an arbitrary $\mathsf{qw}\mathcal{B}_{m,n}$-module is not defined in general. In particular, it is not defined on some $\mathsf{qw}\mathcal{B}_{m,n}$-Specht modules, that contain $D(\lambda')$, $\lambda' \notin \mathcal{C}\mathrm{r}_{m,n}$ as a subquotient. For $\lambda \in \mathcal{C}\mathrm{r}_{m,n}$ we define a Specht module over $\mathsf{X}_{m,n}$ (abusing notation we use the same symbol $S(\lambda)$ for it) as a factor of corresponding $\mathsf{qw}\mathcal{B}_{m,n}$-Specht module $S(\lambda)$ over all subquotients $D(\lambda')$ with $\lambda' \notin \mathcal{C}\mathrm{r}_{m,n}$.

Similarly we let $K(\lambda)$ denote the projective cover for $\mathsf{X}_{m,n}$-module $S(\lambda)$. This projective cover is a subquotient of $\mathsf{qw}\mathcal{B}_{m,n}$ projective module $K(\lambda)$.

Assuming the Conjecture 3.5, we have the equality of multiplicities $\tilde{d}_{\lambda,\mu} = d_{\lambda,\mu}$ for $\mathsf{X}_{m,n}$ in analogy with (4.3). Using [58] and Proposition 4.3.3, we have the following Theorem. We write down the structure of the Loewy graphs for $\mathsf{X}_{m,n}$-projective modules (analogously to the formulas (2.15)–(2.17) for $\mathcal{U}_q s\ell(2|1)$-projective modules). They are oriented graphs where arrows mean the action of the algebra $\mathsf{X}_{m,n}$. States from the subquotient at the beginning of an arrow are mapped to the states in the subquotient at the end of an arrow and (possibly) in the subquotients further the arrows. Investigation of $\mathbf{Ext^1}$ spaces for the algebra $\mathsf{X}_{m,n}$ and the detailed action of all $\mathsf{X}_{m,n}$-generators on projective modules are beyond the scope of this paper.

Theorem 4.3.5. *For $\lambda \in \mathcal{C}r_{m,n}$, $\lambda \notin \mathcal{A}t_{m,n}$, the projective module over $\mathsf{X}_{m,n}$ coincides with the simple module: $K(\lambda) = D(\lambda)$. For $\lambda \in \mathcal{A}t_{m,n}$, we have the following structure of projective modules over $\mathsf{X}_{m,n}$*

for $m > n$:

$$K(A_s^a) = \begin{array}{ccccc} & & D(A_s^a) & & \\ & \swarrow & & \searrow & \\ D(A_{s-1}^a) & & & & D(A_{s+1}^a) \\ & \searrow & & \swarrow & \\ & & D(A_s^a) & & \end{array}, \qquad 2 \leqslant s \leqslant n-1, \quad a \geqslant 1,$$

$$K(A_1^a) = \begin{array}{ccccc} & & D(A_1^a) & & \\ & \swarrow & \downarrow & \searrow & \\ D(A_2^a) & & D(A_0^a) & & D(B_2^a) \\ & \searrow & \downarrow & \swarrow & \\ & & D(A_1^a) & & \end{array}, \qquad a \geqslant 2, \quad n \geqslant 2,$$

$$K(A_1^1) = \quad \begin{array}{ccccc} & & D(A_1^1) & & \\ & \swarrow & \downarrow & \searrow & \\ D(A_2^1) & & D(A_0^1) & & D(C_1^1) \\ & \searrow & \downarrow & \swarrow & \\ & & D(A_1^1) & & \end{array} , \qquad n \geqslant 3,$$

$$K(A_n^a) = \quad \begin{array}{c} D(A_n^a) \\ \downarrow \\ D(A_{n-1}^a) \\ \downarrow \\ D(A_n^a) \end{array} , \qquad a \geqslant 1, \quad n \geqslant 1,$$

$$K(A_0^a) = \quad \begin{array}{c} D(A_0^a) \\ \downarrow \\ D(A_1^a) \end{array} , \qquad a \geqslant 1, \quad n \geqslant 1,$$

$$K(B_s^a) = \quad \begin{array}{ccc} & D(B_s^a) & \\ \swarrow & & \searrow \\ D(B_{s-1}^a) & & D(B_{s+1}^a) \\ \searrow & & \swarrow \\ & D(B_s^a) & \end{array} , \qquad 2 \leqslant s \leqslant \min(a, n) - 1 ,$$

$$K(B_a^a) = \quad \begin{array}{ccc} & D(B_a^a) & \\ \swarrow & & \searrow \\ D(B_{a-1}^a) & & D(C_a^a) \\ \searrow & & \swarrow \\ & D(B_a^a) & \end{array} , \qquad 2 \leqslant a \leqslant n - 2 ,$$

$$K(B_a^a) = \quad \begin{array}{c} D(B_a^a) \\ \downarrow \\ D(B_{a-1}^a) \\ \downarrow \\ D(B_a^a) \end{array} , \qquad a = n - 1, \quad n \geqslant 3,$$

$$K(B_n^a) = \begin{array}{c} D(B_n^a) \\ \downarrow \\ D(B_{n-1}^a) \\ \downarrow \\ D(B_n^a) \end{array}, \qquad a \geqslant n, \quad n \geqslant 2,$$

$$K(C_s^a) = \begin{array}{ccc} & D(C_s^a) & \\ \swarrow & & \searrow \\ D(C_{s-1}^a) & & D(C_{s+1}^a) \\ \searrow & & \swarrow \\ & D(C_s^a) & \end{array}, \qquad a+1 \leqslant s \leqslant n-3, \quad a \geqslant 1,$$

$$K(C_a^a) = \begin{array}{ccc} & D(C_a^a) & \\ \swarrow & & \searrow \\ D(B_a^a) & & D(C_{a+1}^a) \\ \searrow & & \swarrow \\ & D(C_a^a) & \end{array}, \qquad 2 \leqslant a \leqslant n-3,$$

$$K(C_1^1) = \begin{array}{ccc} & D(C_1^1) & \\ \swarrow & & \searrow \\ D(A_1^1) & & D(C_2^1) \\ \searrow & & \swarrow \\ & D(C_1^1) & \end{array}, \qquad n \geqslant 4,$$

$$K(C_{n-2}^a) = \begin{array}{c} D(C_{n-2}^a) \\ \downarrow \\ D(C_{n-3}^a) \\ \downarrow \\ D(C_{n-2}^a) \end{array}, \qquad 1 \leqslant a \leqslant n-3,$$

$$K(C^{n-2}_{n-2}) = \begin{array}{c} D(C^{n-2}_{n-2}) \\ \downarrow \\ D(B^{n-2}_{n-2}) \\ \downarrow \\ D(C^{n-2}_{n-2}) \end{array}, \qquad n \geqslant 3,$$

for $m = n$:

$$K(A^0_0) = \begin{array}{c} D(A^0_0) \\ \downarrow \\ D(C^0_0) \end{array}, \qquad n \geqslant 2,$$

$$K(C^0_0) = \begin{array}{ccccc} & & D(C^0_0) & & \\ & \swarrow & \downarrow & \searrow & \\ D(\hat{C}^0_1) & & D(A^0_0) & & D(C^0_1) \\ & \searrow & \downarrow & \swarrow & \\ & & D(C^0_0) & & \end{array}, \qquad n \geqslant 3,$$

$$K(C^0_s) = \begin{array}{ccccc} & & D(C^0_s) & & \\ & \swarrow & & \searrow & \\ D(C^0_{s-1}) & & & & D(C^0_{s+1}) \\ & \searrow & & \swarrow & \\ & & D(C^0_s) & & \end{array}, \qquad 1 \leqslant s \leqslant n-3,$$

$$K(C^0_{n-2}) = \begin{array}{c} D(C^0_{n-2}) \\ \downarrow \\ D(C^0_{n-3}) \\ \downarrow \\ D(C^a_{n-2}) \end{array}, \qquad n \geqslant 3.$$

Structure of projective modules $K(\hat{C}^0_s), 0 \leqslant s \leqslant n-2$ for $m = n$ and all projective modules for $m < n$ can be obtained from this using the formula (4.6).

We note that the Loewy length of the projective modules coincide with the Loewy length given in Corollary 4.2 in [76].

4.4. The restriction functors

4.4.1. There are two natural embeddings between quantum walled Brauer algebras (see [57])

$$\mathsf{qw}\mathcal{B}_{m-1,n} \to \mathsf{qw}\mathcal{B}_{m,n}, \qquad \mathsf{qw}\mathcal{B}_{m,n-1} \to \mathsf{qw}\mathcal{B}_{m,n}. \tag{4.7}$$

The first embedding acts by identification of the corresponding generators $\mathcal{E}$, $g_1, g_2, \dots g_{m-2}$, $h_1, h_2, \dots h_{n-1}$. The second embedding acts by identification of the generators $\mathcal{E}$, $g_1, g_2, \dots g_{m-1}$, $h_1, h_2, \dots h_{n-2}$. These two maps induce two restriction functors $\mathrm{res}^{m,n}_{m-1,n}$ and $\mathrm{res}^{m,n}_{m,n-1}$ from the category of $\mathsf{qw}\mathcal{B}_{m,n}$-modules to the categories of $\mathsf{qw}\mathcal{B}_{m-1,n}$ and $\mathsf{qw}\mathcal{B}_{m,n-1}$-modules respectively.

Let $\mathrm{add}(\mu)$ be the set of boxes for a partition μ, which can be added singly to μ such that the result $\mu + \square$ is a partition. Let $\mathrm{rem}(\mu)$ be a set of boxes which can be removed from μ such that $\mu/\square$ is a partition.

In what follows the sign $\biguplus$ denotes the non-direct sum of modules. Following [57], where the classical case $q = 1$ is considered, we have for modules over $\mathsf{qw}\mathcal{B}_{m,n}$.

Proposition 4.4.2. *For $\lambda \in \Lambda_{m,n}(f)$ with $n \geqslant 1$ we have*

$$res^{m,n}_{m,n-1} S(\lambda) = \biguplus_{\square \in rem(\lambda^R)} S(\lambda^L, \lambda^R - \square), \quad for\ f = 0,$$

$$res^{m,n}_{m,n-1} S(\lambda) = \biguplus_{\square \in add(\lambda^L)} S(\lambda^L + \square, \lambda^R) \biguplus \biguplus_{\square \in rem(\lambda^R)} S(\lambda^L, \lambda^R - \square), \quad for\ f > 0.$$

This statement is valid for the algebra $\mathsf{qw}\mathcal{B}_{m,n}$ with either generic or special parameters. For $\mathsf{qw}\mathcal{B}_{m,n}$ with generic parameters all $\biguplus$ become direct sums.

As a consequence of the previous statement and Proposition 4.3.3 we have for modules over $\mathsf{X}_{m,n}$.

Proposition 4.4.3. *For $\lambda \in \Lambda_{m,n}(f) \bigcap \mathcal{C}r_{m,n}$ with $n \geqslant 1$ we have*

$$res^{m,n}_{m,n-1} S(\lambda) = \bigoplus_{\square \in rem(\lambda^R)} S(\lambda^L, \lambda^R - \square), \quad for\ f = 0,$$

$$res^{m,n}_{m,n-1} S(\lambda) = \biguplus_{\square \in add(\lambda^L), (\lambda^L + \square, \lambda^R) \in \mathcal{C}r_{m,n-1}} S(\lambda^L + \square, \lambda^R) \biguplus \biguplus_{\square \in rem(\lambda^R)} S(\lambda^L, \lambda^R - \square),$$

$for\ f > 0.$

We formulate an important Conjecture about restriction of the projective modules. The statements 4.4.5 and 4.4.6 formulated as theorems are based on the Conjecture.

Conjecture 4.4.4. *Restriction for projective module $K(\lambda)$ over algebra $\mathsf{X}_{m,n}$ is a sum of projective modules.*

Theorem 4.4.5. *Consider $n \geqslant 1$. For $\lambda \in \mathcal{A}t_{m,n}$ the restrictions for projective modules $K(\lambda)$ over the algebra $\mathsf{X}_{m,n}$ are*

$$res^{m,n}_{m,n-1} K(A^a_s) = K(A^{a+1}_s) \oplus D[(a, 1^{s+1}), (s)] \oplus 2D[(a, 1^s), (s-1)]$$
$$\oplus D[(a, 1^{s-1}), (s-2)], \quad 2 \leqslant s \leqslant n-1, \quad a \geqslant 1,$$

$$res^{m,n}_{m,n-1} K(A_1^a) = K(A_1^{a+1}) \oplus D[(a,1^2),(1)] \oplus 2D[(a,1),\varnothing] \oplus D[(a,2),(1)],$$
$$a \geqslant 2, n \geqslant 2,$$
$$res^{m,n}_{m,n-1} K(A_1^1) = K(A_1^2) \oplus D[(1^3),(1)] \oplus 2D[(1^2),\varnothing], \quad n \geqslant 2,$$
$$res^{m,n}_{m,n-1} K(A_n^a) = D(A_{n-1}^{a+1}) \oplus 2D[(a,1^n),(n-1)] \oplus D[(a,1^{n-1}),(n-2)], \quad n \geqslant 2,$$
$$res^{m,n}_{m,n-1} K(A_0^a) = K(A_0^{a+1}) \oplus D[(a,1),\varnothing], \quad a \geqslant 1, \quad n \geqslant 1,$$
$$res^{m,n}_{m,n-1} K(A_0^0) = K(A_0^1), \quad n \geqslant 1,$$

$$res^{m,n}_{m,n-1} K(B_s^a) = K(B_s^{a+1}) \oplus D[(a,s+1),(1^s)] \oplus 2D[(a,s),(1^{s-1})]$$
$$\oplus D[(a,s-1),(1^{s-2})], \quad 2 \leqslant s \leqslant \min(a,n)-1,$$
$$res^{m,n}_{m,n-1} K(B_a^a) = K(B_a^{a+1}) \oplus 2D[(a,a),(1^{a-1})] \oplus D[(a,a-1),(1^{a-2})],$$
$$2 \leqslant a \leqslant n-1,$$
$$res^{m,n}_{m,n-1} K(B_n^a) = D(B_{n-1}^{a+1}) \oplus 2D[(a,n),(1^{n-1})] \oplus D[(a,n-1),(1^{n-2})], \ 2 \leqslant n \leqslant a,$$

$$res^{m,n}_{m,n-1} K(C_s^a) = K(C_s^{a+1}) \oplus D[(s+2,a+1),(1^{s+2})] \oplus 2D[(s+1,a+1),(1^{s+1})]$$
$$\oplus D[(s,a+1),(1^s)], \quad a+2 \leqslant s \leqslant n-3,$$
$$res^{m,n}_{m,n-1} K(C_a^a) = K(B_{a+1}^{a+1}) \oplus D[(a+2,a+1),(1^{a+2})] \oplus D[(a,a),(1^{a-1})],$$
$$2 \leqslant a \leqslant n-3,$$
$$res^{m,n}_{m,n-1} K(C_{a+1}^a) = K(C_{a+1}^{a+1}) \oplus D[(a+3,a+1),(1^{a+3})]$$
$$\oplus 2D[(a+2,a+1),(1^{a+2})], \quad a \leqslant n-4,$$
$$res^{m,n}_{m,n-1} K(C_{n-2}^a) = D(C_{n-3}^{a+1}) \oplus 2D[(n-1,a+1),(1^{n-1})]$$
$$\oplus D[(n-2,a+1),(1^{n-2})], \quad a \leqslant n-4,$$
$$res^{m,n}_{m,n-1} K(C_{n-2}^{n-3}) = D(B_{n-2}^{n-2}) \oplus 2D[(n-1,n-2),(1^{n-1})], \quad n \geqslant 3,$$
$$res^{m,n}_{m,n-1} K(C_{n-2}^{n-2}) = K(B_{n-1}^{n-1}) \oplus D[(n-2,n-2),(1^{n-3})], \quad n \geqslant 3,$$
$$res^{m,n}_{m,n-1} K(C_1^1) = K(B_2^2) \oplus D[(3,2),(1^3)] \oplus D[(1^2),\varnothing], \quad n \geqslant 4,$$
$$res^{m,n}_{m,n-1} K(C_0^0) = K(A_1^1) \oplus D[(2,1),(1^2)] \oplus D[(1^3),(1^2)], \quad n \geqslant 3,$$

$$res^{m,n}_{m,n-1} K(\hat{A}_s^a) = K(\hat{A}_s^{a-1}) \oplus D[(s+1),(a,1^s)] \oplus 2D[(s),(a,1^{s-1})]$$
$$\oplus D[(s-1),(a,1^{s-2})], \quad 2 \leqslant s \leqslant m-1, \quad a \geqslant 2,$$
$$res^{m,n}_{m,n-1} K(\hat{A}_1^a) = K(\hat{A}_1^{a-1}) \oplus D[(2),(a,1)] \oplus 2D[(1),(a)] \oplus D[(1^2),(a,1)],$$
$$a \geqslant 2, \quad m \geqslant 2,$$
$$res^{m,n}_{m,n-1} K(\hat{A}_m^a) = K(\hat{A}_m^{a-1}) \oplus 2D[(m),(a,1^{m-1})] \oplus D[(m-1),(a,1^{m-2})],$$
$$m \geqslant 2, \quad a \geqslant 2,$$
$$res^{m,n}_{m,n-1} K(\hat{A}_s^1) = K(C_{s-1}^0) \oplus D[(s+1),(1^{s+1})]$$
$$\oplus 2D[(s),(1^s)] \oplus D[(s-1),(1^{s-1})], \quad 2 \leqslant s < m,$$
$$res^{m,n}_{m,n-1} K(\hat{A}_m^1) = D(C_{m-2}^0) \oplus 2D[(m),(1^m)] \oplus D[(m-1),(1^{m-1})], \quad m \geqslant 2,$$
$$res^{m,n}_{m,n-1} K(\hat{A}_1^1) = K(C_0^0) \oplus D[(2),(1^2)] \oplus 2D[(1),(1)], \quad m \geqslant 2,$$
$$res^{m,n}_{m,n-1} K(\hat{A}_0^a) = K(\hat{A}_0^{a-1}) \oplus D[(1),(a)], \quad a \geqslant 1, \quad m \geqslant 1,$$

$$
\begin{aligned}
&res^{m,n}_{m,n-1} K(\hat{B}^a_s) = K(\hat{B}^{a-1}_s) \oplus D[(1^{s+1}), (a,s)] \oplus 2D[(1^s), (a,s-1)] \oplus D[(1^{s-1}), \\
&\quad (a,s-2)], \quad 2 \leqslant s \leqslant \min(a,m) - 1, \\
&res^{m,n}_{m,n-1} K(\hat{B}^a_a) = K(\hat{C}^{a-1}_{a-1}) \oplus 2D[(1^a), (a,a-1)] \oplus D[(1^{a-1}), \\
&\quad (a,a-2)], \quad 2 \leqslant a \leqslant m-1, \\
&res^{m,n}_{m,n-1} K(\hat{B}^a_m) = K(\hat{B}^{a-1}_m) \oplus 2D[(1^m), (a,m-1)] \oplus D[(1^{m-1}), (a,m-2)], \\
&\quad 2 \leqslant m < a, \\
&res^{m,n}_{m,n-1} K(\hat{B}^m_m) = D(\hat{B}^{m-1}_{m-1}) \oplus 2D[(1^m), (m,m-1)] \oplus D[(1^{m-1}), (m,m-2)], \\
&\quad m \geqslant 2,
\end{aligned}
$$

$$
\begin{aligned}
&res^{m,n}_{m,n-1} K(\hat{C}^a_s) = K(\hat{C}^{a-1}_s) \oplus D[(1^{s+3}), (s+1,a+1)] \oplus 2D[(1^{s+2}), (s,a+1)] \\
&\quad \oplus D[(1^{s+1}), (s-1,a+1)], \quad a+2 \leqslant s \leqslant m-3, \quad a \geqslant 1, \\
&res^{m,n}_{m,n-1} K(\hat{C}^0_s) = K(A^1_{s+1}) \oplus D[(1^{s+3}), (s+1,1)] \oplus 2D[(1^{s+2}), (s,1)] \\
&\quad \oplus D[(1^{s+1}), (s-1,1)], \quad 2 \leqslant s \leqslant m-3, \\
&res^{m,n}_{m,n-1} K(\hat{C}^a_a) = K(\hat{C}^{a-1}_a) \oplus D[(1^{a+3}), (a+1,a+1)] \oplus D[(1^a), (a,a-1)], \\
&\quad 1 \leqslant a \leqslant m-3, \\
&res^{m,n}_{m,n-1} K(\hat{C}^a_{a+1}) = K(\hat{C}^{a-1}_{a+1}) \oplus D[(1^{a+4}), (a+2,a+1)] \oplus 2D[(1^{a+3}), \\
&\quad (a+1,a+1)], \quad 1 \leqslant a \leqslant m-4, \\
&res^{m,n}_{m,n-1} K(\hat{C}^a_{m-2}) = K(\hat{C}^{a-1}_{m-2}) \oplus 2D[(1^m), (m-2,a+1)] \oplus D[(1^{m-1}), \\
&\quad (m-3,a+1)], \quad 1 \leqslant a \leqslant m-4, \\
&res^{m,n}_{m,n-1} K(\hat{C}^0_1) = K(A^1_2) \oplus D[(1^4), (2,1)] \oplus 2D[(1^3), (1,1)], \quad m \geqslant 4, \\
&res^{m,n}_{m,n-1} K(\hat{C}^0_{m-2}) = K(A^1_{m-1}) \oplus 2D[(1^m), (m-2,1)] \oplus D[(1^{m-1}), (m-3,1)], \\
&\quad m \geqslant 4, \\
&res^{m,n}_{m,n-1} K(\hat{C}^{m-3}_{m-2}) = K(\hat{C}^{m-4}_{m-2}) \oplus 2D[(1^m), (m-2,m-2)], \quad m \geqslant 4, \\
&res^{m,n}_{m,n-1} K(\hat{C}^{m-2}_{m-2}) = K(\hat{C}^{m-3}_{m-2}) \oplus D[(1^{m-2}), (m-2,m-3)], \quad m \geqslant 3.
\end{aligned}
$$

where we imply $(0) = (1^0) = \varnothing$ *and* $(s,0) = (s)$,

Proof. We discuss the case $K(A^a_s)$ for $2 \leqslant s \leqslant n-1$, $a \geqslant 1$. Other cases are similar. The projective module $K(A^a_s)$ has a filtration by two atypical Specht modules, so one can write it as a non-direct sum

$$K(A^a_s) = S(A^a_s) \biguplus S(A^a_{s-1}).$$

Applying Proposition 4.4.3 one obtains the sum of simple and atypical Specht modules:

$$
\begin{aligned}
\mathrm{res}^{m,n}_{m,n-1} K(A^a_s) &= \mathrm{res}^{m,n}_{m,n-1} \left(S(A^a_s) \biguplus S(A^a_{s-1}) \right) \\
&= S(A^{a+1}_s) \biguplus S[(a,1^{s+1}), (s)] \biguplus S[(a,1^s), (s-1)] \\
&\quad \biguplus S(A^{a+1}_{s-1}) \biguplus S[(a,1^s), (s-1)] \biguplus S[(a,1^{s-1}), (s-2)].
\end{aligned}
$$

In this sum only two modules are atypical, other modules are simple

$$\operatorname{res}^{m,n}_{m,n-1} K(A^a_s) = S(A^{a+1}_s) \biguplus S(A^{a+1}_{s-1}) \biguplus D[(a,1^{s+1}),(s)] \biguplus 2D[(a,1^s),(s-1)] \biguplus D[(a,1^{s-1}),(s-2)].$$

These two atypical Specht modules are glued uniquely into a projective module, thus

$$\operatorname{res}^{m,n}_{m,n-1} K(A^a_s) = K(A^{a+1}_s) \oplus D[(a,1^{s+1}),(s)] \oplus 2D[(a,1^s),(s-1)] \oplus D[(a,1^{s-1}),(s-2)]. \qquad \square$$

To formulate the next theorem we introduce notation $a' = |m-n+1|$.

Theorem 4.4.6. *Consider $\lambda \in \Lambda_{m,n}(f) \bigcap \mathcal{C}r_{m,n}$ for $n \geqslant 1$. The restrictions for simple modules $D(\lambda)$ over the algebra $\mathsf{X}_{m,n}$ are*

For $\lambda \in \mathcal{A}t_{m,n}$:

$$\begin{aligned}
&res^{m,n}_{m,n-1} D(A^a_s) = D(A^{a+1}_s) \oplus D[(a,1^s),(s-1)], \quad a \geqslant 1, \quad 1 \leqslant s \leqslant n-1,\\
&res^{m,n}_{m,n-1} D(A^a_n) = D[(a,1^n),(n-1)], \quad a \geqslant 1, \quad n \geqslant 1,\\
&res^{m,n}_{m,n-1} D(A^a_0) = D(A^{a+1}_0), \quad a \geqslant 0, \quad n \geqslant 0,\\
&res^{m,n}_{m,n-1} D(B^a_s) = D(B^{a+1}_s) \oplus D[(a,s),(1^{s-1})], \quad 1 \leqslant s \leqslant n-1, \quad s \leqslant a,\\
&res^{m,n}_{m,n-1} D(B^a_n) = D[(a,n),(1^{n-1})], \quad 1 \leqslant n \leqslant a,\\
&res^{m,n}_{m,n-1} D(C^a_s) = D(C^{a+1}_s) \oplus D[(s+1,a+1),(1^{s+1})], \quad a+1 \leqslant s \leqslant n-3, \quad a \geqslant 0,\\
&res^{m,n}_{m,n-1} D(C^a_{n-2}) = D[(n-1,a+1),(1^{n-1})], \quad 0 \leqslant a \leqslant n-3,\\
&res^{m,n}_{m,n-1} D(C^a_a) = D(B^{a+1}_{a+1}), \quad 0 \leqslant a \leqslant n-2,
\end{aligned}$$

$$\begin{aligned}
&res^{m,n}_{m,n-1} D(\hat{A}^a_s) = D(\hat{A}^{a-1}_s) \oplus D[(s),(a,1^{s-1})], \quad 1 \leqslant s \leqslant m, \quad a \geqslant 2,\\
&res^{m,n}_{m,n-1} D(\hat{A}^a_0) = D(\hat{A}^{a-1}_0), \quad a \geqslant 1, \quad m \geqslant 0,\\
&res^{m,n}_{m,n-1} D(\hat{A}^1_s) = D(C^0_{s-1}) \oplus D[(s),(1^s)], \quad 1 \leqslant s \leqslant m-1,\\
&res^{m,n}_{m,n-1} D(\hat{A}^1_m) = D[(m),(1^m)], \quad m \geqslant 1,\\
&res^{m,n}_{m,n-1} D(\hat{B}^a_s) = D(\hat{B}^{a-1}_s) \oplus D[(1^s),(a,s-1)], \quad 2 \leqslant s \leqslant m, \quad s < a,\\
&res^{m,n}_{m,n-1} D(\hat{B}^a_a) = D(\hat{C}^{a-1}_{a-1}) \oplus D[(1^a),(a,a-1)], \quad 2 \leqslant a \leqslant m-1,\\
&res^{m,n}_{m,n-1} D(\hat{B}^m_m) = D[(1^m),(m,m-1)], \quad 1 \leqslant m,\\
&res^{m,n}_{m,n-1} D(\hat{C}^a_s) = D(\hat{C}^{a-1}_s) \oplus D[(1^{s+2}),(s,a+1)], \quad a+1 \leqslant s \leqslant m-2, \quad a \geqslant 1,\\
&res^{m,n}_{m,n-1} D(\hat{C}^a_a) = D(\hat{C}^{a-1}_a), \quad 1 \leqslant a \leqslant m-2,\\
&res^{m,n}_{m,n-1} D(\hat{C}^0_s) = D(A^1_{s+1}) \oplus D[(1^{s+2}),(s,1)], \quad 1 \leqslant s \leqslant m-2.
\end{aligned}$$

For $\lambda \notin \mathcal{A}t_{m,n}$ first we list all exceptional cases (the generic rule will be given below):

$$\begin{aligned}
&res^{m,n}_{m,n-1} D[(a',1^{s-1}),(s)] = K(A^{a'}_s) \oplus D[(a'+1,1^{s-1}),(s)], \quad 1 \leqslant s \leqslant n-1,\\
&\quad a' \geqslant 1,\\
&res^{m,n}_{m,n-1} D[(a',s),(1^{s+1})] = K(B^{a'}_{s+1}) \oplus D[(a'+1,s),(1^{s+1})], \quad 1 \leqslant s \leqslant a'-1,\\
&\quad s \leqslant n-2,
\end{aligned}$$

$$res^{m,n}_{m,n-1} D[(s,a'+1),(1^{s+2})] = K(C^{a'}_s) \oplus D[(s,a'+2),(1^{s+2})],$$
$$a'+2 \leqslant s \leqslant n-3,$$
$$res^{m,n}_{m,n-1} D[(a'+1,a'+1),(1^{a'+3})] = K(C^{a'}_{a'+1}), \quad a' \leqslant n-4,$$
$$res^{m,n}_{m,n-1} D[(s),(a',1^{s+1})] = K(\hat{A}^{a'}_{s+1}) \oplus D[(s),(a'-1,1^{s+1})], \quad 0 \leqslant s \leqslant m-1,$$
$$a' \geqslant 2,$$
$$res^{m,n}_{m,n-1} D[(s),(1^{s+2})] = K(\hat{A}^{1}_{s+1}) \oplus D[(s,1),(1^{s+2})], \quad 1 \leqslant s \leqslant m-1,$$
$$res^{m,n}_{m,n-1} D[(1^{s-1}),(a',s)] = K(\hat{B}^{a'}_{s}) \oplus D[(1^{s-1}),(a'-1,s)], \quad 2 \leqslant s \leqslant a'-1,$$
$$s \leqslant m,$$
$$res^{m,n}_{m,n-1} D[(1^{a'-1}),(a',a')] = K(\hat{B}^{a'}_{a'}), \quad 1 \leqslant a' \leqslant m,$$
$$res^{m,n}_{m,n-1} D[(1^{s}),(s,a'+1)] = K(\hat{C}^{a'}_{s-1}) \oplus D[(1^{s}),(s,a')], \quad a'+1 \leqslant s \leqslant m-1,$$

where we imply $(0) = (1^0) = \varnothing$ *and* $(s,0) = (s)$.

For $\lambda \notin \mathcal{A}t_{m,n}$ *the generic rule is:*

for $f = 0$

$$res^{m,n}_{m,n-1} D(\lambda) = \sum_{\square \in rem(\lambda^R)} D(\lambda^L, \lambda^R - \square),$$

for $f > 0$

$$res^{m,n}_{m,n-1} D(\lambda) = \bigoplus_{\square \in add(\lambda^L),(\lambda^L+\square,\lambda^R) \in \mathcal{C}r_{m,n-1}} D(\lambda^L+\square,\lambda^R) \oplus \bigoplus_{\square \in rem(\lambda^R)} D(\lambda^L,\lambda^R-\square).$$

Proof. If $\lambda \notin \mathcal{A}\mathrm{t}_{m,n}$ then $D(\lambda) = S(\lambda)$, and the proof follows from 4.4.3 similarly to the proof of Theorem 4.4.5.

Now we consider $\lambda \in \mathcal{A}\mathrm{t}_{m,n}$. We discuss only $D(A^a_s)$ for $a \geqslant 1$, $1 \leqslant s \leqslant n-1$, other cases are similar. We prove that

$$\mathrm{res}^{m,n}_{m,n-1} D(A^a_s) = D(A^{a+1}_s) \oplus D\big[(a,1^s),(s-1)\big], \quad a \geqslant 1, s \leqslant n-1,$$

by induction on s. First, we prove the induction base for $s = n-1$, then we check the induction step from s to $s-1$. The $\mathsf{X}_{m,n}$-module $S(A^a_n)$ is simple: $S(A^a_n) = D(A^a_n)$, so we have from 4.4.3

$$\mathrm{res}^{m,n}_{m,n-1} D(A^a_n) = \mathrm{res}^{m,n}_{m,n-1} S(A^a_n) = S\big[(a,1^n),(n-1)\big] = D\big[(a,1^n),(n-1)\big]. \tag{4.8}$$

According to 4.4.3 we have for $s < n$

$$\mathrm{res}^{m,n}_{m,n-1} S(A^a_s) = S(A^{a+1}_s) \oplus D\big[(a,1^{s+1}),(s)\big] \oplus D\big[(a,1^s),(s-1)\big]. \tag{4.9}$$

We write $\mathsf{X}_{m,n}$-Specht modules as a non-direct sum $S(A^a_s) = D(A^a_s) \biguplus D(A^a_{s+1})$ for $s < n$. The $\mathsf{X}_{m,n-1}$-module $S(A^{a+1}_{n-1}) = D(A^{a+1}_{n-1})$, so from (4.9) for $s = n-1$ we get

$$\mathrm{res}^{m,n}_{m,n-1} \Big(D(A^a_{n-1}) \biguplus D(A^a_n) \Big)$$
$$= D(A^{a+1}_{n-1}) \biguplus D\big[(a,1^n),(n-1)\big] \biguplus D\big[(a,1^{n-1}),(n-2)\big].$$

Now having in mind (4.8) we get the induction base

$$\mathrm{res}^{m,n}_{m,n-1} D(A^a_{n-1}) = D(A^{a+1}_{n-1}) \biguplus D\big[(a, 1^{n-1}), (n-2)\big].$$

We also note that $\mathsf{X}_{m,n-1}$ module $S(A^{a+1}_s) = D(A^{a+1}_s) \biguplus D(A^{a+1}_{s+1})$ for $s < n-1$, so from (4.9) we get

$$\mathrm{res}^{m,n}_{m,n-1} \Big(D(A^a_s) \biguplus D(A^a_{s+1}) \Big)$$
$$= D(A^{a+1}_s) \biguplus D(A^{a+1}_{s+1}) \biguplus D\big[(a, 1^s), (s-1)\big] \biguplus D\big[(a, 1^{s+1}), (s)\big],$$

and now the induction step is straightforward. □

Remark 4.4.7. The second restriction functor $\mathrm{res}^{m,n}_{m-1,n}$ can be calculated from the first one. Actually

$$\mathrm{res}^{m,n}_{m-1,n} K(\lambda) = \hat{G}\, \mathrm{res}^{n,m}_{n,m-1} \hat{G} K(\lambda), \tag{4.10}$$
$$\mathrm{res}^{m,n}_{m-1,n} D(\lambda) = \hat{G}\, \mathrm{res}^{n,m}_{n,m-1} \hat{G} D(\lambda). \tag{4.11}$$

We can also make generalization to the $\mathsf{qw}\mathcal{B}_{m,n}$ modules.

Conjecture 4.4.8. *Consider the algebra $\mathsf{qw}\mathcal{B}_{m,n}$ with special parameter $\theta = -(-\frac{\delta}{\gamma})^{M-N}$ for $M \neq N$. Let $\lambda \in \Lambda_{m,n}$ be an (M,N)-cross bipartition, then $res^{m,n}_{m,n-1} D(\lambda)$ contains only subquotients $D(\lambda')$ for which $\lambda' \in \Lambda_{m,n-1}$ is an (M,N)-cross bipartition.*

In other words, the restriction functor for $\mathsf{qw}\mathcal{B}_{m,n}$ with special parameters preserves the class of all (M,N)-cross bipartitions. We note that the case $M = N$ requires additional investigation.

In particular we have the next important consequence for $M = 2,\ N = 1$.

Conjecture 4.4.9. *For $\lambda \in \mathcal{C}r_{m,n}$ the restrictions $res^{m,n}_{m,n-1} D(\lambda)$ for simple modules over $\mathsf{qw}\mathcal{B}_{m,n}$ with $\theta = \frac{\delta}{\gamma}$ are explicitly given by the formulas from Theorem 4.4.6 without any changes.*

This conjecture was directly checked for all $\mathsf{qw}\mathcal{B}_{m,n}$-modules whenever $m + n \leqslant 8$.

5. The mixed tensor product as a bimodule

We introduce new notation in order to simplify the formula for the bimodule decomposition.

5.1. Notation

We introduce the notation $\bar{\mathcal{Z}}^p_{t,r}$ for simple $\mathcal{U}_q s\ell(2|1)$ modules:

$$\bar{\mathcal{Z}}^p_{t,r} = \mathcal{Z}^{1,(-1)^p}_{t+r,r}, \quad r \neq 0,$$
$$\bar{\mathcal{Z}}^p_{t,0} = \mathcal{Z}^{1,(-1)^{p+1}}_{t+1,0}, \quad t \geqslant 0.$$

We also introduce the notation $\bar{\mathcal{R}}^p_{t,r}$ for projective covers of atypical modules $\bar{\mathcal{Z}}^p_{t,r}$. Namely,

$$\bar{\mathcal{R}}^p_{0,r} = \mathcal{R}^{1,(-1)^p}_{r,r}, \quad r \geqslant 1,$$
$$\bar{\mathcal{R}}^p_{t,0} = \mathcal{R}^{1,(-1)^{p+1}}_{t+1,0}, \quad t \geqslant 0.$$

Typical modules $\bar{\mathcal{Z}}^p_{t,r}$ coincide with their projective covers, so we do not introduce any new notation for them. We rewrite the formulas (2.15)–(2.17) in the new notation:

$$\begin{array}{ccc} & \bar{\mathcal{Z}}^p_{t,0} & \\ \swarrow & & \searrow \\ \bar{\mathcal{R}}^p_{t,0} = \bar{\mathcal{Z}}^{p+1}_{t+1,0} & & \bar{\mathcal{Z}}^{p-1}_{t-1,0}, \\ \searrow & & \swarrow \\ & \bar{\mathcal{Z}}^p_{t,0} & \end{array} \qquad \begin{array}{ccc} & \bar{\mathcal{Z}}^p_{0,t} & \\ \swarrow & & \searrow \\ \bar{\mathcal{R}}^p_{0,t} = \bar{\mathcal{Z}}^{p+1}_{0,t+1} & & \bar{\mathcal{Z}}^{p-1}_{0,t-1}, \\ \searrow & & \swarrow \\ & \bar{\mathcal{Z}}^p_{0,t} & \end{array} \qquad t \geqslant 1, \tag{5.1}$$

and the exceptional case is

$$\begin{array}{ccc} & \bar{\mathcal{Z}}^p_{0,0} & \\ \swarrow & & \searrow \\ \bar{\mathcal{R}}^p_{0,0} = \bar{\mathcal{Z}}^{p+1}_{1,0} & & \bar{\mathcal{Z}}^{p-1}_{0,1}. \\ \searrow & & \swarrow \\ & \bar{\mathcal{Z}}^p_{0,0} & \end{array} \tag{5.2}$$

Then the dimensions are:

$$\begin{aligned} \dim \bar{\mathcal{R}}^p_{r,0} = \dim \bar{\mathcal{R}}^p_{0,r} = 8r + 4, \quad r > 0, \\ \dim \bar{\mathcal{R}}^p_{0,0} = 8. \end{aligned}$$

5.2. The bimodule is a direct sum of subbimodules

$$\mathcal{T}_{m,n} = T^s_{m,n} \oplus T^{\mathrm{at}}_{m,n}, \tag{5.3}$$

where the $T^s_{m,n}$ part is the direct sum of simple $\mathsf{X}_{m,n} \boxtimes \mathcal{U}_q s\ell(2|1)$-bimodules, and $T^{\mathrm{at}}_{m,n}$ is an indecomposable $\mathsf{X}_{m,n} \boxtimes \mathcal{U}_q s\ell(2|1)$-bimodule. Each subquotient in $T^s_{m,n}$ contains a typical $\mathcal{U}_q s\ell(2|1)$-module and a typical $\mathsf{X}_{m,n}$-module, and each subquotient in $T^{\mathrm{at}}_{m,n}$ contains an atypical $\mathcal{U}_q s\ell(2|1)$-module and an atypical $\mathsf{X}_{m,n}$-module. We call $T^s_{m,n}$ the semisimple part and $T^{\mathrm{at}}_{m,n}$ the atypical part.

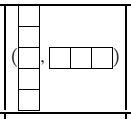
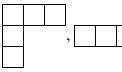
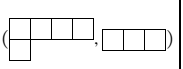
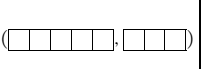
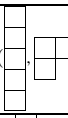
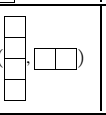
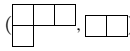
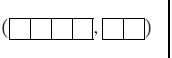
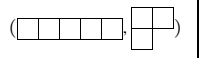
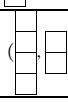
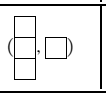
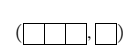
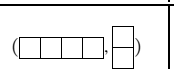
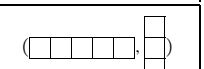
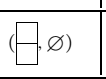
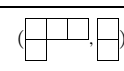
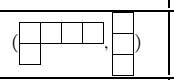
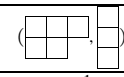
5.2.1. Examples

Before giving a general formula for the decomposition of $\mathcal{T}_{m,n}$ in 5.3, we illustrate the structure of the semisimple part $T^s_{m,n}$ with two examples. $T^s_{m,n}$ has the structure

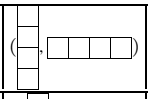
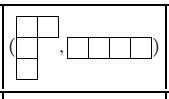
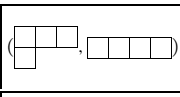
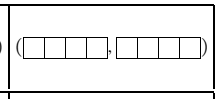
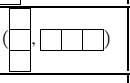
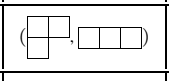
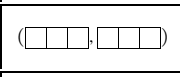
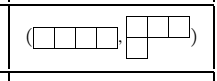
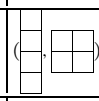
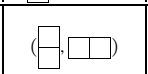
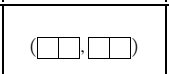
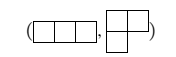
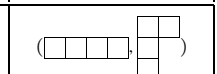
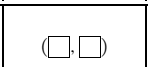
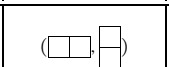
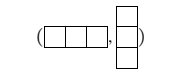
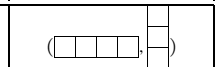
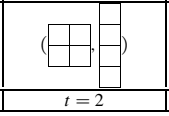
$$T^s_{m,n} = \bigoplus_{t,r} D(\lambda_{m,n}(t,r)) \boxtimes \bar{\mathcal{Z}}^{p(t,r)}_{t,r} \tag{5.4}$$

For given m, n, we represent the sum in (5.4) as a table of bipartitions $\lambda_{m,n}(t,r)$ in coordinates (t,r). All parts of the sum outside the table vanish, and 0 in the table means that the corresponding submodule in (5.4) vanishes.

For $m = 5$ and $n = 3$, the table of bipartitions $\lambda_{5,3}(t, r)$ reads

$r = 5$	0		0			
$r = 4$			0			
$r = 3$			0			
$r = 2$	0		0			0
$r = 1$	0	0	0		0	0
	$t = -2$	$t = -1$	$t = 0$	$t = 1$	$t = 2$	$t = 3$

For $m = 4$ and $n = 4$, the table of bipartitions $\lambda_{4,4}(t, r)$ reads

$r = 4$	0	0				
$r = 3$	0	0				
$r = 2$		0				
$r = 1$	0	0				
$r = 0$	0	0	0	0	0	0
$r = -1$	0	0	0		0	0
	$t = -1$	$t = 0$	$t = 1$	$t = 2$	$t = 3$	$t = 4$

5.2.2. In the next Theorem, we give explicit formulas for the decomposition of $\mathcal{T}_{m,n}$ for $m \geqslant n$; the case $m < n$ can be easily recovered from $m > n$ using operation $\hat{G}$ interchanging m with n

$$\mathcal{T}_{n,m} = \hat{G}\mathcal{T}_{m,n}.$$

The operation is involutive, $\hat{G}^2 = 1$, and additive, $\hat{G}(X \oplus Y) = \hat{G}(X) \oplus \hat{G}(Y)$. It acts on the indecomposable summands in the semisimple part $T^s_{m,n}$ by the formula

$$\hat{G}\Big(D\big[\lambda^L, \lambda^R\big] \boxtimes \bar{\mathcal{Z}}^p_{t,r}\Big) = \hat{G}\Big(D\big[\lambda^L, \lambda^R\big]\Big) \boxtimes \hat{G}\Big(\bar{\mathcal{Z}}^p_{t,r}\Big), \tag{5.5}$$

where the action $\hat{G}\Big(D\big[\lambda^L, \lambda^R\big]\Big)$ is defined in (4.5) and

$$\hat{G}\bar{\mathcal{Z}}^p_{t,r} = \bar{\mathcal{Z}}^p_{r,t}. \tag{5.6}$$

When applied to the atypical part $T^{\mathrm{at}}_{m,n}$, the operation $\hat{G}$ acts on each simple subquotient by the formula (5.5) and does not change the structure of the Loewy graph.

Theorem 5.3. *The $\mathsf{X}_{m,n} \boxtimes \mathcal{U}_q s\ell(2|1)$-bimodule decomposition of $\mathcal{T}_{m,n}$, $m \geqslant n$, has the form $\mathcal{T}_{m,n} = T^{s}_{m,n} \oplus T^{at}_{m,n}$ with the semisimple part*

$m > n$:

$$
\begin{aligned}
T^{s}_{m,n} = {} & \bigoplus_{s=1}^{n} \bigoplus_{\substack{k=1,\\ k\neq a}}^{a+s} D[(k, 1^{s-k+a}),\ (s)] \boxtimes \bar{\mathcal{Z}}^{s+k+a+1}_{k-a,s+a} \oplus \\
& \bigoplus_{s=a+2}^{m} \bigoplus_{k=1}^{s-a-1} D[(s),\ (k, 1^{s-k-a})] \boxtimes \bar{\mathcal{Z}}^{s+k+a+1}_{s-a,k+a} \oplus \\
& \bigoplus_{s=1}^{n-1} \bigoplus_{k=1}^{\min(s,\, n-s)} D[(1^{s+k+a}),\ (s,k)] \boxtimes \bar{\mathcal{Z}}^{s+k+a}_{1-k-a,s+a} \oplus \\
& \bigoplus_{s=a+1}^{m-1} \bigoplus_{\substack{k=1,\\ k\neq a+1}}^{\min(s,\, m-s)} D[(s,k),\ (1^{s+k-a})] \boxtimes \bar{\mathcal{Z}}^{s+k+a}_{s-a,1-k+a} \oplus \\
& \bigoplus_{k=1}^{\lfloor \frac{a}{2} \rfloor} \bigoplus_{s=k}^{a-k} D[(s,k,1^{a-s-k}),\ \varnothing] \boxtimes \bar{\mathcal{Z}}^{s+k+a}_{s-a,1-k+a} \oplus \\
& \bigoplus_{s=\lfloor \frac{a}{2} \rfloor+1}^{a-1} \bigoplus_{k=1-s+a}^{\min(s,\, m-s)} D[(s,k),\ (1^{s+k-a})] \boxtimes \bar{\mathcal{Z}}^{s+k+a}_{s-a,1-k+a},
\end{aligned}
$$

$m = n$:

$$
\begin{aligned}
T^{s}_{m,m} = {} & \bigoplus_{s=1}^{m} \bigoplus_{k=1}^{s} D[(k, 1^{s-k}),\ (s)] \boxtimes \bar{\mathcal{Z}}^{s+k+1}_{k,s} \oplus \\
& \bigoplus_{s=2}^{m} \bigoplus_{k=1}^{s-1} D[(s),\ (k, 1^{s-k})] \boxtimes \bar{\mathcal{Z}}^{s+k+1}_{s,k} \oplus \\
& \bigoplus_{s=2}^{m-1} \bigoplus_{k=2}^{\min(s,\, m-s)} D[(1^{s+k}),\ (s,k)] \boxtimes \bar{\mathcal{Z}}^{s+k}_{1-k,s} \oplus \\
& \bigoplus_{s=1}^{m-1} \bigoplus_{k=2}^{\min(s,\, m-s)} D[(s,k),\ (1^{s+k})] \boxtimes \bar{\mathcal{Z}}^{s+k}_{s,1-k},
\end{aligned}
$$

and the atypical part $T^{at}_{m,n}$ is given by Figs. 2–6 in Appendix A.

5.4. Verification

To check the decomposition formula for the bimodule we make two powerful verifications using formulas for tensor product decompositions for $\mathcal{U}_q s\ell(2|1)$ modules and restrictions for $\mathsf{X}_{m,n}$ modules. We check that $\mathcal{T}_{m,n} \otimes \overline{\mathbf{3}}$ coincides with $\mathrm{res}^{m,n+1}_{m,n}\, \mathcal{T}_{m,n+1}$ as $\mathcal{U}_q s\ell(2|1)$-module in

the first verification and as $\mathsf{X}_{m,n}$-module in the second one. In order to do this we introduce two Grothendieck (forgetful) functors $\mathbb{P}$ and $\mathbb{Q}$.

We define the Grothendieck functor $\mathbb{P}$ on the category of $\mathcal{U}_q s\ell(2|1)$-modules which maps an indecomposable module into a direct sum of its simple subquotients. The functor $\mathbb{P}$ on any $\mathcal{U}_q s\ell(2|1)$-module is known from 2.4. For example

$$\mathbb{P}\bar{\mathcal{R}}^p_{t,0} = 2\bar{\mathcal{Z}}^p_{t,0} \oplus \bar{\mathcal{Z}}^{p+1}_{t+1,0} \oplus \bar{\mathcal{Z}}^{p-1}_{t-1,0}, \quad t \geqslant 1,$$
$$\mathbb{P}\bar{\mathcal{Z}}^p_{t,r} = \bar{\mathcal{Z}}^p_{t,r}, \quad \forall p, t, r.$$

We define the other Grothendieck functor $\mathbb{Q}$ on the category of $\mathsf{X}_{m,n}$ modules which maps an indecomposable module into a direct sum of its simple subquotients. The functor $\mathbb{Q}$ on any $\mathsf{X}_{m,n}$-module is known from 4.3.5. For example

$$\mathbb{Q}K(A^1_1) = 2D(A^1_1) \oplus D(A^1_2) \oplus D(A^1_0) \oplus D(C^1_1), \quad n \geqslant 3,$$
$$\mathbb{Q}D(\lambda) = D(\lambda), \quad \forall m, n, \lambda.$$

The functors $\mathbb{P}$ and $\mathbb{Q}$ do not change semisimple part of the bimodule:

$$\mathbb{P}T^s_{m,n} = \mathbb{Q}T^s_{m,n} = T^s_{m,n},$$

because semisimple part is a direct sum of simple bimodules.

5.4.1. As $\mathcal{U}_q s\ell(2|1)$ module

The action of $\mathbb{Q}$ on the atypical part $T^{\mathrm{at}}_{m,n}$ has the form

***m > n*:**

$$\mathbb{Q}T^{\mathrm{at}}_{m,n} = \bigoplus_{s=1}^{n} D(A^a_s) \boxtimes \bar{\mathcal{R}}^s_{0,a+s-1} \oplus \bigoplus_{s=2}^{\min(a,n)} D(B^a_s) \boxtimes \bar{\mathcal{R}}^s_{0,a-s+1} \oplus \bigoplus_{s=a}^{n-2} D(C^a_s) \boxtimes \bar{\mathcal{R}}^{s+1}_{s-a,0} \oplus D(A^a_0) \boxtimes \bar{\mathcal{Z}}^1_{0,a},$$

***m = n*:**

$$\mathbb{Q}T^{\mathrm{at}}_{m,m} = \bigoplus_{s=1}^{m-2} D(\hat{C}^0_s) \boxtimes \bar{\mathcal{R}}^{s-1}_{0,s} \oplus \bigoplus_{s=0}^{m-2} D(C^0_s) \boxtimes \bar{\mathcal{R}}^{s-1}_{s,0} \oplus D(A^0_0) \boxtimes \bar{\mathcal{Z}}^1_{0,0}.$$

We introduce the notation $\overline{\mathcal{T}_{m,n}} = \mathbb{Q}\mathcal{T}_{m,n}$. The following relation must hold:

$$\overline{\mathcal{T}_{m,n}} \otimes \overline{\mathbf{3}} = \mathbb{Q}\,\mathrm{res}^{m,n+1}_{m,n}\,\overline{\mathcal{T}_{m,n+1}}. \tag{5.7}$$

Because $\overline{\mathcal{T}_{m,n}}$ has the form $\overline{\mathcal{T}_{m,n}} = \bigoplus D \boxtimes \mathcal{R} \oplus D \boxtimes \mathcal{Z}$, we can calculate $\overline{\mathcal{T}_{m,n}} \otimes \overline{\mathbf{3}}$ using formulas from 3.1. Because $\overline{\mathcal{T}_{m,n+1}}$ contains as subquotients only modules $D(\lambda)$ for

$\lambda \in \mathcal{C}r_{m,n+1}$, we can calculate $\mathrm{res}_{m,n}^{m,n+1} \overline{\mathcal{T}_{m,n+1}}$ using formulas from 4.4.6, and then apply the functor $\mathbb{Q}$. We have checked the validity of relation 5.7 for all m, n whenever $m + n \leqslant 25$.

5.4.2. As $\mathsf{X}_{m,n}$ module

The action of $\mathbb{P}$ on the atypical part $T^{\mathrm{at}}_{m,n}$ has the form

$m > n$:

$$\begin{aligned}\mathbb{P}T^{\mathrm{at}}_{m,n} = &\bigoplus_{s=1}^{n} K(A^a_s) \boxtimes \bar{\mathcal{Z}}^s_{0,a+s-1} \oplus \\ &\bigoplus_{s=2}^{\min(a,\,n)} K(B^a_s) \boxtimes \bar{\mathcal{Z}}^s_{0,a-s+1} \oplus \\ &\bigoplus_{s=a}^{n-2} K(C^a_s) \boxtimes \bar{\mathcal{Z}}^{s+1}_{s-a,0} \\ &\oplus D(A^a_n) \boxtimes \bar{\mathcal{Z}}^{n+1}_{0,m} \\ &\oplus PT^{\mathrm{right}}_{m,n},\end{aligned}$$

where

$$PT^{\mathrm{right}}_{m,n} = \begin{cases} 0, & n = 0, \\ D(B^a_n) \boxtimes \bar{\mathcal{Z}}^{n+1}_{0,m-2n}, & 1 \leqslant n \leqslant \frac{m}{2}, \\ D(B^a_a) \boxtimes \bar{\mathcal{Z}}^{a+1}_{0,0}, & n = \frac{m+1}{2}, \quad n \geqslant 2, \\ D(C^a_{n-2}) \boxtimes \bar{\mathcal{Z}}^{n}_{2n-m-1,0}, & \frac{m}{2} + 1 \leqslant n \leqslant m-1, \end{cases}$$

$m = n \geqslant 2$:

$$\begin{aligned}\mathbb{P}T^{\mathrm{at}}_{m,m} = &\bigoplus_{s=1}^{m-2} K(\hat{C}^0_s) \boxtimes \bar{\mathcal{Z}}^{s-1}_{0,s} \oplus \\ &\bigoplus_{s=0}^{m-2} K(C^0_s) \boxtimes \bar{\mathcal{Z}}^{s-1}_{s,0} \\ &\oplus D(\hat{C}^0_{m-2}) \boxtimes \bar{\mathcal{Z}}^{m-2}_{0,m-1} \\ &\oplus D(C^0_{m-2}) \boxtimes \bar{\mathcal{Z}}^{m-2}_{m-1,0},\end{aligned}$$

$m = n = 1$:

$$\mathbb{P}T^{\mathrm{at}}_{1,1} = D(A^0_0) \boxtimes \bar{\mathcal{Z}}^1_{0,0}.$$

We introduce the notation $\underline{\mathcal{T}_{m,n}} = \mathbb{P}\mathcal{T}_{m,n}$. The following relation must hold:

$$\mathbb{P}\left(\underline{\mathcal{T}_{m,n}} \otimes \overline{\mathbf{3}}\right) = \mathrm{res}_{m,n}^{m,n+1} \underline{\mathcal{T}_{m,n+1}}. \tag{5.8}$$

Because $\underline{\mathcal{T}_{m,n}}$ has the form $\underline{\mathcal{T}_{m,n}} = \bigoplus K \boxtimes \mathcal{Z} \bigoplus D \boxtimes \mathcal{Z}$, we can calculate $\underline{\mathcal{T}_{m,n}} \otimes \overline{\mathbf{3}}$ using formulas from 3.1. Because $\underline{\mathcal{T}_{m,n+1}}$ contains as subquotients only modules $K(\lambda)$ and $D(\lambda)$ for $\lambda \in \mathcal{C}r_{m,n+1}$, we can calculate $\mathrm{res}_{m,n}^{m,n+1} \underline{\mathcal{T}_{m,n+1}}$ using formulas from 4.4.6 and 4.4.5 and

then apply the functor $\mathbb{P}$. We have checked the validity of relation 5.8 for all m, n whenever $m + n \leqslant 25$.

6. Conclusion

In the present work we have studied the $\mathcal{U}_q s\ell(2|1)$ mixed tensor product and found its decomposition as a bimodule over $\mathsf{X}_{m,n} \boxtimes \mathcal{U}_q s\ell(2|1)$. These results are the basis for a further study of the $\mathcal{U}_q s\ell(2|1)$-spin-chain and appropriate LCFT.

The next step is studying the mixed tensor product with parameter q at the root of unity. We expect the appearance of the Lusztig limit of algebra $\mathcal{U}_q s\ell(2|1)$ in that case. We anticipate that $\mathsf{X}_{m,n}$ will remain the centralizer of $\mathcal{L}\mathcal{U}_q s\ell(2|1)$ on the mixed tensor product and some triplet extension of $\mathsf{X}_{m,n}$ will be the centralizer of $\mathcal{U}_q s\ell(2|1)$.

Natural ways for further developments of the results presented in this paper:

(1) Describe explicitly the algebra $\mathsf{X}_{m,n}$ and identify it with some quotient of $\mathsf{qw}\mathcal{B}_{m,n}$. Similar problem is posed the algebras $\mathsf{X}_{m,n}^{M,N}$ of $\mathcal{U}_q s\ell(M|N)$-endomorphisms.
(2) Describe the structure of Specht and projective $\mathsf{X}_{m,n}^{M,N}$-modules and perhaps $\mathsf{qw}\mathcal{B}_{m,n}$-modules. This problem becomes significantly more complicated when parameter q is a root of unity.
(3) Figure out the restriction functor on all simple and projective modules of the algebra $\mathsf{qw}\mathcal{B}_{m,n}$.
(4) Classify Ext^1 spaces for modules over the algebra $\mathsf{X}_{m,n}$ and describe explicitly the action of $\mathsf{X}_{m,n}$-generators on the basis of projective modules $K(\lambda)$. The solution to this problem will allow one to describe explicitly the $\mathsf{X}_{m,n}$-action in the bimodule $\mathcal{T}_{m,n}$.

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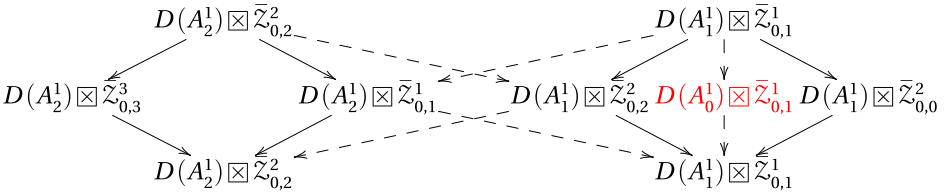
Appendix A. Atypical part of the bimodule

In this section we represent the structure of Loewy graph for the indecomposable bimodule $T^{\mathrm{at}}_{m,n}$, see 5.3. Detailed investigation of $\mathsf{X}_{m,n}$ action on these bimodules are beyond the scope of this paper. See paper [12], where the spin chain over $\mathcal{U}_q s\ell(2)$ is investigated for comparison.

In each vertex of the graph there is some subquotient $D(\lambda) \boxtimes \bar{\mathcal{Z}}^p_{t,r}$. The meaning of the arrows is the same as in 4.3.5. On the figures the action of algebra $\mathcal{U}_q s\ell(2|1)$ is denoted by solid lines, and the action of $\mathsf{X}_{m,n}$ is denoted by dash lines.

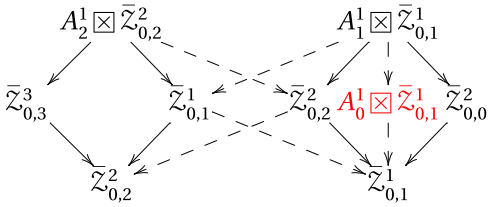
The subquotients connected by dash lines have the same $\mathcal{U}_q s\ell(2|1)$ module as a tensor multiplier. The subquotients connected by solid lines have the same $\mathsf{X}_{m,n}$ module as a tensor multiplier.To simplify the figures we omit $\mathsf{X}_{m,n}$ multiplier where it does not cause inconsistency. We also do not write symbol D each time, and write only λ for simple module $D(\lambda)$.

For example, the bimodule for $T^{\text{at}}_{3,2}$ is



(A.1)

We use shorthand notation for $T^{\text{at}}_{3,2}$:



(A.2)

We mark in red the subquotient where the figure has irregular form. (For interpretation of the colors in this figure, the reader is referred to the web version of this article.)

The structure of $T^{\text{at}}_{m,n}$ for the case $1 \leqslant n \leqslant \frac{m}{2}$ is shown in Fig. 2.

The case $\frac{m}{2} + 1 \leqslant n \leqslant m - 2$ is shown in Fig. 3.

The case $n = \frac{m+1}{2}, n \geqslant 2$ is shown in Fig. 4.

The case $n = m - 1, n \geqslant 1$ is shown in Fig. 5.

The case $n = m, n \geqslant 2$ is shown in Fig. 6.

Two exceptional cases are:

$$T^{\text{at}}_{m,0} = D(A_0^m)\boxtimes\bar{\mathcal{Z}}^1_{0,m},$$
$$T^{\text{at}}_{1,1} = D(A_0^0)\boxtimes\bar{\mathcal{Z}}^1_{0,0}.$$

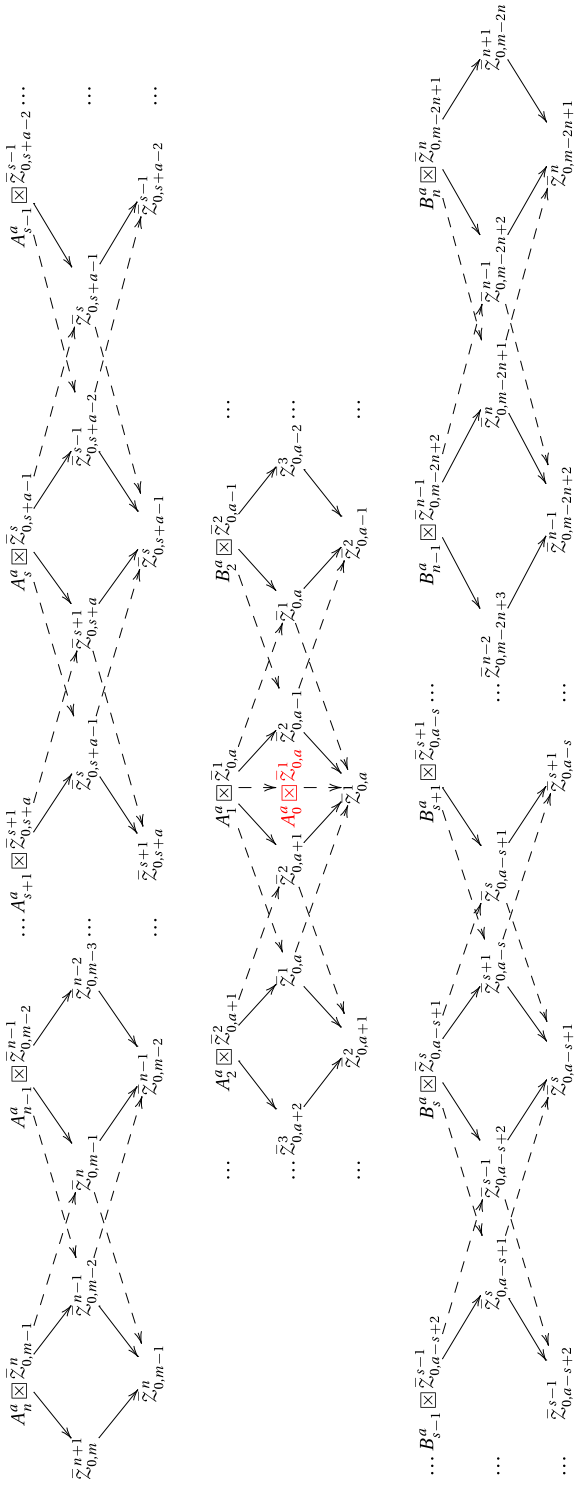


Fig. 2. $T^{at}_{m,n}$, $1 \leqslant n \leqslant \frac{m}{2}$.

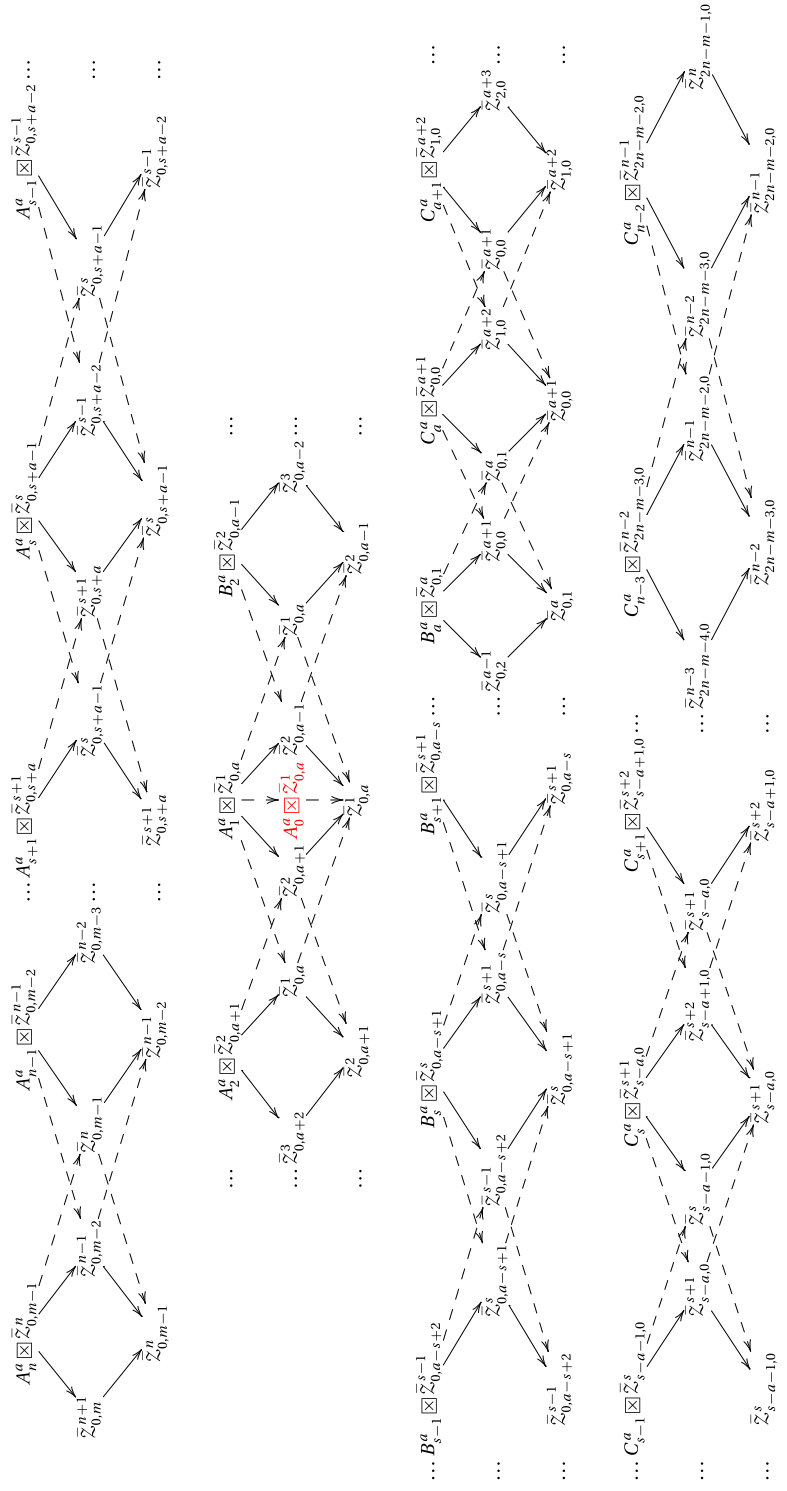


Fig. 3. $T^{dt}_{m,n}$, $\frac{m}{2}+1 \leqslant n \leqslant m-2$.

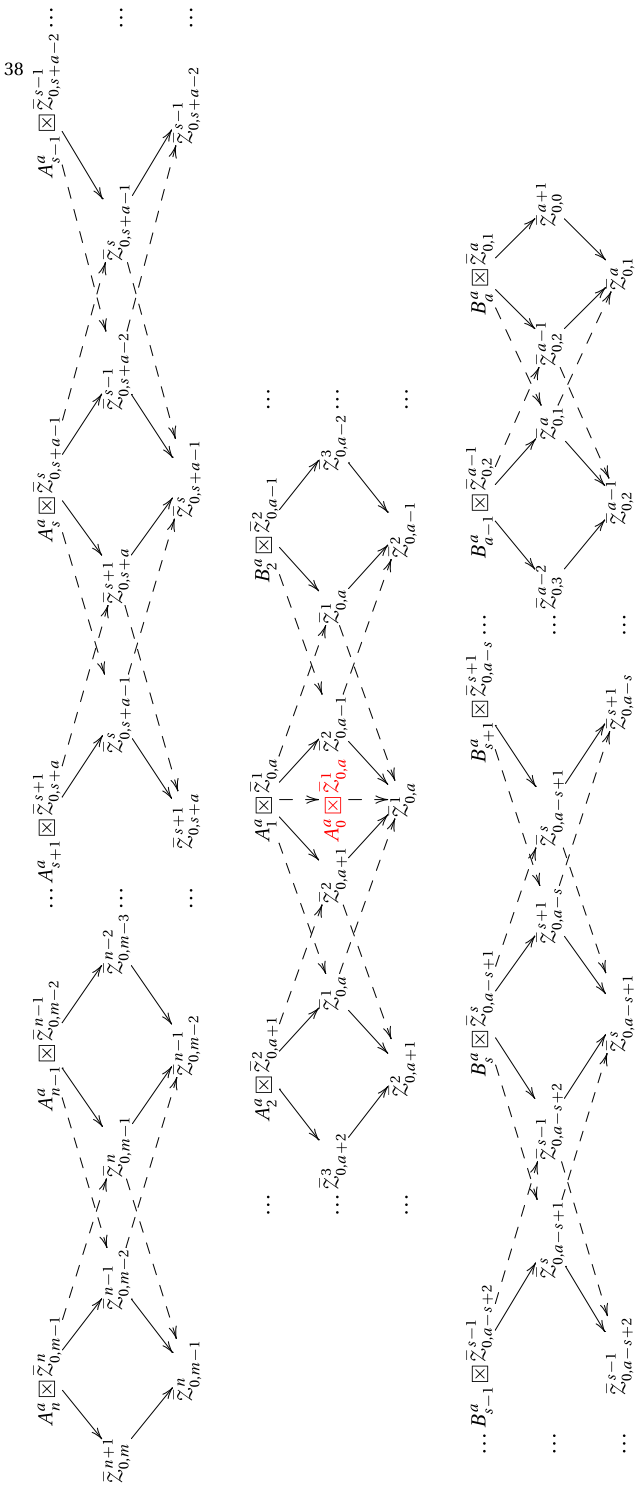


Fig. 4. $T^{al}_{m,n}, n = \frac{m+1}{2}, n \geqslant 2$ $(a = n-1)$.

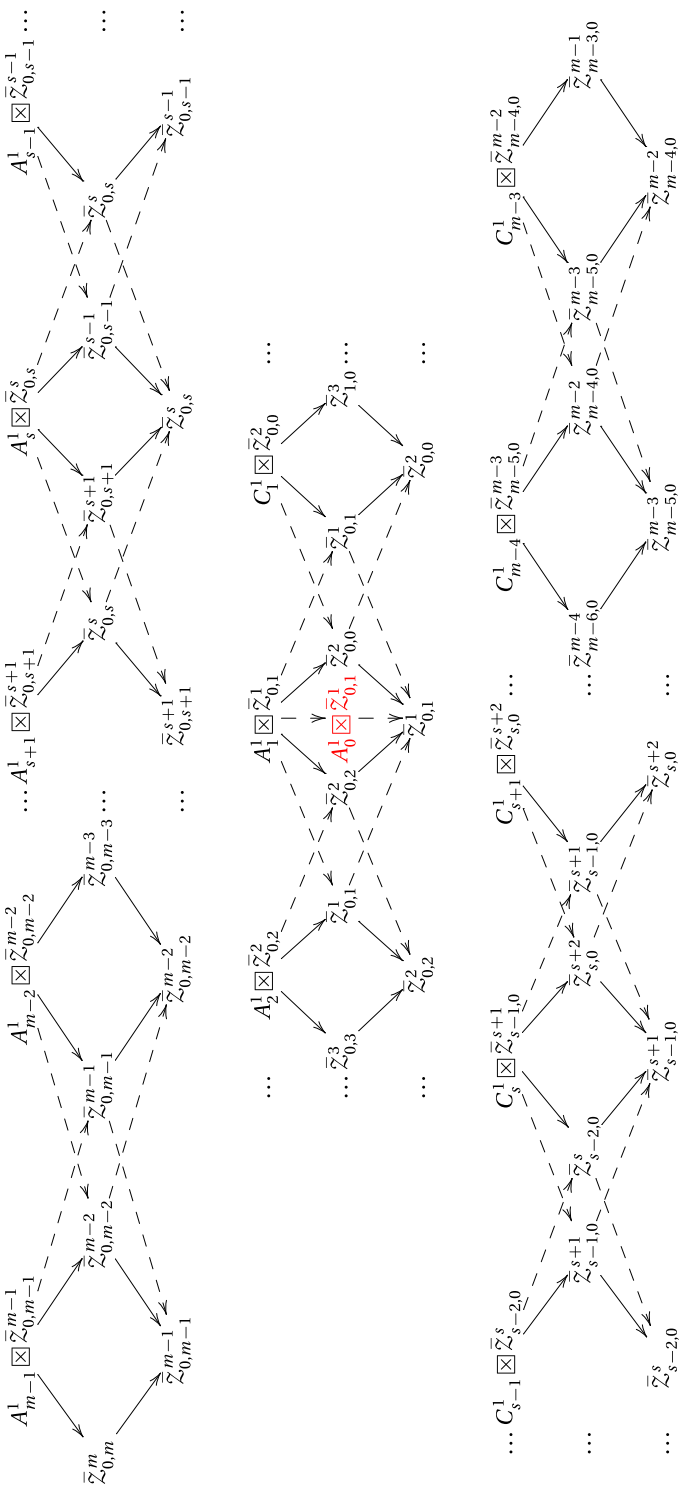


Fig. 5. $T^{at}_{m,m-1}$, $m \geqslant 2$.

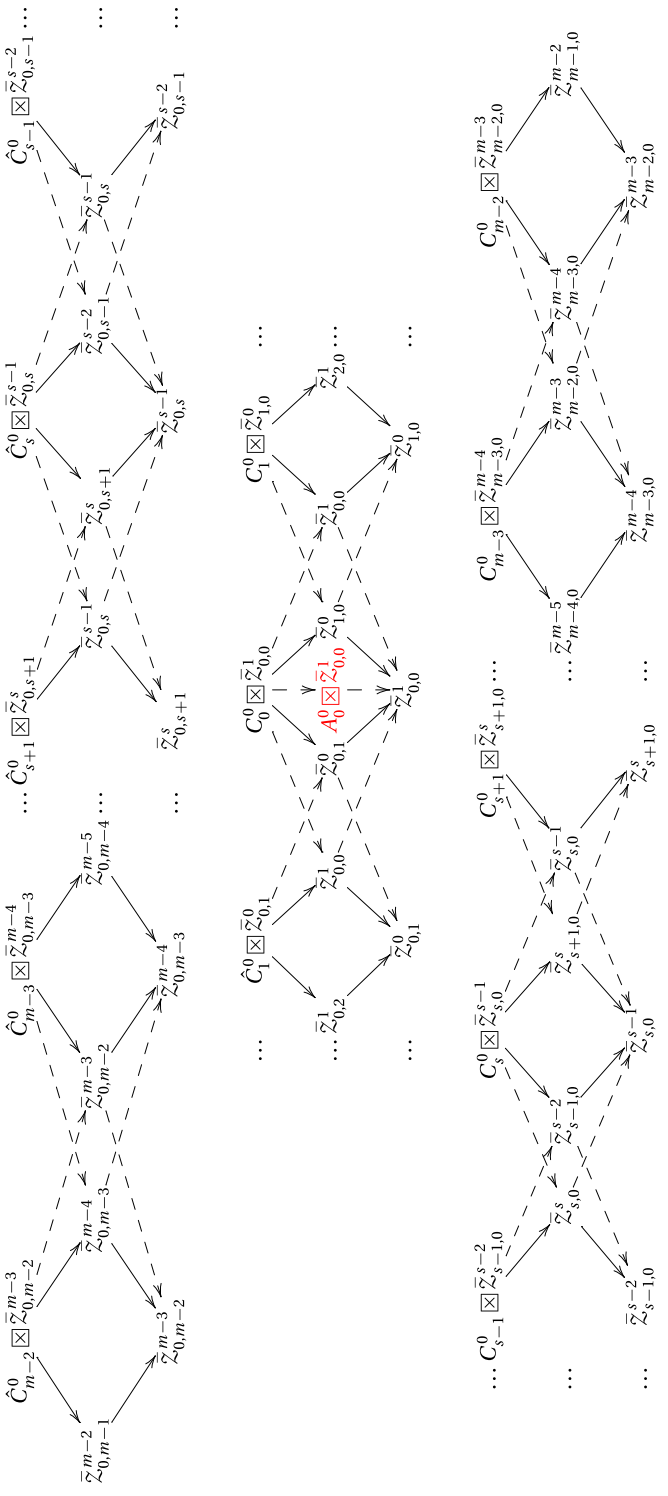


Fig. 6. $T^{at}_{m,m}$, $m \geqslant 2$.

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