# OPTIMIZATION OF COLLISION AMPLITUDES UNDER CONSTRAINTS

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## 1. ASSUMPTIONS AND OBJECTIVES

### (a) Introduction

These lectures will describe methods for optimization under constraints. The methods will be illustrated by applications to collision amplitudes, both at asymptotic energies and at finite energies. The applications will include the use of integral constraints.

Starting from the Mandelstam representation, Froissart (1961) deduced that the total cross section $\sigma_{tot}(s)$ for the collision of two particles having c.m. energy $s^{\frac{1}{2}}$ must obey the following bound as $s \rightarrow \infty$,

$$\sigma_{tot}(s) \quad \leq \quad \text{constant}\,[\ell n(s/s_o)]^2 \tag{1.1}$$

where $s_o$ is a constant.

Since 1961 there have been many extensions of the idea of obtaining bounds on physical quantities from basic assumptions without any use of special theoretical models. These developments have been dominated by the work of Martin and his many associates and collaborators. It has been extensively surveyed in the review articles by Martin (1969) and by others listed in the list of references. These reviews contain detailed references to most original papers in this area of research, so these lecture notes will contain only minimal references to original papers.

Assumptions: The basic results of quantum field theory which are commonly used are:

(i) Unitarity

(ii) Analyticity in the Lehmann-Martin ellipse

(iii) Dispersion relations for collision amplitudes F (s,t) at fixed momentum transfer in a limited range of t.

In addition it is useful to make phenomenological assumptions about collision amplitudes, cross sections or scattering lengths. These lead to bounds or constraints between different physical (or analytically continued) quantities; examples will be given in later lectures.

Notation: For equal mass particles with c.m. momentum k,

$$s = 4\left(m^2 + k^2\right) \quad , \quad t = -2k^2\left(1 - \cos\vartheta\right). \tag{1.2}$$

We shall also be discussing unequal mass collisions but shall not give detailed formulae in those cases. The collision amplitude F (s,t) will be normalised so that

$$\frac{d\sigma}{dt} = \frac{4\pi\,|F(s,t)|^2}{s\,(s-4m^2)} \tag{1.3}$$

$$\sigma_{tot}(s) = \frac{4\pi}{k^2}\sum_0^\infty (2\ell+1)\,\mathrm{Im}\, f_\ell(s) = \frac{4\pi}{k s^{1/2}}\,\mathrm{Im}\, F(s,0) \tag{1.4}$$

Within the Lehmann-Martin ellipse

$$F(s,t) = \frac{s^{1/2}}{k}\sum_0^\infty (2\ell+1)\, f_\ell(s)\, P_\ell(\cos\vartheta) \tag{1.5}$$

Unitarity requires that partial waves satisfy

$$|f_\ell(s)|^2 \le \mathrm{Im}\, f_\ell(s) \le 1 \tag{1.6}$$

Writing $f_\ell = r_\ell + i a_\ell$ , the unitarity constraint can be written

$$a_\ell - a_\ell^2 - r_\ell^2 \ge 0 \tag{1.7}$$

(b) Objectives of Optimization

I shall indicate briefly some of the objectives that have been achieved and return later to a discussion of further results.

Dispersion Relations. Using unitarity (1.6), the partial wave expansion (1.5) and the properties of Legendre polynomials, Jin and Martin have shown that the dispersion relation for F(s,t) at fixed t requires no more than two subtractions,

$$|F(s,t)| \leq (s/s_0)^2, \quad (t < 4m^2). \tag{1.8}$$

High energy bounds. The Froissart bound (1.1) holds with the constant equal to $4\pi / (t_0 - \varepsilon)$, where $t_0 = 4m^2$ and $\varepsilon$ may be arbitrarily small.

Integrated bounds have been obtained for averaged total cross sections by Common and Yndurain. These bounds will be discussed in section 4 of these lecture notes. Various bounds have been obtained on amplitudes and on differential cross-sections at fixed t. These bounds can be extended to inelastic reactions and to inclusive reactions as described in the review by Roy (1972).

Absorptive parts of elastic amplitudes

MacDowell and Martin and later Singh and Roy obtained a variety of bounds on ImF(s,t) at high energies. In section 3 some of these bounds will be considered as well as related bounds at finite energies.

Bounds on crossed-channel amplitudes

The integrated bounds of Common and Yndurain mentioned above, lead naturally to restictions on amplitudes in the crossed channel. For example, knowing N scattering results one can deduce restrictions on $N\bar{N} \longrightarrow \pi\pi$ amplitudes. These will be discussed in section 4 of these notes.

Theorems on particle-antiparticle amplitudes

The Pomeranchuk theorem has several variants; the following form is due to Martin. If F denotes a forward amplitude and

$$\lim_{s \to \infty} \frac{F(ab \to ab) - F(\bar{a}b \to \bar{a}b)}{s \ln (s/s_0)} \to 0 \tag{1.8}$$

and if the following limit exists,

$$\lim \left[ \sigma_{tot}(ab) - \sigma_{tot}(\bar{a}b) \right] \tag{1.9}$$

then this limit (1.9) must be zero.

Numerous related theorems arise if it is assumed that (1.9) is not zero as was indicated by the early results from Serpukhov on total cross sections. Some of these theorems are discussed in the reviews by

Eden (1971) and Roy (1972).

## 2. Optimization under Constraints

References: M. Aoki (1971), M.R. Hestenes (1966) and M.B. Einhorn and R. Blankenbecler (1971).

### (a) Formulation and terminology

Objective function: The function that we wish to optimize is called the objective function. Other names used include Lagrangian, criterion function, utility, cost or penalty functions. To be definite we will describe conditions required to minimise the objective function. This function depends on a set of real variables $x_1, x_2, \dots x_n$, and it will be denoted by

$$f(x) = f(x_1, x_2 \dots x_n) \tag{2.1}$$

Constraints: We consider equality constraints and inequality constraints. The equality constraints will be written

$$f_\alpha(x) = 0, \quad \alpha = 1,2, \dots p. \tag{2.2}$$

The inequality constraints will be written

$$g_\beta(x) \geqslant 0, \quad \beta = 1,2, \dots q. \tag{2.3}$$

Any point $x = (x_1, x_2, \dots x_n)$ that satisfies the constraints is called a feasible point and the set of such points is called the feasible set S.

Tangent cone: the set of all (unit length) half lines h, originating at a point $x_o$ in S and tangent to a curve in S is called the tangent cone to S at $x_o$.

Increments: the increment of f along a small vector v through a feasible point $x_o$ is called the first differential of f and is denoted by

$$\delta f = f'(x_o, v) = (f'(x_o), v) = \sum \frac{\partial f}{\partial x_i} v_i \tag{2.4}$$

The second differential is defined by

$$f''(x_o, v) = \sum \frac{\partial^2 f}{\partial x_i \partial x_j} v_i v_j \tag{2.5}$$

Regular points of S (equality constraints only):
Let $x_o$ be a feasible point and let k be a unit vector satisfying

$$(f'_\alpha\ (x_o), k) = 0, \quad \text{for all} \tag{2.6}$$

If every k satisfying (2.8) lies in the tangent cone C at $x_o$, then $x_o$ is a regular point of S.

Normal points of S: If the gradients $f'(x_o)$ are linearly independent, $x_o$ is a normal point. Every normal point is a regular point (Hestenes 1966).

Inequality constraints: The definitions of normal points and regular points extend to inequality constraints if we divide these into ineffective or interior constraints $\beta$ in I $(x_o)$, and effective or boundary constraints $\beta$ in $B(x_o)$, defined by

$$I(x_o) = \{\beta \mid g_\beta(x_o) > 0\} \tag{2.7}$$

$$B(x_o) = \{\beta \mid g_\beta(x_o) = 0\} \tag{2.8}$$

(b) Minimization with Equality Constraints

The standard method of Lagrange multipliers determines all local minima that are regular points. It is summarised by the following two theorems:

Theorem 1

Let $x_o$ be a regular point of S and let $x_o$ be a local minimum of f(x) on S.

Then:

(i) there exist multipliers $\lambda_\alpha$ such that

$$L'(x_o) = \frac{\partial L}{\partial x_i} = 0, \quad i = 1, 2, \ldots n \tag{2.9}$$

where the auxiliary function L is defined by

$$L(x) = f + \sum \lambda_\alpha f_\alpha \ . \tag{2.10}$$

(ii) for a minimum

$$L''(x_o,h) \geqslant 0 \qquad (2.11)$$

for all h in the tangent cone at $x_o$,

(iii) if $x_o$ is normal, the multipliers $\lambda_\alpha$ are unique.

Theorem 2

If (2.9) is satisfied and if $L''(x_o,h)$ is strictly positive for all h in the tangent cone at $x_o$, then $x_o$ is a local minimum of f(x).

(c) Interpretation of the Lagrange Multipliers

The Lagrange multipliers can be interpreted as sensitivity coefficients with respect to small changes in the constraints. Thus if $x^*$ denotes a local minimum of f(x), with equality constraints,

$$b_\alpha - f_\alpha(x) = 0 \quad , \quad \alpha = 1, 2, \dots p \qquad (2.12)$$

Then

$$\frac{\partial f}{\partial x_i^*} - \sum_\alpha \lambda_\alpha \frac{\partial f_\alpha}{\partial x_i^*} = 0 \qquad (2.13)$$

From (2.12) and (2.13) one finds

$$\frac{\partial f(x^*)}{\partial b_\gamma} = \lambda_\gamma \quad . \qquad (2.14)$$

Thus $\lambda_\gamma$ is the rate of change in the stationary value $f(x^*)$ with respect to small changes in the parameter $b_\gamma$ in the constraints (2.12). In practice the sign of $\lambda_\gamma$ may be intuitively obvious from this interpretation.

(d) Linear Programming

Linear Programming problems are those in which both the objective function and the constraints are linear in the variables.

Standard form: Minimise the objective function

$$\phi(x) = \sum c_i x_i + d \qquad (2.15)$$

subject to

$$A x = b \ , \ x \geq 0 \tag{2.16}$$

where A is an (m,n) matrix of rank m, (m<n) and

$$x^T = (x_1 \ \cdots \ x_n) \ , \ b^T = (b_1 \cdots b_m) \ ; \tag{2.17}$$

Extreme points: The feasible set S is the inside of a convex polyhedron (possibly extending to infinity). The extreme points of the linear objective function

$$\phi = \sum c_i x_i + d \tag{2.18}$$

occur at the vertices of the polyhedron. Thus they are members of the set of vectors x in which (n-m) of the $x_i$ are zero. Thus with m constraints only m of the $x_i$ are non-zero.

Lagrangian method: The Lagrangian associated with the standard linear programming problem is

$$L(x, \lambda) = \sum c_i x_i + d + \sum \lambda_\alpha \left[ b_\alpha - (A x)_\alpha \right] \tag{2.19}$$

subject to $x \geqslant 0$.

(e) Minimisation with Inequality Constraints

Consider the minimisation of f(x) subject to the constraints

$$f_\alpha (x) = 0 \ , \ \alpha = 1, 2, \ldots p, \tag{2.20}$$

$$g_\beta (x) \geqslant 0 \ , \ \beta = 1, 2, \ldots q . \tag{2.21}$$

For any feasible $x_o$, let $I(x_o)$ be the set of indices $\beta$ for which $g_\beta(x_o) > 0$ and $B(x_o)$ be those $\beta$ for which $g_\beta(x_o) = 0$.

Theorem 3

Let $x_o$ be a regular point and a local minimum in the feasible set S. Then,

(i) There exist multipliers $\lambda_\alpha$, and $\mu_\beta \leqslant 0$ such that

$$L'(x_o) = 0 \tag{2.22}$$

where

$$L(x_o) = f + \sum \lambda_\alpha f_\alpha + \sum \mu_\beta g_\beta \tag{2.23}$$

(ii) If $\beta$ is in $I(x_o)$ we may choose $\mu_\beta = 0$.

(iii) Let $S_1$ be the subset of S for which $g_\beta(x) = 0$ for all $\beta$ in $B(x_o)$.

Then

$$L''(x_o, h) \geqslant 0 \tag{2.24}$$

for all h in the tangent cone of $S_1$ at $x_o$.

(iv) If $x_o$ is a normal point, the multipliers are unique.

<u>Theorem 4</u>

If (2.22) is satisfied and if (instead of (2.24))

$$L''(x_o, h) > 0 \tag{2.25}$$

for all h in the tangent cone $S_1$ at $x_o$, then $x_o$ is a local minimum of f(x).

Sometimes when L is linear in a coordinate, theorem 4 may not apply. Then one could use the following obvious theorem.

<u>Theorem 5</u>

If $f'(x_o, h) > 0$ for all h in the tangent cone at $x_o$, then $x_o$ is a local minimum of f(x).

## 3. Collision Amplitudes under Constraints

### (a) Linear Constraints at High Energies

For our first example to illustrate the general method we consider the problem discussed by Singh (1971b), but using the Lagrangian method as suggested by Hodgkinson (unpublished). Let A(s,t) = ImF be the imaginary part of a scattering amplitude. Consider its maximum value in

$$0 < t \leq 4m^2 = t_1 \tag{3.1}$$

Define $w = 1 + 2t/(s-4)$, $w_1 = 1 + 2t_1/(s-4)$. Then we have to maximise

$$A(s,t) = \sum_0^\infty (2\ell+1) \; a_\ell P_\ell \; (w) \tag{3.2}$$

subject to the constraints

$$a_\ell \geq 0, \text{ for } \ell = 0,1,2,\ .. \tag{3.3}$$

$$\sum_0^\infty (2\ell+1) a_\ell P_\ell(w_1) = (k/s^{\frac{1}{2}}) A(s,t_1) = A_1 \tag{3.4}$$

(where $A_1 < (s/s_o)^2$ follows from the Jin-Martin bound),

$$\sum_0^\infty (2\ell+1) a_\ell = \frac{k^2 \sigma(tot)}{4\pi} = (k/s^{\frac{1}{2}}) A(s,0) = A_o. \tag{3.5}$$

The method of linear programming tells us that the maximum of A(s,t) will occur when all $a_\ell$ are zero except for two, $a_p$ and $a_q$ say. It does not at once inform us that p+l = q. This can be seen from the Lagrangian for the constrained problem.

$$\begin{aligned} L = & \sum (2\ell+1) a_\ell P_\ell(w) + \alpha [A_o - \sum (2\ell+1) a_\ell] \\ & + \beta [A_1 - \sum (2\ell+1) a_\ell P_\ell(w_1)] \\ & + \sum (2\ell+1) \lambda_\ell a_\ell \end{aligned} \tag{3.6}$$

It is obvious that max A(s,t) will increase if either $A_o$ or $A_1$ is increased, hence $\alpha > 0$ and $\beta > 0$. Also since $\lambda_\ell$ are inequality multipliers $\lambda_\ell \geqq 0$ for a maximum (from theorem 3). The differential of L is zero at a maximum,

$$\frac{1}{(2\ell+1)} \frac{\partial L}{\partial a_\ell} = 0 = P_\ell(w) - \alpha - \beta P_\ell(w_1) + \lambda_\ell. \tag{3.7}$$

Let I denote the interior subset and $B_o$ the boundary subset of solutions of (3.7),

$$I : a_\ell > 0, \quad \lambda_\ell = 0, \ P_\ell(w) = \alpha + \beta P_\ell(w_1) \tag{3.8}$$

$$B_o : a_\ell = 0, \quad \lambda_\ell = \alpha + \beta P_\ell(w_1) - P_\ell(w) > 0 \tag{3.9}$$

From (3.8) and (3.9) it is obvious that $a_\ell$ is zero except for $a_p$, $a_{p+1}$ .

The value of p can be found from the constraints (3.4) and (3.5). For large values of s, p is also large and one can use

$$P_p(w_1) \sim P_{p+1}(w_1) \sim I_o(y_1) \sim \frac{\exp(y_1)}{(2\pi y_1)^{1/2}} \qquad (3.10)$$

where $y_1 = L(t_1/k^2)^{1/2}$ . This gives

$$p \leq \frac{1}{4} s^{1/2} \ln\left[s/(s_o^2 \sigma_{tot})\right] . \qquad (3.11)$$

Substituting in (3.2) one obtains Singh's result, namely

$$\frac{A(s,t)}{A_o} \leq \frac{A^{max}(s,t)}{A_o} \leq I_o(y) . \qquad (3.12)$$

The above method extends readily to the use of non-linear unitarity $a_\ell - a_\ell^2 > 0$ instead of the linear form (3.3).

(b) Phenomenological Constraints at Finite Energy

Singh and Roy (1970) bounded A(s,t) given by (3.2) in the region $t < 0$, using the partial wave series for $\sigma$(total) and for $\sigma$(elastic) as constraints. In the region $|t| < 0.1$ (GeV/c)$^2$, they found

$$\max \left| \frac{A(s,t)}{A(s,0)} \right|^2 \approx \left(\frac{d\sigma}{dt}\right) \Big/ \left(\frac{d\sigma}{dt}\right)_{t=0} \qquad (3.13)$$

At high energies the Singh-Roy bound given by the left hand side of (3.13) exceeds the data by only about 10% in $|t| < 0.1$ (GeV/c)$^2$, but for larger $|t|$ it differs from the data by more than an order of magnitude.

Savit, Einhorn and Blankenbecler (1971) imposed the additional constraint that partial waves decrease monotonically $a_{\ell+2} \leq a_\ell$ . However, they found, only a slight improvement was obtained.

Jacobs et al (1970) imposed instead, an additional constraint that fixed a phenomenological value of A(s,t) at a physical value of $t \approx -0.1$ (GeV/c)$^2$. This was found to extend the region in which max A(s,t) was close to the data in the sense of (3.13). For the constraint at larger values of $|t|$ oscillations are induced in max A(s,t) and its agreement with the data becomes poorer.

Hahn and Hodgkinson (1971) impose an additional constraint at $t > 0$ instead of $t < 0$. Thus they use (3.4) with Jin-Martin bound replaced by a phenomenological value of $A_1$ .

This value is obtained by extrapolating from experimental data in $t < 0$ to $t = t_1 = 4m^2$, using

$$A(s,t) = A_o \exp\left[\tfrac{1}{2}bt + \tfrac{1}{2}ct^2\right] \tag{3.14}$$

The resulting auxiliary objective function (or Lagrangian) is given by (3.2) and

$$\begin{aligned} L = A(s,t) &+ \alpha\left[A_o - \sum(2\ell+1)a_\ell\right] \\ &+ \beta\left[A_1 - \sum(2\ell+1)a_\ell P_\ell(w_1)\right] \\ &+ \frac{1}{2a}\left[\sigma_{e\ell} - \sum(2\ell+1)(a_\ell^2+r_\ell^2)\right] \\ &+ \sum(2\ell+1)\lambda_\ell(a_\ell - a_\ell^2 - r_\ell^2) \end{aligned} \tag{3.15}$$

For a maximum of $A(s,t)$, $r_\ell = 0$ and

$$P_\ell(z) - \alpha - \beta P_\ell(w_1) - (a_\ell/a) + \lambda_\ell(1-2a_\ell) = 0, \tag{3.16}$$

$$\lambda_\ell \geqslant 0, \quad (1/a) + 2\lambda_\ell \geqslant 0. \tag{3.17}$$

where $-1 < z < 1$ and $w_1 > 1$.
There are three classes of $\ell$ values for the solutions of these equations,

$$\begin{aligned} &\ell \in I : \lambda_\ell = 0,\ (a_\ell/a) = P_\ell(z) - \alpha - \beta P_\ell(w_1). \\ &\ell \in B_o : a_\ell = 0,\ \lambda_\ell = \alpha + \beta P_\ell(w_1) - P_\ell(z) \geqslant 0 . \\ &\ell \in B_1 : a_\ell = 1,\ \lambda_\ell = P_\ell(z) - (1/a) - \alpha - \beta P_\ell(w_1). \end{aligned} \tag{3.18}$$

The parameters are evaluated from the equality constraints that cause each of the square brackets in (3.15) to be zero. The solutions for $a_\ell$ give a partial wave profile for $A^{max}(s,t)$, that depends on both s and t.

The normalised bound U is defined by

$$U(s,t) = A^{max}(s,t)/A_o \tag{3.19}$$

One can similarly obtain a lower bound,

$$L(s,t) = A^{min}(s,t)/A_o \tag{3.20}$$

The resulting upper and lower bounds obtained by Hahn and Hodgkinson are compared with experiment in Fig. 1, which also shows the Singh-Roy bound, in pion-nucleon scattering. It is remarkable how close U(s,t) comes to the data, when one recalls that it involves only one constraint (at $t = 4m^2$) additional to $\sigma_{tot}$ and $\sigma_{el}$. It will of course be noted that the use of the ratio (3.19) compensates partly for the fact U(s,t) involves only the imaginary part of the amplitude.

## 4. Integral Constraints

### (a) Bounds on Averaged $\pi\pi$ Cross Sections

Reference: Common and Yndurain (1971), Roy (1972) and Steven (1972).

We shall consider $\pi^0 \pi^0$ scattering, using units with $m_\pi = 1$. From the Froissart-Gribov formula for partial waves in the t channel, evaluated in the threshold limit t = 4, we obtain the scattering length $\alpha_2^t$. This gives

$$a = \frac{15\pi}{8} \alpha_2^t = \int_4^\infty \frac{A(s,4)}{2s^3} ds \qquad (4.1)$$

In terms of s channel partial waves,

$$A(s,t) = 2(s^{1/2}/k) \sum (2\ell+1) a_\ell(s) P_\ell(z) \qquad (4.2)$$

$$\sigma_{tot}(s) = 2 \left(\frac{4\pi}{k^2}\right) \sum (2\ell+1) a_\ell \qquad (4.3)$$

where the factor 2 in (4.2) and (4.3) comes from the identity of the pions.

We will assume that the scattering length $\alpha_2^t$ is known, for example by considering the effect of $\pi\pi$ scattering on $\pi$N dispersion relations using experimental $\pi$N data (Morgan and Shaw 1969). We then maximise the averaged total cross-section $\bar{\sigma}_{tot}$ defined by

$$\Sigma_T = \frac{\bar{\sigma}_{tot}(s_1, s_2)}{32\pi} = \int_{s_1}^{s_2} \sum (2\ell+1) a_\ell(s) \frac{q(s)\, ds}{(s-4)} \qquad (4.4)$$

where q(s) is a chosen weight function, and $a_\ell(s)$ are subject to the constraint (4.1) with (4.2), and the unitarity constraint

$$a_\ell - a_\ell^2 - r_\ell^2 \geq 0. \qquad (4.5)$$

The auxiliary objective function is

$$L = \Sigma_T + D(s_1,s_2)\,\Big[\, a - \int_4^\infty (ks^{5/2})^{-1} \sum (2\ell+1)\, a_\ell P_\ell(w)\, ds \Big]$$
$$+ \sum (2\ell+1) \int_4^\infty \left(2ks^{5/2}\right)^{-1} \lambda_\ell(s,s_1,s_2)\,(a_\ell - a_\ell^2 - r_\ell^2)\, ds \qquad (4.6)$$

where $w = 1 + 8/(s-4)$.
From the general theory considered earlier, for a maximum,

$$\lambda_\ell(s,s_1,s_2) \geqslant 0 \qquad (4.7)$$

$$\lambda_\ell = 0 \quad \text{if} \quad 0 < a_\ell(s) < 1 \qquad (4.8)$$

Taking functional derivatives of L, we obtain

$$\frac{\partial L}{\partial r_\ell} = 0 = \lambda_\ell(s,s_1,s_2)\, r_\ell(s) \qquad (4.9)$$

$$\frac{\partial L}{\partial a_\ell} = 0 = \frac{\theta(s-s_1)\,\theta(s_2-s)\, q(s)}{(s-4)} - \frac{D(s_1,s_2)\, P_\ell(w)}{k\, s^{5/2}} + \frac{\lambda_\ell(s,s_1,s_2)(1-2a_\ell)}{2k\, s^{5/2}} \qquad (4.10)$$

In general when $\lambda_\ell = 0$ the solutions for $a_\ell$ and $r_\ell$ are indeterminate, but solutions exist to (4.10) only for discrete values of s. The contribution of the corresponding values of $a_\ell$ can make no contribution to $\bar{\sigma}_{tot}$ so they can be ignored. Exceptions to this general rule occur only for certain choices of weight function. For example, Roy (1972) considers

$$q(s) = C(k/s^{5/2}) : \qquad (4.11)$$

Then (4.10) can be satisfied for $\ell = 0$ for all s by choice of D. The problem reduces in this case to optimising $\bar{\sigma}_{tot}$ when only $a_0(s)$ is non-zero. One finds that if the scattering length $\alpha_2^t$ is not too large, one obtains the bound

$$\frac{15\pi}{8}\alpha_2^t > \frac{1}{8\pi}\int_{s_1}^{s_2} \sigma_{tot}(s)\, \frac{k\,ds}{s^{5/2}} \qquad (4.12)$$

Roy (1972) finds that bounds of this type are approached to within a factor 2 by currently accepted scattering lengths and cross-sections.

The corresponding bounds for more general weight functions have been evaluated by Steven(1972). The method follows from (4.10) ignoring the class I in which $0 < a_\ell < 1$, and considering only the classes $B_o : a_\ell = 0$, and $B_1 : a_\ell = 1$.

Blankenbecler and Savit (1972) have introduced a modification to the above method by assuming the $\pi\pi$ amplitude and its partial waves to be known for energies less than a known constant C. Above this energy they assume a particular functional form for $\sigma_{tot}$ and they take $\sigma_{e\ell} \leq \frac{1}{2}\sigma_{tot}$ for energy greater than C. These constraints lead to a lower bound on the scattering length. The requirement that this lower bound is less than the experimental value then sets a constraint on the assumed functional form for $\sigma_{tot}$. In the particular case where $\sigma_{tot}$ is taken to be a constant equal to $\sigma(\infty)$ above $s = 25\ m_\pi^2$, they find that $\sigma(\infty)$ is less than about 40 mb, which compares well with the factorization estimate of 15-20 mb.

(b) Pion-Nucleon Amplitudes

Reference: Common and Yndurain (1971 and 1972),
Kolanowski and Lukaszuk (1972), Steven (1972).

The above techniques for bounding averaged cross-sections have been extended by the above authors to pion-nucleon scattering. They also invert the argument to obtain bounds on the amplitudes in the crossed channel. Thus (4.12) provides a lower bound on the $\pi\pi$ scattering length if the total cross-section is assumed to be known, for example by putting in phenomenological $\pi\pi$ resonances.

In the case of $\pi N$ scattering the total cross-sections are known from experiment. Then the formalism leads to bounds in the crossed channel, namely

$$\pi\pi \rightarrow N\bar{N} \tag{4.13}$$

Common and Yndurain (1972) introduce the additional feature that the annihilation cross-section $N\bar{N} \rightarrow \pi\pi$ is constrained via unitarity by the $\pi\pi$ elastic amplitude,

$$|f_\ell(N\bar{N} \rightarrow \pi\pi)|^2 \leq \mathrm{Im} f_\ell(\pi\pi) - |f_\ell(\pi\pi)|^2 \tag{4.14}$$

This observation gives an immediate gain of a factor 4 in the analogue of the Froissart bound for the total annihilation cross-section

$\sigma(N\bar{N} \to \pi\pi)$. It can also be used in two further ways. One way is to obtain a local bound at finite energies on the annihilation cross-section when $\sigma_{tot}(\pi\pi)$ is assumed to be known. The latter constrains the partial wave series involving $\mathrm{Im}f_{\ell}(\pi\pi)$ and via (4.14) it constrains the partial wave expansion of $\sigma(N\bar{N} \to \pi\pi)$. The local value of $\sigma_{tot}(\pi\pi)$ may be taken from a Regge model since the energy for $N\bar{N}$ annihilation exceeds 2 GeV.

The constraint (4.14) is most relevant at low energies near the annihilation threshold. In this region (above 2 GeV) the bound may be used to test models for extrapolation of experimental values for $\sigma(N\bar{N} \to \pi\pi)$. These experimental values are obtained at energies of several GeV where the bounds are rather weak. At lower energies the bounds provide a cut-off to the region where a Regge model extrapolation for $\sigma(N\bar{N} \to \pi\pi)$ might be applicable. The results of Common and Yndurain are summarised in Fig. 2 which is adapted from their (1972) paper.

Finally Common and Yndurain use the Froissart-Gribov expression for the $\pi\pi$ scattering length to constrain the $\pi\pi$ amplitude as in (4.1) and (4.2). This then leads via (4.14) to constraints on the energy-averaged partial wave expansion of $\sigma(N\bar{N} \to \pi\pi)$. These energy averaged bounds are not very tight presumably because (4.14) is a weak constraint at most energies, since it is clear from experimental results and Regge models that $f_{\ell}(N\bar{N} \to \pi\pi)$ should tend to zero as the energy increases.

(c) Other applications of optimization theory

Within particle physics as further experimental evidence becomes available there will be more scope for studying more inequalities based on phenomenological constraints. This should apply particularly to inequalities involving spin parameters in two-body reactions and to inequalities involving multiparticle production and inclusive reactions. From a more theoretical viewpoint it is valuable to use inequalities to limit the effects on bootstrap calculations of unknown couplings to inelastic processes (Ciulli et al. (1972)).

Outside particle physics optimization theory forms an important part of many studies of complex systems whether in operational research, systems analysis or control theory. One of the central problems in its use for environmental or social applications arises from the conflict between desirable objective functions. This conflict, coupled with the multitude of influences, the paucity of data and the varying time scales of observation of change, leads to problems of such magnitude that the particle theorist may pause to reflect on the simplicity of his own problems of understanding the fundamental laws of physics.

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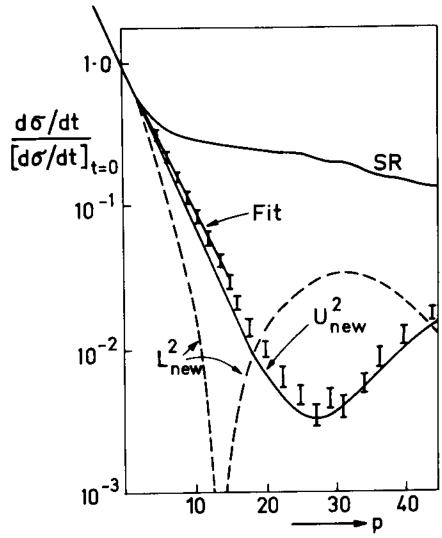


Fig. 1

Comparison with experiments (for $\pi^- p$ elastic at 2 GeV/c)of the Hahn-Hodgkinson semi-phenomenological bounds U = max ImF s,t), and L = min ImF(s,t). The parameter $\rho$ is proportional to t, the range shown being about 1 $(\text{GeV/c})^2$. Their fit to the data that led to the constraint at $4m_\pi^2$ is also shown. The Singh-Roy bound SR follows from unitarity and $\sigma_{el}$ ,$\sigma_{tot}$ only

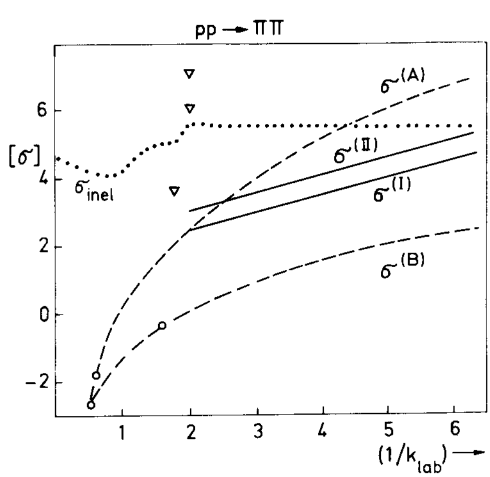


Fig. 2

The bounds of Common and Yndurain, compared with two extrapolations (A) and (B) of experimental values of $(p\bar{p} \rightarrow \pi\pi)$. The broken lines denote the extrapolations. The contiuous lines denote the bounds using two different Regge parametrisations I and II for the $\pi\pi$ amplitude. The dotted line denotes the total inelastic cross section for $p\bar{p}$. The triangles denote integrated bounds using different scattering lengths, the top one being that normally accepted