

On the motion of compact objects in relativistic viscous fluids

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ABSTRACT: We present a world-line effective field theory of compact objects moving relativistically through a viscous fluid. The theory is valid when velocity gradients are small compared to the inverse size of the object. Working within the EFT eliminates the need to solve a boundary value problem by turning all interactions between the fluid and the object into a source term in the action. We use the EFT to derive the relativistic equations of motion for a compact object immersed in a viscous fluid in a curved background, when the relative velocity of the object and the fluid is small compared to the speed of light.

KEYWORDS: astrophysical fluid dynamics, gravitational waves / theory, astrophysical black holes, neutron stars

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1 Introduction

Solving the Navier-Stokes equations presents a significant challenge which, for realistic three dimensional cases, at present, can only be attacked numerically. The inclusion of immersed dynamical objects considerably compounds this challenge as do relativistic and gravitational effects that arise, e.g., when simulating binary inspirals. The most severe challenge presented by the addition of dynamical objects is a set of boundary conditions that must be updated at each time step [1]. To quote from the review on Smoothed Particle Hydrodynamics (SPH) [2], “*the treatment of boundary conditions is certainly one of the most difficult technical points of the SPH method, so that it still remains a very active research topic.*” This challenge is even more pronounced in grid-based simulations. This paper aims to introduce a formalism that will mitigate the computational hurdles presented by this problem, all within a first principles approximation. Our formalism also facilitates analytic calculations when non-linearities can be treated as perturbations.

Let us start with the simple case of an infinitely stiff spherical body moving through a viscous fluid. We will expand the ratio R/r , with r being the typical scale for fluid gradients and R being the object’s size. In this limit it is natural to work within the confines of a Point Particle Effective Field Theory (PPEFT) which has been utilized to great success in various contexts including gravitational wave production from binary inspirals [3, 4], the interactions

between defects on fluctuating membranes [5] and Casimir forces on cogs [6] and monopole catalysis [7]. In these cases, the EFT is used to systematically generate analytic solutions from first principles. However, in the present context, an EFT will be used to simplify numerical approximations which in some cases may be intractable otherwise. It will also allow to write down relativistic equations of motion for compact objects in a viscous fluid.

The EFT starts with a world-line action for a point particle, and corrections due to finite size effects are accounted for by adding higher dimensional terms. These terms are accompanied by coefficients whose values encapsulate the short distance physics which has been coarse grained. In the context of our problem of interest here, the power of the EFT is manifested by the fact that we can effectively eliminate the need to impose boundary conditions at each time step. All of boundary effects in the EFT are captured by the introduction of an effective coupling in the action, whose value can be fixed by a matching calculation, as will be discussed below, which can, in general, be done analytically since the matching coefficient can be extracted for a fixed (symmetric) fluid configuration.

Since we are interested in immersing the particle in a fluid, the full EFT will include a bulk action for the fluid which will couple to the point particle. We will also show how viscosity affects the motion of the body in a generally covariant way at the level of the action using the in-in formalism. The bulk action in the in-in formalism was presented in [8],¹ and we will generalize this result to allow for viscous interactions between the body and the fluid. This will allow us to write down a generally covariant equation of motion for a body in a viscous fluid which, to our knowledge, has not appeared in the literature.²

For Stokes flows (viscosity dominated) the point particle approximation is often performed using what is called “Stokeslets” [11] which corresponds to point multipoles the strength of which can be matched to some full theory calculation. We can think of the EFT as a generalization of Stokeslets in that the EFT need not be viscosity dominated, the EFT is fully relativistic, and perhaps most importantly, will include non-linear effects on the world-line which is not included in any Stokeslet formulation.

2 The fluid EFT

The Lagrangian approach³ to fluid dynamics is usually based upon one of two variable choices which can be interpreted as two distinct frames. In the Eulerian variables $\phi^I(x, t)$ can be thought of as labeling the fluid element that is sitting at x and time t , while the Lagrangian frame describes the physics in terms of $X^I(\phi^I, t)$ which gives the position at time t of the particle whose label is given by ϕ^I . In this paper we will be working in the Eulerian frame where the comoving volume form is given by

$$\Omega(\phi) = \frac{1}{3!} \epsilon^{IJK} d\phi^I \wedge d\phi^J \wedge d\phi^K \quad (2.1)$$

We can build an algebraically conserved one form current

$$J = *\Omega, \quad (2.2)$$

¹For other approaches to relativistic viscous hydrodynamics [9].

²Although the form of the equation was clearly anticipated by [10].

³There is a long history of Lagrangian approaches to the field theory of fluids, for references see [12]. For a more modern discussion see [13, 14].

and in the non-relativistic limit we may associate $d * J = 0$ with the conservation of particle number.⁴ J^2 is the square of the (rest frame) density and can be written as $J^2 = \rho^2 \equiv \det(B_{IJ})$ where $B_{IJ} = \partial_\mu \phi^I \partial^\mu \phi^J$. The fluid four velocity u^μ is defined as a vector field which preserves the value of the label ϕ^I , i.e. $u \cdot \partial \phi^I = 0$, and is normalized so that in the fluids local rest frame $u^\mu = (1, \vec{0})$, i.e. $u^\mu = J^\mu / \sqrt{J^2}$ is the dimensionless four velocity normalized to one and we work in units where $c = 1$.

A fluid is defined in terms of an action which is invariant under volume preserving spatial diffeomorphisms,

$$\phi^I \rightarrow \phi'^I \quad (2.3)$$

with

$$\left| \text{Det} \frac{\partial \phi'^I}{\partial \phi^J} \right| = 1. \quad (2.4)$$

This symmetry defines a fluid. The power counting is such that one derivative on the field is leading order and subsequent derivatives are suppressed by powers of the UV cut-off, which in this hydrodynamic approximation is the mean-free-path. The leading order action is written in the convenient form

$$S_f = \int d^4x \rho U(\rho^{-1}), \quad (2.5)$$

where U is some unknown function.

Typically in an EFT we would expand in some background solution allowing us to generate a polynomial action with a set of Wilson coefficients that need to be fixed by matching. But when working in hydrodynamics we are interested in the Cauchy problem, not just small oscillations around a ground state. Furthermore, it does not behoove us to calculate the Euler-Lagrange equations in terms of the Euler variables ϕ^I . Instead, we use the fact the hydrodynamic equations of motion are equivalent to stress energy conservation where the stress tensor follows from the action

$$T^{\mu\nu} = u^\mu u^\nu (\rho U - U') + g^{\mu\nu} U'. \quad (2.6)$$

Considering the fluid at rest allows us to identify ρU with the energy density \mathcal{E} . Moreover, comparison with a perfect fluid form

$$T^{\mu\nu} = u^\mu u^\nu (\mathcal{E} + P) - g^{\mu\nu} P \quad (2.7)$$

allows us identify the pressure as well, $U' = -P$ and $U = \frac{\mathcal{E}}{\rho}$.

3 The point particle EFT in a perfect fluid

Typically world-line effective theories consider the dynamics of an object around some fixed background with a set of isometries. Here we wish to push the formalism a little further by considering the case where the object moves in a time-evolving background, i.e. a fluid flow.

⁴For relativistic fluids the conservation law becomes that of the entropy current.

In the world-line approximation, we assume that the gradients in the fluid are small compared to the inverse size of the particle. Conservatively the EFT will be under best control for laminar flow/low Reynolds number where the velocity gradients near the surface of the body are relatively small. This would seem to pose a challenge to cases where the fluid boundary conditions are such that the velocity of the fluid is discontinuous. However, since the viscosity (η)⁵ smooths out abrupt changes in velocity, we would expect the velocity gradients to scale as

$$\frac{dv}{dr} \sim \frac{1}{\eta}. \quad (3.1)$$

We will also consider the case of potential flow of a perfect fluid, under situations where the flow is laminar. Away from the body, where the EFT is valid, all gradients die off with higher powers of $1/r$.

To make our theory consistent with covariance, the object must be deformable. We could decide to treat the deformations as new world-line degrees of freedom, or if we assume that the time scale for the changes in the fluid velocity field is large compared to signal crossing time in the interior, we can account for the effect of these changes by including terms in the action which describe the, effectively instantaneous, response due the changing shape of the body. In the case of the gravitational world-line theory such terms are tidal response operators [15] proportional to the Weyl tensor squared and are suppressed by derivatives of the gravitational field. In the present context such finite size effects will be down by higher order gradients of the fluid field.

To build the EFT we parameterize the particle's world-line via $x^\mu(\lambda)$. The relevant symmetries are: world-line reparameterization, Lorentz invariance and volume preserving diffeomorphisms acting on the fluid field, ϕ^I . The most general action which satisfies these symmetries, and is leading order in the derivative expansion, is of the form

$$S_p = -M \int d\lambda \sqrt{\dot{x}^2} F\left(\frac{\dot{x} \cdot u}{\sqrt{\dot{x}^2}}, \rho\right) \quad (3.2)$$

where F is a dimensionless unknown function, and M is the object's mass. We notice that the combination

$$\gamma = \frac{\dot{x} \cdot u}{\sqrt{\dot{x}^2}} = 1 + (\vec{u} - \dot{\vec{x}})^2 f(\vec{u}, \dot{\vec{x}}), \quad (3.3)$$

is approximately one when the relative velocities vanish, and we will use this fact to expand F around this kinematic point.

From this action, we can extract the stress-energy tensor,

$$T^{\mu\nu} = M \int d\lambda \frac{\delta^4(x - x(\lambda))}{\sqrt{-g}} \left[\frac{\dot{x}^\mu \dot{x}^\nu}{\sqrt{\dot{x}^2}} F(\gamma, \rho) + \sqrt{\dot{x}^2} (u^\mu u^\nu - g^{\mu\nu}) \rho \frac{\partial F(\gamma, \rho)}{\partial \rho} + \sqrt{\dot{x}^2} \frac{\partial F(\gamma, \rho)}{\partial \gamma} \left(\frac{\dot{x}^\mu u^\nu + u^\mu \dot{x}^\nu}{\sqrt{\dot{x}^2}} - \gamma u^\mu u^\nu - \gamma \frac{\dot{x}^\mu \dot{x}^\nu}{\dot{x}^2} \right) \right]. \quad (3.4)$$

⁵For a perfect fluid the generation/non-generation of a boundary layer leads to D'Alembert's paradox discussed below.

Note that for the case of the fluid we were able to match for an unknown function (U) using covariance (2.7) and the known form for the stress energy tensor for a fluid. We could try to follow the same process in this case but now we have four tensor structures (as opposed to two in the case of the fluid) and even if we assume an internal equilibrium it's not clear how to match these functions. To gain predictive power we will expand the function F around $\gamma = 1$.⁶ In light of eq. (3.3), this expansion corresponds to one in small relative velocities which we may consider as a generic configuration. Moreover this limit overlaps with the non-relativistic limit where we will be performing our matching. In general, we will put our results in relativistic form.

In the non-relativistic limit the fluid's particle density will be approximately proportional to its energy density,

$$\mathcal{E}(\rho) = \mu\rho + \mathcal{O}(c^{-2}), \quad (3.5)$$

where μ is the mass of the underlying fluid particles. The object will respond primarily to the mass/energy density and current of the fluid, as opposed to the particle density. We then introduce the appropriate dimensionless variable,

$$\hat{\rho} = \frac{\rho\mu}{\rho_{ob}}, \quad (3.6)$$

where ρ_{ob} is the density of the object. $\hat{\rho}$ sets the magnitude the of drag forces. Without loss of generality we can then consider our action to be a function of $\hat{\rho}$ rather than ρ .

The relativistic stress energy tensor then takes the form

$$\begin{aligned} T^{\mu\nu} = M\delta^3(x - x(t)) & \left[\frac{\dot{x}^\mu \dot{x}^\nu}{\sqrt{\dot{x}^2}} (F(1, \hat{\rho}) - F'(1, \hat{\rho})) + (\dot{x}^\mu u^\nu + u^\mu \dot{x}^\nu - \gamma u^\mu u^\nu \sqrt{\dot{x}^2}) F'(1, \hat{\rho}) \right. \\ & \left. + \sqrt{\dot{x}^2} (u^\mu u^\nu - g^{\mu\nu}) \left(\hat{\rho} \frac{\partial F(1, \hat{\rho})}{\partial \hat{\rho}} + (\gamma - 1) \hat{\rho} \frac{\partial F'(1, \hat{\rho})}{\partial \hat{\rho}} \right) \right] + \mathcal{O}((\gamma - 1)^2), \end{aligned} \quad (3.7)$$

where prime denotes derivative with respect to the first argument. The four-velocities in the lab frame are given by

$$\dot{x}^\mu = \frac{(1, \vec{v})}{(1 - \vec{v}^2)^{1/2}} \quad \text{and} \quad u^\mu = \frac{(1, \vec{u})}{(1 - \vec{u}^2)^{1/2}} \quad \text{and} \quad \gamma = \frac{1 - \vec{u} \cdot \vec{v}}{(1 - \vec{u}^2)^{1/2} (1 - \vec{v}^2)^{1/2}} \quad (3.8)$$

In this limit the action is then given as

$$S_p = -M \int d\lambda \sqrt{\dot{x}^2} \left(F(1, \hat{\rho}) + F'(1, \hat{\rho}) (\gamma - 1) + \mathcal{O}((1 - \gamma)^2) \right) \quad (3.9)$$

3.1 Matching

To match $F(1, \hat{\rho})$ and $F'(1, \hat{\rho})$ we consider equating T_{00} in the full and effective theory. We are free to match in the non-relativistic limit since Lorentz symmetry will allow us to lift the result to the full relativistic action. Consider T_{00} in the static limit, $u = v = 0$

$$\begin{aligned} \int_V T_{00}^{\text{EFT}} &= M_{fl} + MF(1, \hat{\rho}) \\ \int_V T_{00}^{\text{full}} &= M_{fl} - \mu\rho V_p + M, \end{aligned} \quad (3.10)$$

⁶This is similar to what happens when considering the motion of a particle in Einstein-Aether theory [16].

where M_{fl} is the total mass of the fluid and V_p is the volume of the ball. The internal energy of the sphere has been absorbed into the mass. Crucially, in the full theory the fluid energy density is integrated only outside the ball, producing the $-\mu\rho V_p$ contribution, while in the EFT the ball has been treated as a point particle and the fluid energy density is integrated over all of space. The overcounting of fluid density in the EFT is precisely accounted for by the matching coefficient,

$$F = 1 - \hat{\rho}. \quad (3.11)$$

To match F' we calculate the force on the particle in both full and effective theory,

$$\frac{d}{dt} \frac{\partial L}{\partial v^i} = M((1 - \hat{\rho})\dot{v}^i + F'(\dot{u}^i - \dot{v}^i)) = MF'(\vec{v} - \vec{u}) \cdot \partial^i \vec{u} \quad (3.12)$$

where we have used eq. (3.11) and the incompressibility condition. In the full theory we can solve the problem for a potential flow around a sphere and the result for the force on a sphere is given by (see e.g. [17])

$$\dot{v}^i = \frac{1}{2}\hat{\rho}(3\dot{u}^i - \dot{v}^i). \quad (3.13)$$

We are immediately struck by the fact that in the full theory there is no net force on the particle in a constant flow. This is known as d'Alembert's paradox.⁷

There seems to be an inconsistency between the full and the effective theory. But we need to take care when interpreting the equations of motion. In the EFT we treat the delta function source as perturbation, so we write $\vec{u} = \vec{u}_0 + \delta\vec{u}$, for some constant background \vec{u}_0 . Then to leading order we can replace \vec{u} by \vec{u}_0 in the r.h.s. of equation (3.12), which then vanishes in agreement with the full theory result. Matching coefficients then yields

$$F' = -\frac{3}{2}\hat{\rho}. \quad (3.14)$$

3.1.1 Solving the modified Euler equation: alternative matching method

As previously stated, the power of the EFT is that it allows us to morph a dynamical boundary value problem into a problem with no boundaries but with localized sources on the right hand side of the fluid equations. In the full fluid theory the fluid equations of motion are equivalent to energy-momentum conservation. In the effective theory, provided that the particle also satisfies its equation of motion, the fluid equations of motion are equivalent to conservation of *total* energy-momentum conservation. The non-zero energy momentum of the particle then appears as localized sources in the continuity and Euler equations.

As another check on our previous matching we will see how the EFT reproduces the full theory fluid flow for a body at rest in a fluid which asymptotically is constant and non-zero in the z-direction in the non-relativistic limit. We will leave $F(1, \hat{\rho})$ and $F'(1, \hat{\rho})$ unfixed in the

⁷Perhaps it would more appropriately called a ‘‘puzzle’’. What is even more interesting phenomena is that while the drag force becomes finite when viscosity is turned on, it does not drop to zero as the viscosity is taken to zero. This is known as the ‘‘dissipative anomaly’’ (see e.g. [18]).

subsequent calculation, and determine their values by matching the fluid flow. Using the fluid and particle energy momentum tensors, eqs. (2.7) and (3.7), we obtain the continuity equation

$$\partial_\mu T^{\mu 0} = \partial_0 \left(\rho_{ob} \hat{\rho} + \delta^3(x) M F \right) + \partial_j \left(u^j \rho_{ob} \hat{\rho} + u^j \delta^3(x) M \hat{\rho} \frac{\partial F}{\partial \hat{\rho}} \right) = 0. \quad (3.15)$$

For static and incompressible flows this reduces to

$$\partial_j u^j = -V_p \frac{\partial F}{\partial \hat{\rho}} \partial_j \left(u^j \delta^3(x) \right), \quad (3.16)$$

where $V_p = M/\rho_{ob}$ is the volume of the body.

Conservation of momentum gives the Euler equation

$$\begin{aligned} \partial_\mu T^{\mu i} = & \partial_0 \left(u^i \left(\rho_{ob} \hat{\rho} + \delta^3(x) M \hat{\rho} \frac{\partial F}{\partial \hat{\rho}} \right) \right) + \partial_j \delta^{ij} \left(P + \delta^3(x) M \hat{\rho} \frac{\partial F}{\partial \hat{\rho}} + \frac{u^2}{2} \delta^3(x) M \hat{\rho} \frac{\partial F}{\partial \hat{\rho}} \right) \\ & + \partial_j \left(u^i u^j \left(\rho_{ob} \hat{\rho} + \delta^3(x) M \hat{\rho} \frac{\partial F}{\partial \hat{\rho}} - \delta^3(x) M F' \right) \right). \end{aligned} \quad (3.17)$$

Before proceeding to solving these equations are finite fluid velocity, let's look at the simplest case, $u^i = 0$. Here there is already a small lesson to be learned. There is an exact solution which has $u^i = 0$, for which the Euler equation only requires that the total stress is homogeneous in space

$$T^{ij} = \delta^{ij} \left(P_{\text{static}} + \delta^3(x) M \hat{\rho} \frac{\partial F}{\partial \hat{\rho}} \right) = \delta^{ij} P_0. \quad (3.18)$$

The Euler equation dictates that fluid pressure P_{static} is the sum of a constant term and a delta function term. The singular term serves only to cancel out the singular stress on the worldline, ensuring that the *total* stress is constant throughout space. Once we turn on finite fluid velocity, we will again find that both the stress on the particle and the solution for the fluid pressure contain distributional singularities but in the total stress these two contributions cancel out, as they must.

Let us now look for static and incompressible solutions to these equations which have finite fluid velocity. To do so, we will treat the particles effects on the flow perturbatively and expand the fluid about the constant solution $u_j = U^j + \delta u^j$, $P = P_0 + \delta P$.

In doing this expansion, we must recognize that terms of the form $\delta^3(x) \delta u^j(x)$ are singular since $\delta u^j(x)$ is itself sourced by the delta function. These singularities are well understood in the context of effective field theory and are readily accounted for by suitable regularization and renormalization of the parameters of the theory. For our purposes, we will implicitly work in an analytic regularization scheme, e.g. dimensional regularization, in which scaleless divergences like $\delta^3(x) \delta u^j(x)$ are automatically renormalized to zero. With this understood, the linearized Euler equation is simply

$$\rho_{ob} \hat{\rho} U^j \partial_j \delta u^i + \partial^i \delta P = M F' U^i U^j \partial_j \delta^3(x) - \frac{1}{2} U^2 M \hat{\rho} \frac{\partial F'}{\partial \hat{\rho}} \partial^i \delta^3(x) - M \hat{\rho} \frac{\partial F}{\partial \hat{\rho}} \partial^i \delta^3(x). \quad (3.19)$$

By contracting the Euler equation with the background velocity U^i , we can identify a Bernoulli equation which specifies a quantity that is constant along the background

streamlines, i.e. along the U^i direction. Since we require the perturbations $\delta P, \delta u^i$ to decay asymptotically, this Bernoulli equation gets lifted to a quantity which is constant over all of space, giving an expression for the pressure

$$\delta P = -\rho_{ob}\hat{\rho}U_j\delta u^j + \delta^3(x)M \left(U^2 F' - \frac{U^2}{2}\hat{\rho}\frac{\partial F'}{\partial \hat{\rho}} - \hat{\rho}\frac{\partial F}{\partial \hat{\rho}} \right). \quad (3.20)$$

Inserting this back into (3.19) we obtain

$$U^j(\partial_j\delta u_i - \partial_i\delta u_j) = \frac{V_p F'}{\hat{\rho}}U^j(U_i\partial_j - U_j\partial_i)\delta^3(x). \quad (3.21)$$

Note that this only determines the vortical part of the flow, $\omega^i = \epsilon^{ijk}\partial_j u_k$, and we'll need to check the continuity equation afterwards to determine whether additional longitudinal parts of u_j need to be added.

In terms of the vorticity the linearized Euler equation takes the simple form,

$$\epsilon_{ijk}U^j\omega^k = \frac{V_p F'}{\hat{\rho}}U^jU_l\epsilon_{ijk}\epsilon^{klm}\partial_m\delta^3(x). \quad (3.22)$$

This determines the vorticity up to terms tangent to U^j . However, since there are no other tensors that we can construct which are odd under parity and point along the U^j direction we can immediately solve for the vorticity

$$\omega^k = \frac{V_p F'}{\hat{\rho}}\epsilon^{klm}U_l\partial_m\delta^3(x). \quad (3.23)$$

We see that the object is acting as a dipolar point source for vorticity. This is quite analogous to the point source of a magnetic dipole, i.e. an infinitesimal current loop. The solution is still effectively potential flow away from the source since there is no ‘‘monopole’’ source for the vorticity. Using the standard Helmholtz decomposition of the flow into the sum of a gradient term and the curl of a vector potential, we can derive the Biot-Savart-like integral solution

$$\begin{aligned} \delta u^i &= -\epsilon^{ijk}\partial_j\nabla^{-2}\omega_k \\ &= -\frac{V_p F'}{\hat{\rho}}\frac{1}{4\pi}\left(\frac{U^i}{|x|^3} - \frac{3(U^j x_j)x^i}{|x|^5}\right). \end{aligned} \quad (3.24)$$

In writing this we've discarded a term proportional to $\delta^3(x)$. Since the solutions to the equation of motion in the EFT are only valid as an expansion in powers of $V_p/|x|^3$, the delta function term is an unphysical artifact. Its appearance here is not surprising (see e.g. [19]), but since we've already dropped other short distance contributions when we performed the perturbative expansion to solve the equation of motion, to remain consistent we must also discard the delta function term.

The solution for potential flow around a static sphere of radius R is elementary and is given by, see e.g. [17]

$$u^i = U^i + \frac{R^3}{2}\left(\frac{U^i}{|x|^3} - \frac{3(U^j x_j)x^i}{|x|^5}\right). \quad (3.25)$$

Matching the EFT to this exact solution we determine

$$F' = -\frac{3}{2}\hat{\rho}, \quad (3.26)$$

in agreement with (3.14) which had been derived by an equation of motion argument.

As previously mentioned, since the Euler equation only determined the vorticity we need to also check that our solution is consistent with the continuity equation (3.16). The careful statement of (3.16) is obtained if we multiply by x^i integrate over some region \mathcal{B} , and integrate by parts to lift the derivative off the delta function, that is

$$\int_{\partial\mathcal{B}} d^2x n_j x^i \delta u^j - \int_{\mathcal{B}} d^3x \delta u^i = V_{ob} \frac{\partial F}{\partial \hat{\rho}} U^i \quad (3.27)$$

It is straightforward to verify that the integral of the dipole solution δu^i over a sphere is zero, and so the bulk integral in (3.27) vanishes. Performing the surface integral over a sphere with constant radius, we find

$$\int_{\partial\mathcal{B}} d^2x n_j x^i \delta u^j = \frac{2}{3} \frac{MF'}{\rho_{ob} \hat{\rho}} U^i. \quad (3.28)$$

Consistency with the continuity equation then requires

$$F' = \frac{3}{2} \hat{\rho} \frac{\partial F}{\partial \hat{\rho}}, \quad (3.29)$$

which implies via (3.26) that

$$\frac{\partial F}{\partial \hat{\rho}} = -1, \quad (3.30)$$

in agreement with (3.11).

As another check we can compute the total pressure, i.e. the isotropic part of the stress tensor

$$T^{ij} \supset \delta^{ij} \left(P + \delta^3(x) M \hat{\rho} \frac{\partial F}{\partial \hat{\rho}} + \frac{u^2}{2} \delta^3(x) M \hat{\rho} \frac{\partial F}{\partial \hat{\rho}} \right). \quad (3.31)$$

Linearizing, and inserting our solutions for δP and δu_i we find the total pressure

$$T^{ij} \supset \delta^{ij} \left(P_0 - \frac{\mu \rho U^2 R^3}{2 r^3} (1 - 3 \cos^2 \theta) \right), \quad (3.32)$$

in agreement with the $\mathcal{O}((R/r)^3)$ pressure known for the exact solution.

3.2 Incompressibility and the point particle limit

One might still wonder how it is that we could assume that the fluid density was constant throughout all of space, given that the density should be zero “inside the object”. As is standard for the incompressible limit, the five unknown quantities u^i, P, ρ are reduced to four by prescribing the profile of the mass density and using the four equations coming from energy-momentum conservation to determine u^i and P . In the full theory a natural

choice is that the density be constant. In the EFT we might expect that the density is constant everywhere except on the location of the particle. An ansatz for the functional form of the density which models this is

$$\mu\rho(x) = \mu\rho_0 - \alpha\mu\rho_0 V_p \delta^3(x), \quad (3.33)$$

with some parameter α to account for the missing fluid. This is not, however, what we did when solving the equations in the previous section, where we simply took the density as a constant. Our assumption that $\alpha = 0$ may seem further invalidated by the fact that the pressure solved for in (3.20) contained a singular piece at the location of the object.

The resolution of this confusion is the fact that the part of the density profile (3.33) which is localized to the worldline can always be absorbed into a redefinition of the matching function $F(\gamma, \rho)$. To see this particularly clearly, notice that the energy momentum tensor (3.4) contains a contribution of the form

$$T^{\mu\nu} \supset M \int d\lambda \frac{\delta^4(x - x(\lambda))}{\sqrt{-g}} \left[\sqrt{\dot{x}^2} (u^\mu u^\nu - g^{\mu\nu}) \hat{\rho} \frac{\partial F(\gamma, \hat{\rho})}{\partial \hat{\rho}} - \gamma \sqrt{\dot{x}^2} \frac{\partial F(\gamma, \hat{\rho})}{\partial \gamma} u^\mu u^\nu \right]. \quad (3.34)$$

These tensor structures are exactly those which appear in the fluid stress tensor (2.7), so we can always redefine $F(\gamma, \hat{\rho})$ to absorb the singular part of the density. Given that F and its derivatives are unknown matching functions that are fit by matching in any case, allowing for a delta function in the density has no effect on any physical prediction. As a check of this statement we have also performed these calculations for general α . The matching coefficient $F(1, \hat{\rho})$ was modified, but $F'(1, \rho)$ was not. While the “fluid” density and pressure depended on α , the computed fluid flows and *total* energy-momentum tensor were ultimately independent of α , as claimed.

3.3 How the EFT reproduces d’Alembert’s paradox

As previously mentioned it is a text book calculation to show that the net force in a stationary potential flow vanishes [17]. This is known as d’Alembert’s paradox, which is the counter-intuitive result that the drag force is zero when the fluid is moving at fixed velocity with respect to the particle. This result typically follows by solving the boundary value problem, and integrating the pressure around the particle. But in the EFT we have dispensed with the boundary value problem. Indeed if we took the full theory solution to the flow and plugged into the r.h.s. of (3.12) we would not find zero. In fact that process would be ill defined as in the EFT we have already coarse grained over scales of order the radius. That is, the details of the flow at the edges have been integrated out. The question arises as to what we take for the velocity gradient at the position of the particle? The proper way to understand this issue follows from considering the meaning of the Euler equation in the EFT where the sphere is now acting like a vorticial point source that has an effective charge proportional to the relative velocity between the particle and the fluid. We may gain insight into this issue by asking the analogous question in the context of a massive particle generating a gravitational field. In that case the field is infinite at the source, and we know how that this corresponds to an infinite renormalization which we may set to zero. That is the field at the position of the particle, *generated by the particle* is taken to be zero. This is analogous

with how the self force is implemented in the EFT approach to the self force problem [20]. This does not mean that there is *never* a force on the point particle. If we had some other external sources (point particle in the EFT) then the field generated by that source will generate a non-zero gradient for the other particle.

Note that even though we fixed the matching coefficients in a non-relativistic context, once we have fixed F and F' we can return to the fully invariant EFT. This is the power of the EFT, we can fix the matching coefficient in the simplest of states, since it is a universal short distance coefficient. This action is valid for any laminar flow despite the fact that we calculate the matching coefficient in a potential flow. This may seem like an insignificant accomplishment given that to the order we are working the incompressible and irrotational nature of the flow is sub-leading. The question then arises whether or not for potential flow (which is solvable) one could deduce the equations of motion by solving for the force on the sphere and then simply treating the sphere as a particle, thus avoiding the need to use the EFT in the first place? The answer is no for the following reasons. It is important to recall that the functions $F(\rho)$ and $F'(\rho)$ should be evaluated on the surface of the sphere when matching, which by assumption, *would* take the same value when extrapolated to the origin, in the point particle picture. The errors incurred in this assumption being subleading in the gradient expansion. We may then match to the case of an incompressible flow. However, once we have determined how the functions F, F' depend upon ρ we may relax the assumption of incompressibility, as ρ is free to vary over distances much larger than the radius of the sphere. If we're interested in higher-order derivative corrections this would entail new matching coefficients.

Thus the EFT has allowed us to lift the force law determined by the potential flow to a non-potential flow. So that in the equations of motion for the incompressible case will differ, in a calculable way

$$M \frac{dv^i}{dt} = \frac{d}{dt} \left(\rho V_p u^i - \frac{1}{2} \rho V_p (v - u)^i \right). \quad (3.35)$$

3.4 The relativistic modified Euler equations

Our expansion remains valid in cases of fully relativistic velocities, provided the relative velocity between the fluid and the compact object remains small to c . This allows for a straightforward modification of the relativistic Euler equation within the same framework. Once again, Conservation of the relativistic energy-momentum tensor allows us to derive the relativistic modified Euler and continuity equations.

$$\partial_\mu T_{fl}^{\mu\nu} = u^\mu u^\nu \partial_\mu (\rho + p) + (\rho + p) u^\mu \partial_\mu u^\nu + (\rho + p) u^\nu \partial_\mu u^\mu - g^{\mu\nu} \partial_\mu p. \quad (3.36)$$

The projector to the plane perpendicular to the fluid velocity is given by

$$h_\beta^\alpha = \delta_\beta^\alpha - u^\alpha u_\beta, \quad (3.37)$$

which allows us to extract the relativistic Euler equation,

$$h_\nu^\alpha \partial_\mu T_{fl}^{\mu\nu} = (\rho + p) u^\mu \partial_\mu u^\alpha - g^{\mu\alpha} \partial_\mu p. \quad (3.38)$$

Finally, conservation of the full energy-momentum tensor leads to the relativistic modified Euler equation,

$$(\rho + p)u^\mu \partial_\mu u^\alpha - g^{\mu\alpha} \partial_\mu p = h_\nu^\alpha \partial_\mu T_{pp}^{\mu\nu}, \quad (3.39)$$

where T_{pp} is given in (3.34). While the relativistic modified continuity equation is obtained by projecting the conservation of the full energy-momentum tensor onto the fluid velocity.

$$u^\mu \partial_\mu \rho + (\rho + p) \partial_\mu u^\mu = u_\nu \partial_\mu T_{pp}^{\mu\nu} \quad (3.40)$$

3.5 Higher order corrections

Let us consider generalizing the action by including derivative corrections. Any term we add will be multiplied by some function of the leading order invariants $\frac{\dot{x} \cdot u}{\sqrt{\dot{x}^2}}$ and ρ . At the one derivative level we have two possibilities is $\partial \cdot u$ and $(\partial_\mu \rho u^\mu)$. However we can use the equations of motion (3.40) to eliminate one of those terms leaving

$$S_1 = \int d\lambda \sqrt{\dot{x}^2} \left[(\partial \cdot u) G_1 \left[\frac{\dot{x} \cdot u}{\sqrt{\dot{x}^2}}, \rho \right] \right]. \quad (3.41)$$

We can again expand G_1 around $\gamma = 1$ before, but it is no longer possible to match using the analytic solution. We may match by calculating numerically. The new equations of motion, in the non-relativistic limit, which correct eq. (3.13) are given by

$$M((1 - \hat{\rho})\dot{v}^i + F'(\dot{u}^i - \dot{v}^i)) = MF'(\vec{v} - \vec{u}) \cdot \partial^i \vec{u} - \partial_i (\partial \cdot u) G_1[1, \rho] - (\partial \cdot u) \frac{\partial G_1[1, \rho]}{\partial \rho} \partial_i \rho. \quad (3.42)$$

To our knowledge this result has not appeared in the literature.

3.6 The coupling to the metric in the Newtonian limit

The action (3.2) defining our EFT is covariant, so describing the interaction with gravity is completely straightforward. In general one might be interested in using this effective theory to study environmental effects on relativistic gravitational dynamics, as can be relevant for astrophysical compact binary inspirals. While such potential applications are interesting, in this work we will illustrate the utility of this EFT in just the simplest gravitational setting, Newtonian gravity. In doing so we will derive known results with relative ease.

For weak gravitational fields, $h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu} \ll 1$, the action simplifies to

$$S_p = -M \int d\lambda \sqrt{\dot{x}^2} F \left(\frac{\dot{x} \cdot u}{\sqrt{\dot{x}^2}}, \rho \right) + \frac{1}{2} \int d^4x \sqrt{-g} T^{\mu\nu} h_{\mu\nu}, \quad (3.43)$$

where the inner products in the first term are now taken with the flat metric. In the non-relativistic limit T^{00} dominates and this reduces to

$$S_p = -M \int dt \left(F(1, \hat{\rho}) \left(1 - \frac{v^2}{2} \right) + \frac{1}{2} F'(1, \hat{\rho}) (\vec{v} - \vec{u})^2 \right) + \frac{MF(1, \hat{\rho})}{2} \int dt h_{00}(x(t)). \quad (3.44)$$

Let us investigate a few consequences of this. First, for simplicity, let's set $\vec{u} = 0$. In doing so it becomes clear that the gravitational and inertial masses differ

$$S = -M \int dt + \int dt \left(M(F - F') \frac{v^2}{2} + MF \frac{h_{00}(x(t))}{2} \right). \quad (3.45)$$

If we insert the values for F, F' determined by matching, and introduce the Newtonian potential $\Phi = -\frac{1}{2}h_{00}$, the equation of motion is then

$$\frac{d^2 x^i}{dt^2} = -\frac{\rho_{ob} - \mu\rho_{fl}}{\rho_{ob} + \frac{1}{2}\mu\rho_{fl}} \partial^i \Phi. \quad (3.46)$$

We see that the gravitational attraction of the particle towards an external source is lessened by the presence of the fluid. In fact, if the fluid has the same mass density as the object then the object will not accelerate towards the gravitating body at all! We can check here the well known physics of a bubble in a uniform gravitational field $\Phi = gz$, $\rho_{ob} = 0$,

$$\frac{d^2 z}{dt^2} = +2g \quad (3.47)$$

seeing that indeed bubbles do rise, and they also accelerate at twice the standard gravitational acceleration of matter.

We are now equipped to understand what Archimedes was thinking some 2000+ years ago. Archimedes must have realized that the presence of a nonzero fluid density spontaneously breaks local Lorentz invariance so that below the symmetry breaking scale the ratio of gravitational to inertial masses is no longer protected from renormalization!

Adding a little more complexity we will work towards a description of the gravitational dynamics of two bodies submerged in fluid. To add a second body we would have two worldlines and two matching functions $F_{1,2}$. In the non-relativistic limit though, since T^{00} dominates we can readily compute the mutual gravitational potential by taking the known potential and substituting the masses for the relevant coefficients here,

$$V_N(|x_1 - x_2|) = -\frac{G(m_1 - \mu\rho V_{p1})(m_2 - \mu\rho V_{p2})}{|x_1 - x_2|}. \quad (3.48)$$

Where we've assumed $\vec{u} = 0$. Again, we see that buoyancy effects modify the mutual attraction. When both objects are more dense than the fluid, they attract. When one object is more dense but the other is less then they repel (again, think Archimedes). Amusingly we've also derived a third result, that two bubbles in a fluid will gravitationally attract one another. Our formalism allows us to reproduce these well known results in a way that allows us to calculate relativistic corrections in a straight forward fashion.

4 Allowing for viscosity

There are many ways to include viscosity in mechanical problems, but the closed time path formalism (or in-in or Keldysh formulation) is probably the best formulated for the EFT. It is more often utilized in out of equilibrium systems, but is also well suited for dissipative systems since it is time asymmetric. The reader interested in the details of the subject may consult [8, 21]. The path and the degrees of freedom are doubled such that we can write the world-line action as

$$\mathbf{S} = \int (L(\dot{x}_+, u_+, B_+) d\lambda_+ - L(\dot{x}_-, u_-, B_-) d\lambda_-) \quad (4.1)$$

The action should be invariant under two separate world-line reparameterization invariances $\text{Diff}_1 \otimes \text{Diff}_1$. It is convenient to work in the Keldysh basis

$$x_a^\mu = x_+^\mu - x_-^\mu \quad x_r^\mu = \frac{1}{2}(x_+^\mu + x_-^\mu), \quad (4.2)$$

with similar definitions implied for the fluid variables. The equations of motion are then determined by varying the action with respect to the x_a and then setting it to zero, so that, classically at least, we only need to consider terms in the action linear x_a . To generate dissipation we add terms to the action which involve both types of variables.⁸ We must do so in a way which is consistent with our symmetry group

$$G = \text{Diff}_1 \otimes \text{Diff}_1 \otimes \text{Diff}_3 \otimes P_4, \quad (4.3)$$

where P_4 is the 4-D Poincare group and Diff_3 are the group of volume preserving diffeomorphisms acting on the fluid elements.

If we try covariantize the action we are immediately struck by the fact that $x_{a,r}$ are not tensorial objects. Moreover, we have two different reparameterization groups for x_- and x_+ . While the action need not depend upon x_r , it will have to depend upon x_a or some covariantization thereof. Intuitively we can guess the relevant tensorial generalization of x_a should be given that it needs to reduce to the $x_+ - x_-$ in the flat space limit. Thus it should be formed from $\partial_\mu \sigma$ where sigma is the squared geodesic distance between x_+ and x_- . As we show in the appendix this is indeed correct.

There exists a methodology for building tensorial objects from two points for a review see [22]. First defining x_r as being the point along the geodesic (parameterized by s) $z^\mu(\lambda, s)$ halfway between x_+ and x_-

$$x_r^\mu(\lambda) = z^\mu \left(\lambda, s = \sqrt{\frac{\sigma}{2}} \right), \quad (4.4)$$

where $\sigma(x_-, x_+)$ is related to the geodesic distance between x_- and x_+ , while

$$x_{a\mu} = \frac{\partial}{\partial x_r^\mu} \sigma(x_r, x_+). \quad (4.5)$$

x_r is invariant under the off-diagonal combination of the two RPIs, whose effect is to change the geodesic curve passing through x_r . We may use this latter symmetry to impose that the tangent of the geodesic is orthogonal to \dot{x}_r . We can then form tensors which are invariant under residual RPI

$$\frac{\dot{x}_r^\mu}{\sqrt{\dot{x}_r^2}} \quad \text{and} \quad \Pi_{\mu\nu} x_a^\mu \equiv X_{a\nu}, \quad (4.6)$$

⁸This would not be the case if we were interested in quantum effects. The general principles for writing down the action are discussed in [8]. Moreover, in this work we are truncating to only the lowest order dissipative terms and not explicitly writing the terms which lead to fluctuations. To be consistent with the thermodynamics of the underlying microscopic theory, one should also include such fluctuation terms with coefficients related to the coefficients of the dissipation terms via ‘‘dynamical KMS symmetry’’, however these can be suppressed at low temperatures.

where $\Pi_{\mu\nu}$ is transverse projector

$$\Pi_{\mu\nu} \equiv \left(g^{\mu\nu} - \frac{\dot{x}_r^\mu \dot{x}_r^\nu}{\dot{x}_r^2} \right). \quad (4.7)$$

At lowest order in derivatives the unique interaction Lagrangian invariant under the symmetries is

$$\mathbf{S}_{\text{int}} = \int \frac{u_r \cdot X_a}{\sqrt{\dot{x}_r^2}} K \left(\frac{\dot{x}_r \cdot u_r}{\sqrt{\dot{x}_r^2}}, \rho_r \right) d\lambda. \quad (4.8)$$

To get the equations of motion we vary with respect to x_a and then set $x_+ = x_-$,

$$m\ddot{x}^\mu = \Pi^{\mu\nu} u_\nu K \left(\frac{\dot{x}_r \cdot u_r}{\sqrt{\dot{x}_r^2}}, \rho \right) \quad (4.9)$$

As in the conservative case we now expand the function K in the small relative velocities and choose the global time as the affine parameter. At next to leading order in velocities we have

$$m\ddot{x}^i = \Pi^{\mu\nu} u_\nu \left(K(1, \rho) + (K'(1, \rho) \left(\frac{\dot{x}_r \cdot u_r}{\sqrt{\dot{x}_r^2}} - 1 \right) + O((\gamma - 1)^2) \right) \quad (4.10)$$

This is the covariant generalization of the Stokes equation, which, to the best of our knowledge, has not previously appeared in the literature. A comprehensive treatment of covariant mechanics was given in [10], where the challenges involved in covariantizing dissipative forces were also discussed.

Now we would like to fix the matching coefficients $K(1, \rho)$ and $K'(1, \rho)$. As before we can match by working in the non-relativistic limit. The leading order equation of motion is

$$m\ddot{\vec{x}} = K(1, \rho)(\vec{u} - \dot{\vec{x}}) \quad (4.11)$$

The well known full theory solution for the force on a sphere in the Stokes limit (small Reynolds number) is given by $\vec{F} = 6\pi\rho\nu(\vec{u} - \dot{\vec{x}})$, where ν is the kinematic viscosity. Thus we find $K(1, \rho) = 6\pi\rho\nu$. To match $K'(1, \rho)$ we would need to calculate the corrections to Stokes law in the full theory. This can always be done numerically.

5 Applications

A primary motivation of this paper is to build a theory which allows one to calculate the corrections to gravitational wave signals from binary inspirals when the constituents are immersed in a fluid. At the outset we have assumed that the gradients in the fluid are small, i.e. that the flow is laminar with small Reynolds number $Re < 2000$. This is not within the regime we would expect for an inspiral. The Reynolds number will scale like $Re = \frac{Rv}{\nu}$. The viscosity (ν) of a fluid scales as $\nu \sim lv_T$, where v_T is the thermal velocity and l the mean free path. Thus to get a sufficiently small Reynolds number we would need the fluid to be extremely hot and dilute, which would imply a small density and therefore any drag force would be negligible. In general, we would expect dynamical friction to be the dominant dissipative force during the inspiral.

Furthermore, we have matched with hard wall boundary conditions which surely are inappropriate for neutron stars or black holes where we expect mass accretion. Indeed, once we can neglect drag forces on the object we no longer need to restrict ourselves to laminar flow, as the forces on the object will be purely gravitational. Given that we are still working in the point particle approximation we may ask if in the turbulent regime we will still be able to capture the effect in a systematic fashion? Accretion and drag forces in supersonic flows were famously studied by Bondi and Hoyle (for review see [23]), who were able to characterize the drag and mass accretion with some simple relations, which, despite some simplifying assumptions, seems to agree with modern simulations. It will be interesting to investigate if one could use the EFT at large Reynolds number and apply accretion rates from the Bondi-Hoyle analysis.

6 Discussion and future directions

In this paper we have developed a point particle EFT formalism which generalizes the gravitational EFT [4] to allow for the coupling of compact objects to fluids for the purposes of reducing the complexity of the coupled Navier Stokes equation. By considering the long wavelength approximation the EFT is capable of describing the motion of a compact object in a fluid as a point particle. There is no need to deal with nettlesome boundary conditions at each step. The physics of the boundary is encapsulated by some new point like interactions which arise in the equations of motion. The strength of these interactions is controlled by a set of constants which can be fixed by performing a matching calculation.

Previous work on the flow passed compact objects is restricted case of Stokes flow. One introduces point forces called “Stokelets” (see e.g. [24]) and chooses the amplitude of the forces to be proportional to the background flow. By working in the EFT, i.e. at the level of an action allows us to generalize this method in multiple ways. We can systematically include non-linearities and we can treat the system relativistically, as we have demonstrated. Thus our formalism could be useful even outside the context of binary inspirals, which was our primary motivation. Additionally, we believe that our formalism for the motion of a compact object in a curved space which includes dissipation is a new result. In particular, our covariant generalization of the Keldysh formalism for viscous motion is new.

We can use the EFT to calculate the forces on the object due to drag, however, the theory is only valid when the flow is laminar, so that the gradients in the fluid density are small compared to the inverse size of the objects. In this case we were able to write down a fully relativistic equation of motion and fixed the coefficients up to fourth order in the velocity difference in the perfect fluid case and to second order in the viscous case. In all cases our results were written completely covariantly allowing for the propagation in a general space-time. The results in this paper can be generalized to include finite size effects, or spin [25].

Our method will be most useful in situations where both post-Newtonian as well as fluid effects are relevant. These conditions are expected to be satisfied for the case of ERMI’s, but the relevance of the fluid on the inspiral will depend upon the specifics of the conditions. In cases where the drag forces are negligible the fluid effects will all be gravitational and the fact that the flow will not be laminar becomes less of an obstacle in that the contact

interaction, and our lack of understanding thereof, becomes irrelevant. However, the mass accretion rate will depend upon the possibly turbulent flow in the regime where we can't trust our results. In this case a phenomenological model, e.g. Bondi-Hoyle-Lyttleton [23], could be used, thus eliminating the need to calculate the flow in the region where the EFT fails. In such cases we may calculate the post-Newtonian corrections due to the fluid as well and thus capture all of the relevant effects.

A Generally covariant Keldysh variables

A.1 Motivation and results

To describe dissipation in a Lagrangian formalism one typically introduces the in-in framework. In this framework the dynamical variables, say (x, ϕ) , are doubled into two copies, and the dynamics of a standard conservative system is written as

$$\mathbf{S}[x_+, x_-, \phi_+, \phi_-] = S[x_+, \phi_+] - S[x_-, \phi_-]. \quad (\text{A.1})$$

The conservative nature of the underlying physics dictates the separation of the action into these two independent terms, ie. there is no coupling between plus and minus variables.

If the x^μ represent some system of interest and the ϕ are a collection of environmental variables, one can integrate out the ϕ variables to obtain an in-in effective action for the x^μ variables [26–28]. Quite generally the coupling to the environment will involve an exchange of conserved quantities such as energy-momentum, and the effective action for the system will reflect that the system itself no longer conserves energy-momentum (or in a quantum theory, the evolution may no longer be unitarity). The signature that the system is effectively “open” is that the effective action will involve coupling between the plus and minus variables, e.g.,

$$\mathbf{S}^{\text{eff}}[x_+, x_-] = S'[x_+] - S'[x_-] + \Phi[x_+, x_-], \quad (\text{A.2})$$

where $\Phi[x_+, x_-]$ need not separate into plus and minus contributions. In practice, the underlying dynamics may be known and the effective theory can be explicitly computed by exactly integrating out the environment. When the exact dynamics are too complicated, or even unknown, one can resort to a “bottom-up” construction of Φ as an effective theory. For details see [8].

Suppose the system of interest is a point particle moving through flat spacetime. One typically utilizes the in-in action by defining Keldysh variables

$$x_r^\mu = \frac{1}{2}(x_+^\mu + x_-^\mu) \quad x_a^\mu = x_+^\mu - x_-^\mu. \quad (\text{A.3})$$

The main convenience provided by these variables is that when we expand the action for small x_a we generate the standard classical equations of motion for x_r . For example, when $\Phi = 0$ we have

$$\mathbf{S}^{\text{eff}}[x_r, x_a] = S'[x_r] - S'[x_r] + \int x_a \frac{\delta S'[x_r]}{\delta x_r} + \mathcal{O}(x_a)^2. \quad (\text{A.4})$$

In a limit where higher order terms in x_a can be neglected, we see that extremizing the action with respect to x_a then imposes the classical equation of motion for x_r . If Φ is non-zero

and contains terms linear in x_a , we can then get modified equations of motion for x_r which are, in general, dissipative. Retaining high-order terms in x_a leads to effective stochastic forces in the equation of motion for x_r .

In this paper we are interested in a generally covariant description so that we may describe a particle experiencing dissipation while moving in curved spacetime. There is no obstruction to introducing plus and minus variables, but there is an immediate confusion about how to implement Keldysh variables. For example the time derivatives $\frac{dx_+^\mu}{d\lambda}$ and $\frac{dx_-^\mu}{d\lambda}$ are vectors living in entirely different tangent spaces, so how can we add and subtract them? In what follows we will provide a geometric construction which allows for a definition of Keldysh variables in curved spacetime. The crucial property possessed by the flat space expression (A.4), which we will require of our curved space construction, is that when the effective action is expanded in powers of x_a , the terms independent of x_a vanish and the remaining terms are local, i.e. given by the integral of an effective Lagrangian on the x_r worldline.

The dynamical variables are $x_\pm(\lambda)$, and the underlying action has independent reparameterization invariance under $\lambda \rightarrow \tilde{\lambda}(\lambda)$ for each worldline. To build our Keldysh variables we need a geometric relationship between points on the two worldlines. Due to the two-fold reparameterization invariance, for a given parameterization of x_- we have an infinite set of physically equivalent choices for mapping a given point $x_-(\lambda)$ to a point on the x_+ worldline. To proceed we will assume that the worldlines are in each other's normal convex neighborhood. That is, for every pair of points on x_- and x_+ there is a unique geodesic linking the two.⁹

Concretely, what we require is that for a given parameterization $x_-(\lambda)$ there exists a family of geodesics which are indexed by λ and parametrized by s , $z^\mu(\lambda, s)$, such that $z^\mu(\lambda, s_0) = x_-^\mu(\lambda)$ and that $z^\mu(\lambda, s_1)$ is a unique point on x_+ for all λ . For a choice of geodesic family $z^\mu(\lambda, s)$ and a parameterization $x_-(\lambda)$ we then have an induced parameterization $x_+(\lambda)$. The two-fold reparameterization invariance of the dynamics will manifest itself in a reparameterization invariance of z^μ in addition to invariance under the choice of geodesic family. For a given point $x_-(\lambda)$ we could have considered any geodesic $z^\mu(\lambda, s)$ which connects to a point on x_+ .

Once a geodesic family has been chosen, we can define Synge's worldfunction $\sigma(x_-, x_+)$. This object is a biscalar on each worldline, and its derivatives generate objects which are bitensors (see [22] for details). This is the object which allows us to link the points along each worldline in an invariant manner to yield tensorial quantities living in the same tangent space. The definition of Synge's worldfunction is

$$\sigma(x_-(\lambda), x_+(\lambda)) = \frac{1}{2}(s_1 - s_0) \int_{s_0}^{s_1} ds g_{\mu\nu}(z^\mu(\lambda, s)) \frac{dz^\mu(\lambda, s)}{ds} \frac{dz^\nu(\lambda, s)}{ds}, \quad (\text{A.5})$$

which is invariant under affine reparameterizations of s , and equal to one-half the squared geodesic distance between $x_-(\lambda)$ and the corresponding point $x_+(\lambda)$. The quantity $\epsilon \equiv g_{\mu\nu}(z^\mu(\lambda, s)) \frac{dz^\mu(\lambda, s)}{ds} \frac{dz^\nu(\lambda, s)}{ds}$ is a constant as a consequence of the geodesic equation, and

⁹We expect a sufficient condition for our construction to be that not all points on the two worldlines are in each other's normal convex neighborhood, but just that there exist collections of open sets, $\mathcal{U}_\pm = \{U_\pm\}$, which cover each worldline and that for every element U_+ there exists an element U_- in its normal convex neighborhood, and that there are sufficient consistency conditions on the overlaps of all these sets.

therefore numerically

$$\sigma(x_-(\lambda), x_+(\lambda)) = \frac{1}{2}(s_1 - s_0)^2 \epsilon. \quad (\text{A.6})$$

For time/space-like geodesics we can choose s to be the proper time/length and $\epsilon = \pm 1$.

Derivatives of this function are tangent to the geodesics

$$\frac{\partial}{\partial x_-^\mu} \sigma(x_-(\lambda), x_+(\lambda)) = -(s_1 - s_0) g_{\mu\nu}(x_-(\lambda)) \frac{dz^\mu(\lambda, s_0)}{ds}. \quad (\text{A.7})$$

An important property, which we will later use, is that this vector squares to the worldfunction

$$g^{\mu\nu}(x_-) \partial_\mu \sigma(x_-, x_+) \partial_\nu \sigma(x_-, x_+) = 2\sigma(x_-, x_+). \quad (\text{A.8})$$

This construction allows us to introduce Keldysh variables. Let us *define* x_r^μ to be the point along the curve $z^\mu(\lambda, s)$ which, for each λ , is 1/2 the geodesic distance from x_- to x_+ . Said differently, if s is taken as the proper length parameter (for space like geodesics)

$$x_r^\mu(\lambda) = z^\mu \left(\lambda, \sqrt{\frac{\sigma}{2}} \right). \quad (\text{A.9})$$

While we started from a description with x_\pm as independent variables, we can now make the choice to use x_r and x_+ as independent variables. Thus $\partial_\mu(x_r, x_+)$ may be taken as a vector living along the worldline x_r which is an independent variable to the curve x_r . We then *define* the Keldysh x_a variable to be

$$x_{a\mu} = \frac{\partial}{\partial x_r^\mu} \sigma(x_r, x_+). \quad (\text{A.10})$$

Whereas x_r^μ is not a vector, just a coordinate, we see that $x_{a\mu}$ is indeed a (co)vector. Using the definition of the worldfunction it is easy to check that

$$g^{\mu\nu} x_{a\mu} x_{a\nu} = 8\sigma(x_r, x_+) = 2\sigma(x_-, x_+) \quad (\text{A.11})$$

which is precisely the squared geodesic distance between x_- and x_+ , in agreement with the standard flat spacetime definition.

As alluded to previously there is a diagonal of $\text{Diff}_1 \otimes \text{Diff}_1$ which corresponds to reparametrizations of $x_r(\lambda)$. The off-diagonal action however will leave x_r invariant and will instead change the curve $z^\mu(\lambda, s)$ passing through a given point $x_r(\lambda)$. With all of this redundancy in the choice of z^μ , we are free to chose $z^\mu(\lambda, s)$ to intersect $x^\mu(r, \lambda)$ orthogonally, for each λ . That is, we are free to choose $\partial_\mu \sigma$ such that $\dot{x}_r^\mu \partial_\mu \sigma = 0$. We can manifest this orthogonality by building the action out of the building blocks

$$\frac{\dot{x}_r^\mu}{\sqrt{\dot{x}_r^2}} \quad \text{and} \quad X_a^\mu \equiv \left(g^{\mu\nu} - \frac{\dot{x}_r^\mu \dot{x}_r^\nu}{\dot{x}_r^2} \right) x_{a\mu}. \quad (\text{A.12})$$

While the former is intuitive, the validity of the latter can be confirmed by noting that this projector is essential to make sure that the four-forces in the equation of motion for x_r^μ preserve the on-shell condition

$$0 = \frac{d}{d\lambda} \dot{x}_r^2 \propto 2\dot{x}_r^\mu \frac{\partial \mathcal{L}}{\partial x_a^\mu}, \quad (\text{A.13})$$

as required by RPI invariance.

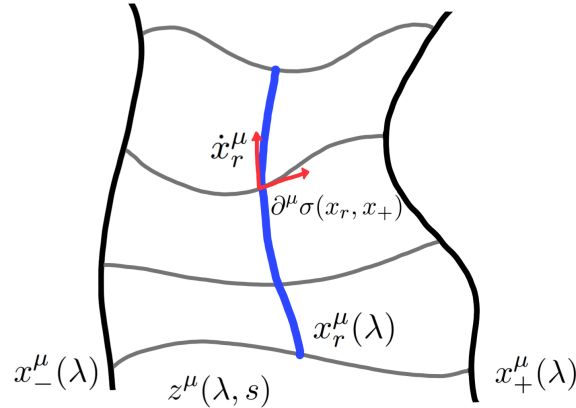


Figure 1. An illustration of our generally covariant Keldysh variable construction. A point on the left curve is labeled $x_-^\mu(\lambda)$, and is connected to a point on the right curve via a geodesic $z^\mu(\lambda, s)$. The intersection point on the right curve is then labeled $x_+(\lambda)$. The curve bisecting the geodesic family $z^\mu(\lambda, s)$ along the midline of proper length is called $x_r^\mu(\lambda)$. For a generic choice of geodesic family the tangent vectors \dot{x}_r^μ and $\partial^\mu \sigma$ at the point $x_r^\mu(\lambda)$ are not necessarily orthogonal, but can be made so by a suitable choice of geodesics $z^\mu(\lambda, s)$.

A.2 Further geometric intuition

To make this discussion more self-contained, here we provide some more geometric details and intuition to explain why Synge's worldfunction is the natural object to use in this construction.

We start with the \pm variables, for which the action is given by integrals of functions along each of the x_\pm worldlines. We desire a curved space generalization of Keldysh variables which will involve an action integrated only over some x_r worldline, with some tensorial quantity x_a^μ living on the x_r worldline. This Keldysh effective action, even in the absence of dissipative and stochastic contributions from $\Phi[x_+, x_-]$, must be able to describe the Lagrangians $L(x_\pm)$ which live on the x_\pm worldlines. The question is then, how can functions at x_\pm be described using objects living on x_r . To answer this, we employ the geodesic family $z^\mu(\lambda, s)$.

Once a geodesic family has been chosen there is a way to describe functions on x_+ via functions on x_- . For an arbitrary smooth function $f(x)$ we have that

$$f(x_+(\lambda)) = f(z(\lambda, s_1)) = \sum_n \frac{(s_1 - s_0)^n}{n!} \frac{d^n}{ds^n} f(z(\lambda, s)) \Big|_{s=s_0}. \quad (\text{A.14})$$

If z^μ is an affine parameterized geodesic then the higher s derivatives take a very simple form

$$\begin{aligned} f(x_+(\lambda)) &= \sum_n \frac{(s_1 - s_0)^n}{n!} \frac{dz^{\mu_1}(\lambda, s_0)}{ds} \dots \frac{dz^{\mu_n}(\lambda, s_0)}{ds} \nabla_{\mu_1} \dots \nabla_{\mu_n} f(x_-(\lambda)) \Big|_{s=s_0} \\ &= \exp \left((s_1 - s_0) \frac{dz^\mu(\lambda, s_0)}{ds} \nabla_\mu \right) f(x_-(\lambda)). \end{aligned} \quad (\text{A.15})$$

Thus, the function $f(x_+(\lambda))$ can be computed using vectors in the tangent space of x_- . In principle any affine parameter s can be used, but the above has a nice s -independent description in terms of the derivative of Synge's worldfunction (A.7),

$$f(x_+) = \exp \left(-g^{\mu\nu}(x_-) \partial_\mu \sigma(x_-, x_+) \nabla_\nu \right) f(x_-), \quad (\text{A.16})$$

where we've suppressed the λ dependence in each function, and the partial derivative is taken with respect to the first argument.

Similarly, once the mid-point worldline x_r has been introduced we can also use the derivatives of Sygne's worldfunction to describe functions at both x_- and x_+ ,

$$\begin{aligned} f(x_+) &= \exp\left(-g^{\mu\nu}(x_r)\frac{\partial}{\partial x_r^\mu}\sigma(x_r, x_+)\nabla_\mu\right)f(x_r), \\ f(x_-) &= \exp\left(-g^{\mu\nu}(x_r)\frac{\partial}{\partial x_r^\mu}\sigma(x_r, x_-)\nabla_\mu\right)f(x_r) \end{aligned} \quad (\text{A.17})$$

Since we've chosen x_r to be equidistant from both of these curves we have that

$$\frac{\partial}{\partial x_r^\mu}\sigma(x_r, x_+) = -\frac{\partial}{\partial x_r^\mu}\sigma(x_r, x_-), \quad (\text{A.18})$$

so that the expressions can be unified as

$$f(x_\pm) = \exp\left(\mp g^{\mu\nu}(x_r)\frac{\partial}{\partial x_r^\mu}\sigma(x_r, x_\pm)\nabla_\mu\right)f(x_r). \quad (\text{A.19})$$

Finally, we can define $x_{a\mu} \equiv \partial_{r\mu}\sigma(x_r, x_\pm)$. We can chose, instead of x_\pm , to take $x_{a\mu}$ and x_r^μ as the independent dynamical variables, with x_r^μ being the coordinates of a worldline and $x_{a\mu}$ being a tensor on the x_r worldline. In terms of these variables we clearly have the ability to describe functions along the x_\pm worldlines via

$$f(x_\pm) = \exp\left(\mp x_a^\mu\nabla_\mu\right)f(x_r). \quad (\text{A.20})$$

We have thus derived purely geometric quantities which possesses the desired properties and reduce to the familiar Keldysh variables in the flat space limit.

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