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Minimal length, generalized uncertainty principle and regularization of singular potentials

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Abstract. The minimal length hypothesis, introduced via a generalized uncertainty principle (GUP), provides a natural ultraviolet cutoff that regularizes divergences and singularities in quantum theory. To illustrate this property, we review key results for three singular interactions: the Coulomb, inverse-square, and Dirac delta potentials. We show that the minimal length acts as an intrinsic cutoff, systematically removing the singularities of these potentials in both relativistic and non-relativistic settings and yielding well-defined spectra. Therefore, the minimal length should be identified as a characteristic scale of the studied system.

1 Introduction

The Planck length ($l_p \approx 10^{-35}m$) emerges as a fundamental minimal scale, below which quantum-gravitational effects dominate and standard notions of measurement lose validity [1]. In this context, the Generalized Heisenberg Uncertainty Principle (GUP) provides a formal framework to account for this constraint [2, 3, 4, 5]. Originating from quantum gravity approaches [6, 2], the minimal-length GUP has been explored across diverse areas of theoretical physics, see, for instance, Refs. [7, 8, 9, 10]. For a critical discussion of conceptual issues and recent developments of the GUP, we refer the reader to Ref. [11]. Alternative versions of the GUP, which incorporate different bounds on position and momentum uncertainties, are outlined in Ref. [12].

Investigations of the GUP at low energies are mainly motivated by three considerations: its intrinsic UV/IR mixing, which may allow high-energy effects to surface at low scales [8]; the proposal that the minimal length may be system-dependent, thus making its formalism particularly relevant for composite systems [4, 13]; and finally, its role as a natural cutoff for ultraviolet singularities [14, 15, 16]. This work addresses the latter aspect by reviewing results for the Schrödinger equation with singular potentials and extending the discussion to relativistic systems. The analysis highlights the regularizing effect of the minimal length on singular interactions and corroborates the proposal that this fundamental length is scale dependent.

Instead of the great physical importance of singular potentials, they are problematic in standard quantum mechanics and often exhibit pathological features such as the non-self-adjointness of the Hamiltonian, the lack of a lower bound for the energy, or the appearance of arbitrary phases in the solutions [19]. To address these issues, advanced methods such as regularization [20], renormalization [21], and self-adjoint extensions [22] are required in order to solve the wave equations and extract the physical energy spectrum. Typical examples of singular interactions in the Schödinger equation include the inverse-square potential $-\alpha/r^2$ ($\alpha > \alpha_{cr}$) [19], the inverse power-law potential $-\alpha/r^n$ ($n > 2$) [19], and the Dirac delta potential $-\alpha\delta^D(x)$ in dimensions $D > 1$ [23]. In relativistic frameworks, further singularities arise: in the Klein–Gordon and Dirac equations, even the Coulomb potential $-\alpha/r$ becomes singular for $\alpha > \alpha_{cr}$ [19, 24], while in the Salpeter equation the Dirac delta potential is singular already in one dimension [25].

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The remainder of this paper is organized as follows. Section 2 introduces the basic elements of the minimal-length formalism based on the GUP. Section 3 discusses the deformed Schrödinger equation with a minimal length, reviewing results obtained for the inverse-square potential $-\alpha/r^2$ [13] and for the Dirac delta potential $-\alpha\delta^D(x)$ [17]. Section 4 analyzes deformed relativistic equations within the minimal-length framework: the Coulomb potential in the K–G equation, focusing in particular on the strong-coupling regime [26], and the one-dimensional Dirac delta potential in the Salpeter equation. Finally, Section 5 summarizes our main results and conclusions.

2 Minimal length and GUP

In this work, we focus on the GUP with a minimal length, expressed as [3, 4],

$$(\Delta x)(\Delta p) \geq \frac{\hbar}{2}\left(1+\beta(\Delta p)^2\right), \qquad \beta > 0. \tag{1}$$

This relation leads to a nonzero minimal position uncertainty, $(\Delta x)_{\min} = \hbar\sqrt{\beta}$, and follows from the modified commutation relation [4]

$$[\widehat{X},\ \widehat{P}] = i\hbar(1+\beta\widehat{P}^2), \tag{2}$$

The position and momentum operators are usually represented as [4]:

$$\widehat{X} = (1+\beta\widehat{p}^2)\widehat{x}, \qquad \widehat{P} = \widehat{p}. \tag{3}$$

To preserve operator symmetry, the scalar product is modified as [4]:

$$\langle \psi \mid \varphi\rangle = \int_{-\infty}^{+\infty} \frac{dp}{1+\beta p^2}\psi^*(p)\varphi(p). \tag{4}$$

The generalization to N dimensions of algebra (2) to imply a GUP with a minimal length is:

$$[\widehat{X}_i, \widehat{P}_j] = i\hbar\left(\delta_{ij}(1+\beta\widehat{P}^2) + \beta^{'}\widehat{P}_i\widehat{P}_j\right), \quad [\widehat{P}_i, \widehat{P}_j] = 0, \tag{5}$$

$$[\widehat{X}_i, \widehat{X}_j] = i\hbar\frac{(2\beta-\beta^{'}) + \beta(2\beta+\beta^{'})\widehat{P}^2}{1+\beta\widehat{P}^2}(\widehat{P}_i\widehat{X}_j - \widehat{P}_j\widehat{X}_i),\ \beta,\beta^{'} > 0. \tag{6}$$

The corresponding GUP implies a minimal length, given by [5]: $(\Delta X_i)_{\min} = \hbar\sqrt{(N\beta+\beta')}$.

The position and momentum operators satisfying Eqs. (6) are commonly represented by [5, 8]:

$$\widehat{X}_i = \left[(1+\beta\widehat{p}^2)\widehat{x}_i + \beta^{'}\widehat{p}_i\widehat{p}_j\widehat{x}_j + i\gamma\hbar\widehat{p}_i\right], \qquad \widehat{P}_i = \widehat{p}_i, \tag{7}$$

where γ is a small positive parameter related to β and $\beta^{'}$. To first order in β, one may also use [34]

$$\widehat{X}_i = \widehat{x}_i + \frac{2\beta-\beta'}{4}(\widehat{p}^2\widehat{x}_i + \widehat{x}_i\widehat{p}^2),\ \widehat{P}_i = \widehat{p}_i(1+\frac{\beta'}{2}\widehat{p}^2), \tag{8}$$

where $\widehat{x}_i$ and $\widehat{p}_i$ satisfy the standard commutators. For $\beta' = 2\beta$, this reduces to

$$\widehat{X}_i = \widehat{x}_i,\ \widehat{P}_i = \widehat{p}_i\left(1+\beta\widehat{p}^2\right), \tag{9}$$

first employed in Ref. [7].

3 Singular potentials in the Schrödinger equation with GUP

3.1 Inverse square potential

The $1/r^2$ potential plays a central role in quantum mechanics and appears in diverse contexts, from Efimov physics and dipole-bound states to black hole and cosmic string backgrounds [16]. Its singularity manifests in the Hamiltonian which is non–self-adjoint, and prevents the definition of a consistent bound-state spectrum. To deal with this potential, conventional approaches, such regularization [27], renormalization [21], or self-adjoint extensions [22], are used. Within the GUP framework, this potential has been studied in momentum space, in [13, 18] and in coordinate space in [16]. It has been shown that the minimal length acts as a natural cutoff. This regularizes the potential and leads to a well-defined discrete spectrum.

In this section we briefly illustrate the singular features of this potential in ordinary quantum mechanics, by using momentum representation, and we show that the potential becomes regular in the GUP formalism.

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3.1.1 Singular Features in ordinary quantum mechanics: an illustration in momentum space For the potential $V(r) = -\alpha/r^2$, the s-wave Schrödinger equation in momentum space admits the physical solution:

$$\psi(p) = AF(\frac{5}{4}+\frac{i}{2}\nu, \frac{5}{4}-\frac{i}{2}\nu, \frac{3}{2}; -\frac{p^2}{k^2}), \tag{10}$$

where: $\nu = \sqrt{\frac{2m\alpha}{\hbar^2} - 1/4}$, $2m\alpha/\hbar^2 > 1/4$, and $k^2 = -2mE$.

Solution (10) is square-integrable both near $p = 0$ and as $p \to \infty$, so integrability alone cannot serve as a quantization condition. Another peculiarity of the potential is the asymptotic behavior of $\psi(p)$ at infinity, which reads:

$$\psi(p) \sim p^{-\frac{5}{2}}\left(Ap^{-i\nu} + Bp^{+i\nu}\right) \sim p^{-\frac{5}{2}}\cos\left(\nu \ln p + \varphi\right). \tag{11}$$

Here, φ is an arbitrary phase. The eigenfunctions are not mutually orthogonal; indeed, the scalar product of $\psi_1(p)$ and $\psi_2(p)$ with eigenvalues k_1 and k_2, is given by [13]:

$$\langle \psi_1 | \psi_2 \rangle = C \sin\left[\nu \ln(\frac{k_1}{k_2})\right] \neq 0. \tag{12}$$

Requiring orthogonality leads to the discrete energy spectrum:

$$E_n \equiv E_2 = E_1 \exp[-\frac{2n\pi}{\nu}], \quad n = 0, \pm 1, ... \tag{13}$$

The orthogonality condition determines the relative, but not the absolute, energy levels. Fixing E_1 yields an infinite sequence of bound states accumulating at zero energy. The potential thus fails to describe short-range interactions and must be regularized. One of the simplest methods is to introduce a cutoff $\Lambda \gg k$, with the boundary condition: $\psi(\Lambda) = 0$, which leads to the following energy spectrum:

$$E_n = -\frac{\Lambda^2}{2m}\exp\frac{2}{\nu}\left[\arg(A) - (n+\frac{1}{2})\pi\right], n = 0, +1, +2, ... \tag{14}$$

This procedure yields a finite ground state. Other regularization approaches can be found in Refs. [27], while certain renormalization approaches are discussed in Refs.[21].

3.1.2 Natural Regularization within the Minimal-Length GUP Framework The physical solution to the deformed Schrödinger equation for the $1/r^2$ potential in momentum space is written in terms of Heun's function as [13]

$$\psi(\xi) = A(1-\xi)H(\xi_0, q, a, b, c, d; \xi), \tag{15}$$

where: $\xi = \frac{(\beta+\beta')p^2}{(\beta+\beta')p^2+1}$, $a = \frac{1}{2}(3-\epsilon-\widetilde{\nu})$, $b = \frac{1}{2}(3-\epsilon+\widetilde{\nu})$, $c = \frac{3}{2}$, $d = 2$, $e = \frac{1}{2}-\epsilon$, $\widetilde{\nu} = \left((\epsilon-1)^2 - \frac{4\kappa}{1-2\omega}\right)^{\frac{1}{2}}$, $q = -\left(\frac{3}{2} + \frac{\kappa}{1-2\omega}\right)$, $\epsilon = \frac{\beta}{\beta+\beta'}$, $\xi_0 = \frac{2\omega}{2\omega-1}$, $\omega = -m(\beta+\beta')E$, $\kappa = \frac{m\alpha}{2\hbar^2}$.

The requirement of square integrability of solution (15) at infinity imposes the spectral equation [13]:

$$H(\frac{2\omega-1}{2\omega}, \frac{2\omega-1}{2\omega}q, a, b, c, e; \frac{2\omega-1}{2\omega}) = 0 \tag{16}$$

In the limit: $\omega = -(\beta+\beta^{'})mE \ll 1$, the Heun's function reduces to a hypergiometric function and Eq. (16) gives the spectrum:

$$E_n = \frac{-1}{4m\beta}\exp\left\{\frac{2}{\nu}\left[\arg(A) - (n+\frac{1}{2})\pi\right]\right\}, \; n = 0, 1, 2, ..., \tag{17}$$

where: $|E_n| \ll \frac{1}{4m\beta}$, $A = \frac{\Gamma(i\nu)}{\Gamma(\frac{5}{4}+i\frac{\nu}{2})\Gamma(\frac{1}{4}+i\frac{\nu}{2})}$.

The spectrum (17) coincides with that obtained using cutoff regularization, Eq. (14), where β plays the role of Λ^{-2}. However, in this case, β has a physical meaning and it is inherently included in the formalism. Eq. (17) shows that E_n is inversely proportional to the minimal length: if β is very small, the ground-state energy exceeds the energy range where non-relativistic quantum mechanics is applicable. The minimal length must be viewed as representing an intrinsic scale of the system.

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3.2 Dirac delta potential

The Dirac delta potential is widely used as a solvable model of short-range interactions with applications in nuclear physics [28], atomic systems [29], and solid-state models [30], , as well as in topological effects such as the Aharonov-Bohm effect [31], and in curved backgrounds like cosmic strings [32]. It also serves as a model for analyzing the features of singular interactions by using self-adjoint, regularization or renormalization methods [21]. This potential has been studied within the GUP formalism in Ref. [33] for $D = 1$, and in Ref. [17] for $D = 1, 2, 3$, where it was shown that the minimal length absorbs the divergences arising for $D \geq 2$.

In this subsection, following Ref. [17], the treatment of the potential in ordinary quantum mechanics is presented, by highlighting its singular behavior for $D \geq 2$, and the regularization of the problem within the GUP framework is then outlined.

3.2.1 Treatment in ordinary quantum mechanics: Divergences and regularization The Schödinger equation in momentum space for the potential, $V(\mathbf{x}) = -\alpha\delta^D(\mathbf{x})$, admits the solution [17]:

$$\psi(\mathbf{p}) = N.\frac{2m\alpha}{(2\pi\hbar)^D}\frac{1}{\mathbf{P}^2 + k^2},\ k^2 = -2mE, \tag{18}$$

where N is a normalization constant, and with the quantization condition:

$$\frac{\Omega_D}{(2\pi\hbar)^D}\int_0^{+\infty}\frac{p^{D-1}}{p^2+k^2}dp = \frac{1}{2m\alpha},\quad \Omega_D = \frac{2(\pi)^{\frac{D}{2}}}{\Gamma\left(\frac{D}{2}\right)}. \tag{19}$$

For $D = 1$, the integral in Eq. (19) is finite and leads to the bound-state energy: $E = -\frac{m\alpha^2}{2\hbar^2}$. However for $D \geq 2$, this integral diverges and one must use regularization and renormalization to extract the energy. For example, for $D = 3$, the introduction of an ultraviolet cutoff $\Lambda \gg 1$, yields: $E \simeq -\frac{1}{2m}\left(\frac{4\pi\hbar^3}{\alpha_R}\right)^2$, where α_R is the renormalized coupling constant defined as: $\frac{1}{\alpha_R} = \frac{1}{\alpha_0} - \frac{\Lambda}{2\pi^2\hbar^3}$.

3.2.2 Treatment within the Minimal-Length formalism: Regularization By using the representation (9), the solution to the deformed Schrödinger equation has the following form [17]:

$$\psi(\mathbf{p}) = N\frac{2m\alpha}{(2\pi\hbar)^D(2\beta\mathbf{p}^4 + \mathbf{p}^2 + k^2)}, \tag{20}$$

where N is a normalization constant, and with the quantization condition:

$$\int_{-\infty}^{+\infty}\frac{d^D\mathbf{p}}{(2\beta\mathbf{p}^4 + \mathbf{p}^2 + k^2)} = \frac{(2\pi\hbar)^D}{2m\alpha}, \tag{21}$$

The integral in Eq (21) is finite even for $D \geq 2$. For instance, for $D = 3$, the binding energy is obtained as: $E \simeq -\left(\frac{\sqrt{m}\alpha}{8\pi\hbar^3\beta}\right)^2$ [17], which arises naturally, without the need for regularization or renormalization.

In the following section, will consider the Dirac delta potential in the context of the relativistic Schrödinger equation (Salpeter equation). We show that in this problem the divergency occurs even in $1D$, and the regularization is unavoidable. We show also that the problem becomes regular, as the nonrelativistic case, in the GUP framework.

4 Singular potentials in relativistic equations with GUP

4.1 Dirac delta potential in the Salpeter equation

Now we address the Dirac delta potential in the relativistic Schrödinger framework (Salpeter equation), where divergences appear even in $1D$, requiring regularization and renormalization [25]. We further show that, as in the nonrelativistic case, the minimal length naturally regularizes the problem.

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4.1.1 Undeformed case: Divergece and regularization Salpeter equation for a spinless particle of mass m interacting with the potential $V(x) = -\alpha\delta(x)$ reads :

$$\left(\sqrt{p^2+m^2}-E\right)\psi(p) - \frac{\alpha}{2\pi\hbar}\int_{-\infty}^{+\infty} dp'\psi(p') = 0, \tag{22}$$

for which the solution is:

$$\psi(p) = N\frac{\alpha}{2\pi\hbar}\frac{1}{\sqrt{P^2+m^2}-E}, \tag{23}$$

with the quantization condition

$$\int_0^{+\infty}\frac{dp}{\sqrt{p^2+m^2}-E} = \frac{\pi\hbar}{\alpha}. \tag{24}$$

One can check that the integral in Eq. (24) diverges logarithmically at large momenta. As in the nonrelativistic case, the divergence can be suppressed by introducing an ultraviolet cutoff Λ at large momenta, which for $\Lambda \gg 1$ gives:

$$\frac{1}{\alpha_R} = \frac{E}{\pi\hbar\sqrt{m^2-E^2}}\left[\frac{\pi}{2} + \arctan\left(\frac{E}{\sqrt{m^2-E^2}}\right)\right], \tag{25}$$

where, α_R is the renormalized coupling constant of the potential, defined by: $\frac{1}{\alpha_R} = \frac{1}{\alpha} - \frac{1}{\pi\hbar}\ln\left(\frac{2\Lambda}{m}\right)$. In the nonrelativistic limit, $E = E_{NR} + m$, with $|E_{NR}| \ll m$; Eq. (25) yields the binding energy $E_{NR} \simeq -\frac{m\alpha_R^2}{2\hbar^2}$. It should be noted that other regularization approaches can also be applied to this problem; see, for instance, Ref. [25].

4.1.2 Deformed case: regularization in the GUP framework By using the representation (3) in momentum space, and taking $c = 1$, the deformed Salpeter equation for the δ potential takes the integral form:

$$\left(\sqrt{p^2+m^2}-E\right)\psi(p) - \frac{\alpha}{2\pi\hbar}\int_{-\infty}^{+\infty}\frac{dp'}{1+\beta p'^2}\psi(p') = 0, \tag{26}$$

which has the solution

$$\psi(p) = N\frac{\lambda}{2\pi\hbar}\frac{1}{\sqrt{p^2+m^2}-E}, \tag{27}$$

where N is a normalization constant. The quantization condition turn to be:

$$\int_{-\infty}^{+\infty}\frac{dp}{\left(\sqrt{p^2+m^2}-E\right)(1+\beta p^2)} = \frac{2\pi\hbar}{\alpha}. \tag{28}$$

The integral in Eq. (28) is now finite. In the limit $\beta \ll 1$, one gets:

$$\frac{1}{\alpha} \simeq \frac{E}{\pi\hbar\sqrt{m^2-E^2}}\left(\frac{\pi}{2} + \arctan\left(\frac{E}{\sqrt{m^2-E^2}}\right)\right) - \frac{1}{2\hbar}\sqrt{\beta}E + \frac{1}{\pi\hbar}\ln\frac{2}{m\sqrt{\beta}}. \tag{29}$$

This result is similar to that obtained with an ultraviolet cutoff before renormalization. In Eq. (29), β is a physical parameter, therefore, applying renormalization would not be appropriate in this context. However, to recover the nonrelativistic binding energy from Eq. (29) in the proper limit, the condition $m\sqrt{\beta} \simeq 2$ (i.e., $(\Delta X)_{\min} \simeq \frac{\lambda_C}{\pi}$, λ_C is the Compton wavelength) must hold. Under these conditions, Eq. (29) reduces to

$$\frac{\hbar}{\alpha} = \sqrt{\frac{m}{-2E_{NR}}} - \frac{m}{2}\sqrt{\beta}, \tag{30}$$

for which solution is:

$$E_{NR} = -\frac{m\alpha^2}{2\hbar^2} + \frac{m^2\alpha^3}{2\hbar^3}\sqrt{\beta} - \frac{3}{8}\frac{m^3\alpha^4}{\hbar^4}\beta + \frac{1}{4}m^4\alpha^5\frac{\beta^{\frac{3}{2}}}{\hbar^5} + O\left(\beta^2\right). \tag{31}$$

This result is similar to that obtained in the Schrödinger equation [17, 33]. We conclude that analyzing this singular problem within the GUP framework not only demonstrates the regularizing role of the minimal length, but also establishes its order of magnitude in the relativistic regime, which turns out to be comparable to the Compton wavelength of the particle. This supports earlier findings regarding the effective interpretation of the minimal length when introduced into low-energy physics [13].

4.2 Coulomb potential in the Klein-Gordon equation

4.2.1 Undeformed case: Illustration of the singularity in momentum space For s-waves, the K-G equation with the potential $V(r)=\frac{-kZe^2}{r}$ admits the following physical solution [26]:

$$\psi\left(p\right)=\frac{A}{p}(1+\frac{ic}{\sqrt{\epsilon}}p)^{-\frac{3}{2}-\mu}F(\frac{3}{2}+\mu,\frac{1}{2}-w+\mu,2\mu+1;\frac{2}{1+\frac{ic}{\sqrt{\epsilon}}p}), \tag{32}$$

where: $w=\frac{KZEe^2}{\hbar c\sqrt{\epsilon}}$, $\mu=\frac{1}{2}\sqrt{1-\left(Z/Z_c\right)^2}$, $Z_c=\frac{k^2e^4}{4\hbar^2c^2}$. At infinity, the K-G equation admits two asymptotic solutions: $\psi_1\sim p^{-\frac{5}{2}-\mu}$ and $\psi_2\sim p^{-\frac{5}{2}+\mu}$. If $Z<Z_c\approx 68$, the physical solution is ψ_1, however if $Z>Z_c$, then, $\psi_1$and ψ_2 display the same asymptotic behavior, and the general solution becomes a linear combination of the two. This leads to the emergence of an arbitrary phase parameter, a typical feature of singular potentials.

For the energy spectrum, it can be obtained by requiring the square integrablility of the solution (32), which gives the condition [26]:

$$1/2-w+\mu=-n, \qquad n=0,1,2..., \tag{33}$$

Eq. (33) leads to the well-known relativistic spectrum of Coulomb potential [24].

It is easily seen that the condition (33) fails when $Z>Z_c$ (strong coupling regime). In this case the problem becomes singular and the use of regularization methods is mandatory [24].

4.2.2 Deformed case: regularizing effect of the minimal length

Zero-energy case To illustrate the regularizing effect of the minimal length, we analyse the deformed K-G equation in the case $E=0$. The use of the representation (7) with: $\ell=0$, $\gamma=0$ and $E=0$, the K-G equation takes the form [26]:

$$\psi''+\frac{2}{p}\left\{\frac{4p^2+2m^2c^2}{p^2+m^2c^2}-\frac{1+\beta'p^2}{1+\left(\beta+\beta'\right)p^2}\right\}\psi'+\left\{\frac{6+\left(10\beta+6\beta'\right)p^2}{\left[1+\left(\beta+\beta'\right)p^2\right]}+\frac{Z^2e^4/\hbar^2c^2}{\left[1+\left(\beta+\beta'\right)p^2\right]^2}\right\}\frac{\psi}{p^2+m^2c^2}=0. \tag{34}$$

At infinity, Eq. (34) admits two asymptotic behaviors: $\psi_1\sim p^{-3-2\beta/(\beta+\beta')}$ and $\psi_2\sim p^{-2}$. It is observed that there is no distinction between the strong- and weak-coupling regimes. The physical solution is ψ_1 regardless of the value of Z, which can be interpreted as a signal of potential regularization.

It has been shown that Eq. (34) is a Heun's differential equation, with the physical solution [26]:

$$\psi\left(\xi\right)=A\left(1-\xi\right)H\left(\xi_0,q,a,b,c,d,\xi\right), \tag{35}$$

where

$$\xi = \frac{(\beta+\beta')p^2}{1+(\beta+\beta')p^2}, a=\frac{1}{2}\left(3-\omega_1-\nu\right), b=\frac{1}{2}\left(3-\omega_1+\nu\right), c=\frac{3}{2}, d=2 \tag{36}$$

$$e = \frac{1}{2}-\omega_1, \omega_1=\frac{2\beta}{\beta+\beta'},\ \nu=\left(\left(\omega_1-1\right)^2-\frac{4\kappa}{1-2\omega_2}\right)^{\frac{1}{2}}, \kappa=\frac{k^2Z^2e^4}{4\hbar^2c^2} \tag{37}$$

$$q = -\left(\frac{3}{2}+\frac{\kappa}{1-2\omega_2}\right), \xi_0=\frac{2\omega_2}{2\omega_2-1}, \omega_2=\frac{1}{2}\left(\beta+\beta'\right)m^2c^2. \tag{38}$$

Nonzero-energy case: Deformed K-G Equation as a generalized Heun's equation In the nonzero energy case, the K-G equation for the Coulomb potential is a generalized Heun's differential equation of the form [26]:

$$\varphi''+\left(\gamma x+\delta x-1+\epsilon_1x-x_1+\epsilon_2x-x_2\right)\varphi'+(abx^2+\rho_1x+\rho_2x(x-1)(x-x_1)(x-x_2))\varphi=0, \tag{39}$$

where:

$$x=\frac{1}{2}-\frac{i}{2}\sqrt{6\beta}p, \omega=\frac{Ze^2E}{\hbar c^2}, \epsilon^2=m^2c^2-E^2/c^2, k=(Ze^2/\hbar c)^2, \tag{40}$$

$$a = 1, b = \frac{7}{3}, \rho_1 = -\frac{7}{3} - \frac{\omega}{3}\sqrt{6\beta}, \rho_2 = \frac{\beta\epsilon^2}{2} + \frac{\omega}{6}\sqrt{6\beta} - \frac{\lambda}{4} + \frac{1}{12}, \tag{41}$$

$$\gamma = \frac{1}{6} + \frac{\omega\sqrt{6\beta}}{2\left(1-6\beta\epsilon^2\right)}, \delta = \frac{1}{6} - \frac{\omega\sqrt{6\beta}}{2\left(1-6\beta\epsilon^2\right)}, \epsilon_1 = 2 + \frac{\omega\left(1-3\beta\epsilon^2\right)}{\left(1-6\beta\epsilon^2\right)\epsilon} \tag{42}$$

$$\epsilon_2 = 2 - \frac{\omega\left(1-3\beta\epsilon^2\right)}{\left(1-6\beta\epsilon^2\right)\epsilon},\ x_1 = \frac{1}{2} + \frac{\epsilon}{2}\sqrt{6\beta},\ x_2 = \frac{1}{2} - \frac{\epsilon}{2}\sqrt{6\beta}. \tag{43}$$

Eq. (39) is a Fuchsian equation with regular singularities at $z = 0, 1, x_1, x_2, \infty$. The analytic solutions of the generalized Heun's equation (39) are not available in the literature, and the corresponding deformed energy spectrum has not yet been derived. It is therefore important to further investigate the nonzero-energy solution in order to extract the energy spectrum and analyze the strong-coupling regime.

5 Conclusions

In this overeview, we highlighted that the GUP formalism with a minimal length provides a natural regularization of singular potentials in both nonrelativistic and relativistic quantum mechanics. We showed that the minimal length acts as an intrinsic ultraviolet cutoff, and that this fundamental scale need not be restricted to the Planck length when addressing singular potentials in quantum mechanics or phenomena in low-energy physics.

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