

# Open problems and results in the group theoretic approach to quantum gravity via the BMS group and its generalizations

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**Abstract.** The Bondi-Metzner-Sachs group  $B$  is the common asymptotic group of all asymptotically flat (lorentzian) space-times, and is the best candidate for the universal symmetry group of General Relativity. However, in quantum gravity, complexified or euclidean versions of General Relativity are frequently considered. McCarthy has shown that there are forty–two generalizations of  $B$  for these versions of the theory and a variety of further ones, either real in any signature, or complex. A firm foundation for quantum gravity can be laid by following through the analogue of Wigner’s programme for special relativity with  $B$  replacing the Poincare group  $P$ . Here the main results which have been obtained so far in this research programme are reported and the more important open problems are stated.

## 1. History generalizations, other approaches and open problems

In 1939 Wigner, having followed a suggestion of Dirac, published a remarkable paper [1] which laid the foundations of quantum theory of fields . This paper greatly influenced also mathematics since it was the first explicit treatment of infinite dimensional representations of a Lie group, namely, of the Poincare group  $P$ .

It is interesting that G.W.Mackey developed his pioneering work on group representations of semi–direct products [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13] by extending Wigner’s work and by using previous results derived by Clifford [14]. Remarkably, Mackey was able to show [2, 15] that Wigner’s pioneering construction, as well as the Stone–von Neumann theorem [16, 17] on the uniqueness of operators satisfying the Heisenberg commutation relations could be derived as consequences of a principal theorem generalizing to weakly (and hence strongly) continuous unitary representations of separable locally compact groups of a well–known theorem for finite groups.

It is noteworthy that the entirely novel discrete results on the spectra of area and volume derived in loop quantum gravity stem from the fact that the Stone–von Neumann theorem cannot be employed any more: The Stone–von Neumann theorem tells us that there is (up to equivalence) precisely one irreducible regular representation of the Weyl algebra generated by the exponentiated position and momentum operators together with their Poisson relations. In loop quantum gravity one of the assumptions of the Stone–von Neumann theorem is not valid and as a result an inequivalent (unique!) representation of the corresponding Weyl algebra is obtained (see e.g. [23]).

Wigner's paper was also deeply satisfying philosophically, being based only on the most firmly established experimental facts and theoretical principles so that his work was free of any tacit or ad hoc assumptions. In particular he based his study on the following premises:

- The set of pure states of an isolated quantum mechanical system was described by the projective space  $\mathcal{P}(H)$  of complex straight lines, through the origin, of a complex Hilbert space  $H$ .
- The relativity principle was expressed as the numerical invariance of all transition probabilities under Poincare transformations of the affine Minkowski space.

With these premises Wigner obtained:

- An exhaustive list of the strongly continuous unitary irreducible representations (IRs) of  $P$ .
- A complete classification of all possible solutions of all possible  $P$ -invariant wave equations without having to find or solve the equations!. These solution sets are precisely the IRs of the universal cover of  $P$ .
- The only known plausible theoretical definition of what a relativistic 'elementary particle' is: An 'elementary particle' is that entity which is described by one of the IRs of  $P$ . Quite unexpectedly, he showed that these IRs are fully parameterized by the mass and the spin of elementary particles. By doing so, he demonstrated in the most dramatic way 'The unreasonable effectiveness of mathematics in the natural sciences'.

In 1962, an intensive study of asymptotically flat axially symmetric space–times, representing bounded gravitational sources emitting gravitational radiation, was undertaken by Bondi, Metzner and Van der Burg [19, 20]. Soon after Sachs [21] showed that their results hold intact if the simplifying assumption of axial symmetry is dropped. This study was preceded by pioneering work of Trautman [22] on boundary conditions appropriate for gravitational radiation. Bondi, van der Burg, Metzner and Sachs analyzed the Einstein equations under the assumption that the spacetime metric admits a power series expansion in a distance from sources (measured along a system of null geodesics) with the leading term being the Minkowski metric. The main result was a simple definition of the total gravitational energy (the Bondi mass), which, due to the Einstein equations, is a nonincreasing function of a retarded time (a similar result was obtained earlier in [22], see [23]). Besides, there was a totally unexpected spin–off: it was found that the set of coordinate transformations which preserve the imposed, gravitational radiation–dictated, boundary conditions form a group, much larger than the Poincare group. the so called Bondi–Metzner–Sachs group  $B$ . Consequently,  $B$  is an unavoidable consequence of the presence of gravitational waves in General Relativity.  $B$  is the same for the whole class of asymptotically flat space–times which share the aforementioned boundary conditions – hence its universality and importance. In this sense,  $B$  is the common asymptotic group of all asymptotically flat (Lorentzian) space–times, and is the best candidate for the universal symmetry group of General Relativity. As such, one may expect  $B$  to play a role in General Relativity similar to the role of the Poincare group  $P$  of Special Relativity. In fact,  $B$  aroused considerable interest as a possible candidate for replacing the Poincare group [24]. A study of unitary representations of the group was initiated by Sachs [24], and taken further by Cantoni [25, 26]. A little later, Komar [27] and Newman [28] suggested the possible relevance of the group to the problem of quantizing the gravitational field. Much later Awada, Gibbons and Shaw found [29] the appropriate generalization of this group for supergravity. Their analysis is valid for *any* space–time dimension.

$B$  was first discovered [20, 21] *not* as a transformation group of (exact) global diffeomorphisms of a fixed manifold, but as a pseudo–group of local diffeomorphisms ('asymptotic isometries') of the asymptotic region of (Lorentzian) space–times which are asymptotically flat in lightlike

future directions. However, Penrose showed with his conformal technique [30] that, by ‘going to infinity’,  $B$  could be interpreted as an (exact) global transformation group  $B \times \mathfrak{S}^+ \rightarrow \mathfrak{S}^+$  of the ‘future null boundary’  $\mathfrak{S}^+$  of the space–times concerned. The importance of groups in Physics stems from the fact that they are automorphism groups of some underlying structure. Penrose showed that this is the case for  $B$ : He gave a geometric structure to  $\mathfrak{S}^+$ , the ‘strong conformal geometry’, such that the classical action  $B \times \mathfrak{S}^+ \rightarrow \mathfrak{S}^+$  is the group of automorphisms of this geometry.

The key observation behind Penrose’s conformal technique is that ‘infinity’ is far away with respect to the physical space–time metric. This means that one needs infinitely many ‘meter sticks’ in succession to ‘get to infinity’. But what if by a conformal rescaling of the metric we replaced these meter sticks by ones that grow in length the farther out we go? Then it might be possible that only a finite number of them suffices to cover an infinite range, provided the growth rate is just right. This is not a new idea. In the theory of functions of one complex variable is common practice to bring infinity to a finite distance by using a conformal transformation. Penrose realized that he could use this type of transformation in General Relativity to map infinity to the boundary of an unphysical space–time which lies at finite distance with respect to the unphysical metric which is conformally related to the metric of the physical space–time; the effectiveness of the conformal technique in General Relativity partially stems from the fact that null geodesics are invariant under conformal rescalings of the metric. The conformal technique was in accord with Penrose’s novel approach to General Relativity whose main ingredient was to take the conformal structure of space–time as fundamental. With Penrose’s conformal technique, roughly speaking, points at infinity, which can be thought of as endpoints of null geodesics, are adjoined to the space–time by means of a conformal rescaling of the metric. The existence of a conformal factor which compresses the points at infinity into a pair of finite null hypersurfaces  $\mathfrak{S}^\pm$  containing the endpoints of all null geodesics, together with a causality assumption and the vacuum (Einstein or Einstein–Maxwell) field equations in the asymptotic region, guarantee [31] that are non–shearing null hypersurfaces with the topology  $R \times S^2$ . Penrose’s analysis provided a basis for later work due to Newman [32], Newman and Penrose [33], Schmidt [34], Winicour [35], and others. These investigations have been crucial to the study of radiation – especially gravitational radiation – and also to the development of several ideas concerning global issues in general relativity. In particular, a great deal of the analysis of black holes, singularities [36], and, more recently, of H–spaces [37] and of asymptotic quantization of zero rest mass fields [31, 38] relies heavily on concepts and techniques – the conformal technique playing a prominent role – introduced originally in the investigation of null infinity.

$B = C(S^2, R) \otimes_T L_+^\uparrow$ , i.e., is the semi–direct product of the group of *globally* defined orientation–preserving conformal transformations of the unit two–sphere, which is isomorphic to the orthochronous homogeneous Lorentz group  $L_+^\uparrow$ , times the abelian normal subgroup  $C(S^2, R)$  of so–called supertranslations (no relation to supersymmetry);  $C^\infty(S^2, R)$  is the set of real–valued functions defined on the unit two–sphere. Supertranslations occur for a direct physical reason [39]. Imagine a family of observers in the far field zone (‘near  $\mathfrak{S}^+$ ’) of a bounded source which emits gravitational radiation, at various angular coordinates  $(\theta, \phi)$  who synchronize their clocks. A gravitational wave passes and they examine their clocks again. They will find that they have become desynchronized, that is, they have passed from a common retarded time  $u$  to one of the form  $u + \alpha(\theta, \phi)$ , where  $\alpha(\theta, \phi) \in C(S^2, R)$ . In Special Relativity, there are no gravitational waves, this sort of desynchronization cannot occur, and therefore one has no need to introduce  $B$ .

Wigner’s work for  $P$ , and the universal property of  $B$  for general relativity, make it reasonable to attempt to lay a similarly firm foundation for quantum gravity, by following through the analogue of Wigner’s programme with  $B$  replacing the group  $P$ . Some years ago McCarthy constructed explicitly [40, 41, 42, 43] the IRs of  $B$  for exactly this purpose. He based his

work on G.W.Mackey's pioneering work on group representations [8, 10]; in fact MacCarthy's construction necessitated an extension to the relevant infinite-dimensional case of Mackey's semi-direct product theory.

One striking difference between  $P$  and  $B$  is as follows. The 'little groups' for  $P$  (from which the IRs are induced) are compact for positive mass squared, but non-compact for zero or negative mass squared. While compact little groups always give discrete spins, the non-compact ones also give *continuous* spins. However, the little groups for  $B$  are *always* compact [44, 45, 46, 47], and this means that the IRs of  $B$  necessarily have only *discrete* spins. Since only discrete spins are observed in nature, this means that the presence of gravity, or General Relativity, via  $B$ , gives a possible explanation that continuous spins are never observed. Consequently, apart from its connection with gravitation, the BMS group seems to be suited to elementary particle physics [41] because the 'elementary particles' it defines have only discrete spins with a finite number of polarization states and because it leaves open the possibility of a coupling, which avoids the pitfall of O'Raifeartaigh's theorem [48], with an internal symmetry group. Evidently representations of the BMS group should be used in the context of an S-matrix theory.

Another striking difference between  $P$  and  $B$  is that, while the little groups of  $P$  are all of *infinite* order (they are all three dimensional connected Lie groups), some of the little groups of  $B$  are *finite*. These finite groups are precisely the (double covering groups of the) symmetry groups of the regular polygons or polyhedra in ordinary Euclidean 3-space. That is, certain IRs of  $B$  are induced from the complex linear IRs of the cyclic or dihedral groups, or symmetry groups of the tetrahedron, cube or icosahedron [42].

Much later Kronheimer classified completely [49, 50] (see also [51, 52]) the ALE gravitational instantons which solve the real nonlinear euclidean self-dual Einstein equations. He showed that the parametrization of their spaces (moduli spaces) intimately involves the same complex linear IRs of these same finite symmetry groups (and not just the groups themselves). However, his description only partially describes the moduli spaces since it still involves *constraint equations* which remain unsolved. Moreover, gravitational instantons arise for either complex or Euclidean space-times, and this made it important to find analogues of  $B$  appropriate to complex space-times, or space-times of different signatures.

The role of the IRs of  $B$  is, however, much less well understood than the role of Wigner's IRs of  $P$ . In order to make this role better understood, and to relate the group theoretical approach more closely to other approaches to quantum gravity, where complexified or euclidean versions of General Relativity are frequently considered, McCarthy constructed analogues of  $B$  for these versions of the theory, and a variety of further ones, either real in any signature, or complex and obtained 42 groups  $G$  [53]. Three of them are complex, whereas the remaining 39 are real. All these groups  $G$  have a semi-direct product structure :

$$G = C(A, R) \circledast_T H, \tag{1}$$

where,  $C(A, R)$  is an Abelian subgroup of  $G$ .  $A$  denotes in all cases a locally compact space.  $C(A, R)$  are the supertranslations, real-valued functions defined on  $A$ . The representation 'T' of  $H$  on  $C(A, R)$  which specifies in each case the semi-direct product is not relevant here. Among these 42 groups more relevant to the purposes of this note are the 'Spi' group  $B_S$  ('Spi' stands for spatial infinity; it rhymes with 'Scri' which denotes null infinity), the Euclidean BMS group  $EB$ , the complex BMS group  $\mathcal{CB}$  and its two real sections : the original BMS group  $B$  and the

Ultrahyperbolic BMS group  $B(2, 2)$ . These groups are given respectively by :

$$B_S = C(S^2 \times I, R) \otimes_T SL(2, C), \quad (2)$$

$$EB = C(S^3, R) \otimes_T (SU(2) \times SU(2)), \quad (3)$$

$$CB = C(S^2 \times S^2, C) \otimes_T (SL(2, C) \times SL(2, C)), \quad (4)$$

$$B = C(S^2, R) \otimes_T SL(2, C), \quad (5)$$

$$B(2, 2) = C_e(S^1 \times S^1, R) \otimes_T (SL(2, R) \times SL(2, R)), \quad (6)$$

where  $I$  and  $S^n$  denote respectively an open interval of  $R$  and the  $n$ -dim sphere. Ashtekar and Hansen [54], in examining both space-like and null infinity for real lorentzian space-times derived a BMS-like group based on the unit space-like hyperboloid.  $B_S$  is easily shown to be isomorphic to the group defined by Ashtekar and Hansen.

$B$  is often interpreted via its classical action  $B \times \mathfrak{S}^+ \rightarrow \mathfrak{S}^+$  on Penrose's future null infinity  $\mathfrak{S}^+$ . This classical action is vital for understanding, for example, the nature of classical linear and angular momentum for radiating space-times. Interestingly enough each of the generalizations  $G$  of  $B$  has an analogous classical action. In fact each group  $G$  has two actions, the first being a 'lifting' of the second 'projective' action. It is the *projective* actions of the groups  $G$  which generalize the Penrose action  $B \times \mathfrak{S}^+ \rightarrow \mathfrak{S}^+$ . The spaces on which the groups act are interpreted not as 'boundaries at infinity', but rather as 'blow-ups' at the origin of parts of punctured flat spaces [55, 56, 57].

Let us pause for a moment and revise the situation in  $4-d$  General Relativity. The asymptotic structure of the gravitational field of isolated systems has been investigated in detail in two regimes; at large null separations from sources and at large space-like separations. At present there are two main approaches to the problem of the asymptotic structure of the gravitational field at large null separations; that of Bondi, Metzner, Van der Burgh [19, 20] and Sachs [21] and that of Penrose [30, 31] in the framework of his conformal technique. An important link between the two approaches was found by Tamburino and Winicour [58] who gave a construction of the Bondi-Sachs coordinates for metrics satisfying Penrose's assumptions and the vacuum Einstein equations.

In the spatial regime, on the other hand, the situation is less satisfactory as far as the geometrical results are concerned. The early work on the subject [59, 60, 61, 62] was based on slicing of spacetime into space and time: the boundary conditions were introduced on the initial data on a Cauchy surface rather than on the 4-geometry as a whole. Furthermore, the status of the asymptotic symmetry group at spatial infinity remained unclear because these frameworks are ill equipped to explore the spacetime supertranslations. Indeed, in the literature on these 3+1 frameworks, one can find conflicting statements even on what the asymptotic symmetry group is.

To overcome these problems, five manifestly covariant approaches were introduced in the late 1970s. In the first of these covariant approaches Geroch [63, 64, 65] geometrized Arnowitt, Deser and Misner's approach by using Penrose's conformal technique. He attached an additional 'point at infinity' to the 3-dim Cauchy surface. The resulting conformally completed manifold has the topology of a 3-sphere. To study further spatial infinity Geroch introduced a 3-dim manifold  $S_G$  (consisting in his formulation of all 'points' at spatial infinity) with an appropriate metric. In the second of these covariant approaches Ashtekar and Hansen [66, 67], unified spatial and null infinity. Their method is a four-dimensional formulation of Geroch's version [63, 64, 65] of the Arnowitt, Deser and Misner's approach. According to this method spatial infinity is brought metrically close by a conformal transformation of the physical metric and is represented by a single point  $i^0$ . This point is attached to the space-time and serves as its spatial boundary. This framework involves a conformal completion of the *entire* space-time, null infinity becoming a null cone with spatial infinity  $i^0$  its vertex. Since spatial infinity is now naturally 'tied' to

null infinity, this framework enables one to establish theorems relating the two regimes. This framework is used, for example, both to formulate and to prove the assertion that the ADM mass is the past limit of the future Bondi mass and the ADM 4-momentum at spatial infinity is indeed the past limit of the Bondi 4-momentum [68, 69]. However, precisely because all points at spatial infinity are now ‘squeezed down’ to a single point the smoothness of the completed manifold fails at this point. So, inevitably, one is forced to deal with complicated differentiable structures there. This circumstance is less satisfactory than that of null infinity  $\mathfrak{S}^\pm$ , which is formulated as a smooth boundary of space-time. To overcome these unwanted features Ashtekar and Hansen propose a suitable ‘blowing up’ of  $i^0$ . The result of this ‘blowing up’ is a 4-dim manifold called Spi constructed from various inextendible space-like curves ‘regular’ at  $i^0$ . Spi has the structure of a principal fibre bundle: The base space is the unit time-like hyperboloid and the structure group is the additive group of reals. But both  $\mathcal{S}_G$  and Spi are somehow artificial and neither of them is a boundary surface of the space-time itself. Thus, e.g., it is meaningless to ask whether or not a curve of the physical space-time has an end point on  $\mathcal{S}_G$  or Spi, or whether a tensor field defined on physical space-time has a limit on  $\mathcal{S}_G$  or Spi.

There does exist a related but coordinate dependent treatment similar to that of Bondi et al. [20] in which the awkward differentiability conditions are avoided. In this approach Beig and Schmidt [70, 71] postulated a Bondi-type expansion of the metric near spatial infinity and showed that Einstein’s vacuum equations give rise to a hierarchy of linear equations for the coefficients in this expansion. They demonstrated that this hierarchy can be completely solved provided the initial data satisfy certain constraints.

In the remaining three covariant approaches spatial infinity now arises as a boundary of spacetime, rather than a single point. Therefore, these approaches are closer in spirit to the Penrose treatment of null infinity. In the third of these approaches Sommers [72] attached to an asymptotically flat space-time at spatial infinity a 3-dim boundary using projective rather than conformal completion. The boundary is assumed to be the unit timelike hyperboloid of the Minkowski metric and is constructed from Minkowski’s space-time as the set of all ‘end points’ of (equivalence classes of) spacelike curves. Unfortunately however these investigations have remained rather incomplete. Thus, in [72] one finds a precise definition of asymptotic flatness only in the weak field limit. For full, non-linear General Relativity, one finds only guidelines as to how these boundary conditions may be modified. The weak field analysis is based on a projective completion of space-time. Detailed calculations show [73] that already in the case of Schwarzschild space-time, these projective techniques run into difficult problems and it is not clear if the programme outlined in [72] can in fact be completed. The fourth covariant approach developed in [74, 75, 76] is more complete in that it does specify the basic definitions in the full, non-linear case. However, almost none of the consequences of these definitions are then worked out: neither the asymptotic symmetries nor the structure of the asymptotic field equations is explored in any detail.

The previous work culminated in the the fifth covariant approach in which Ashtekar and Romano [73] completed the programme initiated in [72] and [74, 75, 76] by extracting the desirable features from the  $i^0$  description [66] and the related coordinate-dependent treatment [70, 71]. In the new framework, spatial infinity arose as a boundary of spacetime. However, Ashtekar and Romano’s framework was somewhat of a hybrid, in that it involved both the conformal and projective structure.

We conclude that at present there are two main approaches to the problem of the asymptotic structure of the gravitational field at large null separations, and three main approaches to the problem of the asymptotic structure of the gravitational field at large space-like separations; at large null separations the two main approaches are that of Bondi, Metzner, Van der Burgh [19, 20] and Sachs [21] and that of Penrose [30, 31] in the framework of his conformal technique; whereas at large space-like separations the four main approaches are that of Beig and Schmidt

[70, 71] which is similar to that of Bondi et al., that of Ashtekhar and Hansen [38] who employ Penrose's conformal technique, that of Sommers [72] and Persides [74, 75, 76] who use a projective technique, and finally that of Ashtekhar and Romano [73] who involve both the conformal and the projective structure. In the Penrose's treatment [30, 31] null infinity is described as smooth boundary of the unphysical space–time; similarly in Sommers', Persides', and, Ashtekhar and Romano's approach spatial infinity is defined as smooth boundary of the unphysical space–time. In both cases the unphysical space–time is conformally related to the original physical space–time.

In Penrose's treatment the asymptotic symmetry group  $B$  arises [77] as the group of diffeomorphisms preserving certain structures which are universal, i.e., which are common to all asymptotically flat space–times, *induced on null infinity from its embedding as a smooth boundary of the unphysical space–time; Penrose's strong conformal geometry is an example of such a universal structure.* Similarly, in Ashtekhar and Romano's approach the asymptotic symmetry group  $B_S$  arises [73] as the group of diffeomorphisms preserving certain universal structures induced on spatial infinity from its embedding as a smooth boundary of the unphysical space–time. In the null regime  $B$  was also derived by Bondi's approach [20], whereas, in the spatial regime  $B_S$  has not been derived by Bondi's approach so far [70, 71]. Ashtekhar and Romano showed [73] that  $B_S = C(S^2 \times I, R) \otimes_T SL(2, C)$ , i.e.,  $B_S$  has a structure similar to the structure of  $B$ . Indeed,  $B_S$  has a unique infinite–dimensional Abelian normal subgroup, the subgroup  $C(S^2 \times I, R)$  of supertranslations. The quotient of  $B_S$  with this subgroup is just the Lorentz group  $SL(2, C)$ :  $B_S$  is the semi–direct product of the supertranslations  $C(S^2 \times I, R)$  and the Lorentz group  $SL(2, C)$ . Finally  $B_S$  admits a preferred  $4-d$  normal subgroup, the subgroup  $\mathcal{T}$  of Spi translations.

The derivation of the aforementioned asymptotic symmetry groups in the spatial and in the null regime is much simpler in the group theoretic approach advocated in [53]. In this derivation no unnatural structures are invoked – no first and second order structures are employed and no 'blowing up' is needed [54], why one should stop at the second order structure, i.e., at the first derivative of the metric, and why one squeezes space–like infinity to a single point to 'blow it up' later? – and, moreover, the construction gives in a single stroke *all possible asymptotic symmetry groups in four dimensions both for real space–times in any signature and for complex space–times.* It is difficult to imagine that there is a simpler and more economical way to obtain these groups. The Einstein equations are not employed explicitly as it is done in the other approaches [54, 73].

Theories attempting to unify the forces often require a higher dimensional space–time. Still important and fundamental even in higher dimensional theories are the notion of an isolated system and the question arises if  $B$  and its 41 generalizations have higher dimensional analogs. In higher dimensions, there are only a few works [29, 78, 79, 80, 81, 82, 83] about asymptotic structure at spatial infinity or null infinity though recently the importance of higher–dimensional black holes is increasing in string theory and TeV gravity scenario [84, 85].

In higher dimensional space–times, the asymptotic structure of the gravitational field at spatial infinity has been studied in [82, 83]. In [82] Shiromizu and Tomizawa investigate asymptotic flatness in the spatial regime in higher dimensions following Ashtekar and Romano [73], whereas in [83] Tanabe, Tanahashi and Shiromizu define asymptotic flatness and investigate asymptotic structure at spatial infinity in higher dimensions, following Ashtekar and Hansen [66]. Much work still needs to be done since the structure of the asymptotic symmetry group is not stated clearly in [82, 83]. In [83] the asymptotic symmetry group is defined as the group of diffeomorphisms which preserve the universal structure at  $i^0$ . It is shown that the asymptotic symmetry group consists of transformations which have infinite translational directions. However the structure of the group is not given. Moreover in [83] it is shown that as in four dimensions, by imposing additional constraints on the behavior of the 'magnetic' part of the Weyl tensor

the supertranslational ambiguity can be removed. Then, the asymptotic symmetry of the space–time reduces to the Poincare symmetry, which is a symmetry of background flat metric, and conserved quantities associated with this Poincare symmetry can be constructed. In [82] it is shown that in higher dimensions the geometric structure of spatial infinity is generally different from its structure in four dimensions: Spatial infinity<sup>3</sup>, is an  $(d - 1)$ –dimensional manifold in a general  $d$ –dimensional space–time. In four dimensions the spatial infinity  $\mathcal{H}$  is homeomorphic to  $S^2 \times I$ , i.e.,  $\mathcal{H}$  is the 3–dim (timelike) hyperboloid of unit space–like directions [73]. The key result in [82] is that in higher dimensions there are many more varieties for the geometric structure of  $\mathcal{H}$  due to the nontrivial  $(d - 1)$ –dimensional Weyl tensor. In fact it appears [82] that this non triviality of the Weyl tensor makes the situation more complicated than one might have naively anticipated: The geometric structure of  $\mathcal{H}$  depends on the number of dimensions  $d > 4$  and also on the class of space–times considered. Therefore, the naive generalization of  $B_S$  to higher dimensions, i.e., of

$$B_S = C(S^2 \times I, R) \otimes_T SL(2, C)$$

to

$$B_S^d = C(S^{d-2} \times I, R) \otimes_T L_+^d,$$

where,  $L_+^d$  is the identity component of the Lorentz group in  $d$  dimensions, is not generally valid except in the very restricted class of static space–times [82]. Much work needs still to be done so that the structure of the asymptotic symmetry group is determined in the spatial regime and in higher directions for the different classes of space–times.

In higher dimensional space–times, the asymptotic structure at spatial infinity can be well-defined [82, 83]. On the other hand, the asymptotic structure at null infinity is not understood completely in higher dimensional space–times [78, 79, 80, 81]. Indeed, by using Penrose’s conformal technique the definition of null infinity can be given only in even dimensions [78, 79, 80, 81]. This situation is to be contradistinguished with the situation at spatial infinity: by using Penrose’s conformal technique the definition of spatial infinity can be given in *any* number of dimensions [83]. The difficulty in the definition of null infinity compared with spatial infinity is due to the presence of the gravitational waves at null infinity. At spatial infinity, since there are no gravitational waves, the asymptotic structure is ‘stationary’ and the total mass and total angular momentum are conserved, while the asymptotic structure at null infinity might be disturbed by gravitational waves. Hence, a stable definition of null infinity against gravitational waves is needed. Such a definition in even dimensions can be given with the employment of Penrose’s conformal embedding method, but cannot be given in odd dimensional space–times [78, 79, 80] since the smoothness of Einstein equations at null infinity cannot be shown. This non–smoothness is related with the fact that in conformal embedding method the conformal factor  $\Omega \sim 1/r$  is introduced and the behavior of gravitational waves near null infinity is of order  $O(\Omega^{(d-2)/2})$  in  $d$  dimensional space–times. The problem comes from the half–integer power of  $\Omega$ . In odd dimensions null infinity is defined not by using Penrose’s conformal embedding method but by using the Bondi coordinates [81]. Neither in even dimensions [78, 79, 80] nor in odd dimensions [81] there is the well known infinite set of angle dependent translational symmetries (supertranslations). Furthermore, it appears [78, 79, 80, 81], that when  $d > 4$  the asymptotic symmetry group in the null regime is the Poincare group in  $d$  dimensions. However the problem of the asymptotic symmetric group in the null regime in more than four dimensions has not been settled: in [29] a different definition of asymptotic flatness is given and this new definition results in a generalization of  $B$  in the null regime in higher dimensions which *does* entail supertranslations.

In the classical regime and in higher dimensions in order to compare the results of the aforementioned other approaches to asymptotic symmetries with the results of the group

theoretic approach advocated here, and in order to make contact with other approaches to quantum gravity where complexified or euclidean versions of General Relativity are frequently considered, it is highly desirable to

- (i) Construct the analogs of  $B$  in higher dimensions in complexified space–times as well as in real space–times of any signature, generalizing the work of McCarthy[53].
- (ii) Search for and construct classical actions of the analogs of  $B$  in higher dimensions in analogy with the action of  $B$ ,  $B \times \mathfrak{S}^+ \rightarrow \mathfrak{S}^+$  on Penrose’s future null infinity.
- (iii) Determine geometric structures for which the classical actions of (ii) are the automorphism groups in analogy with Penrose’s ‘strong conformal geometry’ for which  $B$  is the automorphism group.

## 2. Wigner–Mackey’s theory in a nutshell

Neither the 42 groups defined in [53] nor their (supersymmetric) generalizations in four and higher dimensional space–times are (locally) compact and there is no complete theory for such groups. Fortunately, we do not need to concern ourselves with the general case but only with the class of groups with the general structure

$$\mathcal{B} = C(A, R) \otimes_T H,$$

where,  $A$  and  $H$  have the same meaning as in (1), and,

- (i)  $C(A, R)$  is abelian normal subgroup of  $\mathcal{B}$
- (ii)  $T$  is an homomorphism

$$T : G \rightarrow \text{Aut}(C(A, R))$$

- (iii)  $C(A, R)$  and  $G$  are topological groups. In the product topology of  $C(A, R) \times G$ ,  $\mathcal{B}$  becomes a topological group. In fact in Wigner–Mackey’s theory it is assumed that in the product topology of  $C(A, R) \times G$ ,  $\mathcal{B}$  becomes a *locally compact* topological group. This is *not* the case for  $B$  and its generalizations and some care is needed in the use of the results of Wigner–Mackey’s theory.

To construct the IRs of these groups it would be sufficient to study just Wigner–Mackey’s theory of induced representations of semi–direct products. We will present the bare essentials of this theory; our treatment will be purely functional analytic leaving aside the, more natural for induced representations, geometrical, fibre bundle dominated scheme [86, 87, 88].

### 2.1. The bare essentials

Let  $\mathcal{A}$  denote  $C(A, R)$ . It is convenient to give a few definitions and state some simple facts:

- (i) **Characters - The Dual Group** The IRs of  $\mathcal{A}$  are one – dimensional and are called characters. They can be given the structure of an abelian group  $\hat{\mathcal{A}}$ , the dual group of  $\mathcal{A}$ , with group operation given by

$$(\chi_1 \chi_2)(\alpha) = \chi_1(\alpha) \chi_2(\alpha),$$

where,  $\chi_1, \chi_2 \in \hat{\mathcal{A}}$ ,  $\alpha \in \mathcal{A}$ .

- (ii)  $\mathcal{A}$  has a natural real vector space structure

$$\begin{aligned} (\chi_1 + \chi_2)(\alpha) &:= \chi_1(\alpha) + \chi_2(\alpha) \\ (\lambda \circ \chi)(\alpha) &:= \lambda \chi(\alpha). \end{aligned}$$

(iii) **Dual Action** The action  $T$  of  $G$  on  $\mathcal{A}$  induces a dual action  $\hat{T}$  of  $G$  on  $\hat{\mathcal{A}}$  defined by

$$(\hat{T}(g)\chi)(\alpha) := \chi(T(g^{-1})\alpha)$$

It is precisely this dual action which determines the structure of the IRs of  $\mathcal{B}$

(iv) **Little Groups**  $L_\chi$  For a *given*  $\chi$  the largest subgroup  $L_\chi$  of  $\mathcal{B}$  which leaves  $\chi$  fixed is called the little group of  $\chi$ , i.e.,

$$L_\chi = \{g \in \mathcal{B} \mid \hat{T}(g)\chi = \chi\}$$

(v) **The Orbit of  $\chi$ , denoted by  $\mathcal{B}_\chi$ , is defined by**

$$\mathcal{B}_\chi = \{g\chi \mid g \in \mathcal{B}\}$$

$\hat{\mathcal{A}}$  is partitioned by the orbits  $\mathcal{B}_\chi$

### 2.1.1. Constructing the Hilbert space $\mathcal{H}_\nu$ in which the IRs of $\mathcal{B}$ are materialized

(i) **There is a natural bijection**

$$\begin{aligned} \mathcal{B}/L_\chi &\longleftrightarrow \mathcal{B}_\chi, \\ &\text{given by,} \\ gL_\chi &\longleftrightarrow g\chi, \end{aligned}$$

where,  $g \in \mathcal{B}$ .

(ii) **The coset space  $\mathcal{B}/L_\chi$  has a unique class of quasi-invariant measures for the  $\mathcal{B}$ -action [86]; let  $\nu$  be one of these**

(iii) **Let  $U$  be a continuous irreducible of the little group  $L_\chi$  on a Hilbert space  $\mathcal{D}$**

(iv) **Let  $\mathcal{H}_\nu$  be the space of functions  $\psi : \mathcal{B} \rightarrow \mathcal{D}$  which satisfy the conditions**

- (a)  $\psi(gh) = U(h^{-1})\psi(g)$
- (b)  $\int_{\mathcal{B}_\chi} \langle \psi(p), \psi(p) \rangle d\nu(p) < \infty$ ,

where,  $h \in L_\chi, g \in \mathcal{B}$ , and, where, the scalar product under the integral sign is that of  $\mathcal{D}$ .

(v) **Remark** The integrand is expressed as a function on  $\mathcal{B}_\chi \longleftrightarrow \mathcal{B}/L_\chi$  since, in view of 1, the integrand is constant on the cosets in  $\mathcal{B}/L_\chi$

(vi)  **$\mathcal{H}_\nu$  is turned into a Hilbert space by introducing the scalar product**

$$\langle \psi_1, \psi_2 \rangle = \int_{\mathcal{B}_\chi} \langle \psi_1(p), \psi_2(p) \rangle d\nu(p)$$

(vii) **Definition of the representation of  $\mathcal{B}$  on  $\mathcal{H}_\nu$**

Define an action of  $\mathcal{B}$  on  $\mathcal{H}_\nu$  by

$$(g_o\psi)(g) = \psi(g_o^{-1}g) \tag{7}$$

$$(\alpha\psi)(g) = [(\hat{T}(g)\chi)(\alpha)]\psi(g), \tag{8}$$

where,  $g, g_o \in G$ , and  $\alpha \in \mathcal{A}$ . This action gives a unitary representation of  $\mathcal{B}$  on  $\mathcal{H}_\nu$  which is continuous whenever  $U$  is.

(viii) **This is precisely the representation of  $\mathcal{B}$  induced from  $\chi$  and the irreducible representation  $U$  of the little group  $L_\chi$**

### 2.1.2. Mackey's Theorems

- (i) **First Theorem :** Given the topological restrictions on  $\mathcal{B} = \mathcal{A} \otimes_T G$  (separability and local compactness), *any* representation of  $\mathcal{B}$ , constructed by the method above, *is irreducible if the representation  $U$  of  $L_\chi$  on  $\mathcal{D}$  is irreducible.*
- (ii) **Main Conclusion :** **An irreducible representation of  $\mathcal{B}$  is obtained for each  $\chi \in \hat{\mathcal{A}}$  and each irreducible representation  $U$  of  $L_\chi$**
- (iii) **Second Theorem :** If  $\mathcal{B} = \mathcal{A} \otimes_T G$  is a regular semi-direct product (i.e.,  $\hat{\mathcal{A}}$  contains a Borel subset which meets each orbit  $G_\chi$  in  $\hat{\mathcal{A}}$  in just one point – **the  $G$ -action is not too pathological when the  $G$ -orbits can be enumerated in some way**) then **all** of its irreducible representations can be obtained in this way

### 3. Physics does not dictate the topology of $\hat{\mathcal{A}}$ and this gives rise to a host of possibilities

Sachs's original derivation [24] required that the supertranslations be twice differentiable, i.e.,  $\mathcal{A} = C^2(S^2, R)$ , and Cantoni, in his investigation of representations [25, 26], gave to this  $\mathcal{A} = C^2(S^2, R)$  the pre-Hilbert topology determined by the area measure of  $S^2$ . McCarthy chose the same topology in his investigation [40, 42], but widened  $\mathcal{A} = C^2(S^2, R)$  to  $\mathcal{A} = L^2(S^2, R)$  i.e., to the square-integrable real-valued functions defined on the two sphere in order to simplify the treatment. The discreteness result of the spin parameterizing the ensuing representations is true with either the Cantoni or McCarthy choices.

The key observation here is that Sachs's original derivation [24] was superceded by that of Penrose [30, 31], who gave a precise and intrinsic derivation of  $B$  as that group of exact conformal motions of the future (or past) null boundary  $\mathfrak{S}^\pm$  of (conformally compactified weakly asymptotically simple) space-times which preserve 'null angles'. Since *truly arbitrary* supertranslation functions describe symmetry transformations in Penrose's sense, *supertranslations need not have some minimum degree of smoothness.*

*It is precisely this arbitrariness in the class of functions allowed in  $\mathcal{A}$  which permits a wide range of choices of 'reasonable' topologies for  $\mathcal{A}$ .* The catch is that the 'smoother' you require the functions in  $\mathcal{A}$  to be, the 'rougher' you can allow the generalized functions in  $\hat{\mathcal{A}}$  to be. This explains why little groups might change when the topology of  $\mathcal{A}$  is modified: The induced representations are associated with the existence of invariant characters, i.e., of elements in  $\hat{\mathcal{A}}$ , the topological dual of  $\mathcal{A}$  (supermomenta in  $B$ , generalizations of the Poincare momenta), which are left invariant by the action of some subgroup (little group) of  $H$ ,  $H = SL(2, C)$  in the case of  $B$ . Little groups are then connected, via  $\hat{\mathcal{A}}$ , to the topology of  $\mathcal{A}$ . Since, e.g., a refinement of the topology may broaden  $\hat{\mathcal{A}}$ , new invariant elements with associated little groups may come into existence. In fact, as we will explain, this is precisely what happens when  $\mathcal{A}$  is endowed with the nuclear topology [44].

As mentioned already, Cantoni, in his investigation of representations [25, 26], gave to  $\mathcal{A} = C^2(S^2, R)$  the pre-Hilbert topology determined by the area measure of  $S^2$ . Cantoni has shown [89] that if  $SL(2, C)$  is given the usual topology, then  $B_2 = C^2(S^2, R) \otimes_T SL(2, C)$  is a non-locally compact group in the product topology of  $\mathcal{A} \times SL(2, C)$ ,  $\mathcal{A}$  now being the group  $C^2(S^2, R)$ . McCarthy worked on two choices and proposed a third one:

#### 3.1. First choice : $B$ in the Hilbert topology

McCarthy in his first study of the representations of  $B$  [40, 42, 43, 46], chose the same topology with the one used by Cantoni [89], but widened  $\mathcal{A} = C^2(S^2, R)$  to  $\mathcal{A} = L^2(S^2, R)$  to simplify the treatment.  $\mathcal{A}$  is endowed with the Hilbert topology as follows [40]:

- Introduce a scalar product into  $\mathcal{A}$

$$\langle \alpha, \beta \rangle = \int_{S^2} \alpha(x)\beta(x)d\mu(x)$$

where,  $x \in S^2$ , and,  $\alpha, \beta \in \mathcal{A}$ .

- With this scalar product,  $\mathcal{A} = L^2(S^2, R)$  becomes a real Hilbert space, and, in the induced metric topology, becomes an (Abelian) topological group.
- With a proof similar to Cantoni's, it can be shown that, when  $\mathcal{A} = L^2(S^2, R)$  is endowed with the aforementioned topology, and  $SL(2, C)$  is given the usual topology, then  $B$  is a non-locally compact group in the product topology of  $\mathcal{A} \times SL(2, C)$ .

The representations are determined by the action of  $SL(2, C)$  on the topological dual of  $\mathcal{A}$ ,  $\hat{\mathcal{A}}$ . It is a fundamental theorem, the Reisz-Frechet theorem, (the Hilbert space representation theorem), that  $\mathcal{A} = L^2(S^2, R)$  and  $\hat{\mathcal{A}}$  are isometrically isomorphic. It is precisely this isomorphism which simplifies the treatment when  $\mathcal{A} = C^2(S^2, R)$  to  $\mathcal{A} = L^2(S^2, R)$ .

### 3.1.1. The more important results and open problems in the Hilbert topology

The four more important results in the Hilbert topology are:

- $B$  irreducibles, i.e.,  $B$ -elementary entities, describe 'elementary particles' [46].
- In the representation theory of  $B$  only compact little groups, hence discrete spins, arise.
- There is strong evidence which suggests that a class of IRs of  $B$ , those induced from non-connected discrete finite little groups, correspond to ALE gravitational instantons [53].
- Piard has proved [90, 91] that the inducing construction for  $B$  [40, 42] is exhaustive despite the fact that  $B$  is *not* locally compact in the Hilbert topology. This makes McCarthy's results even more important.

The more important open problem in the Hilbert topology is:

- Find the physical interpretation of the IRs of  $B$ , in particular of those induced from non-connected discrete finite little groups and establish and clarify the connection with the ALE gravitational instantons.

### 3.2. Second choice : $B$ in the nuclear topology

McCarthy in his subsequent study of the representations of  $B$  [44, 92] chose  $\mathcal{A} = C^\infty(S^2, R)$ , i.e., chose  $\mathcal{A}$  to be all the smooth real-valued functions on  $S^2$  and not all the square-integrable functions on  $S^2$ . This choice by no means determines the topology uniquely. However, a natural choice for vector spaces whose elements are 'smooth' in some sense, e.g., for the set of smooth functions on a compact manifold, is the nuclear topology. The nuclear topology is determined by the following notion of convergence: a sequence of functions  $a_k$  on the sphere is said to converge to zero, if and only if, the functions  $a_k$  together with all their derivatives of all orders, converge uniformly to zero over the sphere. The expectation is that in a finer topology, the nuclear is finer than the Hilbert, more representations become continuous, and some have non-compact little groups, hence, possibly, continuous spins.

#### 3.2.1. The more important results and open problems in the nuclear topology

The two more important results in the nuclear topology are:

- It appears [44] that the extra representations in the nuclear topology (which may have continuous spins) describe 'scattering' states, and the remaining ones are identified with bound states, corresponding to elementary particles.

- The particles appropriate to  $B$  have discrete spins irrespective of the choice of topology, Hilbert or nuclear.

The two more important open problems in the nuclear topology are:

- Some of the IRs of  $B$  in the nuclear topology are induced from non–compact infinite discrete subgroups of  $SL(2, C)$  [44], which are far from being known!
- It is not known if the inducing construction for  $B$  in the nuclear topology [44] is exhaustive, i.e., a proof corresponding to the one given by Piard [90, 91] for  $B$  in the Hilbert topology, is lacking for  $B$  in the nuclear topology.

### 3.3. Third choice: $\hat{\mathcal{A}}$ becomes enlarged to the space of real hyperfunctions $\mathcal{Y}(S^2, R)$ on the 2–sphere $S^2$

If  $\mathcal{A}$  is taken to consist of real analytic  $C^\omega(S^2, R)$  functions rather than  $C^\infty(S^2, R)$  functions then, with an appropriately fine topology, the dual space  $\hat{\mathcal{A}}$  becomes enlarged to the space of real hyperfunctions  $\mathcal{Y}(S^2, R)$  on the 2–sphere  $S^2$ . The space of hyperfunctions is larger than the space of distributions on  $S^2$ . This new topology on  $\mathcal{A} = C^\omega(S^2, R)$ , and the associated hyperfunctional supermomenta, are probably more ‘physical’ than the nuclear topology and the associated distributional supermomenta [93]. It seems that the category of hyperfunctions is more appropriate than that of distributions for discussing  $S$ –matrix theory. Since the representation theory of  $B$  in nuclear topology strongly suggests [44] that the physical situations being analyzed are related to scattering problems, the present discussion of  $\mathcal{A} = C^\omega(S^2, R)$  seems to imply that one should really consider hyperfunctional supermomenta, or more generally, hyperfunctional solutions to Einstein’s equations.

Interestingly, quantum field theories in which fields are smeared by hyperfunctions show a non local behaviour and the density of states can have a non polynomial growth. This might in principle allow to recover bulk locality, in the exploration of the holographic principle in asymptotically flat spacetimes via the BMS group [94], although one should consider hyperfunctional solutions to the Einstein equations. More remarkably, if one assumes that the high energy behaviour of the density of states in the bulk is dominated by black holes, the exponential growth of states which suggests an intrinsic degree of non locality, might be explained by working with hyperfunctions.

## 4. Some more open problems

Some more open problems within this research program in the quantum regime are the following:

- One of the main objectives of the research program expounded here is to find the IRs of  $B$  and its generalizations [53] in four and higher dimensions in Hilbert spaces [95, 96, 97, 98, 99, 100]. These Hilbert space representations are related to elementary particles and quantum gravity via gravitational instantons. Thus, in the spirit of Klein’s Erlangen programme, the geometries, symmetries and elementary quantum systems can all be studied from a single point of view.
- Prove the connection with instantons – ALE, complex and hyperbolic – in all dimensions. This should shed some light in the structure of the quantum gravity vacuum at least in the semi–classical regime.
- Explore if the holographic principle of t’Hooft and Susskind is realized in asymptotically–flat space–times. Two routes have been pursued: In one of them the holographic boundary is the future null infinity and the QFT at the boundary is ‘B–invariant’ [94], in the other, the symmetry algebra of asymptotically flat space–times at null infinity in 4 dimensions is taken to be the semi–direct sum of supertranslations with infinitesimal *local* conformal transformations [101] and not, as usually done, with the Lorentz algebra.

Try a third route: to put the holoscreen at spatial infinity, since then the analogy with AdS/CFT correspondence is better. This presupposes to find first the IRs of  $B_S$ .

- Alternatively, find the IRs of  $B$  when  $\hat{A}$  becomes enlarged to the space of real hyperfunctions and explore if the holographic principle of t'Hooft and Susskind is realized in asymptotically flat space–times by putting the holographic screen at  $\mathfrak{S}^+$  and by employing the IRs of  $B$  in this new topology, where the supermomenta are hyperfunctions, for the fields at  $\mathfrak{S}^+$ .
- $B$  and its analogs result as asymptotic symmetry groups of Einstein–Maxwell isolated systems. It is highly desirable to find the corresponding groups in four and higher dimensions for Einstein–Maxwell–Yang–Mills isolated systems. This will reveal the structure of the quantum gravity–yang–mills vacuum and hopefully some conclusions will be drawn for the cosmological constant at least in the semi–classical regime.

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