Discrete Directions in the Standard Model

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Abstract : A single generation of leptons (resp. of quarks) can be described by the smallest irreducible non typical (resp. typical) representation of the Lie superalgebra SU(2|1). Here, the "super" qualifier refers to left-right parity. It is shown that reducible indecomposable representations describe mixing between families.The smallest representations of this kind are of several types. One describes the mixing of each leptonic generation with a corresponding right neutrino. Another describes either the coupling between generations of extended leptonic families of the previous kind or the coupling between generations of quarks. This leads then to particular parametrizations for the Yukawa couplings and gives constraints between masses and mixing matrices (for instance a relation between Cabibbo angle and quark masses). The bosonic sector of the Standard Model can itself be described in terms of a generalised Yang-Mills field incorporating both the usual Yang-Mills fields of $SU(2) \times U(1)$ and the complex doublet of Higgs fields.

Keywords : Standard Model, Lepton Masses, Quark Masses, Cabibbo angle, Kobayashi-Maskawa, Neutrino Mixing, Higgs, Lie Superalgebras, Non-commutative geometry.

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0. Introduction

Properties of the fermions in the Standard Model (gauge couplings to the $SU(2)$ group of weak isospin, to the $U(1)$ group of weak hypercharge and Yukawa couplings to the Higgs fields) are recovered, when there is no mixing, by assuming that individual families of leptons and quarks are described by irreducible representations of the Lie superalgebra $SU(2|1)$. A simultaneous description of the different generations and of their mixings can be done by using the reducible indecomposable representations of this superalgebra. This was already discussed in [1]. In this reference the role of $SU(2|1)$ in the bosonic sector was also discussed. Here,we summarise the situation of the fermionic sector. The "super" qualifier refers here to a Z_2-symmetry that has nothing to do with usual "super-symmetry" since it refers to transformations that do not exchange bosons and fermions but describes the content and mixings between the worlds of right and left particles (it may help two think of these two worlds as parallel universes or as the two components of a non-connected Space-Time, as in [3]).

1. The Lie superalgebra $SU(2|1)$

$SU(2|1)$ is a finite dimensional simple Lie superalgebra of dimension 8. It can be explicitely defined in terms of 3×3 matrices of supertrace zero, the grading operator being diag(1,1,-1). It also coincides (is isomorphic) with the orthosymplectic Lie superalgebra $Osp(2|2)$. The same Lie superalgebra could also be defined as the algebra of derivations of a Grassmann algebra with two generators. We call I_1, I_2, I_3 and Y the generators of its even part and $\Omega_\pm$, $\Omega'_\pm$ those of the odd part. The "even" part of this superalgebra coincides with the Lie algebra of $SU(2) \times U(1)$. The "odd" part has real dimension 4 (complex dimension 2) and is itself a representation space (a complex doublet) for the even part. The (super) commutation relations are given as follows (with $I_\pm \doteq \frac{I_1 \pm i I_2}{\sqrt{2}}$).

$$
\begin{array}{ccc}
[I_3, I_\pm] = \pm I_\pm & [I_+, I_-] = I_3 & [Y, I_\pm] = [Y, I_3] = 0 \\
[Y, \Omega_\pm] = -\Omega_\pm & [Y, \Omega'_\pm] = \Omega'_\pm & [I_3, \Omega_\pm] = \pm\frac{1}{2}\Omega_\pm \\
[I_3, \Omega'_\pm] = \pm\frac{1}{2}\Omega'_\pm & [I_\pm, \Omega_\mp] = \frac{1}{\sqrt{2}}\Omega_\pm & [I_\pm, \Omega'_\mp] = \frac{-1}{\sqrt{2}}\Omega'_\pm \\
[I_\pm, \Omega_\pm] = [I_\pm, \Omega'_\pm] = 0 & \{\Omega_\pm, \Omega_\pm\} = \{\Omega_\pm, \Omega_\mp\} = 0 & \{\Omega'_\pm, \Omega'_\pm\} = \{\Omega'_\pm, \Omega'_\mp\} = 0 \\
\{\Omega_\pm, \Omega'_\pm\} = \sqrt{2} I_\pm & \{\Omega_\pm, \Omega'_\mp\} = \pm I_3 + Y/2 &
\end{array}
$$

Because it is a super-algebra, its representation spaces can be decomposed into an "even" subspace and an "odd" subspace (Z_2-grading). Even generators leave invariant separately the right and left subspaces. Odd generators mix the two subspaces. Even generators are associated with usual gauge fields -valued in $Lie(SU(2) \times U(1))$ - whereas odd generators are associated with Higgs fields. It is natural to expect that elementary fermions (leptons and quarks) should be associated with representations that are, in some sense " the smallest ones". This is indeed the case. Usually one describes elementary particles in terms of the smallest irreducible representations (fundamental representations) of Lie algebras. But a Lie superalgebra is not, strictly speaking, a Lie algebra, and, in the case of $SU(2|1)$, two new features arise.

The first is that there exist two kinds of representations, the typical and the non-typical ones. In the typical case, the dimension of the "even" subspace is equal to the dimension of the "odd" subspace. This means that there is an equal number of degrees of freedom for the left and right sectors. In the non-typical case, this number is different for the two chirality sectors. The mere existence of non-typical representation can be interpreted in terms of parity violation. We shall see that the smallest non-typical representation (call it $[\ell]$) describes a massive charged Dirac lepton and a massless (left) neutrino –for instance the electron and its neutrino. We shall see that the smallest typical representation (call it $[q]$) describes a single generation of quarks –for instance the up ad down quarks.

The other new feature is that reducible representations are not necessarily fully reducible: reducible indecomposable representations do exist. In our approach the existence of several generations is linked to the existence of reducible indecomposable representations. It is natural to restrict our attention to the "smallest" ones, in particular to those that contain the typical or non-typical irreducible representations of smallest dimensions as direct summands. Being indecomposable, they have a taste of "elementarity", but, being reducible, they justify that, in some limit, one can build a reasonable theory while forgetting the particles associated with the complement of the direct summand. $SU(2|1)$ has many kinds of reducible indecomposable representations. As far as quarks and leptons are concerned, the most important ones are the following –the precise matrix realization of generators is given in the appendix.

$$\begin{aligned} &1)\ \ [\ell] \supset\!\!\!\!+ [1] \\ &2)\ \ ([\ell] \supset\!\!\!\!+ [1]) \supset\!\!\!\!+ ([\ell] \supset\!\!\!\!+ [1]) \supset\!\!\!\!+ ([\ell] \supset\!\!\!\!+ [1]) \\ &3)\ \ [q] \supset\!\!\!\!+ [q] \supset\!\!\!\!+ [q] \end{aligned} \tag{1}$$

The first describes the coupling of a usual single leptonic family with its right neutrino ($[1]$ denotes the trivial representation), the second describes the coupling of three such extended families. The last describes the couplings of three quark generations. In this last case the Cartan subalgebra is not diagonal. Taking into account a necessary factor three for color, the whole fermionic content –with massive neutrinos as well as mixing bewteen generations, both in the quark and leptonic sectors– is therefore described by the sum of one representation of type 2 and three identical representations of type 3. We shall justify this claim in the following. There are other kinds of indecomposable representations (some of them are described in [2]). But the ones displayed above are, in a sense, more fundamental than others since they involve the smallest irreducible representations.

The representations that we consider here are not hermitian representations. Most of the time, in Lagrangian models describing a multiplet of particles, it is not necessary to add explicitely the contribution of a corresponding multiplet of antiparticles because both contributions are equal and the only requirement is to add the hermitian conjugate of the interaction term in order to build a real Lagrangian. In the present case, this is not so. This can be traced back to the existence of a superalgebra (anti)-automorphism that cannot be implemented by hermitian conjugacy. The net result is that one has to add to the lagrangian both contributions coming from a multiplet and its conjugate multiplet. These two contributions are different. In the case of several families of quarks described by a reducible indecomposable representation, the generator Y itself is not hermitian and, being non-diagonal, would lead to flavour changing neutral currents. The fact of adding both contributions coming from the representations describibing quarks and anti-quarks precisely cancels these unwanted currents. A description of these particular representations involve a priori several complex parameters that cannot all be made real when there are more than two generations. This, together with the fact that we have to add the two –unequal– contributions coming from quarks and anti-quarks, is an algebraic interpretation at the root of existence of CP-violation. Only the expression of generators describing representations $[\ell]$, $[q]$ *etc.* are given in the appendix. Those describing the conjugate representations $[\overline{\ell}]$, $[\overline{q}]$ *etc.* can be obtained from the previous ones by changing the sign of the off-diagonal blocks. For instance, in the case of $[\overline{\ell}]$ by changing $\frac{1}{\kappa}$ into $\frac{-1}{\kappa}$.

2. $SU(2|1)$ and the bosonic sector of the Standard Model

The bosonic fields of the Standard Model are described by an $SU(2)$ gauge field, a $U(1)$ gauge field, and one complex doublet of Higgs fields. From the previous description of $SU(2|1)$, we see that the whole family of bosonic fields builds up a representation space for the Lie superalgebra $SU(2|1)$ which can be identified with the adjoint representation itself. This representation is of dimension 8 and denoted by $[\mathcal{A}]$. The branching rule of this adjoint representation with respect to $Lie(SU(2) \times U(1))$ is

$$[\mathcal{A}] \longrightarrow (I=0)_{y=0} \oplus (I=1)_{y=0} \oplus (I=\frac{1}{2})_{y=1} \oplus (I=\frac{1}{2})_{y=-1} \tag{2}$$

where y denotes the eigenvalue (hypercharge) of Y. The adjoint representation contains therefore an $SU(2)$ triplet of zero hypercharge, a singlet of zero hypercharge, and two isodoublets of weak hypercharge -1 and $+1$. The first two $SU(2)$ representations are therefore naturally associated with usual gauge fields $\overrightarrow{W}$ and W_8 (usually called B in the literature), the last two with the Higgs field $\Phi = \begin{pmatrix} \phi_0 \\ \phi_+ \end{pmatrix}$ and its charged conjugated $\overline{\Phi} = (\overline{\phi}_0, \overline{\phi}_-)$. The adjoint representation is a typical representation. It can be seen that the whole Lagrangian for the bosonic sector of the Standard Model (Yang-Mills term and Higgs potential) can be obtained from the curvature of a generalized connection. This was first described by [3] in terms of the non commutative geometry of a non-connected space, then in [4] where a formalism resting on the use of algebraic superconnections was introduced. The introduction of the Lie superalgebra $SU(2|1)$, in relation with the previous approach was done in [1]. We do not use either of the above formalisms in the present paper and take only the appearance of representations of $SU(2|1)$ in the Standard Model as an observational fact. Let us only mention that generalized gauge field incorporating both Yang-Mills and Higgs field can be decomposed along the generators of $SU(2|1)$ as follows.

$$\mathcal{A} = i(\sqrt{2}\,\overrightarrow{W}\overrightarrow{T} + \frac{W_8}{\sqrt{6}}Y + \frac{\phi_0}{\mu}\Omega_+ + \frac{\phi_+}{\mu}\Omega_- + \frac{\overline{\phi}_0}{\mu}\Omega'_- + \frac{\overline{\phi}_-}{\mu}\Omega'_+) \tag{3}$$

The constant μ is an arbitrary mass, and the numerical constants that appear in this expression only insure that the final kinetic term of gauge fields in the Lagrangian is correctly normalized. The mass generation (symmetry breaking) is associated with the generators Ω_+ and Ω'_-. The curvature itself is obtained from $\mathcal{A}$ by using a d operator acting not only on the fields but also on the matrices. This curvature contains also a constant term that can be interpreted as the contribution of a background connection.

One recovers the whole bosonic part of the Standard Model by taking the trace (not the supertrace) of the square of the curvature. Although it is not the purpose of the present article, we would like to stress that the approach followed to get the bosonic part of the Lagrangian is not at all a gauging of the super-algebra $SU(2|1)$. Like with other Lie superalgebras, this would lead to serious positivity problems –as it was discussed and recognized many times in the past ([5]).

3. Elementary fermions as representations of $SU(2|1)$

To characterize a representation (irreducible or reducible indecomposable), it is enough to know the matrices describing action of the odd generators. Action of the even generators I_3 and Y can then be obtained from the commutation rules given in the first section. The explicit expressions for odd generators are given with compact notations in the appendix. The action of Ω_+, for instance, is gotten from those formulae by replacing the symbol Ω_+ by 1 and Ω_- Ω'_+ Ω'_- by zero. The dimensionless constants κ, α, β, *etc.* that appear there are arbitrary. They reflect the arbitrariness coming into the normalization of scalar products defined in the different representation spaces appearing in the branching rule of $SU(2|1)$ versus $SU(2) \times U(1)$. Representations involving different sets of parameters are equivalent (although not unitarily in general). These constants are a priori complex, but they can usually made real by chiral rotations of the fermionic fields. The three generations case requires however more care. Yukawa couplings beween Higgs fields and a fermionic multiplet Ψ are written as follows.

$$\overline{\Psi}\left(\phi_0\Omega_+ + \phi_+\Omega_- + \overline{\phi}_0\Omega'_- + \overline{\phi}_-\Omega'_+\right)\Psi + h.c. \tag{4}$$

The mass term corresponding to a given representation is obtained by replacing ϕ_0 and $\overline{\phi_0}$ by 1 and ϕ_+, $\overline{\phi}_-$ by 0. Only generators Ω_+ and Ω'_- contribute to it. As already discussed, the whole contribution of a given family to the Lagrangian density should be gotten by adding to [4] another term of the same kind associated with the representation describing the corresponding antiparticules.

Leptons. In the Standard Model, a single generation of leptons is described by a left $SU(2)$ doublet ($I = \frac{1}{2}$) of (weak) hypercharge $y = -1$ and a right singlet ($I = 0$) of hypercharge $y = -2$. The direct sum of these two representations is a reducible representation of $Lie(SU(2) \times U(1))$ but an irreducible representation of the Lie superalgebra $SU(2|1)$ – actually, the fundamental representation. It is the smallest non typical irreducible representation. As explained before, "non typical" means that the number of left handed fields –here two– is not equal to the number of right handed fields –here one. Let us define the graded hypercharge as follows. For a left Pauli spinor, it is equal to the hypercharge and for a right Pauli spinor, it is equal to the opposite of the hypercharge. Then it is clear that, for a given leptonic family, the sum of graded hypercharges vanishes : $2 \times (-1) - 1 \times (-2) = 0$. We called $[\ell]$ this 3-dimensional representation. Under $SU(2) \times U(1)$ we have the branching rule

$$[\ell] \longrightarrow (I = \frac{1}{2})_{y=-1} \oplus (I = 0)_{y=-2} \tag{5}$$

Quarks. In the Standard Model, a single generation of quarks is described by a left $SU(2)$ doublet ($I = \frac{1}{2}$) of hypercharge $y = \frac{1}{3}$ and by two right singlets ($I = 0$) of hypercharges $y = -\frac{2}{3}$ and $y = \frac{4}{3}$. The direct sum of these three representations is a reducible representation of $Lie(SU(2) \times U(1))$ but an irreducible representation of the Lie superalgebra $SU(2|1)$. We call it $[q]$. Here again, the graded sum of hypercharges (the sum of graded hypercharges) vanishes : $2 \times (\frac{1}{3}) - 1 \times (\frac{-2}{3}) - 1 \times (\frac{4}{3}) = 0$. This the smallest typical irreducible representation. It is "typical" since the number of left handed fields –here two– is equal to the number of right handed fields. Under $SU(2) \times U(1)$ we have the branching rule

$$[q] \longrightarrow (I = \frac{1}{2})_{y=\frac{1}{3}} \oplus (I = 0)_{y-1=\frac{-2}{3}} \oplus (I = 0)_{y+1=\frac{4}{3}} \tag{6}$$

The value $y = \frac{1}{3}$ used in the above representation is however not imposed by representation theory of $SU(2|1)$ alone. Every other value (except $y = \pm 1$) leads to a typical representation of the same type. The particular value $\frac{1}{3}$ is gotten by imposing the constraint of cancellation of anomalies along with a factor three for color.

Mixing of irreducible representations. Starting with the observation that one recaptures all the aspects of the Standard Model (nothing less, nothing more) by postulating that single fermionic families of leptonic or quark type are described by irreducible representations of $SU(2|1)$ and that their coupling to the Higgs fields is described as above by using the odd generators of $SU(2|1)$, we postulate that mixing between different generations of quarks or of leptons should be described by the smallest non fully reducible representations.

Right neutrinos. In the "minimal" version of the Standard Model, the neutrino is only left-handed and does not have any right-handed partner. This is at the origin of parity violation in nature. It is possible however to introduce, by hand, a right handed partner. In order for the theory to be still compatible with experience (evidence of parity violation in weak interactions), the coupling of this right-handed neutrino to the gauge fields should be extremely weak. It is usually described as an $SU(2) \times U(1)$ singlet ($I = 0$ and $y = 0$). Its possible observability comes only from the fact that it is coupled to the other fermions by Yukawa couplings involving the Higgs field. This induces a trilinear coupling between this right-handed neutrino, the other leptons and the longitudinal part of the gauge bosons. From the non-zero value of the neutral Higgs field in the vacuum, one also gets a direct coupling to its left-handed partner and therefore a mass term for the neutrino itself (Dirac mass term). The smallest reducible indecomposable representation of the Lie superalgebra $SU(2|1)$ involving the fundamental representation has dimension 4 and can be written as the sum (not a direct sum) of the leptonic multiplet $[\ell]$ –describing e_L, ν_L and e_R– and of the trivial representation [1] –describing ν_R. The fact that $[\ell] \not\oplus [1]$ is not a direct sum (matrix elements of odd generators beween ν_R and e_L, ν_L and e_R do not vanish) implies existence of the Yukawa couplings previously described. Explicit expressions for odd generators are given in the appendix.

*Mixing between leptonic generations.*If the electronic neutrino is massive, the same property should be expected for those associated respectively with the muon and the tau family. Moreover, one can consider the possibility of mixing between leptonic families. This would lead in particular to violation of the three leptonic numbers. Several mechanisms have been proposed to describe this theroretically. Since we are presenting a unified description of all elementary fermions, it is natural to look for a mechanism suggested by the reducible indecomposable representations of the Lie superalgebra $SU(2|1)$.

Such a mechanism exists and is unique. It comes from the fact that the representation $([\ell]\supset\!\!\!\!+\,[1])\supset\!\!\!\!+\,([\ell]\supset\!\!\!\!+\,[1])\supset\!\!\!\!+\,([\ell]\supset\!\!\!\!+\,[1])$ does exist. Explicit expressions for odd generators are again given in the appendix. Notice that it is <u>*not*</u> possible to build a reducible indecomposable representation involving only single copies of the leptonic representation $[\ell]$. It is necessary, in order to build something indecomposable involving several generations, to take into account the right handed partners of the neutrinos. Both masses for leptons and mixing matrix can be gotten from the modulus and phase of the matrix describing the Yukawa interaction.

*Mixing between quark generations.*In the Standard Model, existence of mixing between generations of quarks is described by introducing arbitrary Yukawa couplings between the different families. Here, it comes from the fact that the reducible indecomposable representation $[q]\supset\!\!\!\!+\,[q]\supset\!\!\!\!+\,[q]$ does exist. This representation is rather special in the sense that, not only it is not a highest weight representations but the Cartan subalgebra (the hypercharge generator) is not diagonal. The fact that such a representation is not a direct sum implies existence of Yukawa couplings mixing the generations. Because of the fact that Y is non diagonal (it contains complex parameters in Jordan position) , one could fear that one gets flavour changing neutral currents. The point is that, after adding the contribution of the representation describing antiparticles, the unwanted contribution cancels out. The number of unknown parameters entering the expression of odd generators –and therefore the matrix of Yukawa couplings– can be chosen smaller than the number of constants describing masses and mixing angles. For this reason, we can obtain non-trivial relations between these observable quantities. For instance, in the case of two generations (the indecomposable representation $[q]\supset\!\!\!\!+\,[q]$ does exist), mass matrices for sectors of charge $2/3$ and $-1/3$, gotten from the expression of odd generators given in the appendix are

$$\frac{\mu}{\alpha}\begin{pmatrix} a & y \\ 0 & a \end{pmatrix} \quad \frac{\mu}{\beta}\begin{pmatrix} b & -y \\ 0 & b \end{pmatrix} \tag{7}$$

where

$$a \doteq \frac{1}{3} + |\alpha|^2 \quad ; \quad b \doteq \frac{2}{3} - |\beta|^2 \quad y \doteq -\epsilon'\beta$$

In this two-generations case, all these parameters can be made real. We have also to add the contribution coming from the representation describing antiquarks. This changes the above mass matrices and in particular the relation between the above constants and the quark masses. One finds

$$\mathcal{M}(2/3) = \mu \begin{pmatrix} \alpha & \frac{x}{\alpha} \\ 0 & \alpha \end{pmatrix} \quad \mathcal{M}(-1/3) = \mu \begin{pmatrix} \beta & \frac{x}{\alpha} \\ 0 & \beta \end{pmatrix} \tag{8}$$

We introduce the diagonal mass matrix

$$\widehat{M}(Q) = \mathcal{U}_L(Q)\mathcal{M}(Q)\mathcal{U}_R(Q)^\dagger \tag{9}$$

The unitary matrices are -in the case of two generations- pure rotation matrices, and their coefficients (angles) depend only on the unknwown parameters a, b and y. One can express these angles solely in terms of ratios of quark masses (the eigenvalues of $\widehat{M}(Q)$). This was already done in [1] and we give here a short derivation of this result.

From equation (8) we get $det(\widehat{M}(Q)) = det(\mathcal{M}(Q))$ and $Tr(\mathcal{M}(Q)) = Tr(\mathcal{U}_L(Q)^\dagger \widehat{M}(Q)\mathcal{U}_R(Q)) = Tr(\widehat{M}(Q)\mathcal{U}_R(Q)\mathcal{U}_L(Q)^\dagger)$. We parametrize the rotation matrices as follows.

$$\mathcal{U}_L(Q) = \begin{pmatrix} \cos(\theta_Q/2) & \sin(\theta_Q/2) \\ -\sin(\theta_Q/2) & \cos(\theta_Q/2) \end{pmatrix} \tag{10}$$

It can be seen that, in this two generations case, $\mathcal{U}_R(Q)$ is gotten from $\mathcal{U}_L(Q)$ by replacing θ_Q by $\pi - \theta_Q$. Therefore

$$\mathcal{U}_R(Q)\mathcal{U}_L(Q)^\dagger = \begin{pmatrix} \sin(\theta_Q) & \cos(\theta_Q) \\ -\cos(\theta_Q) & \sin(\theta_Q) \end{pmatrix} \tag{11}$$

From the equation for the determinant, we get $det\left(\widehat{M}\left(\frac{2}{3}\right)\right) = m_u \; m_c = \mu^2\alpha^2$ and $det\left(\widehat{M}\left(-\frac{1}{3}\right)\right) = m_d \; m_s = \mu^2\beta^2$. From the equation for the trace we get $(m_u + m_c)\sin\left(\theta_{2/3}\right) = 2\mu\alpha$ and $(m_d + m_s)\sin\left(\theta_{-1/3}\right) = 2\mu\beta$. Therefore

$$\sin\left(\theta_{2/3}\right) = \frac{2\sqrt{m_u \; m_c}}{m_u + m_c} \quad \sin\left(\theta_{-1/3}\right) = \frac{2\sqrt{m_d \; m_s}}{m_d + m_s} \tag{12}$$

and the Cabibbo angle is equal to

$$\theta_c = \frac{\theta_{2/3} - \theta_{-1/3}}{2} \tag{13}$$

This relation can be approximated (in radians) by

$$|\theta_c| \approx \sqrt{\frac{m_d}{m_s}} + \sqrt{\frac{m_u}{m_c}} \tag{14}$$

The observation that $|\theta_c|$ can be fitted by $\sqrt{\frac{m_d}{m_s}}$ was done many years ago and was then obtained in [7] using *ad hoc* conjectured mass matrices (different from those obtained above from representation theory of $SU(2|1)$).

In the case of two generations, the number of independent constants entering the Standard Model is equal to 5, four masses and one angle. The case of three generations can be handled similarly. In this last case, we have six masses and four independent parameters in the Kobayashi-Maskawa matrix (three moduli and one phase), therefore 10 constants. The two equations giving the eigenvalues are now cubic rather than quadratic. This leads to relations between the parameters that are less "simple" than in the two generation case. Here again one can take a smaller number of independent parameters in $SU(2|1)$ representations. The results turn out to be rather sensitive to the precise values of <u>all</u> quark masses –remember that there is a factor of the order of 10^{10} between the square of the top mass and the square of the up mass! A detailed discussion of the three-generations case is done in [6].

Concluding remarks

The emergence of irreducible representations of $SU(2|1)$ (typical and non-typical) in the Standard Model is here presented as an empirical fact. Postulating this property amounts to recover the fact that fermions have to appear in specific $SU(2)$ doublets or singlets of given chirality and appropriate hypercharge. In the Standard Model, the values of those hypercharges is fixed by experiment (we know the electric charges of quarks and leptons) whereas, in the present approach, their values are uniquely determined by algebraic constraints. The fact that irreducible representations of this superalgebra could describe quarks and leptons was already noticed in [8][9].

The new observation here is that reducible indecomposable representations of the same Lie superalgebra provide a clue with respect to the problem of generations, both for quarks and leptons –including the phenomena of neutrino masses and neutrino oscillations. It also provides a pattern of elementary fermions in agreement with all the features of the Standard Model. The corresponding algebraic constraints appearing in the mixing matrices (the matrices of odd generators expressing the indecomposability of these representations) lead to a possible diminution of the number of arbitrary parameters of the Standard Model and to relations that seem to agree with experiment. All this analysis is however carried at the classical level. We know that, in the framework of perturbative quantum field theories, a relation (or a numerical value) appearing at the classical level has no definitive meaning unless it is shown that the quantum field theory should preserve this particular relation. We do not know yet what happens at the quantum level but the whole approach seems to have many nice features, therefore there is a hope that some of them should remain after proper quantization. The gauging –in a non-commutative way– of the discrete symmetry expressing our freedom of chosing between left and right is precisely described by Higgs fields. It was shown (cf. [3] and also [4] where a rather different formalism is used) that taking acount such a gauging leads to the whole bosonic sector of the Standard Model, including its Higgs potential with symmetry breaking. $SU(2|1)$ appears also as a global symmetry of the free bosonic sector. The final total Lagrangian is invariant under $SU(2) \times U(1)$ as usual (with spontaneous symmetry breaking to $U(1)$). All these features are rather intriguing and the approach deserves certainly more understanding.

Appendix

We give here the odd generators for the representations used in the text.

The leptonic and quark representations $[\ell]$ and $[q]$.

$$[\ell] = \mu \begin{pmatrix} 0_{2\times 2} & \begin{pmatrix} \Omega_+/\kappa \\ \Omega_-/\kappa \end{pmatrix} \\ (\,\kappa\Omega'_- \quad \kappa\Omega'_+\,) & 0 \end{pmatrix} \qquad [q] = \mu \begin{pmatrix} 0 & 0 & \frac{1}{3\alpha}\Omega_+ & \frac{2}{3\beta}\Omega'_+ \\ 0 & 0 & \frac{1}{3\alpha}\Omega_- & -\frac{2}{3\beta}\Omega'_- \\ \alpha\Omega'_- & \alpha\Omega'_+ & 0 & 0 \\ -\beta\Omega_- & \beta\Omega_+ & 0 & 0 \end{pmatrix}$$

The representation $[\ell] \oplus [1]$

$$\mu \begin{pmatrix} 0_{2\times 2} & \begin{pmatrix} \Omega_+/\kappa \\ \Omega_-\kappa \end{pmatrix} & \begin{pmatrix} \eta\Omega'_+ \\ -\eta\Omega'_- \end{pmatrix} \\ \begin{matrix} (\,\kappa\Omega'_- \quad \kappa\Omega'_+\,) \\ (\,0 \quad 0\,) \end{matrix} & 0_{2\times 2} & \end{pmatrix}$$

The representation $[q] \oplus [q] \oplus [q]$.

It is described by 12×12 matrices. Each block appearing below refers to a 4×4 matrix.

$$\mu \begin{pmatrix} A & B' & B''' \\ 0 & A & B'' \\ 0 & 0 & A \end{pmatrix}$$

With

$$A = \begin{pmatrix} 0 & C_1 \\ C_2 & 0 \end{pmatrix} \qquad B' = \begin{pmatrix} 0 & C'_1 \\ 0 & 0 \end{pmatrix} \qquad B'' = \begin{pmatrix} 0 & C''_1 \\ 0 & 0 \end{pmatrix}$$

$$B''' = \begin{pmatrix} 0 & C'''_1 \\ 0 & 0 \end{pmatrix} \quad C_1 \doteq C_1(\varepsilon, \gamma) = \begin{pmatrix} \gamma\Omega_+ & \varepsilon\Omega'_+ \\ \gamma\Omega_- & -\varepsilon\Omega'_- \end{pmatrix} \quad C_2 = \begin{pmatrix} \alpha\Omega'_- & \alpha\Omega'_+ \\ -\beta\Omega_- & \beta\Omega_+ \end{pmatrix}$$

$$C'_1 = C_1(\varepsilon', \gamma') \qquad C''_1 = C_1(\varepsilon'', \gamma'') \qquad C'''_1 = C_1(\varepsilon''', \gamma''')$$

These generators obey the commutation relations of $SU(2|1)$ provided the following constraints are satisfied.

$$\varepsilon\beta = \tfrac{2}{3} \quad ; \quad \gamma\alpha = \tfrac{1}{3} \quad ; \quad \epsilon'\beta = -\gamma'\alpha$$

The representation $([\ell] \oplus [1]) \oplus ([\ell] \oplus [1]) \oplus ([\ell] \oplus [1])$

$$\mu \begin{pmatrix} A_1 & B' & B''' \\ 0 & A_2 & B'' \\ 0 & 0 & A_3 \end{pmatrix}$$

With

$$A_i = \begin{pmatrix} 0 & C^i_1 \\ C^i_2 & 0 \end{pmatrix} \qquad B' = \begin{pmatrix} 0 & C'_1 \\ 0 & 0 \end{pmatrix} \qquad B'' = \begin{pmatrix} 0 & C''_1 \\ 0 & 0 \end{pmatrix}$$

$$B''' = \begin{pmatrix} 0 & C'''_1 \\ 0 & 0 \end{pmatrix} \quad C^i_1 \doteq C_1(1/\kappa_i, \eta_i) = \begin{pmatrix} \Omega_+/\kappa_i & \eta_i\Omega'_+ \\ \Omega_-/\kappa_i & -\eta_i\Omega'_- \end{pmatrix} \quad C^i_2 = \begin{pmatrix} \kappa_i\Omega'_- & \kappa_i\Omega'_+ \\ 0 & 0 \end{pmatrix}$$

$$C'_1 = C_1(0, \eta') \qquad C''_1 = C_1(0, \eta'') \qquad C'''_1 = C_1(0, \eta''')$$

References

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