Energy inequalities in integrable quantum field theory

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Referat (abstract):

Negative energy densities are an abundant and necessary feature of quantum field theory (QFT) and may lead to surprising measurable effects. Some of these stand in contrast to classical physics, so that the accumulation of negative energy, also in quantum field theory, must be subject to some constraints. One class of such constraints is commonly referred to as quantum energy inequalities (QEI). These are lower bounds on the averaged stress-energy tensor which have been established quite generically in quantum field theory, however, mostly excluding models with self-interaction.

A rich but mathematically tractable class of interacting models are those subject to integrability. In this thesis, we give an overview of the construction of integrable models via the inverse scattering approach, extending previous results on the characterization of local observables to models with more than one particle species and inner degrees of freedom.

We apply these results to the stress-energy tensor, leading to a characterization of the stress-energy tensor at one-particle level. In models with simple interaction, where the S-matrix is independent of the particles' momenta, this suffices to construct the full stress-energy tensor and provide a state-independent QEI. In models with generic interaction, we obtain QEIs at the one-particle level and find that they substantially constrain the choice of reasonable stress-energy tensors, in some cases fixing it uniquely.

Author's note:

Most parts of this thesis have been included in the preprint

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which appears in the bibliography as [BCM23] and has been submitted to publication in *Annales Henri Poincaré*. New, in comparison to the preprint, are the results from Chapter 3 as well as most of the material in Chapter 2, Section 5.1, and Section 5.2, and all of the material in the appendices. Throughout the rest of the document there are significant verbatim inclusions of the preprint although in many cases the contents of the preprint were substantially expanded and supplemented by detailed computations.

The author's contribution to the preprint encompasses parts of the conceptualization, providing the computations and proofs, and writing of the drafts; all of that was under supervision and with continuous input by H. Bostelmann and D. Cadamuro. In Chapter 6, the idea for the state-independent QEI result for constant scattering functions and the style of its presentation is due to H. Bostelmann. The conceptualization of the project on QEIs at one-particle level and an exemplaric computation on the decomposition of form factors is due to D. Cadamuro. Review of the preprint drafts was done by all authors together including substantial editing by H. Bostelmann and myself.

Declaration:

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Contents

Chapter 1

Introduction

The realm of quantum physics has many peculiar features which tend to challenge our intuition trained by the everyday experience of classical mechanics. One such feature, though not limited to quantum physics per sé, is the appearance of negative energies: While the positivity of the total energy in a system, also in quantum physics, is a hallmark of stability, locally, energy may be negative. Heuristically, we may think of this as a consequence of Heisenberg's uncertainty principle, where a narrow localization in space and time has to allow for a wide uncertainty in momentum and energy so that positivity may be violated statistically. In this regard, it may not even be surprising that negative energy densities are an abundant and even necessary feature of quantum field theory [EGJ65] and that the energy density at a point can become arbitrarily negative; see, e.g., [Few06, Sec. 2].

The presence of negative energies indicates that the system is in a state which has locally lower energy than the global ground state of the system, usually the vacuum, and is thus sometimes termed a "sub-vacuum" effect. This may lead to bizarre effects: Understanding energy as measuring the tendency to "act" in some way, negative energy indicates rather the tendency to "be acted upon" in some way. For instance, a source of negative energy radiation may gain energy instead of losing it, and a chunk of negative energy may slow down an oscillation rather than stimulating it or cool a thermodynamical system rather than heating it. In more generality, one may say that negative energies may cause an effect which is the reverse of our classical intuition. For the detection of such atypical processes, and thus negative energy, many proposals exist involving effects such as reduced atomic decay rates [FR11] or increases in the speed of light pulses in nonlinear materials [DLF19]. These proposals are about to get within experimental reach but have not yet been measured. We refer to Section 5 of [For10] and references therein for a brief discussion of the detection of "sub-vacuum" effects.

As has been stressed, such effects stand in contrast to classical physics, so that—since classical physics emerges from quantum physics—the extent of negative energies has to be limited for physically reasonable models of quantum physics. Such limits are also needed because an infinite energy sink, for instance implemented by unconstrained accumulation of negative energy, clearly renders a system unstable. Of interest are here not only the constraints on magnitude (how much negative energy can be gathered?) but, given energy's tendency to balance out over time, also on duration (how long can negative energy be sustained?). Since the total energy, i.e., the energy density averaged over the whole space, has to be positive, there has to be also a constraint on the scaling in space.

Mathematically, all these questions can be summarized into whether and in what form local averages of the energy density are bounded from below. We refer to such a bound as quantum energy inequality (QEI) which may take the following form: For the energy density T^{00} at a spacetime point x averaged over a positive test function g^2 the inequality

$$\langle\varphi, \int dx\, g(x)^2 T^{00}(x)\varphi\rangle \geq -c_g\|\varphi\|^2, \tag{1.1}$$

holds for a suitably large set of state vectors φ with a constant $c_g > 0$ which does not depend on φ. If such a bound holds, the previous questions can be answered by choosing specific profiles for the averaging function g. Here we give an example and otherwise refer to the review [Few12, Sec. 1.3]: Choosing g to be supported in a compact region and normalized, the l.h.s. of (1.1) gives a decent measure for the actual energy in that region whereas c_g measures the maximum amount of negative energy that could be "gathered" in that region (and in that class of states). Varying the scale of that region in time and space allows for assessing the duration with which the negative energy density can be sustained and the scaling (in space) with which positivity is restored.

It may also happen that the limit where g becomes constant in time restores positivity, which is referred to as averaged weak energy condition (AWEC). Looking at components of $T^{\mu\nu}$ in a null direction instead, one obtains the weaker averaged null energy condition (ANEC) which is expected to hold quite generically [KS20, Sec. 4.3] although mathematically precise evidence is mostly limited to 1+1d Minkowski space [Ver00; FH05]. Connections to quantum information and entropy were discussed more recently in form of the quantum null energy condition (QNEC) which is expected to be a consequence of ANEC [Bou+16; CF20].

In the following, we will focus on a few more aspects of QEIs relevant to this thesis and refer the interested reader to the reviews [Few06; Rom06; Ver08; For10; Few12; KS20] and references therein. Starting with the original work by Lawrence Ford in the late 70s [For78], which shows how a certain QEI is necessary to prevent a violation of the second law of thermodynamics, QEIs have a rich history. In the meantime, they have been established quite generically in linear QFTs, including QFTs on curved spacetimes as well as in 1+1d conformal QFTs; see, in particular, the reviews [Ver08, Sec. 3], [Few12, Sec. 1.4], and [KS20, Sec. 3]. Also, they have gained significant importance in semiclassical gravity, where the expectation value of $T^{\mu\nu}$ appears on the right-hand side of the Einstein equations. In this context, QEIs can yield constraints on exotic spacetime geometries such as wormholes and warp drives and lead to generalized singularity theorems extended from classical results in general relativity; see, in particular, the reviews [Rom06] and [KS20, Sec. 5].

From this perspective, the field of QEIs appears to be already close to complete. There is, however, still an important gap to be filled: It is not yet known how much the presence of self-interaction (i.e., having a nontrivial scattering matrix between particles) affects the results mentioned before. While free field QEIs should apply when sampling times[1] are larger than the interaction time scale [Rom06], it would be useful to analyse this more carefully and there are situations where this fails and interaction plays a significant role. For instance, consider the Casimir effect, where two conducting plates are brought very close together, resulting in an attractive force between the plates. The effect can be explained by the presence of a negative energy density in the confined region due to the boundary conditions posed by the conducting plates. Provided the setup can be sustained for arbitrarily high energies or infinitely long, this would also violate typical QEIs.A natural interpretation for the first would be that the failure of the QEI is because of unrealistic boundary conditions and that the positive energy contributions required to sustain the Casimir setup must be included [OG03; GO04]. The second point, however, implies that the AWEC fails (but note the ANEC). For QEIs it indicates that QEIs in the presence of self-interaction have to be weaker than QEIs in linear QFTs or that they have to be interpreted in a relative sense, i.e., as difference inequalities in comparison with another state (here the Casimir ground state).

Some generic lower bounds of the energy density including interacting models, but weaker than (1.1), can be obtained from operator product expansions [BF09] or recently using Tomita-Takesaki modular theory [MPV22]. Concrete results in models with self-interaction are rare, though.

The situation is better when specializing to the class of 1+1d integrable models. In these models, the scattering operator or S-matrix is constrained to be particle number conserving and factorizing but nonetheless allows for a large class of interactions; see, e.g., [KW78; ZZ79; AAR01]. Factorizing here means that the scattering process decomposes into a chain of elementary scattering processes corresponding to the interaction of two particles. A reversal of the perspective then leads to a generic construction procedure for such models: As a starting point, fix the "supposed-to-be" particle spectrum and elementary processes. Then, under the given constraints for the scattering, this completely fixes the S-matrix and is sufficient information to define the integrable model. This is known as the inverse scattering approach and has been shown to yield agreement with perturbatively defined QFTs in many

[1] I.e., the time scale of the support of the averaging function.

cases; see the last cited references again.

A QEI in this context was first established in the Ising model [BCF13]. Also, a QEI at one-particle level (i.e., where (1.1) holds for one-particle states φ) has been obtained for the class of models with one scalar particle type and no bound states [BC16]. The class of integrable models is much richer, though – they can also describe several particle species with a more complicated scattering matrix between them or particles with inner degrees of freedom; further, these particles may form bound states[2]. Aside from a recent result where a QEI is proven for the sine-Gordon model [FC22] such models have not been treated yet.

The construction of local observables in integrable models, for instance of the stress-energy tensor, typically follows the so-called form factor program; see, e.g., [Smi92; BFK08]. The first step of this program is the inverse scattering approach as described above. The second step is a list of equations, the form factor equations, expected to be equivalent to locality of the observable in question. This equivalence is also called local commutativity theorem and was treated in the physics literature by [KS89; Smi92; Las94; Que99] along with investigations in specific models, e.g., [KW78; BK02; BFK06]. Later, rigorous proofs were given which establish equivalence for models with one scalar particle type or more general models but less strict so-called wedge-localization of the observable, in both cases excluding bound states [Lec07; BC15; AL17]. The derived equations pertain to truncated momentum space correlation functions of the observable and are termed form factors. The last step of this program, the definition of the (Wightman) n-point functions, involves an infinite Fourier-type series. While its convergence can be argued at the heuristic level, it is a long-standing problem to mathematically show this; a proof for the easiest non-trivial model, the sinh-Gordon model, was only given recently [Koz21; Koz22]. An alternative approach was developed using operator-algebraic methods. A rigorous construction was given, first for models with only one scalar particle type [Lec07], and later extended to models including several particle species and particles with inner degrees of freedom [LS14; AL17] so that these models are amenable to a mathematical analysis.

[2] Bound states are understood as poles of the scattering matrix within the so-called physical strip. See [Bab+99, Sec. 2.2] for further details.

Structure of the thesis

The primary focus of this thesis is exploring QEIs in the presence of self-interaction, focussing on the class of integrable models in quantum field theory. The main results extend QEIs obtained in [BCF13; BC16] to a generic class of models, including several particle species, inner degrees of freedom, and bound states. As a secondary point we extend results on constructive aspects of such models.

To begin with, in Chapter 2, we will review the construction of integrable models in 1+1d via the inverse scattering method, including results on asymptotic completeness of these models, and establish the mathematical framework for the following parts. We will also point out the connection of this framework to algebraic quantum field theory. In Chapter 3 we introduce the concept of form factor equations which characterize the momentum space correlations of local operators in models with factorizing scattering. We extend here preceding mathematical results on the local commutativity theorem for one- and two-particle form factors to a setup with several particle species and inner degrees of freedom (Thm. 3.2.1, Prop. 3.2.2).

Further, in Chapter 4, we will explain the decomposition of form factors into an observable-specific and a model-dependent part involving the so-called "minimal solution". We supplement the decomposition by establishing existence of minimal solutions for a large class of models and reviewing a recipe to obtain them in practice. Of crucial importance for later, we derive the asymptotic growth of the minimal solution depending directly on the properties of the scattering function (Prop. 4.2.6).

We proceed in Chapter 5 with the analysis and description of the particular local operator–the stress-energy tensor $T^{\mu\nu}$ (also called energy-momentum tensor)–including its 00-component, the energy density T^{00}. For QEIs it central to know what form these operators take (confer Eq. (1.1)). The classical Lagrangian is often used as heuristic guidance; however, if one takes an inverse scattering approach to integrable models, starting by prescribing the scattering function, then a classical Lagrangian may not even be available in all cases. Instead, we will restrict the possible form of the energy density starting from generic physical assumptions (such as the continuity equation, but initially disregarding QEIs). The abstract assumptions will be transferred to the one-particle level, which will be our main focus later, and will be shown to fix $T^{\mu\nu}$ up to a polynomial factor (Thm. 5.3.1, Prop. 5.3.4).

We then ask whether QEIs can hold given such a stress-energy tensor. In the following two chapters, we present the two main results:

In Chapter 6, for a class of models with *rapidity-independent* scattering function, with a canonical choice of energy density, we establish a QEI in states of arbitrary particle number (Theorem 6.3.3).

In Chapter 7, for generic scattering functions, we give necessary and sufficient criteria for QEIs to hold *at one-particle level* (Thm. 7.1.1); it turns out that the existence of QEIs critically depends on the large-rapidity behaviour of the two-particle form factor F_2 of the energy density. We conclude this chapter by connecting F_2 more directly to the properties of the model at hand, thereby obtaining a recipe for QEIs at one-particle level to hold in generic models.

In Chapter 8, we apply our results to several concrete examples, namely, to the Bullough-Dodd model (Sec. 8.2) which has bound states, to the Federbush model (Sec. 8.3) as an interacting model with rapidity-independent scattering function, and to the $O(n)$ nonlinear sigma model (Sec. 8.4) which features several particle species. In particular, we investigate how QEIs further restrict the choice of the stress-energy tensor in these models, sometimes fixing it uniquely.

Lastly, in Chapter 9, we will discuss the results obtained in this thesis, mention unfinished work obtained during the PhD project, and suggest future research directions.

Chapter 2

Constructive aspects of integrable quantum field theories

The aim of this chapter is to give a general and rigorous description of an integrable QFT model focussing on constructive aspects. This will be the foundation of the following chapters. The method of construction is the inverse scattering method, which starts by specifying the model in terms of its scattering data. The scattering data consists of the model's particle spectrum and interactions which can be represented, respectively, by the one-particle little space and the two-to-two-particle scattering function (as will be introduced later). That this scattering function fully captures the dynamics of the model is a special feature of integrable QFT models in 1+1d, where the S-matrix factorizes and is fully determined by its two-to-two-particle component.

The structure of this chapter will be the following: We start with specifying the particle content of the model—the particle spectrum—allowing immediately for the construction of the one-particle state space of the model (first quantization) (Sec. 2.2). The next step is to specify the particle's interactions via their two-to-two-particle scattering function (Sec. 2.3) which is the central input for the construction of the full state space (second quantization) (Sec. 2.4).

Important properties like asymptotic completeness (Sec. 2.5) and the form factor series (Chapter 3) follow. We will also briefly connect the construction given here to the framework of algebraic quantum field theory (Sec. 2.6). Further background material like details on Poincaré group representations in 1+1d, discrete symmetries, and bound states and proofs of some statements from the main text are deferred to Appendix A.

2.1 General notation

We will work on 1+1d Minkowski space $(\mathbb{M}, g)$ and choose the Minkowski metric to be $g = \mathrm{diag}(+1, -1)$ by convention. The Minkowski inner product will be denoted by $p.x = g_{\mu\nu}p^\mu x^\nu$. A single parameter, called *rapidity*, conveniently parametrizes the mass shell on $\mathbb{M}$. In this parameterization, the momentum at rapidity θ is given by $p^0(\theta; m) := m \operatorname{ch} \theta$ and $p^1(\theta; m) := m \operatorname{sh} \theta$, where $m > 0$ denotes the mass. We will use θ, η, λ to denote real and ζ to denote complex rapidities. Introducing the open and closed strips, $\mathbb{S}(a,b) := \mathbb{R} + i(a,b)$ and $\mathbb{S}[a,b] := \mathbb{R} + i[a,b]$, respectively, the region $\mathbb{S}[0,\pi]$ will be of particular significance and is referred to as the *physical strip*.

As test function spaces, we denote with $\mathcal{D}(\Omega)$ the space of smooth compactly supported functions on Ω and with $\mathcal{S}(\Omega)$ the space of smooth rapidly decaying (or Schwartz')functions on Ω. If necessary, we supplement a specification of the space of values in typical fashion. Concerning the Fourier transform, we adopt the convention that for a function $f \in \mathcal{S}(\mathbb{R})$ its Fourier transform is given by

$$\tilde{f}(k) := \int dx\, f(x) e^{ikx} \tag{2.1}$$

extended by continuity to functions $f \in L^1(\mathbb{R})$ or $f \in L^2(\mathbb{R})$. For functions $f \in \mathcal{S}(\mathbb{M})$ we adopt the convention

$$\tilde{f}(p) := \int d^2x\, f(x) e^{ip.x} \tag{2.2}$$

with analogous extensions to larger function spaces.

In the following, let $\mathcal{K}$ be a finite-dimensional complex Hilbert space with inner product $(\cdot,\cdot)$, linear in the second position. We denote its extension to $\mathcal{K}^{\otimes 2}$ as $(\cdot,\cdot)_{\mathcal{K}^{\otimes 2}}$ and the induced norm as $\|\cdot\|_{\mathcal{K}^{\otimes 2}}$; i.e., for $v_i, w_i \in \mathcal{K}, i = 1,2$, we have $(v_1 \otimes v_2, w_1 \otimes w_2)_{\mathcal{K}^{\otimes 2}} = (v_1,w_1)(v_2,w_2)$. For computations, it will be convenient to choose an orthonormal basis $\{e_\alpha\}, \alpha \in \{1,...,d_{\mathcal{K}}\}$, where $d_{\mathcal{K}} \in \mathbb{N}$ denotes the dimension of $\mathcal{K}$. In this basis, we denote $v \in \mathcal{K}^{\otimes m}$ and $w \in \mathcal{B}(\mathcal{K}^{\otimes n}, \mathcal{K}^{\otimes m})$ in vector and tensor notation, using multi-indices $\boldsymbol{\alpha} \in \{1,...,d_{\mathcal{K}}\}^m, \boldsymbol{\beta} \in \{1,...,d_{\mathcal{K}}\}^n$, by

$$v^{\boldsymbol{\alpha}} := (e_{\boldsymbol{\alpha}}, v), \quad w^{\boldsymbol{\alpha}}_{\boldsymbol{\beta}} := (e_{\boldsymbol{\alpha}}, we_{\boldsymbol{\beta}}). \tag{2.3}$$

For adjoints (we will use $*$ to denote them), we have $(w^*)^{\boldsymbol{\beta}}_{\boldsymbol{\alpha}} = \overline{w^{\boldsymbol{\alpha}}_{\boldsymbol{\beta}}}$.

Operators on $\mathcal{K}$ or $\mathcal{K}^{\otimes 2}$ will be denoted by uppercase Latin letters. This also applies to vectors in $\mathcal{K}^{\otimes 2}$, which are identified with operators on $\mathcal{K}$ as follows: For an antilinear involution $J \in \mathcal{B}(\mathcal{K})$ (to be fixed later), the map $A \mapsto \hat{A}$ defined by

$$\forall u,v \in \mathcal{K}: \quad (u, \hat{A}v) := (u \otimes Jv, A)_{\mathcal{K}^{\otimes 2}} \tag{2.4}$$

yields a vector space isomorphism between $\mathcal{K}^{\otimes 2}$ and $\mathcal{B}(\mathcal{K})$. Some of its properties are collected in the appendix (Lemma A.6.1). Sometimes we use the special element $I_{\otimes 2} \in \mathcal{K}^{\otimes 2}$ defined by $\widehat{I_{\otimes 2}} = \mathbb{1}_{\mathcal{K}}$. For an arbitrary orthonormal basis $\{e_\alpha\}_\alpha$ of $\mathcal{K}$ it is explicitly given by

$$I_{\otimes 2} = \sum_\alpha e_\alpha \otimes Je_\alpha. \tag{2.5}$$

We also introduce the *flip operator* $\mathbb{F} \in \mathcal{B}(\mathcal{K}^{\otimes 2})$ given by $\mathbb{F}(u \otimes v) = v \otimes u$ $(u,v \in \mathcal{K})$. $I_{\otimes 2}$ is invariant under the action of $\mathbb{F}$ and of $U^{\otimes 2}$ for any $U \in \mathcal{B}(\mathcal{K})$ with U unitary or anti-unitary and $[U,J] = 0$; also, we have $\|I_{\otimes 2}\|_{\mathcal{K}} = \sqrt{d_{\mathcal{K}}}$ (Lemma A.6.2).

2.2 Particle spectrum and one-particle space

The *particle spectrum* consists of a finite index set labeling particle types $\mathfrak{I}$, a collection $\{(m_i, s_i, q_i)\}_{i\in\mathfrak{I}}$ of characteristic quantities for each particle type: the particle's mass m_i, spin s_i, and charge q_i under the group of global symmetries $\mathcal{G}$.

These quantities classify the particles' transformation behaviour under the symmetries of the model. Mass m and spin s label the positive-energy unitary representations $U_{[m,s]}$ of the proper orthochronous Poincaré group which are constructed in analogy to the Wigner classification of relativistic particles in 1+3d and defined explicitly later on. As opposed to the 1+3d case, these representations are reducible unless $s = 0$ which is explained in more detail in Appendix A.1. The charge q labels inequivalent unitary irreducible representations V_q of $\mathcal{G}$ on a Hilbert space $\mathcal{K}_q$. We suppose further that there is an involution acting on $\mathfrak{I}$ as $i \mapsto \bar{i}$ such that $m_{\bar{i}} = m_i$, $s_{\bar{i}} = s_i$ and $q_{\bar{i}} = \bar{q}_i$, where $\bar{q}$ labels the (complex) conjugate[1] representation $V_{\bar{q}} = \overline{V_q}$ with respect to q acting on the (complex) conjugate space $\mathcal{K}_{\bar{q}} = \overline{\mathcal{K}_q}$. The linear mapping between $\mathcal{K}_q$ and $\mathcal{K}_{\bar{q}}$ defines the *charge conjugation map* C satisfying $C = C^{-1} = C^*$.

In the *presence of bound states* we will supplement the particle spectrum by a set of fusion rules $\mathfrak{F} \subset \{ij \to k : i, j, k \in \mathfrak{I}\}$ which collects feasible fusion processes of the model. Here $ij \to k$ means that particle type i and j fuse to bound state type k. A model without bound states corresponds to $\mathfrak{F} = \emptyset$. A more detailed definition of this additional structure and its interpretation will be deferred to Appendix A.5.

We restrict here to finitely many massive particles, $m > 0$, with half-integer spin, $s \in \frac{1}{2}\mathbb{N}_0$. Note that in general, m and s can be nonnegative real numbers, where continuous spin is a special feature of 1+1d. Also, we will restrict to $\mathcal{G}$ being a compact Lie group—a standard assumption in QFT.

A central ingredient to our framework will be:

Definition 2.2.1. A *(one-particle) little space* $(\mathcal{K}, V, J, M)$ (with global symmetry group $\mathcal{G}$) is given by a finite dimensional Hilbert space $\mathcal{K}$, a unitary representation V of a compact Lie group $\mathcal{G}$ on $\mathcal{K}$, an antiunitary involution J on $\mathcal{K}$, and a linear operator M on $\mathcal{K}$ with strictly positive spectrum. We further assume that $V(g)$, J, and M commute pairwise.

The construction of the little space corresponding to a given particle spectrum is straightforward: Let J_q denote the antilinear conjugation on $\mathcal{K}_q$ resulting from a combination of charge and complex conjugation. Let further $M_i = m_i \mathbb{1}_{\mathcal{K}_{q_i}}$ for each

[1] The conjugate representation is also known as the contragradient or dual representation. By unitarity of V_q, we have $V_q^{-1,t} = \overline{V_q}$.

$i \in \mathfrak{I}$. Then the corresponding little space is

$$(\mathcal{K}, V, J, M) = \oplus_{i\in\mathfrak{I}}(\mathcal{K}_{q_i}, V_{q_i}, J_{q_i}, M_i), \tag{2.6}$$

where $\oplus$ refers to the direct sum of the tuple's constituents, namely, Hilbert spaces as well as representations and operators acting upon them. From now on we will proceed with an abstract little space.

The little space represents the discrete remnant of the one-particle state space, the full one-particle (state) space is recovered by boosts. So given some little space $(\mathcal{K}, V, J, M)$, we define the *one-particle space* $\mathcal{H}_1 := L^2(\mathbb{R}, \mathcal{K}) \cong L^2(\mathbb{R}) \otimes \mathcal{K}$, on which we consider the (anti-)unitary operators

$$(U_1(x,\lambda)\varphi)(\theta) := e^{ip(\theta;M).x}\varphi(\theta-\lambda), \quad (x,\lambda) \in \mathcal{P}_+^\uparrow, \tag{2.7}$$

$$(U_1(j)\varphi)(\theta) := J\varphi(\theta), \tag{2.8}$$

$$(V_1(g)\varphi)(\theta) := V(g)\varphi(\theta), \quad g \in \mathcal{G}, \tag{2.9}$$

for any $\varphi \in \mathcal{H}_1$.

This defines a unitary strongly continuous representation of the proper Poincaré group $\mathcal{P}_+$ and of $\mathcal{G}$, where the antiunitary $U_1(j)$ is the CPT operator, representing spacetime reflection: Note here that by antilinearity and by $[M, J] = 0$ one has $U_1(x,\lambda)U_1(j) = U_1(j)U_1(-x,\lambda)$. The (one-particle) generators of the group of translations and of the group of boosts are given by $p(\cdot; M)$ and $-i\frac{d}{d\theta}$, respectively.[2]

We will denote the dimension of $\mathcal{K}$ by $d_{\mathcal{K}}$. Further, we will denote the spectrum of the mass operator M as $\mathfrak{M} \subset (0,\infty)$ and its spectral projections as $E_m, m \in \mathfrak{M}$.

Remark 2.2.2. (Convention for charge conjugation) For a given basis $\{e_\alpha\}$ of $\mathcal{K}$ we may introduce the charge conjugated basis $\tilde{e}_{\bar{\alpha}} := Je_\alpha$. For convenience, we will use the charge conjugated basis instead of the original one whenever barred indices appear, i.e., for $v \in \mathcal{K}$, $v^{\bar{\alpha}}$ denotes $(\tilde{e}_{\bar{\alpha}}, v)$ instead of $(e_{\bar{\alpha}}, v)$. In this context, we extend $e_\alpha \mapsto \tilde{e}_{\bar{\alpha}}$ by linearity to a unitary matrix which we denote with $C \in \mathcal{B}(\mathcal{K})$ and refer to as *charge conjugation matrix*. With our convention for the barred indices, we have that $C_\alpha^\beta = \delta_\alpha^{\bar{\beta}}$.

2.3 The scattering function

Scattering in integrable models is tightly constrained by the existence of an infinite set of conservation laws and therefore obeys a number of simplifying properties: Each scattering process is particle-number conserving and factorizes into two-to-two-particle scattering processes. Moreover, the set of incoming momenta coincides

[2]In both cases these are unbounded operators which are essentially self-adjoint on the domain of smooth compactly supported functions.

with the set of outgoing momenta. Due to these constraints, scattering in integrable models is fully determined in terms of its two-to-two-particle scattering function. The latter will be the central ingredient for the construction of a model and this section is devoted to describe it axiomatically; specifying also some important special cases.

Physical considerations require the scattering function to satisfy properties like unitarity, crossing symmetry, and the Yang-Baxter equation; the general axiomatic theory is well-known and given in [Iag93]. For integrable models these properties pass down to the two-particle scattering function and standard textbook accounts are found in [Mus10, Chap. 17], [Dor98, Chap. 3], and [AAR01, Chap. 8]. Basis-independent formulations are found for instance in [Bis12] and [AL17, Defn. 2.1].

In contrast to higher dimensional theories, a two-particle scattering process in 1+1d is fully parametrized by a single parameter. Standard choices for incoming/outgoing particles with momenta p_1 and p_2 (resp., rapidities θ_1 and θ_2) are the Mandelstam variable $s = (p_1 + p_2)^2$ or the absolute value of the rapidity difference $\theta = |\theta_1 - \theta_2|$. Their relation is given by the formula

$$s = (p_1 + p_2)^2 = m_1^2 + m_2^2 + 2m_1 m_2 \operatorname{ch} \theta \tag{2.10}$$

and we choose θ as our preferred parameter. For further details on the translation between the two descriptions see, e.g., [Lec06, Sec. 3.1].

As the central object to define the interaction of the model we introduce the S-function (also referred to as the auxillary scattering function [Bab+99, Eq. (2.7)]). It is closely related to the two-particle scattering function of the model, differing from it only in the presence of fermions or anyons by a “statistics factor” as will be seen in Section 2.5. Anyons are particles with exotic statistics which appear only in the context of 1+1 and 1+2d QFT.

Definition 2.3.1. Let $(\mathcal{K}, V, J, M)$ be a one-particle little space. A meromorphic function $S : \mathbb{C} \to \mathcal{B}(\mathcal{K}^{\otimes 2})$ with no poles on the real line is called *S-function* iff for all $\zeta, \zeta' \in \mathbb{C}$ the following holds:

(S1) Unitarity: $S(\bar{\zeta})^* = S(\zeta)^{-1}$.

(S2) Hermitian analyticity: $S(\zeta)^{-1} = S(-\zeta)$.

(S3) CPT-invariance: $J^{\otimes 2} \mathbb{F} S(\zeta) \mathbb{F} J^{\otimes 2} = S(\zeta)^*$.

(S4) Yang-Baxter equation: For $\mathbb{1} = \mathbb{1}_{\mathcal{K}}$,
$(S(\zeta) \otimes \mathbb{1})(\mathbb{1} \otimes S(\zeta + \zeta'))(S(\zeta') \otimes \mathbb{1}) = (\mathbb{1} \otimes S(\zeta'))(S(\zeta + \zeta') \otimes \mathbb{1})(\mathbb{1} \otimes S(\zeta))$.

(S5) Crossing symmetry:
$(u_1 \otimes u_2, S(i\pi - \zeta)\, v_1 \otimes v_2)_{\mathcal{K}^{\otimes 2}} = (Jv_1 \otimes u_1, S(\zeta)\, v_2 \otimes Ju_2)_{\mathcal{K}^{\otimes 2}}, \quad u_i, v_i \in \mathcal{K}.$

(S6) Translational invariance:
$(E_m \otimes E_{m'})S(\zeta) = S(\zeta)(E_{m'} \otimes E_m), \quad m, m' \in \mathfrak{M}.$

(S7) $\mathcal{G}$ invariance:
$\forall\, g \in \mathcal{G} : \quad [S(\zeta), V(g)^{\otimes 2}] = 0.$

Properties (S1) and (S2) recombine to $S(\zeta)S(-\zeta) = \mathbb{1}_{\mathcal{K}^{\otimes 2}}$ and $S(\zeta) = S(-\bar{\zeta})^*$. In a basis, these two as well as (S3) and (S5), respectively, amount to the following conditions:

$$\begin{aligned} S^{\gamma\delta}_{\rho\sigma}(\zeta)S^{\rho\sigma}_{\alpha\beta}(-\zeta) &= \delta^{\gamma}_{\alpha}\delta^{\delta}_{\beta}, & S^{\gamma\delta}_{\alpha\beta}(\zeta) &= \overline{S^{\alpha\beta}_{\gamma\delta}(-\bar{\zeta})}, \\ S^{\gamma\delta}_{\alpha\beta}(\zeta) &= S^{\bar{\beta}\bar{\alpha}}_{\bar{\delta}\bar{\gamma}}(\zeta), & S^{\gamma\delta}_{\alpha\beta}(i\pi - \zeta) &= S^{\bar{\alpha}\gamma}_{\beta\bar{\delta}}(\zeta). \end{aligned} \tag{2.11}$$

See Lemma A.3.2 in the appendix for a proof of this statement. Note also that there are different conventions concerning the placement of the tensor indices on S. For instance, compared to the convention adopted here, which agrees with [Bab+99; AL17], one has that [Smi92] twists the lower indices, i.e. $S^{\gamma\delta}_{\alpha\beta} = (S_{Smi92})^{\gamma\delta}_{\beta\alpha}$.

There are a number of important subclasses of S-functions which we will treat:

Definition 2.3.2. An S-function S is called

- *regular* iff $\kappa(S) := \sup\{\kappa \geq 0 : S \restriction_{\mathbb{S}(-\kappa,\kappa)}$ *is analytic and bounded.*$\} > 0$
- *diagonal* iff there exists an orthonormal basis $\{e_\alpha\}$ of $\mathcal{K}$ and $\mathbb{C}$-valued coefficients $s_{\alpha\beta}(\zeta)$ such that $S(\zeta)$ for all $\zeta \in \mathbb{C}$ is of the form[a]
$$S(\zeta) = \sum_{\alpha\beta} s_{\alpha\beta}(\zeta)|e_\beta \otimes e_\alpha)(e_\alpha \otimes e_\beta|. \tag{2.12}$$
- *scalar* iff $d_{\mathcal{K}} = 1$
- *constant* iff $S(\zeta)$ is independent of ζ
- *k-invariant*[b] with $k \in \{c, p, t, cp, ct, pt, cpt\}$ iff $S(\zeta) = S_k(\zeta)$ for all $\zeta \in \mathbb{C}$, where
$$(S_c)^{\gamma\delta}_{\alpha\beta} = S^{\bar{\gamma}\bar{\delta}}_{\bar{\alpha}\bar{\beta}}, \quad (S_p)^{\gamma\delta}_{\alpha\beta} = S^{\delta\gamma}_{\beta\alpha}, \quad (S_t)^{\gamma\delta}_{\alpha\beta} = S^{\alpha\beta}_{\gamma\delta}, \tag{2.13}$$
and the others by concatenation, e.g., $S_{cp} = (S_c)_p$.

[a] We use here standard braket notation where $|v)$ denotes a vector in $\mathcal{K}$ and $(v|$ a linear functional on $\mathcal{K}$ such that $(v|w) = (v, w)$.

[b] In written text we use upper case K=C,P,T,... instead of k.

Let us briefly comment on these subclasses: The regular class excludes certain more exotic particle spectra and interactions: First, the analyticity in a finite strip rules out infinitely many poles approaching the real line. Note here that poles within

$\mathbb{S}(-\pi, 0)$ or $\mathbb{S}(0, \pi)$, respectively, correspond to *resonances* (unstable particles with a finite lifetime) or *bound states* [Wei95]. Second, this also excludes[3] the presence of infintely many "masses" leading to a divergence of the thermodynamical partition function [Lec15, Sec. 3.3]. Lastly, the boundedness in a finite strip[4] implies the absence of exotic factors in the scattering functions like $\zeta \mapsto e^{ia\,\mathrm{sh}\,\zeta}$, $a > 0$, which have an essential singularity at infinity. The diagonal class corresponds to models where the particle spectrum is non-degenerate, i.e., where each particle is distinguished by its mass and its charge under $\mathcal{G}$; as a consequence, the scattering function is completely diagonal and S takes the form (2.12) [Mus10, Sec. 17.4]. The scalar class allows for one scalar particle type only. In this case, $\mathcal{G}$ can be taken to be either trival or equal to $\mathbb{Z}_2$ and S is automatically diagonal. The constant subclass refers to very simple types of interactions independent of the rapidity of the scattering particles. It is still larger than the class of free models and contains for instance the Federbush model which will be treated in Section 8.3. Finally, the discrete symmetries C-, P-, T-, etc., are present in many models, not only integrable ones. For details we refer to Section A.2.

For these subclasses the axioms from Definition 2.3.1 take an easier form

Proposition 2.3.3. *(a) The class of diagonal scattering functions consists of meromorphic functions of the form* (2.12) *(for some choice of an ONB of $\mathcal{K}$) with no poles on the real line, where the coefficients $s_{\alpha\beta}(\zeta)$ satisfy*

$$s_{\alpha\beta}(-\zeta) = s_{\beta\alpha}(\zeta)^{-1} = \overline{s_{\beta\alpha}(\bar{\zeta})}, \quad s_{\alpha\beta}(\zeta) = s_{\bar{\alpha}\bar{\beta}}(\zeta) = s_{\bar{\beta}\alpha}(i\pi - \zeta), \tag{2.14}$$

and $(V(g)^{\otimes 2})^{\gamma\delta}_{\alpha\beta} s_{\gamma\delta}(\zeta) = s_{\alpha\beta}(\zeta)$ for $g \in \mathcal{G}$. In this case $s_{\alpha\beta} = S^{\beta\alpha}_{\alpha\beta}$ and all other components of S vanish. Diagonal S-functions are automatically PT- and C-invariant.

(b) The class of scalar scattering functions consists of meromorphic functions $\zeta \mapsto s(\zeta) \in \mathbb{C}$ with no poles on the real line and

$$s(-\zeta) = s(\zeta)^{-1} = \overline{s(\bar{\zeta})} = s(\zeta + i\pi). \tag{2.15}$$

(c) The class of constant scattering functions consists of unitary self-adjoint matrices $S \in \mathcal{B}(\mathcal{K}^{\otimes 2})$ which commute with $V(g)^{\otimes 2}$ and $J^{\otimes 2}\mathbb{F}$ and satisfy (S4), (S5), and (S6).

[3]Note that this is also excluded by assuming finite-dimensionality of the little space.

[4]See, e.g., [Kar+77], where boundedness in a strip is taken as a technical assumption in case of the Thirring model and [Mit77] for a brief discussion of its validity.

Proof. (a): Given (2.12), we find

$$\begin{aligned} S^{\gamma\delta}_{\alpha\beta}(\zeta) &= (e_\gamma \otimes e_\delta, S(\zeta) e_\alpha \otimes e_\beta)_{\mathcal{K}^{\otimes 2}} \\ &= \sum_{\rho,\sigma} s_{\rho\sigma}(\zeta)(e_\gamma \otimes e_\delta, e_\sigma \otimes e_\rho)(e_\rho \otimes e_\sigma, e_\alpha \otimes e_\beta) \\ &= s_{\alpha\beta}(\zeta)\delta^\delta_\alpha \delta^\gamma_\beta. \end{aligned} \tag{2.16}$$

The only non-vanishing components of S are therefore given by $S^{\beta\alpha}_{\alpha\beta} = s_{\alpha\beta}$ and it is easy to adapt (2.11) and (S7) to this special case yielding (2.14). The Yang-Baxter equation (S4) and translational invariance (S6) are automatically satisfied for diagonal scattering functions. Concerning the Yang-Baxter equation one has for $\zeta'' = \zeta + \zeta'$ that

$$\begin{aligned} (S(\zeta) \otimes 1_{\mathcal{K}})(1_{\mathcal{K}} \otimes S(\zeta''))(S(\zeta') \otimes 1_{\mathcal{K}})^{\gamma\beta\alpha}_{\alpha\beta\gamma} &= s_{\beta\gamma}(\zeta)s_{\alpha\gamma}(\zeta'')s_{\alpha\beta}(\zeta') \\ &= s_{\alpha\beta}(\zeta')s_{\alpha\gamma}(\zeta'')s_{\beta\gamma}(\zeta) \\ &= (1_{\mathcal{K}} \otimes S(\zeta'))(S(\zeta'') \otimes 1_{\mathcal{K}})(1_{\mathcal{K}} \otimes S(\zeta))^{\gamma\beta\alpha}_{\alpha\beta\gamma}; \end{aligned} \tag{2.17}$$

and all other components vanish. It remains to prove PT- and C-invariance. By definition, PT-invariance amounts to $S^{\gamma\delta}_{\alpha\beta} = S^{\beta\alpha}_{\delta\gamma}$ which is trivial if $\gamma = \beta$ and $\delta = \alpha$. C-invariance follows by CPT-invariance.

(b): For $d_{\mathcal{K}} = 1$, S has a single component and is automatically diagonal. Thus (2.14) reduces to (2.15) and (S7) becomes trivial.

(c): For constant $S = S(\zeta) = S(0)$, it holds that $S = S^* = S^{-1}$ due to conditions (S1) and (S2) reduce to . $J^{\otimes 2}\mathbb{F}S\mathbb{F}J^{\otimes 2} = S$ is equivalent to $[S, J^{\otimes 2}\mathbb{F}] = 0$. □

It will be relevant for later (Secs. 4.1,7.3) that generic S-functions can be decomposed into eigenvalues. For real arguments, $S(\theta) \in \mathcal{B}(\mathcal{K}^{\otimes 2})$ is unitary and hence diagonalizable; this extends to complex arguments by analyticity:

Proposition 2.3.4. *Let S be an S-function and $D(S)$ its domain of analyticity. Then there exists $k \in \mathbb{N}$ and a discrete set $\Delta(S) \subset D(S)$ such that the number of distinct eigenvalues of $S(\zeta)$ is k for all $\zeta \in D(S) \setminus \Delta(S)$ and strictly less than k for all $\zeta \in \Delta(S)$. Further, for any simply connected domain $D_0 \subset D(S) \setminus \Delta(S)$ there exist analytic functions $s_i : D_0 \to \mathbb{C}$, and analytic projection-valued functions $P_i : D_0 \to \mathcal{B}(\mathcal{K}^{\otimes 2})$, $i = 1, ..., k$ such that for all $\zeta \in D_0$*

$$S(\zeta) = \sum_{i=1}^{k} s_i(\zeta) P_i(\zeta) \tag{2.18}$$

and:

(a) $s_1(\zeta), ..., s_k(\zeta)$ coincide with the eigenvalues of $S(\zeta)$ and $P_1(\zeta), ..., P_k(\zeta)$ coincide with the projectors onto the respective eigenspaces.

In particular, $P_i(\zeta)P_j(\zeta) = \delta_{ij}P_i(\zeta)$ *for* $i, j = 1, ..., k$*.*

(b) *If* $-\zeta \in D_0$ *one has* $s_i(-\zeta) = s_i(\zeta)^{-1}$ *and* $P_i(-\zeta) = P_i(\zeta)$*.*

(c) *If* $\bar{\zeta} \in D_0$ *one has* $\overline{s_i(\bar{\zeta})} = s_i(\zeta)^{-1}$ *and* $P_i(\bar{\zeta}) = P_i(\zeta)^*$*.*

(d) *Each* P_i *satisfies CPT-invariance,* $P_i(\zeta) = J^{\otimes 2}\mathbb{F}P_i(\zeta)^*\mathbb{F}J^{\otimes 2}$*, translational invariance,* $(E_m \otimes E_{m'})P_i(\zeta) = P_i(\zeta)(E_{m'} \otimes E_m)$ *for all* $m, m' \in \mathfrak{M}$*, and* $\mathcal{G}$*-invariance,* $[P_i(\zeta), V(g)^{\otimes 2}] = 0, g \in \mathcal{G}$*.*

The decomposition is unique up to relabeling.

Proof. For the eigenvalue decomposition of a matrix-valued analytic function, see [Par78, Theorem 4.8] or [Kat95, Chapter 2]. Restricting S to its domain of analyticity $D(S)$ we can apply the theorem from the first-named reference: For some $k \in \mathbb{N}$ there exists a discrete set $\varDelta(S) \subset D(S)$ such that the number of discrete eigenvalues of $S(\zeta)$ is k for all $\zeta \in D(S) \setminus \varDelta(S)$ and strictly less than k for all $\zeta \in \varDelta(S)$. Further, for any simply connected domain $D_0 \subset D(S) \setminus \varDelta(S)$ we obtain analytic functions $s_i : D_0 \to \mathbb{C}$ and $P_i, N_i : D_0 \to \mathcal{B}(\mathcal{K}^{\otimes 2})$ for $i = 1, ..., k$ such that for each $\zeta \in D_0$

$$S(\zeta) = \sum_{i=1}^{k}(s_i(\zeta)P_i(\zeta) + N_i(\zeta))$$

is the unique Jordan decomposition of $S(\zeta)$ with distinct eigenvalues $s_i(\zeta)$, eigenprojectors $P_i(\zeta)$ and eigennilpotents $N_i(\zeta)$, $i = 1, ..., k$. Let us enlarge D_0 to $\tilde{D}_0$ within $D(S) \setminus \varDelta(S)$ such that $\tilde{D}_0 \cap \mathbb{R} \subset \mathbb{R}$ is open and non-empty and such that $\tilde{D}_0$ is still simply connected; this is always possible since $\mathbb{C} \setminus D(S)$ and $\varDelta(S)$ are discrete, i.e., countable and without finite accumulation points. Since $S(\theta)$ for $\theta \in \mathbb{R}$ is unitary and therefore semisimple we find that $N_i \restriction \tilde{D}_0 \cap \mathbb{R} = 0$. Since N_i is analytic, this implies $N_i = 0$. From the properties of the Jordan decomposition we further infer that $P_i(\zeta)P_j(\zeta) = \delta_{ij}P_i(\zeta)$, $i, j = 1, ..., k$. This concludes the proof of property (a). The properties (b)–(d) are implied by the corresponding properties of S, namely (S1)-(S3), (S6), (S7), in a straightforward manner:

Inverting the eigendecomposition of S (using orthonormality of the projectors by Item (a)) one obtains $S(\zeta)^{-1} = \sum_{i=1}^{k} s_i(\zeta)^{-1}P_i(\zeta)$. By (S2) one has $S(-\zeta) = S(\zeta)^{-1}$ which for $-\zeta \in D_0$ (using again orthonormality of the projectors) yields $s_i(-\zeta) = s_i(\zeta)^{-1}$ and $P_i(-\zeta) = P_i(\zeta)$, proving Item (b). Item (c) follows analogously using $S(\bar{\zeta})^* = S(\zeta)^{-1}$ by (S1) and (S2). Item (d) is inferred by the properties (S3), (S6), and (S7) also analogously but even simpler since the properties modify just the projectors P_i. □

Note that the s_i (within any domain D_0 from above) satisfy all the properties of a scalar S-function except for crossing symmetry (S5). Specifically, these are the properties (S1) and (S2), since (S3), (S4), (S6), and (S7) are trivially satisfied in the scalar setting.

Remark 2.3.5. In typical examples, the decomposition in (2.18) can be extended to all of $\mathbb{C}$ if one allows for meromorphic s_i and P_i. This applies in particular to models with constant eigenprojectors which includes all models with constant or diagonal S-functions, the other examples treated in Chapter 8, and other typical models such as the sine-Gordon or Gross-Neveu model.

2.4 Full state space

From the preceding data—one-particle little space $(\mathcal{K}, V, J, M)$ and S-function S—the full interacting state space can be constructed. The construction is a generalization of the second quantization of a one-particle state space for a free field theory. In this generalization, the symmetrized (or anti-symmetrized) Fock space and the creators and annihilators are replaced by S-symmetric variants. Note here that the presence of a Fock-like structure for the interacting state space goes in line with the desired property that the interaction processes conserve the number of particles. Historically, the S-symmetric creators and annihilators were found first in [ZZ79; Fad80] and named ZF operators thereafter. The full construction of S-symmetrized second quantization was then given in [LM95]. We give a brief overview of the construction following also [LS14; Lec15]. We start by introducing the interacting state space.

Interacting state space Given a one particle space $(\mathcal{H}_1, U_1, V_1)$ with $\mathcal{H}_1 = L^2(\mathbb{R}, \mathcal{K})$ (Sec. 2.2), let $\hat{\mathcal{H}} := \oplus_{n=0}^{\infty} \mathcal{H}_1^{\otimes n}$ denote the unsymmetrized Fock space over $\mathcal{H}_1$. For each $n \in \mathbb{N}$ a function $\Psi_n \in \mathcal{H}_1^{\otimes n}$ is referred to as *S-symmetric* iff it satisfies for all $\boldsymbol{\theta} \in \mathbb{R}^n$ and $k \in \{1, .., n-1\}$,

$$S(\theta_{k+1} - \theta_k)_{k,k+1} \Psi_n(\theta_1, ..., \theta_{k+1}, \theta_k, ..., \theta_n) = \Psi_n(\theta_1, ..., \theta_k, \theta_{k+1}, ..., \theta_n). \tag{2.19}$$

Here, the subscript $k, k+1$ indicates that $S(\theta)$ acts on the tensor components k and $k+1$. Then the *S-symmetrized Fock space* [LM95; LS14] is given by

$$\mathcal{H}_S = \oplus_{n=0}^{\infty} \mathcal{H}_{S,n}, \quad \mathcal{H}_{S,n} = \{\Psi \in \mathcal{H}_1^{\otimes n} : \Psi \text{ is } S\text{-symmetric}\} \tag{2.20}$$

with $\mathcal{H}_{S,1} = \mathcal{H}_1$ and $\mathcal{H}_{S,0} = \mathbb{C}$. $\mathcal{H}_{S,n}$ is naturally a closed subspace of $\mathcal{H}_1^{\otimes n}$ and $\mathcal{H}_S$ of $\hat{\mathcal{H}}$, so let $\mathcal{P}_{S,n} : \mathcal{H}_1^{\otimes n} \to \mathcal{H}_{S,n}$ and $\mathcal{P}_S : \hat{\mathcal{H}} \to \mathcal{H}_S$ denote the corresponding orthogonal projections. For a state $\Psi \in \mathcal{H}_S$ the component in $\mathcal{H}_{S,n}$ will be denoted by Ψ_n and referred to as an *n-particle state*. The particle number operator N is given by $(N\Psi)_n := n\Psi_n$, accordingly. The Fock vacuum is given by $\Omega \in \mathcal{H}_S$ with

$(\Omega)_n = \delta_{n0}$ for $n \in \mathbb{N}_0$. Of technical importance is the subspace of *finite particle states*

$$\mathcal{H}_S^{\mathrm{f}} = \left\{ \Psi \in \mathcal{H}_S : \Psi_n = 0 \textit{ for large enough } n \right\} = \cup_{n\in\mathbb{N}_0} \mathcal{H}_{S,n} \tag{2.21}$$

which defines a closed dense subspace of $\mathcal{H}_S$.

Operators acting on $\mathcal{H}_1$, in particular the symmetry representations, extend to $\mathcal{H}_S$ by an S-symmetrized variant of standard second quantization. We quote [Lec15] with slight adaptations and minor additions:

Proposition 2.4.1. *Take arbitrary $(x, \lambda) \in \mathcal{P}_+^\uparrow$, $g \in \mathcal{G}$, $\Psi \in \mathcal{H}_S$, and let $\leftarrow$ denote the reversion*[a] *of the tensor components of $\mathcal{H}_1^{\otimes n}$. Then*

$$U_S(x, \lambda) = \mathcal{P}_S \oplus_{n=0}^\infty U_1(x, \lambda)^{\otimes n} \, \mathcal{P}_S, \tag{2.22}$$

$$V_S(g) = \mathcal{P}_S \oplus_{n=0}^\infty V_1(g)^{\otimes n} \, \mathcal{P}_S, \tag{2.23}$$

$$(U(j)\Psi)_n = U_1(j)^{\otimes n} \, \overleftarrow{\Psi}_n, \tag{2.24}$$

defines a strongly continuous, unitary, positive-energy representation of $\mathcal{P}_+ \times \mathcal{G}$ on $\mathcal{H}_S$ with a (unique) invariant vector Ω. Some particular consequences are that $U(j)U_S(x, \lambda) = U_S(-x, \lambda)U(j)$, that $U(j) = U(j)^{-1}$, and that $V_S(g)$ commutes with $U_S(x, \lambda)$ and $U(j)$.

[a] I.e., we reverse the order of arguments and the order of the $\mathcal{K}$-tensor components.

The generators of translations and boosts—we will refer to them as the *total energy-momentum operator* P^μ and the *boost generator* K—are given by S-twisted second quantization of $p(\cdot; M)$ and $-i\frac{d}{d\lambda}$, accordingly. For instance, P^μ acts simply as

$$(P^\mu \Psi)_n(\boldsymbol{\theta}) = P^\mu(\boldsymbol{\theta})\Psi_n(\boldsymbol{\theta}), \quad P^\mu(\boldsymbol{\theta}) = \sum_{j=1}^n p(\theta_j, M_j), \tag{2.25}$$

for $\Psi \in \mathcal{H}_S$ and where M_j is the (one-particle) mass operator M acting on the j-th tensor component of $\mathcal{K}^{\otimes n}$. This concludes the construction of the interacting state space.

ZF operators The S-twisted creators $z_S^\dagger$ and annihilators z_S (or ZF operators) operate on the interacting state space we have just constructed. We will often use $z_S^\sharp$ to represent both, z_S and $z_S^\dagger$, and define them as operator-valued distributions $\varphi \mapsto z_S^\sharp(\varphi)$ with domain $\mathcal{H}_1 = L^2(\mathbb{R}, \mathcal{K})$ via

$$(z_S^\dagger(\varphi)\Psi)_n := \sqrt{n}\,\mathcal{P}_S(\varphi \otimes \Psi_{n-1}), \quad z_S(\varphi) := (z_S^\dagger(\varphi))^*, \quad \varphi \in \mathcal{H}_1. \tag{2.26}$$

In particular, $z_S^\dagger(\varphi)\Omega = \varphi$ and $z_S(\varphi)\Omega = 0$.

Also, products of z_S and $z_S^\dagger$ can be linearly extended in tensor powers of $\mathcal{H}_1$, i.e., for $m, n \in \mathbb{N}_0$, $i \in \{1, .., m\}$, $j \in \{1, .., n\}$, and $\varphi_i, \chi_j \in \mathcal{H}_1$, we have

$$z_S^{\dagger m} z_S^n(\varphi_1 \otimes ... \otimes \varphi_m; \chi_1 \otimes ... \otimes \chi_n) = z_S^\dagger(\varphi_1)...z_S^\dagger(\varphi_m) z_S(\chi_1)...z_S(\chi_n). \tag{2.27}$$

Let us summarize a few properties of z_S and $z_S^\dagger$, quoting again [Lec15] with slight adaptations:

Proposition 2.4.2. *Let $\varphi, \chi \in \mathcal{H}_1$ and $\Psi \in \mathcal{H}_S^{\mathrm{f}}$ be arbitrary.*

(a) $z_S^\sharp(\varphi)$ is in general unbounded, but well-defined on $\mathcal{H}_S^{\mathrm{f}}$, and $z_S(\varphi)^* \supset z_S^\dagger(\varphi)$.

(b) For $(x,\lambda) \in \mathcal{P}_+^\uparrow$ and $g \in \mathcal{G}$, we have

$$\begin{aligned} U_S(x,\lambda) z_S^\sharp(\varphi) U_S(x,\lambda)^{-1} &= z_S^\sharp(U_1(x,\lambda)\varphi), \\ V_S(g) z_S^\sharp(\varphi) V_S(g)^{-1} &= z_S^\sharp(V_1(g)\varphi), \\ U(j) z_S^\sharp(\varphi) U(j) z_S^\dagger(\Psi)\Omega &= z_S^\dagger(\Psi) z_S^\sharp(U_1(j)\varphi)\Omega. \end{aligned} \tag{2.28}$$

(c) Relative to the particle number operator N, one has bounds

$$\|z_S(\varphi)\Psi\| \le \|\varphi\| \|\sqrt{N}\Psi\|, \quad \|z_S^\dagger(\varphi)\Psi\| \le \|\varphi\| \|\sqrt{N-1}\Psi\|. \tag{2.29}$$

(d) z_S, $z_S^\dagger$ form a representation of the ZF algebra with S-function S: They satisfy

$$\begin{aligned} z_S^\dagger z_S^\dagger((1 - S_\leftarrow)(\varphi \otimes \chi)) &= 0, \\ z_S z_S((1 - U_1(j)^{\otimes 2} S_\leftarrow U_1(j)^{\otimes 2})(\varphi \otimes \chi)) &= 0, \\ z_S z_S^\dagger(\varphi \otimes \chi) - z_S^\dagger z_S((1 \otimes U_1(j)) S_\leftarrow^{i\pi}(U_1(j)\varphi \otimes \chi)) &= \langle \varphi, \chi \rangle \, \mathbb{1}, \end{aligned} \tag{2.30}$$

where $S^{i\pi} := S(i\pi + \cdot)$ and $S_\leftarrow f(\zeta_1,\zeta_2) := S(\zeta_2 - \zeta_1) f(\zeta_2,\zeta_1)$ for a $\mathcal{K}^{\otimes 2}$-valued function in two arguments.

In a basis, the ZF algebra relations (2.30) become

$$\begin{aligned} &z_{S,\alpha}^\dagger(\theta) z_{S,\beta}^\dagger(\eta) - S_{\alpha\beta}^{\gamma\delta}(\theta - \eta) z_{S,\gamma}^\dagger(\eta) z_{S,\delta}^\dagger(\theta) = 0 \\ &z_{S,\alpha}(\theta) z_{S,\beta}(\eta) - S_{\delta\gamma}^{\beta\alpha}(\theta - \eta) z_{S,\gamma}(\eta) z_{S,\delta}(\theta) = 0 \\ &z_{S,\alpha}(\theta) z_{S,\beta}^\dagger(\eta) - S_{\beta\delta}^{\alpha\gamma}(\eta - \theta) z_{S,\gamma}^\dagger(\eta) z_{S,\delta}(\theta) = \delta_{\alpha\beta}\delta(\theta - \eta). \end{aligned} \tag{2.31}$$

This is shown in Appendix A.3.

Remark 2.4.3. (Free creation- and annihilation operators) For $S_{\alpha\beta}^{\gamma\delta} = \pm\delta_\alpha^\delta \delta_\beta^\gamma$ or equivalently $S = \pm\mathbb{F}$, these relations reduce to the canonical commutation, resp., anticommutation relations. In this case, the ZF operators $z_{S,\alpha}^\sharp$ are just the ordinary creators and annihilators $a_{\pm,\alpha}^\sharp$ of a free model with $d_\mathcal{K}$ real bosons (+) or fermions (-). The construction of the state space reduces to the standard one, a fully symmetrized/antisymmetrized Fock space $\mathcal{H}_\pm := \mathcal{H}_{S=\pm\mathbb{F}}$ over $\mathcal{H}_1$.

The ZF operators fulfill a similar role as the free annihilators and creators: They solve the 'one-particle problem' and generate the space of states and operators. The first property means that for each state $\varphi \in \mathcal{H}_1$ there exists an operator A such that $\varphi = A\Omega$; for example $A = z_S^\dagger(\varphi)$. The second property means that expressions of the form $A_1 ... A_n \Omega$ with $A_j = z_S^\dagger(\varphi_j)$, $\varphi_j \in \mathcal{S}(\mathbb{R}, \mathcal{K})$, and $n \in \mathbb{N}$ form a total subset[5] of $\mathcal{H}_S$. This follows from the identity

$$\mathcal{P}_S(\varphi_1 \otimes ... \otimes \varphi_n) = \tfrac{1}{\sqrt{n!}} z_S^\dagger(\varphi_1) ... z_S^\dagger(\varphi_n)\Omega \tag{2.32}$$

which is straightforward to check from the definition of $z_S^\dagger$ in (2.26); confer the proof in Appendix A.6. Due to these two properties, $z_S^\dagger$ will play an important role in the construction and analysis of local operators (Chap. 3).

Particle statistics We also introduce a *statistics matrix* $\sigma \in \mathcal{B}(\mathcal{K}^{\otimes 2})$ with $\sigma_{\alpha\beta}^{\gamma\delta} = \sigma_{\alpha\beta}\delta_\alpha^\delta \delta_\beta^\gamma$ in order to treat particles satisfying different asymptotic exchange relations or "statistics". In our setup, which includes bosons and fermions, we have coefficients $\sigma_{\alpha\beta} \in \{\pm 1\}$ which satisfy $\sigma_{\alpha\beta} = -1$ if both, α and β, correspond to fermionic states and $\sigma_{\alpha\beta} = +1$ in all other cases, i.e., where at least α or β corresponds to a bosonic state. This implies also that $\sigma_{\alpha\beta} = \sigma_{\alpha\beta}^{-1} = \overline{\sigma_{\alpha\beta}}$ and $\sigma_{\alpha\beta} = \sigma_{\bar{\alpha}\beta} = \sigma_{\alpha\bar{\beta}}$ which makes σ a constant diagonal S-function[6]. As a result, we may apply the above construction and obtain the standard Fock space $\mathcal{H}_\sigma$ over $\mathcal{H}_1$ with a Bose/Fermi-grading which is symmetrized/antisymmetrized depending on σ. The corresponding creators and annihilators will be denoted by $a_{\sigma,\alpha}^\sharp$. Note here that for arbitrary statistics matrix σ and S-function S also $\zeta \mapsto \sigma S(\zeta)$ defines a valid S-function.

S-symmetry For explicit computations, we will need more specific information on S-symmetry and the related projection $\mathcal{P}_S$ which was introduced rather abstractly. To begin with, we quote a result from the literature:

Proposition 2.4.4 ([LM95; LS14; AL17]). *Let the elementary transpositions be denoted by $\pi_k \in \mathfrak{S}_n$, $k = 1, ..., n-1$, where π_k is the permutation that exchanges k and $k+1$. Then the map $\pi_k \mapsto D_n^{\pi_k}$,*

$$(D_n^{\pi_k}\psi)(\boldsymbol{\theta}) = S(\theta_{k+1} - \theta_k)_{k,k+1}\psi(\theta_1, ..., \theta_{k+1}, \theta_k, ..., \theta_n), \tag{2.33}$$

generates a unitary representation of the permutation group $\mathfrak{S}_n$ on $\mathcal{H}_1^{\otimes n}$. Moreover,

[5] A subset of a linear space is termed total if its linear span is dense in the ambient space.

[6] As a special feature of 1+1d (roughly speaking due to the lack of rotations) anyons can appear. Those are particles with exotic "statistics"; in this setup their statistics (matrix) can be an arbitrary constant S-function rather than just having $\sigma_{\alpha\beta} \in \{\pm 1\}$ (see, e.g., [Smi90]). However, they will not be considered in this thesis.

denoting this representation by $\mathfrak{S}_n \ni \tau \mapsto D_n^\tau$, *one has*

$$\mathcal{P}_{S,n} = \frac{1}{n!} \sum_{\tau \in \mathfrak{S}_n} D_n^\tau. \tag{2.34}$$

Remark 2.4.5. Note that $\mathcal{P}_S$ commutes with $U(j)$, $U_S(x,\lambda)$, and $V_S(g)$. To show this, by construction of $\mathcal{P}_S$, it suffices to show that $S_\leftarrow := D_2^{\pi_1}$ commutes with the respective operators at the two-particle level. Commutativity with these operators at the two-particle level is implemented by CPT-invariance (S3), translational invariance (S6), and $\mathcal{G}$-invariance (S7), respectively (Lemma A.6.4, appendix). In case that S is addtionally k-invariant for $k \in \{c, p, t, cp, ct, pt\}$ one also has that $\mathcal{P}_S$ commutes with $U(k)$.

For later, it will be useful to give D_n^τ an explicit form:

Corollary 2.4.6. *For each* $n \in \mathbb{N}$, $\boldsymbol{\theta} \in \mathbb{R}^n$, *and* $\tau \in \mathfrak{S}_n$ *there exists a unitary operator* $S^\tau(\boldsymbol{\theta})$ *on* $\mathcal{K}^{\otimes n}$ *such that*

$$(D_n^\tau \psi)(\boldsymbol{\theta}) = S^\tau(\boldsymbol{\theta})\psi(\boldsymbol{\theta}^\tau), \quad \boldsymbol{\theta}^\tau := (\theta_{\tau(1)}, ..., \theta_{\tau(n)}),\ \psi \in \mathcal{H}_1^{\otimes n}, \tag{2.35}$$

and

(a) $S^{\mathrm{id}}(\boldsymbol{\theta}) = \mathbb{1}_{\otimes n}$, *where* id *denotes the trivial permutation,*

(b) $S^{\tau\circ\rho}(\boldsymbol{\theta}) = S^\tau(\boldsymbol{\theta})S^\rho(\boldsymbol{\theta}^\tau)$, *for arbitrary* $\rho \in \mathfrak{S}_n$,

(c) $(S^\tau(\boldsymbol{\theta}))^{-1} = S^{\tau^{-1}}(\boldsymbol{\theta}^\tau)$.

Proof. The existence of $S^\tau(\boldsymbol{\theta})$ is by construction of D_n^τ; decompose τ into elementary transpositions, apply the representation property $D_n^{\tau_1\circ\tau_2} = D_n^{\tau_1}D_n^{\tau_2}, \tau_{1/2} \in \mathfrak{S}_n$, and use that by definition of D_n^τ (2.33), $S^{\pi_k}(\boldsymbol{\theta}) = S(\theta_{k+1} - \theta_k)_{k,k+1}$. Property (a) is immediate using $D_n^{\mathrm{id}} = \mathbb{1}_{\mathcal{H}_1^{\otimes n}}$. Using again the representation property for $\tau_1 = \tau$ and $\tau_2 = \rho$, (b) follows: For arbitrary $\psi \in \mathcal{H}_1^{\otimes n}$

$$\begin{aligned} S^{\tau\circ\rho}(\boldsymbol{\theta})\psi(\boldsymbol{\theta}^{\tau\circ\rho}) &= D_n^{\tau\circ\rho}\psi(\boldsymbol{\theta}) \\ &= D_n^\tau D_n^\rho \psi(\boldsymbol{\theta}) \\ &= S^\tau(\boldsymbol{\theta})D_n^\rho\psi(\boldsymbol{\theta}^\tau) \\ &= S^\tau(\boldsymbol{\theta})S^\rho(\boldsymbol{\theta}^\tau)\psi((\boldsymbol{\theta}^\tau)^\rho) \\ &= S^\tau(\boldsymbol{\theta})S^\rho(\boldsymbol{\theta}^\tau)\psi(\boldsymbol{\theta}^{\tau\circ\rho}). \end{aligned} \tag{2.36}$$

Property (c) is a consequence of (a) and (b) taking $\rho = \tau^{-1}$. □

Necessary and sufficient for the previous two results is that the properties (S1), (S2), and (S4) hold.

Corollary 2.4.7. *Let $1 \leq i < j \leq n$, $n \in \mathbb{N}$. Let further $\pi_{i,j} \in \mathfrak{S}_n$ denote the shift permutation $i \to j$, i.e., $\pi_{i,j}(i) = j$ and $\pi_{i,j}(k) = k-1$ for $i < k \leq j$ and $\pi_{i,j}(k) = k$ for $k > j$ and $k < i$. Then*

$$S^{\pi_{i,j}}(\boldsymbol{\theta}) = \prod_{k=i}^{j-1} S(\theta_j - \theta_k)_{k,k+1} := S(\theta_j - \theta_{j-1})_{j-1,j} ... S(\theta_j - \theta_{i+1})_{i+1,i+2} S(\theta_j - \theta_i)_{i,i+1}. \tag{2.37}$$

Note that the ordering in (2.37) is relevant, however, other orderings are possible due to the Yang-Baxter relation (S4).

Proof. For $n = 2$, $\pi_{1,2} = \pi_1$ is the elementary transposition and by definition (2.33), $S^{\pi_1}(\boldsymbol{\theta}) = S(\theta_2 - \theta_1)_{1,2}$. Proceeding by induction we assume the validity of (2.37) for n and prove it for $n+1$: Here the cases $|i-j| \leq n-1$ for $1 \leq i < j \leq n+1$ are already covered, since $\pi_{i,j}$ can be treated as an element of $\mathfrak{S}_n$ which is a subgroup of $\mathfrak{S}_{n+1}$. Thus, it remains to show the hypothesis for $\pi_{1,n+1}$.

First, note that $\pi_{1,n+1} = \pi_{2,n+1} \circ \pi_{1,2}$ and $\pi_{1,2} = \pi_1$. Then, using Corollary 2.4.6(b) twice, we find for $\boldsymbol{\theta} \in \mathbb{R}^{n+1}$ that

$$S^{\pi_{1,n+1}}(\boldsymbol{\theta}) = S^{\pi_{2,n+1}}(\boldsymbol{\theta}) S^{\pi_1}(\boldsymbol{\theta}^{\pi_{2,n+1}}). \tag{2.38}$$

By induction hypothesis,

$$S^{\pi_{2,n+1}}(\boldsymbol{\theta}) = \prod_{k=2}^{n+1} S(\theta_{n+1} - \theta_k)_{k,k+1}, \tag{2.39}$$

and by Definition (2.33)

$$S^{\pi_1}(\boldsymbol{\theta}^{\pi_{2,n+1}}) = S(\theta_{\pi_{2,n+1}(2)} - \theta_1)_{1,2} = S(\theta_{n+1} - \theta_1)_{1,2}. \tag{2.40}$$

As a result this yields

$$\begin{aligned} S^{\pi_{1,n+1}}(\boldsymbol{\theta}) &= \left(\prod_{k=2}^{n+1} S(\theta_{n+1} - \theta_k)_{k,k+1} \right) S(\theta_{n+1} - \theta_1)_{1,2} \\ &= \prod_{k=1}^{n+1} S(\theta_{n+1} - \theta_k)_{k,k+1}. \end{aligned} \tag{2.41}$$

□

Improper rapidity eigenstates The class of improper rapidity eigenstates generates $\mathcal{H}_S$ and provides a convenient "basis" for explicit computations. The improper rapidity eigenstates are defined as follows.

$$\begin{aligned} |\boldsymbol{\theta}_{\boldsymbol{\alpha}}\rangle_S &:= \tfrac{1}{\sqrt{n!}} z^\dagger_{S,\alpha_1}(\theta_1) ... z^\dagger_{S,\alpha_n}(\theta_n)\Omega, \\ \langle \boldsymbol{\theta}_{\boldsymbol{\alpha}}|_S &:= \tfrac{1}{\sqrt{n!}} \langle \Omega | \, z_{S,\alpha_n}(\theta_n) ... z_{S,\alpha_1}(\theta_1), \end{aligned} \quad n \in \mathbb{N}, \boldsymbol{\theta} \in \mathbb{R}^n, \boldsymbol{\alpha} \in \{1, ..., d_{\mathcal{K}}\}^n, \tag{2.42}$$

is to be read as a formal notation for vector-valued distributions: Having $\varphi_j \in \mathcal{H}_1$, $j = 1, ..., n$, those are given by

$$\varphi_1 \otimes ... \otimes \varphi_n \mapsto \begin{array}{ccc} \frac{1}{\sqrt{n!}} z_S^\dagger(\varphi_1)...z_S^\dagger(\varphi_n)\Omega & = & \int d\boldsymbol{\theta}\ |\boldsymbol{\theta}_{\boldsymbol{\alpha}}\rangle_S\, \varphi_1^{\alpha_1}(\theta_1)...\varphi_n^{\alpha_n}(\theta_n) \\ \frac{1}{\sqrt{n!}} \langle\Omega|\, z_S(\varphi_n)...z_S(\varphi_1) & = & \int d\boldsymbol{\theta}\ \langle\boldsymbol{\theta}_{\boldsymbol{\alpha}}|_S\, \varphi_{1,\alpha_1}(\theta_1)...\varphi_{n,\alpha_n}(\theta_n) \end{array}. \tag{2.43}$$

These states satisfy the following properties:

Proposition 2.4.8. *For arbitrary $\boldsymbol{\theta}, \boldsymbol{\eta} \in \mathbb{R}^n$ the expressions defined in (2.42) satisfy*

(a) S-symmetry: For any $\tau \in \mathfrak{S}_n$ we have

$$|\boldsymbol{\theta}_{\boldsymbol{\alpha}}\rangle_S = |\boldsymbol{\theta}^\tau_{\boldsymbol{\beta}}\rangle_S\, ((S^\tau(\boldsymbol{\theta}))^{-1})^{\boldsymbol{\beta}}_{\boldsymbol{\alpha}}, \quad \langle\boldsymbol{\theta}_{\boldsymbol{\alpha}}|_S = (S^\tau(\boldsymbol{\theta}))^{\boldsymbol{\alpha}}_{\boldsymbol{\beta}}\, \langle\boldsymbol{\theta}^\tau_{\boldsymbol{\beta}}|_S\,. \tag{2.44}$$

(b) orthonormality (up to ordering):

$$\langle\boldsymbol{\theta}_{\boldsymbol{\alpha}}|\boldsymbol{\eta}_{\boldsymbol{\beta}}\rangle_S = \frac{1}{n!}\sum_{\tau\in\mathfrak{S}_n} S^\tau(\boldsymbol{\theta})^{\boldsymbol{\alpha}}_{\boldsymbol{\beta}}\delta(\boldsymbol{\theta}^\tau - \boldsymbol{\eta}) = \frac{1}{n!}\sum_{\tau\in\mathfrak{S}_n}((S^\tau(\boldsymbol{\eta}))^{-1})^{\boldsymbol{\alpha}}_{\boldsymbol{\beta}}\delta(\boldsymbol{\theta} - \boldsymbol{\eta}^\tau) \tag{2.45}$$

and orthogonality for an unequal number of arguments.

(c) completeness: On the unsymmetrized Fock space $\hat{\mathcal{H}}$ one has

$$\mathcal{P}_S = \sum_{n\in\mathbb{N}} \int d^n\boldsymbol{\lambda}\ |\boldsymbol{\lambda}_{\boldsymbol{\alpha}}\rangle_S\, \langle\boldsymbol{\lambda}_{\boldsymbol{\alpha}}|_S = \sum_{n\in\mathbb{N}} n! \int\limits_{\lambda_{\tau(1)}>...>\lambda_{\tau(n)}} d^n\boldsymbol{\lambda}\ |\boldsymbol{\lambda}_{\boldsymbol{\alpha}}\rangle_S\, \langle\boldsymbol{\lambda}_{\boldsymbol{\alpha}}|_S\,, \tag{2.46}$$

where the last equality is for some fixed $\tau \in \mathfrak{S}_n$.

(d) Poincaré-covariance:

$$U_S(a,\lambda)\,|\boldsymbol{\theta}_{\boldsymbol{\alpha}}\rangle_S = e^{ip_{\boldsymbol{\alpha}}(\boldsymbol{\theta}).a}\,|(\boldsymbol{\theta} + \lambda\mathbf{1})_{\boldsymbol{\alpha}}\rangle_S\,, \quad (a,\lambda) \in \mathcal{P}_+^\uparrow, \tag{2.47}$$

where $p_{\boldsymbol{\alpha}}(\boldsymbol{\theta}) = \sum_{i=1}^n p(\theta_i; m_{\alpha_i})$ and $\mathbf{1} = (1, ..., 1) \in \mathbb{R}^n$. In particular,

$$P^\mu\,|\boldsymbol{\theta}_{\boldsymbol{\alpha}}\rangle_S = p^\mu_{\boldsymbol{\alpha}}(\boldsymbol{\theta})\,|\boldsymbol{\theta}_{\boldsymbol{\alpha}}\rangle_S\,. \tag{2.48}$$

(e) CPT-covariance: For $\psi \in \mathcal{H}_{S,n}$,

$$U(j)\,|\boldsymbol{\theta}_{\boldsymbol{\alpha}}\rangle_S = J^{\boldsymbol{\beta}}_{\boldsymbol{\alpha}}\,|\overleftarrow{\boldsymbol{\theta}}_{\overleftarrow{\boldsymbol{\beta}}}\rangle^{cc}_S = |\overleftarrow{\boldsymbol{\theta}}_{\overleftarrow{\overline{\boldsymbol{\alpha}}}}\rangle^{cc}_S\,, \tag{2.49}$$

where the "cc" superscript denotes the antilinear distribution

$$\psi \mapsto \int d\boldsymbol{\theta}\,|\boldsymbol{\theta}_{\boldsymbol{\alpha}}\rangle_S\,\overline{\psi^{\boldsymbol{\alpha}}(\boldsymbol{\theta})}, \quad \psi \in \mathcal{H}_1^{\otimes n}.$$

The proof is given in Appendix A.4.

2.5 Asymptotic completeness; closing the circle

It is time to motivate the construction given so far and to check its consistency. The starting points of the construction were a particle spectrum and a two-particle interaction (the S-function) expected to describe an integrable model with a factorizing S-matrix. Now, given the constructed state space, it is possible to derive a scattering theory from it which should show the anticipated features connected to integrability and the input data. Concretely, we expect that the resulting scattering theory is asymptotically complete (all states of the model are describable as scattering states) and the resulting collision operator—the S-matrix—is particle-number conserving and factorizes into a product of two-particle scattering processes with a scattering function closely related to the S-function. This was proven rigorously before in specific sub cases: the scalar bosonic case [Lec06; Lec07], the tensor bosonic case [LS14], and the scalar fermionic case [BC21]. Therefore, here, we skip the rigorous derivation of the scattering states and instead start with the explicit form for the Moeller operators directly and show that they indeed give rise to an S-matrix which has the properties mentioned before. Also, we will take the opportunity to choose a slightly different presentation than in the references above, working with improper rapidity eigenstates (as introduced in (2.42)) in close connection to the physics literature of the form factor community.

Given a physical Hilbert space $\mathcal{H}_{\text{phys}}$ together with a unitary representation of the Poincaré group, a scattering theory consists of the identification of states in the physical Hilbert space with asymptotic states, i.e., incoming and outgoing particle configurations. The underlying idea is that the asymptotic states are separated by large distances and thus—for sufficiently fast decaying interactions—can be treated as isolated from each other, which makes the particle picture well-defined. The identifications correspond to two isometric embeddings $W_{\text{in/out}} : \mathcal{H}_{\text{in/out}} \to \mathcal{H}_{\text{phys}}$ referred to as Moeller operators. If the model is entirely captured by its scattering theory, we suppose that all these spaces are isomorphic: $\mathcal{H}_{\text{in}} \cong \mathcal{H}_{\text{phys}} \cong \mathcal{H}_{\text{out}}$ which is also known as *asymptotic completeness*.

Definition 2.5.1. The map $\hat{\mathbf{S}} : \mathcal{H}_{\text{in}} \to \mathcal{H}_{\text{out}}$ given by $\hat{\mathbf{S}} = W_{\text{out}}^* W_{\text{in}}$ is called *S-matrix.*[a]

[a]Note that there are many different conventions for the choice of mapping directions of the Moeller operators as well as the definition of the S-matrix. E.g, the combination $W_{\text{in}} W_{\text{out}}^*$ might be referred to as S-matrix, too; in this case acting on the interacting state space $\mathcal{H}_{\text{phys}}$.

In our setup, we have $\mathcal{H}_{\text{in}} = \mathcal{H}_{\text{out}} = \mathcal{H}_\sigma$ and $\mathcal{H}_{\text{phys}} = \mathcal{H}_S$, where σ and S are the statistics matrix and the S-function from the preceding section. It is convenient that $\mathcal{H}_\sigma$ and $\mathcal{H}_S$ are subspaces of the unsymmetrized Fock space $\hat{\mathcal{H}}$. Let $\varphi_i \in \mathcal{D}(\mathbb{R}, \mathcal{K})$ for $i = 1, ..., n$ and introduce the partial ordering $\varphi_i \succ \varphi_j :\Leftrightarrow \operatorname{supp} \varphi_i > \operatorname{supp} \varphi_j$.

The Moeller operators are then defined via

$$\begin{aligned} W_{\text{in}}\, \mathcal{P}_\sigma\,(\varphi_1 \otimes \ldots \otimes \varphi_n) &:= \mathcal{P}_S\,(\varphi_1 \otimes \ldots \otimes \varphi_n)\,, \quad \varphi_1 \succ \ldots \succ \varphi_n, \\ W_{\text{out}}\, \mathcal{P}_\sigma\,(\varphi_1 \otimes \ldots \otimes \varphi_n) &:= \mathcal{P}_S\,(\varphi_1 \otimes \ldots \otimes \varphi_n)\,, \quad \varphi_1 \prec \ldots \prec \varphi_n. \end{aligned} \tag{2.50}$$

and extended to $\mathcal{H}_\sigma$ by linearity. Equivalently, on improper rapidity eigenstates,

$$W_{\text{in/out}}\,|\boldsymbol{\theta}_{\boldsymbol{\alpha}}\rangle_\sigma = |\boldsymbol{\theta}_{\boldsymbol{\alpha}}\rangle_S\,, \quad \text{for } \boldsymbol{\theta} = \boldsymbol{\theta}^{\text{in/out}} \tag{2.51}$$

where "in" and "out" denote the permutations of $\boldsymbol{\theta}$ which put it in descending, resp., ascending order, i.e., $\theta_{\text{in}(1)} > \ldots > \theta_{\text{in}(n)}$ and $\theta_{\text{out}(1)} < \ldots < \theta_{\text{out}(n)}$. Note that implicitly, "in" and "out" depend on $\boldsymbol{\theta}$.

Leaving away the rigorous verification by Haag-Ruelle scattering theory (see references given above), this definition for the Moeller operators is in accordance with intuition on scattering theory: Particles $1, \ldots, n$ threaded along a line (space is $\mathbb{R}$ in 1+1d) from left to right with in-(out-)ordered rapidities will isolate from each other upon evolution to the past (future) without any collision; allowing for identification of interacting and free states asymptotically.

The Moeller operators have the following properties:

Lemma 2.5.2. *$W_{\text{in/out}} : \mathcal{H}_\sigma \to \mathcal{H}_S$ defined by (2.50) are Hilbert-space isomorphisms, which intertwine with the representation of the Poincaré group, i.e.,*

$$W_{\text{in/out}} U_\sigma(a,\lambda) = U_S(a,\lambda) W_{\text{in/out}}, \qquad W_{\text{in/out}} U_\sigma(j) = U_S(j) W_{\text{out/in}}. \tag{2.52}$$

Proof. Use "ex" to either mean "in" or "out". First, we will show that W_{ex} is a surjective and norm-preserving linear map, thus a Hilbert-space isomorphism [Con07, Prop. I.5.2]: Equation (2.50) or, equivalently, (2.51) defines W_{ex} on a total subset of $\mathcal{H}_\sigma$ and maps onto a total subset of $\mathcal{H}_S$ (Prop. A.6.3). As a consequence, its extension by continuity and linearity is defined on all of $\mathcal{H}_\sigma$ and maps onto $\mathcal{H}_S$. Moreover, W_{ex} is norm-preserving: By Proposition 2.4.8(b) for $\boldsymbol{\theta} = \boldsymbol{\theta}^{\text{ex}}$ and $\boldsymbol{\eta} = \boldsymbol{\eta}^{\text{ex}}$,

$$\langle \boldsymbol{\theta}_{\boldsymbol{\alpha}}|_\sigma\, W_{\text{ex}}^* W_{\text{ex}}\,|\boldsymbol{\eta}_{\boldsymbol{\beta}}\rangle_\sigma = \langle \boldsymbol{\theta}_{\boldsymbol{\alpha}}|\boldsymbol{\eta}_{\boldsymbol{\beta}}\rangle_S = \delta(\boldsymbol{\theta} - \boldsymbol{\eta})\delta^{\boldsymbol{\alpha}}_{\boldsymbol{\beta}} = \langle \boldsymbol{\theta}_{\boldsymbol{\alpha}}|\boldsymbol{\eta}_{\boldsymbol{\beta}}\rangle_\sigma\,. \tag{2.53}$$

Second, we will prove Poincaré- and CPT-invariance as given in (2.52) by proving it on a total subset of $\mathcal{H}_\sigma$: Proposition 2.4.8(d) implies that for $\boldsymbol{\theta} = \boldsymbol{\theta}^{\text{ex}}$ we have that

$$\begin{aligned} W_{\text{ex}} U_\sigma(a,\lambda)\,|\boldsymbol{\theta}_{\boldsymbol{\alpha}}\rangle_\sigma &= e^{ip_{\boldsymbol{\alpha}}(\boldsymbol{\theta}).a} W_{\text{ex}}\,|(\boldsymbol{\theta} + \lambda\mathbf{1})_{\boldsymbol{\alpha}}\rangle_\sigma \\ &= e^{ip_{\boldsymbol{\alpha}}(\boldsymbol{\theta}).a}\,|(\boldsymbol{\theta} + \lambda\mathbf{1})_{\boldsymbol{\alpha}}\rangle_S = U_S(a,\lambda)\,|\boldsymbol{\theta}_{\boldsymbol{\alpha}}\rangle_S = U_S(a,\lambda) W_{\text{ex}}\,|\boldsymbol{\theta}_{\boldsymbol{\alpha}}\rangle_\sigma\,. \end{aligned} \tag{2.54}$$

Item (e) of the same proposition implies that for $\boldsymbol{\theta} = \boldsymbol{\theta}^{\text{ex}}$ and $\boldsymbol{\eta} = \boldsymbol{\eta}^{\overline{\text{ex}}}$, where $\overline{\text{in}} := \text{out}$ and $\overline{\text{out}} := \text{in}$,

$$U_S(j) W_{\text{ex}}\,|\boldsymbol{\theta}_{\boldsymbol{\alpha}}\rangle_\sigma = U_S(j)\,|\boldsymbol{\theta}_{\boldsymbol{\alpha}}\rangle_S = |\overleftarrow{\boldsymbol{\theta}}_{\overleftarrow{\overline{\boldsymbol{\alpha}}}}\rangle_S = W_{\overline{\text{ex}}}\,|\overleftarrow{\boldsymbol{\theta}}_{\overleftarrow{\overline{\boldsymbol{\alpha}}}}\rangle_\sigma = W_{\overline{\text{ex}}} U_\sigma(j)\,|\boldsymbol{\theta}_{\boldsymbol{\alpha}}\rangle_\sigma\,. \tag{2.55}$$

□

This concludes proving that the definition of the Moeller operators is compatible with asymptotic completeness. It remains to show that the S-matrix has the expected features. In this regard we establish that the S-matrix is indeed factorizing and particle number conserving and connect its elementary factors, the two-to-two-particle scattering function, to the S-function.

Proposition 2.5.3 (S-matrix). *The S-matrix $\hat{\mathbf{S}}$ is given by*

$$(\hat{\mathbf{S}}\Psi)_n(\boldsymbol{\theta}) = S^{(n)}(\boldsymbol{\theta})\Psi_n(\boldsymbol{\theta}), \quad S^{(n)}(\boldsymbol{\theta}) = \sigma^{\mathrm{out}} S^{\iota}(\boldsymbol{\theta}^{\mathrm{out}})\sigma^{\mathrm{in}^{-1}}, \quad \Psi \in \mathcal{H}_\sigma, \tag{2.56}$$

where $\iota \in \mathfrak{S}_n$ is the inversion permutation given by $\iota(k) = n+1-k$ for $k = 1,\dots,n$. In the case that the S-function S commutes with the statistics matrix σ, i.e., $[S(\zeta),\sigma] = 0$ for all $\zeta \in \mathbb{C}$, one has

$$S^{(n)}(\boldsymbol{\theta}) = \sigma^{\iota} S^{\iota}(\boldsymbol{\theta}^{\mathrm{out}}) = (\sigma S)^{\iota}(\boldsymbol{\theta}^{\mathrm{out}}). \tag{2.57}$$

Explicitly, denoting $\theta_{ij} := \theta_i - \theta_j$, we have that

$$S^{\iota}(\boldsymbol{\theta}) = \prod_{i=2}^{n}\prod_{j=1}^{i-1} S(\theta_{ij})_{n+j-i,n+j-i+1}, \tag{2.58}$$

where the product order goes as

$$\begin{aligned}&\Big(S(\theta_{n(n-1)})_{1,2}S(\theta_{n(n-2)})_{2,3}\dots S(\theta_{n1})_{n-1,n}\Big)\\ &\times\Big(S(\theta_{(n-1)(n-2)})_{2,3}\dots S(\theta_{(n-1)1})_{n-1,n}\Big)\dots\big(S(\theta_{32})_{n-2,n-1}S(\theta_{31})_{n-1,n}\big)\,S(\theta_{21})_{n-1,n}.\end{aligned}$$

Proof. First, let us note that $\iota = \mathrm{in}\circ\mathrm{out}^{-1} = \mathrm{out}\circ\mathrm{in}^{-1}$ and that according to Proposition 2.4.8(a) $\langle\boldsymbol{\theta}_{\boldsymbol{\alpha}}|_S = S^{\mathrm{ex}}(\boldsymbol{\theta})^{\boldsymbol{\beta}}_{\boldsymbol{\alpha}}\langle\boldsymbol{\theta}^{\mathrm{ex}}_{\boldsymbol{\beta}}|_S$ and, analogously, $\langle\boldsymbol{\theta}_{\boldsymbol{\alpha}}|_\sigma = (\sigma^{\mathrm{ex}})^{\boldsymbol{\beta}}_{\boldsymbol{\alpha}}\langle\boldsymbol{\theta}^{\mathrm{ex}}_{\boldsymbol{\beta}}|_\sigma$. For simplicity, let us suppress the tensor indices for the next computation, i.e., $\langle\boldsymbol{\theta}|_S = S^{\mathrm{ex}}(\boldsymbol{\theta})\langle\boldsymbol{\theta}^{\mathrm{ex}}|_S$ and so on. Taking also into account the definitions of $\hat{\mathrm{S}}$ (Defn. 2.5.1) and W_{ex} (Eq. (2.50)), for $\Psi \in \mathcal{H}_S$ and $\boldsymbol{\theta} \in \mathbb{R}^n$ we have:

$$\begin{aligned}(\hat{\mathrm{S}}\,\Psi)_n(\boldsymbol{\theta}) &= \langle\boldsymbol{\theta}|_\sigma\,\hat{\mathrm{S}}\,|\Psi\rangle = \langle\boldsymbol{\theta}|_\sigma\,W^*_{\mathrm{out}}W_{\mathrm{in}}\,|\Psi\rangle\\ &= \sigma^{\mathrm{out}}\,\langle\boldsymbol{\theta}^{\mathrm{out}}|_\sigma\,W^*_{\mathrm{out}}W_{\mathrm{in}}\,|\Psi\rangle\\ &= \sigma^{\mathrm{out}}\,\langle\boldsymbol{\theta}^{\mathrm{out}}|_S\,W_{\mathrm{in}}\,|\Psi\rangle\\ &= \sigma^{\mathrm{out}}S^{\iota}(\boldsymbol{\theta}^{\mathrm{out}})\,\langle(\boldsymbol{\theta}^{\mathrm{out}})^{\iota}|_S\,W_{\mathrm{in}}\,|\Psi\rangle\\ &= \sigma^{\mathrm{out}}S^{\iota}(\boldsymbol{\theta}^{\mathrm{out}})\,\langle\boldsymbol{\theta}^{\iota\circ\mathrm{out}}|_S\,W_{\mathrm{in}}\,|\Psi\rangle\\ &= \sigma^{\mathrm{out}}S^{\iota}(\boldsymbol{\theta}^{\mathrm{out}})\,\langle\boldsymbol{\theta}^{\mathrm{in}}|_S\,W_{\mathrm{in}}\,|\Psi\rangle\\ &= \sigma^{\mathrm{out}}S^{\iota}(\boldsymbol{\theta}^{\mathrm{out}})\,\langle\boldsymbol{\theta}^{\mathrm{in}}|_\sigma\,|\Psi\rangle\\ &= \sigma^{\mathrm{out}}S^{\iota}(\boldsymbol{\theta}^{\mathrm{out}})\sigma^{\mathrm{in}^{-1}}\,\langle\boldsymbol{\theta}|_\sigma\,|\Psi\rangle\\ &= \sigma^{\mathrm{out}}S^{\iota}(\boldsymbol{\theta}^{\mathrm{out}})\sigma^{\mathrm{in}^{-1}}\Psi_n(\boldsymbol{\theta}); \qquad \text{proving (2.56).}\end{aligned} \tag{2.59}$$

Assuming commutativity of S and σ, all the factors of the latter in (2.56) can be moved to the left. Equation (2.57) follows upon

$$\sigma^{\mathrm{out}}\sigma^{\mathrm{in}^{-1}} = \sigma^{\mathrm{out}\circ\mathrm{in}^{-1}} = \sigma^{\iota},$$

where we use Corollary 2.4.6(b) and $\iota = \mathrm{out}\circ\mathrm{in}^{-1}$.

Lastly, we will prove (2.58) by induction in n. For $n = 2$, $\iota = \pi_1$ so that $S^{\iota}(\theta) = S^{\pi_1}(\theta) = S(\theta_2 - \theta_1)$ by definition (Eq. (2.33)). Assuming (2.58) to be valid for n, we will show its validity for $n+1$: First, note that $\iota_{n+1} = \pi_{1,n+1}\circ(\mathrm{id}_1\otimes\iota_n)$, where $\pi_{i,j}$ denotes the shift permutation, which was defined in Corollary 2.4.7. Thus for $(\boldsymbol{\theta},\lambda)\in\mathbb{R}^{n+1}$, using repeatedly Corollary 2.4.6(b),

$$S^{\iota_{n+1}}(\boldsymbol{\theta},\lambda) = S^{\pi_{1,n+1}}(\boldsymbol{\theta},\lambda)S^{\mathrm{id}_1\otimes\iota_n}((\boldsymbol{\theta},\lambda)^{\pi_{1,n+1}}). \tag{2.60}$$

By Corollary 2.4.7,

$$S^{\pi_{1,n+1}}(\boldsymbol{\theta},\lambda) = \prod_{j=1}^{n} S(\lambda-\theta_j)_{j,j+1} \tag{2.61}$$

and for the other factor

$$S^{\mathrm{id}_1\otimes\iota_n}((\boldsymbol{\theta},\lambda)^{\pi_{1,n+1}}) = S^{\mathrm{id}_1\otimes\iota_n}(\lambda,\boldsymbol{\theta}) = \mathbb{1}_{\mathcal{K}}\otimes S^{\iota_n}(\boldsymbol{\theta}) = \prod_{i=2}^{n}\prod_{j=1}^{i-1}\mathbb{1}_{\mathcal{K}}\otimes S(\theta_{ij})_{n+j-i,n+j-i+1}, \tag{2.62}$$

where the last equality is by induction assumption. As a result, using $\mathbb{1}_{\mathcal{K}}\otimes S(\theta)_{i,j} = S(\theta)_{i+1,j+1}$ and defining $\boldsymbol{\eta} = (\boldsymbol{\theta},\lambda)$,

$$\begin{aligned} S^{\iota_{n+1}}(\boldsymbol{\theta},\lambda) &= \prod_{j=1}^{n} S(\lambda-\theta_j)_{j,j+1}\prod_{i=2}^{n}\prod_{j=1}^{i-1} S(\theta_{ij})_{n+1+j-i,n+1+j-i+1} \\ &= \prod_{i=n+1}^{n+1}\prod_{j=1}^{i-1} S(\eta_{n+1}-\eta_j)_{n+1+j-i,n+1+j-i+1}\prod_{i=2}^{n}\prod_{j=1}^{i-1} S(\eta_{ij})_{n+1+j-i,n+1+j-i+i} \\ &= \prod_{i=2}^{n+1}\prod_{j=1}^{i-1} S(\eta_{ij})_{n+1+j-i,n+1+j-i+1}, \end{aligned} \tag{2.63}$$

which concludes the proof. □

The expressions for the S-matrix are in agreement with [Kar79b, (A.1)]; the expressions in the reference are for $\theta_1 > ... > \theta_n$, where in $=$ id and out $= \iota$ so that (2.56) and the first equality in (2.57) agree with each other (without an assumption on commutativity of S and σ). Also, it becomes apparent that the S-matrix is indeed of factorizing form and is given as a product of two-particle scattering functions of all the participating one-particle states. More precisely, the elementary building block is the two-particle scattering function which evaluates to[7]

$$S^{(2)}(\theta_1,\theta_2)^{\alpha_1\alpha_2}_{\beta_1\beta_2} = S^{(2)}(\theta_1-\theta_2)^{\alpha_1\alpha_2}_{\beta_1\beta_2} = \begin{cases}\sigma_{\alpha_1\alpha_2}S^{\alpha_2\alpha_1}_{\beta_1\beta_2}(\theta_1-\theta_2), & \theta_1\geq\theta_2\\ S^{\alpha_1\alpha_2}_{\beta_2\beta_1}(\theta_2-\theta_1)\sigma_{\beta_1\beta_2} & \theta_1<\theta_2.\end{cases} \tag{2.64}$$

[7]Note that on the r.h.s. of (2.64) we do not sum over indices.

We see, that in the bosonic case, where $\sigma = \mathbb{1}$, scattering function and S-function (as introduced in Defn. 2.3.1) are the same. In the presence of fermionic states, the scattering function and the S-function differ precisely by the statistics matrix which describes the exchange statistics of the asymptotic particles. This perspective can be expected to hold also for more general, so-called *anyonic* statistics, where the corresponding σ can be any constant S-function (see, e.g., [Smi90]). The analysis from above would hold for any such σ, however, the definition of the Moeller operators (Eq. (2.50)) would have to be argued by an anyonic version of the Haag-Ruelle scattering scheme (or similar). As anyons are beyond the scope of this document, this will not be discussed in further detail.

2.6 Connection to algebraic quantum field theory

Algebraic quantum field theory, also known as local quantum physics (for a standard text book account we refer to [Haa92]), considers algebras of local observables $\mathcal{A}(\mathcal{O})$ associated with spacetime regions $\mathcal{O}$ as the fundamental physical description of a model. A motivation is given by the fact that a model may be described by different sets of fields which, however, all lead to the same S matrix (Borchers classes [Bor60]) so that a distinction of models based on which fields appear in their description may be misleading. In other words, similar to the choice of a coordinate system for a differential manifold, the choice of fields for describing the model is rather conventional than foundational. From this viewpoint, quantum fields and states are secondary objects obtained from the primary objects (the local algebras) by representation on a concrete Hilbert space. The principle of locality refers to commutativity of algebra elements which are causally separated from each other, i.e., for any finite region $\mathcal{O}$ with causal complement $\mathcal{O}'$ in $\mathbb{M}$ we require

$$\mathcal{A}(\mathcal{O}') \subset \mathcal{A}(\mathcal{O})', \tag{2.65}$$

where the prime on the algebra denotes its commutant[8]. This property lies at the heart of the formalism and implements Einstein causality which states that there should be no "action at a distance" meaning that upon causal separation the operators should be statistically independent.

Supplemented with additional conditions like Poincaré covariance, the framework of algebraic quantum field theory provides one of the standard axiomatic descriptions of quantum field theory, the Haag-Kastler axioms, as, e.g., formulated in [Haa92, Sec. III.1]. In the following, we will sketch the connection between algebraic quantum field theory and the framework developed in the preceding sections,

[8]The algebra of bounded operators which commute with the algebra in question.

concluding with the realization of the Haag-Kastler axioms for a large subclass of integrable models.

As a preliminary, we introduce wedges and double cones as subregions of Minkowski space. A right (left) wedge with tip at $x \in \mathbb{M}$ is given by $\mathcal{W}_x$ ($\mathcal{W}'_x$), where $\mathcal{W}_x = x + \mathcal{W}_R$, $\mathcal{W}_R := \{x \in \mathbb{M} : |x^0| < x^1\}$, and where $\mathcal{O}'$ denotes the causal complement of $\mathcal{O}$ within $\mathbb{M}$. A double cone between the points $x, y \in \mathbb{M}$ is given by $\mathcal{O}_{x,y} := \mathcal{W}_x \cap \mathcal{W}'_y$. These regions are illustrated in Figure 2.1.

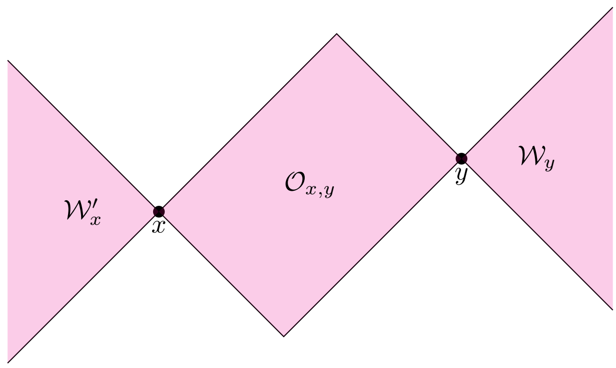

Figure 2.1: Illustration in 1+1d Minkowski space of wedge regions $\mathcal{W}'_x$ and $\mathcal{W}_y$ with the double cone region $\mathcal{O}_{x,y} = \mathcal{W}_x \cap \mathcal{W}'_y$ given as the intersection of their causal complements.

Mathematically, a convenient minimalistic set of data to construct an algebraic quantum field theory (in 1+1d) is given by a *Borchers triple*.

Definition 2.6.1. Given a separable Hilbert space $\mathcal{H}$, a Borchers triple $(\mathcal{M}, U, \Omega)$ consists of a von Neumann algebra $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$, a unitary representation U of the translation group $(\mathbb{M}, +)$ on $\mathcal{H}$, and a vector $\Omega \in \mathcal{H}$ such that

1. U is strongly continuous, of positive energy[a], and has Ω as its unique (up to a phase) invariant vector,
2. $\mathcal{M}_x := U(x)\mathcal{M}U(x)^* \subset \mathcal{M}$ for all $x \in \mathcal{W}_R$ (right wedge),
3. Ω is cyclic and separating for $\mathcal{M}$.

[a] Let P^μ, $\mu = 0, 1$ denote the self-adjoint generators of U, then for U to have *positive energy* means $P^2 \geq 0, P^0 > 0$ on the common invariant domain of P^μ; cf. Section A.1.

Given such a triple $(\mathcal{M}, U, \Omega)$, a net indexed by double cones $\mathcal{O}_{x,y} = \mathcal{W}_x \cap \mathcal{W}'_y$ is given by

$$\mathcal{A}(\mathcal{O}_{x,y}) := \mathcal{M}_x \cap \mathcal{M}'_y. \tag{2.66}$$

Note that this can easily be generalized to arbitrary open regions $\mathcal{O}$ by approximation with double cones. The thus defined net $\mathcal{A}$ is local and $\mathcal{P}_+$-covariant (under a certain extension of U) [Bor92, Sec. III]. In case that $(\mathcal{M}, \Omega)$ satisfies the so-called modular nuclearity condition, the intersection defined in (2.66) is also large enough such that Ω is cyclic and separating for $\mathcal{A}(\mathcal{O}_{x,y})$ (Reeh-Schlieder property) [BL04]. These properties taken together, imply then that the net $\mathcal{A}$ implements an algebraic quantum field theory in terms of the Haag-Kastler axioms.

In order to connect this to our context, we will give the construction of the Borchers triple which turns out to satisfy modular nuclearity and therefore gives rise to an algebraic quantum field theory in the sense illustrated before. To begin with, consider $\mathcal{H} = \mathcal{H}_S$ as constructed in Section 2.4 and introduce the field

$$\Phi_S(f) = z_S^\dagger(f^+) + z_S(U_1(j)f^-), \quad f \in \mathcal{S}(\mathbb{M}, \mathcal{K}), \tag{2.67}$$

where $f^\pm(\theta) := \tilde{f}(\pm p(\theta; M))$ for $\theta \in \mathbb{R}$ with $\tilde{f}$ denoting the Fourier transform of f; confer (2.2). Moreover, we introduce the "reflected" field

$$\Phi'_S(f) = U(j)\Phi_S(U_1(j)f)U(j), \quad f \in \mathcal{S}(\mathbb{M}, \mathcal{K}). \tag{2.68}$$

Then Ω is a cyclic vector with respect to the polynomial field algebra generated by either field Φ_S or Φ'_S and both $\Phi_S(f)$ and $\Phi'_S(f)$ define essentially self-adjoint unbounded operators on $\mathcal{H}_S^{\mathrm{f}}$ which are covariant under $\mathcal{P}_+^\uparrow$ and $\mathcal{G}$ with respect to U_S and V_S, respectively. However, unless $S = \mathbb{F}$, these operators are not strictly local, as for $f, g \in \mathcal{S}(\mathbb{M}, \mathcal{K})$ with spacelike separated supports,

$$[\Phi_S(f), \Phi_S(g)]\Psi = [\Phi'_S(f), \Phi'_S(g)]\Psi = 0, \quad \Psi \in \mathcal{H}_S^{\mathrm{f}} \tag{2.69}$$

holds iff $S = \mathbb{F}$ [LS14, Prop. 3.1]. Nonetheless the fields have a remnant localization property, they are relatively *wedge-local*, i.e., for $f \in \mathcal{S}(\mathcal{W}'_x, \mathcal{K})$ and $g \in \mathcal{S}(\mathcal{W}_y, \mathcal{K})$ such that x is to the left of y, i.e., $x - y \in \mathcal{W}'$, it holds that

$$[\Phi_S(f), \Phi'_S(g)]\Psi = 0, \quad \Psi \in \mathcal{H}_S^{\mathrm{f}}. \tag{2.70}$$

This localization property indicates the interpretation of $\Phi_S(f)$ generating operations localized in a (shifted and smeared) left wedge and $\Phi'_S(g)$, analogously, in a right wedge. An algebra of bounded operators $\mathcal{M}$ is obtained by considering all bounded functions of the field Φ_S and is generated by its exponentials. In the end, the Borchers triple $(\mathcal{M}, U, \Omega)$ is given by

$$\mathcal{M} := \{\exp(i\Phi'_S(f)^{\mathrm{cl}}),\ f = U_1(j)f \in \mathcal{S}(\mathcal{W}_R, \mathcal{K})\}'', \quad U := U_S(\cdot, 0), \quad \Omega := \Omega_S, \tag{2.71}$$

where $\Phi_S(f)^{\mathrm{cl}}$ denotes the closure of $\Phi_S(f)$. As a consequence,

$$\mathcal{M}' = \{\exp(i\Phi_S(f)^{\mathrm{cl}}),\ f = U_1(j)f \in \mathcal{S}(\mathcal{W}_R, \mathcal{K})\}''. \tag{2.72}$$

Modular nuclearity is satisfied at least for the free model $S(\zeta) = \mathbb{F}$ [BDL90] and for regular S-functions with no poles in the physical strip which have $S(0) = -\mathbb{F}$ and satisfy an intertwining property [AL17]. As a consequence, the net $\mathcal{A}$ constructed as above indeed defines an algebraic quantum field theory model.

At the end, let us note that modular nuclearity also has various other consequences, for example, that $\mathcal{A}$ is weakly additive and that it satisfies Haag duality for wedge and double cone regions $\mathcal{O}$ [Lec07],

$$\mathcal{A}(\mathcal{O}') = \mathcal{A}(\mathcal{O})', \tag{2.73}$$

a stronger version of (2.65). Also, modular nuclearity implies the compactness of $\mathcal{G}$ so that for non-compact symmetry groups other methods would be needed.

In the presence of fermions and global gauge symmetries one may define a larger net of algebras, the field net, and consider the algebra of local observables as a subnet. The field net can be constructed in the same way as described before for the net of local observables but will be non-local in general. For fermions the Borchers triple receives an additional grading which distinguishes bosons and fermions. The resulting field net is only twisted-local (implementing that fermionic fields anticommute) and the algebra of local observables is obtained as the "bosonic" subalgebra (see, e.g., [BW76]). In the presence of a global gauge symmetry the field net is non-local but the algebra of local observables is obtained as the gauge invariant subalgebra. For fermions, it is also proven, that the field net satisfies similar properties as stated above for the net of local algebras [BC21].

Chapter 3

Locality and the form factor series

In the last chapter, we presented the construction of integrable models via the inverse scattering method. We concluded with a Haag-Kastler net describing local observables as operators that are simultaneously localized in a left- and a right-wedge (Sec. 2.6). We found that, while the condition of modular nuclearity is, in principle, sufficient to guarantee that the net constructed in such a way is "large" enough, in practice, it would be helpful to have more explicit information on the net of local observables. In particular, in view of obtaining quantum energy inequalities, we will need to study the smeared stress-energy tensor and need to identify it within the abstract algebra we have obtained.

The most common approach to treat local observables in our framework (1+1d integrable models) is the so-called form factor program [Smi92; BFK08] which we briefly reviewed in the introduction (Chap. 1). In its usual formulation, it involves the construction of Wightman n-point functions by an infinite series. The summation runs over a certain family of truncated momentum space correlation functions, the so-called *form factors*. The convergence of this series is expected to be generically very fast, as indicated by numerical analysis [CM93; DM95; DC98][1] and heuristic arguments (cf. the discussion in [Smi92, Chap. 10]). However, mathematical results were not obtained until recently in the examples of the Ising model [BC19] and the sinh-Gordon model [Koz21; Koz22]. This shall be fine for us here for two reasons: The abstract existence of (sufficiently many) local observables was presented in the previous chapter (Sec. 2.6). Moreover, we will focus on expectation values in finite-particle states where convergence issues are absent.

In integrable models, one expects a one-to-one correspondence between a local operator and its form factors subject to a number of conditions which go under the name "form factor equations". The equivalence between locality and the form factor equations, known as the local commutativity theorem and first established in [KS89][Smi92, Sec. 2], is rigorously proven in the scalar case ($d_{\mathcal{K}} = 1$) for models with bosons [BC15] and fermions [BC21]. In more generality—including models with several particle species and inner degrees of freedom ($d_{\mathcal{K}} > 1$), with semilocal[2] oper-

[1] Confer also [EK05, Sec. 2.5] for a brief overview and additional references.

[2] We mean here allowing a constant phase factor to appear in the exchange relation under spacelike separation. This is for instance necessary to treat anyons but may also appear for other fields, like solitonic fields. E.g., [BFK06; Del09], treat this case.

ators and with bound states, i.e., poles of the S-function in the physical strip—this is expected to hold [KW78; Smi92; BFK08] but not under complete mathematical control. In the case $d_{\mathcal{K}} > 1$ and excluding bound states, there is only a derivation of the form factors for wedge-local observables with appropriate weaker conditions and without reconstruction of the observable from the form factors [AL17]. Models with bound states have been treated in [Que99; BK02; BFK06; CT15; CT17], and we refer to the discussion in Appendix A.5 for more details.

In this chapter, we will first introduce the form factors and their relation with locality, providing the form factor equations in generality but without proof (Sec. 3.1). After that, we restrict to the relevant case for the following chapters—the one- and two-particle form factors. Under this restriction, we will provide a proof of the local commutativity theorem (Sec. 3.2) and show the form factor's transformation properties under the symmetries of the model and conjugation (Sec. 3.3). The proof of the local commutativity theorem excludes bound states. However, it treats otherwise generic models with a regular scattering function, generalizing previous results as mentioned above. In the transformation properties we include explicit expressions for invariant observables and derivatives of observables at the level of form factors (Sec. 3.3.1). A brief literature survey for different conventions on the form factors is deferred to Appendix B.

3.1 Locality and the form factor series

In this section, we outline the correspondence between local operators and their form factors in the full generality of our framework. This outline will have a somewhat prototypical character, skipping some mathematical details since parts of the presentation are, in this generality, not covered by mathematical proofs. However, in the next section, we will back it up by providing a partial proof restricting to one- and two-particle form factors. The general situation (excluding bound states) should follow using our methods which originate from [BC15; AL17]. Note also that the presented form factor equations align with the physics literature; for instance [BFK08, Sec. 3].

To begin with, recall the state space $\mathcal{H}_S$, the ZF operators $z_S^\sharp$, their tensor powers $z_S^{\dagger m} z_S^n$, and the rapidity eigenstates $|\boldsymbol{\theta}_\alpha\rangle_S$ introduced in Section 2.4.

Then, generically, any operator A on $\mathcal{H}_S$ can be expanded in powers of $z_S^\dagger$ and z_S and we expect a series of the form

$$A = \sum_{n=0}^{\infty} A_n, \qquad A_n = \sum_{k=0}^{n} \frac{1}{k!(n-k)!}\, z_S^{\dagger k} z_S^{n-k}(f_{k,n-k}), \tag{3.1}$$

for a suitable family of distributions $f_{m,n}$ and which holds at least in expectation values for a suitable class of states. The family of distributions is in a one-to-one-correspondence with A and is given by the truncated[3] momentum space correlation functions of A. More specifically, the $f_{m,n}$ have distributional kernels $(f_{m,n})_{\boldsymbol{\alpha\beta}}(\boldsymbol{\theta};\boldsymbol{\eta})$ which arise by extending

$$\sqrt{m!n!}\,C_{\boldsymbol{\alpha}\boldsymbol{\alpha}'}\,\langle\boldsymbol{\theta}_{\boldsymbol{\alpha}'}|_S\,A\,|\overleftarrow{\boldsymbol{\eta}}_{\overleftarrow{\boldsymbol{\beta}}}\rangle_S\,,\quad \theta_i\neq\eta_j,\,1\leq i\leq m, 1\leq j\leq n, \tag{3.2}$$

to coinciding rapidities $\theta_i=\eta_j$ in a certain "truncated" way. For details we refer to [BC15, Sec. 3.2][AL17, Chap. 4]. We shall note here that there are various conventions for the form factors in the literature. We here closely follow the conventions in [BC15] extending it to $d_{\mathcal{K}}>1$. Other conventions differ from this one by a change of the order of rapidities or the constant coefficients in front of the expression in (3.2). For a more detailed comparison with other conventions, see Appendix B.

Now, given that A is to some degree localizable, the $f_{m,n}$ will satisfy corresponding analyticity properties. In particular, for A localized in some finite open region for each $n\in\mathbb{N}_0$ we expect all $\{f_{k,n-k}\}_{k=0,\dots,n}$ to be distributional boundary values of a single meromorphic function $F_n:\mathbb{C}^n\to\mathcal{K}^{\otimes n}$:

$$f_{k,n-k}(\boldsymbol{\theta},\boldsymbol{\eta})=(\mathbb{1}_k\otimes J^{\otimes(n-k)})F_n(\boldsymbol{\theta}+i\mathbf{0},\boldsymbol{\eta}+i\boldsymbol{\pi}-i\mathbf{0}), \tag{3.3}$$

with $i\mathbf{0}$ indicating the distributional limit from within a certain region. The F_n depend linearly on A and the family $\{F_n\}_{n\in\mathbb{N}_0}$ is in one-to-one correspondence with A.

The F_n are known as the *form factors* of A which satisfy a number of well-known properties, the *form factor equations* which we will state at the end of this section. In line with the literature, we will call F_n the *n-particle* form factor; though note that expectation values in n-particle *states* generically have contributions from all zero- to $2n$-particle form factors.

Combining (3.1) and (3.3), we see that the symbols A_n (for local A) depend on F_n only; we may write $A_n[F_n]$ to clarify the dependence. The resulting series summing the F_n is termed the *form factor series*. As discussed above, this series is expected to converge fast in many cases, but a proof in any generality is absent.

In the remainder of this section, we provide precise notions of regular high-energy behaviour and locality as well as a definite version of the form factor equations. For these we expect the local commutativity theorem to hold, i.e., a one-to-one correspondence between local A with regular high-energy behaviour and a family of functions $\{F_n\}_{n\in\mathbb{N}_0}$ satisfying the form factor equations. Overall, this will set the stage for the proof in the next section and provide the means to discuss the stress-energy tensor at the level of form factors in the following chapters.

[3] This is sometimes also referred to as "connected" or "contracted".

Energy-bounded quadratic forms The convergence issues for the form factor series may be ignored by treating (3.1) in the sense of a quadratic form on a "nice" domain $\mathcal{D}_S$, i.e., the equality holds evaluated in expectation values $\langle \Psi, \cdot \Psi \rangle$ where $\Psi \in \mathcal{D}_S$, and where $\mathcal{D}_S$ is contained in the finite particle states so that the infinite sum over $n \in \mathbb{N}_0$ collapses to a finite one. Introducing, the energy norms

$$\|\Psi\|_k := \|(1+P^0)^k \Psi\|, \quad k \in \mathbb{Z}, \quad \Psi \in \mathcal{H}_S, \tag{3.4}$$

a possible choice for $\mathcal{D}_S$, which we use in the following, is the space of energy bounded finite particle states

$$\mathcal{D}_S := \mathcal{H}_S^{\mathrm{f}} \cap \mathcal{H}_S^{\mathrm{eb}}, \quad \mathcal{H}_S^{\mathrm{eb}} := \{\Psi \in \mathcal{H}_S : \forall k \in \mathbb{Z} : \|\Psi\|_k < \infty\}. \tag{3.5}$$

In the series (3.1) for suitably chosen F_n we can regard each $A_n[F_n]$ separately, as an operator on $\mathcal{D}_S$. For example, for $n = 1$ if F_1 and $F_1(\cdot + i\pi)$ are square-integrable, one has $F_1, JF_1(\cdot + i\pi) \in \mathcal{H}_1$ so that $z_S^\dagger(F_1)$ and $z_S(JF_1(\cdot + i\pi))$, and thus also A_1, define unbounded operators on $\mathcal{D}_S$ (Prop. 2.4.2). It is also expected (while not a priori clear) that (3.1) defines a local operator for suitable F_n. While slightly improper, this motivates to choose operator notation for A:

Remark 3.1.1 (Notation for quadratic forms). For a quadratic form $A = A[\cdot]$ on some domain $\mathcal{D}$ and $\varphi, \chi \in \mathcal{D}$ we denote

$$\langle \varphi, A\chi \rangle := A[\varphi, \chi] := \tfrac{1}{4}(A[\varphi+\chi] - A[\varphi-\chi] + iA[\varphi - i\chi] - iA[\varphi + i\chi]). \tag{3.6}$$

For the operator-form-factor correspondence to hold, we will need some regularity assumption on the operator. A typical assumption is a restriction on the high-energy behaviour of A:

Definition 3.1.2. A quadratic form $A : \mathcal{D}_S \times \mathcal{D}_S \to \mathbb{C}$ is referred to as *(polynomially) energy-bounded* iff there exists a $k \in \mathbb{Z}$ such that

$$\|A\|_k := \|(1+P^0)^{-k} A (1+P^0)^{-k}\| < \infty. \tag{3.7}$$

$\mathcal{Q}$ will denote the class of polynomially energy-bounded quadratic forms.

Such energy bounds are expected to hold for Wightman-type quantum fields and were studied in detail in [FH81] and [BW92, Chaps. 12-14]. The assumption of polynomial energy bounds is in particular mild enough to include the total energy-momentum operator P^μ (for $k = \frac{1}{2}$) and, desirably, the smeared stress-energy tensor $T^{\mu\nu}(f)$; confer Remark 5.1. The operator-form-factor correspondence can also be formulated for operators with almost exponential growth in the energy which was done in [BC15].

Locality We shall define precisely what we mean by A being local. Following the notion of locality presented in Section 2.6 for an arbitrary open region $\mathcal{O} \subset \mathbb{M}$, we consider $A \in \mathcal{Q}$ as *localized (in $\mathcal{O}$) relative to the observables* iff A is affiliated with $\mathcal{A}(\mathcal{O})$ meaning that all bounded functions[4] of A are elements of $\mathcal{A}(\mathcal{O})$. Note here that for bounded A, "affiliated with" reduces to "element of". As we have learned in Section 2.6, the wedge-local algebras are generated by the fields Φ_S and Φ'_S and the local algebras by intersections of these wedge-algebras. So, as we will argue below (Rem. 3.1.4), locality relative to the observables implies a weaker notion of locality involving commutativity with the fields Φ_S and Φ'_S. This notion of locality will be sufficient for our purposes and can be given as follows:

Definition 3.1.3. Let $x, y \in \mathbb{M}$ be arbitrary. $A \in \mathcal{Q}$ is referred to as *wedge-local*

$$\text{in } \mathcal{W}_x \quad \text{iff} \quad \forall f \in \mathcal{S}(\mathcal{W}'_x, \mathcal{K}) : \quad [A, \Phi_S(f)] = 0, \tag{3.8}$$

$$\text{in } \mathcal{W}'_x \quad \text{iff} \quad \forall f \in \mathcal{S}(\mathcal{W}'_x, \mathcal{K}) : \quad [A, \Phi'_S(f)] = 0. \tag{3.9}$$

A is referred to as *local* in a double cone $\mathcal{O}_{x,y}$ iff A is wedge-local in $\mathcal{W}_x$ and $\mathcal{W}'_y$. For an arbitrary open finite region $\mathcal{O} \subset \mathbb{M}$ we refer to A as local in $\mathcal{O}$ iff A is local in some double cone contained in $\mathcal{O}$.

A few comments on this notion of locality are in order. First, note that for $f \in \mathcal{S}(\mathbb{M}, \mathcal{K})$, the ZF operators $z^\dagger_S(f^+)$ and $z_S(Jf^-)$ leave $\mathcal{D}_S$ invariant, so that products of A with the ZF operators from the left or right and thus the commutators in (3.8) and (3.9) are well-defined. Second, note that for A to be local in $\mathcal{W}'_x$ is equivalent to $U(j)A^*U(j)$ being wedge-local in $\mathcal{W}_{-x}$ which is compatible with $U(j)$ implementing the spacetime reflection. Note also that this notion of locality is compatible with the representation of the translation group: For $y \in \mathbb{M}$ take $U_S(y) := U_S(y, 0)$ and $A(y) := U_S(y)AU_S(y)^{-1}$. Then we have that A is wedge-local in $\mathcal{W}_x$ iff $A(y)$ is wedge-local in $\mathcal{W}_{x+y}$. This is because for $f \in \mathcal{S}(\mathcal{W}'_x, \mathcal{K})$ one has

$$\begin{aligned}
[A(y), \Phi_S(f)] &= [U_S(y)AU_S(y)^{-1}, \Phi_S(f)] \\
&= U_S(y)[A, U_S(y)^{-1}\Phi_S(f)U_S(y)]U_S(y)^{-1} \\
&= U_S(y)[A, \Phi_S(f_{-y})]U_S(y)^{-1},
\end{aligned} \tag{3.10}$$

where f_{-y} defined as $f_{-y}(x) := f(x - y)$ has now support in $\mathcal{W}'_{x+y}$. Analogously, A is wedge-local in $\mathcal{W}'_x$ iff $A(y)$ is wedge-local in $\mathcal{W}'_{x+y}$.

We conclude this part by arguing briefly how locality relative to the observables implies Definition 3.1.3:

[4] For self-adjoint A the spectral theorem allows to define bounded functions of A. In case that A is not selfadjoint we may use the polar decomposition of its closure and demand that the unitary factor and bounded functions of the positive factor are elements of $\mathcal{A}(\mathcal{O})$.

Remark 3.1.4. To begin with, we abbreviate "relative to the observables" to "r.t.o." and introduce that two operators $A, B \in \mathcal{Q}$ commute strongly iff all their bounded functions commute and weakly on $\mathcal{D}_{AB} \subset \mathcal{H}_S$ iff $[A, B]\mathcal{D}_{AB} = 0$ (here $\mathcal{D}_{AB}$ is usually taken to be a common dense invariant domain).

Now, because of Haag-duality we have that $\mathcal{A}(\mathcal{O}) = \mathcal{A}(\mathcal{O}')'$ for $\mathcal{O}$ being a wedge or double cone region; confer (2.73). Therefore, a bounded function of A is in $\mathcal{A}(\mathcal{O})$ precisely if it commutes with elements localized in the causal complement. As a consequence, A is localized in a wedge $\mathcal{W}_x$ r.t.o. iff its bounded functions commute with the elements of $\mathcal{A}(\mathcal{W}_x') = \mathcal{M}'$, which is generated by the bounded functions of $\Phi_S(f)$, $f \in \mathcal{S}(\mathcal{W}_x, \mathcal{K})$; confer (2.72). Thus we infer that A is localized in $\mathcal{W}_x$ r.t.o. iff A commutes strongly with $\Phi_S(f)$. Analogously, A is localized in a wedge $\mathcal{W}_x'$ r.t.o. iff A commutes strongly with $\Phi_S'(f)$ and A is localized in a double cone region r.t.o. iff A commutes strongly with both; confer also (2.71) and (2.66) and note here that the causal complement of a double cone $\mathcal{O}_{x,y}$ is given by $\mathcal{O}_{x,y}' = \mathcal{W}_x' \cup \mathcal{W}_y$; confer Figure 2.1.

As, of course, strong commutativity implies weak commutativity on any domain, locality relative to the observables implies the notion formulated in Defintion 3.1.3. While the converse statement is generally not true (for $d_{\mathcal{K}} = 1$ a possible converse is given in [Cad13, Prop.4.4(ii)]), we should note that if A is bounded, the distinction becomes irrelevant, and the statements are entirely equivalent.

The form factor equations Finally, we define the form factor equations in detail. In models which have bound states or fermions, we have to introduce a few more objects, though: In the presence of bound states, the form factors have additional poles. In order to describe them we use concepts defined in Appendix A.5; for the definition of the fusion rules $\mathcal{F} = \{ij \to k\}$, the fusion angles $\theta_{(ij)}^k$, and the bound state intertwiners $\Gamma_k^{ij} : \mathcal{K} \to \mathcal{K}^{\otimes 2}$ we refer to that section. Without bound states, one has $\mathcal{F} = \emptyset$ so that (F1b) (below) is absent. The form factor equations also depend on the statistics of the particles. So in case that there are fermions in the model we introduce $\sigma_{1A} \in \mathcal{B}(\mathcal{K}^{\otimes n})$ (for any n) which multiplies with -1 if A and the first tensor component are fermionic and with 1 otherwise. For $\boldsymbol{\zeta} \in \mathbb{C}^n$ let us denote $\boldsymbol{\zeta}_{jn} = (\zeta_j, ..., \zeta_n)$. Recall also the charge conjugation matrix $C \in \mathcal{B}(\mathcal{K})$ (Sec. 2.2) and introduce $\check{C} \in \mathcal{K}^{\otimes 2}$ which yields C applying $\hat{}$ as introduced in (2.4).

The form factor equations then amount to:

Definition 3.1.5. A family $\{F_n\}_{n\in\mathbb{N}_0}$ of distributions $F_n \in \mathcal{D}'(\mathbb{C}^n, \mathcal{K}^{\otimes n})$ satisfies the form factor equations corresponding to $A \in \mathcal{Q}$ iff, for all n,

(F1) (analytic structure) F_n is meromorphic on all of $\mathbb{C}^n$.
On $\mathcal{T}_n := \{\boldsymbol{\zeta} \in \mathbb{C}^n : \operatorname{Im}\zeta_n + 2\pi > \operatorname{Im}\zeta_1 > \ldots > \operatorname{Im}\zeta_n\}$, F_n has poles whenever

(F1a) (kinematic poles) $\zeta_2 - \zeta_1 = i\pi$. These poles have the residue

$$\operatorname*{res}_{\zeta_2-\zeta_1=i\pi} F_n(\boldsymbol{\zeta}) = -\tfrac{1}{2\pi i}\left(\mathbb{1} - \sigma_{1A}^n \prod_{p=3}^n (\mathbb{F}S(\zeta_2-\zeta_p))_{2,p}\right)(\check{C} \otimes F_{n-2}(\boldsymbol{\zeta}_{3n})).$$

(F1b) (bound state poles) $\zeta_2 - \zeta_1 = i\theta^k_{(ij)}$ for all $ij \to k \in \mathfrak{F}$ with fusion angle $0 < \theta^k_{(ij)} < \pi$. These poles have the residue

$$\operatorname*{res}_{\zeta_2=\zeta_1+i\theta^k_{(ij)}} F_n(\boldsymbol{\zeta}) = \tfrac{1}{\sqrt{2\pi}}(\Gamma^{ij}_k \otimes \mathbb{1}_{n-2})F_{n-1}(\zeta_1 - i\theta^k_{ij}, \boldsymbol{\zeta}_{3n}).$$

and no more poles than required by consistency with (F2) and (F3).

(F2) (S-symmetry) $F_n(\boldsymbol{\zeta}) = S^\tau(\boldsymbol{\zeta})F_n(\boldsymbol{\zeta}^\tau), \quad \tau \in \mathfrak{S}_n$.

(F3) (S-periodicity) $F_n(\zeta_1 + 2\pi i, \boldsymbol{\zeta}_{2n}) = (\sigma_{1A}\mathbb{F})^{\pi_{1,n}}\, F_n(\boldsymbol{\zeta}_{2n}, \zeta_1)$.

(F4) (bounds) There exist constants $a, b, r \geq 0$ such that for all $\boldsymbol{\zeta} \in \mathcal{T}_n$ with $|\operatorname{Re}(\zeta_i - \zeta_j)| \geq r$ for all $1 < i < j < n$ it holds that

$$||F_n(\boldsymbol{\zeta})||_{\mathcal{K}^{\otimes n}} \leq a \exp\Big(b \sum_{j=1}^n |\operatorname{Re}\zeta_j|\Big).$$

As stated before, these equations are expected (and in a some models proven) to be equivalent to $A \in \mathcal{Q}$ being local (in the sense of Defn. 3.1.3). In case that A satisfies additional properties like hermiticity or Poincaré covariance, we can also derive related conditions on its form factors. In the following sections, we will prove such properties of the form factors in detail and restricting to the case $n \leq 2$ which will be the relevant one for our later analysis.

3.2 Local commutativity theorem for one- and two-particle form factors

Let us consider an operator $A \in \mathcal{Q}$, which is localized in a wedge or a double cone (Defn. 3.1.3). This section aims to construct the n-particle form factors of A in the sense of proving that local operators yield form factors satisfying the form factor equations (Defn. 3.1.5). For simplicity, we will restrict our presentation to the case $n \leq 2$ as we will only need this case later on. However, the presented methods should generalize to $n \in \mathbb{N}$ in a possibly complicated but nonetheless straightforward manner. We will obtain form factor equations in agreement with those in Definition 3.1.5 (for $n \leq 2$, $\mathfrak{F} = \emptyset$) and prove that A can be reconstructed at one-particle level by the series given in (3.1). The ideas for the proof are based on preceding results: We will closely follow [Cad13; BC15], where the n-particle form factors were constructed for all $n \in \mathbb{N}_0$ restricting to a scalar S-function without bound state poles ($d_{\mathcal{K}} = 1, \mathfrak{F} = \emptyset$). It is also stated there how A is reconstructed from the form factors at finite particle level. A generalization to $d_{\mathcal{K}} > 1$ was presented in [AL17], however, studying only wedge-local A (also excluding bound state poles) and without reconstruction of A.

The main results of this section are:

Theorem 3.2.1. *(Two-particle form factor equations) Assume a model with a regular S-function S which has no poles in the physical strip. Then if $A \in \mathcal{Q}$ with $\langle \Omega, A\Omega \rangle = 0$ is localized in a double cone there exists a function $F_2 : \mathbb{C}^2 \to \mathcal{K}^{\otimes 2}$ which satisfies the form factor equations at two-particle level without bound states (Defn. 3.1.5, $n = 2, \mathfrak{F} = \emptyset$). In particular,*

(F2.1) (analytic structure) F_2 is meromorphic on all of $\mathbb{C}^2$ and analytic on the tube $|\operatorname{Im} \zeta_1 - \zeta_2| < 2\pi + \kappa$ for any $\kappa < \kappa(S)$,

(F2.2) (S-symmetry) $F_2(\boldsymbol{\zeta}) = S(\zeta_2 - \zeta_1) F_2(\overleftarrow{\boldsymbol{\zeta}})$,

(F2.3) (S-periodicity) $F_2(\zeta_1, \zeta_2 + i2\pi) = \sigma_{1A} \mathbb{F} F_2(\zeta_2, \zeta_1)$,

(F2.4) (bounds) there is a constant $c \geq 0$ such that for large enough $k \in \mathbb{Z}$ one has for all $\boldsymbol{\zeta} \in \mathbb{R}^2 + i\pi\mathbb{Z}^2$ that $||F_2(\boldsymbol{\zeta})||_{\mathcal{K}^{\otimes 2}} \leq c(|\operatorname{ch} \zeta_1|^k + |\operatorname{ch} \zeta_2|^k)\|A\|_k$;

and F_2 is such that for arbitrary $\varphi, \chi \in \mathcal{H}_1$

$$\langle \varphi, A\chi \rangle = \int d\theta d\eta \, (\varphi(\theta) \otimes J\chi(\eta), F_2(\theta, \eta + i\pi))_{\mathcal{K}^{\otimes 2}}, \tag{3.11}$$

and for arbitrary $\psi \in \mathcal{H}_{S,2}$

$$\langle \psi, A\Omega \rangle = \int d\boldsymbol{\theta}\, (\psi(\boldsymbol{\theta}), F_2(\boldsymbol{\theta}))_{\mathcal{K}^{\otimes 2}}, \quad \langle \Omega, A\psi \rangle = \int d\boldsymbol{\theta}\, (J^{\otimes 2}\psi(\boldsymbol{\theta}), F_2(\boldsymbol{\theta} + i\boldsymbol{\pi}))_{\mathcal{K}^{\otimes 2}}. \tag{3.12}$$

Proposition 3.2.2. *(One-particle form factor equations) If* $A \in \mathcal{Q}$ *is localized in a double cone, then there exists a function* $F_1 : \mathbb{C} \to \mathcal{K}$ *which satisfies the form factor equations at one-particle level (Defn. 3.1.5,* $n = 1$*). In particular,* F_1 *is an analytic function which is periodic under shifts of* $2\pi i$ *and satisfies* $\|F_1(\cdot + i\lambda)\|_2 < \infty$ *for all* $\lambda \in \pi\mathbb{Z}$*. Moreover,* F_1 *is such that for arbitrary* $\varphi \in \mathcal{H}_1$

$$\langle \varphi, A\Omega \rangle = \int d\theta (\varphi(\theta), F_1(\theta)), \quad \langle \Omega, A\varphi \rangle = \int d\theta (J\varphi(\theta), F_1(\theta + i\pi)). \tag{3.13}$$

Note here that the assumption $\langle \Omega, A\Omega \rangle = 0$ in Theorem 3.2.1 is for simplicity. It holds, in particular, for $A = T^{\mu\nu}(f)$ with f being an arbitrary test function; confer Remark 5.1. Note also that the conditions provided here specify a larger analyticity region than the generic form factor equations presented in the preceding section. This is a consequence of the lack of kinematic poles for $n = 1, 2$.

The remainder of this section will be devoted to proving these results. The proof strategy also applies to higher particle numbers. It can be given as follows: First, we will introduce distributions $f_{m,n}$ which map m- and n-particle states to expectation values of A in these states. The properties of the $f_{m,n}$ follow straightforwardly from those of the ZF operators $z_S^\sharp$ and of A. These distributions can be represented as boundary values of analytic functions, say $F_{m,n}$. For A localized in a wedge, one finds that these analytic functions can be matched at each level of $m + n$ so that we may write F_{m+n} instead; later this will denote the $m+n$-particle form factor. It already satisfies S-symmetry and analyticity for imaginary parts in a certain simplex region. Now, if A is localized in a double cone it is localized in a left and a right wedge, or equivalently, A and $U(j)A^*U(j)$ are localized in a right wedge (confer discussion below Defn. 3.1.3). Therefore one obtains two families of distributions $f_{m,n}^{[A]}$ and $f_{m,n}^{[U(j)A^*U(j)]}$ which both give rise to analytic functions in the simplex region mentioned above, say F_{m+n} and $F_{m+n}^\dagger$. It turns out that at the boundaries of the simplex region F_{m+n} and $F_{m+n}^\dagger$ can be matched (using CPT-invariance (S3)). It is then expected that a consistent meromorphic continuation to all of $\mathbb{C}^{m+n}$ arises which satisfies the properties in Definition 3.1.5.

To start with, we will introduce distributions $f_{m,n} : \mathcal{H}_1^{\otimes m} \times \mathcal{H}_1^{\otimes n} \to \mathbb{C}$ for $m, n, m + n \in \{0, 1, 2\}$. To keep the notation light, we will mostly denote the ZF operators as $z^\sharp$ omitting the subscript S. For $m = n = 0$ we have a constant, which we assume to be vanishing in the following

$$f_{0,0} := \langle \Omega, A\Omega \rangle = 0. \tag{3.14}$$

For $m+n=1$ we extend the expressions

$$f_{1,0}(\varphi) := \langle \Omega, z(\varphi)A\Omega \rangle\,, \quad f_{0,1}(\varphi) := \langle \Omega, Az^\dagger(\varphi)\Omega \rangle\,, \quad \varphi \in \mathcal{D}(\mathbb{R}, \mathcal{K}), \tag{3.15}$$

by linearity and continuity to $\mathcal{H}_1$. For $m+n=2$ and $\varphi, \chi \in \mathcal{D}(\mathbb{R}, \mathcal{K})$ we define

$$\begin{aligned} f_{2,0}(\varphi \otimes \chi) &:= \langle \Omega, z(\chi)z(\varphi)A\Omega \rangle\,, \\ f_{1,1}(\varphi; \chi) &:= \langle \Omega, z(\varphi)Az^\dagger(\chi)\Omega \rangle\,, \\ f_{0,2}(\varphi \otimes \chi) &:= \langle \Omega, Az^\dagger(\chi)z^\dagger(\varphi)\Omega \rangle\,, \end{aligned} \tag{3.16}$$

and extend $f_{2,0}$, $f_{1,1}$, and $f_{0,2}$ by linearity and continuity to $\mathcal{H}_1^{\otimes 2}$. Note here that the assumption $f_{0;0} = 0$ is irrelevant for all $f_{m,n}$ except for $f_{1,1}$ which would have an additional term "$-\langle \varphi, \chi \rangle f_{0,0}$". We may sometimes write $f_{m,n}^{[A]}$ to indicate the dependence on A.

Lemma 3.2.3. *For $m, n, m+n \in \{0,1,2\}$ and $\varphi \in \mathcal{H}_1^{\otimes m}, \chi \in \mathcal{H}_1^{\otimes n}$ we have:*

(a) Let $z^{\sharp m}$ denote the m-th tensor power of $z^\sharp$ as introduced in (2.27) and let $\overleftarrow{\varphi}$ denote φ with reversed tensor components in $\mathcal{H}_1^{\otimes m}$, i.e., φ with reversed $\mathcal{K}$-components and order of arguments. Then,

$$f_{m,n}(\varphi; \chi) = \langle \Omega, z^m(\overleftarrow{\varphi})Az^{\dagger n}(\overleftarrow{\chi})\Omega \rangle\,. \tag{3.17}$$

(b) Having $U = U_S(x, \lambda)$ for any $(x, \lambda) \in \mathcal{P}_+^\uparrow$ one has

$$f_{m,n}^{[UAU^{-1}]}(\varphi; \chi) = f_{m,n}^{[A]}((U_1^{-1})^{\otimes m}\varphi; (U_1^{-1})^{\otimes n}\chi).$$

(c)

$$f_{m,n}^{[U(j)A^*U(j)]}(\varphi; \chi) = f_{n,m}^{[A]}((U_1(j))^{\otimes n}\chi; (U_1(j))^{\otimes m}\varphi).$$

(d) Specializing to $m=2, n=0$ or $m=0, n=2$ we have

$$f_{m,n}(\varphi; \chi) = f_{m,n}(S_\leftarrow\varphi; U_1(j)^{\otimes 2}S_\leftarrow U_1(j)^{\otimes 2}\chi),$$

where $S_\leftarrow\varphi(\boldsymbol{\theta}) := S(\theta_2 - \theta_1)\varphi(\theta_2, \theta_1)$.

Proof. (a) is clear by definition. For (b) let $k \in \mathbb{N}_0$ and consider U as given above. Then using Proposition 2.4.2(b) for $\varphi_1, ..., \varphi_k \in \mathcal{D}(\mathbb{R}, \mathcal{K})$ we find

$$\begin{aligned} Uz^{\sharp k}(\varphi_1 \otimes ... \otimes \varphi_k)U^{-1} &= Uz^\sharp(\varphi_1)U^{-1}...Uz^\sharp(\varphi_k)U^{-1} \\ &= z^\sharp(U_1\varphi_1)...z^\sharp(U_1\varphi_k) \\ &= z^{\sharp k}(U_1^{\otimes k}(\varphi_1 \otimes ... \otimes \varphi_k)). \end{aligned} \tag{3.18}$$

Since $U\Omega = \Omega$, confer Proposition 2.4.1, we have that

$$
\begin{aligned}
f_{m,n}^{[UAU^{-1}]}(\varphi;\chi) &= \langle \Omega, z^m(\overleftarrow{\varphi})UAU^{-1}z^{\dagger n}(\overleftarrow{\chi})\Omega\rangle \\
&= \langle U\Omega, z^m(\overleftarrow{\varphi})UAU^{-1}z^{\dagger n}(\overleftarrow{\chi})U\Omega\rangle \\
&= \langle \Omega, U^{-1}z^m(\overleftarrow{\varphi})UAU^{-1}z^{\dagger n}(\overleftarrow{\chi})U\Omega\rangle \\
&= \langle \Omega, z^m((U_1^{-1})^{\otimes m}\,\overleftarrow{\varphi})Az^{\dagger n}((U_1^{-1})^{\otimes n}\,\overleftarrow{\chi})\Omega\rangle \\
&= f_{m,n}^{[A]}((U_1^{-1})^{\otimes m}\varphi;(U_1^{-1})^{\otimes n}\chi).
\end{aligned}
\tag{3.19}
$$

Concerning Item (c) we recall the properties of $U(j)$: Due to Proposition 2.4.1 we have that $U(j) = U(j)^\dagger = U(j)^{-1}$ is antilinear and $U(j)\varphi = U_1(j)^{\otimes m}\,\overleftarrow{\varphi}$. In particular, it follows that $U(j)\Omega = \Omega$. Also, by CPT-invariance (S3), one has that $U(j)\,\mathcal{P}_S = \mathcal{P}_S\,U(j)$ (Rem. 2.4.5). Therefore,

$$
\begin{aligned}
\sqrt{m!}^{-1}U(j)z^{\dagger m}(\overleftarrow{\varphi})\Omega &= U(j)\,\mathcal{P}_S\,\varphi \\
&= \mathcal{P}_S\,U(j)\varphi \\
&= \mathcal{P}_S\,U_1(j)^{\otimes m}\,\overleftarrow{\varphi} \\
&= \sqrt{m!}^{-1}z^{\dagger m}(U_1(j)^{\otimes m}\varphi)\Omega.
\end{aligned}
\tag{3.20}
$$

Applying these properties, we find

$$
\begin{aligned}
f_{m,n}^{[A]}(\varphi;\chi) &= \langle \Omega, z^m(\overleftarrow{\varphi})Az^{\dagger n}(\overleftarrow{\chi})\Omega\rangle \\
&= \langle A^* z^{\dagger m}(\varphi)\Omega, z^{\dagger n}(\overleftarrow{\chi})\Omega\rangle \\
&= \langle U(j)^2A^*U(j)^2z^{\dagger m}(\varphi)\Omega, z^{\dagger n}(\overleftarrow{\chi})\Omega\rangle \\
&= \langle U(j)z^{\dagger n}(\overleftarrow{\chi})\Omega, U(j)A^*U(j)U(j)z^{\dagger m}(\varphi)\Omega\rangle \\
&= \langle z^{\dagger n}(U_1(j)^{\otimes n}\chi)\Omega, U(j)A^*U(j)z^{\dagger m}(U_1(j)^{\otimes m}\,\overleftarrow{\varphi})\Omega\rangle \\
&= \langle \Omega, z^n(U_1(j)^{\otimes n}\,\overleftarrow{\chi})U(j)A^*U(j)z^{\dagger m}(U_1(j)^{\otimes m}\,\overleftarrow{\varphi})\Omega\rangle \\
&= f_{n,m}^{[U(j)A^*U(j)]}(U_1(j)^{\otimes n}\chi;U_1(j)^{\otimes m}\varphi).
\end{aligned}
\tag{3.21}
$$

Item (d) follows from the ZF algebra relations; confer (2.30). We start with the case $m = 2, n = 0$: As a prerequisite, note that using unitarity (S1), hermitian analyticity (S2), and CPT invariance (S3)

$$
\begin{aligned}
S_{\leftarrow}\,\overleftarrow{\varphi}\,(\boldsymbol{\theta}) &= S(\theta_2-\theta_1)\,\overleftarrow{\varphi}\,(\theta_2,\theta_1) \\
&= S(\theta_2-\theta_1)\mathbb{F}\varphi(\theta_1,\theta_2) \\
&= S(\theta_1-\theta_2)^*\mathbb{F}\varphi(\theta_1,\theta_2) \\
&= \mathbb{F}J^{\otimes 2}S(\theta_1-\theta_2)J^{\otimes 2}\varphi(\theta_1,\theta_2) \\
&= \mathbb{F}(U_1(j)^{\otimes 2}S_{\leftarrow}U_1(j)^{\otimes 2}\varphi)(\overleftarrow{\boldsymbol{\theta}}) \\
&= \overleftarrow{U_1(j)^{\otimes 2}S_{\leftarrow}U_1(j)^{\otimes 2}\varphi}\,(\boldsymbol{\theta}).
\end{aligned}
\tag{3.22}
$$

Therefore, by the ZF algebra relation on z and (3.22),

$$\begin{aligned} f_{2,0}(\varphi) &= \langle \Omega, z^2(\overleftarrow{\varphi}) A\Omega \rangle \\ &= \langle \Omega, z^2(U_1(j)^{\otimes 2} U_1(j)^{\otimes 2}\, \overleftarrow{\varphi}) A\Omega \rangle \\ &= \langle \Omega, z^2(U_1(j)^{\otimes 2} S_{\leftarrow} U_1(j)^{\otimes 2}\, \overleftarrow{\varphi}) A\Omega \rangle \\ &= \langle \Omega, z^2(\overleftarrow{S_{\leftarrow}\varphi}) A\Omega \rangle \\ &= f_{2,0}(S_{\leftarrow}\varphi). \end{aligned} \tag{3.23}$$

The case $m = 0, n = 2$ follows by the ZF algebra relation on $z^\dagger$: Using again (3.22) we find

$$\begin{aligned} f_{0,2}(\chi) &= \langle \Omega, A z^{\dagger 2}(\overleftarrow{\chi}) \Omega \rangle \\ &= \langle \Omega, A z^{\dagger 2}(S_{\leftarrow}\, \overleftarrow{\chi}) \Omega \rangle \\ &= \langle \Omega, A z^{\dagger 2}(\overleftarrow{U_1(j)^{\otimes 2} S_{\leftarrow} U_1(j)^{\otimes 2}\chi}) \Omega \rangle \\ &= f_{0,2}(U_1(j)^{\otimes 2} S_{\leftarrow} U_1(j)^{\otimes 2}\chi). \end{aligned} \tag{3.24}$$

□

The next step is to construct the form factors F_1 and F_2 for wedge-localized A. As part of the argument, we will repeatedly use the following lemma:

Lemma 3.2.4. *Assume a model with a regular S-function that has no poles in the physical strip. Let $A \in \mathcal{Q}$ be wedge-local in the right wedge $\mathcal{W}_R$, and fix arbitrary $\Psi_{1/2} \in \mathcal{D}_S$. Define functionals $K, K^\dagger : \mathcal{D}(\mathbb{R}, \mathcal{K}) \to \mathbb{C}$ by*

$$K(\varphi) := \langle \Psi_1, [z(\varphi), A]\Psi_2 \rangle, \quad K^\dagger(\varphi) := \langle \Psi_1, [A, z^\dagger(\varphi)]\Psi_2 \rangle. \tag{3.25}$$

Then there exists an analytic function $\hat{K} : \mathbb{S}(-\pi, 0) \to \mathcal{K}$ with continuous L^2-boundary values such that

$$K(\varphi) = \int d\theta (\varphi(\theta), \hat{K}(\theta)), \quad K^\dagger(\varphi) = \int d\theta (J\varphi(\theta), \hat{K}(\theta - i\pi)). \tag{3.26}$$

Moreover, for arbitrary $k \in \mathbb{N}$ there exists $c_k > 0$ such that for any $0 \leq \lambda \leq \pi$:

$$\|\hat{K}(\cdot + i\lambda)\|_2 \leq c_k \|\Psi_1\|_{2,k} \|\Psi_2\|_{2,k} \|A\|_k. \tag{3.27}$$

Proof. The proof is easily adapted from [AL17, Lemma 4.1]; note the change of sign in $K^\dagger$ and that instead of a bounded operator, we consider a quadratic form (or unbounded operator) which requires a modification in the resulting norms. □

Remark 3.2.5. For Ψ_1, Ψ_2 being zero-particle states, Lemma 3.2.4 holds for arbitrary S-function (bound state poles and regularity are irrelevant in this case).

As the one-particle case (Prop. 3.2.2) is much simpler to prove, we will start with that:

Corollary 3.2.6. *For $A \in \mathcal{Q}$ localized in a right wedge, there is an analytic function $F_1 : \mathbb{S}(-\pi, 0) \to \mathcal{K}$ with continuous L^2-boundary values such that for $\varphi \in \mathcal{H}_1$*

$$f_{1,0}(\varphi) = \int d\theta(\varphi(\theta), F_1(\theta)), \quad f_{0,1}(\varphi) = \int d\theta(J\varphi(\theta), F_1(\theta - i\pi)). \tag{3.28}$$

Proof. For concreteness, take A to be localized in $\mathcal{W}_x$, $x \in \mathbb{M}$. Then by Lemma 3.2.3 for $m = 1, n = 0$ or $m = 0, n = 1$ we have

$$f^{[A]}_{m,n}(\varphi) = f^{[A(-x)]}_{m,n}(U_1(-x)\varphi), \tag{3.29}$$

where $A(-x) := U_S(-x)AU_S(-x)^{-1}$. Since A is localized in the wedge $\mathcal{W}_x$ we have that $A(-x)$ is localized in the (untranslated) right wedge $\mathcal{W}_R$ (Sec. 3.1, paragraph on locality). Because $z(\varphi)\Omega = 0$ it follows that

$$f^{[A]}_{1,0}(\varphi) = \langle \Omega, z(\varphi)A\Omega \rangle = \langle \Omega, [z(\varphi), A]\Omega \rangle \,, \tag{3.30}$$
$$f^{[A]}_{0,1}(\varphi) = \langle \Omega, Az^\dagger(\varphi)\Omega \rangle = \langle \Omega, [A, z^\dagger(\varphi)]\Omega \rangle \,, \tag{3.31}$$

so that Lemma 3.2.4 (using Rem. 3.2.5) is applicable to the functionals $K = f^{[A(-x)]}_{1,0}$ and $K^\dagger = f^{[A(-x)]}_{0,1}$ with $\Psi_1 = \Psi_2 = \Omega$. As a result, one obtains

$$f^{[A(-x)]}_{1,0}(\varphi) = \int d\theta(\varphi(\theta), \hat{K}(\theta)), \quad f^{[A(-x)]}_{0,1}(\varphi) = \int d\theta(J\varphi(\theta), \hat{K}(\theta - i\pi)). \tag{3.32}$$

Using (3.29) it follows that

$$f^{[A]}_{1,0}(\varphi) = \int d\theta(U_1(-x)\varphi(\theta), \hat{K}(\theta)) = \int d\theta(\varphi(\theta), e^{iP(\theta).x}\hat{K}(\theta)) \tag{3.33}$$

and, analogously, using also $U_1(j)U_1(-x) = U_1(x)U_1(j)$,

$$f^{[A]}_{0,1}(\varphi) = \int d\theta(J\varphi(\theta), e^{-iP(\theta).x}\hat{K}(\theta - i\pi)) = \int d\theta(J\varphi(\theta), e^{iP(\theta - i\pi).x}\hat{K}(\theta - i\pi)). \tag{3.34}$$

Recall here that $(U_1(j)\varphi)(\theta) = J\varphi(\theta)$ according to (2.8).

To summarize, we have that $F_1(\zeta) := e^{iP(\zeta).x}\hat{K}(\zeta)$ has the required properties. Note here that the translational factor for $\zeta \in \mathbb{R} + i\pi\mathbb{Z}$ is unitary and thus norm-preserving so that for any $\lambda \in \{-\pi, 0\}$ and some $k \in \mathbb{N}_0$ one has

$$\|F_1(\cdot + i\lambda)\|_2 \le \|\hat{K}(\cdot + i\lambda)\|_2 \le c_k \|\Omega\|^2_{2,k}\|A\|_k = c_k\|A\|_k < \infty. \tag{3.35}$$

□

For now, localization of A was used only for a right wedge, say $\mathcal{W}_x$. However, in case that A is localized in a double cone, say $\mathcal{O}_{x,y}$, it is also localized in the left wedge $\mathcal{W}'_y$ or, equivalently, $U(j)A^*U(j)$ is localized in the right wedge $\mathcal{W}_y$. So, for

$n = 1$, applying Corollary 3.2.6 also to $U(j)A^*U(j)$ we obtain an analytic function $F_1^\dagger : \mathbb{S}(-\pi, 0) \to \mathcal{K}$ with continuous L^2-boundary values such that

$$f_{1,0}^{[U(j)A^*U(j)]}(\varphi) = \int d\theta (\varphi(\theta), F_1^\dagger(\theta)), \quad f_{0,1}^{[U(j)A^*U(j)]}(\varphi) = \int d\theta (J\varphi(\theta), F_1^\dagger(\theta - i\pi)). \tag{3.36}$$

This allows us to conclude the proof of the one-particle case:

Proof of Proposition 3.2.2. To begin with, define

$$F_1^{[A]}(\zeta) := F_1(\zeta), \quad \zeta \in \mathbb{S}(-\pi, 0), \qquad F_1^{[A]}(\zeta) := F_1^\dagger(\zeta - i\pi), \quad \zeta \in \mathbb{S}(0, \pi), \tag{3.37}$$

and

$$F_1^{[A]}(\zeta + i2\pi\mathbb{Z}) := F_1^{[A]}(\zeta), \quad \zeta \in \mathbb{S}(-\pi, 0) \cup \mathbb{S}(0, \pi). \tag{3.38}$$

This defines $F_1^{[A]}$ on $\mathbb{C}$ up to the boundaries $\mathbb{R} + i\pi\mathbb{Z}$. However, the boundary limits all correspond to boundary limits of F_1 and $F_1^\dagger$ on $\mathbb{R}$ or $\mathbb{R} + i\pi$ which exist in terms of a L^2-function due to Corollary 3.2.6. Thus F_1 will extend to all of $\mathbb{C}$ provided that the boundary limits do not depend on the direction of approach. Let $i0$ denote a right-sided limit in $\mathbb{R}$. Then the independent consistency conditions are $F_1^{[A]}(\theta + i0) = F_1^{[A]}(\theta - i0)$ and $F_1^{[A]}(\theta + i\pi - i0) = F_1^{[A]}(\theta + i\pi + i0)$, $\theta \in \mathbb{R}$, and amount to

$$F_1(\theta) = F_1^\dagger(\theta - i\pi), \quad F_1(\theta - i\pi) = F_1^\dagger(\theta), \quad \theta \in \mathbb{R}. \tag{3.39}$$

In view of (3.28) and (3.36), this is equivalent to: For all $\varphi \in \mathcal{H}_1$

$$\begin{aligned} f_{1,0}^{[A]}(\varphi) &= \int d\theta (\varphi(\theta), F_1(\theta)) \\ &= \int d\theta (\varphi(\theta), F_1^\dagger(\theta - i\pi)) \\ &= \int d\theta (J^2\varphi(\theta), F_1^\dagger(\theta - i\pi)) \\ &= \int d\theta (J(U_1(j)\varphi)(\theta), F_1^\dagger(\theta - i\pi)) \\ &= f_{0,1}^{[U(j)A^*U(j)]}(U_1(j)\varphi) \end{aligned} \tag{3.40}$$

and, analogously,

$$f_{0,1}^{[A]}(\varphi) = f_{1,0}^{[U(j)A^*U(j)]}(U_1(j)\varphi). \tag{3.41}$$

which were proven in Item (c) of Lemma 3.2.3.

As a result, $F_1^{[A]}$ extends to an analytic function on all of $\mathbb{C}$ which is $2\pi i$-periodic. The required bounds transfer from F_1 and $F_1^\dagger$ directly to $F_1^{[A]}$. □

As our next step, we treat the two-particle case, i.e., we prove Theorem 3.2.1. Similar to before, we will define $F_2^{[A]}$ on various regions and show that these definitions consistently extend to the full complex plane by matching values at the

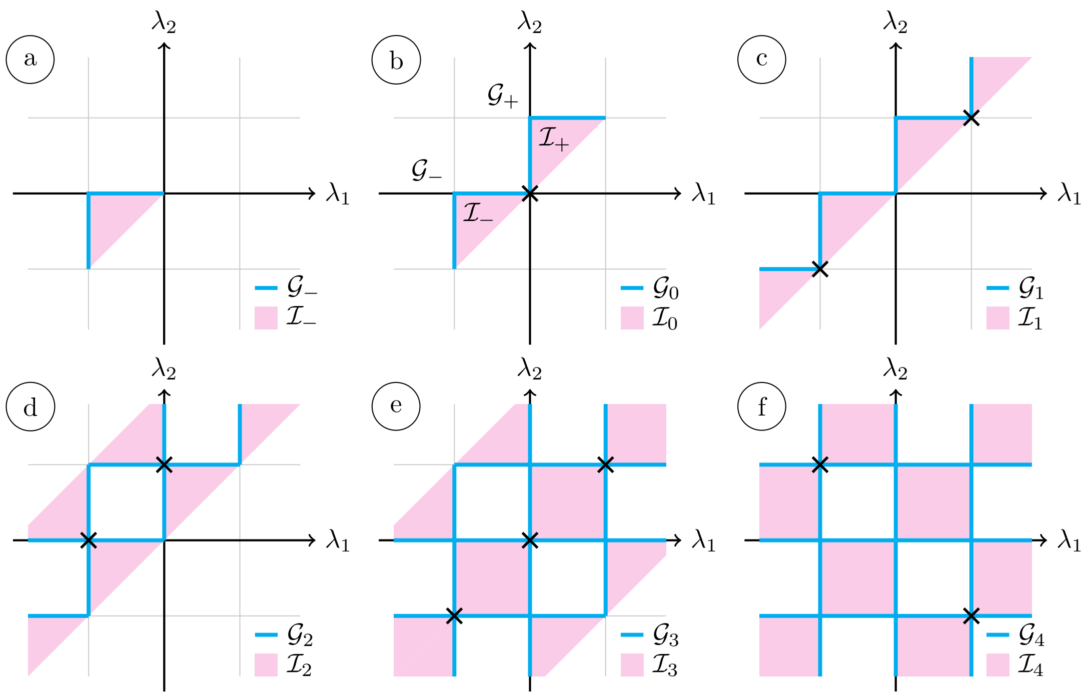

Figure 3.1: This is a schematic depiction of the analytic continuation process used in the proof of Theorem 3.2.1. The thin grey grid has a lattice spacing of π and corresponds to the lattice $\mathbb{L}_2$ as defined in the text. For $i = -, +, 0, 1, 2, 3, 4$ the regions $\mathcal{I}_i$ and $\mathcal{G}_i$ are depicted by magenta contours and blue lines, respectively. The black crosses mark the nodes $\mathcal{B}_i \cap \overline{\mathcal{G}}_{i-1}$ where the consistency conditions on the extension $F_2^{[A]}$ are imposed (for the crosses in Figure b take $\mathcal{G}_{-1} := \mathcal{G}_-$).

boundaries of the respective domains. Since this case is geometrically more involved, we have a schematic depiction of these regions and the continuation process in Figure 3.2. The regions are defined in the following way:

For a subregion $\mathcal{R}$ of $\mathbb{R}^2$, let $\mathcal{R}^{\text{fl}}$ and $\overleftarrow{\mathcal{R}}$ denote the regions obtained by pointwise application of the transformations mapping $\boldsymbol{\lambda} = (\lambda_1, \lambda_2)$ to $\boldsymbol{\lambda}^{\text{fl}} := (\lambda_2 - 2\pi, \lambda_1)$ and $\overleftarrow{\boldsymbol{\lambda}} := (\lambda_2, \lambda_1)$, respectively. Then define

$$
\begin{aligned}
\mathcal{I}_+ &= \{\boldsymbol{\lambda} : 0 < \lambda_1 < \lambda_2 < \pi\}, \\
\mathcal{I}_- &= \{\boldsymbol{\lambda} : -\pi < \lambda_1 < \lambda_2 < 0\}, \\
\mathcal{I}_0 &= \mathcal{I}_+ \cup \mathcal{I}_-, \\
\mathcal{I}_1 &= \mathcal{I}_0 + 2\boldsymbol{\pi}\mathbb{Z}, \\
\mathcal{I}_2 &= \mathcal{I}_1 \cup \mathcal{I}_1^{\text{fl}}, \\
\mathcal{I}_3 &= \mathcal{I}_2 \cup \overleftarrow{\mathcal{I}}_2, \\
\mathcal{I}_4 &= \mathcal{I}_3 + 2\pi(1, -1)\mathbb{Z}.
\end{aligned}
\tag{3.42}
$$

Important for the analysis will be the part of the boundary of these regions which lies on the lattice $\mathbb{L}_2 := \mathbb{R} \times \pi\mathbb{Z} \cup \pi\mathbb{Z} \times \mathbb{R}$:

$$\overline{\mathcal{G}}_i := \partial\mathcal{I}_i \cap \mathbb{L}_2, \quad \mathcal{G}_i := \overline{\mathcal{G}}_i \setminus \pi\mathbb{Z}^2, \quad \mathcal{B}_i := \overline{\mathcal{G}}_i \cap \pi\mathbb{Z}^2, \quad i = +,-,0,1,2,3,4 \tag{3.43}$$

so that $\overline{\mathcal{G}}_i$ denotes the closure of $\mathcal{G}_i$ and is obtained by adding the nodes: $\overline{\mathcal{G}}_i = \mathcal{G}_i \cup \mathcal{B}_i$. The nodes will play a distinguished role as the points where the consistency conditions on the extensions defining $F_2^{[A]}$ will be imposed. Lastly, for any region $\mathcal{R} \subset \mathbb{R}^2$ we introduce the tube region $\mathcal{T}(\mathcal{R}) := \mathbb{R}^2 + i\mathcal{R} \subset \mathbb{C}^2$. Explicitly, we have for instance $\overline{\mathcal{G}}_- = \{-\pi\} \times [-\pi, 0] \cup [-\pi, 0] \times \{0\}$ so that $\mathcal{T}(\overline{\mathcal{G}}_-) = (\mathbb{R} - i\pi) \times \mathbb{S}[-\pi, 0] \cup \mathbb{S}[-\pi, 0] \times \mathbb{R}$ and $\mathcal{B}_- = \{(-\pi, -\pi), (-\pi, 0), (0, 0)\}$.

For the following, we will use repeatedly a result for Cauchy-Riemann(CR) distributions on tubes [BC15, Sec. 3.1]: A *CR distribution* on $\mathcal{T}(\mathcal{G})$, where $\mathcal{G}$ is any of $\mathcal{G}_i$, $i = -,+,0,1,2,3,4$, is a function analytic on the edges of $\mathcal{G}$ and a distribution in the remaining (real) variable. Further, one requires that all the boundary values on the (corresponding) nodes $\mathcal{B}$ exist as distributional boundary values and are independent of the directions of approach within $\mathcal{G}$. Then, we will use repeatedly:

Lemma 3.2.7 (Part of Lemma 3.1 in [BC15])**.** *Let $\mathcal{G}$ be a connected graph, and F a CR distribution on $\mathcal{T}(\mathcal{G})$. Then, F extends to an analytic function on the interior of the convex hull of $\overline{\mathcal{G}}$.*

Note here that $\mathcal{I}_i$ is always contained the interior of the convex hull of $\overline{\mathcal{G}}_i$ but that the latter becomes much larger for increasing i, eventually covering all of $\mathbb{C}^2$ for $i = 4$.

We are now ready, to tackle the two-particle case. Similar to the one-particle case, we connect the distributions $f_{2,0}$, $f_{1,1}$, and $f_{0,2}$ via an analytic function F_2 by applying Lemma 3.2.4:

Lemma 3.2.8. *For $A \in \mathcal{Q}$ localized in a right wedge, there is an analytic function $F_2 : \mathcal{T}(\mathcal{I}_-) \to \mathcal{K}^{\otimes 2}$ with continuous boundary values at $\mathcal{T}(\overline{\mathcal{G}}_-)$ such that for $\varphi, \chi \in \mathcal{H}_1$*

$$f_{2,0}(\varphi \otimes \chi) = \int (\varphi(\theta) \otimes \chi(\eta), F_2(\theta, \eta))_{\mathcal{K}^{\otimes 2}}\, d\theta d\eta, \tag{3.44}$$

$$f_{1,1}(\chi; \varphi) = \int (J\varphi(\theta) \otimes \chi(\eta), F_2(\theta - i\pi, \eta))_{\mathcal{K}^{\otimes 2}}\, d\theta d\eta, \tag{3.45}$$

$$f_{0,2}(\varphi \otimes \chi) = \int (J\varphi(\theta) \otimes J\chi(\eta), F_2(\theta - i\pi, \eta - i\pi))_{\mathcal{K}^{\otimes 2}}\, d\theta d\eta. \tag{3.46}$$

Further, there is a constant $c \geq 0$ such that for large enough $k \in \mathbb{N}_0$ one has

$$\|F_2(\boldsymbol{\zeta})\|_{\mathcal{K}} \leq c\,(|\operatorname{ch}\zeta_1|^k + |\operatorname{ch}\zeta_2|^k)\|A\|_k, \quad \boldsymbol{\zeta} \in \mathcal{T}(\mathcal{B}_-). \tag{3.47}$$

Proof. For simplicity, we prove the statement first for the untranslated wedge $\mathcal{W}_R$ and comment at the end how to prove the statement for translated wedges:

Due to $z(\chi)\Omega = 0$ we find

$$f_{2,0}(\varphi \otimes \chi) = \langle \Omega, z(\chi)z(\varphi)A\Omega\rangle = \langle z^\dagger(\chi)\Omega, [z(\varphi), A]\Omega\rangle\,, \tag{3.48}$$

$$f_{1,1}(\chi;\varphi) = \langle \Omega, z(\chi)Az^\dagger(\varphi)\Omega\rangle = \langle z^\dagger(\chi)\Omega, [A, z^\dagger(\varphi)]\Omega\rangle\,. \tag{3.49}$$

Thus for $\chi \in \mathcal{D}(\mathbb{R},\mathcal{K})$ choosing $\Psi_1 = z^\dagger(\chi)\Omega = \chi$ and $\Psi_2 = \Omega$ the functionals $\varphi \mapsto f_{2,0}(\varphi\otimes\chi)$, resp., $\varphi \mapsto f_{1,1}(\varphi;\chi)$, have the form K, resp., $K^\dagger$, of Lemma 3.2.4. Applying the lemma we obtain an analytic function $\hat{K}_\chi : \mathbb{S}(-\pi,0) \to \mathcal{K}$ such that

$$f_{2,0}(\varphi\otimes\chi) = \int d\theta(\varphi(\theta), \hat{K}_\chi(\theta)), \quad f_{1,1}(\chi;\varphi) = \int d\theta(J\varphi(\theta), \hat{K}_\chi(\theta - i\pi)). \tag{3.50}$$

Then we apply Riesz' representation theorem (note here that for arbitrary $-\pi \leq \lambda \leq 0$ one has $\|\hat{K}_\chi(\cdot + i\lambda)\|_2 \leq c_k\|A\|_k\|\chi\|_{2,k}$ and that $\hat{K}_\chi$ is antilinear in χ). Thus there exists a function $\hat{K} : \mathbb{S}(-\pi,0)\times\mathbb{R} \to \mathcal{K}^{\otimes 2}$ such that

$$f_{2,0}(\varphi\otimes\chi) = \int d\theta d\eta(\varphi(\theta)\otimes\chi(\eta), \hat{K}(\theta,\eta))_{\mathcal{K}^{\otimes 2}}, \tag{3.51}$$

$$f_{1,1}(\chi;\varphi) = \int d\theta d\eta(J\varphi(\theta)\otimes\chi(\eta), \hat{K}(\theta - i\pi,\eta))_{\mathcal{K}^{\otimes 2}}, \tag{3.52}$$

which is analytic in the first variable and which satisfies for all $-\pi \leq \lambda \leq 0$ that

$$\|(\theta,\eta) \mapsto \hat{K}(\theta + i\lambda, \eta)(1 + P^0(\eta))^{-k}\|_2 \leq c_k\|A\|_k. \tag{3.53}$$

Equation (3.51) yields (3.44) and (3.52) yields (3.45) for $F_2(\zeta,\eta) := \hat{K}(\zeta,\eta)$, $\zeta \in \mathbb{S}(-\pi,0)$, $\eta\in\mathbb{R}$.

The analytic continuation in η is obtained analogously: Due to $z(\varphi)\Omega = 0$,

$$f_{1,1}(\chi;\varphi) = \langle \Omega, z(\chi)Az^\dagger(\varphi)\Omega\rangle = \langle \Omega, [z(\chi), A]z^\dagger(\varphi)\Omega\rangle \tag{3.54}$$

$$f_{0,2}(\varphi\otimes\chi) = \langle \Omega, Az^\dagger(\chi)z^\dagger(\varphi)\Omega\rangle = \langle \Omega, [A, z^\dagger(\chi)]z^\dagger(\varphi)\Omega\rangle\,. \tag{3.55}$$

Then for $\varphi \in \mathcal{D}(\mathbb{R},\mathcal{K})$ choosing $\Psi_1 = \Omega$ and $\Psi_2 = z^\dagger(\varphi)\Omega = \varphi$ the functionals $\chi \mapsto f_{1,1}(\chi;\varphi)$, resp., $\chi \mapsto f_{0,2}(\varphi\otimes\chi)$, have the form K, resp., $K^\dagger$, of Lemma 3.2.4. Application of the lemma yields an analytic function $\hat{K}^\dagger_\varphi : \mathbb{S}(-\pi,0) \to \mathcal{K}$ such that

$$f_{1,1}(\chi;\varphi) = \int d\eta(\chi(\eta), \hat{K}^\dagger_\varphi(\eta)), \quad f_{0,2}(\varphi\otimes\chi) = \int d\eta(J\chi(\eta), \hat{K}^\dagger_\varphi(\eta - i\pi)). \tag{3.56}$$

Analogous to before, Riesz' representation theorem applied to $\hat{K}_\varphi$ (with linearity in φ) yields a $\hat{K}^\dagger : \mathbb{R}\times\mathbb{S}(-\pi,0) \to \mathcal{K}^{\otimes 2}$ such that

$$f_{1,1}(\chi;\varphi) = \int d\theta d\eta(J\varphi(\theta)\otimes\chi(\eta), \hat{K}^\dagger(\theta,\eta))_{\mathcal{K}^{\otimes 2}}, \tag{3.57}$$

$$f_{0,2}(\varphi\otimes\chi) = \int d\theta d\eta(J\varphi(\theta)\otimes J\chi(\eta), \hat{K}^\dagger(\theta,\eta - i\pi))_{\mathcal{K}^{\otimes 2}}, \tag{3.58}$$

which is analytic in the second variable and which satisfies for all $-\pi \leq \lambda \leq 0$ that

$$\|(\theta,\eta) \mapsto \hat{K}^\dagger(\theta, \eta + i\lambda)(1 + P^0(\theta))^{-k}\|_2 \leq c_k\|A\|_k. \tag{3.59}$$

Now, we can define $F_2(\theta - i\pi, \zeta) := \hat{K}^\dagger(\theta, \zeta)$, $\zeta \in \mathbb{S}(-\pi, 0)$ so that F_2 extends continuously to $\mathcal{T}(\mathcal{G}_-)$ iff the boundary values of F_2 at $\mathcal{T}(\{(-\pi, 0)\})$ agree. This means that

$$\hat{K}(\theta - i\pi, \eta) = F_2(\theta - i\pi + i0, \eta) = F_2(\theta - i\pi, \eta - i0) = \hat{K}^\dagger(\theta, \eta) \tag{3.60}$$

holds (where $i0$ denotes the right-sided distributional limit within $\mathbb{R}$) and is a consequence of (3.52) and (3.57).

The bounds for F_2 on $\overline{\mathcal{G}}_i$ can be transferred from the bounds on $\hat{K}$ and $\hat{K}^\dagger$ (Eqs. (3.53) and (3.59)): First note that

$$\|(1 + P^0(\theta))\|_{\mathcal{K}} \leq 1 + m_+ \operatorname{ch}\theta \leq (1 + m_+) \operatorname{ch}\theta, \quad m_+ := \max \mathfrak{M}, \tag{3.61}$$

and therefore there must be a constant $c_k' \geq 0$ such that for all $-\pi \leq \lambda \leq 0$ and $\theta, \eta \in \mathbb{R}$

$$\|K(\theta + i\lambda, \eta)\|_{\mathcal{K}} \leq c_k((1 + m_+) \operatorname{ch}\theta)^k \|A\|_k \leq c_k'(\operatorname{ch}\theta)^k \|A\|_k, \tag{3.62}$$

$$\|K^\dagger(\theta, \eta + i\lambda)\|_{\mathcal{K}} \leq c_k((1 + m_+) \operatorname{ch}\theta)^k \|A\|_k \leq c_k'(\operatorname{ch}\theta)^k \|A\|_k. \tag{3.63}$$

concluding with the estimate in (3.47). Lemma 3.2.7 then yields that F_2 extends to an analytic function on $\mathcal{T}(\mathcal{I}_-)$ since $\mathcal{I}_-$ is the interior of the convex hull of $\overline{\mathcal{G}}_-$.

It remains to comment on the case, where A is localized in a translated wedge $\mathcal{W}_x$, $x \in \mathbb{M}$. In this case, $F_2^{[A]}(\boldsymbol{\zeta}) = e^{iP(\boldsymbol{\zeta}).x} F_2^{[A(-x)]}(\boldsymbol{\zeta})$ follows analogously to the proof of Corollary 3.2.6 using also Lemma 3.2.3(b). The additional factor is analytic and does not modify the bounds in (3.62) and (3.63) because for $\boldsymbol{\zeta} \in \mathcal{T}(\mathcal{B}_-)$ one has $|e^{iP(\boldsymbol{\zeta}).x}| = 1$ so that one may draw the same conclusion as before. □

For A being localized in a double cone, the procedure to prove Theorem 3.2.1 is analogous to the case $n = 1$ but more difficult, involving additional extension steps. The regions involved in the consecutive extension steps are illustrated in Figure 3.2.

Now we are ready to prove the main result:

Proof of Thm. 3.2.1. As A and $U(j)A^*U(j)$ are localized in a right wedge, we find, applying Lemma 3.2.4, analytic functions $F_2, F_2^\dagger : \mathcal{T}(\mathcal{I}_-) \to \mathcal{K}^{\otimes 2}$ with continuous boundary values at $\mathcal{T}(\overline{\mathcal{G}}_-)$ such that F_2 satisfies (3.44)–(3.46) and $F_2^\dagger$ the analogous equations, namely, that for all $\varphi, \chi \in \mathcal{H}_1$

$$f_{2,0}^{[U(j)A^*U(j)]}(\varphi \otimes \chi) = \int \Big(\varphi(\theta) \otimes \chi(\eta), F_2^\dagger(\theta, \eta)\Big)_{\mathcal{K}^{\otimes 2}} d\theta d\eta, \tag{3.64}$$

$$f_{1,1}^{[U(j)A^*U(j)]}(\chi; \varphi) = \int \Big(J\varphi(\theta) \otimes \chi(\eta), F_2^\dagger(\theta - i\pi, \eta)\Big)_{\mathcal{K}^{\otimes 2}} d\theta d\eta, \tag{3.65}$$

$$f_{0,2}^{[U(j)A^*U(j)]}(\varphi \otimes \chi) = \int \Big(J\varphi(\theta) \otimes J\chi(\eta), F_2^\dagger(\theta - i\pi, \eta - i\pi)\Big)_{\mathcal{K}^{\otimes 2}} d\theta d\eta. \tag{3.66}$$

Step 0 Then we can define $F_2^{[A]}$ on $\mathcal{T}(\mathcal{I}_0)$ (Fig. 3.2 a→b) in the following way:

$$F_2^{[A]}(\boldsymbol{\zeta}) := F_2(\boldsymbol{\zeta}), \quad \boldsymbol{\zeta} \in \mathcal{T}(\mathcal{I}_-), \qquad F_2^{[A]}(\boldsymbol{\zeta}) := F_2^\dagger(\boldsymbol{\zeta} - i\boldsymbol{\pi}), \quad \boldsymbol{\zeta} \in \mathcal{T}(\mathcal{I}_+) \tag{3.67}$$

which extends continuously to $\mathcal{T}(\mathcal{G}_0)$ since F_2 and $F_2^\dagger$ have continuous boundary values on $\mathcal{T}(\overline{\mathcal{G}}_-)$. For it to extend to $\mathcal{T}(\overline{\mathcal{G}}_0)$ with $\overline{\mathcal{G}}_0 = \overline{\mathcal{G}}_+ \cup \overline{\mathcal{G}}_-$ the limits have to agree at the intersecting boundary $\mathcal{T}(\{(0,0)\})$ independent of the direction of approach. In view of Lemma 3.2.7 it is actually sufficient to check directional independence for approach from within $\mathcal{G}_+$ and $\mathcal{G}_-$. This amounts to

$$F_2^{[A]}(\theta - i0, \eta) = F_2(\theta - i0, \eta) \overset{!}{=} F_2^\dagger(\theta - i\pi, \eta + i0 - i\pi) = F_2^{[A]}(\theta, \eta + i0), \tag{3.68}$$

for all $\theta, \eta \in \mathbb{R}$ and where $i0$ denotes the right-sided distributional boundary limit on $\mathbb{R}$. Due to the bounds on F_2 and $F_2^\dagger$ these limits exist, and we may simply check

$$F_2(\boldsymbol{\theta}) = F_2^\dagger(\boldsymbol{\theta} - i\boldsymbol{\pi}), \quad \boldsymbol{\theta} \in \mathbb{R}^2. \tag{3.69}$$

This relation is equivalent to

$$f_{2,0}^{[A]}(\varphi \otimes \chi) = f_{0,2}^{[U(j)A^*U(j)]}(U_1(j)\varphi \otimes U_1(j)\chi), \quad \varphi, \chi \in \mathcal{H}_1 \tag{3.70}$$

as by (3.44) and (3.66)

$$\begin{aligned} f_{2,0}^{[A]}(\varphi \otimes \chi) &= \int d\boldsymbol{\theta}\, (\varphi(\theta_1) \otimes \chi(\theta_2), F_2(\boldsymbol{\theta}))_{\mathcal{K}^{\otimes 2}} \\ &= \int d\boldsymbol{\theta}\, \left(\varphi(\theta_1) \otimes \chi(\theta_2), F_2^\dagger(\boldsymbol{\theta} - i\boldsymbol{\pi})\right)_{\mathcal{K}^{\otimes 2}} \\ &= \int d\boldsymbol{\theta}\, \left(J^2\varphi(\theta_1) \otimes J^2\chi(\theta_2), F_2^\dagger(\boldsymbol{\theta} - i\boldsymbol{\pi})\right)_{\mathcal{K}^{\otimes 2}} \\ &= f_{0,2}^{[U(j)A^*U(j)]}(U_1(j)\varphi \otimes U_1(j)\chi). \end{aligned} \tag{3.71}$$

This last identity was already proven in Item (c) of Lemma 3.2.3 so that we can conclude with $F_2^{[A]}$ being defined on $\mathcal{T}(\mathcal{I}_0)$ with continuous boundary values on $\mathcal{T}(\overline{\mathcal{G}}_0)$ and bounds of the form (F2.4) for $\boldsymbol{\zeta}$ on that boundary.

The remainder of the proof will consist of a number of similar steps progressively extending $F_2^{[A]}$ to larger and larger regions eventually covering all of $\mathbb{C}^2$. Let us label each step by $i = 1, 2, 3, 4$. Then as an input in each of these steps we start with $F_2^{[A]}$ being well-defined on $\mathcal{T}(\mathcal{I}_{i-1})$ having continuous boundary values. Next, we define $F_2^{[A]}$ on $\mathcal{T}(\mathcal{I}_i)$ and check consistency at the points $\mathcal{T}(\mathcal{B}_i)$ meaning that, using again Lemma 3.2.7, the limit values at these points are independent from the direction of approach within $\mathcal{G}_i$. As a result, we have that $F_2^{[A]}$ is well-defined on $\mathcal{T}(\mathcal{I}_i)$ having continuous boundary values such that the input to start with in the next step is given. Finally, note that often the value of $F_2^{[A]}$ at a point in $\mathcal{T}(\mathcal{B}_i)$ and the values in the point's neighbourhood within $\mathcal{G}_i$ are determined by a corresponding neighbourhood within $\mathcal{G}_{i-1}$ (or at least partially within). As a consequence, it will be actually sufficient to check a much smaller set of consistency conditions. In the following, we will therefore only state the extension definitions, the required new consistency conditions, and their proof.

Step 1 We extend $F_2^{[A]}$ to $\mathcal{T}(\mathcal{I}_1)$ (Fig. 3.2 b→c) by

$$F_2^{[A]}(\boldsymbol{\zeta} + i2\boldsymbol{\pi}\mathbb{Z}) := F_2^{[A]}(\boldsymbol{\zeta}), \quad \boldsymbol{\zeta} \in \mathcal{T}(\mathcal{I}_0). \tag{3.72}$$

It is sufficent to check consistency at $\mathcal{T}(\{(-\pi,-\pi)\})$ which amounts to

$$F_2^\dagger(\boldsymbol{\theta}) = F_2(\boldsymbol{\theta} - i\boldsymbol{\pi}). \tag{3.73}$$

This is equivalent to

$$f_{2,0}^{[U(j)A^*U(j)]}(\varphi \otimes \chi) = f_{0,2}^{[A]}(U_1(j)\varphi \otimes U_1(j)\chi) \tag{3.74}$$

as can be seen by replacing $A \leftrightarrow U(j)A^*U(j)$ and $F_2 \leftrightarrow F_2^\dagger$ in (3.71). This relation, as the one in Step 0, was shown in Lemma 3.2.3(c).

Step 2 We extend $F_2^{[A]}$ to $\mathcal{T}(\mathcal{I}_2)$ (Fig. 3.2 c→ d) by

$$F_2^{[A]}(\zeta_2 - i2\pi, \zeta_1) := \mathbb{F} F_2^{[A]}(\boldsymbol{\zeta}), \quad \boldsymbol{\zeta} \in \mathcal{T}(\mathcal{I}_1). \tag{3.75}$$

It is sufficient to check consistency at $\mathcal{T}(\{(-\pi,0)\})$ which amounts to

$$F_2(\theta_1 - i\pi, \theta_2) = \mathbb{F} F_2^\dagger(\theta_2 - i\pi, \theta_1). \tag{3.76}$$

This is equivalent to $f_{1,1}^{[A]}(\chi;\varphi) = f_{1,1}^{[U(j)A^*U(j)]}(U_1(j)\chi; U_1(j)\varphi)$ for all $\varphi, \chi \in \mathcal{H}_1$ as by (3.45) and (3.65)

$$\begin{aligned}
f_{1,1}^{[A]}(\chi;\varphi) &= \int d\theta d\eta\, (J\varphi(\theta) \otimes \chi(\eta), F_2(\theta - i\pi, \eta))_{\mathcal{K}^{\otimes 2}} \\
&= \int d\theta d\eta\, \Big(J\varphi(\theta) \otimes \chi(\eta), \mathbb{F} F_2^\dagger(\eta - i\pi, \theta)\Big)_{\mathcal{K}^{\otimes 2}} \\
&= \int d\theta d\eta\, \Big(\chi(\eta) \otimes J\varphi(\theta), F_2^\dagger(\eta - i\pi, \theta)\Big)_{\mathcal{K}^{\otimes 2}} \\
&= \int d\theta d\eta\, \Big(\chi(\theta) \otimes J\varphi(\eta), F_2^\dagger(\theta - i\pi, \eta)\Big)_{\mathcal{K}^{\otimes 2}} \\
&= \int d\theta d\eta\, \Big(J(U_1(j)\chi)(\theta) \otimes (U_1(j)\varphi)(\eta), F_2^\dagger(\theta - i\pi, \eta)\Big)_{\mathcal{K}^{\otimes 2}} \\
&= f_{1,1}^{U(j)A^*U(j)]}(U_1(j)\chi; U_1(j)\varphi).
\end{aligned} \tag{3.77}$$

Also this relation was proven in Item (c) of Lemma 3.2.3.

Step 3 To continue $F_2^{[A]}$ to $\mathcal{T}(\mathcal{I}_3)$ (Figure 3.2 d→ e) we define

$$F_2^{[A]}(\overleftarrow{\boldsymbol{\zeta}}) := S(\zeta_1 - \zeta_2) F_2^{[A]}(\boldsymbol{\zeta}), \quad \boldsymbol{\zeta} \in \mathcal{T}(\mathcal{I}_2). \tag{3.78}$$

Note here that this extension is analytic, even at the boundaries: By assumption S has no poles within the physical strip $\mathbb{S}(0,\pi)$ and by regularity of S (Defn. 2.3.2) even not in the slightly larger strip $\mathbb{S}(-\kappa, \pi+\kappa)$ for any $0 < \kappa < \kappa(S)$. Then for

$\boldsymbol{\zeta} \in \mathcal{T}(\overline{\mathcal{I}}_2)$ one has $0 \leq \mathrm{Im}(\zeta_1 - \zeta_2) \leq \pi$ such that $S(\zeta_1 - \zeta_2)$ is analytic in that region.

It is sufficient to check consistency at $\mathcal{T}(\{(0,0),(-\pi,-\pi)\})$ which amounts to

$$F_2(\boldsymbol{\theta}) = S(\theta_2 - \theta_1)F_2(\overleftarrow{\boldsymbol{\theta}}), \quad F_2(\boldsymbol{\theta} - i\pi) = S(\theta_2 - \theta_1)F_2(\overleftarrow{\boldsymbol{\theta}} - i\pi), \tag{3.79}$$

and analogous relations for $F_2^\dagger$. Equation (3.79) is equivalent to

$$f_{2,0}(\varphi \otimes \chi) = f_{2,0}(S_\leftarrow(\varphi \otimes \chi)) \quad \text{and} \quad f_{0,2}(\varphi \otimes \chi) = f_{0,2}(U_1(j)^{\otimes 2} S_\leftarrow U_1(j)^{\otimes 2}(\varphi \otimes \chi)) \tag{3.80}$$

due to

$$\begin{aligned} f_{2,0}(\varphi \otimes \chi) &= \int d\boldsymbol{\theta}\, (\varphi(\theta_1) \otimes \chi(\theta_2), F_2(\boldsymbol{\theta}))_{\mathcal{K}^{\otimes 2}} \\ &= \int d\boldsymbol{\theta}\, (\varphi(\theta_1) \otimes \chi(\theta_2), S(\theta_2 - \theta_1)F_2(\overleftarrow{\boldsymbol{\theta}}))_{\mathcal{K}^{\otimes 2}} \\ &= \int d\boldsymbol{\theta}\, (S(\theta_2 - \theta_1)^\dagger(\varphi(\theta_1) \otimes \chi(\theta_2)), F_2(\overleftarrow{\boldsymbol{\theta}}))_{\mathcal{K}^{\otimes 2}} \\ &= \int d\boldsymbol{\theta}\, (S(\theta_1 - \theta_2)^\dagger(\varphi(\theta_2) \otimes \chi(\theta_1)), F_2(\boldsymbol{\theta}))_{\mathcal{K}^{\otimes 2}} \\ &= \int d\boldsymbol{\theta}\, (S(\theta_2 - \theta_1)(\varphi(\theta_2) \otimes \chi(\theta_1)), F_2(\boldsymbol{\theta}))_{\mathcal{K}^{\otimes 2}} \\ &= \int d\boldsymbol{\theta}\, ((S_\leftarrow(\varphi \otimes \chi))(\boldsymbol{\theta}), F_2(\boldsymbol{\theta}))_{\mathcal{K}^{\otimes 2}} \\ &= f_{2,0}(S_\leftarrow(\varphi \otimes \chi)) \end{aligned} \tag{3.81}$$

and

$$\begin{aligned} f_{0,2}(\varphi \otimes \chi) &= \int d\boldsymbol{\theta}\, (J\varphi(\theta_1) \otimes J\chi(\theta_2), F_2(\boldsymbol{\theta} - i\boldsymbol{\pi}))_{\mathcal{K}^{\otimes 2}} \\ &= \int d\boldsymbol{\theta}\, (J\varphi(\theta_1) \otimes J\chi(\theta_2), S(\theta_2 - \theta_1)F_2(\overleftarrow{\boldsymbol{\theta}} - i\boldsymbol{\pi}))_{\mathcal{K}^{\otimes 2}} \\ &= \int d\boldsymbol{\theta}\, (S(\theta_2 - \theta_1)(J\varphi(\theta_2) \otimes J\chi(\theta_1)), F_2(\boldsymbol{\theta} - i\boldsymbol{\pi}))_{\mathcal{K}^{\otimes 2}} \\ &= \int d\boldsymbol{\theta}\, (J^{\otimes 2}J^{\otimes 2}S(\theta_2 - \theta_1)J^{\otimes 2}(\varphi(\theta_2) \otimes \chi(\theta_1)), F_2(\boldsymbol{\theta} - i\boldsymbol{\pi}))_{\mathcal{K}^{\otimes 2}} \\ &= \int d\boldsymbol{\theta}\, (J^{\otimes 2}(U_1(j)^{\otimes 2}S_\leftarrow U_1(j)^{\otimes 2}(\varphi \otimes \chi))(\boldsymbol{\theta}), F_2(\boldsymbol{\theta} - i\boldsymbol{\pi}))_{\mathcal{K}^{\otimes 2}} \\ &= f_{2,0}(U_1(j)^{\otimes 2}S_\leftarrow U_1(j)^{\otimes 2}(\varphi \otimes \chi)). \end{aligned} \tag{3.82}$$

Also, we can replace F_2 with $F_2^\dagger$, however, without obtaining new relations. The equations in (3.80) were proven in Item (d) of Lemma 3.2.3.

Step 4 We obtain $\mathcal{I}_4$ from $\mathcal{I}_3$ by applying arbitrary shifts $2\pi\mathbb{Z}(1,-1)^t$ (Figure 3.2 e→f). Thus for all $n \in \mathbb{N}$ and $\boldsymbol{\zeta} \in \mathcal{T}(\mathcal{I}_3)$ let us define

$$F_2^{[A]}(\boldsymbol{\zeta} + i2\pi n(0,1)) := \prod_{j=1}^{n}(\mathbb{F}S(\zeta_1 - \zeta_2 - i2\pi(n-j)))F_2^{[A]}(\boldsymbol{\zeta}), \tag{3.83}$$

$$F_2^{[A]}(\boldsymbol{\zeta} - i2\pi n(0,1)) := \prod_{j=1}^{n}(S(\zeta_2 - \zeta_1 - i2\pi(n-j))\mathbb{F})F_2^{[A]}(\boldsymbol{\zeta}), \tag{3.84}$$

where the order of the product goes as $\prod_{j=1}^{n} c_j := c_1 \cdot \ldots \cdot c_n$. This extends $F_2^{[A]}$ to $\mathcal{T}(\mathcal{I}_4)$ and automatically implements consistency at the boundaries as it implies (straightforwardly by induction on n) that

$$F_2^{[A]}(\boldsymbol{\zeta} + i2\pi(0,1)) = \mathbb{F}S(\zeta_1 - \zeta_2)F_2^{[A]}(\boldsymbol{\zeta}), \quad \boldsymbol{\zeta} \in \mathcal{T}(\overline{\mathcal{G}}_4). \tag{3.85}$$

Note however, that the extension (3.83) in general hits poles of S whenever $-\pi < \mathrm{Im}(\zeta_1 - \zeta_2) < 0$ and analogously for (3.84) whenever $0 < \mathrm{Im}(\zeta_1 - \zeta_2) < \pi$. Note here that $S(\zeta + i2\pi n)$ for arbitrary $n \in \mathbb{Z}$ has a similar pole structure as $S(\zeta)$: By hermitian analyticity (S2) and crossing symmetry (S5) we have that

$$S(\zeta + i\pi n) = (S(\zeta + i\pi(n+1))^{\mathrm{cr}})^{-1}, \tag{3.86}$$

where $(u_1 \otimes u_2, O^{\mathrm{cr}} v_1 \otimes v_2)_{\mathcal{K}^{\otimes 2}} := (Jv_1 \otimes u_1, O\, v_2 \otimes Ju_2)_{\mathcal{K}^{\otimes 2}}$. Thus, $S(\zeta + i2\pi n)$ is singular when $S(\zeta)$ is, only that the residues might have a different tensor structure. As a result, for $\boldsymbol{\zeta} \in \mathcal{T}(\overline{\mathcal{I}}_4)$ analyticity holds whenever $\mathrm{Im}\,\zeta_2 + 2\pi n > \mathrm{Im}\,\zeta_1 > \mathrm{Im}\,\zeta_2 - \pi + 2\pi n$ or $\mathrm{Im}\,\zeta_1 + 2\pi n > \mathrm{Im}\,\zeta_2 > \mathrm{Im}\,\zeta_1 - \pi + 2\pi n$ or slightly enlarged regions by regularity.

Step 5 Finally, to use Lemma 3.2.7 again, we need to restrict $F_2^{[A]}$ to the connected component of the analyticity region established in Step 4. This evaluates to $\boldsymbol{\zeta} \in \mathcal{T}(\mathcal{G}_4)$ with

$$|\,\mathrm{Im}\,\zeta_1 - \zeta_2| < 2\pi + \kappa \tag{3.87}$$

for any $\kappa < \kappa(S)$ with $\kappa(S)$ being the maximal extension of the regular strip; confer Definition 2.3.2. Applying the lemma we obtain $F_2^{[A]}$ for any $\boldsymbol{\zeta} \in \mathbb{C}^2$ satisfying (3.87). We may then use (3.83) and (3.84), now for arbitrary $\boldsymbol{\zeta}$, to define $F_2^{[A]}$ on all of $\mathbb{C}^2$ as a meromorphic function. This conludes the proof of (F2.1).

Equations (3.75) and (3.78) are compatible with (3.83) and (3.84) and extend by meromorphy to $\mathbb{C}^2$ implying Items (F2.2) and (F2.3). Since all the operations applied to $F_2^{[A]}(\boldsymbol{\zeta})$ in the extension steps—confer (3.67), (3.72), (3.75), (3.78), (3.83), (3.84)—are bounded for $\boldsymbol{\zeta} \in \mathcal{T}(\mathbb{L}_2)$, the bounds for $\mathcal{T}(\mathcal{B}_-)$ (3.47) transfer to that region implying (F2.4). □

3.3 Transformation properties of the form factors

In many cases it is interesting to know how transformations of an operator A reflect on the form factors. E.g., imposing invariance of A under a certain transformation should yield additional constraints at the level of form factors. In this section we will briefly argue how the one- and two-particle form factor transform under conjugation $A \mapsto A^*$, gauge symmetry $A \mapsto V_S(g)AV_S(g)^{-1}$, $g \in \mathcal{G}$, and discrete transformations

$A \mapsto U(k)AU(k)^{-1}$, where $k \in \{c, p, t, ..., j = cpt\}$. For the sake of completeness we include also proper orthochronous Poincaré transformations which were partially treated in the preceding section.

The main result is

Proposition 3.3.1. *Let $A \in \mathcal{Q}$ be localized in a double cone, $n \in \{1,2\}$, and $F_n^{[A]}$ be the n-particle form factor with respect to A as constructed in Proposition 3.2.2 or Theorem 3.2.1. Then we have for all $\boldsymbol{\zeta} \in \mathbb{C}^n$ that*

(a) $F_n^{[A^*]}(\boldsymbol{\zeta}) = \mathbb{F}_n J^{\otimes n} F_n^{[A]}(\overleftarrow{\boldsymbol{\zeta}} + i\boldsymbol{\pi})$, *where* $\mathbb{F}_1 = \mathbb{1}$ *and* $\mathbb{F}_2 = \mathbb{F}$.

(b) $F_n^{[U_S(x,\lambda)AU_S(x,\lambda)^{-1}]}(\boldsymbol{\zeta}) = e^{iP(\boldsymbol{\zeta}).x} F_n^{[A]}(\boldsymbol{\zeta} - \lambda\mathbf{1})$ *for all* $(x, \lambda) \in \mathcal{P}_+^\uparrow$

(c) $F_n^{[U(k)AU(k)^{-1}]}(\boldsymbol{\zeta}) = U(k)_n F_n^{[A]}(\boldsymbol{\zeta})$ *for all* $k \in \{c, p, cp\}$ *and*
$F_n^{[U(k)AU(k)^{-1}]}(\boldsymbol{\zeta}) = U(k)_n F_n^{[A]}(\bar{\boldsymbol{\zeta}})$ *for all* $k \in \{t, ct, pt, j = cpt\}$

(d) $F_n^{[V_S(g)AV_S(g)^{-1}]}(\boldsymbol{\zeta}) = V(g)^{\otimes n} F_n^{[A]}(\boldsymbol{\zeta})$ *for all* $g \in \mathcal{G}$.

These transformation properties will be inferred by the transformation properties of the distributions $f_{m,n}$, which were introduced in (3.14)–(3.16). These properties then transfer to $F_{m+n}^{[A]}(\boldsymbol{\zeta})$ via (3.28) and (3.44)–(3.46), however, with arguments restricted to the respective boundaries, i.e., $\boldsymbol{\zeta} \in \mathbb{R} \cup \mathbb{R} + i\pi$ for $m + n = 1$ and $\boldsymbol{\zeta} \in \mathcal{T}(\overline{\mathcal{G}}_-)$ for $m+n = 2$. Following the extension procedures employed in the proofs of Proposition 3.2.2 and Theorem 3.2.1 these properties then extend to similar ones being valid on all of $\mathbb{C}^{m+n}$.

First, we need the transformation behaviour of $f_{m,n}$:

Lemma 3.3.2. *Let $A \in \mathcal{Q}$ be arbitrary, $m, n, m + n \in \{0, 1, 2\}$, and $f_{m,n}$ be as defined in (3.17). Then we have for all $\varphi \in \mathcal{H}_{S,m}, \chi \in \mathcal{H}_{S,n}$ that*

(a) $f_{m,n}^{[A^*]}(\varphi; \chi) = \overline{f_{n,m}^{[A]}(\overleftarrow{\chi}; \overleftarrow{\varphi})}$,

(b) $f_{m,n}^{[U_S(x,\lambda)AU_S(x,\lambda)^{-1}]}(\varphi; \chi) = f_{m,n}^{[A]}((U_1(x, \lambda)^{-1})^{\otimes m}\varphi; (U_1(x, \lambda)^{-1})^{\otimes n}\chi)$
for all $(x, \lambda) \in \mathcal{P}_+^\uparrow$,

(c) $f_{m,n}^{[U(k)AU(k)^{-1}]}(\varphi; \chi) = f_{m,n}^{[A]}(U(k)^{-1}\varphi; U(k)^{-1}\chi)$ *for all* $k \in \{c, p, cp\}$ *and*
$f_{m,n}^{[U(k)AU(k)^{-1}]}(\varphi; \chi) = \overline{f_{m,n}^{[A]}(U(k)^{-1}\varphi; U(k)^{-1}\chi)}$ *for all* $k \in \{t, ct, pt, j = cpt\}$,
where the cases $m = 2$ or $n = 2$ require that S is k-invariant (Defn. 2.3.2).

(d) $f_{m,n}^{[V_S(g)AV_S(g)^{-1}]}(\varphi; \chi) = f_{m,n}^{[A]}(V_1(g)^{-1}\varphi; V_1(g)^{-1}\chi)$ *for all* $g \in \mathcal{G}$.

Proof. (a):

$$
\begin{aligned}
f_{m,n}^{[A^*]}(\varphi;\chi) &= \langle \Omega, z^m(\overleftarrow{\varphi}) A^* z^{\dagger n}(\overleftarrow{\chi})\Omega\rangle \\
&= \langle z^{\dagger m}(\varphi)\Omega, A^* z^{\dagger n}(\overleftarrow{\chi})\Omega\rangle \\
&= \langle A z^{\dagger m}(\varphi)\Omega, z^{\dagger n}(\overleftarrow{\chi})\Omega\rangle \\
&= \langle z^n(\chi) A z^{\dagger m}(\varphi)\Omega, \Omega\rangle \\
&= \overline{\langle \Omega, z^n(\chi) A z^{\dagger m}(\varphi)\Omega\rangle} \\
&= \overline{f_{n,m}^{[A]}(\overleftarrow{\chi};\overleftarrow{\varphi})}.
\end{aligned}
\tag{3.88}
$$

(b) was already proven in Item (b) of Lemma 3.2.3 and (d) follows in exactly the same way by replacing U with $V_S(g)$. It remains to show (c): For $U(k)$ linear we have that

$$
\begin{aligned}
f_{m,n}^{[U(k)AU(k)^{-1}]}(\varphi;\chi) &= \langle \Omega, z^m(\overleftarrow{\varphi}) U(k)AU(k)^{-1} z^{\dagger n}(\overleftarrow{\chi})\Omega\rangle \\
&= \langle U(k)^{-1} z^{\dagger m}(\varphi)\Omega, AU(k)^{-1} z^{\dagger n}(\overleftarrow{\chi})\Omega\rangle \\
&= \langle z^{\dagger m}(U(k)^{-1}\varphi)\Omega, A z^{\dagger n}(U(k)^{-1}\,\overleftarrow{\chi})\Omega\rangle \\
&= \langle \Omega, z^{m}(U(k)^{-1}\,\overleftarrow{\varphi}\; A z^{\dagger n}(U(k)^{-1}\,\overleftarrow{\chi})\Omega\rangle \\
&= f_{m,n}^{[A]}(U(k)^{-1}\varphi; U(k)^{-1}\chi).
\end{aligned}
\tag{3.89}
$$

Note here that we used

$$
\sqrt{n!}^{-1} U(k)^{-1} z^{\dagger n}(\chi)\Omega = U(k)^{-1}\,\mathcal{P}_{S,n}\,\overleftarrow{\chi} = \mathcal{P}_{S,n}\,U(k)^{-1}\,\overleftarrow{\chi} = \sqrt{n!}^{-1} z^{\dagger n}(U(k)^{-1}\chi)\Omega.
\tag{3.90}
$$

in the third equality of (3.89); this holds provided that $[U(k), \mathcal{P}_{S,n}] = 0$ which is valid for k-invariant S due to Remark 2.4.5.

For antilinear $U(k)$ the computation runs analogously only that an additional complex conjugation has to appear. □

Lemma 3.3.3. *Let $A \in \mathcal{Q}$ be localized in a wedge. Then for $m, n, m+n \in \{0,1,2\}$ and $\boldsymbol{\theta} \in \mathbb{R}^m, \boldsymbol{\eta} \in \mathbb{R}^n$ we have that*

(a) $F_{m+n}^{[A^*]}(\boldsymbol{\eta} - i\boldsymbol{\pi}, \boldsymbol{\theta}) = J^{\otimes m+n}\mathbb{F}_{m+n}\mathbb{F}_m\mathbb{F}_n F_{m+n}^{[A]}(\overleftarrow{\boldsymbol{\theta}} - i\boldsymbol{\pi}, \overleftarrow{\boldsymbol{\eta}})$, where $\mathbb{F}_{0/1} = \mathbb{1}$ and $\mathbb{F}_2 = \mathbb{F}$.

(b) $F_{m+n}^{[U_S(x,\lambda)AU_S(x,\lambda)^{-1}]}(\boldsymbol{\eta} - i\boldsymbol{\pi}, \boldsymbol{\theta}) = e^{i(P(\boldsymbol{\theta})-P(\boldsymbol{\eta})).x} F_{m+n}^{[A]}(\boldsymbol{\eta} - i\boldsymbol{\pi} - \lambda\mathbf{1}, \boldsymbol{\theta} - \lambda\mathbf{1})$ for all $(x,\lambda) \in \mathcal{P}_+^\uparrow$,

(c) $F_{m+n}^{[U(k)AU(k)^{-1}]}(\boldsymbol{\eta} - i\boldsymbol{\pi}, \boldsymbol{\theta}) = ((J^{\otimes n}U(k)_n J^{\otimes n} \otimes U(k)_m)F_{m+n}^{[A]})(\boldsymbol{\eta} - i\boldsymbol{\pi}, \boldsymbol{\theta})$ for all $k \in \{c, p, t, cp, ct, pt, cpt\}$, where the cases $m = 2$ or $n = 2$ require that S is k-invariant (Defn. 2.3.2).

(d) $F_{m+n}^{[V_S(g)AV_S(g)^{-1}]}(\boldsymbol{\eta} - i\boldsymbol{\pi}, \boldsymbol{\theta}) = V(g)^{\otimes m+n} F_{m+n}^{[A]}(\boldsymbol{\eta} - i\boldsymbol{\pi}, \boldsymbol{\theta})$ for all $g \in \mathcal{G}$.

Proof. Equations (3.28), (3.44)–(3.46) which connect $f_{m,n}^{[A]}$ and $F_{m+n}^{[A]}$ can be summarized as: For all $\varphi \in \mathcal{H}_{S,m}, \chi \in \mathcal{H}_{S,n}$,

$$f_{m,n}^{[A]}(\varphi;\chi) = \int d^m\boldsymbol{\theta} d^n\boldsymbol{\eta} (J^{\otimes n}\chi(\boldsymbol{\eta}) \otimes \varphi(\boldsymbol{\theta}), F_{m+n}^{[A]}(\boldsymbol{\eta} - i\boldsymbol{\pi}, \boldsymbol{\theta}))_{\mathcal{K}^{\otimes m+n}}. \tag{3.91}$$

Now it is straightforward to derive Items (a)-(d). We do it exemplarically for Items (a) and (b): If A is localized in a wedge, so are A^* and $U_S(x,\lambda)AU_S(x,\lambda)^{-1}$. In the following let $\varphi \in \mathcal{H}_{S,m}, \chi \in \mathcal{H}_{S,n}$ be arbitrary. Item (a) of Lemma 3.3.2 yields $f_{m,n}^{[A^*]}(\varphi;\chi) = \overline{f_{n,m}^{[A]}(\overleftarrow{\chi};\overleftarrow{\varphi})}$ for all φ,χ as above which is equivalent to

$$F_{m+n}^{[A^*]}(\boldsymbol{\theta} - i\boldsymbol{\pi}, \boldsymbol{\eta}) = J^{\otimes m+n}\mathbb{F}_{m+n}\mathbb{F}_m\mathbb{F}_n F_{m+n}^{[A]}(\boldsymbol{\eta} - i\boldsymbol{\pi}, \boldsymbol{\theta})$$

as

$$\begin{aligned}
f_{m,n}^{[A^*]}(\varphi;\chi) &= \int d\boldsymbol{\theta} d\boldsymbol{\eta} (J^{\otimes n}\chi(\boldsymbol{\eta}) \otimes \varphi(\boldsymbol{\theta}), F_{m+n}^{[A^*]}(\boldsymbol{\eta} - i\boldsymbol{\pi}, \boldsymbol{\theta}))_{\mathcal{K}^{\otimes m+n}} \\
&\stackrel{!}{=} \int d\boldsymbol{\theta} d\boldsymbol{\eta} (J^{\otimes n}\chi(\boldsymbol{\eta}) \otimes \varphi(\boldsymbol{\theta}), J^{\otimes m+n}\mathbb{F}_{m+n}\mathbb{F}_m\mathbb{F}_n F_{m+n}^{[A]}(\overleftarrow{\boldsymbol{\theta}} - i\boldsymbol{\pi}, \overleftarrow{\boldsymbol{\eta}}))_{\mathcal{K}^{\otimes m+n}} \\
&= \int d\boldsymbol{\theta} d\boldsymbol{\eta} (\overleftarrow{\varphi}(\boldsymbol{\theta}) \otimes J^{\otimes n}\overleftarrow{\chi}(\boldsymbol{\eta}), J^{\otimes m+n} F_{m+n}^{[A]}(\boldsymbol{\theta} - i\boldsymbol{\pi}, \boldsymbol{\eta}))_{\mathcal{K}^{\otimes m+n}} \\
&= \int d\boldsymbol{\theta} d\boldsymbol{\eta} \overline{(J^{\otimes m}\overleftarrow{\varphi}(\boldsymbol{\theta}) \otimes \overleftarrow{\chi}(\boldsymbol{\eta}), F_{m+n}^{[A]}(\boldsymbol{\theta} - i\boldsymbol{\pi}, \boldsymbol{\eta}))_{\mathcal{K}^{\otimes m+n}}} \\
&= \overline{f_{n,m}^{[A]}(\overleftarrow{\chi},\overleftarrow{\varphi})}.
\end{aligned} \tag{3.92}$$

Concerning (b) we have due to Lemma 3.3.2(b) that

$$f_{m,n}^{[U_S(x,\lambda)AU_S(x,\lambda)^{-1}]}(\varphi;\chi) = f_{m,n}^{[A]}((U_1(x,\lambda)^{-1})^{\otimes m}\varphi; (U_1(x,\lambda)^{-1})^{\otimes n}\chi) \tag{3.93}$$

for all φ,χ as above which is equivalent to

$$F_{m+n}^{[U_S(x,\lambda)AU_S(x,\lambda)^{-1}]}(\boldsymbol{\eta} - i\boldsymbol{\pi}, \boldsymbol{\theta}) = e^{i(P(\boldsymbol{\theta})-P(\boldsymbol{\eta})).x} F_{m+n}^{[A]}(\boldsymbol{\eta} - i\boldsymbol{\pi} - \lambda\mathbf{1}, \boldsymbol{\theta} - \lambda\mathbf{1}) \tag{3.94}$$

as

$$\begin{aligned}
&f_{m,n}^{[U_S(x,\lambda)AU_S(x,\lambda)^{-1}]}(\varphi;\chi) \\
&= \int d\boldsymbol{\theta} d\boldsymbol{\eta} (J^{\otimes n}\chi(\boldsymbol{\eta}) \otimes \varphi(\boldsymbol{\eta}), F_{m+n}^{[U_S(x,\lambda)AU_S(x,\lambda)^{-1}]}(\boldsymbol{\eta} - i\boldsymbol{\pi}, \boldsymbol{\theta}))_{\mathcal{K}^{\otimes m+n}} \\
&\stackrel{!}{=} \int d\boldsymbol{\theta} d\boldsymbol{\eta} (J^{\otimes n}\chi(\boldsymbol{\eta}) \otimes \varphi(\boldsymbol{\theta}), (e^{i(P(\boldsymbol{\theta})-P(\boldsymbol{\eta})).x} F_{m+n}^{[A]}(\boldsymbol{\eta} - i\boldsymbol{\pi} - \lambda\mathbf{1}, \boldsymbol{\theta} - \lambda\mathbf{1}))_{\mathcal{K}^{\otimes m+n}} \\
&= \int d\boldsymbol{\theta} d\boldsymbol{\eta} (J^{\otimes n}\chi(\boldsymbol{\eta}) \otimes \varphi(\boldsymbol{\theta}), ((U_1(-x,\lambda)^{\otimes n} \otimes U_1(x,\lambda)^{\otimes m})F_{m+n}^{[A]})(\boldsymbol{\eta} - i\boldsymbol{\pi}, \boldsymbol{\theta}))_{\mathcal{K}^{\otimes m+n}} \\
&= \int d\boldsymbol{\theta} d\boldsymbol{\eta} (J^{\otimes n}(U_1(x,\lambda)^{-1})^{\otimes n}\chi(\boldsymbol{\eta}) \otimes (U_1(x,\lambda)^{-1})^{\otimes m}\varphi(\boldsymbol{\theta}), F_{m+n}^{[A]}(\boldsymbol{\eta} - i\boldsymbol{\pi}, \boldsymbol{\theta}))_{\mathcal{K}^{\otimes m+n}} \\
&= f_{m,n}^{[A]}((U_1(x,\lambda)^{-1})^{\otimes m}\varphi; (U_1(x,\lambda)^{-1})^{\otimes n}\chi).
\end{aligned} \tag{3.95}$$

For Item (d) we also use that $[J, V(g)] = 0$ (Defn. 2.2.1). ∎

The relations proven in Lemma 3.3.3 can be extended to $\mathbb{S}[0,\pi]$ for $m+n=1$ and to $\mathcal{T}(\overline{\mathcal{I}}_-)$ for $m+n=2$ as presented before in Corollary 3.2.6 and Lemma 3.2.8, respectively. If A is localized in a double cone we can even extend them to the whole of $\mathbb{C}^{m+n}$ as before in (the proofs of) Proposition 3.2.2 and Theorem 3.2.1, respectively. To follow this whole procedure again would be intricate and little inspiring. Thus we will here at the cost of leaving a little gap of faith for the reader not present a full proof but simply state that the relations given in Proposition 3.3.1 agree with those in Lemma 3.3.3 when restricted to $\mathbb{R}\cup\mathbb{R}+i\pi$ $(m+n=1)$ or $\mathcal{T}(\overline{\mathcal{G}}_-)$ $(m+n=2)$.

3.3.1 Form factors of invariant operators and derivatives

It is our aim now to state very specific conditions on the one- and two-particle form factor of A implied by its invariance or covariance under some transformation and also to relate the form factor of the derivative of A. The results will be used in the following chapter when A is given as the smeared stress-energy tensor.

Proposition 3.3.4. *Let A be localized in a double cone and $F_n = F_n^{[A]}$ its n-particle form factor for $n = 1, 2$. Then*

(a) If A is hermitian, i.e, $A = A^*$, then

$$F_1(\zeta) = JF_1(\bar{\zeta} + i\pi), \tag{3.96}$$
$$F_2(\boldsymbol{\zeta}) = J^{\otimes 2}\mathbb{F}F_2(\overleftarrow{\bar{\boldsymbol{\zeta}}} + i\boldsymbol{\pi}). \tag{3.97}$$

(b) Let a family of such A be denoted by $A^{\boldsymbol{\mu}}, \boldsymbol{\mu} = \mu_1 ... \mu_k, k \in \mathbb{N}_0$ and transform as a Lorentz k-tensor, i.e., $U_S(0,\lambda)A^{\boldsymbol{\mu}}U_S(0,\lambda)^{-1} = (\Lambda(\lambda)^{\otimes k}A)^{\boldsymbol{\mu}}$, then for all $\lambda \in \mathbb{C}$,

$$(\Lambda(\lambda)^{\otimes k}F_1(\zeta))^{\boldsymbol{\mu}} = F_1^{\boldsymbol{\mu}}(\zeta - \lambda), \tag{3.98}$$
$$(\Lambda(\lambda)^{\otimes k}F_2(\boldsymbol{\zeta}))^{\boldsymbol{\mu}} = F_2^{\boldsymbol{\mu}}(\boldsymbol{\zeta} - \lambda\mathbf{1}). \tag{3.99}$$

In particular, if $k = 0$ then F_1 is a constant and F_2 depends only on the difference $\zeta_1 - \zeta_2$.

(c) If A is CPT-invariant, i.e., $A = U(j)AU(j)$, then

$$F_1(\zeta) = JF_1(\bar{\zeta}), \tag{3.100}$$
$$F_2(\boldsymbol{\zeta}) = J^{\otimes 2}\mathbb{F}F_2(\overleftarrow{\bar{\boldsymbol{\zeta}}}). \tag{3.101}$$

(d) *If A is $\mathcal{G}$-invariant, i.e., $A = V_S(g)AV_S(g)^{-1}$ for $g \in \mathcal{G}$ then we have that*

$$F_1(\zeta) = V(g)F_1(\zeta), \tag{3.102}$$

$$F_2(\boldsymbol{\zeta}) = V(g)^{\otimes 2}F_2(\boldsymbol{\zeta}). \tag{3.103}$$

(e) *If A is C-,P-, or T-invariant, i.e., $A = U(k)AU(k)^{-1}$ for $k = c, p, t$ then*

$$C\colon F_1(\zeta) = CF_1(\zeta), \quad P\colon F_1(\zeta) = F_1(-\zeta), \quad T\colon F_1(\zeta) = \overline{F_1(-\overline{\zeta})}, \tag{3.104}$$

$$C\colon F_2(\boldsymbol{\zeta}) = C^{\otimes 2}F_2(\boldsymbol{\zeta}), \quad P\colon F_2(\boldsymbol{\zeta}) = \mathbb{F}F_2(-\overleftarrow{\boldsymbol{\zeta}}), \quad T\colon F_2(\boldsymbol{\zeta}) = \overline{F_2(-\overline{\boldsymbol{\zeta}})}, \tag{3.105}$$

where C is the charge conjugation matrix and $C\overline{F} = JF$.

Proof. The proof is straightforwardly implied by the relations given in Proposition 3.3.1. For Item (b) we note that the relation extends to complex λ by analyticity of F_1 and F_2 in the respective regions and uniqueness of the analytic continuation. For Item (e) we also employ the implementations of the representations for the discrete symmetries $U_n(k)$ which were studied in Section A.2 and set the phase factors to one. □

Proposition 3.3.5. *Let A be localized in a double cone and $F_n^{[A]}$ its n-particle form factor for $n = 1, 2$. If A is weakly differentiable*[a] *on $\mathcal{D}_S \times \mathcal{D}_S$ with respect to $U_S(x, 0)$ then for all $\boldsymbol{\zeta} \in \mathbb{C}^n$ we have that*

$$F_n^{[\partial_\mu A]}(\boldsymbol{\zeta}) = -iP_\mu(\boldsymbol{\zeta})F_n^{[A]}(\boldsymbol{\zeta}), \quad \mu = 0, 1. \tag{3.106}$$

[a] This means that for $x \in \mathbb{M}$ the limit $s \to 0$ on $s^{-1}(U_S(sx, 0)AU_S(sx, 0)^{-1} - A)$ exists weakly, i.e., in matrix elements for a class of states.

Proof. Let $e_\mu, \mu = 0, 1$ denote the standard basis of $\mathbb{M}$. Then

$$\begin{aligned} F_n^{[\partial_\mu A]}(\boldsymbol{\zeta}) &= \lim_{s\to 0} F_n^{[s^{-1}(U_S(se_\mu,0)AU_S(se_\mu,0)^{-1}-A)]}(\boldsymbol{\zeta}) \\ &= \lim_{s\to 0} s^{-1}\left(F_n^{[U_S(se_\mu,0)AU_S(se_\mu,0)^{-1}]}(\boldsymbol{\zeta}) - F_n^{[A]}(\boldsymbol{\zeta})\right) \\ &= \lim_{s\to 0} s^{-1}\left(e^{-iP(\boldsymbol{\zeta}).se_\mu} - 1\right) F_n^{[A]}(\boldsymbol{\zeta}) \\ &= \lim_{s\to 0} s^{-1}\left(e^{-isP_\mu(\boldsymbol{\zeta})} - 1\right) F_n^{[A]}(\boldsymbol{\zeta}) \\ &= -iP_\mu(\boldsymbol{\zeta})F_n^{[A]}(\boldsymbol{\zeta}). \end{aligned} \tag{3.107}$$

□

Chapter 4

Structure of form factors and the minimal solution

In this chapter we will classify the structure of the two-particle form factors in more detail (Sec. 4.1). The structure classification consists of the eigendecomposition of the S-function and for each S-function eigenvalue of a well-known factorization of the form factors into a polynomial, characterizing the observable, times a model-dependent factor which is independent of the observable. The model-dependent factor consists of bound state factors and the so-called *minimal solution*. The bound state factors are fixed by the poles of the S-function eigenvalue within the physical strip.

We supplement the factorization result by establishing existence of the minimal solution for a large class of S-functions (Sec. 4.2). As a byproduct, but of crucial importance for later, we obtain an estimate for the asymptotic growth of the minimal solution. This will play a role in Chapters 7 and 8 to decide the validity of QEIs in one-particle states in generic models. The results are based on a well-known integral transform which represents the minimal solution in terms of the so-called *characteristic function*. To conclude this chapter, we provide a concrete procedure to obtain the characteristic function (Sec. 4.3).

4.1 Classification of two-particle form factors

As the starting point of this section, we will introduce the minimal solution more thoroughly and derive some of its properties including uniqueness for a given S-function eigenvalue. The minimal solution is a well-known concept in the form factor program which plays an essential role in the description of the observables of the model [KW78]. It is uniquely defined as the "most regular" solution to Watson's equations with respect to a given S-function eigenvalue. Watson's equations are the complex-valued two-particle form factor equations of a scalar observable. We make this more precise below. The mathematical treatment is basically taken from [BC16] adding slight generalizations to S-functions with bound state poles. For a less rigorous but informative treatment we refer to [KW78].

After that, we prove the decomposition of the form factor into an operator-dependent and a model-dependent part for S-functions with constant eigenprojectors. This factorization is well known: A treatment of the scalar case (no eigendecomposition necessary) appears for instance in [KW78; FMS93; BC16]. More general cases are often stated without a derivation (e.g., [Del09; BFK10]) and usually refer to a diagonal S-function, where the eigendecomposition is given and non-degenerate due to diagonality; confer Definition 2.3.2.

The minimal solution and its properties To begin with, we intend to analyze eigenvalues s of some matrix-valued S-function; thus s will denote a $\mathbb{C}$-valued function from now on. Central to the section is:

Lemma 4.1.1. *Let $s : \mathbb{C} \to \mathbb{C}$ be a meromorphic function with no poles on the real line. Then there exists at most one meromorphic function $f : \mathbb{C} \to \mathbb{C}$ such that*

(a) *f has no poles and no zeroes in $\mathbb{S}[0, \pi]$, except for a first-order zero at 0 in case that $s(0) = -1$,*

(b) $\exists a, b, r > 0\, \forall |\operatorname{Re}\zeta| \geq r, \operatorname{Im}\zeta \in [0, \pi] : \quad |\log|f(\zeta)|| \leq a + b|\operatorname{Re}\zeta|,$

(c) $f(i\pi + \zeta) = f(i\pi - \zeta),$

(d) $f(\zeta) = s(\zeta)f(-\zeta),$

(e) $f(i\pi) = 1.$

If such a function exists, we will refer to it as *the minimal solution* $f_{s,\min}$ with respect to s. Due to (d), a necessary condition for existence is the relation $s(-\zeta) = s(\zeta)^{-1}$ for all $\zeta \in \mathbb{C}$.

Proof of Lemma 4.1.1. Assume that there are two functions f_A, f_B with the stated properties. Then the meromorphic function $g(\zeta) := f_A(\zeta)/f_B(\zeta)$ has neither poles nor zeroes in $\mathbb{S}[0, 2\pi]$ and satisfies $g(\zeta) = g(-\zeta) = g(\zeta + 2\pi i)$; this implies that $q := g \circ \mathrm{ch}^{-1}$ is well-defined and entire. The asymptotic estimates (b) for $|\log|f_{A/B}||$ imply an analogous estimate for $|\log|g|| = |\log|f_A| - \log|f_B||$ by the triangle inequality. Thus q is polynomially bounded at infinity and therefore a polynomial. However, since q does not have zeroes, it must be a constant with $q = q(-1) = 1$ due to $g(i\pi) = 1$. Hence $f_A = f_B$. □

Corollary 4.1.2. *If in addition $\overline{s(\bar{\zeta})} = s(\zeta)^{-1}, \zeta \in \mathbb{C}$, then it holds that*

$$f_{\min}(\zeta) = \overline{f_{\min}(-\bar{\zeta})}. \tag{4.1}$$

Proof. Since $\overline{s(-\bar{\zeta})} = s(\zeta)$ it is clear that $\zeta \mapsto \overline{f_{\min}(-\bar{\zeta})}$ satisfies the same properties (a)–(e) as $f_{\min}$. By uniqueness they have to be equal. □

Corollary 4.1.3. *For $n \in \mathbb{N}$ let $s_1, ..., s_n : \mathbb{C} \to \mathbb{C}$ be meromorphic functions such that their minimal solutions $f_{j,\min}$ exist. Then the minimal solution with respect to $\zeta \mapsto s_\Pi(\zeta) = \prod_{j=1}^n s_j(\zeta)$ exists and is given by*

$$\zeta \mapsto f_{\Pi,\min}(\zeta) = (-i \operatorname{sh} \tfrac{\zeta}{2})^{-2\lfloor \frac{k}{2} \rfloor} \prod_{j=1}^{n} f_{j,\min}(\zeta), \tag{4.2}$$

where $k = |\{j = 1, ..., n : s_j(0) = -1\}|$.

Proof. One easily checks that (4.2) satisfies conditions (b)–(e) of Lemma 4.1.1 with respect to s_Π. Also, counting the order of zeroes at 0 on the r.h.s. yields $k - 2\lfloor \frac{k}{2} \rfloor$, which evaluates to 1 for odd k (when $s_\Pi(0) = -1$) and to 0 otherwise (when $s_\Pi(0) = +1$), thus establishing condition (a). □

We now apply these results to classify "non-minimal" solutions, having more zeroes or poles than allowed by condition (a):

Lemma 4.1.4. *Let $f : \mathbb{C} \to \mathbb{C}$ be a meromorphic function which satisfies properties (c)-(e) of Lemma 4.1.1 with respect to some meromorphic function s, and suppose*

$$\exists a, b, r > 0\, \forall |\operatorname{Re}\zeta| \geq r, \operatorname{Im}\zeta \in [0, \pi] : \quad |f(\zeta)| \leq a \exp b|\operatorname{Re}\zeta|. \tag{4.3}$$

Assume further that the minimal solution $f_{s,\min}$ with respect to s exists. Then there is a unique rational function q with $q(-1) = 1$ such that

$$f(\zeta) = q(\operatorname{ch}\zeta) f_{s,\min}(\zeta). \tag{4.4}$$

In particular, if f has no poles in $S[0, \pi]$, then q is a polynomial.

Proof. Since the pole set of the meromorphic function f has no finite accumulation points, and its intersection with $\mathbb{S}[0, 2\pi]$ must be located in a compact set due to (c) and the estimate (4.3), this intersection must be finite. Now, define $\zeta \mapsto g(\zeta) := f(\zeta)/f_{s,\min}(\zeta)$ which satisfies $g(\zeta) = g(-\zeta) = g(\zeta + 2\pi i)$. Then, analogous to the proof of Lemma 4.1.1, there exists a meromorphic function $q = g \circ \operatorname{ch}^{-1}$ which is polynomially bounded at infinity and has finitely many poles. Thus it is a rational function. Lastly, note that $q(-1) = 1$ due to $g(i\pi) = 1$. □

Decomposition of the two-particle form factor For simplicity we treat a Lorentz-invariant observable $A \in \mathcal{Q}$ such that its two-particle form factor $F_2^{[A]}$ can be identified as $F_2^{[A]}(\zeta_1, \zeta_2) = F(\zeta_1 - \zeta_2)$ with some function $F : \mathbb{C} \to \mathcal{K}^{\otimes 2}$ according to Proposition 3.3.4(b). Moreover, we make two simplifying assumptions: First, that F is flip-invariant, i.e., $\mathbb{F}F(\zeta) = F(\zeta)$ for all $\zeta \in \mathbb{C}$ which is a consequence of A being parity-invariant and holds due to Proposition 3.3.4(e). Second, that the S-function S has constant eigenprojectors, i.e., where the eigendecomposition (Prop. 2.3.4) extends to the whole of $\mathbb{C}$ with the eigenprojectors P_i independent of the rapidity. To the best of the author's knowledge this assumption is satisfied in most physically relevant models but it holds at least for constant and diagonal S-functions and many other typical examples (Rem. 2.3.5). Note that we include bound states in our analysis and therefore supplement the form factor equations at two-particle level proven in Theorem 3.2.1 by the relations for the bound state residues (F1b)

$$\underset{\zeta_2=\zeta_1+i\theta^k_{(ij)}}{\mathrm{res}} F_2(\zeta_1,\zeta_2) = \tfrac{1}{\sqrt{2\pi}}\Gamma^{ij}_k F_1(\zeta_1 - i\theta^k_{ij}) \tag{4.5}$$

for all $ij \to k \in \mathfrak{F}$ with fusion angle $0 < \theta^k_{(ij)} < \pi$ and bound state intertwiner Γ^{ij}_k (Appendix A.5). For Lorentz-invariant A with F as specified above, we thus obtain

$$\underset{\zeta=i\theta^k_{(ij)}}{\mathrm{res}} F(\zeta) = u^{ij}_k, \tag{4.6}$$

where $u^{ij}_k \in \mathcal{K}$ is a constant vector given by $\frac{1}{\sqrt{2\pi}}\Gamma^{ij}_k F_1$.

In this setup, we are able to decompose $F(\zeta)$ into separate eigenspaces $P_i\mathcal{K}^{\otimes 2}$ and in each of them F factorizes into a rational function Q in $\mathrm{ch}\,\zeta$ times the minimal solution $f_{i,\mathrm{min}}$ with respect to the S-function's eigenvalue s_i. Since Q is rational, in a basis, it decomposes into a fraction of two polynomials. The requirements of (4.6) fix the poles of Q and thus the form of the denominator. As a result, the only piece of F which contains information on A is the polynomial in the numerator.

Proposition 4.1.5 (Decomposition of F). *Assume an S-function S which has constant eigenprojectors $P_i, i \in \{1,...,k\}$ and is parity-invariant, i.e., $[S,\mathbb{F}] = 0$. Let further $F : \mathbb{C} \to \mathcal{K}^{\otimes 2}$ denote a meromorphic function with no poles on the real line and which satisfies*

$$F(\zeta) = S(\zeta)F(-\zeta), \quad F(\zeta+i\pi) = F(-\zeta+i\pi), \quad F(\zeta) = \mathbb{F}F(-\zeta) \quad F(i\pi) = I_{\otimes 2},$$

and that there exists $a,b,r > 0$ such that for all $|\mathrm{Re}\,\zeta| \geq r$ and $\mathrm{Im}\,\zeta \in [0,\pi]$ one has $\|F(\zeta)\| \leq a\exp(b|\mathrm{Re}\,\zeta|)$. Provided that the minimal solutions with respect to the eigenvalues of S exist, then F is of the form

$$F(\zeta) = \sum_{i=1}^{k} Q_i(\mathrm{ch}\,\zeta) f_{i,\mathrm{min}}(\zeta), \tag{4.7}$$

where $x \mapsto Q_i(x) \in P_i\mathcal{K}^{\otimes 2}$ is a rational polynomial function with $\sum_{i=1}^k Q_i(-1) = I_{\otimes 2}$ and $\mathbb{F}Q_i(x) = Q_i(x)$ for all $x \in \mathbb{C}$.

Proof. Define $F_i(\zeta) := P_iF(\zeta)$. The components of the F_i in some basis satisfy the properties of a non-minimal solution as treated in Lemma 4.1.4: First, note that

$F_i(\zeta) = s_i(\zeta)F_i(-\zeta)$ as

$$\begin{aligned} F_i(\zeta) &= P_i F(\zeta) \\ &= P_i S(\zeta) F(-\zeta) \\ &= s_i(\zeta) P_i F(-\zeta) \\ &= s_i(\zeta) F_i(-\zeta). \end{aligned} \tag{4.8}$$

Second, due to $\|P_i\| \leq 1$, for some $a, b, r > 0$ we have that

$$\forall |\operatorname{Re}\zeta| \geq r, \operatorname{Im}\zeta \in [0,\pi]: \quad \|F_i(\zeta)\| \leq \|F(\zeta)\| \leq a \exp(b|\operatorname{Re}\zeta|). \tag{4.9}$$

Third, using that $[P_i, \mathbb{F}] = 0$ due to $[S, \mathbb{F}] = 0$, we have

$$\begin{aligned} F_i(\zeta + i\pi) &= P_i F(\zeta + i\pi) \\ &= P_i F(-\zeta + i\pi) \\ &= F_i(-\zeta + i\pi). \end{aligned} \tag{4.10}$$

Choosing an ONB $\{e_\alpha\}$ of $\mathcal{K}$ we obtain that $F_i^{\alpha\beta}$ is $\mathbb{C}$-valued and satisfies the conditions of Lemma 4.1.4 with respect to s_i for each α, β. Thus $F_i^{\alpha\beta}(\zeta) = q_i^{\alpha\beta}(\operatorname{ch}\zeta) f_{i,\min}(\zeta)$ with rational polynomial functions $q_i^{\alpha\beta}$.

Defining $Q_i(x) = q_i^{\alpha\beta}(x)\, e_\alpha \otimes e_\beta$ one obtains

$$F(\zeta) = \sum_{i=1}^{k} F_i(\zeta) = \sum_{i=1}^{k} Q_i(\operatorname{ch}\zeta) f_{i,\min}(\zeta). \tag{4.11}$$

For $\zeta = i\pi$, $F(i\pi) = I_{\otimes 2}$ and $f_{i,\min}(i\pi) = 1$ imply that $\sum_{i=1}^{k} Q_i(-1) = I_{\otimes 2}$. Note that by construction $q_i^{\alpha\beta} = q_i^{\beta\alpha}$ so that $\mathbb{F} Q_i(x) = Q_i(x)$ for all $x \in \mathbb{C}$. $\square$

4.2 Existence of the minimal solutions and asymptotic growth

In this section we establish the existence of a common integral representation of the minimal solution for a large class of (eigenvalues of) regular S-functions, namely those satisfying the hypothesis of Theorem 4.2.1 below. As a byproduct, but of crucial importance for later, we obtain an explicit formula for the asymptotic growth of the minimal solution (Proposition 4.2.6). The integral transform is well-known and has been employed before in many concrete models, e.g., sinh-Gordon [FMS93], $SU(n)$-Gross-Neveu [BFK10], and $O(n)$-nonlinear sigma model [BFK13]. Also the existence of the integral representations was argued before in [KW78], but without giving explicit assumptions. General results on the asymptotic growth of the minimal solution, based on this integral representation, are new to the best of the authors knowledge.

For $\mathbb{C}$-valued functions s and f, the integral expressions of interest are formally given by

$$f[s](t) := \frac{i}{\pi} \int_0^\infty s'(\theta)s(\theta)^{-1} \cos(\pi^{-1}\theta t)d\theta, \tag{4.12}$$

$$s_f(\zeta) := \exp\left(-2i \int_0^\infty f(t) \sin\frac{\zeta t}{\pi} \frac{dt}{t}\right), \tag{4.13}$$

$$m_f(\zeta) := \exp\left(2 \int_0^\infty f(t) \sin^2 \frac{(i\pi - \zeta)t}{2\pi} \frac{dt}{t \operatorname{sh} t}\right); \tag{4.14}$$

we will give conditions for their well-definedness below. $f[s]$ will be referred to as the *characteristic function*[1] with respect to s. For a large class of functions s, in particular having in mind eigenvalues of S-functions, the functions $s_{f[s]}$ and $m_{f[s]}$ will agree with s and $f_{s,\min}$ respectively.

For the following let us agree to call a function f on $\mathbb{R}$ *exponentially decaying* iff

$$\exists a, b, r > 0 \,\forall |t| \geq r : \quad |f(t)| \leq a \exp(-b|t|); \tag{4.15}$$

analogously for functions on $[0, \infty)$. A function f on a strip $\mathbb{S}(-\epsilon, \epsilon)$ will be called *uniformly* L^1 if $f(\cdot + i\lambda) \in L^1(\mathbb{R})$ for every $\lambda \in (-\epsilon, \epsilon)$, with the L^1 norm uniformly bounded in λ. A function $s : \mathbb{C} \to \mathbb{C}$ will be called *regular* iff it is analytic and bounded in a strip $\mathbb{S}(-\epsilon, \epsilon)$ for some $\epsilon > 0$. This agrees with the notion of regularity of a scalar S-function (Defn. 2.3.2).

Now, we are ready to state the main result:

Theorem 4.2.1. *Let $s : \mathbb{C} \to \mathbb{C}$ be a meromorphic function with no poles on the real line, satisfying $s(\zeta)^{-1} = s(-\zeta)$, and regularity. Suppose that $r_s(\zeta) := is'(\zeta)/s(\zeta)$ is uniformly L^1 on some strip $\mathbb{S}(-\epsilon, \epsilon)$. Then $f[s] \in C([0, \infty), \mathbb{R})$ is exponentially decaying. If further $f[s] \in C^2([0, \delta))$ for some $\delta > 0$, then the minimal solution with respect to s exists.*

In more detail, under these assumptions $s_{f[s]}$ and $m_{f[s]}$ are well-defined, non-vanishing and analytic on $\mathbb{S}(-\epsilon, \epsilon)$ and $\mathbb{S}(-\epsilon, 2\pi + \epsilon)$, respectively. The meromorphic continuations of $s_{f[s]}$ and $m_{f[s]}$ to all of $\mathbb{C}$ exist. In case that $s(0) = 1$, we have $s_{f[s]} = s$ and $m_{f[s]} = f_{s,\min}$. In case that $s(0) = -1$, we have $s_{f[s]} = -s$ and $m_{f[s]} = f_{-s,\min}$; and $f_{s,\min}(\zeta) = (-i \operatorname{sh} \frac{\zeta}{2}) f_{-s,\min}(\zeta)$.

The examples treated in Chapter 8 fulfill this condition: In particular, for products of S-functions of sinh-Gordon type (Eq. (8.1)), r_s is exponentially decaying (on

[1] Differing conventions for $f[s]$ are found in the literature. In the form factor programm community, one mostly takes $2f[s]$ as the characteristic function: Compare formulas (4.13)–(4.14) with, e.g., [FMS93, Eq. (4.10)–(4.11)] or [KW78, Eq. (2.18)–(2.19)], but noting a typo in Eq. (2.19) there.

a strip) and $f[s]$ is actually smooth on $[0, \infty)$ (Eq. (8.3)). For the eigenvalues of the S-function of the $O(n)$-NLS model, $s \in \{s_0, s_+, s_-\}$, we find $r_s(\theta + i\lambda) \lesssim \theta^{-2}$, $|\theta| \to \infty$, uniformly in $\lambda \in [-\epsilon, \epsilon]$ for some $\epsilon > 0$ and again, that $f[s]$ is smooth on $[0, \infty)$ (cf. Section 4.3).

We give the proof of the theorem in several steps. To begin with, we have:

Lemma 4.2.2. *Let r_s be uniformly L^1 on a strip $\mathbb{S}(-\epsilon, \epsilon)$. Then $f[s] : \mathbb{R} \to \mathbb{R}$ is an even, bounded, and continuous function which decays exponentially.*

Proof. Since r_s restricted to $\mathbb{R}$ is L^1-integrable, its Fourier transform $\widetilde{r_s}$ is bounded, continuous and vanishes towards infinity. As r_s is even,

$$r_s(-\theta) = is'(-\theta)s(\theta) = -i(s(\theta)^{-1})'s(\theta) = is'(\theta)s(\theta)^{-1} = r_s(\theta), \tag{4.16}$$

also $\widetilde{r_s}$ is even and we have

$$f[s](t) = \tfrac{i}{\pi} \int_0^\infty r_s(\theta) \cos(\pi^{-1}\theta t) dt = \tfrac{i}{2\pi} \int r_s(\theta) e^{i\pi^{-1}\theta t} dt = \tfrac{i}{2\pi} \widetilde{r_s}(t). \tag{4.17}$$

Let now $0 < \lambda < \epsilon$ be arbitrary. By assumption, $r_s(\cdot + i\lambda)$ is L^1-integrable as well; and by the translation property of the Fourier transform,

$$\widetilde{r_s}(t) = e^{-\lambda t} \tfrac{1}{2\pi} \widetilde{r_s(\cdot + i\lambda)}(t), \tag{4.18}$$

where $\widetilde{r_s(\cdot + i\lambda)}(t)$ vanishes for $|t| \to \infty$ due to the Riemann-Lebesgue lemma. Thus $\widetilde{r_s}$ is exponentially decaying towards $+\infty$, and since it is even also towards $-\infty$. □

We continue by establishing some basic properties of s_f and m_f including existence:

Lemma 4.2.3. *Let $f \in C([0, \infty), \mathbb{R})$ be exponentially decaying. Then the functions s_f and m_f are well-defined by the integral expressions (4.13) and (4.14). Further they are non-vanishing and analytic on $\mathbb{S}(-\epsilon, \epsilon)$ and $\mathbb{S}(-\epsilon, 2\pi + \epsilon)$, respectively, for some $\epsilon > 0$.*

Proof. By assumption there exist positive constants $a, r, \epsilon > 0$ such that $|f(t)| < a \exp(-\epsilon\pi^{-1}t)$ for all $t \geq r$. By the triangle inequality, $|\sin \pi^{-1}\zeta t| = \frac{1}{2}|e^{i\pi^{-1}\zeta t} + e^{-i\pi^{-1}\zeta t}| \leq \exp(\pi^{-1}|\operatorname{Im}\zeta|t)$ for all $t \geq 0$. Thus for $|\operatorname{Im}\zeta| < \epsilon$ one has that $f(t)t^{-1}\sin(\pi^{-1}\zeta t)$ is exponentially decaying. Also, this function is continuous (including at $t = 0$ because of the first-order zero of the sine function at zero). By similar arguments, the same holds for its derivative with respect to ζ. In particular, $t \mapsto f(t)t^{-1}\sin(\pi^{-1}\zeta t)$ and its ζ-derivative are absolutely integrable for all $\zeta \in \mathbb{S}(-\epsilon, \epsilon)$. As a consequence, s_f is well-defined and analytic on $\mathbb{S}(-\epsilon, \epsilon)$. Since s_f is given by an exponential, it does not vanish.

The argument for m_f runs analogously. The estimate from above gives

$$|\sin^2(2\pi)^{-1}(i\pi-\zeta)t| \leq \exp(\pi^{-1}|\operatorname{Im}(i\pi-\zeta)|t)$$

for all $t \geq 0$. Thus

$$t \mapsto f(t)(t\operatorname{sh} t)^{-1}\sin^2((2\pi)^{-1}(i\pi-\zeta)t)$$

is exponentially decaying for $|\operatorname{Im}\zeta-\pi| < \pi+\epsilon$. It is further continuous (including at $t=0$ because of the second-order zero of the sine-function at zero). Together with similar properties of the ζ-derivative, it follows that m_f is well-defined, analytic, and non-vanishing in the region $\mathbb{S}(-\epsilon, 2\pi+\epsilon)$. □

Lemma 4.2.4. *Let $f \in C([0,\infty),\mathbb{R})$ be exponentially decaying. Then $s_f(0) = 1$ and $m_f(i\pi) = 1$. Moreover, there exists $\epsilon > 0$ such that for all $\zeta \in \mathbb{S}(-\epsilon,\epsilon)$,*

$$s_f(\zeta)^{-1} = s_f(-\zeta) \tag{4.19}$$

and

$$m_f(\zeta) = s_f(\zeta)m_f(-\zeta), \quad m_f(i\pi+\zeta) = m_f(i\pi-\zeta). \tag{4.20}$$

Proof. $s_f(0) = 1$ and $m_f(i\pi) = 1$ is immediate by definition. Next, take $\zeta \in \mathbb{S}(-\epsilon,\epsilon)$ with an ϵ from Lemma 4.2.3. Then

$$-\sin(\pi^{-1}\zeta t) = \sin(\pi^{-1}(-\zeta)t) \tag{4.21}$$

implies that $s_f(\zeta)^{-1} = s_f(-\zeta)$. Similarly,

$$\sin^2\frac{\zeta t}{2\pi} = \sin^2\frac{-\zeta t}{2\pi} \tag{4.22}$$

implies that $m_f(i\pi+\zeta) = m_f(i\pi-\zeta)$. Lastly, the relation

$$\sin^2\frac{(i\pi-\zeta)t}{2\pi} - \sin^2\frac{(i\pi+\zeta)t}{2\pi} = -i\operatorname{sh}(t)\sin\frac{\zeta t}{\pi} \tag{4.23}$$

implies that

$$\begin{aligned}\log\frac{m_f(\zeta)}{m_f(-\zeta)} &= 2\int_0^\infty f(t)\left(\sin^2\frac{(i\pi-\zeta)t}{2\pi} - \sin^2\frac{(i\pi+\zeta)t}{2\pi}\right)\frac{dt}{t\operatorname{sh} t}\\ &= -2i\int_0^\infty f(t)\sin(\pi^{-1}\zeta t)\frac{dt}{t} = \log s_f(\zeta),\end{aligned} \tag{4.24}$$

which concludes the proof. □

Corollary 4.2.5. *For $c \in \mathbb{R}$ and exponentially decaying $f, g \in C([0,\infty),\mathbb{R})$,*

(a) $s_{cf} = (s_f)^c, \quad m_{cf} = (m_f)^c,$

(b) $s_{f+g} = s_f s_g, \quad m_{f+g} = m_f m_g,$

(c) $s_f = s_g \Leftrightarrow f = g, \quad m_f = m_g \Leftrightarrow f = g.$

Proof. (a) and (b) are immediate since $\log s_f$ and $\log m_f$ are linear in f by definition. In (c), we only need to show "$\Rightarrow$", and by the previous parts we may asssume $g = 0$, with $s_0 = m_0 = 1$. Now if $s_f = 1$, we compute from (4.13) for any $\lambda \in \mathbb{R}$,

$$0 = \frac{d}{d\lambda} \log s_f(\lambda) = -2i \frac{d}{d\lambda} \int_0^\infty f(\pi t) \sin(\lambda t) \frac{dt}{t} = -2i \int_0^\infty f(\pi t) \cos(\lambda t)\, dt, \tag{4.25}$$

hence $f = 0$ by the inversion formula for the Fourier cosine transform.

If $m_f = 1$, we use (4.20) to conclude that $s_f = 1$, which implies $f = 0$ as seen earlier. $\square$

Proposition 4.2.6 (Asymptotic estimate)**.** *Let $f \in C([0,\infty),\mathbb{R})$ be exponentially decaying and $C^2([0,\delta))$ for some $\delta > 0$. Let $f_0 := f(0)$ and $f_1 := f'(0)$, where $'$ refers to the half-sided derivative. Then there exist constants $0 < c \le c'$ and $r > 0$ such that*

$$\forall |\operatorname{Re}\zeta| \ge r, \operatorname{Im}\zeta \in [0, 2\pi]: \quad c \le \frac{|m_f(\zeta)|}{|\operatorname{Re}\zeta|^{f_1} \exp|\operatorname{Re}\zeta|^{f_0/2}} \le c'. \tag{4.26}$$

Proof. In the following let $z := (i\pi - \zeta)/2\pi$ with $x := |\operatorname{Re} z| > 0$ and $y := |\operatorname{Im} z| \le \frac{1}{2}$. Then

$$\begin{aligned} \operatorname{Re}\log m_f(\zeta) &= 2 \int_0^\infty \frac{f(t)}{t \operatorname{sh} t} \operatorname{Re}\sin^2(zt) dt \\ &= \int_0^\infty \frac{f(t)}{t \operatorname{sh} t} (1 - \cos 2xt \operatorname{ch} 2yt) dt =: I(z). \end{aligned} \tag{4.27}$$

The aim is to show that $|I(z) - f_0 \pi x - f_1 \log x|$ is uniformly bounded in $z \in \mathbb{S}[-\frac{1}{2}, \frac{1}{2}]$, as this implies the bound (4.26) by monotonicity of the exponential function. To begin with, note that the integrand of (4.27) for $t \ge 1, y \le \frac{1}{2}$, is uniformly bounded by $f(t)(t \operatorname{sh} t)^{-1}(1 + \operatorname{ch} t)$. This is integrable on $[1,\infty)$ by the exponential decay of f. The integral over $[1,\infty)$ is thus bounded uniformly in z by a constant c_0.

As further preliminaries let us note that

$$\left| \int_0^1 ((t \operatorname{sh} t)^{-1} - t^{-2})(1 - \cos 2xt \operatorname{ch} 2yt) f(t) dt \right| \le c_1, \tag{4.28}$$

$$\left| \int_0^1 (\mathrm{ch}(2yt) - 1) t^{-2} \cos 2xt f(t) dt \right| \leq c_2, \tag{4.29}$$

where c_1, c_2 are constants independent of x and y. This is implied by mean-value-estimates using regularity of the functions $(t \,\mathrm{sh}\, t)^{-1} - t^{-2}$ and $(\mathrm{ch}(2yt) - 1)t^{-2}$ also at $t = 0$, where t and y are evaluated on compact ranges while x appears only in the argument of the cosine-function. The same reasoning allows us to estimate

$$\left| \int_0^1 (f(t) - f_0 - f_1 t) t^{-2} (1 - \cos 2xt) dt \right| \leq c_3, \tag{4.30}$$

where we apply Taylor's theorem to $f \in C^2([0, \delta))$ to argue regularity at $t = 0$.

In order to apply the estimates, we expand the integrand of $I(z)$ as follows:

$$\begin{aligned}
&\frac{f(t)}{t \,\mathrm{sh}\, t}(1 - \cos 2xt \,\mathrm{ch}\, 2yt) dt \\
&\quad = ((t \,\mathrm{sh}\, t)^{-1} - t^{-2}) f(t)(1 - \cos 2xt \,\mathrm{ch}\, 2yt) + t^{-2} f(t)(1 - \cos 2xt \,\mathrm{ch}\, 2yt) \\
&\quad = ((t \,\mathrm{sh}\, t)^{-1} - t^{-2}) f(t)(1 - \cos 2xt \,\mathrm{ch}\, 2yt) - t^{-2} f(t) \cos 2xt (\mathrm{ch}\, 2yt - 1) \\
&\qquad\qquad + t^{-2} f(t)(1 - \cos 2xt) \\
&\quad = ((t \,\mathrm{sh}\, t)^{-1} - t^{-2}) f(t)(1 - \cos 2xt \,\mathrm{ch}\, 2yt) - t^{-2} f(t) \cos 2xt (\mathrm{ch}\, 2yt - 1) \\
&\qquad + t^{-2}(f(t) - f_0 - f_1 t)(1 - \cos 2xt) + t^{-2}(f_0 + f_1 t)(1 - \cos 2xt).
\end{aligned} \tag{4.31}$$

Then applying (4.28)-(4.30) and the triangle inequality yields

$$|I(z) - J(x)| \leq c_0 + c_1 + c_2 + c_3, \tag{4.32}$$

where

$$J(x) := \int_0^1 (f_0 + f_1 t) \frac{1 - \cos 2xt}{t^2} dt = f_0 \Big(-1 + \cos 2x + 2x \,\mathrm{Si}(2x) \Big) + f_1 \,\mathrm{Cin}(2x) \tag{4.33}$$

in terms of the standard sine and cosine integral functions. Since these have the asymptotics $\mathrm{Si}(x) = \frac{\pi}{2} + \mathcal{O}(x)$, $\mathrm{Cin}(x) = \log x + \mathcal{O}(1)$ as $x \to \infty$ [Nis, §6.12(ii)], one finds constants $r, c > 0$ such that

$$\forall x \geq r : \quad |I(z) - f_0 \pi x - f_1 \log x| \leq c. \tag{4.34}$$

□

With the asymptotic estimate for m_f we can now prove the main result:

Proof of Theorem 4.2.1. First, consider $s(0) = 1$. By Lemma 4.2.2, $f[s]$ is exponentially decaying and hence by Lemma 4.2.3, $s_{f[s]}$ is well-defined, and analytic and nonvanishing on a small strip. Combining (4.12) and (4.13) with the inversion formula for the Fourier cosine transform, we find for $\lambda \in \mathbb{R}$,

$$\frac{d}{d\lambda} \log s_{f[s]}(\lambda) = -\frac{2i}{\pi} \int_0^\infty f[s](t) \cos \frac{\lambda t}{\pi} dt = \frac{s'(\lambda)}{s(\lambda)} = \frac{d}{d\lambda} \log s(\lambda). \tag{4.35}$$

Since also $s_{f[s]}(0) = 1 = s(0)$, we conclude that $s_{f[s]} = s$ first on the real line, and then as meromorphic functions.

Further by Lemma 4.2.4, $f := m_{f[s]}$ is analytic and non-vanishing on the physical strip $\mathbb{S}[0,\pi]$, satisfies $f(i\pi) = 1$, and for some $\epsilon > 0$,

$$f(\zeta) = s(\zeta)f(-\zeta), \quad f(i\pi+\zeta) = f(i\pi-\zeta), \quad \zeta \in \mathbb{S}(-\epsilon,\epsilon); \tag{4.36}$$

in fact, using these relations we can extend f as a meromorphic function to all of $\mathbb{C}$. Also, Proposition 4.2.6 yields the asymptotic estimate in Lemma 4.1.1(b). In summary, Lemma 4.1.1 applies to f, hence $f = f_{s,\min}$ is the unique minimal solution with respect to s.

In the case $s(0) = -1$, one finds $s_{f[s]} = s_{f[-s]} = -s$ by the above; also, $m_{f[s]} = m_{f[-s]}$ is the minimal solution with respect to $-s$, and from Corollary 4.1.3, we have $f_{s,\min}(\zeta) = -i \operatorname{sh} \frac{\zeta}{2} m_{f[-s]}(\zeta)$. □

4.3 Computing a characteristic function

In this section, we present a method to explicitly compute characteristic functions (as defined in the preceding section) for a certain class of S-functions. The method is known but only briefly described in [Kar+77]. We illustrate it here using the eigenvalues of the S-function of the $O(n)$-nonlinear sigma model, i.e., s_i for $i = \pm, 0$ (see Definition 8.4.1 and below). First, we present the general method; second, we check the examples $f[s_\pm]$ against the literature; lastly, we compute $f[s_0]$.

The method applies to S-function eigenvalues which are given as a product of Gamma functions; see [Bab+99, Appendix C] for some typical examples. While this product can be infinite in general, we restrict here to finite products, which suffice for our purposes. Specifically, let s be of the form

$$s(\theta) = \frac{\prod_{x\in A_+} \Gamma(x + \frac{\theta}{\lambda\pi i}) \prod_{x\in A_-} \Gamma(x - \frac{\theta}{\lambda\pi i})}{\prod_{x\in A_+} \Gamma(x - \frac{\theta}{\lambda\pi i}) \prod_{x\in A_-} \Gamma(x + \frac{\theta}{\lambda\pi i})}, \tag{4.37}$$

where $\lambda > 0$ and $A_\pm$ are finite subsets of $(0,\infty)$ such that $|A_+| = |A_-|$. It is straightforward to check that this indeed defines a regular $\mathbb{C}$-valued S-function, apart from crossing symmetry which can only be satisfied for $A_+ = A_- = \emptyset$ or infinite products.

Lemma 4.3.1. *The characteristic function with respect to s as in (4.37) is*

$$t \mapsto f[s](t) = \frac{1}{1-e^{-\lambda t}} \left(\sum_{x\in A_-} - \sum_{x\in A_+} \right) e^{-\lambda x t}. \tag{4.38}$$

Proof. Since $f[s]$ is linear in $\log s$, it suffices to consider the case $A_+ = \{x_+\}$, $A_- = \{x_-\}$. Using Malmstén's formula (see, e.g., [Bat53, Sec. 1.9])

$$\log \Gamma(z) = \int_0^\infty \left(z - 1 - \frac{1 - e^{-(z-1)t}}{1 - e^{-t}} \right) \frac{e^{-t}}{t} dt, \quad \operatorname{Re} z > 0, \tag{4.39}$$

we find

$$\frac{d}{d\theta} \log s(\theta) = \int_0^\infty \frac{\left(e^{\frac{\theta t}{\lambda \pi i}} + e^{\frac{-\theta t}{\lambda \pi i}}\right)\left(e^{-x_+ t} - e^{-x_- t}\right)}{\lambda \pi i (1 - e^{-t})} dt = -\frac{2i}{\pi} \int_0^\infty \underbrace{\frac{e^{-\lambda x_- t} - e^{-\lambda x_+ t}}{1 - e^{-\lambda t}}}_{=:g(t)} \cos \frac{\theta t}{\pi} dt. \tag{4.40}$$

By definition in (4.12), $f[s]$ is given as the Fourier cosine transform of $s(\theta)^{-1} \frac{d}{d\theta} s(\theta) = \frac{d}{d\theta} \log s(\theta)$; its inversion formula yields that $f[s] = g$ since g is clearly integrable. □

Example 4.3.2 (Eigenvalues $s_\pm$)**.** Referring to Definition 8.4.1 and to (8.34), the eigenvalues $s_\pm$ of the $O(n)$-nonlinear sigma model can be written as

$$s_\pm(\theta) = (b \pm c)(\theta) = h_\pm(\theta) b(\theta) \tag{4.41}$$

with

$$b(\theta) = s(\theta) s(i\pi - \theta), \quad s(\theta) = \frac{\Gamma\left(\frac{\nu}{2} + \frac{\theta}{2\pi i}\right) \Gamma\left(\frac{1}{2} + \frac{\theta}{2\pi i}\right)}{\Gamma\left(\frac{1+\nu}{2} + \frac{\theta}{2\pi i}\right) \Gamma\left(\frac{\theta}{2\pi i}\right)}, \tag{4.42}$$

$$h_\pm(\theta) = \frac{\theta \mp i\pi\nu}{\theta} = \mp \frac{\frac{\nu}{2} \mp \frac{\theta}{2\pi i}}{\frac{\theta}{2\pi i}} = \mp \frac{\Gamma(1 + \frac{\nu}{2} \mp \frac{\theta}{2\pi i}) \Gamma(\frac{\theta}{2\pi i})}{\Gamma(\frac{\nu}{2} \mp \frac{\theta}{2\pi i}) \Gamma(1 + \frac{\theta}{2\pi i})}, \tag{4.43}$$

where we used $z = \Gamma(z+1)/\Gamma(z)$ in order to represent $h_\pm$ in terms of Γ. As a result,

$$s_\pm(\theta) = \mp \frac{\Gamma(\frac{1 \mp 1}{2} + \frac{\nu}{2} + \frac{\theta}{2\pi i}) \Gamma(\frac{1}{2} + \frac{\theta}{2\pi i}) \Gamma(\frac{1}{2} + \frac{\nu}{2} - \frac{\theta}{2\pi i}) \Gamma(1 - \frac{\theta}{2\pi i})}{\Gamma(\frac{1}{2} + \frac{\nu}{2} + \frac{\theta}{2\pi i}) \Gamma(1 + \frac{\theta}{2\pi i}) \Gamma(\frac{1 \mp 1}{2} + \frac{\nu}{2} - \frac{\theta}{2\pi i}) \Gamma(\frac{1}{2} - \frac{\theta}{2\pi i})}, \tag{4.44}$$

which is of the form (4.37) for $\lambda = 2$, $A_+ = \{\frac{1}{2}, \frac{1 \mp 1}{2} + \frac{\nu}{2}\}$, and $A_- = \{1, \frac{1}{2} + \frac{\nu}{2}\}$. Due to Lemma 4.3.1 we find

$$f[-s_+](t) = \frac{1}{1 - e^{-2t}} \left(e^{-2t} + e^{-(\nu+1)t} - e^{-t} - e^{-\nu t} \right) = -\frac{1 + e^{(1-\nu)t}}{e^t + 1} \tag{4.45}$$

$$f[s_-](t) = \frac{1}{1 - e^{-2t}} \left(e^{-2t} + e^{-(\nu+1)t} - e^{-t} - e^{-(\nu+2)t} \right) = \frac{e^{-\nu t} - 1}{e^t + 1}. \tag{4.46}$$

This agrees with [BFK13, Eq. (2.11)] and [KW78, Eq. (5.7)]. We read off

$$f[-s_+](t) = -1 + \tfrac{\nu}{2} t + \mathcal{O}(t^2), \quad t \to 0; \tag{4.47}$$

$$f[s_-](t) = -\tfrac{\nu}{2} t + \mathcal{O}(t^2), \quad t \to 0. \tag{4.48}$$

Example 4.3.3 (Eigenvalue $-s_0$)**.** Referring again to Definition 8.4.1 and to (8.34), the eigenvalue s_0 of the $O(n)$-nonlinear sigma model can be written as

$$s_0(\theta) = h_0(\theta)b(\theta) \tag{4.49}$$

with

$$h_0(\theta) = \frac{\theta^2 + i\pi(1+\nu)\theta - \nu\pi^2}{\theta(\theta - i\pi)} = -\frac{(\frac{1}{2} + \frac{\theta}{2\pi i})(\frac{\nu}{2} + \frac{\theta}{2\pi i})}{\frac{\theta}{2\pi i}(\frac{1}{2} - \frac{\theta}{2\pi i})}. \tag{4.50}$$

Using again $z = \Gamma(z+1)/\Gamma(z)$, we find

$$s_0(\theta) = -\frac{\Gamma(1 + \frac{\nu}{2} + \frac{\theta}{2\pi i})\Gamma(\frac{3}{2} + \frac{\theta}{2\pi i})\Gamma(\frac{1}{2} + \frac{\nu}{2} - \frac{\theta}{2\pi i})\Gamma(1 - \frac{\theta}{2\pi i})}{\Gamma(\frac{1}{2} + \frac{\nu}{2} + \frac{\theta}{2\pi i})\Gamma(1 + \frac{\theta}{2\pi i})\Gamma(1 + \frac{\nu}{2} - \frac{\theta}{2\pi i})\Gamma(\frac{3}{2} - \frac{\theta}{2\pi i})}, \tag{4.51}$$

which is of the form (4.37) for $\lambda = 2$, $A_+ = \{\frac{3}{2}, 1 + \frac{\nu}{2}\}$, and $A_- = \{1, \frac{1}{2} + \frac{\nu}{2}\}$. Due to Lemma 4.3.1 we find

$$f[-s_0](t) = \frac{1}{1 - e^{-2t}} \left(e^{-2t} + e^{-(1+\nu)t} - e^{-3t} - e^{-(2+\nu)t} \right) = \frac{e^{-t} + e^{-\nu t}}{e^t + 1} \tag{4.52}$$

and conclude

$$f[-s_0](t) = 1 - (1 + \tfrac{\nu}{2})t + \mathcal{O}(t^2), \quad t \to 0. \tag{4.53}$$

Chapter 5

The stress-energy tensor

This chapter analyses what form the stress-energy tensor $T^{\mu\nu}$ (also referred to as energy-momentum tensor) and, in particular, the energy density T^{00} can take in our setup. A definite expression for the energy density is central to the QEI results which we obtain in Chapters 6 and 7. Since our models do not necessarily arise from a classical Lagrangian, we study the stress-energy tensor using a "bootstrap" approach: We require a list of physically motivated properties for $T^{\mu\nu}$ and study which freedom of choice remains. As a first step, we write down generic properties expected to characterize the stress-energy tensor by first principles (Sec. 5.1). After that, we restrict our attention to the stress-energy tensor at one-particle level. We motivate this to be the relevant object for the following chapters and show that the generic properties of $T^{\mu\nu}$ imply a similar but simpler set of conditions characterizing the stress-energy tensor at one-particle level (Sec. 5.2). Note here that the stress-energy tensor at one-particle level is fully determined by its two-particle form factor $F_2^{\mu\nu}$ (cf. Thm. 3.2.1 in the preceding chapter). We will derive the general form for such $F_2^{\mu\nu}$ and thereby classify stress-energy tensors at one-particle level up to a polynomial degree of freedom (Sec. 5.3). Similarly, also the stress-energy tensor evaluated between a zero- and a one-particle state is fully determined by the one-particle form factor $F_1^{\mu\nu}$ (cf. Prop. 3.2.2 in the preceding chapter). We find that $F_1^{\mu\nu}$ is completely fixed up to a constant but of inferior importance for QEIs.

5.1 The stress-energy tensor from first principles

This chapter collects and briefly discusses the characteristic features of a generic stress-energy tensor. Most importantly, the stress-energy tensor should serve as a *local* generator of the translations, i.e., provide a conserved current which integrates to the *global* generator of the translations—the total energy-momentum operator P^μ. The existence of such a local generator is, of course, motivated by close analogy to Noether's theorem of classical mechanics [Noe18], which asserts the existence of a local current and, upon spacelike integration, a conserved charge for any continuous global symmetry. While the existence of local conserved charges (i.e., charges associated with finite regions) is established in quantum field theory under quite generic assumptions [BDL86], the existence of a (point-)local current is not given in general (see, e.g., [BCM22]). However, our main focus will be the one-particle level, where the existence of $T^{\mu\nu}$ can be argued more directly so that we shall not dwell on these matters here.

Another important fact is that the stress-energy tensor of a model might not be unique. For instance, in the context of Lagrangian field theories, the stress-energy tensor appears in different versions. While the classical stress-energy tensor arising as a consequence of Noether's theorem is often referred to as *canonical stress-energy tensor* or simply as *energy-momentum tensor*, it is well-known that it may fail to be a symmetric Lorentz tensor (e.g., for fields with spin) and to be conserved on curved spacetimes [Wal84, Appendix E.1]. This is problematic in general relativity where $T^{\mu\nu}$ appears on the r.h.s. of Einstein's equation. The same applies to the expectation value of the quantized stress-energy tensor in semiclassical gravity which is an important context for QEIs. The *Hilbert or metrical stress-energy tensor* provides a resolution to this problem. It is classically given by variation of (the matter part of) the action with respect to the spacetime metric (confer Equation (C.4) in the appendix). In quantum field theory, adding renormalization as usual, an analogous stress-energy tensor can be given; this at least for scalar bosonic theories including perturbative interaction [HW05]. While classically, the metrical stress-energy tensor is essentially unique, the quantized operator may be ambiguous due to renormalization. Also, for both classical and quantized models, restricting back to flat spacetime, different gravitational models may become physically equivalent and the metrical procedure may yield a family of stress-energy tensors. For instance, the free scalar field on curved spacetime depends on a parameter, the curvature coupling. Restricting to flat spacetime then yields a one-parameter family of physically equivalent stress-energy tensors (see Appendix C.1). This freedom of choice is also present from the Noetherian point of view, where the canonical stress-energy tensor may be modified by a boundary term which does not spoil the integration to P^μ. From this perspective, the stress-energy tensor associated to a model is not necessarily unique even if the model admits a Lagrangian. And we will later see that, indeed, there is some freedom of choice (Thm. 5.3.1, Prop. 5.3.4 below).

For the specific properties which $T^{\mu\nu}$ is supposed to satisfy, we follow [Ver00; FV03; MPV22], which all specify a set of axioms for the stress-energy tensor (first reference) or the energy density (second and third reference) that is argued to apply in generic situations. We generalize the setup by including global gauge symmetries and are more specific by treating symmetries under discrete spacetime transformations. The notable features are: $T^{\mu\nu}$ is given as a Wightman field with (a certain) regular high energy behaviour (t1) which is local (t2) and hermitian (t5). Also, $T^{\mu\nu}$ should behave covariantly under proper Poincaré transformations as a CPT-invariant symmetric 2-tensor (t6), (t7), (t8). Most importantly $T^{\mu\nu}$ provides a conserved current generating the translations locally, meaning that $T^{\mu\nu}$ is divergence-free (t9) and that the total energy-momentum operator P^μ is obtained upon integration of $T^{\mu 0}$ along a constant time hyperplane. We formulate this carefully and sufficient for our purposes by supposing the identity to hold as a weak limit in compactly supported finite-particle states (t10). Lastly, we demand that $T^{\mu\nu}$ is invariant under the action of $\mathcal{G}$ (t11) and, optionally, covariant under parity inversion (t12).

Definition 5.1.1. A stress-energy tensor $T^{\mu\nu}, \mu, \nu = 0, 1$, is a family of distributions on $\mathbb{M}$, $f \mapsto T^{\mu\nu}(f)$, which map from $\mathcal{D}(\mathbb{M})$ to quadratic forms on $\mathcal{D}_S \times \mathcal{D}_S$ and satisfies for all $f \in \mathcal{D}(\mathbb{M})$ that

(t1) Regularity: $T^{\mu\nu}(f) \in \mathcal{Q}$ is energy-bounded (Defn. 3.1.2) with

$$\|T^{\mu\nu}(f)\|_k = \|(1+P^0)^{-k} T^{\mu\nu}(f)(1+P^0)^{-k}\| \leq c\|f\|_{L^1}$$

for some $c > 0$ and large enough $k \in \mathbb{Z}$.

(t2) Locality: $T^{\mu\nu}(f)$ is localized in $\operatorname{supp} f$ (Def. 3.1.3), i.e., $T^{\mu\nu}(f)$ commutes weakly with $\Phi_S(g)$ and $\Phi'_S(g)$ on $\mathcal{D}_S$ whenever $\operatorname{supp} f - \operatorname{supp} g \subset \mathcal{W}_R$.

(numeration skip)

(t5) Hermiticity: $T^{\mu\nu}(f)^* = T^{\mu\nu}(Jf)$.

(t6) Symmetry: $T^{\mu\nu}(f) = T^{\nu\mu}(f)$.

(t7) Poincaré covariance: $U_S(a,\lambda) T^{\mu\nu}(f) U_S(a,\lambda)^{-1} = \Lambda(\lambda)^\mu_{\ \mu'} \Lambda(\lambda)^\nu_{\ \nu'} T^{\mu'\nu'}(f_{(a,\lambda)})$, where $\Lambda(\lambda) := \begin{pmatrix} \mathrm{ch}(\lambda) & \mathrm{sh}(\lambda) \\ \mathrm{sh}(\lambda) & \mathrm{ch}(\lambda) \end{pmatrix}$ and $f_{(a,\lambda)}(y) := f(\Lambda(\lambda)^{-1}(y-a))$.

(t8) CPT-invariance: $U(j) T^{\mu\nu}(f) U(j)^{-1} = T^{\mu\nu}(f_j)$, where $f_j(x) = Jf(-x)$.

(t9) Continuity equation: $\partial_\mu T^{\mu\nu}(f) = 0$.

(t10) Density property: For $\varphi, \chi \in \mathcal{D}_{S,n}$ and $n \in \mathbb{N}$, we have

$$\lim_{k\to\infty} \langle \varphi, (P^\mu - T^{\mu 0}(f_k))\chi \rangle = 0, \quad \text{where } f_k(t,x) = f_0(t) h_k(x)$$

such that $\int f_0(t)dt = 1$ and $\lim_{k\to\infty} h_k = 1$.

(t11) $\mathcal{G}$ invariance: $V_S(g) T^{\mu\nu}(f) V_S(g)^{-1} = T^{\mu\nu}(f), \quad g \in \mathcal{G}$.

It is called *parity-covariant* if, in addition, (t12) Parity covariance: $U(p)T^{\mu\nu}(f)U(p)^{-1} = (I_p)^{\mu}_{\mu'}(I_p)^{\nu}_{\nu'}T^{\mu'\nu'}(f_p)$, where $I_p = \mathrm{diag}(1,-1)$ and $f_p(x) = f(I_p x)$.

A few comments on the (mathematical) interpretation of these properties are in order. First, note that in slight abuse of notation we write $T^{\mu\nu}(f)$ as an operator; confer Remark 3.1.1. For the properties to hold in the sense of a quadratic form on $\mathcal{D}_S \times \mathcal{D}_S$ then means that they hold when evaluated within expectation values $\langle \Psi, \cdot \Psi \rangle$ for $\Psi \in \mathcal{D}_S$. Locality (t2) is to be interpreted in the sense introduced in the preceding chapter (Defn. 3.1.3). It is implied by locality relative to the observables (Rem. 3.1.4) which is imposed for instance in the guiding reference [MPV22]. We will here not assume this stronger notion of locality since it is not necessary for what follows later. For (t9) the derivative is meant in the sense of distributions which always exists and is defined as $\partial_\mu T^{\mu\nu}(f) := -T^{\mu\nu}(\partial_\mu f)$. We conclude this section by commenting on a number of points: The high-energy behaviour or regularity of $T^{\mu\nu}$ (t1), the existence of a point-local field $T^{\mu\nu}(x)$, details for the motivation of (t10), the vanishing of the zero-point energy, $\langle \Omega, T^{\mu\nu}(f)\Omega \rangle = 0$, and finally a setup for proving QEIs. For the discussion, the C^∞-domain of P^0 denoted by $C^\infty(P^0) := \bigcap_{n\in\mathbb{N}} \mathrm{dom}((P^0)^n)$ will constitute an important class of vectors which is dense in $\mathcal{H}_S$.

High-energy behaviour of the stress-energy tensor To argue that the polynomial energy bounds (t1) are a reasonable assumption for $T^{\mu\nu}(f)$ let us first take a look at $A = P^\mu$. Then the energy bound (3.7) holds quite clearly for $k \geq \frac{1}{2}$ since $\|(1+P^0)^{-\frac{1}{2}} P^0 (1+P^0)^{-\frac{1}{2}}\| = \|P^0(1+P^0)^{-1}\| \leq 1$ and $P^1 \leq P^0$.

Now, in view of the density property (t10) where $T^{\mu 0}$ "integrated over the whole space" gives P^μ, it appears reasonable that finite integrals $T^{\mu\nu}(f)$ are bounded at least by higher moments of P^μ [Ver00]. In fact this is necessary as soon as $C^\infty(P^0)$ is in the domain of $T^{\mu\nu}(f)$ [BW92, Prop. 12.4.10].

Pointlocal stress-energy tensor Due to the polynomial energy bounds (t1), $T^{\mu\nu}(f)$ has a smooth kernel when interpreted as a quadratic form on the dense domain $C^\infty(P^0)$. This means that for all $\mu, \nu = 0, 1$ and $\varphi, \chi \in C^\infty(P^0)$ there exists a smooth function $x \mapsto \langle \varphi, T^{\mu\nu}(x)\chi \rangle$ such that for all $f \in \mathcal{D}(\mathbb{M})$:

$$\langle \varphi, T^{\mu\nu}(f)\chi \rangle = \int dx\, f(x)\, \langle \varphi, T^{\mu\nu}(x)\chi \rangle\,. \tag{5.1}$$

Moreover, for large enough $k \in \mathbb{N}_0$, $(1+P^0)^{-k} T^{\mu\nu}(x)(1+P^0)^{-k}$ defines a bounded operator on $\mathcal{H}$ which is uniformly bounded in x and $T^{\mu\nu}(x)$ defines a bounded operator from $\mathcal{H}_{-k}$ to $\mathcal{H}_k$, where $\mathcal{H}_j$, $j \in \mathbb{Z}$, is obtained by completion of $C^\infty(P^0)$ with respect to the modified inner product $\langle \varphi, \chi \rangle_j := \langle (1+P^0)^{-j}\varphi, (1+P^0)^{-j}\chi \rangle$.

Thus, a posteriori, (5.1) holds also for more general vectors $\varphi \in \mathcal{H}_{-k}$ and $\chi \in \mathcal{H}_k$. These properties are a consequence of generic results on polynomially energy bounded fields and can be drawn from the discussions in [FH81], [BW92, Chaps. 12-14], and in particular from [BW92, Prop. 14.3.4]. We may then transfer the other properties of $T^{\mu\nu}(f)$ (t2)–(t12) to $T^{\mu\nu}(x)$ establishing a point-local stress-energy tensor. We will do so in more detail for the stress-energy tensor at one-particle level in Section 5.2.

Another consequence of (t1) is that we may also define $T^{\mu\nu}(f)$ for arbitrary $f \in L^1(\mathbb{M})$. To see this, take (5.1) and observe that for k as above,

$$\langle\varphi, T^{\mu\nu}(f)\chi\rangle = \int dx\, f(x)\, \langle(1+P^0)^k\varphi, (1+P^0)^{-k}T^{\mu\nu}(x)(1+P^0)^{-k}(1+P^0)^k\chi\rangle \tag{5.2}$$

is well defined for arbitrary $\varphi \in \mathcal{H}_k, \chi \in \mathcal{H}_{-k}$ by the uniform boundedness of $(1+P^0)^{-k}T^{\mu\nu}(x)(1+P^0)^{-k}$. For $f = g^2$ with $g \in \mathcal{S}(\mathbb{M})$ this will be useful in Section 5.1.

Discussion of (t10) Now, we are ready to discuss (t10). To begin with, note that $\mathcal{D}_{S,n} \subset C^\infty(P^0)$ so that on these vectors, (t10) is well-defined and we have that,

$$\langle\varphi, P^\mu\chi\rangle = \lim_{k\to\infty} \langle\varphi, T^{\mu0}(f_k)\chi\rangle = \int d(t,x)\, f_0(t)\, \langle\varphi, T^{\mu0}(t,x)\chi\rangle\,. \tag{5.3}$$

Choosing for f_0 a sequence of functions constituting an approximate identity[1] for some $t_0 \in \mathbb{R}$, we may even infer that

$$\langle\varphi, P^\mu\chi\rangle = \int dx\ \langle\varphi, T^{\mu0}(t_0,x)\chi\rangle \tag{5.4}$$

which is close to what we would classically expect. Note here that existence of the limit follows by the existence of the point-local stress-energy tensor for the given class of vectors; for details we refer to the proof of [BW92, Prop. 14.3.4]. The existence of the integral in (5.4) is intricate to argue, though, and we will refrain from it here but give a proof for the case $n = 1$ in the following section.

In order to avoid such problems, one may employ a weaker notion than (t10), used in particular in [Ver00; MPV22]. This weaker notion of the density property, avoids a definition of the pointlocal stress-energy tensor and the integral over the whole space in (5.4) by writing (t10) as a commutator identity with localized algebra elements so that the spatial integral collapses to a finite integral over the localization region of the respective algebra element. We will not use this weaker density notion here, since the validity of (5.4) in compactly supported states with a fixed finite particle number appears to be reasonable (we validate it later for $n = 1$ which is

[1] An approximate identity for $t_0 \in \mathbb{R}$ is a sequence of functions $(f_k)_{k\in\mathbb{N}}$, where $\int f_k(t)dt = 1$ and such that $\lim_{k\to\infty} \int f_k(t)g(t)dt = g(t_0)$ for all functions g continuous around t_0.

sufficient for our purposes). For the interested reader, we present how (t10) implies the weaker density property in Appendix C.2.

Vanishing of the zero-point energy The famous vanishing of the zero-point energy reads

$$\langle \Omega, T^{\mu\nu}(f)\Omega\rangle = 0 \tag{5.5}$$

for arbitrary $f \in \mathcal{D}(\mathbb{M})$. It directly follows from the density property (t10) for $\varphi = \chi = \Omega$. In this case (5.4) implies that

$$\langle \Omega, P^0\Omega\rangle = 0 \stackrel{!}{=} \int dx^1 \ \langle \Omega, T^{00}(x)\Omega\rangle \,. \tag{5.6}$$

However, due to Ω being invariant with respect to U the integrand on the r.h.s. evaluates to the constant $\langle \Omega, T^{00}(0)\Omega\rangle$ which thus has to vanish for the integral to be finite. This also applies for the other components of $T^{\mu\nu}$: Lorentz covariance (t6) and symmetry (t7) imply that $\langle \Omega, T^{\mu\nu}(0)\Omega\rangle$ is a symmetric Lorentz two-tensor invariant under $\Lambda(\lambda)^{\otimes 2}$ for all $\lambda \in \mathbb{R}$. The only such tensors are of the form $cg^{\mu\nu}$ for some constant c which has to vanish due to the argument given before.

Note though, that there are indications that in interacting models with non-perturbative effects the vacuum expectation value may not vanish identically but be a small non-vanishing constant exponentially suppressed in the coupling constant(s); see, e.g., [HH14]. On the other hand, in models with finite renormalization (e.g., the sine-Gordon model [FC22]) this cannot happen. Moreover, for the purpose of QEI results the zero-point energy corresponds to a mere shift of the QEI bound, so that it is easy to adjust for this case (Sec. 7.2). Thus we will assume the vanishing of the zero-point energy in the following.

The stress-energy tensor for QEIs So far, we have treated $A = T^{\mu\nu}(f)$ with $f \in \mathcal{D}(\mathbb{M})$. For the QEI results following in the next two chapters we have a slightly different setup, namely we consider

$$A = \int dt\, g^2(t) T^{\mu\nu}(t,0) \tag{5.7}$$

with real-valued $g \in \mathcal{S}(\mathbb{R})$ as our operator (or quadratic form) of interest, where integration is understood to be weakly on $\mathcal{D}_{S,n} \times \mathcal{D}_{S,n}$ for any $n \in \mathbb{N}$; existence of this integral is clarified in Remark 5.1 due to $\mathcal{D}_{S,n} \subset C^\infty(P^0)$.

The setup is motivated as follows: First, it turns out later, similar to the free situation, that smearing in space is not necessary to obtain a QEI so that for simplicity we restrict to time-averaged QEIs. The square form of the (real-valued) test function takes care of two requirements, namely, that the averaging function needs to be positive and that the square root of the averaging function is also smooth.

The latter is a typical requirement to derive QEIs for linear QFTs and we will also need it for the results presented later on. Whether this requirement is of technical or physical nature has not been clarified yet [FV03, Remark (ii) after Thm. 4.1]. Lastly, it is unnecessarily strict to demand compact support of the averaging function, and the Schwartz class provides a convenient generalization.

5.2 The stress-energy tensor at one-particle level

In the remainder of this chapter we will deal with the form factors of the smeared stress-energy tensor $A = T^{\mu\nu}(f)$ for some fixed $\mu, \nu \in \{0, 1\}$ and $f \in \mathcal{D}(\mathbb{M})$. We will focus mostly on its two-particle form factor and briefly treat the one-particle form factor but ignore higher particle numbers. The motivation for this is the following:

(a) in some models the stress-energy tensor has only the two-particle coefficient, i.e., $F_n^{[A]} = 0$ for $n \neq 2$ (see Chapter 6)

(b) one-particle expectation values which will be partly our focus (see Chapter 7) are determined solely by the coefficients $F_n^{[A]}$ for $n \leq 2$

(c) the coefficients with $n < 2$ are not essential for QEI results since they yield only bounded contributions to the expectation values of A (Sec. 7.2) and it can be easily adjusted for them where it is needed.

In this section we will prove that the generic properties of the full stress-energy tensor (Defn. 5.1.1) imply a simpler set of conditions at the one-particle level (Defn. 5.2.1 below). These conditions will be characterized in terms of $F_2^{[A]}$ or, more precisely, in terms of the two-particle form factor $F_2^{[T^{\mu\nu}(x)]}$ of the point-local field $T^{\mu\nu}(x)$. Since we have not explicitly developed the form factor equations for point-local operators, instead we may fix $F_2^{\mu\nu}(\boldsymbol{\zeta}; x)$ by requiring

$$F_2^{[T^{\mu\nu}(f)]}(\boldsymbol{\zeta}) = \int dx\, f(x) F_2^{\mu\nu}(\boldsymbol{\zeta}; x), \tag{5.8}$$

for all $f \in \mathcal{D}(\mathbb{M})$, and identify $F_2^{[T^{\mu\nu}(x)]}(\boldsymbol{\zeta}) = F_2^{\mu\nu}(\boldsymbol{\zeta}; x)$. Existence of such $F_2^{\mu\nu}(\cdot; x)$ is guaranteed by Remark 5.1 and details will be given in the proof of Proposition 5.2.2 below. Therein, we will also show that, assuming $F_0^{[A]} = 0$, the expectation value of the (smeared) stress-energy tensor in one-particle states $\varphi, \chi \in \mathcal{H}_1 \cap \mathcal{D}_S$ is then given by

$$\langle \varphi, T^{\mu\nu}(f)\chi \rangle = \int d\theta d\eta\, dx\, f(x) \left(\varphi(\theta) \otimes J\chi(\eta), F_2^{\mu\nu}(\theta, \eta + i\pi; x)\right)_{\mathcal{K}^{\otimes 2}}. \tag{5.9}$$

We may then characterize a stress-energy tensor at one-particle level as follows:

Definition 5.2.1. Assume a little space $(\mathcal{K}, V, J, M)$, an S-function S, and a subset $\mathfrak{P} \subset \mathbb{S}(0,\pi)$. Then a *stress-energy tensor at one-particle level* (with poles $\mathfrak{P}$) is formed by functions $F_2^{\mu\nu} : \mathbb{C}^2 \times \mathbb{M} \to \mathcal{K}^{\otimes 2}$, $\mu, \nu = 0, 1$, which for arbitrary $\boldsymbol{\zeta} = (\zeta_1, \zeta_2) \in \mathbb{C}^2$, $x \in \mathbb{M}$ satisfy

(T1) Regularity: There exist constants $a, b, r \geq 0$ and $\kappa < \kappa(S)$ such that for all $|\operatorname{Re}(\zeta_2 - \zeta_1)| \geq r$ and $|\operatorname{Im}(\zeta_1 - \zeta_2)| \leq 2\pi + \kappa$ it holds that

$$\max_{\mu,\nu} ||F_2^{\mu\nu}(\zeta_1, \zeta_2; x)||_{\mathcal{K}^{\otimes 2}} \leq a \exp b \left(|\operatorname{Re}\zeta_1| + |\operatorname{Re}\zeta_2|\right).$$

(T2) Analyticity: $F_2^{\mu\nu}(\boldsymbol{\zeta}; x)$ is meromorphic in $\zeta_2 - \zeta_1$, where the poles within $\mathbb{S}(0,\pi)$ are all first-order and $\mathfrak{P}$ denotes the set of poles in that region.

(T3) S-symmetry: $F_2^{\mu\nu}(\boldsymbol{\zeta}; x) = S(\zeta_2 - \zeta_1) F_2^{\mu\nu}(\overleftarrow{\boldsymbol{\zeta}}; x)$.

(T4) S-periodicity: $F_2^{\mu\nu}(\boldsymbol{\zeta}; x) = \mathbb{F} F_2^{\mu\nu}(\zeta_2, \zeta_1 + i2\pi; x)$.

(T5) Hermiticity: $F_2^{\mu\nu}(\boldsymbol{\zeta}; x) = \mathbb{F} J^{\otimes 2} F_2^{\mu\nu}(\overleftarrow{\bar{\boldsymbol{\zeta}}} + i\boldsymbol{\pi}; x)$.

(T6) Symmetry: $F_2^{\mu\nu} = F_2^{\nu\mu}$.

(T7) Poincaré covariance: For all $\lambda \in \mathbb{C}$ and $a \in \mathbb{M}$ it holds that

$$\Lambda(\lambda)^{\otimes 2} F_2(\boldsymbol{\zeta}; \Lambda(\lambda)x + a) = e^{iP(\boldsymbol{\zeta}).a} F_2(\boldsymbol{\zeta} - (\lambda, \lambda); x), \quad \Lambda(\lambda) := \begin{pmatrix} \operatorname{ch}(\lambda) & \operatorname{sh}(\lambda) \\ \operatorname{sh}(\lambda) & \operatorname{ch}(\lambda) \end{pmatrix}.$$

(T8) CPT invariance: $F_2^{\mu\nu}(\boldsymbol{\zeta}; x) = \mathbb{F} J^{\otimes 2} F_2^{\mu\nu}(\overleftarrow{\bar{\boldsymbol{\zeta}}}; -x)$.

(T9) Continuity equation: $P_\mu(\boldsymbol{\zeta}) F_2^{\mu\nu}(\boldsymbol{\zeta}; x) = 0$.

(T10) Normalization: $E_m^{\otimes 2} F_2^{0\mu}(\theta, \theta + i\pi; x) = \delta^{0\mu} \frac{m^2}{2\pi} E_m^{\otimes 2} I_{\otimes 2}$ for all $m \in \mathfrak{M}$.

(T11) $\mathcal{G}$ invariance: $F_2^{\mu\nu}(\boldsymbol{\zeta}; x) = V(g)^{\otimes 2} F_2^{\mu\nu}(\boldsymbol{\zeta}; x), \quad g \in \mathcal{G}$.

It is called *parity-covariant* if, in addition,

(T12) Parity covariance: $F_2^{\mu\nu}(\boldsymbol{\zeta}; x) = (I_p)^\mu_{\mu'} (I_p)^\nu_{\nu'} F_2^{\mu'\nu'}(-\boldsymbol{\zeta}; I_p x)$, $I_p = \operatorname{diag}(1, -1)$.

Property (T7) implies that for any $f \in \mathcal{S}(\mathbb{M})$,

$$\int dx\, f(x) F_2^{\mu\nu}(\boldsymbol{\theta}; x) = \tilde{f}(P(\boldsymbol{\theta})) F_2^{\mu\nu}(\boldsymbol{\theta}; 0) \quad \text{where } \tilde{f}(p) = \int dx f(x) e^{ip.x}. \tag{5.10}$$

To motivate the definition of the stress-energy tensor at one-particle level we will now show that these conditions indeed follow from the definition of the full stress-energy tensor (Defn. 5.1.1). While the properties of the stress-energy tensor at one-particle level are expected to hold also in the presence of bound states we will exclude bound states in the derivation. This is because we also excluded them in the treatment of form factors in the preceding chapter.

In situations with bound states, we rely on condition (F1b) (Defn. 3.1.5) which asserts a first-order pole at $\zeta_1 - \zeta_2 = i\theta$ in case the model admits a fusion angle θ (App. A.5). Note that for our purposes it suffices to state the position of poles via the pole set $\mathfrak{P}$ since the order is fixed to be first-order in all cases and the value of the residue is not fixed aside from symmetry considerations: Condition (F1b) relates the bound state residue to the one-particle form factor which, however, has a free constant prefactor; confer [MS94].

Proposition 5.2.2. *Assume a model with a regular S-function S which has no poles in the physical strip and with a stress-energy tensor $T^{\mu\nu}$ (Defn. 5.1.1) satisfying $\langle \Omega, T^{\mu\nu}(f)\Omega\rangle = 0$ for all $f \in \mathcal{D}(\mathbb{M})$. Then its two-particle form factors $F_2^{\mu\nu}(\boldsymbol{\zeta}; x)$, $\mu, \nu \in \{0, 1\}$, form a stress-energy tensor at one-particle level without poles in the sense that expectation values in one-particle states have the form (5.9) and the conditions specified in Definition 5.2.1 hold for $\mathfrak{P} = \emptyset$.*

Proof. Let $f \in \mathcal{D}(\mathbb{M})$ be arbitrary. Then by (t1) and (t2) $T^{\mu\nu}(f) \in \mathcal{Q}$ is polynomially energy bounded and localized in a double cone. Also, we have $\langle \Omega, T^{\mu\nu}(f)\Omega\rangle = 0$ by assumption. Thus we can apply Theorem 3.2.1 and obtain a family of analytic functions $F_2^{\mu\nu}[f] : \mathbb{C}^2 \to \mathcal{K}^{\otimes 2}$ which satisfies the two-particle form factor equations without bound states (Defn. 3.1.5, $n = 2$, $\mathfrak{F} = \emptyset$) and which is such that

$$\langle \varphi, T^{\mu\nu}(f)\chi\rangle = \int d\theta d\eta\, (\varphi(\theta) \otimes J\chi(\eta), F_2^{\mu\nu}[f](\theta, \eta + i\pi))_{\mathcal{K}^{\otimes 2}} \tag{5.11}$$

for arbitrary $\varphi, \chi \in \mathcal{H}_1$.

Next, we establish that there are smooth functions $F_2^{\mu\nu}(\boldsymbol{\zeta}; x)$, $\mu, \nu \in \{0, 1\}$ such that

$$F_2^{\mu\nu}[f](\boldsymbol{\zeta}) = \int dx\, f(x) F_2^{\mu\nu}(\boldsymbol{\zeta}; x). \tag{5.12}$$

For that, we have due to (F2.4) a constant $a \geq 0$ such that for large enough $k \in \mathbb{Z}$ and for all $\boldsymbol{\zeta} \in \mathbb{R}^2 + i\pi\mathbb{Z}^2$,

$$\|F_2^{\mu\nu}[f](\boldsymbol{\zeta})\|_{\mathcal{K}^{\otimes 2}} \leq a(|\operatorname{ch}\zeta_1|^k + |\operatorname{ch}\zeta_2|^k)\|T^{\mu\nu}(f)\|_k. \tag{5.13}$$

Note here that by a maximum modulus principle (as in [Cad13, Eq. (C.6)]) these bounds may be extended to the analyticity region of $F_2^{\mu\nu}[f]$, i.e., to $|\operatorname{Im}(\zeta_1 - \zeta_2)| < 2\pi + \kappa$ with $\kappa < \kappa(S)$; confer (F2.1) of Theorem 3.2.1.

Morever, by (t1), we have for some $c \geq 0$ and for large enough k (the same as above) that

$$\|T^{\mu\nu}(f)\|_k \leq c\, \|f\|_{L^1}. \tag{5.14}$$

As a result, denoting $\mu_- = \min \mathfrak{M}$,

$$\begin{aligned}&\|((1+p^0(\zeta_1;M))^{-k}\otimes(1+p^0(\zeta_2;M))^{-k})F_2^{\mu\nu}[f](\boldsymbol{\zeta})\|_{\mathcal{K}^{\otimes 2}}\\&\le|\operatorname{ch}\zeta_1|^{-k}|\operatorname{ch}\zeta_2|^{-k}\mu_-^{-2k}\|F_2^{\mu\nu}[f](\boldsymbol{\zeta})\|_{\mathcal{K}^{\otimes 2}}\end{aligned} \tag{5.15}$$

is uniformly bounded in f, so that there exists a function $F_2^{\mu\nu}(\boldsymbol{\zeta};x)$ such that (5.12) holds and which satisfies (T1). In more detail, $F_2^{\mu\nu}(\boldsymbol{\zeta};x)$ can be constructed as a limit value of $F_2^{\mu\nu}[f]$ by choosing a sequence $(f_n)_n$ of test functions constituting an approximate identity for x. The existence of the limit is guaranteed by the uniform boundedness in f for the energy-damped expression (5.15). Also, the properties of $F_2^{\mu\nu}[f](\boldsymbol{\zeta})$ which are due to (F2.1)–(F2.3) of Theorem 3.2.1 straightforwardly transfer to $F_2^{\mu\nu}(\boldsymbol{\zeta};x)$ resulting in (T2)–(T4) (for $\mathfrak{P}=\emptyset$).

Exemplarically, we show (c)⇒(T3): Starting with

$$F_2^{\mu\nu}[f](\boldsymbol{\zeta}) - S(\zeta_2-\zeta_1)F_2^{\mu\nu}[f](\overleftarrow{\boldsymbol{\zeta}}) = 0, \tag{5.16}$$

due to (5.12) it follows that

$$\int dx\, f(x)\left(F_2^{\mu\nu}(\boldsymbol{\zeta};x) - S(\zeta_2-\zeta_1)F_2^{\mu\nu}(\overleftarrow{\boldsymbol{\zeta}};x)\right) = 0 \tag{5.17}$$

for all $f\in\mathcal{D}(\mathbb{M})$ which implies (T3).

Most of the properties (t5)–(t12) transfer analogously to (T5)–(T12). In more detail, since $T^{\mu\nu}(f)^* = T^{\mu\nu}(Jf)$ (t5) we have that

$$J^{\otimes 2}\mathbb{F}F_2^{\mu\nu}[f](\boldsymbol{\zeta}) = F_2^{\mu\nu}[Jf](\overleftarrow{\boldsymbol{\zeta}}+i\boldsymbol{\pi})$$

due to Proposition 3.3.4(a). Restricting to f with $f = Jf$ we can infer (T5). That $T^{\mu\nu}(f)$ is symmetric under exchange of μ and ν (t6) clearly transfers to $F_2^{\mu\nu}$ by (5.11) and (5.12). Thus (T6) holds. Moreover, (t7), (t8), and (t11) imply (T7), (T8), and (T11), respectively, by virtue of Proposition 3.3.4(b)–(d). Similarly, (T9) holds by (t9) and Proposition 3.3.5 and, optionally, (T12) by (t12) and Proposition 3.3.4(e).

It remains to prove that (T10) holds. For this it is convenient to employ the improper momentum eigenstates (Sec. 2.4) and use the calculus of generalized functions (as, e.g., introduced in [Bog+90, Chap. 2]). First, note that

$$\langle\theta_\alpha|P^\mu|\eta_\beta\rangle = p^\mu(\theta;m_\alpha)\delta_{\alpha\beta}\delta(\theta-\eta). \tag{5.18}$$

On the other hand, by (t10)

$$\begin{aligned}\langle\theta_\alpha|P^\mu|\eta_\beta\rangle &= \int ds\ \langle\theta_\alpha|T^{\mu 0}(0,s)|\eta_\beta\rangle\\&= \int ds\ \langle\theta_\alpha|U_S(se_1,0)T^{\mu 0}(0)U_S(se_1,0)^{-1}|\eta_\beta\rangle\\&= \langle\theta_\alpha|T^{\mu 0}(0)|\eta_\beta\rangle\int ds\, e^{is(p^1(\theta;m_\alpha)-p^1(\eta;m_\beta))}\\&= F_{2,\alpha\beta}^{\mu 0}(\theta,\eta+i\pi)\,2\pi\delta(p^1(\theta;m_\alpha)-p^1(\eta;m_\beta)).\end{aligned} \tag{5.19}$$

Here, we used covariance of the eigenstates $|\theta_\alpha\rangle$ under translations (Prop. 2.4.8(d)) and that $\langle\theta_\alpha|A|\eta_\beta\rangle = F^{[A]}_{2,\alpha\beta}(\theta,\eta+i\pi)$ for A with $\langle\Omega, A\Omega\rangle = 0$ which follows from

$$\int d\theta d\eta\, \varphi_\alpha(\theta)\,\langle\theta_\alpha|A|\eta_\beta\rangle\,\chi^\beta(\eta) = \langle\varphi, A\chi\rangle = \int (\varphi(\theta)\otimes J\chi(\eta), F^{[A]}_2(\theta,\eta+i\pi)); \quad (5.20)$$

confer the definition of the rapidity eigenstates in (2.42) or (2.43) and the defining property (3.11) for the two-particle form factor in Theorem 3.2.1.

Now, for $m = m_\alpha = m_\beta$ we have that

$$p^1(\theta;m) - p^1(\eta;m) = m(\operatorname{sh}\theta - \operatorname{sh}\eta) = 2m\operatorname{sh}\rho\operatorname{sh}\tfrac{\tau}{2},$$

where $\rho = \frac{\theta+\eta}{2}$ and $\tau = \theta - \eta$. Thus

$$\begin{aligned}\delta(p^1(\theta;m) - p^1(\eta;m)) &= (2m\operatorname{sh}\rho\,\tfrac{\partial}{\partial\tau}\operatorname{sh}\tfrac{\tau}{2})^{-1}\delta(\tau)\\ &= (m\operatorname{sh}\rho\operatorname{ch}\tfrac{\tau}{2})^{-1}\delta(\tau)\\ &= (m\operatorname{sh}\rho)^{-1}\delta(\tau).\end{aligned} \quad (5.21)$$

Combining (5.18), (5.19), and (5.21) yields that

$$F^{\mu 0}_{2,\alpha\beta}(\theta,\theta+i\pi) = \frac{1}{2\pi}p^0(\theta;m_\alpha)p^\mu(\theta;m_\alpha)\delta_{\alpha\beta}; \quad (5.22)$$

or, in more abstract notation, that

$$E^{\otimes 2}_m F^{\mu 0}_2(\theta,\theta+i\pi) = \frac{1}{2\pi}p^0(\theta;m)p^\mu(\theta;m)E^{\otimes 2}_m I_{\otimes 2} \quad (5.23)$$

for all $m \in \mathfrak{M}$ (T10). □

To conclude this section, note that although the definition of the stress-energy tensor at one-particle level is motivated by Proposition 5.2.2, the definition may be read independently of the existence of a full stress-energy tensor (Defn. 5.1.1). Instead, finding suitable functions $F^{\mu\nu}_2(\boldsymbol{\zeta};x)$ obeying the properties of a stress-energy tensor at one-particle level, we may use (5.9) to *define* $T^{\mu\nu}(f)$ as a quadratic form at one-particle level, i.e., on $\mathcal{H}_1 \cap \mathcal{D}_S$. For such $T^{\mu\nu}$ we may infer QEI results at the one-particle level (see Chapter 7).

More generally, we may also use the series expansion (3.1) to define $T^{\mu\nu}$ for arbitrary particle numbers, i.e., as a quadratic form on $\mathcal{D}_S$. This, however, requires the whole family $\{F_n\}_{n\in\mathbb{N}_0}$ as an input and usually infinitely many elements have to be non-zero for the form factor series to define a local operator. In special situations, though, where the S-function is constant, it is sufficent to consider $F^{[A]}_n = 0$ for $n \neq 2$ resulting in a local operator on $\mathcal{D}_S$ (see Chapter 6).

5.3 Characterization at one-particle level

In the preceding section we provided and motivated a number of conditions which the stress-energy tensor at one-particle level is ought to satisfy. We are now in a position to characterize the generic form it may take.

Theorem 5.3.1. *F_2 forms a stress-energy tensor at one-particle level (with poles $\mathfrak{P}$) iff it is of the form*

$$F_2^{\mu\nu}(\zeta_1, \zeta_2; x) = \frac{M^{\otimes 2}}{2\pi} \mathcal{L}^{\mu\nu}(P(\boldsymbol{\zeta})) e^{iP(\boldsymbol{\zeta}).x} F(\zeta_2 - \zeta_1), \quad \boldsymbol{\zeta} = (\zeta_1, \zeta_2) \in \mathbb{C}^2, \tag{5.24}$$

where

$$\mathcal{L}^{\mu\nu}(p) := \frac{-p^\mu p^\nu + g^{\mu\nu} p^2}{p^2} \tag{5.25}$$

and $F : \mathbb{C} \to \mathcal{K}^{\otimes 2}$ is a meromorphic function which satisfies for all $\zeta \in \mathbb{C}$ that

(a) $\exists a, b, r > 0 \, \forall |\operatorname{Re} \zeta| \geq r : \quad \|F(\zeta)\|_{\mathcal{K}^{\otimes 2}} \leq a \exp(b|\operatorname{Re} \zeta|)$;

(b) $F \restriction \mathbb{S}[0, \pi]$ *has exactly the poles* $\mathfrak{P}$;

(c) $F(\zeta) = S(\zeta) F(-\zeta)$;

(d) $F(\zeta + i\pi) = \mathbb{F} F(-\zeta + i\pi)$;

(e) $F(\zeta + i\pi) = J^{\otimes 2} F(\bar{\zeta} + i\pi)$;

(f) $F = V(g)^{\otimes 2} F$ *for all* $g \in \mathcal{G}$;

(g) $E_m^{\otimes 2} F(i\pi) = E_m^{\otimes 2} I_{\otimes 2}$ *for all* $m \in \mathfrak{M}$.

It is parity covariant iff, in addition,

(h) $F(\zeta + i\pi) = F(-\zeta + i\pi)$ *or, equivalently,* $F = \mathbb{F} F$.

Remark 5.3.2. As can be seen from the proof, it is sufficient to require (T10) for $\mu = 0$; the case $\mu = 1$ is automatic.

Proof of Theorem 5.3.1. Assume F_2 to satisfy (T1)–(T12). By Poincaré covariance (T7), it is given by

$$F_2(\boldsymbol{\zeta}; x) = e^{iP(\boldsymbol{\zeta}).x} \Lambda\left(-\tfrac{\zeta_1+\zeta_2}{2}\right)^{\otimes 2} F_2(-\tfrac{\zeta_2-\zeta_1}{2}, \tfrac{\zeta_2-\zeta_1}{2}; 0). \tag{5.26}$$

Define $G^{\mu\nu}(\zeta) := F_2^{\mu\nu}(-\frac{\zeta}{2}, \frac{\zeta}{2}; 0)$ and observe that the conditions (T1) to (T3), (T8), and (T11) imply that G is meromorphic with pole set $\mathfrak{P}$ when restricted to $\mathbb{S}[0, \pi]$ and that for all $\mu, \nu = 0, 1$,

$$\begin{aligned} &\forall |\operatorname{Re} \zeta| \geq r : \|G^{\mu\nu}(\zeta)\|_{\mathcal{K}^{\otimes 2}} \leq a \exp(b|\operatorname{Re} \zeta|), && G^{\mu\nu}(\zeta) = S(\zeta) G^{\mu\nu}(-\zeta), \\ &G^{\mu\nu}(\zeta + i\pi) = \mathbb{F} J^{\otimes 2} G^{\mu\nu}(-\bar{\zeta} + i\pi), && G^{\mu\nu}(\zeta) = V(g)^{\otimes 2} G^{\mu\nu}(\zeta). \end{aligned} \tag{5.27}$$

Omit the Minkowski indices for the moment. Then combining (T5) and (T8) we obtain $F_2(\boldsymbol{\zeta};x) = F_2(\boldsymbol{\zeta}+i\boldsymbol{\pi};-x)$ and thus $G(\zeta) = G^\pi(\zeta)$, where $G^\pi(\zeta) := F_2(-\frac{\zeta}{2}+i\pi, \frac{\zeta}{2}+i\pi;0)$. Combining (T4) with the preceding equality, we obtain $G(\zeta+i\pi) = \mathbb{F}G^\pi(-\zeta+i\pi) = \mathbb{F}G(-\zeta+i\pi)$. Moreover, by (T5), we have $G(\zeta+i\pi) = \mathbb{F}J^{\otimes 2}G(-\bar{\zeta}+i\pi) = J^{\otimes 2}G(\bar{\zeta}+i\pi)$. If we demand (T12), this implies $G(\zeta+i\pi) = G(-\zeta+i\pi)$ and with the preceding properties also $G(\zeta) = \mathbb{F}G(\zeta)$. In summary, each $G^{\mu\nu}(\zeta)$, $\mu,\nu = 0,1$ satisfies properties (a)–(f), and possibly (h), analogously.

Concerning (g), due to the continuity equation (T9), we have

$$(M_1+M_2)\operatorname{ch}(\zeta)G^{0\nu}(2\zeta) + (M_1-M_2)\operatorname{sh}(\zeta)G^{1\nu}(2\zeta) = 0. \tag{5.28}$$

where $M_1 := M\otimes \mathbb{1}_{\mathcal{K}}$ and $M_2 := \mathbb{1}_{\mathcal{K}}\otimes M$. Multiplying by the inverses of M_1+M_2 and $\operatorname{ch}\zeta$ (both are invertible, the latter as a meromorphic function) we find

$$G^{0\nu}(2\zeta) = \frac{-M_1+M_2}{M_1+M_2}\operatorname{th}(\zeta)G^{1\nu}(2\zeta), \quad \nu = 0,1. \tag{5.29}$$

Defining $\operatorname{tr} G := g_{\mu\nu}G^{\mu\nu} = G^{00} - G^{11}$, we obtain

$$G^{\mu\nu}(\zeta) = \frac{1}{s(\zeta)^2-1}\begin{pmatrix} s(\zeta)^2 & s(\zeta) \\ s(\zeta) & 1\end{pmatrix}^{\mu\nu} \operatorname{tr} G(\zeta) \tag{5.30}$$

with

$$s(\zeta) := \tfrac{-M_1+M_2}{M_1+M_2}\operatorname{th}\tfrac{\zeta}{2} = \tfrac{P^0(-\zeta/2,\zeta/2)}{P^1(-\zeta/2,\zeta/2)}. \tag{5.31}$$

On the other hand, we have

$$\mathcal{L}^{\mu\nu}(p) = \frac{-1}{p^2}\begin{pmatrix} p^1p^1 & p^0p^1 \\ p^0p^1 & p^0p^0\end{pmatrix}^{\mu\nu} = \frac{1}{s^2-1}\begin{pmatrix} s^2 & s \\ s & 1\end{pmatrix}^{\mu\nu}, \quad s := p^0/p^1. \tag{5.32}$$

so that

$$G^{\mu\nu}(\zeta) = \mathcal{L}^{\mu\nu}(P(-\tfrac{\zeta}{2},\tfrac{\zeta}{2}))\operatorname{tr} G(\zeta). \tag{5.33}$$

Due to the normalization property (T10) we have

$$E_m^{\otimes 2}F_2^{00}(\theta,\theta+i\pi;x) = \frac{m^2}{2\pi}\operatorname{ch}^2\theta E_m^{\otimes 2}I_{\otimes 2}. \tag{5.34}$$

On the other hand, by Poincaré covariance (T7) we have that

$$F_2(0,i\pi;0) = \Lambda(\tfrac{i\pi}{2})^{\otimes 2}F_2(-\tfrac{i\pi}{2},\tfrac{i\pi}{2};0), \quad \Lambda(\tfrac{i\pi}{2}) = \begin{pmatrix} 0 & i \\ i & 0\end{pmatrix} \tag{5.35}$$

so that

$$F_2^{00}(0,i\pi;0) = (\Lambda(\tfrac{i\pi}{2})^0_1)^2F_2^{11}(-\tfrac{i\pi}{2},\tfrac{i\pi}{2};0) = -F_2^{11}(-\tfrac{i\pi}{2},\tfrac{i\pi}{2};0) = -G^{11}(i\pi). \tag{5.36}$$

Next, define $H_m^{\mu\nu}(\zeta) = E_m^{\otimes 2} G^{\mu\nu}(\zeta)$ such that equating (5.34) and (5.36) yields

$$H_m^{11}(i\pi) = -\frac{m^2}{2\pi} E_m^{\otimes 2} I_{\otimes 2}. \tag{5.37}$$

Now, by definition of $s(\zeta) \propto M_1 - M_2$ (5.31) and since $E_m^{\otimes 2}(M_1 - M_2) = 0$, we have that $E_m^{\otimes 2} s(\zeta) = 0$ for arbitrary $m \in \mathfrak{M}$. Then, in view of (5.30), we find that $H_m^{\mu\nu}(\zeta) = 0$ unless $\mu = \nu = 1$, where we have $H_m^{11}(\zeta) = -\operatorname{tr} H_m(\zeta)$. Thus we infer using (5.37) that $\operatorname{tr} H_m(i\pi) = \frac{m^2}{2\pi} E_m^{\otimes 2} I_{\otimes 2}$. Define now

$$F(\zeta) := \left(\tfrac{M^{\otimes 2}}{2\pi}\right)^{-1} \operatorname{tr} G(\zeta) \tag{5.38}$$

which implies $E_m^{\otimes 2} F(i\pi) = E_m^{\otimes 2} I_{\otimes 2}$. Since $M^{\otimes 2}$ commutes with all $S(\zeta)$, $\mathbb{F}$, J and $V(g)$, we find that F satisfies properties (a)–(g), plus (h) in the parity-covariant case. We have thus shown (5.24) for arguments of the form $(-\zeta/2, \zeta/2; x)$. That (5.24) holds everywhere now follows from (T7) together with the identity

$$\mathcal{L}^{\mu\nu}(P(\boldsymbol{\zeta})) = \Lambda\left(-\tfrac{\zeta_1+\zeta_2}{2}\right)^{\mu}_{\ \mu'} \Lambda\left(-\tfrac{\zeta_1+\zeta_2}{2}\right)^{\nu}_{\ \nu'} \mathcal{L}^{\mu'\nu'}(P(-\tfrac{\zeta_2-\zeta_1}{2}, \tfrac{\zeta_2-\zeta_1}{2})). \tag{5.39}$$

The identity is a consequence of $p(\theta+\lambda; m) = \Lambda(\lambda) p(\theta; m)$ and the defining expression for $\mathcal{L}^{\mu\nu}(p)$ (5.25).—The converse direction, to show that (5.24) satisfies (T1) to (T11) (and (T12) provided that (h)) is straightforward. □

Let us call $X \in \mathcal{K}^{\otimes 2}$ *diagonal in mass* if

$$(E_m \otimes E_{m'}) X = 0 \quad \text{for all } m \neq m'. \tag{5.40}$$

Equivalently, $\hat{X}$ commutes with M; confer (2.4) and Lemma A.6.1. On such X, all of M_1, M_2 and $(M \otimes M)^{1/2}$ act the same and in a slight abuse of notation we will use M to denote any of these. If F has this property, i.e., $F(\zeta)$ has it for all $\zeta \in \mathbb{C}$, then the above result simplifies:

Corollary 5.3.3. *Assume that F is diagonal in mass, or equivalently, that $\operatorname{tr} F_2(\cdot; x)$ is diagonal in mass for some x. Then*

$$F_2^{\mu\nu}(\zeta_1, \zeta_2 + i\pi; x) = e^{i(P(\zeta_1) - P(\zeta_2)).x} G_{\text{free}}^{\mu\nu}(\tfrac{\zeta_1+\zeta_2}{2}) F(\zeta_2 - \zeta_1 + i\pi) \tag{5.41}$$

with

$$G_{\text{free}}^{\mu\nu}(\zeta) := \frac{M^2}{2\pi} \begin{pmatrix} \operatorname{ch}^2 \zeta & \operatorname{sh}\zeta \operatorname{ch}\zeta \\ \operatorname{sh}\zeta \operatorname{ch}\zeta & \operatorname{sh}^2 \zeta \end{pmatrix}^{\mu\nu}. \tag{5.42}$$

The energy density, in particular, becomes

$$F_2^{00}(\theta, \eta + i\pi; x) = \frac{M^2}{2\pi} \operatorname{ch}^2\left(\frac{\theta+\eta}{2}\right) e^{i(P(\theta) - P(\eta)).x} F(\eta - \theta + i\pi). \tag{5.43}$$

Proof. On $X \in \mathcal{K}^{\otimes 2}$ which is diagonal in mass we can simplify

$$P(\zeta_1, \zeta_2 + i\pi)X = \big(p(\zeta_1; M) - p(\zeta_2; M)\big)X \tag{5.44}$$

$$= M \begin{pmatrix} \operatorname{ch}\zeta_1 - \operatorname{ch}\zeta_2 \\ \operatorname{sh}\zeta_1 - \operatorname{sh}\zeta_2 \end{pmatrix} X \tag{5.45}$$

$$= 2M \operatorname{sh}\tfrac{\zeta_1 - \zeta_2}{2} \begin{pmatrix} \operatorname{sh}\frac{\zeta_1+\zeta_2}{2} \\ \operatorname{ch}\frac{\zeta_1+\zeta_2}{2} \end{pmatrix} X \tag{5.46}$$

$$= -2i \operatorname{sh}\tfrac{\zeta_1 - \zeta_2}{2} p(\tfrac{\zeta_1+\zeta_2+i\pi}{2}; M)X. \tag{5.47}$$

Then note that $\mathcal{L}(\lambda p) = \mathcal{L}(p)$ for any $\lambda \in \mathbb{C}$ and $p \in \mathbb{M}$ so that

$$\mathcal{L}(P(\zeta_1, \zeta_2 + i\pi))X = \mathcal{L}(-2i \operatorname{sh}\tfrac{\zeta_1-\zeta_2}{2} p(\tfrac{\zeta_1+\zeta_2+i\pi}{2}; M))X = \mathcal{L}(p(\tfrac{\zeta_1+\zeta_2+i\pi}{2}; M))X. \tag{5.48}$$

Thus defining $G^{\mu\nu}_{\text{free}}(\zeta) = \mathcal{L}^{\mu\nu}(p(\zeta + \frac{i\pi}{2}))$ yields the proposed form of F_2 (5.41). Equation (5.42) follows by (5.32) and

$$p(\theta; M)^2 = (p^0(\theta; M))^2 - (p^1(\theta; M))^2 = M^2(\operatorname{ch}^2\theta - \operatorname{sh}^2\theta) = M^2.$$

□

Let us remark now, that F as appearing in (5.24) or (5.41) is actually the two-particle form factor of the trace of the stress-energy tensor $g_{\mu\nu}T^{\mu\nu}$. That this form factor only depends on the difference of the rapidities is a consequence of the trace being invariant under Lorentz transformations (Prop. 3.3.4(b)). We saw that the other factors determining $F_2^{\mu\nu}$, the exponential factor and $G^{\mu\nu}_{\text{free}}$, are basically required by covariance under Poincaré transformations and do not carry much model specific information apart from the dependence on the mass spectrum. On the other hand, F depends directly on the interaction and the model's particle spectrum via the properties (b)–(h) listed in Theorem 5.3.1. In that regard, it will be important to have more information on F.

To that end, recall from Section 4.1 that F—at least for parity-invariant models (Prop. 4.1.5)—can be decomposed into an observable-specific part Q and a model-dependent part consisting of the minimal solution and, if present, factors representing the bound state poles:

Proposition 5.3.4. *Given a model with a parity-invariant S-function S with eigenvalues s_i and constant eigenprojectors P_i, $i \in \{1, ..., k\}$, such that the minimal solutions $f_{i,\min}$ with respect to s_i exist. Then a parity-invariant diagonal-in-mass stress-energy tensor at one-particle level with poles $\mathfrak{P}$ (Defn. 5.2.1) is of the form (5.24) with F given by*

$$F(\zeta) = \sum_{i=1}^{k} Q_i(\operatorname{ch}\zeta) \frac{f_{i,\min}(\zeta)}{d_i(\operatorname{ch}\zeta)}. \tag{5.49}$$

Here d_i, Q_i are polynomials: d_i is $\mathbb{C}$-valued and normalized as $d_i(-1) = 1$, has only first-order zeroes, and the zero-set of $d_i \circ \operatorname{ch}$ restricted to $\mathbb{S}(0, \pi)$ agrees with $\mathfrak{P}$. Q_i takes values in $P_i\mathcal{K}^{\otimes 2}$ and satisfies for all $x \in \mathbb{C}$ that

1. $Q_i(x) = V(g)^{\otimes 2} Q_i(x)$
2. $Q_i(x) = J^{\otimes 2} Q_i(\bar{x})$
3. $Q_i(x) = \mathbb{F} Q_i(x)$
4. $Q_i(-1) = P_i I_{\otimes 2}$.

In case that $\mathfrak{P} \subset i(0, \pi)$, d_i has only real coefficients.

Proof. This is a direct consequence of Theorem 5.3.1 and of Proposition 4.1.5. □

The canonical stress-energy tensor of the free bosonic and fermionic models It is instructive to specialize the above discussion to free models: For a single free particle species of mass m, either a spinless boson ($S = 1$) or a Majorana fermion ($S = -1$), we have $\mathcal{K} = \mathbb{C}$, $Jz = \bar{z}$, $M = m1_{\mathbb{C}}$, $\mathcal{G} = \mathbb{Z}_2$, and $V(\pm 1) = \pm 1_{\mathbb{C}}$. The canonical expressions for the stress-energy tensor at one-particle level appear for instance in [BCF13; BC15] and are given by

$$F^{\mu\nu}_{2,\text{free},+}(\theta, \eta + i\pi; x) = G^{\mu\nu}_{\text{free}}\left(\tfrac{\theta+\eta}{2}\right) e^{i(p(\theta;m)-p(\eta;m)).x}, \tag{5.50}$$

$$F^{\mu\nu}_{2,\text{free},-}(\theta, \eta + i\pi; x) = \operatorname{ch}\tfrac{\theta-\eta}{2} F_{2,\text{free},+}(\theta, \eta + i\pi; x) \tag{5.51}$$

for the bosonic (+) and the fermionic (-) case, respectively; these conform to Definition 5.2.1, including parity covariance. Theorem 5.3.1 applies with $F_+(\zeta) = f_{+,\min}(\zeta) = 1$ and $F_-(\zeta) = f_{-,\min}(\zeta) = -i\operatorname{sh}\frac{\zeta}{2}$. Proposition 5.3.4 applies with

$k = 1$, $Q_1 = d_1 = 1$. Moreover, note that $F_n^{[T^{\mu\nu}(x)]} = 0$ for $n \neq 2$ for these examples (special case of Prop. 6.1.2).

One-particle form factor of the stress-energy tensor In some models, the one-particle form factor of the stress-energy tensor, F_1, is non-zero; in particular in models with bound states, where F_1 is linked to the residues of F_2 ((F1b) in Defn. 3.1.5 or 4.5 in Section 4.1). The general form of $F_1^{\mu\nu}(\zeta; x) := F_1^{[T^{\mu\nu}(x)]}(\zeta)$ can be determined analogous to Theorem 5.3.1. In this case the continuity equation, $P_\mu(\zeta) F_1^{\mu\nu}(\zeta; x) = 0$, implies that $F_1^{0\nu}(0; x) = 0$. Poincaré covariance yields that $F_1^{\mu\nu}(\zeta; x) = e^{ip(\zeta;M).x} \Lambda(-\zeta)^\mu_1 \Lambda(-\zeta)^\nu_1 F_1^{11}(0; 0)$ (Prop. 3.3.4(b)). As a result,

$$F_1^{\mu\nu}(\zeta; x) = e^{ip(\zeta;M).x} \begin{pmatrix} \operatorname{sh}^2 \zeta & -\operatorname{sh}\zeta \operatorname{ch}\zeta \\ -\operatorname{sh}\zeta \operatorname{ch}\zeta & \operatorname{ch}^2 \zeta \end{pmatrix} F_1, \tag{5.52}$$

where $F_1 \in \mathcal{K}$ is constant. Hermiticity and $\mathcal{G}$-invariance imply $F_1 = JF_1 = V(g)F_1$ for all $g \in \mathcal{G}$ (Prop. 3.3.4(a)+(d)). The analogues of the other conditions in Theorem 5.3.1 are automatically satisfied apart from, optionally, parity covariance which implies $F_1^\alpha = \eta_\alpha F_1^\alpha$, where η_α is the parity phase. Thus, in this case, F_1^α can only be non-zero if $\eta_\alpha = 1$.

Boost generator It is often neglected to check that the stress-energy tensor also generates the boosts. Here we verify that, at least at the one-particle level, this is automatic by the conditions we have imposed before: (t1), (t2), (t7), (t10), and (t9) (or implied conditions at the one-particle level). With a suitable mathematical interpretation, it should hold that:

$$K = \int (x^0 T^{01}(x) - x^1 T^{00}(x)) dx^1 = -\int s T^{00}(0, s) ds, \tag{5.53}$$

where at the r.h.s. we have set $x_0 = 0$ by invariance under time translations. Recall that the one-particle generator of boosts is given by $K = -i\frac{d}{d\theta}$ (Sec. 2.2). Thus, in one-particle states $\varphi, \chi \in \mathcal{D}_S \cap \mathcal{H}_1$, the l.h.s. of (5.53) amounts to

$$\langle \varphi, K\chi \rangle = -i \langle \varphi, \chi' \rangle, \tag{5.54}$$

where χ' denotes the derivative of χ. A cumbersome but straightforward computation which we defer to Appendix C.3 shows that also

$$-\int s \langle \varphi, T^{00}(0, s)\chi \rangle ds = -i \langle \varphi, \chi' \rangle ; \tag{5.55}$$

at least for φ, χ from within a single mass sector and under a likely assumption.

Chapter 6

State-independent QEI for constant scattering functions

In this section, we specialize to scattering functions which are constant, i.e., independent of rapidity, and which have a parity-invariant diagonal. This class of models contains linear QFTs with bosons and fermions (presumably, also with anyons) but also models with a simple interaction like the Federbush model (Sec. 8.3).

The section will provide a canonical candidate for the stress-energy tensor in this class of models (Sec. 6.1), a generic estimate for two-particle form factors of a certain factorizing form (Sec. 6.2), and the derivation of a state-independent QEI (Sec. 6.3) together with a discussion of the result (Sec. 6.4).

6.1 Candidate for the stress-energy tensor

Let S denote the S-function. In the constant case (S1) and (S2) imply that the S-function $S \in \mathcal{B}(\mathcal{K}^{\otimes 2})$ is a unitary and self-adjoint matrix, hence has the form $S = P_+ - P_-$ in terms of its eigenprojectors $P_\pm$ for eigenvalues ± 1. An S-function is referred to as having a *parity-invariant diagonal* if

$$\forall \zeta \in \mathbb{C}: \quad [S(\zeta), \mathbb{F}] I_{\otimes 2} = 0. \tag{6.1}$$

Clearly, parity-invariant S-functions have a parity-invariant diagonal: These satisfy $[S, \mathbb{F}] = 0$ according to (2.13). Also, all diagonal S-functions (Defn. 2.3.2) have a parity-invariant diagonal as will be shown in Lemma 6.4.1. Written in a basis, (6.1) amounts to

$$\sum_\gamma \left(S^{\alpha\beta}_{\gamma\bar{\gamma}} - S^{\beta\alpha}_{\gamma\bar{\gamma}} \right) = 0. \tag{6.2}$$

The setup yields two important simplifications. First, a good candidate for the stress-energy tensor at one-particle level can be given very explicitly:

Proposition 6.1.1. *Assume a constant S-function S which has a parity-invariant diagonal. Then*

$$F(\zeta) := \left(P_+ - i \operatorname{sh} \tfrac{\zeta}{2} P_- \right) I_{\otimes 2} \tag{6.3}$$

satisfies the conditions (a) to (h) from Thm. 5.3.1 with respect to S. Thus $F_2^{\mu\nu}$ as given in (5.24) is a parity-covariant stress-energy tensor at one-particle level which *is diagonal in mass (Cor. 5.3.3).*

Proof. To begin with, let us note that $\mathbb{F} I_{\otimes 2} = J^{\otimes 2} I_{\otimes 2} = V(g)^{\otimes 2} I_{\otimes 2} = I_{\otimes 2}$ for all $g \in \mathcal{G}$ and that $\|I_{\otimes 2}\|_{\mathcal{K}}^2 = d_{\mathcal{K}}$, all due to Lemma A.6.2. Now, we show step by step that properties (a)–(h) hold:

First, using $\|P_\pm I_{\otimes 2}\|_{\mathcal{K}} \leq \|P_\pm\| \|I_{\otimes 2}\|_{\mathcal{K}} = \|I_{\otimes 2}\|_{\mathcal{K}} = \sqrt{d_{\mathcal{K}}}$, we may estimate that

$$\forall |\operatorname{Re}\zeta| \geq r : \quad \|F(\zeta)\|_{\mathcal{K}} \leq (1 + |\operatorname{sh}\tfrac{\zeta}{2}|)\sqrt{d_{\mathcal{K}}} \leq \tfrac{\sqrt{d_{\mathcal{K}}}}{2}(1 + 2e^{-\frac{r}{2}} + e^{-r})e^{\frac{1}{2}|\operatorname{Re}\zeta|}.$$

Thus (a) is satisfied.

Second, F is clearly analytic; thus (b) is satisfied with $\mathfrak{P} = \emptyset$.

Third, (c) holds since

$$\begin{aligned} F(\zeta) &= \left(P_+ - i \operatorname{sh}\tfrac{\zeta}{2} P_-\right) I_{\otimes 2} \\ &= \left(P_+ P_+ - i \operatorname{sh}\tfrac{\zeta}{2} P_- P_-\right) I_{\otimes 2} \\ &= (P_+ - P_-)\left(P_+ + i \operatorname{sh}\tfrac{\zeta}{2} P_-\right) I_{\otimes 2} \\ &= S\left(P_+ - i \operatorname{sh}\tfrac{-\zeta}{2} P_-\right) I_{\otimes 2} \\ &= S F(-\zeta). \end{aligned} \tag{6.4}$$

We used here that, since $P_\pm$ are eigenprojectors, they satisfy $P_\pm^2 = P_\pm$ and $P_+ P_- = P_- P_+ = 0$.

Fourth,

$$F(\zeta + i\pi) = \left(P_+ + \operatorname{ch}\tfrac{\zeta}{2} P_-\right) I_{\otimes 2} = F(-\zeta + i\pi). \tag{6.5}$$

Now, note that an operator which commutes with S automatically also commutes with $P_\pm$. This is due to its representation as $P_\pm = \frac{1}{2}(1 \pm S)$ and will be used repeatedly in the following. As $[S, \mathbb{F}] I_{\otimes 2} = 0$ holds by assumption one has also $[P_\pm, \mathbb{F}] I_{\otimes 2} = 0$. This implies that

$$\begin{aligned} \mathbb{F} F(\zeta + i\pi) &= \mathbb{F}\left(P_+ + \operatorname{ch}\tfrac{\zeta}{2} P_-\right) I_{\otimes 2} \\ &= \left(P_+ + \operatorname{ch}\tfrac{\zeta}{2} P_-\right) \mathbb{F} I_{\otimes 2} \\ &= \left(P_+ + \operatorname{ch}\tfrac{\zeta}{2} P_-\right) I_{\otimes 2} \\ &= F(\zeta + i\pi). \end{aligned} \tag{6.6}$$

We used here that $\mathbb{F} I_{\otimes 2} = I_{\otimes 2}$. Equations (6.5) and (6.6) imply (d) and (h).

Fifth, CPT invariance of S (S3) implies $[S, J^{\otimes 2}\mathbb{F}] = 0$; and together with $[S, \mathbb{F}] I_{\otimes 2} = 0$ it implies $[J^{\otimes 2}, S] I_{\otimes 2} = 0$. Thus

$$\begin{aligned} J^{\otimes 2} F(\bar{\zeta} + i\pi) &= J^{\otimes 2}\left(P_+ + \operatorname{ch}\tfrac{\bar{\zeta}}{2} P_-\right) I_{\otimes 2} \\ &= \left(P_+ + \operatorname{ch}\tfrac{\zeta}{2} P_-\right) J^{\otimes 2} I_{\otimes 2} \\ &= \left(P_+ + \operatorname{ch}\tfrac{\zeta}{2} P_-\right) I_{\otimes 2} \\ &= F(\zeta + i\pi). \end{aligned} \tag{6.7}$$

We used here that J is antilinear and that $J^{\otimes 2} I_{\otimes 2} = I_{\otimes 2}$.

Analogously, one also proves $F(\zeta) = V(g)^{\otimes 2} F(\zeta)$ for all $g \in \mathcal{G}$ by using (S7), $[S, V(g)^{\otimes 2}] = 0$, and $V(g)^{\otimes 2} I_{\otimes 2} = I_{\otimes 2}$. Lastly, using $\mathbb{1} = P_+ + P_-$, we have

$$F(i\pi) = (P_+ + P_-) I_{\otimes 2} = I_{\otimes 2}$$

implying (g). □

As a second simplification, for constant diagonal S the form factor equations for F_n $(n > 2)$ simplify significantly and the candidate from above is sufficient to define a full stress-energy tensor, i.e., a stress-energy tensor for states with arbitrary particle number. We support this by proving the following proposition:

Proposition 6.1.2. *Assume a model with a constant S-function and no bound states. Then the family $\{F_n\}_{n\in\mathbb{N}_0}$ satisfies the form factor equations corresponding to $A \in \mathcal{Q}$, provided that $F_n = 0$ for $n \neq 2$ and F_2 is the two-particle form factor corresponding to A and of the form $F_2(\boldsymbol{\zeta}) = F(\boldsymbol{\zeta}) I_{\otimes 2}$ for some complex-valued function F.*

Proof. Recall the residue formula connecting F_n with F_{n-2} (F1a)

$$\operatorname*{res}_{\zeta_2-\zeta_1=i\pi} F_n(\boldsymbol{\zeta}) = -\tfrac{1}{2\pi i}\left(\mathbb{1} - \sigma^n_{1A} \prod_{p=3}^{n} (\mathbb{F}S(\zeta_2-\zeta_p))_{2,p}\right)(\check{C} \otimes F_{n-2}(\boldsymbol{\zeta}_{3n})). \tag{6.8}$$

Note that in the absence of bound states, this equation provides the only connection between form factors with different particle numbers and all the other conditions (F2), (F3), and (F4) are linear in each F_n, separately. As a result, setting $F_n = 0$ for all $n \neq 2$ is consistent provided that the r.h.s. of (6.8) vanishes for $n = 4$. Specializing to that case and constant S-function, (6.8) becomes

$$\operatorname*{res}_{\zeta_2-\zeta_1=i\pi} F_4(\boldsymbol{\zeta}) = -\tfrac{1}{2\pi i}\left(\mathbb{1} - (\mathbb{F}S)_{2,3}(\mathbb{F}S)_{2,4}\right)(\check{C} \otimes F_2(\zeta_3,\zeta_4)). \tag{6.9}$$

Note here that the statistics factor σ^n_{1A} evaluates to 1 independent of the statistics. In a basis, according to Lemma 6.5.1 this evaluates to

$$\begin{aligned}\operatorname*{res}_{\zeta_2-\zeta_1=i\pi} F_4(\boldsymbol{\zeta})^{\boldsymbol{\beta}} &= -\tfrac{1}{2\pi i}\left(\mathbb{1} - (\mathbb{F}S_{2,3})(\mathbb{F}S)_{2,4}\right)^{\boldsymbol{\beta}}_{\boldsymbol{\alpha}} C^{\alpha_1\alpha_2} F_2(\zeta_3,\zeta_4)^{\alpha_3\alpha_4} \\ &= -\tfrac{1}{2\pi i}(\delta^{\boldsymbol{\beta}}_{\boldsymbol{\alpha}} - \delta^{\beta_1}_{\alpha_1} S^{\beta_3\beta_2}_{\gamma\alpha_3} S^{\beta_4\gamma}_{\alpha_2\alpha_4}) C^{\alpha_1\alpha_2} F_2(\zeta_3,\zeta_4)^{\alpha_3\alpha_4}.\end{aligned} \tag{6.10}$$

Now, due to $F_2(\zeta_3,\zeta_4) = F(\zeta_3,\zeta_4) I_{\otimes 2}$ and $(I_{\otimes 2})^{\alpha\beta} = \delta^{\alpha,\bar{\beta}}$, we have that $F_2(\zeta_3,\zeta_4)^{\alpha_3\alpha_4}$ is proportional to $\delta^{\alpha_3,\bar{\alpha}_4}$. Moreover, due to crossing symmetry (S5) we have $S^{\beta_4\gamma}_{\alpha_2\alpha_4} = S^{\gamma\bar{\alpha}_4}_{\bar{\beta}_4\alpha_2}$; confer (2.11). As a result,

$$\begin{aligned}\delta^{\beta_1}_{\alpha_1} S^{\beta_3\beta_2}_{\gamma\alpha_3} S^{\beta_4\gamma}_{\alpha_2\alpha_4} \delta^{\alpha_3,\bar{\alpha}_4} &= \delta^{\beta_1}_{\alpha_1} S^{\beta_3\beta_2}_{\gamma\alpha_3} S^{\gamma\bar{\alpha}_4}_{\bar{\beta}_4\alpha_2} \delta^{\alpha_3,\bar{\alpha}_4} \\ &= \delta^{\beta_1}_{\alpha_1} S^{\beta_3\beta_2}_{\gamma\alpha_3} S^{\gamma\alpha_3}_{\bar{\beta}_4\alpha_2} \\ &= \delta^{\beta_1}_{\alpha_1} \delta^{\beta_3}_{\bar{\beta}_4} \delta^{\beta_2}_{\alpha_2} \\ &= \delta^{\boldsymbol{\beta}}_{\boldsymbol{\alpha}} \delta^{\alpha_3,\bar{\alpha}_4}\end{aligned} \tag{6.11}$$

so that the r.h.s. of (6.10) vanishes. Since F_2 was chosen to be a solution to the form factor equations at two-particle level, this concludes the proof. □

Recalling the expansion (3.1) of a local operator A, then its lowest-order terms are

$$A_0[F_0] = F_0 \mathbb{1}, \tag{6.12}$$

$$A_1[F_1] = z_S^\dagger(F_1) + z_S(JF_1(\cdot + i\pi)), \tag{6.13}$$

$$A_2[F_2] = \frac{1}{2} z_S^\dagger z_S^\dagger(F_2) + z_S^\dagger z((1 \otimes J)F_2(\cdot, \cdot + i\pi)) + \frac{1}{2} z_S z_S(J^{\otimes 2} F_2(\cdot + i\pi, \cdot + i\pi)). \tag{6.14}$$

Provided that the equivalence of the form factor equations (Defn. 3.1.5) and locality of A holds—which is likely but not proven here for states with generic particle numbers—then in view of our candidate $F_2^{\mu\nu}$ (6.3) fulfilling the assumptions of Proposition 6.1.2,

$$T^{\mu\nu}(x) := A_2[F_2^{\mu\nu}(\cdot; x)] \tag{6.15}$$

defines a local observable (upon smearing with some test function). In this simple case locality might also be checked by direct computation from (T1)–(T4). Moreover, properties (T5)–(T12) imply that $T^{\mu\nu}$ is hermitian, is a symmetric covariant two-tensor-valued field with respect to $U_S(x, \lambda)$, integrates to the total energy-momentum operator $P^\mu = \int ds\, T^{\mu 0}(t, s)$ (at least weakly on $\mathcal{D}_{S,n}$, $n \in \mathbb{N}$), and is conserved, $\partial_\mu T^{\mu\nu} = 0$. Hence $T^{\mu\nu}$ is a good candidate for the stress-energy tensor of the interacting model. This expression is in agreement with the expression for the free canonical stress-energy tensor; cf. (5.50) and (5.51). Also for the Federbush model it agrees with the candidate proposed in [CF01, Sec. 4.2.3]; confer also Section 8.3.

6.2 A generic estimate

For the $T^{\mu\nu}$ defined in the preceding section (Eq. (6.15)), we aim to establish a QEI result. Our main technique is a generic estimate—applicable to arbitrary S-functions—for two-particle form factors of a specific factorizing form.

Lemma 6.2.1. *Let S be a (not necessarily constant) S-function and $A \in \mathcal{Q}$. Let $h : \mathbb{S}(0, \pi) \to \mathcal{K}$ be analytic with L^2 boundary values at $\mathbb{R}$ and $\mathbb{R} + i\pi$. For*

$$f := \mathcal{P}_S(h \otimes Jh(\bar{\cdot} + i\pi)), \tag{6.16}$$

we have in the sense of quadratic forms on $\mathcal{D}_S \times \mathcal{D}_S$,

$$A_2[f] \geq -\frac{1}{2} \|h(\cdot + i\pi)\|_2^2 \mathbb{1}. \tag{6.17}$$

Proof. Introduce $h' := Jh(\bar{\cdot} + i\pi)$ and note that $h'' = h$ because of

$$h''(\zeta) = Jh'(\bar{\zeta} + i\pi) = J^2 h(\overline{\bar{\zeta} + i\pi} + i\pi) = h(\zeta).$$

To have a light notation for computations let us write $z^\sharp = z_S^\sharp$ and use J to denote both, the operator on $\mathcal{K}$ and on $\mathcal{H}_1$ (denoted in the main text by J and $U_1(j)$). All of the following computations should be understood in the sense of quadratic forms on $\mathcal{D}_S \times \mathcal{D}_S$.

Using $z(h)^* = z^\dagger(h)$, we expand

$$\begin{aligned} A_1[h]A_1[h]^* &= (z^\dagger(h) + z(h'))(z^\dagger(h) + z(h'))^* \\ &= (z^\dagger(h) + z(h'))(z(h) + z^\dagger(h')) \\ &= z^\dagger z^\dagger(h \otimes h') + zz^\dagger(h'^{\otimes 2}) + z^\dagger z(h^{\otimes 2}) + zz(h' \otimes h). \end{aligned} \tag{6.18}$$

Using the ZF algebra relations (2.30), one may replace

$$z^\dagger z^\dagger(h \otimes h') = z^\dagger z^\dagger(\tfrac{1}{2}(1 + S_\leftarrow)(h \otimes h')) = z^\dagger z^\dagger(\mathcal{P}_S(h \otimes h')), \tag{6.19}$$

$$zz(h' \otimes h) = zz(\tfrac{1}{2}(1 + J^{\otimes 2} S_\leftarrow J^{\otimes 2})(h' \otimes h)) = zz(\mathcal{P}_{J^{\otimes 2} S J^{\otimes 2}}(h' \otimes h)), \tag{6.20}$$

$$zz^\dagger(h'^{\otimes 2}) = z^\dagger z((1 \otimes J)S_\leftarrow^{i\pi}(J \otimes 1)h'^{\otimes 2}) + \|h'\|_2^2 \mathbb{1}, \tag{6.21}$$

and obtain

$$\begin{aligned} A_1[h]A_1[h]^* = z^\dagger z^\dagger[\mathcal{P}_S(h \otimes h')] + z^\dagger z[h^{\otimes 2} + (1 \otimes J)S_\leftarrow^{i\pi}(J \otimes 1)h'^{\otimes 2}] \\ + zz[\mathcal{P}_{J^{\otimes 2} S J^{\otimes 2}}(h' \otimes h)] + \|h'\|_2^2 \mathbb{1}. \end{aligned} \tag{6.22}$$

On the other hand, looking at $A_2[f]$ (6.14) with $f = \mathcal{P}_S(h \otimes h')$ we find

$$\begin{aligned} 2A_2[f] &= z^\dagger z^\dagger[f] + 2z^\dagger z[(1 \otimes J)f(\cdot, \cdot + i\pi)] + zz[J^{\otimes 2} f(\cdot + i\pi, \cdot + i\pi)] \\ &= z^\dagger z^\dagger[\mathcal{P}_S(h \otimes h')] + z^\dagger z[2(1 \otimes J)\, \mathcal{P}_S(h \otimes h')(\cdot, \cdot + i\pi)] \\ &\quad + zz[J^{\otimes 2}\, \mathcal{P}_S(h \otimes h')(\cdot + i\pi, \cdot + i\pi)]. \end{aligned} \tag{6.23}$$

Now, using

$$\begin{aligned} &2(1 \otimes J)\, \mathcal{P}_S(h \otimes h')(\theta_1, \theta_2 + i\pi) \\ =&(1 \otimes J)(1 + S_\leftarrow)(h \otimes h')(\theta_1, \theta_2 + i\pi) \\ =&h(\theta_1) \otimes Jh'(\theta_2 + i\pi) + (1 \otimes J)S(\theta_2 - \theta_1 + i\pi)(h(\theta_2 + i\pi) \otimes h'(\theta_1)) \\ =&h(\theta_1) \otimes Jh'(\theta_2 + i\pi) + (1 \otimes J)S_\leftarrow^{i\pi}(J \otimes 1)(Jh(\theta_2 + i\pi) \otimes h'(\theta_1)) \\ =&h(\theta_1) \otimes h(\theta_2) + (1 \otimes J)S_\leftarrow^{i\pi}(J \otimes 1)(h'(\theta_2) \otimes h'(\theta_1)) \\ =&\left(h^{\otimes 2} + (1 \otimes J)S_\leftarrow^{i\pi}(J \otimes 1)h'^{\otimes 2}\right)(\boldsymbol{\theta}), \end{aligned} \tag{6.24}$$

and

$$\begin{aligned} & J^{\otimes 2}\,\mathcal{P}_S(h\otimes h')(\boldsymbol{\theta}+i\boldsymbol{\pi}) \\ \overset{(S2)}{=} & J^{\otimes 2}\tfrac{1}{2}(1+S_{\leftarrow})(h\otimes h')(\boldsymbol{\theta}+i\pi) \\ = & J^{\otimes 2}\tfrac{1}{2}(1+S_{\leftarrow})(Jh'\otimes Jh)(\boldsymbol{\theta}) \\ = & J^{\otimes 2}\tfrac{1}{2}(1+S_{\leftarrow})J^{\otimes 2}(h'\otimes h)(\boldsymbol{\theta}) \\ = & \tfrac{1}{2}(1+J^{\otimes 2}S_{\leftarrow}J^{\otimes 2})(h'\otimes h)(\boldsymbol{\theta}) \\ = & \mathcal{P}_{J^{\otimes 2}SJ^{\otimes 2}}(h'\otimes h)(\boldsymbol{\theta}), \end{aligned} \tag{6.25}$$

upon (6.23) we find

$$2A_2[f]=z^\dagger z^\dagger(\mathcal{P}_S(h\otimes h'))+z^\dagger z(h^{\otimes 2}+(1\otimes J)S_{\leftarrow}^{i\pi}(J\otimes 1)h'^{\otimes 2})+zz(\mathcal{P}_{J^{\otimes 2}SJ^{\otimes 2}}(h'\otimes h)). \tag{6.26}$$

Comparing (6.22) and (6.26) we obtain

$$A_1[h]A_1[h]^\dagger = 2A_2[\mathcal{P}_S(h\otimes h')]+\|h'\|_2^2\mathbb{1}. \tag{6.27}$$

Lastly, note that the l.h.s is positive as a quadratic form, implying the result. □

6.3 Derivation of the QEI

Our approach is to decompose F_2^{00} into sums and integrals over terms of the factorizing type (6.16) with positive coefficients, then applying the estimate (6.17) to each of them.

To that end, we will call a vector $X\in\mathcal{K}^{\otimes 2}$ *positive* if

$$\forall u\in\mathcal{K}:(u\otimes Ju,X)\geq 0. \tag{6.28}$$

This is equivalent to X being a sum of mutually orthogonal vectors of the form $e\otimes Je$ with positive coefficients:

Lemma 6.3.1. *A vector $X\in\mathcal{K}^{\otimes 2}$ is positive iff there exist $r\in\{0,...,d_{\mathcal{K}}\}$, coefficients $c_\alpha>0$, and orthonormal vectors $e_\alpha\in\mathcal{K}$ for $\alpha=1,..,r$ such that*

$$X=\sum_{\alpha=1}^{r}c_\alpha e_\alpha\otimes Je_\alpha. \tag{6.29}$$

Proof. Vectors of the form $e\otimes Je$ with $e\in\mathcal{K}$ are certainly positive since

$$(u\otimes Ju,e\otimes Je)=(u,e)(e,u)=|(u,e)|^2\geq 0$$

and remain positive when summed with positive coefficients. Conversely, given a positive X, we note that $\hat{X}\in\mathcal{B}(\mathcal{K})$ is a positive matrix, as $(u,\hat{X}u)=(u\otimes Ju,X)\geq 0$.

Its eigendecomposition[1] is thus of the form $\hat{X} = \sum_{\alpha=1}^{r} c_\alpha |e_\alpha)(e_\alpha|$ for some $r \in \{0, ..., d_{\mathcal{K}}\}$, $c_\alpha > 0$, and orthonormal vectors $e_\alpha \in \mathcal{K}$ with $\alpha = 1, .., r$. Finally, note that due to

$$(u, |e)(e|v) = (u, e)(e, v) = (u, e)(Jv, Je) = (u \otimes Jv, e \otimes Je)$$

for arbitrary $u, v \in \mathcal{K}$ one has $\widehat{e \otimes Je} = |e)(e|$. As a consequence, X is of the required form. □

We also recall the notion of a vector *diagonal in mass* (Eq. (5.40)). Now we establish our master estimate as follows:

Lemma 6.3.2. *Fix $n \in \{0, 1\}$. Suppose that $X \in \mathcal{K}^{\otimes 2}$ is positive, diagonal in mass, and satisfies $SX = (-1)^n X$. Let $h : \mathbb{S}(0, \pi) \to \mathbb{C}$ be analytic with continous boundary values at $\mathbb{R}$ and $\mathbb{R}+i\pi$ such that $|h(\zeta)| \leq a \exp(b|\operatorname{Re}\zeta|)$ for some $a, b > 0$. Let $g \in \mathcal{D}_{\mathbb{R}}(\mathbb{R})$. Set*

$$F_2 := \operatorname{Symm}_S \left(\boldsymbol{\zeta} \mapsto h(\zeta_1)\overline{h(\bar{\zeta}_2 + i\pi)}(\operatorname{ch}\zeta_1 - \operatorname{ch}\zeta_2)^n \widetilde{g^2}(P_0(\boldsymbol{\zeta}))X \right). \tag{6.30}$$

Then, in the sense of quadratic forms on $\mathcal{D}_S \times \mathcal{D}_S$,

$$A_2[F_2] \geq -\int_0^\infty \frac{d\nu}{4\pi}(2\nu)^n \left(I_{\otimes 2}, M\left(N_+(\nu, M) + N_-(\nu, M)\right) X\right)_{\mathcal{K}^{\otimes 2}} \mathbb{1}, \tag{6.31}$$

where the integral is convergent and where

$$N_\pm(\nu, m) = \|h(\cdot + \tfrac{1\pm1}{2} i\pi)\tilde{g}(p_0(\cdot; m) + m\nu)\|_2^2. \tag{6.32}$$

Proof. Since X is diagonal in mass, we have $X = \sum_{m\in\mathfrak{M}} E_m^{\otimes 2} X$. $E_m^{\otimes 2} X$ shares the assumed properties with X; it is positive, diagonal in mass, and satisfies $SE_m^{\otimes 2} = (-1)^n E_m^{\otimes 2} X$ as well as $[S, \mathbb{F}]E_m^{\otimes 2} X = 0$. The latter two properties are inferred from $[\mathbb{F}, E_m^{\otimes 2}] = 0$ and $[S, E_m^{\otimes 2}] = 0$ which is due to (S6). As a consequence, we may assume without loss of generality that $X = E_m^{\otimes 2} X$.

To begin with, we collect three facts: By positivity of X and Lemma 6.3.1 we obtain

$$X = \sum_{\alpha=1}^{r} c_\alpha\, e_\alpha \otimes Je_\alpha \tag{6.33}$$

with $r \in \mathbb{N}$, $c_\alpha > 0$ and orthonormal vectors $e_\alpha \in \mathcal{K}$, $\alpha = 1, .., r$.

Second, there is the convolution formula[2] ($n \in \{0, 1\}$, $p_1, p_2 \in \mathbb{C}$),

$$(p_1 - p_2)^n \widetilde{g^2}(p_1 + p_2) = \int \frac{d\nu}{2\pi}(2\nu)^n \tilde{g}(p_1 - \nu)\overline{\tilde{g}(-\bar{p}_2 - \nu)} \tag{6.34}$$

[1] Note that all positive semidefinite matrices are diagonalizable.

[2] For $n = 0$ this is a standard theorem of Fourier analysis. A proof of this statement for $p, p' \in \mathbb{R}$ and $n = 1$ is found in [FV02, Lemma 6.1].

which we will apply for $p_1 = m\,\mathrm{ch}\,\zeta_1$ and $p_2 = m\,\mathrm{ch}\,\zeta_2$.

Third, since g is real-valued, it holds that

$$\overline{\tilde{g}(\bar{p})} = \tilde{g}(-p). \tag{6.35}$$

Now, let

$$h^+_{\nu,\alpha}(\zeta) = h(\zeta)\tilde{g}(p_0(\zeta) - \nu)e_\alpha, \qquad h^-_{\nu,\alpha}(\zeta) = \overline{h^+_{-\nu,\alpha}(\bar{\zeta} + i\pi)} \tag{6.36}$$

and let $f^\pm_{\nu,\alpha}$ relate to $h^\pm_{\nu,\alpha}$ as in (6.16):

$$f^\pm_{\nu,\alpha} = \mathcal{P}_S(h^\pm_{\nu,\alpha} \otimes J h^\pm_{\nu,\alpha}(\bar{\cdot} + i\pi)). \tag{6.37}$$

Then taking into account the three facts (6.33), (6.34), and (6.35) we compute

$$\begin{aligned}
&h(\zeta_1)\overline{h(\bar{\zeta}_2 + i\pi)}(\mathrm{ch}\,\zeta_1 - \mathrm{ch}\,\zeta_2)^n \widetilde{g^2}(P_0(\boldsymbol{\zeta}))X \\
=&h(\zeta_1)\overline{h(\bar{\zeta}_2 + i\pi)} m^{-n}(m\,\mathrm{ch}\,\zeta_1 - m\,\mathrm{ch}\,\zeta_2)^n \widetilde{g^2}(m\,\mathrm{ch}\,\zeta_1 + m\,\mathrm{ch}\,\zeta_2)X \\
\overset{(6.33)}{=}& \sum_{\alpha=1}^r c_\alpha\, h(\zeta_1)\overline{h(\bar{\zeta}_2 + i\pi)} m^{-n}(m\,\mathrm{ch}\,\zeta_1 - m\,\mathrm{ch}\,\zeta_2)^n \widetilde{g^2}(m\,\mathrm{ch}\,\zeta_1 + m\,\mathrm{ch}\,\zeta_2)(e_\alpha \otimes J e_\alpha) \\
\overset{(6.34)}{=}& \sum_{\alpha=1}^r c_\alpha \int \frac{d\nu}{2\pi} (\tfrac{2\nu}{m})^n h(\zeta_1)\overline{h(\bar{\zeta}_2 + i\pi)}\tilde{g}(m\,\mathrm{ch}\,\zeta_1 - \nu)\overline{\tilde{g}(-m\,\mathrm{ch}\,\bar{\zeta}_2 - \nu)}(e_\alpha \otimes J e_\alpha) \\
\overset{(6.35)}{=}& \sum_{\alpha=1}^r c_\alpha \int \frac{d\nu}{2\pi} (\tfrac{2\nu}{m})^n \left(h(\zeta_1)\tilde{g}(m\,\mathrm{ch}\,\zeta_1 - \nu)e_\alpha \otimes J h(\bar{\zeta}_2 + i\pi)\tilde{g}(-m\,\mathrm{ch}\,\bar{\zeta}_2 - \nu)e_\alpha\right) \\
\overset{(6.36)}{=}& \sum_{\alpha=1}^r c_\alpha \int \frac{d\nu}{2\pi} (\tfrac{2\nu}{m})^n \left(h^+_{\nu,\alpha}(\zeta_1) \otimes J h^+_{\nu,\alpha}(\bar{\zeta}_2 + i\pi)\right).
\end{aligned} \tag{6.38}$$

Next, define $f^\pm_\nu := \sum_{\alpha=1}^r c_\alpha f^\pm_{\nu,\alpha}$ and apply $\mathcal{P}_S$ to (6.38). We then find

$$\begin{aligned}
F_2(\boldsymbol{\zeta}) &= \sum_{\alpha=1}^r c_\alpha \int \frac{d\nu}{2\pi} (\tfrac{2\nu}{m})^n f^+_{\nu,\alpha}(\boldsymbol{\zeta}) \\
&= \int \frac{d\nu}{2\pi} (\tfrac{2\nu}{m})^n f^+_\nu(\boldsymbol{\zeta}) \\
&= \int_0^\infty \frac{d\nu}{2\pi} (\tfrac{2\nu}{m})^n \left(f^+_\nu(\boldsymbol{\zeta}) + (-1)^n f^+_{-\nu}(\boldsymbol{\zeta})\right) \\
&= \int_0^\infty \frac{d\nu}{2\pi} (\tfrac{2\nu}{m})^n \left(f^+_\nu(\boldsymbol{\zeta}) + f^-_\nu(\boldsymbol{\zeta})\right),
\end{aligned} \tag{6.39}$$

where the latter equality used $(-1)^n f^+_{-\nu} = f^-_\nu$; see Lemma 6.5.2.

Applying the Lemma 6.2.1 to each $f^\pm_{\nu,\alpha}$ in (6.39), and rescaling $\nu \to m\nu$, we obtain that

$$A_2[f^\pm_\nu] = \sum_{\alpha=1}^r c_\alpha A_2[\mathcal{P}_S(h^\pm_{\nu,\alpha} \otimes J h^\pm_{\nu,\alpha}(\bar{\cdot} + i\pi))] \geq -\tfrac{1}{2} \sum_{\alpha=1}^r c_\alpha \|h^\pm_{\nu,\alpha}(\cdot + i\pi)\|_2^2 \mathbb{1}. \tag{6.40}$$

That the lemma is applicable can be argued as follows: For fixed ν and α $h^{\pm}_{\nu,\alpha}$ is analytic everywhere and L^2 on $\mathbb{R}$ and $\mathbb{R}+i\pi$ since $\zeta \mapsto h(\zeta)\tilde{g}(m\operatorname{ch}\zeta - \nu)$ decays rapidly on $\mathbb{R}$ and $\mathbb{R}+i\pi$ in $\operatorname{ch}\operatorname{Re}\zeta$. This is due to $|h(\zeta)| \leq a\exp b|\operatorname{Re}\zeta|$ for some $a, b > 0$ and the rapid decay of $\tilde{g}$.

Now, we modify (6.40) slightly to comply with the form of the final result. Since $\|e_\alpha\| = 1$ and $|\tilde{g}(-p)| \overset{(6.35)}{=} |\tilde{g}(p)|$ for real p, we simplify

$$\begin{aligned}\|h^{+}_{m\nu,\alpha}(\cdot + i\pi)\|_2 &= \|h(\cdot + i\pi)\tilde{g}(-m\operatorname{ch}\cdot - m\nu)\|_2 \\ &= \|h(\cdot + i\pi)\tilde{g}(m\operatorname{ch}\cdot + m\nu)\|_2 = \sqrt{N_+(\nu, m)}\end{aligned} \tag{6.41}$$

and, similarly,

$$\|h^{-}_{m\nu,\alpha}(\cdot + i\pi)\|_2 = \|h(\cdot)\tilde{g}(-m\operatorname{ch}\cdot - m\nu)\|_2 = \sqrt{N_-(\nu, m)}. \tag{6.42}$$

Also, according to Lemma 6.5.3, $\sum_{\alpha=1}^r c_\alpha = (I_{\otimes 2}, X)$, so that (6.40) yields

$$A_2[f^{\pm}_{m\nu}] \geq -\tfrac{1}{2}(I_{\otimes 2}, X)N_{\pm}(\nu, m)\mathbb{1}. \tag{6.43}$$

Integrating (6.43) with $\sum_{\pm}\int\limits_0^\infty \frac{d\nu}{2\pi}(2\nu)^n m$ then yields

$$A_2[F_2] \geq -\int\limits_0^\infty \frac{d\nu}{4\pi}(2\nu)^n (I_{\otimes 2}, m\,(N_+(\nu, m) + N_-(\nu, m))\,X)\mathbb{1}. \tag{6.44}$$

A suggestive replacement $m \to M$ yields the result (for the case $X = E_m^{\otimes 2}X$). Note here that the integration in ν can be exchanged with taking the expectation value $\langle \Psi, A_2[\cdot]\Psi\rangle$. Evidently, this is allowed since for $\Psi \in \mathcal{D}_S$ the series (3.1) is actually finite and the integrations involved in taking the expectation value are over compact regions.

Concerning finiteness of the r.h.s. of (6.44), its integrand is bounded according to

$$\begin{aligned}&\left|\frac{(2\nu)^n}{4\pi}(I_{\otimes 2}, m\,(N_+(\nu, m) + N_-(\nu, m))\,X)\right| \\ \leq &\frac{m(2\nu)^n}{4\pi}\|I_{\otimes 2}\|_{\mathcal{K}^{\otimes 2}}\|X\|_{\mathcal{K}^{\otimes 2}}\,(N_+(\nu, m) + N_-(\nu, m)) \\ \leq &\frac{m(2\nu)^n}{4\pi}\sqrt{d_{\mathcal{K}}}\,\|X\|_{\mathcal{K}^{\otimes 2}}\,(N_+(\nu, m) + N_-(\nu, m))\,.\end{aligned} \tag{6.45}$$

Now, note that

$$N_+(\nu, m) + N_-(\nu, m) = \int d\theta\,(|h(\theta)|^2 + |h(\theta + i\pi)|^2)|\tilde{g}(m\operatorname{ch}\theta + m\nu)|^2. \tag{6.46}$$

By assumption $|h(\theta)| \leq a\exp(b|\theta|) \leq a(\operatorname{ch}\theta)^b$ for some $a, b > 0$, the same for $|h(\theta+i\pi)|$, and by Lemma 6.5.4 we know that $\nu^n(\operatorname{ch}\theta)^k|\tilde{g}(m\operatorname{ch}\theta+m\nu)|^2$ is integrable in (θ, ν) over $\mathbb{R}\times[0, \infty)$ for any $k \in \mathbb{N}$. In conclusion, the θ- and ν-integrals defining the r.h.s of (6.44) and (6.46) can be exchanged by Fubini-Tonelli's theorem and are finite. $\square$

Now, we can formulate

Theorem 6.3.3 (QEI for constant S-functions). *Consider a constant S-function $S \in \mathcal{B}(\mathcal{K}^{\otimes 2})$ with parity-invariant diagonal, i.e., $[S, \mathbb{F}]I_{\otimes 2} = 0$ and denote its eigenprojectors with respect to its eigenvalues ± 1 by $P_\pm$. Suppose that $P_\pm 1_{\mathcal{K}^{\otimes 2}}$ are both positive. Then for the energy density $T^{00}(x)$ defined in (6.15) and any $g \in \mathcal{D}_\mathbb{R}(\mathbb{R})$, one has in the sense of quadratic forms on $\mathcal{D}_S \times \mathcal{D}_S$:*

$$T^{00}(g^2) \geq -\left(I_{\otimes 2}, (W_+(M)P_+ + W_-(M)P_-)I_{\otimes 2}\right)_{\mathcal{K}^{\otimes 2}} \mathbb{1}, \tag{6.47}$$

where

$$W_\pm(m) := \tfrac{m^3}{4\pi^2} \int_1^\infty ds\, |\tilde{g}(ms)|^2 w_\pm(s) < \infty \tag{6.48}$$

and $w_\pm(s) := s\sqrt{s^2-1} \pm \log(s + \sqrt{s^2-1})$.

The proof of this theorem makes use of a decomposition of the energy density into a sum of five terms which are suitable to apply the master estimate. This is in analogy to decompositions of the free field which have been employed in earlier works [FE98; FM03; Daw06]. The energy density for the free bosonic field decomposes into three terms $T^{00} = \frac{1}{2}\left(:(\partial^0\phi)^2: + :(\partial^1\phi)^2: + m^2{:}\phi^2{:}\right)$. The energy density for the free fermionic field decomposes into two terms.

Proof. We use Lemma 6.3.2 five times: With $h_1(\zeta) = \operatorname{ch}\zeta$, $h_2(\zeta) = \operatorname{sh}\zeta$, $h_3(\zeta) = 1$ (all with $n = 0$ and $X = P_+ I_{\otimes 2}$) and $h_4(\zeta) = \operatorname{ch}\frac{\zeta}{2}$, $h_5(\zeta) = \operatorname{sh}\frac{\zeta}{2}$ (these with $n = 1$ and $X = P_- I_{\otimes 2}$). The chosen functions and vectors are suitable for the lemma: Clearly, the $h_i(\zeta)$ satisfy the exponential bounds and analyticity properties. The vectors $P_\pm I_{\otimes 2}$ are positive by assumption and satisfy $SP_\pm I_{\otimes 2} = \pm P_\pm I_{\otimes 2}$ by construction. Diagonality in mass is shown easily using $P_\pm = \frac{1}{2}(1 \pm S)$ and property (S6): For $m \neq m'$ we have $(E_m \otimes E_{m'})P_\pm I_{\otimes 2} = (E_m \otimes E_{m'})\frac{1}{2}(1 \pm S)I_{\otimes 2} = \frac{1}{2}((E_m \otimes E_{m'}) \pm S(E_{m'} \otimes E_m))I_{\otimes 2} = 0$.

Summation of the five terms "F_2" resulting from (6.30) by insertion of $n = n_i$, $X = X_i$, and $h = h_i$, yields, using

$$\overline{\operatorname{ch}(\bar\zeta + i\pi)} = -\operatorname{ch}\zeta, \quad \overline{\operatorname{sh}(\bar\zeta + i\pi)} = -\operatorname{sh}\zeta, \quad \overline{\operatorname{sh}\tfrac{\bar\zeta + i\pi}{2}} = -i\operatorname{ch}\tfrac{\zeta}{2}, \quad \overline{\operatorname{ch}\tfrac{\bar\zeta + i\pi}{2}} = -i\operatorname{sh}\tfrac{\zeta}{2},$$

and ignoring $\widetilde{g^2}(P_0(\boldsymbol{\zeta}))$ for the moment, that

$$\begin{aligned}
&\left(\sum_{j=1}^{3} h_j(\zeta_1)\overline{h_j(\bar{\zeta}_2+i\pi)}\right)P_+ I_{\otimes 2} + \left(\sum_{j=4}^{5} h_j(\zeta_1)\overline{h_j(\bar{\zeta}_2+i\pi)}\right)(\operatorname{ch}\zeta_1 - \operatorname{ch}\zeta_2)P_- I_{\otimes 2} \\
&= (1-\operatorname{ch}\zeta_1\operatorname{ch}\zeta_2 - \operatorname{sh}\zeta_1\operatorname{sh}\zeta_2)P_+ I_{\otimes 2} - i(\operatorname{ch}\tfrac{\zeta_1}{2}\operatorname{sh}\tfrac{\zeta_2}{2} + \operatorname{sh}\tfrac{\zeta_1}{2}\operatorname{ch}\tfrac{\zeta_2}{2})(\operatorname{ch}\zeta_1-\operatorname{ch}\zeta_2)P_- I_{\otimes 2} \\
&= (1-\operatorname{ch}(\zeta_1+\zeta_2))P_+ I_{\otimes 2} - i\operatorname{sh}\tfrac{\zeta_1+\zeta_2}{2}(\operatorname{ch}\zeta_1-\operatorname{ch}\zeta_2)P_- I_{\otimes 2} \\
&= 2\left(-\operatorname{sh}^2\tfrac{\zeta_1+\zeta_2}{2}P_+ I_{\otimes 2} - i\operatorname{sh}^2\tfrac{\zeta_1+\zeta_2}{2}\operatorname{sh}\tfrac{\zeta_1-\zeta_2}{2}P_- I_{\otimes 2}\right) \\
&= -2\operatorname{sh}^2\tfrac{\zeta_1+\zeta_2}{2}\left(P_+ I_{\otimes 2} - i\operatorname{sh}\tfrac{\zeta_2-\zeta_1}{2}P_- I_{\otimes 2}\right).
\end{aligned} \tag{6.49}$$

Recalling that $G^{00}_{\text{free}}\left(\frac{\zeta-i\pi}{2}\right) = -\frac{M^2}{2\pi}\operatorname{sh}^2\frac{\zeta}{2}$ and $F(\zeta) = (P_+ - i\operatorname{sh}\frac{\zeta}{2}P_-)I_{\otimes 2}$, we find that (6.49) multiplied by $\frac{M^2}{4\pi}\widetilde{g^2}(P_0(\boldsymbol{\zeta}))$ yields

$$G^{00}_{\text{free}}\left(\tfrac{\zeta_1+\zeta_2-i\pi}{2}\right)\widetilde{g^2}(P_0(\boldsymbol{\zeta}))F(\zeta_2-\zeta_1) = \widetilde{g^2}(P_0(\boldsymbol{\zeta}))F_2^{00}(\boldsymbol{\zeta};0) = \int dt\, g^2(t)F_2^{00}(\boldsymbol{\zeta};(t,0)) \tag{6.50}$$

so that the expression for the time-smeared energy density at one-particle level (6.15) is obtained.

From Lemma 6.3.2 we thus obtain[3]

$$\begin{aligned}
T^{00}(g^2) &= \int dt\, g^2(t) A_2[F_2^{00}(\cdot;(t,0))] \\
&\geq -\sum_{i=1}^{5}\sum_{\pm}\int_0^\infty \frac{d\nu}{16\pi^2}(2\nu)^{n_i}\left(I_{\otimes 2}, M^3 N_{\pm,i}(\nu,M)P_{s_i}I_{\otimes 2}\right)_{\mathcal{K}^{\otimes 2}}\mathbb{1}.
\end{aligned} \tag{6.51}$$

Here $s_i := (-1)^{n_i}$. We compute

$$\begin{aligned}
&\sum_{i=1}^{5}\sum_{\pm}\int_0^\infty \frac{d\nu}{16\pi^2}(2\nu)^{n_i}M^3\|h_i(\cdot+\tfrac{1\pm 1}{2}i\pi)\tilde{g}(P_0(\theta)+M\nu)\|_2^2 P_{s_i} \\
&= \frac{M^3}{8\pi^2}\int_0^\infty d\nu \int_{-\infty}^{\infty} d\theta\, |\tilde{g}(P_0(\theta)+M\nu)|^2\left((1+\operatorname{ch}^2\theta+\operatorname{sh}^2\theta)P_+ + 2\nu(\operatorname{ch}^2\tfrac{\theta}{2}+\operatorname{sh}^2\tfrac{\theta}{2})P_-\right) \\
&= \frac{M^3}{4\pi^2}\int_0^\infty d\nu \int_{-\infty}^{\infty} d\theta\, |\tilde{g}(P_0(\theta)+M\nu)|^2\left(\operatorname{ch}^2\theta P_+ + \nu\operatorname{ch}\theta P_-\right).
\end{aligned} \tag{6.52}$$

From here we proceed by changing the order of integration, and then substitute $(\theta,\nu)\to(s=\operatorname{ch}\theta+\nu, t=s-\nu)$ according to

$$\int_{\mathbb{R}\times[0,\infty)} d(\theta\times\nu) = \int_{[1,\infty)\times[0,s-1]}\frac{2d(s\times\nu)}{\sqrt{(s-\nu)^2-1}} = \int_{[1,\infty)\times[1,s]}\frac{2d(s\times t)}{\sqrt{t^2-1}}. \tag{6.53}$$

Note here again that $\nu^n(\operatorname{ch}\theta)^k|\tilde{g}(m\operatorname{ch}\theta+m\nu)|^2$ is integrable in $(\boldsymbol{\theta},\nu)$ for any $n\in\{0,1\}$, $k\in\mathbb{N}$ due to Lemma 6.5.4 so that changing the order of integration is allowed by integrability of the positive integrand and Tonelli's theorem.

[3]The change of order of A_2 and integration in t is allowed. This is again due to the decomposition of $\langle\Psi, A_2[\cdot]\Psi\rangle$ for $\Psi\in\mathcal{D}_S$ into finite sums and integrals over compact regions.

Proceeding with (6.52) and using known integral expressions

$$w_+(s) = \int_1^s \frac{2t^2 dt}{\sqrt{t^2-1}} = s\sqrt{s^2-1} + \log(s+\sqrt{s^2-1}), \tag{6.54}$$

$$w_-(s) = \int_1^s \frac{2(s-t)t dt}{\sqrt{t^2-1}} = s\sqrt{s^2-1} - \log(s+\sqrt{s^2-1}), \tag{6.55}$$

we obtain

$$\begin{aligned} &\frac{M^3}{4\pi^2}\int_1^\infty ds|\tilde{g}(Ms)|^2 \left(\int_1^s dt \left(\frac{2t^2}{\sqrt{t^2-1}}P_+ + \frac{2(s-t)t}{\sqrt{t^2-1}}P_-\right)\right) \\ =&\frac{M^3}{4\pi^2}\int_1^\infty ds|\tilde{g}(Ms)|^2 \left(w_+(s)P_+ + w_-(s)P_-\right) \\ =&W_+(M)P_+ + W_-(M)P_-. \end{aligned} \tag{6.56}$$

Note that $W_\pm(m) < \infty$ either as a consequence of Lemma 6.3.2 or by direct observation: $w_\pm(s)$ grow at most as s^2 for $s \to \infty$ and $\widetilde{g^2}(ms)$ provides rapid decay in s. □

6.4 Discussion of the QEI

In this section, we briefly discuss the range of validity of the QEI result (Thm. 6.3.3) and compare it with known results for non-interacting models. We will show that the QEI obtained here applies to a wide range of models: In addition to models where QEI results were obtained before—the free Bose field [FE98], the free Fermi field [Daw06], the Ising model [BCF13]—the result is applicable also to combinations of these models and the fermionic variant of the Ising model (see, e.g., [BC21]). It also applies to the Federbush model (and generalizations of it as in [Tan14]): Although the Federbush model's S-function is not parity invariant, it has a parity invariant diagonal and (6.3) yields a valid (parity invariant) candidate for the energy density, i.e., it satisfies all the properties (a) to (h). This is due to the fact that the candidate depends only on the parity-invariant part of the S-function. The candidate is in agreement with [CF01, Sec. 4.2.3]. For further details on the Federbush model see Section 8.3.

The QEI result is independent of the statistics of the particles; it depends only on the mass spectrum and the S-function. The aspect of particle statistics comes into play when computing the scattering function from the S-function (Prop. 2.5.3, Eq. (2.64)); it also enters the form factor equations for local operators; see, e.g., [BC21, Sec. 6]. However, in the equations for F_2 relevant for our analysis, the "statistics factors" occur only in even powers, so that our assumptions on the stress-energy tensor—specifically, properties (c) and (d)—are appropriate in both, models with bosons and fermions.

Let us clarify first, that the QEI result applies to models with constant diagonal S-functions:

Lemma 6.4.1. *All diagonal S-functions (Defn. 2.3.2) have a parity-invariant diagonal, i.e.,* $[S, \mathbb{F}]I_{\otimes 2} = 0$. *The assumptions of Theorem 6.3.3 are met whenever the model has a constant diagonal S-function.*

Proof. For a diagonal S-function S (not necessarily constant) to have a parity-invariant diagonal (Eq. (6.2)) is equivalent to $s_{\alpha\bar{\alpha}} = s_{\bar{\alpha}\alpha}$. This holds by (S3); confer Proposition 2.3.3(a). Also by that proposition a constant diagonal S-function satisfies $s_{\alpha\bar{\alpha}} = s_{\bar{\alpha}\alpha} = s_{\alpha\bar{\alpha}}^{-1}$. Thus $s_{\alpha\bar{\alpha}} \in \{\pm 1\}$. Using $P_\pm = \frac{1}{2}(1 \pm S)$, we find

$$\begin{aligned}(P_\pm I_{\otimes 2})^{\alpha\beta} &= \tfrac{1}{2}(\delta^\alpha_\gamma \delta^\beta_\delta \pm S^{\alpha\beta}_{\gamma\delta})\delta^{\gamma\bar{\delta}} \\ &= \tfrac{1}{2}\left(\delta^\alpha_\gamma \delta^\beta_{\bar{\gamma}} \pm s_{\gamma\bar{\gamma}} \delta^\alpha_{\bar{\gamma}} \delta^\beta_\gamma\right) \\ &= \tfrac{1}{2}\left(1 \pm s_{\alpha\bar{\alpha}}\right) \delta^\alpha_\gamma \delta^\beta_{\bar{\gamma}} \\ &= |\{s_{\alpha\bar{\alpha}} = \pm 1\}| \delta^{\alpha\bar{\beta}}.\end{aligned} \tag{6.57}$$

Thus $P_\pm I_{\otimes 2} = \sum_{\alpha: s_{\alpha\bar{\alpha}} = \pm 1} |e_\alpha \otimes J e_\alpha)$ which is clearly positive (Lemma 6.3.1). □

Next, let us clarify the relation of the QEI result to previous results for non-interacting models. Suppose that the S-function is diagonal and that the model has a single mass sector with $M = m\mathbb{1}_\mathcal{K}$ for some $m > 0$ (or looking just at states within this mass sector). Then the lower bound of Theorem 6.3.3 simplifies to

$$T^{00}(g^2) \geq -\frac{m}{4\pi^2} \int_1^\infty ds \, |m\tilde{g}(ms)|^2 \left(C_+ w_+(s) + C_- w_-(s)\right) \mathbb{1}, \tag{6.58}$$

where $C_\pm = |\{\alpha : s_{\alpha\bar{\alpha}} = \pm 1\}|$ counts the positive/negative eigenvalues of $S\mathbb{F}$ which lie on the "diagonal".

In Table 6.4, we survey known and new QEI results for the models mentioned in the beginning. In more generality, (6.58) will also hold for all combinations of these models (by simply adding up $C_\pm$) and to variants of the same models with changed statistics (a switch between bosonic and fermionic statistics exchanges C_+ and C_- for all affected degrees of freedom). In that regard, also generalized Federbush-type models are included in the result. Such a model combines two free models, e.g., a Dirac fermion and a complex boson, with a Federbush-type interaction (Sec. 8.3).

Model	C_+	C_-	reference for the QEI result
free scalar boson	1	0	[FE98]
free Majorana fermion	0	1	"new"
massive Ising	0	1	[BCF13]
free complex boson	2	0	"new"
free Dirac fermion	0	2	[Daw06]
traditional Federbush	0	4	new

Table 6.1: Known QEI results together with new ones. The "new" results have not been explicitly treated in 1+1d but are reasonably clear from the existing literature.

6.5 Supplementary computations

Lemma 6.5.1. *Let $S \in \mathcal{B}(\mathcal{K}^{\otimes 2})$ then*

$$\left(\prod_{j=3}^{n} (\mathbb{F}S)_{1,j}\right)^{\boldsymbol{\beta}}_{\boldsymbol{\alpha}} = \delta^{\beta_1}_{\alpha_1} S^{\beta_3\beta_2}_{\gamma_1\alpha_3} S^{\beta_4\gamma_1}_{\gamma_2\alpha_4} \ldots S^{\beta_{n-1}\gamma_{n-4}}_{\gamma_{n-3}\alpha_{n-1}} S^{\beta_n\gamma_{n-3}}_{\alpha_2\alpha_n}.$$

Proof. Starting with the l.h.s., there is no action on the first tensor component, so that we can split off $\delta^{\beta_1}_{\alpha_1}$. Let us also abbreviate $T = \mathbb{F}S$ so that $T^{\gamma\delta}_{\alpha\beta} = S^{\delta\gamma}_{\alpha\beta}$. Then we compute

$$\begin{aligned}
\left(\prod_{j=2}^{n-1} T_{1,j}\right)^{\beta_2\ldots\beta_n}_{\alpha_2\ldots\alpha_n} &= (T_{1,2}T_{1,3}\ldots T_{1,n-2}T_{1,n-1})^{\beta_2\ldots\beta_n}_{\alpha_2\ldots\alpha_n} && (6.59)\\
&= (T_{1,2}T_{1,3}\ldots T_{1,n-2})^{\beta_2\ldots\beta_{n-1}}_{\gamma_{n-3}\alpha_3\ldots\alpha_{n-1}} \, T^{\gamma_{n-3}\beta_n}_{\alpha_2\alpha_n} && (6.60)\\
&= (T_{1,2}T_{1,3}\ldots T_{1,n-3})^{\beta_2\ldots\beta_{n-2}}_{\gamma_{n-4}\alpha_3\ldots\alpha_{n-2}} \, T^{\gamma_{n-4}\beta_{n-1}}_{\gamma_{n-3}\alpha_{n-1}} T^{\gamma_{n-3}\beta_n}_{\alpha_2\alpha_n} && (6.61)\\
&= \ldots && (6.62)\\
&= T^{\beta_2\beta_3}_{\gamma_1\alpha_3} T^{\gamma_1\beta_4}_{\gamma_2\alpha_4} \ldots T^{\gamma_{n-4}\beta_{n-1}}_{\gamma_{n-3}\alpha_{n-1}} T^{\gamma_{n-3}\beta_n}_{\alpha_2\alpha_n} && (6.63)
\end{aligned}$$

implying the to-be-proven expression. Note the shift in the j-index due to the $\delta^{\beta_1}_{\alpha_1}$ which was split off. □

Lemma 6.5.2. *Under the assumptions of Lemma 6.3.2,*

$$(-1)^n f^+_{-\nu} = f^-_{\nu}.$$

Proof. For the following computation we define $h_{g,\nu}(\zeta) := h(\zeta)g(p_0(\zeta)-\nu)$ such that $h^+_{\nu,\alpha} = h_{g,\nu}(\cdot)e_\alpha$ and $h^-_{\nu,\alpha} = \overline{h_{g,-\nu}(\bar{\cdot} + i\pi)}e_\alpha$. Also, we remark that for two $\mathbb{C}$-valued functions h_i, $i = 1, 2$, we understand $h_1 \otimes h_2$ as the map

$$\boldsymbol{\zeta} \mapsto (h_1 \otimes h_2)(\boldsymbol{\zeta}) := h_1(\zeta_1)h_2(\zeta_2).$$

Then

$$
\begin{aligned}
&(-1)^n f^+_{-\nu} \\
&= (-1)^n \sum_{\alpha=1}^{r} c_\alpha \, \mathcal{P}_S \left(h^+_{-\nu,\alpha} \otimes J h^+_{-\nu,\alpha}(\cdot + i\pi)) \right) \\
&= (-1)^n \tfrac{1}{2} \left(h_{g,-\nu} \otimes \overline{h_{g,-\nu}(\cdot + i\pi)} + (-1)^n \overline{h_{g,-\nu}(\cdot + i\pi)} \otimes h_{g,-\nu} \right) \sum_{\alpha=1}^{r} c_\alpha (e_\alpha \otimes J e_\alpha) \\
&= (-1)^n \tfrac{1}{2} \left(h_{g,-\nu} \otimes \overline{h_{g,-\nu}(\cdot + i\pi)} + (-1)^n \overline{h_{g,-\nu}(\cdot + i\pi)} \otimes h_{g,-\nu} \right) X \\
&= \tfrac{1}{2} \left((-1)^n h_{g,-\nu} \otimes \overline{h_{g,-\nu}(\cdot + i\pi)} + \overline{h_{g,-\nu}(\cdot + i\pi)} \otimes h_{g,-\nu} \right) X \\
&= \tfrac{1}{2} \left(\overline{h_{g,-\nu}(\cdot + i\pi)} \otimes h_{g,-\nu} + (-1)^n h_{g,-\nu} \otimes \overline{h_{g,-\nu}(\cdot + i\pi)} \right) X \\
&= \sum_{i=1}^{r} c_\alpha \tfrac{1}{2} \left(\overline{h_{g,-\nu}(\cdot + i\pi)} \otimes h_{g,-\nu} + (-1)^n h_{g,-\nu} \otimes \overline{h_{g,-\nu}(\cdot + i\pi)} \right) e_\alpha \otimes J e_\alpha \\
&= \sum_{i=1}^{r} c_\alpha \; \mathcal{P}_S \left(h^-_{\nu,\alpha} \otimes J h^-_{\nu,\alpha}(\cdot + i\pi) \right) \\
&= f^+_\nu .
\end{aligned}
\tag{6.64}
$$

We used here that $SX = (-1)^n X$. □

Lemma 6.5.3. *For $X = \sum_{\alpha=1}^{r} c_\alpha e_\alpha \otimes J e_\alpha$ with $r \in \{1, ..., d_\mathcal{K}\}$, we have*

$$\sum_{\alpha=1}^{r} c_\alpha = (I_{\otimes 2}, X).$$

Proof. Using $I_{\otimes 2} = \sum_{\beta=1}^{d_\mathcal{K}} (e_\beta \otimes J e_\beta)$ for some completion of $\{e_\alpha\}, \alpha = 1, .., r$ to a full ONB $\{e_\beta\}, \beta = 1, .., d_\mathcal{K}$ one obtains

$$\left(\sum_{\alpha=1}^{r} c_\alpha \right) = \sum_{\alpha,\beta}^{r} c_\alpha \delta^2_{\alpha\beta} = \sum_{\alpha,\beta}^{r} c_\alpha (e_\alpha \otimes J e_\alpha, e_\beta \otimes J e_\beta) = (I_{\otimes 2}, X).$$

□

Lemma 6.5.4. *For all $k \in \mathbb{N}$ and $g \in \mathcal{S}(\mathbb{R})$, $(\operatorname{ch}\theta)^k |\tilde{g}(\operatorname{ch}\theta + \nu)|^2$ is absolutely integrable in (θ, ν) over $\mathbb{R} \times [0, \infty)$.*

Proof. Transforming the integral, using $s = \operatorname{ch}\theta + \nu$ ($1 \le s < \infty$, $0 \le \nu \le s - 1$) and $t := s - \nu$, yields

$$
\begin{aligned}
&\int_{\mathbb{R}\times[0,\infty)} d(\theta \times \nu) (\operatorname{ch}\theta)^k |\tilde{g}(\operatorname{ch}\theta + \nu)|^2 \\
&\qquad = 2 \int_1^\infty \frac{ds}{\sqrt{(s-\nu)^2 - 1}} \int_0^{s-1} d\nu (s-\nu)^k |\tilde{g}(s)|^2 = \int_1^\infty ds \int_1^s \frac{t^k dt}{\sqrt{t^2 - 1}} |\tilde{g}(s)|^2 .
\end{aligned}
\tag{6.65}
$$

Then $w(s) := |\int_1^s \frac{t^k dt}{\sqrt{t^2-1}}| \lesssim s^{k+1}$ for $s \to \infty$, thus $|w(s)||\tilde{g}(s)|^2$ decays rapidly in s. □

Chapter 7

QEIs at one-particle level for generic scattering functions

This chapter aims to give necessary and sufficient conditions for QEIs at one-particle level in general integrable models, including models with several particle species, inner degrees of freedom, and bound states. The result will apply, in particular, to models with a regular S-function with (up to) first-order poles in the physical strip. Also, we will focus mainly on parity-invariant models.

The observable of interest for the (time-averaged) QEIs is the energy density $T^{00}(g^2)$ at a space point $x = x_0$ (we will take $x_0 = 0$ without loss of generality) and averaged in time by g^2, where $g \in \mathcal{S}_{\mathbb{R}}(\mathbb{R})$. Chapter 5 shows that expectation values of the stress-energy tensor $T^{\mu\nu}(x)$ in one-particle states are entirely determined by its two-particle form factor $F_2^{\mu\nu}(\cdot; x)$ or, due to Poincaré covariance, even by the two-particle form factor of $T(0) := g_{\mu\nu}T^{\mu\nu}(0)$. We may represent it as a matrix-valued function $\hat{F} : \mathbb{C} \to \mathcal{B}(\mathcal{K})$, fixed by the relation

$$\langle \varphi, T(0)\chi \rangle = \int d\theta d\eta \, \varphi_\alpha(\theta) \hat{F}(\theta - \eta)^\alpha{}_\beta \chi^\beta(\eta), \quad \varphi, \chi \in \mathcal{D}(\mathbb{R}, \mathcal{K}), \tag{7.1}$$

where the indices may label particle types and inner degrees of freedom.

For our results, we will assume that $F_2^{\mu\nu}$ is parity-covariant, which implies that $\hat{F}$ is self-adjoint (for real arguments). In this case, we will obtain an almost classification of whether QEIs hold depending on the asymptotic growth of $\hat{F}$: A QEI of the form

$$\langle \varphi, T^{00}(g^2)\varphi \rangle \geq -c_g \|\varphi\|_2^2 \tag{7.2}$$

for all $\varphi \in \mathcal{D}(\mathbb{R}, \mathcal{K})$ and a constant c_g (not dependent on φ) holds if the eigenvalues of $\hat{F}(\zeta)$, for ζ in a strip around $\mathbb{R}$ and $|\operatorname{Re}\zeta| \to \infty$, *all grow strictly slower* than $\frac{1}{4}\exp|\operatorname{Re}\zeta|$. It cannot hold if *one of them grows strictly faster*. The case where some of the eigenvalues grow *like* $\frac{1}{4}\exp|\operatorname{Re}\zeta|$ is inconclusive.

The derivation of this result is based on methods which were developed in [BC16] for the scalar case $d_{\mathcal{K}} = 1$. We extend these methods here to $d_{\mathcal{K}} > 1$ and by including bound states which implies that we analyze a meromorphic matrix-valued function $\hat{F}$ instead of an analytic complex-valued one.

For a given model, it is desirable to link $\hat{F}$ directly with the properties of the model. We will establish this link, thereby providing a recipe for obtaining QEIs in generic models. In Chapter 8, we will illustrate the essential features of this recipe in concrete examples including linear QFTs, the Bullough-Dodd model, the Federbush model, and the $O(n)$-nonlinear sigma model.

The remainder of this chapter is as follows: We will first derive the QEI result depending on the asymptotic growth of $\hat{F}$ (Sec. 7.1), followed by a brief discussion on how to extend the scope of the result (Sec. 7.2), and conclude with the recipe to obtain QEIs in generic models (Sec. 7.3). We defer a broader discussion of the QEI result to the conclusion in Chapter 9 so that we can take into account the findings in explicit examples.

7.1 Derivation of the QEI at one-particle level

Assume that we have a diagonal-in-mass stress-energy tensor at one-particle level $F_2^{\mu\nu}$. Then the expectation values of the time-averaged energy density $T^{00}(g^2)$ are, combining (5.9) with Corollary 5.3.3, given by

$$\langle \varphi, T^{00}(g^2)\varphi\rangle = \int d\theta d\eta \, \mathrm{ch}^2 \frac{\theta+\eta}{2}\Big(\varphi(\theta), \frac{M^2}{2\pi}\widetilde{g^2}(p_0(\theta;M)-p_0(\eta;M))\hat{F}(\eta-\theta)\varphi(\eta)\Big) \tag{7.3}$$

for $\varphi \in \mathcal{D}_S \cap \mathcal{H}_1$. Note here that we relate F from Chapter 5 with $\hat{F}$ by the identity

$$(u \otimes Jv, F(\zeta + i\pi))_{\mathcal{K}^{\otimes 2}} = (u, \hat{F}(\zeta)v), \quad u, v \in \mathcal{K}. \tag{7.4}$$

We ask whether the quadratic form defined by (7.3) is bounded below. In fact, this can be characterized in terms of the asymptotic behaviour of $\hat{F}$:

Theorem 7.1.1. *Let $F_2^{\mu\nu}$ be a parity-covariant stress-energy tensor at one-particle level which is diagonal in mass and $\hat{F}$ be given according to (7.4) and Corollary 5.3.3. Then:*

(a) Suppose there exists $u \in \mathcal{K}$ with $\|u\|_{\mathcal{K}} = 1$, and $c > \frac{1}{4}$ such that

$$\exists r > 0 \, \forall |\theta| \geq r: \quad |(u, \hat{F}(\theta)u)| \geq c \exp|\theta|. \tag{7.5}$$

Then for all $g \in \mathcal{S}_{\mathbb{R}}(\mathbb{R})$, $g \neq 0$ there exists a sequence $(\varphi_j)_j$ in $\mathcal{D}(\mathbb{R}, \mathcal{K})$ with $\|\varphi_j\|_2 = 1$, such that

$$\langle \varphi_j, T^{00}(g^2)\varphi_j\rangle \xrightarrow{j\to\infty} -\infty. \tag{7.6}$$

(b) Suppose there exists $c < \frac{1}{4}$ such that

$$\exists \epsilon, r > 0 \, \forall |\operatorname{Re}\zeta| \geq r, |\operatorname{Im}\zeta| \leq \epsilon: \quad \|\hat{F}(\zeta)\|_{\mathcal{B}(\mathcal{K})} \leq c \exp|\operatorname{Re}\zeta|. \tag{7.7}$$

Then for all $g \in \mathcal{S}_{\mathbb{R}}(\mathbb{R})$ there exists $c_g > 0$ such that for all $\varphi \in \mathcal{D}(\mathbb{R}, \mathcal{K})$,

$$\langle \varphi, T^{00}(g^2)\varphi \rangle \geq -c_g \|\varphi\|_2^2. \tag{7.8}$$

The rest of this section is devoted to the proof of Theorem 7.1.1, which we develop separately for the two parts (a) and (b). We first note that $\hat{F}$ satisfies:

Lemma 7.1.2. *For parity-invariant $T^{00}(g^2)$ and under the identity (7.4), $\hat{F}$ is a meromorphic function which satisfies*

$$\hat{F}(\zeta) = \hat{F}(-\zeta), \tag{7.9}$$
$$\hat{F}(\zeta) = \hat{F}(\bar{\zeta})^*, \tag{7.10}$$
$$\hat{F}(0) = \mathbb{1}_{\mathcal{K}}. \tag{7.11}$$

Note here that these identities are implied by S-periodicity and parity-invariance for (7.9), by S-periodicity and CPT-invariance for (7.10), and by normalization for (7.11).

Proof. By Theorem 5.3.1 including parity covariance (respectively parity invariance of the 00-component), F satisfies the properties

$$F(\zeta + i\pi) = F(-\zeta + i\pi), \tag{7.12}$$
$$F(\zeta) = \mathbb{F}F(\zeta), \tag{7.13}$$
$$F(\zeta + i\pi) = J^{\otimes 2}F(\bar{\zeta} + i\pi), \tag{7.14}$$
$$F(i\pi) = I_{\otimes 2}. \tag{7.15}$$

Using the identification (7.4) and the identities from Lemma A.6.1, in particular, that $\widehat{J^{\otimes 2}\mathbb{F}F(\zeta)} = \widehat{F(\zeta)}^*$ and $\widehat{I_{\otimes 2}} = \mathbb{1}_{\mathcal{K}}$, this is equivalent to the identities above. □

Now the strategy for part (a) closely follows [BC16, Proposition 4.2], but with appropriate generalizations for matrix-valued rather than complex-valued $\hat{F}$.

Proof of Theorem 7.1.1(a). Fix a smooth, even, real-valued function χ with support in $[-1, 1]$. Then for $\rho > 0$ define $\chi_\rho(\theta) := \rho^{-1/2}\|\chi\|_2^{-1}\chi(\rho^{-1}\theta)$, so that χ_ρ has support in $[-\rho, \rho]$ and is normalized with respect to $\|\cdot\|_2$. Define

$$\varphi_j(\theta) := \tfrac{1}{\sqrt{2}}(\chi_{\rho_j}(\theta - j) + s\,\chi_{\rho_j}(\theta + j))M^{-1}u,$$

where $s \in \{\pm 1\}$ and $(\rho_j)_j$ is a null sequence with $0 < \rho_j < 1$; both will be specified later. The φ_j, thus defined, have norm of at most m_-^{-1}, where $m_- := \min \mathfrak{M}$.

Defining $\hat{F}_g(\theta,\eta) := \widetilde{g^2}(p_0(\theta;M) - p_0(\eta;M))\hat{F}(\eta-\theta)$, Equation (7.3) yields

$$\begin{aligned}
&\langle \varphi_j, T^{00}(g^2)\varphi_j\rangle \\
&= \int d\theta d\eta \, \mathrm{ch}^2\, \tfrac{\theta+\eta}{2} \Big(\varphi_j(\theta), \frac{M^2}{2\pi}\hat{F}_g(\theta,\eta)\varphi_j(\eta)\Big) \\
&= \int \frac{d\theta d\eta}{4\pi}\, \mathrm{ch}^2\, \tfrac{\theta+\eta}{2} \Big((\chi_{\rho_j}(\theta-j)\chi_{\rho_j}(\eta-j) + \chi_{\rho_j}(\theta+j)\chi_{\rho_j}(\eta+j)) \\
&+ s\chi_{\rho_j}(\theta-j)\chi_{\rho_j}(\eta+j) + s\chi_{\rho_j}(\theta+j)\chi_{\rho_j}(\eta-j)\Big)(u, \hat{F}_g(\theta,\eta)u).
\end{aligned} \tag{7.16}$$

Here we used that χ_{ρ_j} is real-valued, that $s^2 = 1$, and the M^2 in the kernel was canceled by the M^{-1} appearing in φ_j.

Now, for each summand we redefine the θ- and η-variables to make use of the symmetries of the integral kernel: For $\chi_{\rho_j}(\theta \mp j)\chi_{\rho_j}(\eta \mp j)$ we substitute $\theta \mapsto \pm(\theta+j)$ and $\eta \mapsto \pm(\eta+j)$ so that we obtain in both cases $\chi_{\rho_j}(\theta)\chi_{\rho_j}(\eta)$ since $\chi_{\rho_j}(-\theta) = \chi_{\rho_j}(\theta)$ by assumption on χ. Similarly, for the summands $\chi_{\rho_j}(\theta \mp j)\chi_{\rho_j}(\eta \pm j)$ we redefine $\theta \mapsto \pm(\theta+j)$ and $\eta \mapsto \mp(\eta+j)$ so that we obtain again in both cases $\chi_{\rho_j}(\theta)\chi_{\rho_j}(\eta)$. As a result, the whole expression becomes

$$\begin{aligned}
&= \int \frac{d\theta d\eta}{4\pi}\, \chi_{\rho_j}(\theta)\chi_{\rho_j}(\eta) \\
&\quad \Big(\mathrm{ch}^2(j + \tfrac{\theta+\eta}{2})((u, \hat{F}_g(\theta+j, \eta+j)u) + (u, \hat{F}_g(-\theta-j, -\eta-j)u)) \\
&\qquad + s\, \mathrm{ch}^2\, \tfrac{\theta-\eta}{2}((u, \hat{F}_g(\theta+j, -\eta-j)u) + (u, \hat{F}_g(-\theta-j, \eta+j)u))\Big).
\end{aligned} \tag{7.17}$$

Finally, since $\hat{F}(\theta) = \hat{F}(-\theta)$ by (7.9) and $p_0(\theta;M) = p_0(-\theta;M)$, we may summarize each of the two terms to obtain

$$\langle \varphi_j, T^{00}(g^2)\varphi_j\rangle = \frac{1}{2\pi}\Big(u, (H_{\chi,j,+} + sH_{\chi,j,-})u\Big) \tag{7.18}$$

with $H_{\chi,j,\pm} := \int d\theta d\eta\, \widetilde{g^2}(Mk_j(\theta,\eta))H_{j,\pm}(\theta,\eta)\chi_{\rho_j}(\theta)\chi_{\rho_j}(\eta)$ and

$$\begin{aligned}
H_{j,+}(\theta,\eta) &= \mathrm{ch}^2(j + \tfrac{\theta+\eta}{2})\hat{F}(\theta-\eta), \\
H_{j,-}(\theta,\eta) &= \mathrm{ch}^2\, \tfrac{\theta-\eta}{2}\hat{F}(2j+\theta+\eta), \\
k_j(\theta,\eta) &= 2\,\mathrm{sh}(j + \tfrac{\theta+\eta}{2})\,\mathrm{sh}\,\tfrac{\theta-\eta}{2}.
\end{aligned}$$

Note here that

$$p_0(\theta \pm j;M) - p_0(\eta \pm j;M) = M(\mathrm{ch}(\theta+j) - \mathrm{ch}(\eta+j)) = 2M\,\mathrm{sh}(j + \tfrac{\theta+\eta}{2})\,\mathrm{sh}\,\tfrac{\theta-\eta}{2}.$$

Next, for large j and for $\theta, \eta \in [-\rho_j, \rho_j]$, we establish the estimates

$$(u, H_{j,+}(\theta,\eta)u) \le \|H_{j,+}(\theta,\eta)\|_{\mathcal{B}(\mathcal{K})} \le (\tfrac{1}{2} + 2c)\left(1 + \tfrac{1}{4}e^{2j}e^{2\rho_j}\right), \tag{7.19}$$

$$s(u, H_{j,-}(\theta,\eta)u) \le -ce^{2j}e^{-2\rho_j}, \tag{7.20}$$

$$|k_j(\theta,\eta)| \leq 12e^j \rho_j. \tag{7.21}$$

Namely for (7.19), due to (7.11) and continuity of $\hat{F}$ restricted to $\mathbb{R}$ as well as $\|\cdot\|_{\mathcal{B}(\mathcal{K})}$, we have for $\theta \to 0$ that $\|\hat{F}(\theta)\|_{\mathcal{B}(\mathcal{K})}$ becomes arbitrarily close to $\|\hat{F}(0)\|_{\mathcal{B}(\mathcal{K})} = 1$. Since $2c + \frac{1}{2} > 1$ by assumption, we may thus also obtain $\|\hat{F}(\theta)\|_{\mathcal{B}(\mathcal{K})} \leq 2c + \frac{1}{2}$ for small enough θ, or equivalently, for large enough j and $\theta \in [-2\rho_j, 2\rho_j]$. Also, $\operatorname{ch}^2 x \leq 1 + \frac{1}{4}e^{2x}$ applied to $x = \rho_j + j$ (note that $\frac{\theta+\eta}{2} \leq \rho_j$) yields the estimate.

For (7.20) one uses $\operatorname{ch}^2 x \geq 1$ along with the estimate $-s(u, \hat{F}(\theta)u) \geq c \exp|\theta|$ for all $|\theta| \geq r$, with suitable choice of $s \in \{\pm 1\}$. The latter statement is implied by hypothesis (7.5) and self-adjointness of $\hat{F}(\theta)$ (7.10): The hypothesis yields $|(u, \hat{F}(\theta)u)| \geq c \exp\theta$ for large enough $\theta > 0$. As $\hat{F}(\theta)$ is self-adjoint, one has that $(u, \hat{F}(\theta)u)$ is real-valued. Thus, either $(u, \hat{F}(\theta)u)$ or $-(u, \hat{F}(\theta)u)$ satisfies the bound. A jump between signs is forbidden since $\hat{F}(\theta)$ is continuous.

For (7.21), see [BC16, Eq. (4.17)].

Now choose $\delta > 0$ so small that $\widetilde{g^2}(m_+ p) \geq \frac{1}{2}\widetilde{g^2}(0) > 0$ for $|p| \leq \delta$, where $m_+ := \max \mathfrak{M}$. Choosing specifically the sequence $\rho_j = \frac{\delta}{12}e^{-j}$, we can combine these above estimates in the integrands of $H_{\chi,j,\pm}$ to give, cf. [BC16, Proof of Proposition 4.2],

$$\Big(u, (H_{\chi,j,+} + sH_{\chi,j,-})u\Big) \leq \frac{\delta}{24}\widetilde{g^2}(0)(ce^{-j} - c'e^j)\Big(\rho_j^{-1/2}\|\chi_{\rho_j}\|_1\Big)^2 \xrightarrow{j\to\infty} -\infty \tag{7.22}$$

with some $c' > 0$, noting that $\rho_j^{-1/2}\|\chi_{\rho_j}\|_1$ is independent of j. □

For part (b), we follow [BC16, Theorem 5.1], but again need to take the operator properties of $\hat{F}$ into account.

Proof of Theorem 7.1.1(b). For fixed $\varphi \in \mathcal{D}(\mathbb{R}, \mathcal{K})$ and $g \in \mathcal{S}_{\mathbb{R}}(\mathbb{R})$, we introduce $X_\varphi := \langle \varphi, T^{00}(g^2)\varphi \rangle$. Our aim is to decompose $X_\varphi = Y_\varphi + (X_\varphi - Y_\varphi)$ with $Y_\varphi \geq 0$ and $|X_\varphi - Y_\varphi| \leq c_g\|\varphi\|_2^2$ in order to conclude $X_\varphi \geq -c_g\|\varphi\|_2^2$. Since $[M, \hat{F}(\zeta)] = 0$ from diagonality in mass, we have $X_\varphi = \sum_{m\in\mathfrak{M}} X_{E_m\varphi}$ and can treat each $E_m\varphi$, $m \in \mathfrak{M}$, separately. Therefore in the following, we assume $M = m\mathbb{1}_{\mathcal{K}}$ without loss of generality.

We now express X_φ as in (7.3) and rewrite the integral as

$$X_\varphi = \frac{m^2}{2\pi}\int_0^\infty\int_0^\infty d\theta d\eta\, \widetilde{g^2}(p_0(\theta) - p_0(\eta))\left(\underline{\varphi}(\theta)^t, \underline{\underline{X}}(\theta,\eta)\underline{\varphi}(\eta)\right), \tag{7.23}$$

where $\underline{\varphi}(\theta) = (\varphi(\theta), \varphi(-\theta))^t$ and

$$\underline{\underline{X}}(\theta,\eta) = \begin{pmatrix} \operatorname{ch}^2\frac{\theta+\eta}{2}\hat{F}(-\theta+\eta) & \operatorname{ch}^2\frac{\theta-\eta}{2}\hat{F}(-\theta-\eta) \\ \operatorname{ch}^2\frac{-\theta+\eta}{2}\hat{F}(\theta+\eta) & \operatorname{ch}^2\frac{\theta+\eta}{2}\hat{F}(\theta-\eta) \end{pmatrix}.$$

Using (7.9) we find $\underline{\underline{X}} = \left(\begin{smallmatrix} A & B \\ B & A \end{smallmatrix}\right)$ with

$$A(\theta,\eta) = \operatorname{ch}^2\frac{\theta+\eta}{2}\hat{F}(\theta-\eta), \quad B(\theta,\eta) = \operatorname{ch}^2\frac{\theta-\eta}{2}\hat{F}(\theta+\eta).$$

Defining $H_\pm = A \pm B$ and $\varphi_\pm(\theta) = \varphi(\theta) \pm \varphi(-\theta)$ we obtain further that

$$(\underline{\varphi}(\theta)^t, \underline{\underline{X}}(\theta,\eta)\underline{\varphi}(\eta)) = \sum_\pm (\varphi_\pm(\theta), H_\pm(\theta,\eta)\varphi_\pm(\eta)). \tag{7.24}$$

Let us define

$$K_\pm(\theta) := \sqrt{|H_\pm(\theta,\theta)|} \in \mathcal{B}(\mathcal{K}), \tag{7.25}$$

where for $O \in \mathcal{B}(\mathcal{K})$, $|O|$ denotes the operator modulus of O and $\sqrt{|O|}$ its (positive) operator square root. Now, analogous to X_φ, introduce Y_φ (replacing $H_\pm(\theta,\eta)$ with $K_\pm(\theta)K_\pm(\eta)$),

$$Y_\varphi := \frac{m^2}{2\pi}\sum_\pm \int_0^\infty\int_0^\infty d\theta d\eta\, \widetilde{g^2}(p_0(\theta)-p_0(\eta))\,(\varphi_\pm(\theta), K_\pm(\theta)K_\pm(\eta)\varphi_\pm(\eta))\,. \tag{7.26}$$

Using the convolution formula (6.34) with $n=0$, $p_1 = p_0(\theta)$, $p_2 = p_0(\eta)$, noting that for real arguments it also holds for $g \in \mathcal{S}_\mathbb{R}(\mathbb{R})$, one finds that

$$\widetilde{g^2}(p_0(\theta)-p_0(\eta)) = \int \frac{d\nu}{2\pi}\tilde{g}(p_0(\theta)+\nu)\overline{\tilde{g}(p_0(\eta)+\nu)} \tag{7.27}$$

so that

$$Y_\varphi = \frac{m^2}{2\pi}\sum_\pm \int \frac{d\nu}{2\pi}\left\|\int d\eta\, \psi_\pm(\eta,\nu)\right\|^2_\mathcal{K} \geq 0, \tag{7.28}$$

where $\psi_\pm(\eta,\nu) := \tilde{g}(p_0(\eta)+\nu)K_\pm(\eta)\varphi_\pm(\eta)$. It remains to show that $|X_\varphi - Y_\varphi| \leq c_g\|\varphi\|_2^2$ for some $c_g \geq 0$. For this, we show that $X_\varphi - Y_\varphi$ is the expectation value of a Hilbert-Schmidt integral operator, namely, that its integral kernel $L_\pm(\theta,\eta) = H_\pm(\theta,\eta) - K_\pm(\theta)K_\pm(\eta)$ is square-integrable. In this case the L^2-norm yields the constant c_g; confer [HS78, §4]. Therefore, it suffices to show that

$$c_g := \sum_\pm \int_0^\infty d\theta \int_0^\infty d\eta |\widetilde{g^2}(p_0(\theta)-p_0(\eta))|^2 \|H_\pm(\theta,\eta) - K_\pm(\theta)K_\pm(\eta)\|^2_{\mathcal{B}(\mathcal{K})} \tag{7.29}$$

is finite.

To that end, let us introduce $\rho = \frac{\theta+\eta}{2}$ and $\tau = \theta - \eta$ with $|\partial(\rho,\tau)/\partial(\theta,\eta)| = 1$ and write $L_\pm$ as $L_\pm(\rho,\tau) := H_\pm(\rho+\frac{\tau}{2},\rho-\frac{\tau}{2}) \pm K_\pm(\rho+\frac{\tau}{2})K_\pm(\rho-\frac{\tau}{2})$. In these coordinates, the integration region in (7.29) is given by $\rho > 0$, $|\tau| < 2\rho$. Let $\rho_0 \geq 1$ and $\theta_0 > 0$ be some constants. The region $\rho \leq \rho_0$ is compact; thus, the integral over this region is finite. The region $\rho > \rho_0, |\tau| > 1$ also gives a finite contribution: Because of

$$|p_0(\theta) - p_0(\eta)| = 2m \operatorname{sh}\tfrac{|\tau|}{2}\operatorname{sh}\rho \geq 2m(1-e^{-2\rho_0})\operatorname{sh}\tfrac{1}{2}\operatorname{ch}\rho \tag{7.30}$$

in this region, $|\widetilde{g^2}(p_0(\theta)-p_0(\eta))|^2$ decays faster than any power of $\operatorname{ch}\rho$, while $\|L_\pm(\rho,\tau)\|^2_{\mathcal{B}(\mathcal{K})}$ cannot grow faster than a finite power of $\operatorname{ch}\rho$ due to our hypothesis (7.7). The remaining region is given by $\rho \geq \rho_0$ and $|\tau| \leq 1$. By (7.7), there exists $0 < c < \frac{1}{4}$ and $r > 0$ such that

$$\forall |\theta| \geq r : \ ||\hat{F}(2\theta)||_{\mathcal{B}(\mathcal{K})} \leq c \exp 2|\theta| \leq 4c\operatorname{ch}^2\theta. \tag{7.31}$$

Moreover, using self-adjointness of $\hat{F}$ (see (7.10)), for arbitrary $\theta \in \mathbb{R}$,

$$H_\pm(\theta,\theta) = \mathrm{ch}^2\,\theta\,\hat{F}(0) \pm \hat{F}(2\theta) \geq \mathrm{ch}^2\,\theta\,\mathbb{1}_{\mathcal{K}} - |\hat{F}(2\theta)|. \tag{7.32}$$

Then due to

$$|\hat{F}(2\theta)| \leq \|\hat{F}\|_{\mathcal{B}(\mathcal{K})}\mathbb{1}_{\mathcal{K}}, \tag{7.33}$$

which holds for any bounded self-adjoint Hilbert-space operator (confer [RS80, Thm. VI.6]), and using (7.31), we have that

$$\forall|\theta| \geq r:\ H_\pm(\theta,\theta) \geq (1-4c)\,\mathrm{ch}^2\,\theta\,\mathbb{1}_{\mathcal{K}}. \tag{7.34}$$

Since $c < \frac{1}{4}$, these $H_\pm(\theta,\theta)$ are positive operators with a uniform spectral gap at 0. As a consequence, together with $H_\pm(\theta,\theta)$, also the maps $\theta \mapsto K_\pm(\theta) = \sqrt{H_\pm(\theta,\theta)}$ are analytic near $[r,\infty)$; see [Kat95, Ch. VII, §5.3][1]. Correspondingly, $L_\pm(\rho,\tau)$ is real-analytic in the region where $\rho \geq \frac{|\tau|}{2} + r$. This contains the region $\{(\rho,\tau) : \rho \geq \rho_0, |\tau| \leq 1\}$ if we choose $\rho_0 \geq \frac{1}{2} + r$.

Now in this region, it can be shown that there exists $a > 0$ such that for any normalized $u \in \mathcal{K}$,

$$\Big|\Big(u, L_\pm(\rho,\tau)u\Big)\Big| \leq \tfrac{1}{2}\tau^2 \sup_{|\xi|\leq 1}\Big|\Big(u, \tfrac{\partial^2}{\partial\xi^2}L_\pm(\rho,\xi)u\Big)\Big| \leq \tfrac{1}{2}a\tau^2\,\mathrm{ch}\,\rho. \tag{7.35}$$

This estimate is based on the fact that $L_\pm(\rho,\tau) = L_\pm(\rho,-\tau)$, and $L_\pm(\rho,0) = 0$ (which also uses positivity of $H_\pm$). The first inequality in (7.35) then follows from Taylor's theorem; the second is an estimate of the derivative by Cauchy's formula, using analyticity of $\hat{F}$ in a strip around $\mathbb{R}$, and repeatedly applying the estimate (7.7), confer [BC16, Proof of Lemma 5.3]. Since (7.7) is an estimate in operator norm, and the other parts of the argument are u-independent, one finds $\|\frac{\partial^2}{\partial\xi^2}L_\pm(\rho,\tau)\|_{\mathcal{B}(\mathcal{K})} \leq a\,\mathrm{ch}\,\rho$ with a constant a.

Finiteness of the integral (7.29) now follows from the estimate (7.35) together with $|\widetilde{g^2}(p_0(\theta) - p_0(\eta))| \leq c'(\tau^4\,\mathrm{ch}^4\,\rho + 1)^{-1}$ for some $c' > 0$; confer [BC16, Proof of Lemma 5.4]. □

[1]The reference has a more general scope than necessary so that we should briefly comment on how to apply the referenced remark. The map $\zeta \mapsto H_\pm(\zeta,\zeta) = \mathrm{ch}^2\,\zeta\,\mathbb{1}_{\mathcal{K}} \pm \hat{F}(2\zeta)$ is analytic by assumption in an open strip around $\mathbb{R}$, say D_0. It satisfies that for all $\zeta \in D_0$, $D(H_\pm(\zeta,\zeta)) = \mathcal{K}$ is independent of ζ and $H_\pm(\zeta,\zeta)u$ is analytic in ζ so that it is of type (A) as specified in the reference. Finally, $H_\pm(\zeta,\zeta)$ has a spectral gap at 0 by inequality (7.34) and for small enough $|\operatorname{Im}\zeta|$. The remark in the reference then tells us that $\zeta \mapsto \sqrt{H_\pm(\zeta,\zeta)} = K_\pm(\zeta)$ is analytic in D_0.

7.2 Extending the scope of the QEI result

Let us comment here briefly on three aspects relevant for the scope of the theorem. First, we briefly discuss how the QEI result can be applied to models without parity invariance, i.e., where the stress-energy tensor or the S-function cannot be assumed to be parity-invariant. Second, we argue how the QEI result is modified when taking expectation values for superpositions of zero- and one-particle states and, third, in case that the vanishing of the zero-point energy is not assumed.

QEIs for parity-breaking models In absence of parity covariance of $F_2^{\mu\nu}$, Theorem 7.1.1 applies at least to the parity-covariant part $F_{2,P}^{\mu\nu}$ of $F_2^{\mu\nu}$, which is given by replacing F with $F_P := \frac{1}{2}(1+\mathbb{F})F$ and has all features of a parity-covariant stress-energy tensor at one-particle level except possibly for S-symmetry (T3), which requires the extra assumption $[S,\mathbb{F}]F = 0$. In any case, since S-symmetry is not used in the proof, Theorem 7.1.1 still applies to $F_{2,P}^{\mu\nu}$. Now, if (7.5) holds for F with u satisfying $Ju = \eta u$ with $\eta \in \mathbb{C}$ and $|\eta| = 1$, it holds for F_P due to $(u, \widehat{F}(\theta)u) = (Ju, \widehat{F}(\theta)Ju) = (u, \widehat{\mathbb{F}F}(\theta)u)$. As a consequence, no QEI can hold for $F_2^{\mu\nu}$. On the other hand, if (7.7) is fulfilled for F (hence for F_P), then a one-particle QEI for $F_2^{\mu\nu}$ holds at least in parity-invariant one-particle states.

QEIs for zero- and one-particle states While Theorem 7.1.1(b) establishes a QEI only at one-particle level, the result usually extends to expectation values in vectors $\Psi = c\,\Omega + \Psi_1$, $c \in \mathbb{C}, \Psi_1 \in \mathcal{H}_1$. Namely,

$$\langle \Psi, T^{00}(g^2)\Psi \rangle = \langle \Psi_1, T^{00}(g^2)\Psi_1 \rangle + 2\,\mathrm{Re}\; c \int (\Psi_1(\theta), \widetilde{g^2}(p_0(\theta;M))F_1(\theta))d\theta, \qquad (7.36)$$

where $F_1 = F_1^{[T^{00}(0)]}$ is the one-particle form factor of the energy density. This F_1 may be nonzero. However, it is of the form $F_1(\zeta;0) = F_1(0)\,\mathrm{sh}^2\,\zeta$ (end of Sec. 5.3). Now, the rapid decay of $\widetilde{g^2}$ implies that

$$F_{1,g} : (\theta \mapsto \widetilde{g^2}(p_0(\theta;M))F_1(\theta))$$

is in $L^2(\mathbb{R}, \mathcal{K})$ and by the Cauchy-Schwarz inequality we have

$$|\langle \Psi_1, F_{1,g} \rangle| \leq \|\Psi_1\|_2 \|F_{1,g}\|_2.$$

As clearly, $2|c|\|\Psi_1\|_2 \leq |c|^2 + \|\Psi_1\|_2^2 = \|\Psi\|_2^2$ due to $(|c| - \|\Psi_1\|_2)^2 \geq 0$, it follows that the additional term in (7.36) is bounded by a finite constant $c_g' := \|F_{1,g}\|_2$ times $\|\Psi\|^2$, i.e., the inequality (7.2) holds with constant c_g replaced by $c_g + c_g'$.

QEIs for non-vanishing zero-point energy Accounting for a non-vanishing zero-point energy consists merely of shifting the stress-energy tensor by a constant $c_{g,\Omega} := \langle \Omega, T^{00}(g^2)\Omega\rangle$, i.e., $T^{00}(g^2) \mapsto T^{00}(g^2) + c_{g,\Omega}$. Due to Poincaré invariance of Ω we know that $c_{g,\Omega} = c_\Omega \int g^2(t)dt$, where $c_\Omega = \langle \Omega, T^{00}(0)\Omega\rangle$. Again, the inequality (7.2) holds with constant c_g replaced by $c_g + c_\Omega \|g\|_2^2$. This also holds for superpositions of zero- and one-particle states, where (7.36) has to be supplemented by an additional term $|c|^2 c_{g,\Omega}$.

QEIs along a timelike wordline The QEI we have obtained here is averaged along a timelike trajectory which is constant in space $x_0 = 0$. For other timelike trajectories, we sketch here that the result applies in similar form. For simplicity, we restrict the discussion to a trajectory with constant timelike tangent vector k^μ such that $k.k = 1$. For the QEI results, we then have to replace $T^{00}(0, t)$ by $k_\mu k_\nu T^{\mu\nu}(x + tk)$ for some $x \in \mathbb{M}$. The x can be ignored due to translational invariance. Since at least at the one-particle level the $\mu\nu$-dependence of $T^{\mu\nu}$ is fully fixed by Poincaré covariance in terms of the free expression $G^{\mu\nu}_{\text{free}}$ (Thm. 5.3.1, Cor. 5.3.3), in the proofs of Thm. 7.1.1, there are two replacements:

$$G^{00}_{\text{free}} \mapsto k_\mu k_\nu G^{\mu\nu}_{\text{free}}, \tag{7.37}$$

$$\widetilde{g^2}(p_0(\theta) - p_0(\eta)) \mapsto \widetilde{g^2}(k^\mu(p_\mu(\theta) - p_\mu(\eta))). \tag{7.38}$$

Both of these modifications, require only few changes in the proofs. For instance, the first replacement amounts to a constant modification when $\rho \to \infty$,

$$k_\mu k_\nu G^{\mu\nu}_{\text{free}}(\rho) = G^{00}_{\text{free}}(\rho)(k_0 - k_1 \operatorname{th} \rho)^2. \tag{7.39}$$

7.3 A general recipe to obtain QEIs at one-particle level

To be able to use the QEI result established above (Thm. 7.1.1) in a given model, it is desirable to link the rather abstract $\hat{F}$ appearing there to the properties of the model at hand, i.e., to its particle spectrum and interactions, and in particular to the model's S-function S. We will briefly outline this recipe here, recalling also some concepts from the preceding chapters and giving reference to the relevant mathematical results.

First, recall that $\hat{F}$ was given rather abstractly as the two-particle form factor of the trace of the stress-energy tensor (7.1) and that the QEI result,

$$\langle \varphi, T^{00}(g^2)\varphi \rangle \geq -c_g \|\varphi\|_2^2 \tag{7.40}$$

for all $g \in \mathcal{S}_{\mathbb{R}}(\mathbb{R}), \varphi \in \mathcal{D}(\mathbb{R}, \mathcal{K})$ and a constant c_g, depended crucially on the asymptotic growth of $\hat{F}$ (Thm. 7.1.1). To analyze this, we recall the structure of $\hat{F}$.

Based on the eigendecomposition of S and assuming constant eigenprojectors (Prop. 2.3.4 and below), we can decompose $\hat{F}$ into eigenspaces with respect to S (Prop. 5.3.4). Say that S has k distinct eigenvalues s_i, then there are polynomials Q_i such that

$$\hat{F}(\zeta) = \sum_{i=1}^{k} Q_i(\operatorname{ch}\zeta)\frac{f_{i,\min}(\zeta)}{d_i(\operatorname{ch}\zeta)}, \tag{7.41}$$

where d_i is a complex-valued polynomial fixed by the bound state poles of s_i and $f_{i,\min}$ is the minimal solution with respect to s_i. Recall here that d_i is completely fixed by the position of the poles of s_i in the physical strip $\mathbb{S}(0,\pi)$ and normalization $d_i(1) = 1$. Recall also that the *minimal solution* for given s_i is unique under mild growth conditions (Lemma 4.1.1). Thus the model-dependent part, consisting of the bound state pole factors d_i and the minimal solutions $f_{i,\min}$, is completely fixed by the properties of the model and independent from $T^{\mu\nu}$. Imposing the expected properties for the stress-energy tensor (Defn. 5.2.1), the Q_i are constrained by CPT- and $\mathcal{G}$-invariance, and normalization due to the density property, $\sum_{i=1}^{k} Q_i(1) = I_{\otimes 2}$. However, the degree of the polynomial and some of their coefficients are unconstrained by our assumptions[2]. Thus, the Q_i classify the freedom of choice for the stress-energy tensor (at one-particle level).

Imposing a QEI of the form (7.2) as an additional physical assumption, gives an upper bound on the asymptotic growth of $\hat{F}$ and thus on the degree of the Q_i. In this manner, we can constrain the freedom of choice for the stress-energy tensor at one-particle level considerably and in some cases fix it uniquely. This will be done in Chapter 8 for a number of specific models showing the most important

[2] To be precise, there is one constraint: The polynomial energy bounds on $T^{\mu\nu}$ imply that the Q_i cannot grow faster than polynomials asymptotically.

features of our result. The models include linear QFTs, the Bullough-Dodd model, the Federbush model, and the $O(n)$-nonlinear sigma model.

What has been skipped so far is that for the classification result on $\hat{F}$, existence of the minimal solutions $f_{i,\min}$ has to be assumed, and for the analysis of the asymptotic growth of $\hat{F}$ knowing the asymptotic growth of the minimal solutions is crucial. Existence can be proven for a large class of (eigenvalues of) S-functions by employing a well-known integral representation (Thm. 4.2.1). For this class, the function

$$f[s] : \mathbb{R} \to \mathbb{R}, \quad t \mapsto f[s](t) := -\tfrac{1}{\pi} \int_0^\infty s'(\theta) s(\theta)^{-1} \cos(\pi^{-1}\theta t) d\theta \tag{7.42}$$

is well-defined and referred to as the *characteristic function* of s. In the case $s(0) = 1$, the minimal solution $f_{\min}$ is then obtained from $f = f[s]$ as the meromorphic continuation of

$$m_f : \mathbb{R} \to \mathbb{C}, \quad \theta \mapsto m_f(\theta) := \exp\left(2 \int_0^\infty f(t) \sin^2 \frac{(i\pi - \theta)t}{2\pi} \frac{dt}{t \operatorname{sh} t} \right). \tag{7.43}$$

For $s(0) = -1$, an additional factor needs to be included.

The asymptotic growth of $f_{\min}$ is then controlled by the Taylor expansion of the characteristic function around zero (Proposition 4.2.6): For a continuous function $f : [0, \infty) \to \mathbb{R}$, which is exponentially decaying at large arguments and second-order differentiable on some interval $[0, \delta], \delta > 0$, and where $f_0 := f(0)$, $f_1 := f'(0)$, the growth of $m_f(\zeta)$ is bounded at large $|\operatorname{Re}\zeta|$ as in

$$\exists 0 < c \leq c',\, r > 0 : \forall |\operatorname{Re}\zeta| \geq r, \operatorname{Im}\zeta \in [0, 2\pi] : \quad c \leq \frac{|m_f(\zeta)|}{|\operatorname{Re}\zeta|^{f_1} \exp|\operatorname{Re}\zeta|^{f_0/2}} \leq c'. \tag{7.44}$$

With this said, we have a recipe for a large class of models to determine whether a one-particle QEI in the sense of Theorem 7.1.1 holds, or no such QEI can hold. Namely, given a decomposition of the form (7.41) we know that for a QEI to hold each summand must grow asymptotically strictly less than $\frac{1}{4} \exp|\operatorname{Re}\zeta|$ for $|\operatorname{Re}\zeta| \to \infty$ and $\operatorname{Im}\zeta$ in a small interval around zero. Since $|\operatorname{ch}\zeta| \leq \exp|\operatorname{Re}\zeta|$, its validity depends then on the degree of the polynomials Q_i and d_i as well as the coefficient f_0 and in special cases on the leading coefficients of Q_i and d_i.

We summarize this recipe and results that are relevant for QEIs from the preceding chapters in the following theorem:

Theorem 7.3.1. *Assume a model with parity-invariant regular S-function S which has an eigendecomposition with constant eigenprojectors P_i and whose eigenvalues s_i, $i \in \{1, ..., k\}$ are such that*

$$r_i := \frac{d}{d\zeta} \log s_i \tag{7.45}$$

is uniformly L^1 in a strip $\mathbb{S}(-\epsilon, \epsilon)$ for some $\epsilon > 0$. Further, we introduce polynomials d_i

$$d_i(x) = \prod_j \frac{x - \operatorname{ch}\zeta_j}{1 - \operatorname{ch}\zeta_j}, \tag{7.46}$$

where j runs over the poles of s_i when restricted to the physical strip $\mathbb{S}(0, \pi)$ which we assume to be finite and of first-order. Under these assumptions,

1. *The minimal solutions $f_{i,\min}$ with respect to s_i exist*

2. *The diagonal-in-mass parity-covariant stress-energy tensor at one-particle level has the form $F_2^{\mu\nu}(\theta, \eta + i\pi; x) = e^{i(P(\theta)-P(\eta)).x} G^{\mu\nu}_{\text{free}}(\frac{\theta+\eta}{2})\hat{F}(\eta - \theta)$, where*

$$\hat{F}(\zeta) = \sum_{i=1}^{k} Q_i(\operatorname{ch}\zeta) \frac{f_{i,\min}(\zeta + i\pi)}{d_i(\operatorname{ch}\zeta)} \tag{7.47}$$

   where the Q_i take values in hermitian matrices on $\mathcal{K}$ which are invariant under the adjoint action of J and $V(g)$ for all $g \in \mathcal{G}$ and are normalized such that $\sum_{i=1}^{k} Q_i(1) = \mathbb{1}_{\mathcal{K}}$. (Note here that some Q_i might be identically zero)

3. *Let $f_{i,0}$ and $f_{i,1}$ denote the zero-th and first-order Taylor coefficients at zero of the characteristic function f_i with respect to s_i and let p_i denote the number of poles of s_i when restricted to the physical strip. Let further $\deg Q_i$ denote the polynomial degree of Q_i. Then a QEI of the form (7.40)*

$$\text{holds if:} \qquad \deg Q_i < 1 - \frac{1}{2} f_{i,0} + p_i \quad \text{for all } i \in \{1, ..., k\}, \tag{7.48}$$

$$\text{cannot hold if:} \qquad \deg Q_i > 1 - \frac{1}{2} f_{i,0} + p_i \quad \text{for some } i \in \{1, ..., k\}. \tag{7.49}$$

4. *Let c_{Q_i} denote the $\mathcal{B}(\mathcal{K})$-norm of the leading coefficient of Q_i and c_i, c_i' constants for which (7.44) holds ($m_f = m_{f_i}$). Then a QEI of the form (7.40) also holds if (7.48) applies with "$\leq$" and for all i with "$=$" in (7.48),*

$$f_{i,1} < 0 \quad \text{or} \quad f_{i,1} = 0 \wedge c_{Q_i} < \frac{2^{\deg Q_i - p_i - 2}}{c_i \prod_j (1 - \operatorname{ch}\zeta_j)}. \tag{7.50}$$

Such a QEI cannot hold if (7.49) applies with "=" and

$$f_{i,1} > 0 \quad \textit{or} \quad f_{i,1} = 0 \wedge c_{Q_i} > \frac{2^{\deg Q_i - p_i - 2}}{c'_i \prod_j (1 - \operatorname{ch} \zeta_j)}. \tag{7.51}$$

Chapter 8

Examples

In this chapter, we will look at concrete examples which illustrate the essential features of the abstract results developed in Chapters 6 and 7. This includes a review of former QEI results (models with one scalar particle type, Sec. 8.1) and an analysis of a model with bound states (Generalized Bullough-Dodd model, Sec. 8.2), an interacting model with a constant scattering function (Federbush model, Sec. 8.3), and a model with several particle species ($O(n)$-nonlinear sigma model, Sec. 8.4).

In summary, we will show that in these examples QEIs at one-particle level hold. When imposed as an additional physical assumption, it significantly reduces the freedom of choice in the class of viable stress-energy tensors. In some cases (free fermion, Ising, $O(n)$-nonlinear sigma), this uniquely fixes the stress-energy tensor at one-particle level. In some models with rapidity-independent scattering like the free fermion and the Ising model this even fixes the full stress-energy tensor for which a state-independent QEI in states of arbitrary particle number holds.

To understand the following sections, recall, that a model is fixed by the specification of its one-particle little space (K, J, V, M) and its S-function S (Chap. 2). Here $\mathcal{K}$ is the little space representing the model's discrete degrees of freedom, J the CPT-operator, V the representation of the global symmetry group $\mathcal{G}$ and M is the mass operator with mass spectrum $\mathfrak{M}$. Recall also, that the general recipe to obtain QEIs is summarized in Section 7.3. The assumptions for Theorem 7.3.1 are met in the following way: All models treated below have constant eigenprojectors (the models with one scalar particle type trivially). All models treated below are regular, have at most first-order poles in the physical strip and, except for the Federbush model, are parity-invariant. That the minimal solutions exist[1] was discussed in Section 4.3. In spite of the argued applicability of Theorem 7.3.1, for concreteness, we will treat each model on its own and give "step-by-step" references to the results from the main text.

8.1 Models with one scalar particle type without bound states

As a first step, we review in our context the known results for models of one scalar particle type and without bound states [BC16]. That is, we consider $\mathcal{K} = \mathbb{C}$, J

[1] I.e., that for each (relevant) eigenfunction s of S, $\frac{d}{d\zeta}s(\zeta)$ is uniformly L^1 in a strip.

the complex conjugation, $\mathfrak{M} = \{m\}$ for the one-particle space, and $\mathfrak{P} = \emptyset$ for the stress-energy tensor, with a scattering function of the form

$$S_{\mathrm{gshG}}(\zeta) = \epsilon \prod_{k=1}^{n} s(\zeta; b_k), \quad s(\zeta; b) := \frac{\operatorname{sh}\zeta - i \sin \pi b}{\operatorname{sh}\zeta + i \sin \pi b}, \tag{8.1}$$

where $\epsilon = \pm 1, n \in \mathbb{N}_0$, and $(b_k)_{k\in\{1,\dots,n\}} \subset i\mathbb{R} + (0,1)$ is a finite sequence in which b_k and $\overline{b_k}$ appear the same number of times. This is the most general regular scalar scattering function [Lec06, Prop. 3.2.2] and might be referred to as generalized sinh-Gordon model.

The minimal solution with respect to $\zeta \mapsto s(\zeta; b)$ is known (see, e.g., [BC16, Eq. (2.5)] or [FMS93, Eq. (4.13)]) and in our context given by

$$f_{b,\min}(\zeta) = (-i \operatorname{sh} \tfrac{\zeta}{2}) m_{f(\cdot;b)}(\zeta) \tag{8.2}$$

with characteristic function

$$f(t; b) := \frac{4 \operatorname{sh} \frac{bt}{2} \operatorname{sh} \frac{(1-b)t}{2} \operatorname{sh} \frac{t}{2} - \operatorname{sh} t}{\operatorname{sh} t}. \tag{8.3}$$

Since $f(t; b) = -1 + \mathcal{O}(t^2)$ for $t \to 0$, it follows that $f_{b,\min}$ is uniformly bounded above and below on $\mathbb{S}[0, 2\pi]$ by Proposition 4.2.6. More quantitatively, $f_{b,\min}(\zeta + i\pi)$ converges uniformly to

$$f^{\infty}_{b,\min} := \lim_{\theta\to\pm\infty} f_{b,\min}(\theta + i\pi) = \exp \int_0^\infty (t \operatorname{sh} t)^{-1}(1 + f(t; b))dt < \infty \tag{8.4}$$

for $|\operatorname{Re}\zeta| \to \infty$ and $|\operatorname{Im}\zeta| \leq \delta$ for any $0 < \delta < \pi$. This can be derived in the following way: Since $g(t) := (t \operatorname{sh} t)^{-1}(1 + f(t; b))$ is exponentially decaying and regular (in particular at $t = 0$), it is integrable and $f^{\infty}_{b,\min}$ is finite. As $\log \operatorname{ch} \frac{\zeta}{2} = 2 \int_0^\infty (t \operatorname{sh} t)^{-1} \sin^2 \frac{\zeta t}{2\pi} dt$ for $|\operatorname{Im}\zeta| < \pi$ one may write $\log f_{b,\min}(\zeta + i\pi) = 2 \int_0^\infty (t \operatorname{sh} t)^{-1}(1 + f(t; b)) \sin^2 \frac{\zeta t}{2\pi} dt$. In the limit $|\operatorname{Re}\zeta| \to \infty$ the parts which are non-constant with respect to ζ vanish due to the Riemann-Lebesgue lemma for $|\operatorname{Im}\zeta| < \pi$; uniformity follows from $g(t) \exp(\pm \frac{t \operatorname{Im}\zeta}{\pi})$ being uniformly L^1-bounded in $|\operatorname{Im}\zeta| \leq \delta$ (see, e.g., proof of Thm. IX.7 in [RS75]).

Next, according to Corollary 4.1.3, the minimal solution with respect to S_{gshG} is given by

$$f_{\mathrm{gshG},\min}(\zeta) = (i \operatorname{sh} \tfrac{\zeta}{2})^{-d(\epsilon,n)} \prod_{k=1}^{n} f_{b_k,\min}(\zeta) \tag{8.5}$$

with $d(+1, n) = 2\lfloor \frac{n}{2} \rfloor$ and $d(-1, n) = 2\lfloor \frac{n-1}{2} \rfloor$. For the stress-energy tensor at one-particle level we obtain (using Corollary 4.1.2, Lemma 4.1.4, and Corollary 5.3.3) that

$$F_2^{\mu\nu}(\zeta_1, \zeta_2 + i\pi) = G^{\mu\nu}_{\mathrm{free}}\left(\tfrac{\zeta_1+\zeta_2}{2}\right) F_q(\zeta_1 - \zeta_2 + i\pi), \quad F_q(\zeta) = q(\operatorname{ch}\zeta) f_{\mathrm{gshG},\min}(\zeta + i\pi) \tag{8.6}$$

with q a polynomial having real-valued coefficients and $q(-1) = 1$.

Let $c := 2^{d(\epsilon,n)-\deg q}|c_q| \prod_{k=1}^n f_{b_k,\min}^\infty$, where c_q is the leading coefficient of q. By the preceding remarks we find that for some c', c'' with $0 < c' < c < c''$ and $\delta, r > 0$:

$$\forall |\operatorname{Re}\zeta| \geq r, |\operatorname{Im}\zeta| \leq \delta : \quad c' \leq \frac{|F_q(\zeta + i\pi)|}{\exp((\deg q - \frac{1}{2}d(\epsilon,n))|\operatorname{Re}\zeta|)} \leq c'', \tag{8.7}$$

where c' and c'' can be chosen arbitrarily close to c for large enough r.

We can therefore conclude by Theorem 7.1.1 that a QEI of the form (7.8) holds if $\deg q < \frac{1}{2}d(\epsilon,n) + 1$ and cannot hold if $\deg q > \frac{1}{2}d(\epsilon,n) + 1$. In case that $\deg q = \frac{1}{2}d(\epsilon,n) + 1$, details of q become relevant. This can only occur if $d(\epsilon,n)$ is even, i.e., $\epsilon = +1$. If here c is less (greater) than $\frac{1}{4}$ then a QEI holds (cannot hold).

8.2 Generalized Bullough-Dodd model

We now consider a class of integrable models which treat a single neutral scalar particle that is its own bound state. The presence of the bound state requires the S-function to have a specific "bound state pole" in the physical strip with imaginary positive residue and to satisfy a bootstrap equation for the self-fusion process. Such S-functions are classified in [CT15, Appendix A]. The Bullough-Dodd model itself (see [AFZ79; FMS93] and references therein) corresponds to the maximally analytic element of this class which is given by $\zeta \mapsto S_{\mathrm{BD}}(\zeta; b) = s(\zeta; -\frac{2}{3})s(\zeta; \frac{b}{3})s(\zeta; \frac{2-b}{3})$ where $b \in (0,1)$ is a parameter of the model. The full class allows for so-called CDD factors [CDD56] and an exotic factor of the form $\zeta \mapsto e^{ia \operatorname{sh}\zeta}, a > 0$.

In Lagrangian QFT, from a one-component field φ and a Lagrangian

$$\mathcal{L}_{\mathrm{BD}} = \tfrac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{m^2}{6g^2}(2e^{g\varphi} + e^{-2g\varphi}) \tag{8.8}$$

one obtains as S-function $S_{\mathrm{BD}}(\cdot; b)$ under the perturbation theoretic correspondence $b = \frac{g^2}{2\pi}(1 + \frac{g^2}{4\pi})^{-1}$ [FMS93]. For more general elements of the described class no Lagrangian is known [CT15].

In our context, we will consider the generalized variant of the model, but for simplicity restrict to finitely many CDD factors and do not include the exotic factor:

Definition 8.2.1. The *generalized Bullough-Dodd model* is specified by the mass parameter $m > 0$ and a finite sequence $(b_k)_{k\in\{1,\ldots,n\}} \subset (0,1) + i\mathbb{R}, n \in \mathbb{N}$, which has an odd number of real elements and where the non-real b_k appear in complex conjugate pairs. The one-particle little space is given by $\mathcal{K} = \mathbb{C}$, $\mathcal{G} = \{e\}$, $V = 1_{\mathbb{C}}$, and $M = m1_{\mathbb{C}}$. J corresponds to complex conjugation. The S-function S_{gBD} is of the form

$$S_{\mathrm{gBD}}(\zeta) = s(\zeta; -\tfrac{2}{3}) \prod_{k=1}^{n} s(\zeta; \tfrac{b_k}{3}) s(\zeta; \tfrac{2-b_k}{3}). \tag{8.9}$$

Clearly, S_{BD} is obtained from S_{gBD} for $n = 1$ and $b_1 = b$. Since S_{gBD} is defined as a product of a finite number of factors of the form $s(\cdot; b)$, its minimal solution exists by Corollary 4.1.3 and amounts to

$$f_{\mathrm{gBD,min}}(\zeta) = (-i \operatorname{sh} \tfrac{\zeta}{2})^{-2n} f_{-2/3,\mathrm{min}}(\zeta) \prod_{k=1}^{n} f_{b_k/3,\mathrm{min}}(\zeta) f_{(2-b_k)/3,\mathrm{min}}(\zeta). \tag{8.10}$$

It enters here that $S_{\mathrm{gBD}}(0) = -1$.

The presence of bound states in the model implies the presence of poles in the form factors of local operators (F1b), in particular also for $F_2^{\mu\nu}$ (Eq. (4.5)). For $F_1^{\mu\nu} \neq 0$ we expect a single first-order pole of $F_2^{\mu\nu}(\zeta, \zeta'; x)$ at $\zeta' - \zeta = i\frac{2\pi}{3}$. In case that $F_1^{\mu\nu} = 0$ we expect $F_2^{\mu\nu}(\zeta, \zeta'; x)$ to have no poles in $\mathbb{S}[0, \pi]$.

Lemma 8.2.2 (Stress tensor in the generalized BD model). *A tensor-valued function $F_2^{\mu\nu} : \mathbb{C}^2 \times \mathbb{M} \to \mathcal{K}^{\otimes 2}$ is a stress-energy tensor at one-particle level with respect to S_{gBD} and $\mathfrak{P} \subset \{i\frac{2\pi}{3}\}$ iff it is of the form*

$$F_2^{\mu\nu}(\theta, \eta + i\pi) = G_{\mathrm{free}}^{\mu\nu}\left(\tfrac{\theta+\eta}{2}\right) e^{i(p(\theta;m)-p(\eta;m)).x} F_q(\eta - \theta + i\pi), \tag{8.11}$$

with

$$F_q(\zeta) = q(\operatorname{ch} \zeta)(-2 \operatorname{ch} \zeta - 1)^{-1} f_{\mathrm{gBD,min}}(\zeta), \tag{8.12}$$

where $f_{\mathrm{gBD,min}}$ is the unique minimal solution with respect to S_{gBD} and where q is a polynomial with real coefficients and $q(-1) = 1$.

Proof. By Theorem 5.3.1 and Corollary 5.3.3, $F_2^{\mu\nu}$ is given by (8.11), where $F : \mathbb{C} \to \mathbb{C}$ satisfies properties (b)-(g) of Theorem 5.3.1 with respect to S_{gBD}. According to Lemma 4.1.4, F is of the form (8.12); the factor $(-2 \operatorname{ch} \zeta - 1)^{-1}$ takes the one possible first-order pole within $\mathbb{S}[0, \pi]$, namely at $i\frac{2\pi}{3}$, into account. That q has real coefficients is a consequence of property (e) and Corollary 4.1.2.

Conversely, it is clear that $F_2^{\mu\nu}$, respectively F, as given above has the properties (b)-(g). □

Theorem 8.2.3 (QEI for the generalized BD model). *Let the stress-energy tensor at one-particle level be given by $F_2^{\mu\nu}$ as in (8.11). Then a QEI of the form*

$$\forall g \in \mathcal{S}_{\mathbb{R}}(\mathbb{R})\, \exists c_g > 0\, \forall \varphi \in \mathcal{D}(\mathbb{R}, \mathcal{K}) : \quad \langle \varphi, T^{00}(g^2)\varphi \rangle \geq -c_g \|\varphi\|_2^2 \tag{8.13}$$

holds if $\deg q < n + 2$ *and cannot hold if* $\deg q > n + 2$. *In the case* $\deg q = n + 2$, *introduce*

$$c := 2^{2n-\deg q}|c_q| f^{\infty}_{-2/3,\min} \prod_{k=1}^{n} f^{\infty}_{b_k/3,\min} f^{\infty}_{(2-b_k)/3,\min}, \tag{8.14}$$

where c_q denotes the leading coefficient of q. If here c is less (greater) than $\frac{1}{4}$ then a QEI holds (cannot hold).

Proof. As the minimal solution $f_{\text{gBD},\min}$ is given as a finite product of factors $\zeta \mapsto (-i \operatorname{sh} \frac{\zeta}{2})$ and $f_{b,\min}$, the asymptotic growth can be estimated analogously to the procedure in Section 8.1. Similar to the estimate (8.7), one obtains for some c' and c'' with $0 < c' < c < c''$ and some $\epsilon, r > 0$:

$$\forall |\operatorname{Re}\zeta| \geq r, |\operatorname{Im}\zeta| \leq \epsilon : \quad c' \leq \frac{|F_q(\zeta + i\pi)|}{\exp((\deg q - n - 1)|\operatorname{Re}\zeta|)} \leq c'', \tag{8.15}$$

where c' and c'' can be chosen arbitrarily close to c for large enough r.

Noting that parity covariance is trivial for $\mathcal{K} = \mathbb{C}$ and applying Theorem 7.1.1 yields the conclusions from above depending on $\deg q$ and c. □

8.3 Federbush model

The Federbush model is a well-studied integrable QFT model with a constant, but non-trivial, scattering function; see [Fed61; STW76; Rui81; Rui82; CF01] and references therein. In Lagrangian QFT, the traditional Federbush model is described in terms of two Dirac fields Ψ_1, Ψ_2 by a Lagrangian density[2]

$$\mathcal{L}_{\text{Fb}} = \sum_{j=1}^{2} \tfrac{1}{2}\bar{\Psi}_j(i\partial\!\!\!/ - m_j)\Psi_j - \lambda\pi\epsilon_{\mu\nu}J_1^{\mu}J_2^{\nu}, \quad J_j^{\mu} = \bar{\Psi}_j\gamma^{\mu}\Psi_j. \tag{8.16}$$

The Federbush model obeys a global $U(1)^{\oplus 2}$ symmetry since $\mathcal{L}_{\text{Fb}}$ is invariant under

$$\Psi_j(x) \mapsto e^{2\pi i\kappa}\Psi_j(x), \quad \Psi_j^{\dagger}(x) \mapsto e^{-2\pi i\kappa}\Psi_j^{\dagger}(x), \quad \kappa \in \mathbb{R},\ j = 1, 2. \tag{8.17}$$

The stress-energy tensor of the model has been computed before [SH78] and its trace (Eq. (44) in the reference) is given by

$$T^{\mu}_{\mu} = \sum_{j=1}^{2} m_j {:}\bar{\Psi}_j\Psi_j{:} \tag{8.18}$$

which agrees with the (trace of the) stress-energy tensor of two free Dirac fermions. Note in particular that it is parity-invariant.

In our framework, the model can be described in the following way:

[2]The fields Ψ_j take values in $\mathbb{C}^2$. $\epsilon_{\mu\nu}$ denotes the antisymmetric tensor with $\epsilon_{01} = -\epsilon_{10} = 1$. Other standard notations are $\bar{\psi}_j := \psi_j^{\dagger}\gamma_0$ and $\partial\!\!\!/ = \gamma^{\mu}\partial_{\mu}$ with anticommuting matrices $\gamma^0, \gamma^1 \in \mathrm{Mat}(2 \times 2, \mathbb{C})$, $[\gamma^{\mu}, \gamma^{\nu}]_+ = 2g^{\mu\nu}$.

Definition 8.3.1. The *Federbush model* is specified by three parameters, the particle masses $m_1, m_2 \in (0,\infty)$ and the coupling parameter $\lambda \in (0,\infty)$. The symmetry group is $\mathcal{G} = U(1)^{\oplus 2}$. The one-particle little space is given by $L = (\mathcal{K}, V, J, M)$ with $L = L_1 \oplus L_2$ and where for $j = 1,2$ we define $\mathcal{K}_j = \mathbb{C}^2$ and

$$V_j(\kappa) = \begin{pmatrix} e^{2\pi i\kappa} & 0 \\ 0 & e^{-2\pi i\kappa} \end{pmatrix}, \quad J_j = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad M_j = m_j \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{8.19}$$

as operators on $\mathcal{K}_j$ where J_j is antilinear and for the choice of basis $\{e_j^{(+)} \equiv (1,0)^t$, $e_j^{(-)} \equiv (0,1)^t\}$. The S-function is denoted by $S_{\mathrm{Fb}} \in \mathcal{B}(\mathcal{K}^{\otimes 2})$. Its only nonvanishing components, enumerated as $\alpha, \beta = 1+, 1-, 2+, 2-$ corresponding to $e_{1/2}^{(\pm)}$, are given by $s_{\alpha\beta} := (S_{\mathrm{Fb}})^{\beta\alpha}_{\alpha\beta}$ with

$$s_{\alpha\beta} = -\begin{pmatrix} 1 & 1 & e^{2\pi i\lambda} & e^{-2\pi i\lambda} \\ 1 & 1 & e^{-2\pi i\lambda} & e^{2\pi i\lambda} \\ e^{-2\pi i\lambda} & e^{2\pi i\lambda} & 1 & 1 \\ e^{2\pi i\lambda} & e^{-2\pi i\lambda} & 1 & 1 \end{pmatrix}_{\alpha\beta}. \tag{8.20}$$

Note that S_{Fb} is a constant diagonal S-function; e.g., $s_{\alpha\beta} = s^*_{\beta\alpha} = s^{-1}_{\beta\alpha}$ imply that S_{Fb} is self-adjoint and unitary. Note also that, $s_{\alpha\beta} = s_{\bar\alpha\bar\beta} \neq s_{\beta\alpha}$, where $\bar\alpha$ corresponds to $\alpha \in \{1+, 1-, 2+, 2-\}$ by flipping plus and minus. These relations correspond to the fact that S_{Fb} is C-, PT- and CPT- but not P- or T-symmetric. However, S_{Fb} has a P-invariant diagonal (in the sense of Equation (6.1)) due to $s_{\alpha\bar\alpha} = s_{\bar\alpha\alpha}$ (or Lemma 6.4.1).

Lemma 8.3.2 (Stress tensor for the Federbush model)**.** *A tensor-valued function $F_2^{\mu\nu} : \mathbb{C}^2 \times \mathbb{M} \to \mathcal{K}^{\otimes 2}$ is a stress-energy-tensor at one-particle level with respect to S_{Fb}, is diagonal in mass (Eq. (5.40)), and has no poles, $\mathfrak{P} = \emptyset$, iff it is of the form*

$$F_2^{\mu\nu}(\theta, \eta + i\pi; x) = G^{\mu\nu}_{\mathrm{free}}\left(\tfrac{\theta+\eta}{2}\right) e^{iP(\theta,\eta+i\pi).x} F(\eta - \theta + i\pi) \tag{8.21}$$

with

$$F(\zeta) = \sum_{j=1}^{2} \left(-i\,\mathrm{sh}(\tfrac{\zeta}{2}) q_j^{\mathrm{s}}(\mathrm{ch}\,\zeta)\, e_j^{(+)} \otimes_{\mathrm{s}} e_j^{(-)} + \mathrm{ch}(\tfrac{\zeta}{2}) q_j^{\mathrm{as}}(\mathrm{ch}\,\zeta)\, e_j^{(+)} \otimes_{\mathrm{as}} e_j^{(-)} \right), \tag{8.22}$$

for $e_j^{(+)} \otimes_{\mathrm{s/as}} e_j^{(-)} := e_j^{(+)} \otimes e_j^{(-)} \pm e_j^{(-)} \otimes e_j^{(+)}$ and where each $q_j^{\mathrm{s/as}}$ is a polynomial with real coefficients and $q_j^{\mathrm{s}}(-1) = 1$.
The stress-energy tensor at one-particle level is parity-covariant iff $q_1^{\mathrm{as}} = q_2^{\mathrm{as}} \equiv 0$.

Proof. By Theorem 5.3.1 and Corollary 5.3.3 we have that (8.21) holds with F

satisfying properties (b)-(g). $U(1)^{\oplus 2}$-invariance, property (f), is equivalent to

$$\forall \zeta \in \mathbb{C}, \boldsymbol{\kappa} \in \mathbb{R}^2, r, s \in \{\pm\}, j, k \in \{1,2\} : \quad \left(1 - e^{2\pi i (r\kappa_j + s\kappa_k)}\right) (e_j^{(r)} \otimes e_k^{(s)}, F(\zeta)) = 0.$$

As a consequence, $(e_j^{(r)} \otimes e_k^{(s)}, F(\zeta)) = 0$ unless $j = k$ and $r = -s$. On the remaining components, S acts like $-\mathbb{F}$, thus

$$F(\zeta) = -\mathbb{F}F(-\zeta) = \mathbb{F}F(2i\pi - \zeta), \qquad \text{which implies} \tag{8.23}$$

$$F(\zeta) = \sum_{j=1}^{2} \left(-i\,\mathrm{sh}(\tfrac{\zeta}{2}) f_j^{\mathrm{s}}(\zeta) e_j^{(+)} \otimes_{\mathrm{s}} e_j^{(-)} + \mathrm{ch}(\tfrac{\zeta}{2}) f_j^{\mathrm{as}}(\zeta) e_j^{(+)} \otimes_{as} e_j^{(-)} \right) \tag{8.24}$$

for some functions $f_j^{\mathrm{s/as}}$, where we have factored out the necessary zeroes due to the relations (8.23). Then from the properties of F we find $f_j^{\mathrm{s/as}} : \mathbb{C} \to \mathbb{C}$ to be analytic and to satisfy

$$f_j^{\mathrm{s/as}}(\zeta) = f_j^{\mathrm{s/as}}(-\zeta) = f_j^{\mathrm{s/as}}(2\pi i - \zeta), \quad f_j^{\mathrm{s}}(i\pi) = 1, \tag{8.25}$$

and $f_j^{\mathrm{as}}(i\pi)$ unconstrained. Moreover, $f_j^{\mathrm{s/as}}$ are regular in the sense of (4.3) of Lemma 4.1.4; the lemma implies that $f_j^{\mathrm{s/as}}(\zeta) = q_j^{\mathrm{s/as}}(\mathrm{ch}\,\zeta)$ with $q_j^{\mathrm{s}}(-1) = 1$. Since $J^{\otimes 2} F(\zeta + i\pi) = F(\bar{\zeta} + i\pi)$, $Je_j^{(\pm)} = -e_j^{(\mp)}$, and by the antilinearity of J, we find that $q_j^{\mathrm{s/as}}(\zeta + i\pi) = \overline{q_j^{\mathrm{s/as}}(\bar{\zeta} + i\pi)}$ such that $q_j^{\mathrm{s/as}}$ have real coefficients.

Parity-invariance of F, i.e., $\mathbb{F}F = F$, is equivalent to $q_j^{\mathrm{as}} = -q_j^{\mathrm{as}}$, thus $q_j^{\mathrm{as}} = 0$, because of $(\mathbb{1} \mp \mathbb{F})\, e_j^{(+)} \otimes_{\mathrm{s/as}} e_j^{(-)} = 0$. □

We see that the stress-energy tensor does not need to be parity-covariant. Concerning QEIs we state:

Theorem 8.3.3 (QEI for the Federbush model)**.** *The parity-covariant part of the stress-energy tensor at one-particle level, given by F_2 in (8.21) with $q_1^{\mathrm{as}} = q_2^{\mathrm{as}} \equiv 0$, satisfies a one-particle-QEI of the form*

$$\forall g \in \mathcal{S}_{\mathbb{R}}(\mathbb{R})\, \exists c_g > 0\, \forall \varphi \in \mathcal{D}(\mathbb{R}, \mathcal{K}) : \quad \langle \varphi, T_P^{00}(g^2)\varphi \rangle \geq -c_g \|\varphi\|_2^2 \tag{8.26}$$

iff $q_1^{\mathrm{s}} = q_2^{\mathrm{s}} \equiv 1$.

The candidate stress-energy tensor given by (6.15) (i.e. for $q_1^{\mathrm{s}} = q_2^{\mathrm{s}} = 1, q_1^{\mathrm{as}} = q_2^{\mathrm{as}} = 0$) satisfies a QEI of the form

$$T^{00}(g^2) \geq -\left(\sum_{j=1}^{2} \frac{m_j^3}{2\pi^2} \int_1^\infty ds |\tilde{g}(m_j s)|^2 w_-(s) \right) \mathbb{1} \tag{8.27}$$

with $w_-(s) = s\sqrt{s^2 - 1} - \log(s + \sqrt{s^2 - 1})$ and in the sense of a quadratic form on $\mathcal{D}_{S_{\mathrm{Fb}}} \times \mathcal{D}_{S_{\mathrm{Fb}}}$.

Proof. In case that one $q_j^{\mathrm{s}} \neq 1$ we have for some $c, r > 0$ that $|q_j^{\mathrm{s}}(\operatorname{ch}\zeta)\operatorname{sh}\frac{\zeta}{2}| \geq ce^{3|\operatorname{Re}\zeta|/2}$ for all $|\operatorname{Re}\zeta| \geq r$. Therefore, no QEI can hold due to Theorem 7.1.1(a) and the remarks in Section 7.2 with $u = e_j^{(+)} \pm e_j^{(-)}$ for some $j \in \{1,2\}$. For $q_1^{\mathrm{s}} = q_2^{\mathrm{s}} \equiv 1$ (and $q_1^{\mathrm{as}} = q_2^{\mathrm{as}} \equiv 0$), Theorem 7.1.1(b) yields (8.26). In that case $F(\zeta) = (-i\operatorname{sh}\frac{\zeta}{2})I_{\otimes 2}$ which coincides with the expression in (6.3) due to $P_+ I_{\otimes 2} = 0$ and $P_- I_{\otimes 2} = I_{\otimes 2}$ (Lemma A.6.2). Since S_{Fb} is constant and diagonal, by Lemma 6.4.1, Theorem 6.3.3 applies and yields (8.27). □

We see that for the Federbush model, requiring a one-particle QEI fixes a unique (parity-covariant part of the) stress-energy tensor at one-particle level that extends – since S_{Fb} is constant – to a dense domain of the full interacting state space. The parity-covariant part is in agreement with preceding results for the stress-energy tensor at one-particle level [CF01, Sec. 4.2.3]. This indicates that the parity-violating part of our expression is indeed not relevant for applications in physics. Our candidate for the full stress-energy tensor has the same trace as in [SH78]. That the respective energy density satisfies a generic QEI is no surprise after all, as the QEI results are solely characterized in terms of the trace of the stress-energy tensor which here agrees with that of two free Dirac fermions (as was indicated also by (8.18)).

8.4 O(n)-nonlinear sigma model

The $O(n)$-nonlinear sigma model is a well-studied integrable QFT model of n scalar fields ϕ_j, $j = 1, ..., n$, that obey an $O(n)$-symmetry. For a review see [AAR01, Secs. 6–7] and references therein. In Lagrangian QFT it can be described by a combination of a free Lagrangian and a constraint

$$\mathcal{L}_{\mathrm{NLS}} = \tfrac{1}{2}\partial_\mu \Phi^t \partial^\mu \Phi, \quad \Phi^t \Phi = \frac{1}{2g}, \quad \Phi = (\phi_1, \ldots, \phi_n)^t, \tag{8.28}$$

where $g \in (0, \infty)$ is a dimensionless coupling constant. Clearly, $\mathcal{L}_{\mathrm{NLS}}$ is invariant for Φ transforming under the vector representation of $O(n)$, i.e.,

$$\Phi(x) \mapsto O\Phi(x), \quad O \in \mathrm{Mat}_{\mathbb{R}}(n \times n), \quad O^t = O^{-1}. \tag{8.29}$$

Note that the model – other than one might expect naively from $\mathcal{L}_{\mathrm{NLS}}$ – describes massive particles. This is known as dynamical mass transmutation; the resulting mass of the $O(n)$-multiplet can take arbitrary positive values depending on a choice of a mass scale and corresponding renormalized coupling constant; see, e.g., [AAR01, Sec. 7.2.1] and [JN88].

In our framework, the model can be described in the following way:

Definition 8.4.1. The $O(n)$-*nonlinear sigma model* is specified by two parameters, the particle number $n \in \mathbb{N}$, $n \geq 3$, and the mass $m > 0$. The one-particle little space $(\mathcal{K}, V, J, M)$ is given by $\mathcal{K} = \mathbb{C}^n$ with V the defining or vector representation of $\mathcal{G} = O(n)$, $M = m\mathbb{1}_{\mathbb{C}^n}$, and where J is complex conjugation in the canonical basis of $\mathbb{C}^n$. The S-function is given by

$$S_{\mathrm{NLS}}(\zeta) := (b(\zeta)\mathbb{1} + c(\zeta)\mathbb{F} + d(\zeta)\mathbb{K})\mathbb{F}, \tag{8.30}$$

where in the canonical basis of $\mathbb{C}^n$

$$\mathbb{1}_{\alpha\beta}^{\gamma\delta} = \delta_\alpha^\gamma \delta_\beta^\delta, \quad \mathbb{F}_{\alpha\beta}^{\gamma\delta} = \delta_\alpha^\delta \delta_\beta^\gamma, \quad \mathbb{K}_{\alpha\beta}^{\gamma\delta} = \delta^{\gamma\delta}\delta_{\alpha\beta}, \quad \alpha, \beta, \gamma, \delta = 1, ..., n, \tag{8.31}$$

$$b(\zeta) = s(\zeta)s(i\pi - \zeta), \quad c(\zeta) = -i\pi\nu\zeta^{-1}b(\zeta), \quad d(\zeta) = -i\pi\nu(i\pi - \zeta)^{-1}b(\zeta), \tag{8.32}$$

and

$$\nu = \tfrac{2}{n-2}, \quad s(\zeta) = \frac{\Gamma\left(\frac{\nu}{2} + \frac{\zeta}{2\pi i}\right)\Gamma\left(\frac{1}{2} + \frac{\zeta}{2\pi i}\right)}{\Gamma\left(\frac{1+\nu}{2} + \frac{\zeta}{2\pi i}\right)\Gamma\left(\frac{\zeta}{2\pi i}\right)}. \tag{8.33}$$

S_{NLS} is the unique maximally analytic element of the class of $O(n)$-invariant S-functions [ZZ78]. Maximal analyticity means here that in the physical strip $\mathbb{S}(0, \pi)$, the S-function has no poles and the minimal amount of zeroes which are compatible with the axioms for an S-function, i.e., (S1)–(S7). Its eigenvalue decomposition is given by

$$S_{\mathrm{NLS}}(\zeta) = \left(s_+(\zeta)\tfrac{1}{2}\left(\mathbb{1} + \mathbb{F} - \tfrac{2}{n}\mathbb{K}\right) + s_-(\zeta)\tfrac{1}{2}\left(\mathbb{1} - \mathbb{F}\right) + s_0(\zeta)\tfrac{1}{n}\mathbb{K}\right)\mathbb{F}, \tag{8.34}$$

with eigenvalues $s_\pm = b \pm c$ and $s_0 = b + c + nd$. The S-function is P-, C-, and T-symmetric and satisfies $S_{\mathrm{NLS}}(0) = -\mathbb{F}$.

As a first step, we establish existence of the minimal solution with respect to s_0 and an estimate of its asymptotic growth:

Lemma 8.4.2. *The minimal solution with respect to s_0 exists and is given by $f_{0,\min}(\zeta) = (-i\,\mathrm{sh}\,\frac{\zeta}{2})m_{f_0}(\zeta)$ with characteristic function*

$$f_0(t) = \frac{e^{-t} + e^{-\nu t}}{e^t + 1}. \tag{8.35}$$

Moreover, there exist $0 < c \leq c'$, $r > 0$ such that

$$\forall |\operatorname{Re}\zeta| \geq r, \operatorname{Im}\zeta \in [0, 2\pi]: \quad c \leq \frac{|f_{0,\min}(\zeta)|}{|\operatorname{Re}\zeta|^{-(1+\frac{\nu}{2})}\exp|\operatorname{Re}\zeta|} \leq c'. \tag{8.36}$$

Proof. The characteristic function $f_0 = f[-s_0]$ is computed in Section 4.3. Clearly, it is smooth and exponentially decaying. Applying Lemma 4.1.1 (uniqueness) and Theorem 4.2.1 (existence) we find that m_{f_0} is well-defined and that $f_{0,\min}$ exists and agrees with the expression claimed. The estimate of (7.44) together with

$$f_0(t) = 1 - (1 + \tfrac{\nu}{2})t + \mathcal{O}(t^2), \quad t \to 0 \tag{8.37}$$

and the estimate

$$\forall |\operatorname{Re}\zeta| \geq r > 0 : \quad (1 - e^{-2r}) \exp|\operatorname{Re}\zeta| \leq |2 \operatorname{sh}\zeta| \leq (1 + e^{-2r}) \exp|\operatorname{Re}\zeta| \tag{8.38}$$

imply (8.36). □

Lemma 8.4.3 (Stress-energy tensor in NLS model)**.** *A tensor-valued function $F_2^{\mu\nu}$: $\mathbb{C}^2 \times \mathbb{M} \to \mathcal{K}^{\otimes 2}$ forms a parity covariant stress-energy tensor at one-particle level with respect to S_{NLS} with no poles, $\mathfrak{P} = \emptyset$, iff it is of the form*

$$F_2^{\mu\nu}(\theta, \eta + i\pi; x) = G_{\mathrm{free}}^{\mu\nu}\left(\tfrac{\theta+\eta}{2}\right) e^{i(p(\theta;m) - p(\eta;m)).x} F(\eta - \theta + i\pi), \tag{8.39}$$

with

$$F(\zeta) = q(\operatorname{ch}\zeta) f_{0,\min}(\zeta) I_{\otimes 2}, \tag{8.40}$$

where $f_{0,\min}$ is the unique minimal solution with respect to the S-matrix eigenvalue s_0 and q is a polynomial with real coefficients with $q(-1) = 1$.

Proof. By Corollary 5.3.3, $F_2^{\mu\nu}$ has the form (8.39) with F satisfying properties (b)-(g) in Theorem 5.3.1. By (f), $F(\zeta)$ is an $O(n)$-invariant 2-tensor for each ζ. The general form of such a tensor is $F(\zeta) = \lambda(\zeta) I_{\otimes 2}$ with $\lambda : \mathbb{C} \to \mathbb{C}$ [ADO87, Sec. 4, case (a)].

Consider now property (c), $F(\zeta) = S_{\mathrm{NLS}}(\zeta) F(-\zeta)$. Taking the scalar product of both sides with $\frac{1}{n} I_{\otimes 2}$ in $(\mathbb{C}^n)^{\otimes 2}$ yields

$$\lambda(\zeta) = \tfrac{1}{n}(I_{\otimes 2}, S_{\mathrm{NLS}}(-\zeta) I_{\otimes 2}) \lambda(-\zeta) = s_0(-\zeta)\lambda(-\zeta) \tag{8.41}$$

by (8.30) and $\mathbb{1} I_{\otimes 2} = \mathbb{F} I_{\otimes 2} = \frac{1}{n} \mathbb{K} I_{\otimes 2}$. Here we used that $\mathbb{F} I_{\otimes 2} = J^{\otimes 2} I_{\otimes 2} = I_{\otimes 2}$ by Lemma A.6.2.

In summary, Lemma 4.1.4 can be applied to λ, which implies that $\lambda(\zeta) = q(\operatorname{ch}(\zeta)) f_{0,\min}(\zeta)$ and thus F has the form (8.40). That q has real coefficients is a consequence of (e) and Corollary 4.1.2.

Conversely, it is clear that $F_2^{\mu\nu}$ as in (8.39), respectively F, has the properties (b)-(g). □

Theorem 8.4.4 (QEI for the NLS model)**.** *The stress-energy tensor at one-particle level given by F_2 in* (8.39) *satisfies*

$$\forall g \in \mathcal{S}_{\mathbb{R}}(\mathbb{R})\, \exists c_g > 0\, \forall \varphi \in \mathcal{D}(\mathbb{R}, \mathcal{K}) : \quad \langle \varphi, T^{00}(g^2)\varphi \rangle \geq -c_g \|\varphi\|_2^2 \tag{8.42}$$

iff $q \equiv 1$.

Proof. Given F_2 as in Lemma 8.4.3 and using $\widehat{I_{\otimes 2}} = \mathbb{1}_{\mathcal{K}}$ (Lemma A.6.2), we have $\|\hat{F}(\zeta)\|_{\mathcal{B}(\mathcal{K})} = |q(\operatorname{ch}\zeta) f_{0,\min}(\zeta)|$. Thus by Lemma 8.4.2 there exist $r > 0$ and $0 < c \leq c'$ such that

$$\forall \zeta \in |\operatorname{Re}\zeta| > r, \operatorname{Im}\zeta \in [0, 2\pi] : \quad c\, t(\zeta) \exp|\operatorname{Re}\zeta| \leq \|\hat{F}(\zeta)\|_{\mathcal{B}(\mathcal{K})} \leq c' t(\zeta) \exp|\operatorname{Re}\zeta| \tag{8.43}$$

with $t(\zeta) = |\operatorname{Re}\zeta|^{-(1+\frac{\nu}{2})}|q(\operatorname{ch}\zeta)|$. Note that for $q \equiv 1$, $t(\zeta)$ is polynomially decaying, whereas for non-constant q, $t(\zeta)$ is exponentially growing. Thus if q is constant ($q \equiv 1$), we have $c' t(\zeta) < \frac{1}{4}$ for large enough $|\operatorname{Re}\zeta|$; and if q is not constant, then $ct(\zeta) > \frac{1}{4}$ for large enough $|\operatorname{Re}\zeta|$. We conclude by Theorem 7.1.1 that a QEI of the form (8.42) holds iff $q \equiv 1$. □

Chapter 9

Conclusion, discussion, and outlook

In this thesis, we have established QEIs in a larger class of 1+1d integrable models than previously known in the literature. In particular, we proved that QEIs for generic states hold in a wide class of models with *constant* scattering functions, including not only the Ising model, as known earlier, but also the Federbush model. Moreover, the class includes combinations and bosonic or fermionic variants of these models. In all of these situations, the form factor F_2 of the energy density determines the entire operator.

Furthermore, we have established necessary and sufficient conditions for QEIs to hold *at one-particle level* in generic models, which may include bound states or several particle species. Also in this case, only F_2 contributes to expectation values of the energy density, and a QEI is decided based on the large-rapidity behaviour of F_2. At the foundation of both results was a characterization of the form of the energy density by first principles. However, we found that those principles constrain a viable candidate for the energy density (at one-particle level) only up to polynomial prefactors (in $\operatorname{ch}\zeta$). As seen in the case of the Bullough-Dodd, the Federbush, and the $O(n)$-nonlinear sigma model, one-particle QEIs can then fix the energy density at one-particle level partially or entirely, in analogy to [BC16].

Even more foundational, we also showed that the local commutativity theorem holds in integrable models with more than one degree of freedom ($d_{\mathcal{K}} > 1$) and limited to one- and two-particle form factors. This was necessary to characterize locality of the stress-energy tensor at one-particle level in terms of the form factor equations.

Our results suggest a number of directions for further investigation, of which we discuss the most relevant ones:

What is the nature of the freedom in the form of the stress-energy tensor? The factors $Q_i(\operatorname{ch}\zeta)$ in the energy density were partially left unfixed by our analysis. The imposed conditions by first principles require only that the Q_i are invariant under the group symmetry and that they are normalized. At least in the scalar case ($\mathcal{K} = \mathbb{C}$), this freedom can be understood as a polynomial in the differential operator $\square = g^{\mu\nu}\partial_\mu\partial_\nu$ acting on $T^{\mu\nu}$: Given a stress-energy tensor $T^{\mu\nu}$, define

$\tilde{T}^{\mu\nu} := q(-1 - \frac{\square}{2M^2})T^{\mu\nu}$ for some polynomial q. Then at one-particle level

$$F_2^{[\tilde{T}^{\mu\nu}(x)]}(\boldsymbol{\zeta}) = q(\mathrm{ch}(\zeta_1 - \zeta_2))F_2^{[T^{\mu\nu}(x)]}(\boldsymbol{\zeta}),$$

and, provided that $q(-1) = 1$, $F_2^{[\tilde{T}^{\mu\nu}(x)]}$ defines another valid candidate for the stress-energy tensor at one-particle level. However, for generic models, q may depend on the particle types and cannot be understood in terms of derivatives only.

In the physics literature, given a concrete model, a few standard methods exist to check the validity of a specific choice of q: In case the model admits a Lagrangian, perturbation theory checks are used, e.g., [BK02; BFK10; BFK13]. In case the model can be understood as a perturbation of a conformal field theory, a scaling degree for the large-rapidity behaviour (conformal dimension) of the stress-energy tensor can be extracted, which fixes the large-rapidity behaviour of F_2, e.g., [Zam86; DSC96; CF01]. The large-rapidity scaling degree is also related to momentum-space clustering properties, which were studied for some integrable models, e.g., [Smi92; KM93; Del04; BFK21]. But in the general case, none of these methods may be available, and other constraints – perhaps from QEIs in states of higher particle number – might need to take their place.

Which other models can be treated with these methods? We performed our analysis of one-particle QEIs in a very generic setting; there are nevertheless some limitations. For one, we employed the extra assumption of parity covariance of the stress-energy tensor. While parity invariance of the scattering function (and therefore covariance of the stress-energy-tensor) is satisfied in many models, it is not fully generic. Nevertheless, a non-parity covariant stress-energy tensor is still subject to constraints by our results; in particular, the necessary condition we gave for a one-particle QEI to hold remains unmodified (see remarks in Section 7.2). We expect a sufficient condition for a one-particle QEI, similar to the one presented in Theorem 7.1.1(b), to apply also in a parity-breaking situation. Some numerical tests indicate this; however, an analytic proof remained elusive.

Another point is the decomposition of the two-particle form factor of the (trace of the) stress-energy tensor F into polynomials and factors which are fixed by the model (including the minimal solutions and pole factors). For generic models, multiple polynomial prefactors can appear (at least one for each eigenvalue of the S-function). In typical models, these are few to begin with, and symmetries exclude many of those prefactors (as was presented for the Federbush or the $O(n)$-nonlinear sigma model). In other situations, however, there might be too many unfixed factors for the QEI to meaningfully constrain them.

Lastly, we should remark that also in the presence of higher-order poles in the scattering function, the poles in the form factors are expected to be of first-order

[BK02; BFK06] so that such models should be tractable with our methods. This includes for instance the $Z(n)$-Ising, sine-Gordon, or Gross-Neveu model. Also generic Toda field theories do not seem to pose additional problems.

Do QEIs hold in states with higher particle numbers? Apart from the case of constant S-functions, we treated one-particle expectation values of the energy density, where only the one- and two-particle form factors contribute. At n-particle level, generically the form factors $F_2, ..., F_{2n}$ all enter the expectation values; these are more challenging to handle since the number of rapidity arguments increases and since additional "kinematic" poles arise at the boundary of the analyticity region that were absent in the case $n = 1$. Proving the local commutativity theorem for higher-particle form factors appears achievable with the methods employed in this thesis, though with considerable additional complexity arising due to the increasd number of variables, the kinematic poles, and the non-commutativity of S-functions.

Concerning QEI results, the case of higher particle numbers, requires new methods: Due to the appearance of the kinematic poles, the subtracted factorizing kernel in the proof of the one-particle result is not well-defined. Various other attempts at finding a decomposition of the energy density at two-particle level into a positive and a bounded kernel remained unsuccessful. Still, we conducted some promising numerical tests for specific examples like the sinh-Gordon and $O(n)$-nonlinear sigma model. These tests agreed with the analytical results at one-particle level and indicated that the two-particle form factor (constrained by the one-particle QEI) also decides the QEI at the two-particle level. Since these results are only indications, and to avoid overloading the scope of the thesis, they are not presented here. Also, we do not expect to obtain numerical results at much higher particle numbers due to computational complexity scaling exponentially with n.

Finito.

Appendix A

Constructive aspects of integrable quantum field theory

A.1 Representation theory of the Poincaré group in 1+1d

As is well known, particles in relativistic quantum theory correspond to positive-energy, projective, unitary, irreducible representations of the proper orthochronous Poincaré group; possibly further extended to the proper or full Poincaré group by discrete symmetries. In 1+1d the classification of those representations is different from the ordinary Wigner classification [Wig39; Bar54]: There are no rotations in 1+1d.

As a consequence, spin, to some degree, becomes conventional: The little group[1], whose representations in higher dimensions are classified by the spin number, is trivial. However, in resemblance to field theory models from higher dimensions a spin number can still be introduced "by hand" with the peculiar feature that it can take arbitrary nonnegative values.

While in higher dimensions projective representations are obtained as faithful representations of the universal (double) covering of the Poincaré group, in 1+1d the covering is trivial as the proper orthochronous Poincaré group is itself simply connected [Bos96]. Therefore, in the following we will treat the Poincaré group itself and its faithful representations[2]. We follow [Bog+90, Secs. 3.1, 7.1, and, 7.2] and [Haa92, Sec. I.3] albeit with necessary adaptations to 1+1d.

The Poincaré group $\mathcal{P}$ is a semidirect product of the translation group $\mathcal{T} = (\mathbb{M}, +)$ and the Lorentz group $\mathcal{L}$ which is defined as the invariance group of the Minkowski metric $g_{\mu\nu}$, i.e., it consists of matrices $\Lambda \in \mathrm{Mat}(2 \times 2, \mathbb{R})$ which satisfy $\Lambda^T g \Lambda = g$. The identity component $\mathcal{L}_+^\uparrow$ of $\mathcal{L}$, referred to as *proper orthochronous*, is

[1]Minkowski vectors p on the mass shell, i.e., with $p^2 = m^2$ for some $m > 0$, have the form $p_\circ = (m, 0)^t$ in its center of mass representation (i.e., $p = \Lambda p_\circ$ for some Lorentz matrix Λ). The little group is the invariance group corresponding to Lorentz transformations of $p_\circ$ which is clearly trivial in 1+1d.

[2]Note that despite triviality of the universal covering there are nonfaithful projective representations. These arise as central extensions of the Poincaré algebra but have limited physical relevance [Bos96] and will not be discussed in the thesis.

selected by $\det \Lambda = 1$ (*proper*) and $\Lambda_{00} \geq 1$ (*orthochronous*). We may parametrize

$$\mathcal{L}_+^\uparrow = \left\{ \Lambda(\lambda) := \begin{pmatrix} \operatorname{ch}\lambda & \operatorname{sh}\lambda \\ \operatorname{sh}\lambda & \operatorname{ch}\lambda \end{pmatrix} : \lambda \in \mathbb{R} \right\} \cong (\mathbb{R}, +). \tag{A.1}$$

The parameter λ is referred to as *rapidity*.

The proper orthochronous Poincaré group $\mathcal{P}_+^\uparrow$ is then given as a semidirect product $\mathcal{T} \rtimes \mathcal{L}_+^\uparrow$ with group operation

$$(x, \lambda) \cdot (y, \mu) = (x + \Lambda(\lambda)y, \lambda + \mu), \quad x, y \in \mathbb{M},\ \lambda, \mu \in \mathcal{L}_+^\uparrow. \tag{A.2}$$

For the full Poincaré group we introduce the group of reflections

$$\mathcal{I} = \{\mathbb{1}_{\mathbb{C}^2}, I_p, I_t, I_{pt}\} \tag{A.3}$$

with

$$I_p = \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix}, \quad I_t = \begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix}, \quad I_{pt} = I_p I_t = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{A.4}$$

and define it as a semidirect product $\mathcal{P}_+^\uparrow \rtimes \mathcal{I}$ with group operation

$$(x, \lambda, I) \cdot (y, \mu, J) = (x + \Lambda(\lambda) I y, \lambda + |I|\mu, IJ), \quad (x, \lambda), (y, \mu) \in \mathcal{L}_+^\uparrow,\ I, J \in \mathcal{I}, \tag{A.5}$$

where $|I|$ denotes the determinant of I.

The Lie algebra of the (proper orthochronous) Poincaré group is generated by real linear combinations of the operators iP^μ and iK which satisfy the commutation relations

$$[P^\mu, P^\nu] = 0, \quad [P^\mu, K] = i\epsilon^\mu{}_\nu P^\nu, \quad [K, K] = 0, \tag{A.6}$$

where $\epsilon^{\mu\nu}$ is the Levi-Civita symbol. The operators P^μ and K (and their concrete realizations in a representation) are referred to as *total energy-momentum operator* and *boost generator*, respectively. In a unitary irreducible representation the Casimir operators P^2 and (for $P^2 \geq 0$) $\epsilon = \operatorname{sgn}(P^0)$ become multiples of the identity whose values determine the representation. The physical representations are selected by the requirement of *positive energy*, i.e., $P^0 \geq 0$. The three available choices are

(a) $P^2 = m^2, \epsilon = +1$ for some $m > 0$,

(b) $P^2 = 0, P^\mu \neq 0, \epsilon = +1$,

(c) $P^0 = P^1 = 0$.

In higher dimensions there is another Casimir operator (formed by the square of the so-called Pauli-Lubanski vector) which distinguishes the different spin numbers but it vanishes in 1+1d. We will restrict our presentation to representations of the type (a).

The unitary irreducible representations of $\mathcal{P}_+^\uparrow$ of type (a) are all equivalent to $(U_{[m]}, L^2(\mathbb{R}))$ for some $m > 0$, where

$$(U_{[m]}(x,\lambda)\varphi)(\theta) = e^{ip(\theta;m).x}\varphi(\theta-\lambda), \quad \varphi \in L^2(\mathbb{R}). \tag{A.7}$$

Here P^μ is given as the multiplication operator with $p^\mu(\theta; m)$ and K is given as differential operator $i\frac{d}{d\theta}$.

In resemblance to representations with spin in higher dimensions we introduce also $(U_{[m,s]}, \oplus_{2s+1}L^2(\mathbb{R}))$ for $m > 0, s \in \frac{1}{2}\mathbb{N}$ by

$$(U_{[m,s]}(x,\lambda)\varphi)(\theta,\sigma) = e^{ip(\theta;m).x}e^{\sigma\lambda}\varphi(\theta-\lambda,\sigma), \quad \varphi \in \oplus_{2s+1}L^2(\mathbb{R}), \tag{A.8}$$

where σ ranges through $\{-s, -s+1, ..., s-1, s\}$. $U_{[m,s]}$ defines a unitary representation of $\mathcal{P}_+^\uparrow$ of type (a) which is, however, clearly reducible: The restriction to one σ-component stays invariant under $U_{[m,s]}$. The resulting one-dimensional irreducible representations are isomorphic to (A.7). Note also that $U_{[m,0]} = U_{[m]}$.

A.2 Discrete symmetries

In addition to CPT-invariance—which is expected to hold generically in quantum field theory and proven in the axiomatic formulation [Wei95, Secs. 3.3, 5.8 and references therein]—many models have additional discrete symmetries. In particular, a model can be invariant under charge conjugation C, space inversion P, time inversion T or some combinations of these. The three of them combine again to the usual CPT-invariance. In this section, we will discuss their representation as unitary or antiunitary operators on the one-particle space and their extensions to the full interacting state space. We find that these operators are uniquely determined up to a phase subject to some constraints but mostly conventional and we will motivate a standard choice for those phases. In addition, we will derive how these operators act on the scattering (or S-)function and define the subclasses of k-invariant S-functions, where k corresponds to one of the discrete symmetries mentioned above (we write the mathematical objects associated to the discrete symmetry in lower case, e.g., t for T-inversion). While the results are in principle well-known, most textbook accounts (e.g., [Car71; Wei95]) focus on the 1+3d case whose adaptation has to be taken with some care: In 1+1d, the spin-statistics theorem is violated so that spin and statistics are not related. In this regard, a simplification occurs: Without loss of generality, we may restrict to discussing spinless Poincaré group representations.

To begin with, let $U_1(k)$ denote the operator on the one-particle space $\mathcal{H}_1 = L^2(\mathbb{R}, \mathcal{K})$ which represents the discrete transformations $k \in \{c, p, t, cp, ct, pt, cpt\}$

and is unitary for $k = c, p, cp$, and antiunitary for $k = t, ct, pt, cpt$[3]. At the level of the Poincaré group, the discrete transformations may implement the representation of the reflection group (A.4). Identifying $c \,\widehat{=}\, \mathbb{1}_{\mathbb{C}^2}$, $p \,\widehat{=}\, I_p = \mathrm{diag}(1, -1)$, and $t \,\widehat{=}\, I_t = \mathrm{diag}(-1, 1)$ the Poincaré group laws as specified in (A.5) then imply

$$U_1(c)U_1(x, \lambda) = U_1(x, \lambda)U_1(c), \tag{A.9}$$

$$U_1(p)U_1(x, \lambda) = U_1(I_p x, -\lambda)U_1(p), \tag{A.10}$$

$$U_1(t)U_1(x, \lambda) = U_1(I_t x, -\lambda)U_1(t). \tag{A.11}$$

For $k = cpt$ we obtain, accordingly,

$$U_1(cpt)U_1(x, \lambda) = U_1(-x, \lambda)U_1(cpt). \tag{A.12}$$

If we demand in addition that $U_1(p)$ and $U_1(t)$ do not modify $\mathcal{K}$—so that these correspond to "pure" parity and time inversion—their action on one-particle rapidity eigenstates[4] follows from (A.9)–(A.11) and for $\alpha \in \{1, ..., d_{\mathcal{K}}\}$, amounts to

$$U_1(c) |\theta_\alpha\rangle = \xi_\alpha C_\alpha^\beta |\theta_\beta\rangle, \quad U_1(p) |\theta_\alpha\rangle = \eta_\alpha |-\theta_\alpha\rangle, \quad U_1(t) |\theta_\alpha\rangle = \zeta_\alpha |-\theta_\alpha\rangle^{cc}, \tag{A.13}$$

where ξ_α, η_α, and ζ_α are phase factors (i.e., c-numbers with $|\xi_\alpha| = |\eta_\alpha| = |\zeta_\alpha| = 1$) and $C \in \mathcal{B}(\mathcal{K})$ is the so-called charge conjugation matrix (Sec. 2.2, i.p., Rem. 2.2.2) which maps between particles and antiparticles:

$$C_\alpha^\beta = \delta_{\bar{\alpha}}^\beta = J_\alpha^\beta, \tag{A.14}$$

where J denotes the antiunitary involution corresponding to the one-particle little space $\mathcal{K}$. As a consequence, we have $C = C^* = C^{-1}$ and $|\theta_{\bar{\alpha}}\rangle = |\theta_\beta\rangle C_\alpha^\beta$.

As will be motivated below, the phase factors have to satisfy $\eta_\alpha^2 = \pm 1$ and $\xi_\alpha^2 \zeta_\alpha^2 = \pm 1$ with $+1$ for α bosonic and -1 for α fermionic but are otherwise unconstrained. As a consequence, quite generically[5] it will be possible to make the following standard choice for the phase factors:

$$\eta_\alpha = \eta_{\bar{\alpha}}, \quad \xi_\alpha = \xi_{\bar{\alpha}}, \quad \zeta_\alpha = \zeta_{\bar{\alpha}} \tag{A.15}$$

as well as

$$\begin{aligned} &\eta_\alpha = \xi_\alpha = \zeta_\alpha = 1 && \text{for bosonic } \alpha, \\ &\eta_\alpha = i, \xi_\alpha = 1, \zeta_\alpha = -i && \text{for fermionic } \alpha. \end{aligned} \tag{A.16}$$

[3] That the latter have to be represented antiunitarily is a consequence of requiring the representation to be of positive energy (cf. Sec. A.1) as argued, e.g., in [Wei95, Sec. 2.6].

[4] They were introduced in (2.42) and below. The one-particle states do not depend on the S-function, so we drop the subscript S.

[5] This is well-known in 1+3d: See, e.g., [Car71, Sec. 6.9] in the self-conjugated case. In 1+1d, this also appears to be well-known (see [Kar79a]), but the author is unaware of a systematic reference. Therefore, a short motivation of the statement follows later.

As a result, the CPT-phase is 1 in both cases.

Let us briefly motivate this standard phase choice: Above we have already implemented (anti-)unitarity of $U_1(k)$ and the group laws between the proper orthochronous Poincaré group and the reflection group. It remains to impose the group laws of the reflection group itself:

$$c^2 = p^2 = t^2 = 1 \quad cp = pc, \quad ct = tc, \quad pt = tp. \tag{A.17}$$

However, for the operators, these laws are required to hold only restricted to physical states. Since fermionic states are not physical—only bilinears are—we find that on these states an additional minus sign can appear when imposing the group laws on the operators.Given the definitions in (A.13), the expressions appearing in (A.17) are given by a straightforward computation:

$$U_1(t)^2 \left|\theta_\alpha\right\rangle = \left|\theta_\alpha\right\rangle, \quad U_1(c)^2 \left|\theta_\alpha\right\rangle = \xi_{\bar\alpha}\xi_\alpha \left|\theta_\alpha\right\rangle, \quad U_1(p)^2 \left|\theta_\alpha\right\rangle = \eta_\alpha^2 \left|\theta_\alpha\right\rangle, \tag{A.18}$$

and

$$U_1(c)U_1(p) \left|\theta_\alpha\right\rangle = \xi_\alpha\eta_\alpha \left|-\theta_{\bar\alpha}\right\rangle, \quad U_1(p)U_1(c) \left|\theta_\alpha\right\rangle = \xi_\alpha\eta_{\bar\alpha} \left|-\theta_{\bar\alpha}\right\rangle, \tag{A.19}$$

$$U_1(c)U_1(t) \left|\theta_\alpha\right\rangle = \xi_\alpha\zeta_\alpha \left|-\theta_{\bar\alpha}\right\rangle^{cc}, \quad U_1(t)U_1(c) \left|\theta_\alpha\right\rangle = \xi_\alpha^*\zeta_{\bar\alpha} \left|-\theta_{\bar\alpha}\right\rangle^{cc}, \tag{A.20}$$

$$U_1(p)U_1(t) \left|\theta_\alpha\right\rangle = \eta_\alpha\zeta_\alpha \left|\theta_\alpha\right\rangle^{cc}, \quad U_1(t)U_1(p) \left|\theta_\alpha\right\rangle = \eta_\alpha^*\zeta_\alpha \left|\theta_\alpha\right\rangle^{cc}. \tag{A.21}$$

For bosonic α, i.e., when the relations (A.17) are represented faithfully, Equations (A.18)–(A.21) imply

$$\eta_\alpha = \eta_{\bar\alpha} \in \{\pm 1\}, \quad \xi_\alpha = \xi_{\bar\alpha}, \quad \zeta_\alpha = \zeta_{\bar\alpha} \tag{A.22}$$

and no other relations. For fermionic α we will not discuss the various options in detail but note that apart from these constraints, the choice for the factors ξ_α, η_α, ζ_α is merely conventional. In particular, note that for a given ζ_α, we can adjust it to an arbitrary new value $\tilde\zeta_\alpha$ by transforming $\left|\theta_\alpha\right\rangle \mapsto \sqrt{\zeta_\alpha\tilde\zeta_\alpha^*}\left|\theta_\alpha\right\rangle$. As a consequence, we may take standard conventions for the phase factors.

The operators $U_1(k), k \in \{c, p, t, cp, ct, pt, cpt\}$, have a natural extension to the full interacting state space: For noninteracting models the operators extend to Fock space as tensor products. The same extension should also apply in the interacting theory when restricting to scattering states; on these the interaction should be negligible (Sec. 2.5). Denoting the extension by $U(k)$, we conventionally fix the action on the vacuum state to be $U(c)\left|\Omega\right\rangle = U(p)\left|\Omega\right\rangle = U(t)\left|\Omega\right\rangle = \left|\Omega\right\rangle$[6]. Scattering states for higher particle numbers are given by $\left|\boldsymbol{\theta}_{\boldsymbol{\alpha}}\right\rangle_{\text{in/out}} := \left|\boldsymbol{\theta}^{\text{in/out}}_{\boldsymbol{\alpha}^{\text{in/out}}}\right\rangle_S$, where $\boldsymbol{\theta}^{\text{in/out}}$,

[6]This is always possible as explained, e.g., in [LW66].

$\boldsymbol{\alpha}^{\text{in/out}}$ denote the tuples $\boldsymbol{\theta}$, $\boldsymbol{\alpha}$, sorted in descending/ascending order with respect to $\boldsymbol{\theta}$; confer (2.51). Following [Wei95, Sec. 3.3] we require

$$U(c)\,|\boldsymbol{\theta}_{\boldsymbol{\alpha}}\rangle_{\text{in/out}} = \xi_{\boldsymbol{\alpha}}\,|\boldsymbol{\theta}_{\bar{\boldsymbol{\alpha}}}\rangle_{\text{in/out}}\,, \tag{A.23}$$

$$U(p)\,|\boldsymbol{\theta}_{\boldsymbol{\alpha}}\rangle_{\text{in/out}} = \eta_{\boldsymbol{\alpha}}\,|{-\boldsymbol{\theta}_{\boldsymbol{\alpha}}}\rangle_{\text{in/out}}\,, \tag{A.24}$$

$$U(t)\,|\boldsymbol{\theta}_{\boldsymbol{\alpha}}\rangle_{\text{in/out}} = \zeta_{\boldsymbol{\alpha}}\,|{-\boldsymbol{\theta}_{\boldsymbol{\alpha}}}\rangle^{cc}_{\text{out/in}}\,, \tag{A.25}$$

where $\xi_{\boldsymbol{\alpha}} = \xi_{\alpha_1} \cdot \ldots \cdot \xi_{\alpha_n}$ and analogously for $\eta_{\boldsymbol{\alpha}}$ and $\zeta_{\boldsymbol{\alpha}}$.

As a next point, we derive how the S-function transforms under $U_1(k)$. For this it suffices to consider $|\boldsymbol{\theta}_{\boldsymbol{\alpha}}\rangle = |\boldsymbol{\theta}_{\boldsymbol{\alpha}}\rangle_S$ with $\boldsymbol{\theta} \in \mathbb{R}^2, \boldsymbol{\alpha} \in \{1, ..., d_{\mathcal{K}}\}^2$ and $\theta_1 > \theta_2$ or $\theta_2 < \theta_1$ (such that $\boldsymbol{\theta} = \boldsymbol{\theta}^{\text{in}}$ and $\boldsymbol{\theta} = \boldsymbol{\theta}^{\text{out}}$, respectively). We will use the shorthand notation $\theta_{12} := \theta_1 - \theta_2$. Proposition 2.4.8(a) for $n = 2$ and $\tau = \pi_1$ implies

$$|\boldsymbol{\theta}_{\boldsymbol{\alpha}}\rangle = S^{\boldsymbol{\beta}}_{\boldsymbol{\alpha}}(\theta_{12})\,|\overleftarrow{\boldsymbol{\theta}}_{\boldsymbol{\beta}}\rangle\,. \tag{A.26}$$

Now, for $\theta_1 \gtrless \theta_2$ we have that

$$U(c)\,|\boldsymbol{\theta}_{\boldsymbol{\alpha}}\rangle = U(c)\,|\boldsymbol{\theta}_{\boldsymbol{\alpha}}\rangle_{\text{in/out}} = \xi_{\boldsymbol{\alpha}}\,|\boldsymbol{\theta}_{\bar{\boldsymbol{\alpha}}}\rangle_{\text{in/out}} = \xi_{\boldsymbol{\alpha}}\,|\boldsymbol{\theta}_{\bar{\boldsymbol{\alpha}}}\rangle\,, \tag{A.27}$$

$$U(p)\,|\boldsymbol{\theta}_{\boldsymbol{\alpha}}\rangle = U(p)\,|\boldsymbol{\theta}_{\boldsymbol{\alpha}}\rangle_{\text{in/out}} = \eta_{\boldsymbol{\alpha}}\,|{-\boldsymbol{\theta}_{\boldsymbol{\alpha}}}\rangle_{\text{in/out}} = \sigma_{\boldsymbol{\alpha}}\eta_{\boldsymbol{\alpha}}\,|{-\overleftarrow{\boldsymbol{\theta}}_{\overleftarrow{\boldsymbol{\alpha}}}}\rangle\,, \tag{A.28}$$

$$U(t)\,|\boldsymbol{\theta}_{\boldsymbol{\alpha}}\rangle = U(t)\,|\boldsymbol{\theta}_{\boldsymbol{\alpha}}\rangle_{\text{in/out}} = \zeta_{\boldsymbol{\alpha}}\,|{-\boldsymbol{\theta}_{\boldsymbol{\alpha}}}\rangle^{cc}_{\text{out/in}} = \zeta_{\boldsymbol{\alpha}}\,|{-\boldsymbol{\theta}_{\boldsymbol{\alpha}}}\rangle^{cc}\,. \tag{A.29}$$

Note here that $\sigma_{\boldsymbol{\alpha}}$ denotes the statistics matrix which appears due to the change of order in the asymptotic states (in/out). For (A.27)–(A.29) an analogous relation to (A.26) should hold with S replaced by its transformed version, say S_k for $k \in \{c, p, t, cp, ct, pt, cpt\}$. We compute

$$U(c)\,|\boldsymbol{\theta}_{\boldsymbol{\alpha}}\rangle = \xi_{\boldsymbol{\alpha}}\,|\boldsymbol{\theta}_{\overline{\boldsymbol{\alpha}}}\rangle = \xi_{\boldsymbol{\alpha}} S^{\overline{\boldsymbol{\beta}}}_{\overline{\boldsymbol{\alpha}}}(\theta_{12})\,|\overleftarrow{\boldsymbol{\theta}}_{\overline{\boldsymbol{\beta}}}\rangle = \xi_{\boldsymbol{\alpha}} S^{\overline{\boldsymbol{\beta}}}_{\overline{\boldsymbol{\alpha}}}(\theta_{12})\xi_{\boldsymbol{\beta}}^{-1} U(c)\,|\overleftarrow{\boldsymbol{\theta}}_{\boldsymbol{\beta}}\rangle\,, \tag{A.30}$$

$$U(p)\,|\boldsymbol{\theta}_{\boldsymbol{\alpha}}\rangle = \sigma_{\boldsymbol{\alpha}}\eta_{\boldsymbol{\alpha}}\,|{-\overleftarrow{\boldsymbol{\theta}}_{\overleftarrow{\boldsymbol{\alpha}}}}\rangle = \sigma_{\boldsymbol{\alpha}}\eta_{\boldsymbol{\alpha}} S^{\overleftarrow{\boldsymbol{\beta}}}_{\overleftarrow{\boldsymbol{\alpha}}}(-\theta_{21})\,|{-\boldsymbol{\theta}_{\overleftarrow{\boldsymbol{\beta}}}}\rangle = \sigma_{\boldsymbol{\alpha}}\eta_{\boldsymbol{\alpha}} S^{\overleftarrow{\boldsymbol{\beta}}}_{\overleftarrow{\boldsymbol{\alpha}}}(\theta_{12})\eta_{\boldsymbol{\beta}}^{-1}\sigma_{\boldsymbol{\beta}}^{-1} U(p)\,|\overleftarrow{\boldsymbol{\theta}}_{\boldsymbol{\beta}}\rangle\,, \tag{A.31}$$

$$U(t)\,|\boldsymbol{\theta}_{\boldsymbol{\alpha}}\rangle = \zeta_{\boldsymbol{\alpha}}\,|{-\boldsymbol{\theta}_{\boldsymbol{\alpha}}}\rangle^{cc} = \zeta_{\boldsymbol{\alpha}} \overline{S^{\boldsymbol{\beta}}_{\boldsymbol{\alpha}}(-\theta_{12})}\,|{-\overleftarrow{\boldsymbol{\theta}}_{\boldsymbol{\beta}}}\rangle^{cc} = \zeta_{\boldsymbol{\alpha}} S^{\boldsymbol{\alpha}}_{\boldsymbol{\beta}}(\theta_{12})\zeta_{\boldsymbol{\beta}}^{-1} U(t)\,|\overleftarrow{\boldsymbol{\theta}}_{\boldsymbol{\beta}}\rangle\,, \tag{A.32}$$

where in the last line we used that $\overline{S(-\theta)^{\boldsymbol{\beta}}_{\boldsymbol{\alpha}}} = \overline{(S(\theta)^*)^{\boldsymbol{\beta}}_{\boldsymbol{\alpha}}} = (S(\theta))^{\boldsymbol{\alpha}}_{\boldsymbol{\beta}}$ by (S1) and (S2). As a consequence, we can identify

$$(S_c)^{\gamma\delta}_{\alpha\beta}(\theta) = \xi^*_\alpha \xi^*_\beta \xi_\gamma \xi_\delta S^{\bar{\gamma}\bar{\delta}}_{\bar{\alpha}\bar{\beta}}(\theta), \tag{A.33}$$

$$(S_p)^{\gamma\delta}_{\alpha\beta}(\theta) = \sigma_{\alpha\beta}\eta^*_\alpha \eta^*_\beta \sigma_{\gamma\delta}\eta_\gamma \eta_\delta S^{\delta\gamma}_{\beta\alpha}(\theta), \tag{A.34}$$

$$(S_t)^{\gamma\delta}_{\alpha\beta}(\theta) = \zeta^*_\alpha \zeta^*_\beta \zeta_\gamma \zeta_\delta S^{\alpha\beta}_{\gamma\delta}(\theta), \tag{A.35}$$

and call an S-function *k-invariant* iff $S = S_k$. For most models, the additional factors (phase and statistics) will be irrelevant. This is for instance the case if the particles obey the standard phase choices described above and the interaction allows only the processes bb→bb, {bf,fb} → {bf,fb}, and ff→ff (b=boson,f=fermion). In this case, the conditions read

$$\begin{aligned} &\mathrm{C-invariance:} && S^{\gamma\delta}_{\alpha\beta} = S^{\bar\gamma\bar\delta}_{\bar\alpha\bar\beta} && \Leftrightarrow S = C^{\otimes 2} S C^{\otimes 2}, && \text{(A.36)}\\ &\mathrm{P-invariance:} && S^{\gamma\delta}_{\alpha\beta} = S^{\delta\gamma}_{\beta\alpha} && \Leftrightarrow S = \mathbb{F} S \mathbb{F}, && \text{(A.37)}\\ &\mathrm{T-invariance:} && S^{\gamma\delta}_{\alpha\beta} = S^{\alpha\beta}_{\gamma\delta} && \Leftrightarrow S = S^t, && \text{(A.38)} \end{aligned}$$

where S^t denotes the transpose of S. The other symmetries are obtained by composition:

$$\begin{aligned} &\mathrm{CP-invariance:} && S = C^{\otimes 2}\mathbb{F} S \mathbb{F} C^{\otimes 2}, && \text{(A.39)}\\ &\mathrm{CT-invariance:} && S = C^{\otimes 2} S^t C^{\otimes 2} = J^{\otimes 2} S^* J^{\otimes 2}, && \text{(A.40)}\\ &\mathrm{PT-invariance:} && S = \mathbb{F} S^t \mathbb{F}, && \text{(A.41)} \end{aligned}$$

and, of course,

$$\mathrm{CPT-invariance:} \quad S = C^{\otimes 2}\mathbb{F} S^t \mathbb{F} C^{\otimes 2} = J^{\otimes 2}\mathbb{F} S^* \mathbb{F} J^{\otimes 2}, \tag{A.42}$$

in accordance with (S3).

A.3 S-function and ZF operators in a basis

All computations here are supposing a given choice of an orthonormal basis $\{e_\alpha\}$ of a 1-particle little space $\mathcal{K}$ using the notational conventions outlined in Section 2.1 and the convention on barred indices for the charge conjugated basis (Rem. 2.2.2). Summing over equal indices is understood unless otherwise stated.

Lemma A.3.1. *For $w \in \mathcal{B}(\mathcal{K}^{\otimes m}, \mathcal{K}^{\otimes n})$ one has $(w^*)^{\boldsymbol\beta}_{\boldsymbol\alpha} = \overline{w^{\boldsymbol\alpha}_{\boldsymbol\beta}}$.*

Proof.

$$(w^*)^{\boldsymbol\beta}_{\boldsymbol\alpha} = (e_{\boldsymbol\beta}, w^* e_{\boldsymbol\alpha}) = (w e_{\boldsymbol\beta}, e_{\boldsymbol\alpha}) = \overline{(e_{\boldsymbol\alpha}, w e_{\boldsymbol\beta})} = \overline{w^{\boldsymbol\alpha}_{\boldsymbol\beta}}.$$

□

Lemma A.3.2. *(S1), (S2), (S3) and (S5) in a basis amount to (2.11).*

Proof. (S1) and (S2) are equivalent to $S(\zeta)S(-\zeta) = 1_{\mathcal{K}^{\otimes 2}}$ and $S(\zeta) = S(-\bar\zeta)^*$. The first relation in a basis amounts to

$$\begin{aligned} 0 &= (e_\gamma \otimes e_\delta, (S(\zeta)S(-\zeta) - 1_{\mathcal{K}}) e_\alpha \otimes e_\beta)\\ &= (e_\gamma \otimes e_\delta, S(\zeta) e_\rho \otimes e_\sigma)(e_\rho \otimes e_\sigma, S(-\zeta) e_\alpha \otimes e_\beta) - \delta^\gamma_\alpha \delta^\delta_\beta \qquad \text{(A.43)}\\ &= S^{\gamma\delta}_{\rho\sigma}(\zeta) S^{\rho\sigma}_{\alpha\beta}(-\zeta) - \delta^\gamma_\alpha \delta^\delta_\beta, \end{aligned}$$

or more briefly, $S^{\gamma\delta}_{\rho\sigma}(\zeta)S^{\rho\sigma}_{\alpha\beta}(-\zeta) = \delta^{\gamma}_{\alpha}\delta^{\delta}_{\beta}$. Due to Lemma A.3.1 the second relation evaluates to $S^{\gamma\delta}_{\alpha\beta}(\zeta) = \overline{S^{\alpha\beta}_{\gamma\delta}(-\bar{\zeta})}$. The relation $S(\zeta) = J^{\otimes 2}\mathbb{F}S(\zeta)^*\mathbb{F}J^{\otimes 2}$ (S3) yields

$$\begin{aligned} S^{\gamma\delta}_{\alpha\beta}(\zeta) &= (e_\gamma \otimes e_\delta, S(\zeta)e_\alpha \otimes e_\beta) \\ &= (e_\gamma \otimes e_\delta, J^{\otimes 2}\mathbb{F}S(\zeta)^*\mathbb{F}J^{\otimes 2}e_\alpha \otimes e_\beta) \\ &= \overline{(\tilde{e}_\delta \otimes \tilde{e}_\gamma, S(\zeta)^*\tilde{e}_\beta \otimes \tilde{e}_\alpha)} \\ &= \overline{(S(\zeta)^*)^{\bar{\delta}\bar{\gamma}}_{\bar{\beta}\bar{\alpha}}}. \end{aligned} \tag{A.44}$$

Using Lemma A.3.1 again yields equality with $S^{\bar{\beta}\bar{\alpha}}_{\bar{\delta}\bar{\gamma}}$. Lastly, (S5) implies that

$$\begin{aligned} S^{\gamma\delta}_{\alpha\beta}(i\pi - \zeta) &= (e_\gamma \otimes e_\delta, S(i\pi - \zeta)e_\alpha \otimes e_\beta) \\ &= (\tilde{e}_\alpha \otimes e_\gamma, S(\zeta)e_\beta \otimes \tilde{e}_\delta) \\ &= S^{\bar{\alpha}\gamma}_{\beta\bar{\delta}}(\zeta). \end{aligned} \tag{A.45}$$

□

The ZF operators $z^{\sharp}_S$ evaluated in a basis are represented by $z^{\sharp}_{S,\alpha}$. The relation between the two is given by

$$z^{\sharp}_S(\varphi) = z^{\sharp}_S(\varphi^\alpha e_\alpha) = z^{\sharp}_{S,\alpha}(\varphi^\alpha), \quad z^{\sharp}_{S,\alpha} := z^{\sharp}_S(\cdot\, e_\alpha). \tag{A.46}$$

It is convenient to express them (formally) also in terms of integral kernels $z^{\sharp}_{S,\alpha}(\theta)$ by

$$z^{\dagger}_{S,\alpha}(\varphi^\alpha) = \int d\theta z^{\dagger}_{S,\alpha}(\theta)\varphi^\alpha(\theta), \quad z_{S,\alpha}(\varphi^\alpha) = \int d\theta z_{S,\alpha}(\theta)\overline{\varphi^\alpha(\theta)}. \tag{A.47}$$

Given this, we find:

Lemma A.3.3. *The ZF algebra relations* (2.30) *in a basis amout to* (2.31).

Proof. Given arbitrary $\varphi, \chi \in \mathcal{H}_1$, this is easily obtained by rewriting:

$$\begin{aligned} 0 &= z^{\dagger}_S z^{\dagger}_S\Big((1 - S_{\leftarrow})(\varphi \otimes \chi)\Big) \\ &= \int d\theta_1 d\theta_2\, z^{\dagger}_{S,\alpha_1}(\theta_1)z^{\dagger}_{S,\alpha_2}(\theta_2)\Big(\varphi^{\alpha_1}(\theta_1)\chi^{\alpha_2}(\theta_2) - S(\theta_2 - \theta_1)^{\alpha_1\alpha_2}_{\beta_2\beta_1}\varphi^{\beta_2}(\theta_2)\chi^{\beta_1}(\theta_1)\Big) \\ &= \int d\theta d\eta\, \Big(z^{\dagger}_{S,\alpha}(\theta)z^{\dagger}_{S,\beta}(\eta) - S(\theta - \eta)^{\gamma\delta}_{\alpha\beta}z^{\dagger}_{\gamma}(\eta)z^{\dagger}_{\delta}(\theta)\Big)\varphi^\alpha(\theta)\chi^\beta(\eta), \\ 0 &= z_S z_S\Big((1 - S'_{\leftarrow})(\varphi \otimes \chi)\Big), \quad S' := U_1(j)^{\otimes 2}SU_1(j)^{\otimes 2} \\ &= \int d\theta_1 d\theta_2\, z_{S,\alpha_1}(\theta_1)z_{S,\alpha_2}(\theta_2)\overline{\Big(\varphi^{\alpha_1}(\theta_1)\chi^{\alpha_2}(\theta_2) - S'(\theta_2 - \theta_1)^{\alpha_1\alpha_2}_{\beta_2\beta_1}\varphi^{\beta_2}(\theta_2)\chi^{\beta_1}(\theta_1)\Big)} \\ &= \int d\theta d\eta\, \Big(z_{S,\alpha}(\theta)z_{S,\beta}(\eta) - \overline{S'(\theta - \eta)^{\gamma\delta}_{\alpha\beta}}z_{S,\gamma}(\eta)z_{S,\delta}(\theta)\Big)\overline{\varphi^\alpha(\theta)\chi^\beta(\eta)}, \end{aligned}$$

$$= \int d\theta d\eta \left(z_{S,\alpha}(\theta) z_{S,\beta}(\eta) - S(\theta-\eta)^{\beta\alpha}_{\delta\gamma} z_{S,\gamma}(\eta) z_{S,\delta}(\theta) \right) \overline{\varphi^\alpha(\theta)\chi^\beta(\eta)},$$

where in the last line we used $S' = \mathbb{F}S^*\mathbb{F}$ by (S3). For the third ZF algebra relation, abbreviating $S^c := (1 \otimes U_1(j))S(i\pi + \cdot)(U_1(j) \otimes 1)$, we first consider that

$$\begin{aligned} S^c_\leftarrow(\varphi\otimes\chi)^{\alpha_1}{}_{\alpha_2}(\boldsymbol{\theta}) &= ((1\otimes U_1(j))S^c_\leftarrow(\varphi\otimes\chi))^{\alpha_1\bar{\alpha}_2}(\boldsymbol{\theta}) \\ &= ((1\otimes U_1(j))S^c(\theta_2-\theta_1)(U_1(j)\otimes 1))^{\alpha_1\bar{\alpha}_2}_{\bar{\beta}_2\beta_1}(U_1(j)\varphi)^{\bar{\beta}_2}(\theta_2)\chi^{\beta_1}(\theta_1) \\ &= S(i\pi+\theta_2-\theta_1)^{\alpha_1\bar{\alpha}_2}_{\bar{\beta}_2\beta_1}\overline{\varphi^{\beta_2}(\theta_2)}\chi^{\beta_1}(\theta_1) \\ &= S(\theta_2-\theta_1)^{\beta_2\alpha_1}_{\beta_1\alpha_2}\overline{\varphi^{\beta_2}(\theta_2)}\chi^{\beta_1}(\theta_1) \end{aligned}$$

which yields

$$\begin{aligned} \langle\varphi,\chi\rangle\,\mathbb{1} &= z_S z_S^\dagger\Big(\varphi\otimes\chi\Big) - z_S^\dagger z_S\Big(S^c_\leftarrow(\varphi\otimes\chi)\Big) \\ &= \int d\theta_1 d\theta_2 \Big(z_{S,\alpha_1}(\theta_1) z^\dagger_{S,\alpha_2}(\theta_2)\overline{\varphi^{\alpha_1}(\theta_1)}\chi^{\alpha_2}(\theta_2) \\ &\qquad - z^\dagger_{S,\alpha_1}(\theta_1) z_{S,\alpha_2}(\theta_2) S(\theta_2-\theta_1)^{\beta_2\alpha_1}_{\beta_1\alpha_2}\overline{\varphi^{\beta_2}(\theta_2)}\chi^{\beta_1}(\theta_1)\Big) \\ &= \int d\theta d\eta \left(z_{S,\alpha}(\theta) z^\dagger_{S,\beta}(\eta) - S(\eta-\theta)^{\alpha\gamma}_{\beta\delta} z^\dagger_{S,\gamma}(\eta) z_{S,\delta}(\theta)\right)\overline{\varphi^\alpha(\theta)}\chi^\beta(\eta). \end{aligned}$$

Finally, for the l.h.s we have $\langle\varphi,\chi\rangle\,\mathbb{1} = \int d\theta d\eta\, \delta_{\alpha\beta}\delta(\theta-\eta)\overline{\varphi^\alpha(\theta)}\chi^\beta(\eta)$ which concludes the argument. □

A.4 Improper rapidity eigenstates

Occasionally, it will be helpful to introduce improper momentum/rapidity eigenstates. Those states are frequently used in the physics literature, in particular by the form factor community (e.g., [Bab+99; BFK08]), where they serve as a convenient "basis" for computations in $\mathcal{H}_S$. We recall here the definitions from the main text:

$$\begin{array}{lcl} |\boldsymbol{\theta}_{\boldsymbol{\alpha}}\rangle_S &:=& \frac{1}{\sqrt{n!}} z^\dagger_{S,\alpha_1}(\theta_1)...z^\dagger_{S,\alpha_n}(\theta_n)\,|\Omega\rangle\,, \\ \langle\boldsymbol{\theta}_{\boldsymbol{\alpha}}|_S &:=& \frac{1}{\sqrt{n!}}\,\langle\Omega|\, z_{S,\alpha_n}(\theta_n)...z_{S,\alpha_1}(\theta_1), \end{array} \quad n\in\mathbb{N}, \boldsymbol{\theta}\in\mathbb{R}^n, \boldsymbol{\alpha}\in\{1,...,d_{\mathcal{K}}\}^n, \tag{A.48}$$

which is to be read as a formal notation for vector-valued distributions; having $\varphi_j \in \mathcal{H}_1$, $j = 1,...,n$, they are given by

$$\varphi_1\otimes...\otimes\varphi_n \mapsto \begin{array}{lcl} \frac{1}{\sqrt{n!}} z_S^\dagger(\varphi_1)...z_S^\dagger(\varphi_n)\Omega &=& \int d\boldsymbol{\theta}\ |\boldsymbol{\theta}_{\boldsymbol{\alpha}}\rangle_S\, \varphi_1^{\alpha_1}(\theta_1)...\varphi_n^{\alpha_n}(\theta_n) \\ \frac{1}{\sqrt{n!}}\,\langle\Omega|\, z_S(\varphi_n)...z_S(\varphi_1) &=& \int d\boldsymbol{\theta}\ \langle\boldsymbol{\theta}_{\boldsymbol{\alpha}}|_S\, \varphi_{1,\alpha_1}(\theta_1)...\varphi_{n,\alpha_n}(\theta_n) \end{array}. \tag{A.49}$$

In particular, for an S-symmetric function $\psi \in \mathcal{H}_{S,n}$ we have

$$\langle\psi\,|\boldsymbol{\theta}_{\boldsymbol{\alpha}}\rangle_S = \psi_{\boldsymbol{\alpha}}(\boldsymbol{\theta}), \quad \langle\boldsymbol{\theta}_{\boldsymbol{\alpha}}|_S\,\psi\rangle = \psi^{\boldsymbol{\alpha}}(\boldsymbol{\theta}); \tag{A.50}$$

confer also (A.65) and (A.66) in the proof below.

Note that the indices on $\langle \boldsymbol{\theta}_{\boldsymbol{\alpha}}|_S$ are kept down for notational purposes (breaking with up/down-index conventions). For later, let us stipulate that $\langle \boldsymbol{\theta}_{\boldsymbol{\alpha}}|\boldsymbol{\eta}_{\beta}\rangle_S := \langle \boldsymbol{\theta}_{\boldsymbol{\alpha}}|_S \, |\boldsymbol{\eta}_{\beta}\rangle_S$.

Proposition A.4.1. *The expressions defined in (A.48) satisfy*

(a) S-symmetry, i.e., for any $\tau \in \mathfrak{S}_n$ we have

$$|\boldsymbol{\theta}_{\boldsymbol{\alpha}}\rangle_S = |\boldsymbol{\theta}^\tau_\beta\rangle_S \, ((S^\tau(\boldsymbol{\theta}))^{-1})^\beta_{\boldsymbol{\alpha}}, \quad \langle \boldsymbol{\theta}_{\boldsymbol{\alpha}}|_S = (S^\tau(\boldsymbol{\theta}))^{\boldsymbol{\alpha}}_\beta \, \langle \boldsymbol{\theta}^\tau_\beta|_S \,. \tag{A.51}$$

(b) orthonormality (up to ordering), i.e., for an equal amount of arguments,

$$\langle \boldsymbol{\theta}_{\boldsymbol{\alpha}}|\boldsymbol{\eta}_\beta\rangle_S = \frac{1}{n!} \sum_{\tau \in \mathfrak{S}_n} S^\tau(\boldsymbol{\theta})^{\boldsymbol{\alpha}}_\beta \delta(\boldsymbol{\theta}^\tau - \boldsymbol{\eta}) = \frac{1}{n!} \sum_{\tau \in \mathfrak{S}_n} ((S^\tau(\boldsymbol{\eta}))^{-1})^{\boldsymbol{\alpha}}_\beta \delta(\boldsymbol{\theta} - \boldsymbol{\eta}^\tau). \tag{A.52}$$

otherwise $\langle \boldsymbol{\theta}_{\boldsymbol{\alpha}}|\boldsymbol{\eta}_\beta\rangle_S = 0$.

(c) completeness, i.e., on unsymmetrized Fock space $\hat{\mathcal{H}}$ one has

$$\mathcal{P}_S = \sum_{n\in\mathbb{N}} \int d^n\boldsymbol{\lambda} \, |\boldsymbol{\lambda}_{\boldsymbol{\alpha}}\rangle_S \, \langle \boldsymbol{\lambda}_{\boldsymbol{\alpha}}|_S = \sum_{n\in\mathbb{N}} n! \int_{\lambda_{\tau(1)}>\ldots>\lambda_{\tau(n)}} d^n\boldsymbol{\lambda} \, |\boldsymbol{\lambda}_{\boldsymbol{\alpha}}\rangle_S \, \langle \boldsymbol{\lambda}_{\boldsymbol{\alpha}}|_S \,, \tag{A.53}$$

where the last equality is for some fixed $\tau \in \mathfrak{S}_n$.

(d) Poincare-covariance, i.e.,

$$U_S(a,\lambda) \, |\boldsymbol{\theta}_{\boldsymbol{\alpha}}\rangle_S = e^{ip_{\boldsymbol{\alpha}}(\boldsymbol{\theta}).a} \, |(\boldsymbol{\theta} + \lambda \mathbf{1})_{\boldsymbol{\alpha}}\rangle_S \,, \quad (a,\lambda) \in \mathcal{P}^\uparrow_+, \tag{A.54}$$

where $p_{\boldsymbol{\alpha}}(\boldsymbol{\theta}) = \sum_{i=1}^n p(\theta_i; m_{\alpha_i})$ and $\mathbf{1} = (1, \ldots, 1) \in \mathbb{R}^n$. In particular,

$$P^\mu \, |\boldsymbol{\theta}_{\boldsymbol{\alpha}}\rangle_S = p^\mu_{\boldsymbol{\alpha}}(\boldsymbol{\theta}) \, |\boldsymbol{\theta}_{\boldsymbol{\alpha}}\rangle_S \,. \tag{A.55}$$

(e) CPT-covariance, i.e., for $\psi \in \mathcal{H}_{S,n}$

$$U(j) \, |\boldsymbol{\theta}_{\boldsymbol{\alpha}}\rangle_S = J^\beta_{\boldsymbol{\alpha}} \, |\overleftarrow{\boldsymbol{\theta}}_{\overleftarrow{\beta}}\rangle^{cc}_S = |\overleftarrow{\boldsymbol{\theta}}^{\overleftarrow{\overline{\alpha}}}\rangle^{cc}_S \,, \tag{A.56}$$

where the "cc" supscript denotes the antilinear distribution

$$\psi \mapsto \int d\boldsymbol{\theta} \, |\boldsymbol{\theta}_{\boldsymbol{\alpha}}\rangle_S \overline{\psi^{\boldsymbol{\alpha}}(\boldsymbol{\theta})}, \quad \psi \in \mathcal{H}_1^{\otimes n}).$$

As can be seen from properties (b) and (c) the improper rapidity eigenstates generate $\mathcal{H}_S$ but are not all orthogonal (different states can have nonvanishing overlap). Orthogonality in this sense is obtained by restricting to specifically ordered states, e.g., $\theta_1 \lesseqgtr \ldots \lesseqgtr \theta_n$ as was done in Section 2.5.

Proof. Take an arbitrary $\psi \in \mathcal{H}_1^{\otimes n}$. By (A.49) and (2.32) we have that

$$\mathcal{P}_S\,\psi = \int d\boldsymbol{\theta}\,|\boldsymbol{\theta}_{\boldsymbol{\alpha}}\rangle_S\,\psi^{\boldsymbol{\alpha}}(\boldsymbol{\theta}). \tag{A.57}$$

On the other hand, by (2.34) and (2.35),

$$\mathcal{P}_S\,\psi(\boldsymbol{\lambda}) = \tfrac{1}{n!}\sum_{\tau\in\mathfrak{S}_n} S^\tau(\boldsymbol{\lambda})\psi(\boldsymbol{\lambda}^\tau) = \int d\boldsymbol{\theta}\left(\tfrac{1}{n!}\sum_{\tau\in\mathfrak{S}_n}\delta(\boldsymbol{\theta}-\boldsymbol{\lambda}^\tau)S^\tau(\boldsymbol{\lambda})_{\boldsymbol{\alpha}}\right)\psi^{\boldsymbol{\alpha}}(\boldsymbol{\theta}) \tag{A.58}$$

such that by comparison (ψ was arbitrary),

$$\begin{aligned}
|\boldsymbol{\theta}_{\boldsymbol{\alpha}}\rangle_S(\boldsymbol{\lambda}) &= \tfrac{1}{n!}\sum_{\tau\in\mathfrak{S}_n}\delta(\boldsymbol{\theta}-\boldsymbol{\lambda}^\tau)S^\tau(\boldsymbol{\lambda})_{\boldsymbol{\alpha}}\\
&= \tfrac{1}{n!}\sum_{\tau\in\mathfrak{S}_n} S^\tau(\boldsymbol{\theta}^{\tau^{-1}})_{\boldsymbol{\alpha}}\delta(\boldsymbol{\theta}^{\tau^{-1}}-\boldsymbol{\lambda})\\
&= \tfrac{1}{n!}\sum_{\tau\in\mathfrak{S}_n}(S^{\tau^{-1}}(\boldsymbol{\theta}))^{-1}_{\boldsymbol{\alpha}}\delta(\boldsymbol{\theta}^{\tau^{-1}}-\boldsymbol{\lambda})\\
&= \tfrac{1}{n!}\sum_{\tau\in\mathfrak{S}_n}(S^{\tau}(\boldsymbol{\theta}))^{-1}_{\boldsymbol{\alpha}}\delta(\boldsymbol{\theta}^{\tau}-\boldsymbol{\lambda}),
\end{aligned} \tag{A.59}$$

where in the third equality we used property (c) of Corollary 2.4.6 and in the fourth equality we used that summation over a group is invariant under inversion.

As a direct consequence of (A.59), using properties (b) and (c) of Corollary 2.4.6 and that summations over a group are invariant under uniform translations (here by a fixed element τ or τ^{-1} of $\mathfrak{S}_n$) we prove property (a):

$$\begin{aligned}
|\boldsymbol{\theta}_{\boldsymbol{\alpha}}\rangle_S(\boldsymbol{\lambda}) &= \tfrac{1}{n!}\sum_{\rho\in\mathfrak{S}_n}(S^{\rho}(\boldsymbol{\theta}))^{-1}_{\boldsymbol{\alpha}}\delta(\boldsymbol{\theta}^{\rho}-\boldsymbol{\lambda})\\
&= \tfrac{1}{n!}\sum_{\rho\in\mathfrak{S}_n}(S^{\tau\circ\rho}(\boldsymbol{\theta}))^{-1}_{\boldsymbol{\alpha}}\delta(\boldsymbol{\theta}^{\tau\circ\rho}-\boldsymbol{\lambda})\\
&= \tfrac{1}{n!}\sum_{\rho\in\mathfrak{S}_n}\left(S^{\tau}(\boldsymbol{\theta})S^{\rho}(\boldsymbol{\theta}^\tau)\right)^{-1}_{\boldsymbol{\alpha}}\delta((\boldsymbol{\theta}^{\tau})^\rho-\boldsymbol{\lambda})\\
&= \tfrac{1}{n!}\sum_{\rho\in\mathfrak{S}_n}\left((S^{\rho}(\boldsymbol{\theta}^\tau))^{-1}(S^{\tau}(\boldsymbol{\theta}))^{-1}\right)_{\boldsymbol{\alpha}}\delta((\boldsymbol{\theta}^{\tau})^\rho-\boldsymbol{\lambda})\\
&= \tfrac{1}{n!}\sum_{\rho\in\mathfrak{S}_n}((S^{\rho}(\boldsymbol{\theta}^\tau))^{-1})_{\boldsymbol{\beta}}\delta((\boldsymbol{\theta}^{\tau})^\rho-\boldsymbol{\lambda})((S^{\tau}(\boldsymbol{\theta}))^{-1})^{\boldsymbol{\beta}}_{\boldsymbol{\alpha}}\\
&= |\boldsymbol{\theta}^\tau_{\boldsymbol{\beta}}\rangle_S(\boldsymbol{\lambda})((S^{\tau}(\boldsymbol{\theta}))^{-1})^{\boldsymbol{\beta}}_{\boldsymbol{\alpha}}.
\end{aligned} \tag{A.60}$$

the according relation with respect to $\langle\boldsymbol{\theta}_{\boldsymbol{\alpha}}|_S$ holds by unitarity of the representation which implies $(S^\tau(\boldsymbol{\theta}))^* = (S^\tau(\boldsymbol{\theta}))^{-1}$.

Concerning property (b), note that from the first equality in (A.59) we obtain

$$|\boldsymbol{\theta}_{\boldsymbol{\alpha}}\rangle_S = \mathcal{P}_S\,e_{\boldsymbol{\alpha}}\delta(\boldsymbol{\theta}-\cdot_n), \tag{A.61}$$

where the projection $\mathcal{P}_S$ is extended to act on generalized functions (by the same expression) and $\cdot_n$ indicates arguments of $\mathbb{R}^n$. Analogously, for $\langle\boldsymbol{\theta}_{\boldsymbol{\alpha}}|_S$ we find

$$\langle\boldsymbol{\theta}_{\boldsymbol{\alpha}}|_S(\boldsymbol{\lambda}) = \langle e_{\boldsymbol{\alpha}}\delta(\boldsymbol{\theta}-\cdot_n)|\,\mathcal{P}_S^*. \tag{A.62}$$

Therefore, having $\boldsymbol{\theta}, \boldsymbol{\eta} \in \mathbb{R}^n$ and using $\mathcal{P}_S^* \mathcal{P}_S = \mathcal{P}_S$,

$$\begin{aligned}
\langle \boldsymbol{\theta}_{\boldsymbol{\alpha}} | \boldsymbol{\eta}_\beta \rangle_S &= \int d\boldsymbol{\lambda} \, \langle \boldsymbol{\theta}_{\boldsymbol{\alpha}} |_S \, (\boldsymbol{\lambda}) \, | \boldsymbol{\eta}_\beta \rangle_S \, (\boldsymbol{\lambda}) \\
&= \int d\boldsymbol{\lambda} (e_{\boldsymbol{\alpha}} | \delta(\boldsymbol{\theta} - \boldsymbol{\lambda})(\mathcal{P}_S^* \mathcal{P}_S \, \delta(\boldsymbol{\eta} - \cdot_n) e_{\boldsymbol{\beta}})(\boldsymbol{\lambda}) \\
&= \int d\boldsymbol{\lambda} (e_{\boldsymbol{\alpha}} | \delta(\boldsymbol{\theta} - \boldsymbol{\lambda})(\mathcal{P}_S \, \delta(\boldsymbol{\eta} - \cdot_n) e_{\boldsymbol{\beta}})(\boldsymbol{\lambda}) \\
&= \tfrac{1}{n!} \sum_{\tau \in \mathfrak{S}_n} \int d\boldsymbol{\lambda} \, \delta(\boldsymbol{\theta} - \boldsymbol{\lambda}) \delta(\boldsymbol{\eta} - \boldsymbol{\lambda}^\tau)(e_{\boldsymbol{\alpha}} | S^\tau(\boldsymbol{\lambda}) | e_{\boldsymbol{\beta}}) \\
&= \tfrac{1}{n!} \sum_{\tau \in \mathfrak{S}_n} \delta(\boldsymbol{\theta}^\tau - \boldsymbol{\eta})(S^\tau(\boldsymbol{\theta}))^{\boldsymbol{\alpha}}_{\boldsymbol{\beta}}.
\end{aligned} \tag{A.63}$$

This can of course also be expressed as

$$\begin{aligned}
\langle \boldsymbol{\theta}_{\boldsymbol{\alpha}} | \boldsymbol{\eta}_\beta \rangle_S &= \tfrac{1}{n!} \sum_{\tau \in \mathfrak{S}_n} (S^\tau(\boldsymbol{\eta}^{\tau^{-1}}))^{\boldsymbol{\alpha}}_{\boldsymbol{\beta}} \delta(\boldsymbol{\theta} - \boldsymbol{\eta}^{\tau^{-1}}) \\
&= \tfrac{1}{n!} \sum_{\tau \in \mathfrak{S}_n} ((S^{\tau^{-1}}(\boldsymbol{\eta}))^{-1})^{\boldsymbol{\alpha}}_{\boldsymbol{\beta}} \delta(\boldsymbol{\theta} - \boldsymbol{\eta}^{\tau^{-1}}) \\
&= \tfrac{1}{n!} \sum_{\tau \in \mathfrak{S}_n} ((S^{\tau}(\boldsymbol{\eta}))^{-1})^{\boldsymbol{\alpha}}_{\boldsymbol{\beta}} \delta(\boldsymbol{\theta} - \boldsymbol{\eta}^{\tau}),
\end{aligned} \tag{A.64}$$

using again that summation over a group is invariant under inversion. For unequal number of arguments it is clear that the $\langle \boldsymbol{\theta}_{\boldsymbol{\alpha}} | \boldsymbol{\eta}_\beta \rangle_S = 0$.

Finally, using (A.64), we have

$$\begin{aligned}
\langle \boldsymbol{\lambda}_{\boldsymbol{\alpha}} |_S \, \varphi \rangle &= \int d\boldsymbol{\eta} \, \langle \boldsymbol{\lambda}_{\boldsymbol{\alpha}} | \boldsymbol{\eta}_\beta \rangle_S \, \varphi^{\boldsymbol{\beta}}(\boldsymbol{\eta}) \\
&= \tfrac{1}{n!} \sum_{\tau \in \mathfrak{S}_n} \int d\boldsymbol{\eta} S^\tau(\boldsymbol{\lambda}))^{\boldsymbol{\alpha}}_{\boldsymbol{\beta}} \delta(\boldsymbol{\lambda}^\tau - \boldsymbol{\eta}) \varphi^{\boldsymbol{\beta}}(\boldsymbol{\eta}) \\
&= \tfrac{1}{n!} \sum_{\tau \in \mathfrak{S}_n} S^\tau(\boldsymbol{\lambda}))^{\boldsymbol{\alpha}}_{\boldsymbol{\beta}} \varphi^{\boldsymbol{\beta}}(\boldsymbol{\lambda}^\tau) \\
&= (\mathcal{P}_S \, \varphi)^{\boldsymbol{\alpha}}(\boldsymbol{\lambda}),
\end{aligned} \tag{A.65}$$

and, correspondingly, using (A.63) and unitarity of $S^\tau(\boldsymbol{\eta})$,

$$\begin{aligned}
\langle \varphi \, | \boldsymbol{\lambda}_\beta \rangle_S &= \int d\boldsymbol{\theta} \, \varphi_{\boldsymbol{\alpha}}(\boldsymbol{\theta}) \, \langle \boldsymbol{\theta}_{\boldsymbol{\alpha}} | \boldsymbol{\lambda}_\beta \rangle_S \\
&= \tfrac{1}{n!} \sum_{\tau \in \mathfrak{S}_n} \int d\boldsymbol{\theta} \, \varphi_{\boldsymbol{\alpha}}(\boldsymbol{\theta}) \delta(\boldsymbol{\theta} - \boldsymbol{\lambda}^\tau)((S^\tau(\boldsymbol{\lambda}))^{-1})^{\boldsymbol{\alpha}}_{\boldsymbol{\beta}} \\
&= \tfrac{1}{n!} \sum_{\tau \in \mathfrak{S}_n} \varphi_{\boldsymbol{\alpha}}(\boldsymbol{\lambda}^\tau)((S^\tau(\boldsymbol{\lambda}))^{-1})^{\boldsymbol{\alpha}}_{\boldsymbol{\beta}} \\
&= \tfrac{1}{n!} \sum_{\tau \in \mathfrak{S}_n} \overline{(S^\tau(\boldsymbol{\lambda}))^{\boldsymbol{\beta}}_{\boldsymbol{\alpha}} \varphi^{\boldsymbol{\alpha}}(\boldsymbol{\lambda}^\tau)} \\
&= \tfrac{1}{n!} \sum_{\tau \in \mathfrak{S}_n} \overline{(S^\tau(\boldsymbol{\lambda}) \varphi(\boldsymbol{\lambda}^\tau))^{\boldsymbol{\beta}}} \\
&= \tfrac{1}{n!} \sum_{\tau \in \mathfrak{S}_n} (S^\tau(\boldsymbol{\lambda}) \varphi(\boldsymbol{\lambda}^\tau))_{\boldsymbol{\beta}} \\
&= (\mathcal{P}_S \, \varphi)_{\boldsymbol{\beta}}(\boldsymbol{\lambda}).
\end{aligned} \tag{A.66}$$

Then for $\varphi, \psi \in \mathcal{H}_1^{\otimes n}$,

$$\begin{aligned}\langle \varphi, \mathcal{P}_S \psi \rangle &= \langle \mathcal{P}_S \varphi, \mathcal{P}_S \psi \rangle \\ &= \int d\boldsymbol{\lambda}\, (\mathcal{P}_S \varphi)_{\boldsymbol{\alpha}}(\boldsymbol{\lambda}) (\mathcal{P}_S \psi)^{\boldsymbol{\alpha}}(\boldsymbol{\lambda}) \\ &= \int d\boldsymbol{\lambda}\, \langle \varphi \,|\boldsymbol{\lambda}_{\boldsymbol{\alpha}}\rangle_S \, \langle \boldsymbol{\lambda}_{\boldsymbol{\alpha}}|_S \, \psi \rangle \\ &= \langle \varphi | \int d\boldsymbol{\lambda}\, |\boldsymbol{\lambda}_{\boldsymbol{\alpha}}\rangle_S \, \langle \boldsymbol{\lambda}_{\boldsymbol{\alpha}}|_S \, |\psi\rangle \,. \end{aligned} \tag{A.67}$$

We note that, clearly, $\langle \boldsymbol{\theta}_{\boldsymbol{\alpha}}|_S \, \mathcal{P}_S \, |\boldsymbol{\eta}_{\boldsymbol{\beta}}\rangle_S = 0$ for an unequal number of arguments. Then the second equality for property (c) follows. We first show it for $\tau = \mathrm{id}$:

$$\begin{aligned}\mathcal{P}_{S,n} &= \int d^n\boldsymbol{\lambda}\, |\boldsymbol{\lambda}_{\boldsymbol{\alpha}}\rangle_S \, \langle \boldsymbol{\lambda}_{\boldsymbol{\alpha}}|_S \\ &= \sum_{\rho \in \mathfrak{S}_n} \int\limits_{\lambda_{\rho^{-1}(1)} > \ldots > \lambda_{\rho^{-1}(n)}} d^n\boldsymbol{\lambda}\, |\boldsymbol{\lambda}_{\boldsymbol{\alpha}}\rangle_S \, \langle \boldsymbol{\lambda}_{\boldsymbol{\alpha}}|_S \\ &= \sum_{\rho \in \mathfrak{S}_n} \int\limits_{\lambda_1 > \ldots > \lambda_n} d^n\boldsymbol{\lambda}\, |\boldsymbol{\lambda}^\rho_{\boldsymbol{\alpha}}\rangle_S \, \langle \boldsymbol{\lambda}^\rho_{\boldsymbol{\alpha}}|_S \\ &= \sum_{\rho \in \mathfrak{S}_n} \int\limits_{\lambda_1 > \ldots > \lambda_n} d^n\boldsymbol{\lambda}\, |\boldsymbol{\lambda}_{\boldsymbol{\beta}}\rangle_S \, (S^\rho(\boldsymbol{\lambda}))^{\boldsymbol{\beta}}_{\boldsymbol{\alpha}} ((S^\rho(\boldsymbol{\lambda}))^{-1})^{\boldsymbol{\alpha}}_{\boldsymbol{\beta}} \, \langle \boldsymbol{\lambda}_{\boldsymbol{\beta}}|_S \\ &= \sum_{\rho \in \mathfrak{S}_n} \int\limits_{\lambda_1 > \ldots > \lambda_n} d^n\boldsymbol{\lambda}\, |\boldsymbol{\lambda}_{\boldsymbol{\beta}}\rangle_S \, \langle \boldsymbol{\lambda}_{\boldsymbol{\beta}}|_S \\ &= n! \int\limits_{\lambda_1 > \ldots > \lambda_n} d^n\boldsymbol{\lambda}\, |\boldsymbol{\lambda}_{\boldsymbol{\alpha}}\rangle_S \, \langle \boldsymbol{\lambda}_{\boldsymbol{\alpha}}|_S \,. \end{aligned} \tag{A.68}$$

Note that property (a) was used in the fourth equation. This proof clearly extends to arbitrary choices of $\tau \in \mathfrak{S}_n$.

Concerning (d) and (e), consider Equations (A.65) and (A.66) as well as Proposition 2.4.1. They imply

$$\begin{aligned}\langle \varphi | U_S(a,\lambda) | \boldsymbol{\theta}_{\boldsymbol{\alpha}}\rangle_S &= \langle U_S(a,\lambda)^* \varphi \,|\boldsymbol{\theta}_{\boldsymbol{\alpha}}\rangle_S \\ &= (U_S(a,\lambda)^* \varphi)_{\boldsymbol{\alpha}}(\boldsymbol{\theta}) \\ &= (U_S(a,\lambda)^{-1} \varphi)_{\boldsymbol{\alpha}}(\boldsymbol{\theta}) \\ &= e^{ip_{\boldsymbol{\alpha}}(\boldsymbol{\theta}).a} \varphi_{\boldsymbol{\alpha}}(\boldsymbol{\theta} + \lambda\mathbf{1}) \\ &= \langle \varphi | e^{ip_{\boldsymbol{\alpha}}(\boldsymbol{\theta}).a} | (\boldsymbol{\theta} + \lambda\mathbf{1})_{\boldsymbol{\alpha}}\rangle_S \,, \end{aligned} \tag{A.69}$$

and, paying attention to the antilinearity of $U(j)$,

$$
\begin{aligned}
\langle\varphi|U(j)|\boldsymbol{\theta}_{\boldsymbol{\alpha}}\rangle_S &= \overline{\langle U(j)^*\varphi|\boldsymbol{\theta}_{\boldsymbol{\alpha}}\rangle_S} \\
&= \overline{\langle U(j)^*\varphi| \cdot e_{\boldsymbol{\alpha}}\rangle} \\
&= \langle \cdot e_{\boldsymbol{\alpha}}|U(j)^*\varphi\rangle \\
&= \langle \cdot e_{\boldsymbol{\alpha}}|\boldsymbol{\theta} \mapsto J^{\otimes n} \overleftarrow{\varphi} \, (\overleftarrow{\boldsymbol{\theta}})\rangle \\
&= \langle \boldsymbol{\theta} \mapsto \overleftarrow{\varphi} \, (\overleftarrow{\boldsymbol{\theta}}), J^{\otimes n} \cdot e_{\boldsymbol{\alpha}}\rangle \\
&= J^{\boldsymbol{\beta}}_{\boldsymbol{\alpha}} \varphi_{\overleftarrow{\boldsymbol{\beta}}}(\overleftarrow{\boldsymbol{\theta}})^{\bar{\cdot}} \\
&= J^{\boldsymbol{\beta}}_{\boldsymbol{\alpha}} \, \langle\varphi| \, \overleftarrow{\boldsymbol{\theta}}_{\overleftarrow{\boldsymbol{\beta}}}\rangle^{cc}_S \, ,
\end{aligned}
\tag{A.70}
$$

where $J^{\boldsymbol{\beta}}_{\boldsymbol{\alpha}} = (e_{\boldsymbol{\beta}}, J^{\otimes n} e_{\boldsymbol{\alpha}})$. □

A.5 Bound states

In this section we explain how to treat bound states in integrable models and adopt it in our framework; in particular, we will supplement the definitions of particle spectrum, one-particle little space, and S-function with additional structure. How to treat bound states in integrable models is well known in the physics community; some notable accounts can be found in [Kar79b], [Bab+99, Sec. 2.2], [BK02, Sec. 6], and [BFK06]. The equivalence of locality and the form factor equations in the presence of bound states is to some degree expected but not under full mathematical control: Arguments in favor but without explicit assumptions have been presented in [BK02] for models with first- and second-order poles in the scattering function—in particular, for the sine-Gordon model—and in [BFK06] for models where the form factors can be analytically continued in the coupling constant to a model without bound states, possibly the $\mathbb{Z}(n)$-Ising model. Some limited mathematical results were obtained in a class of models with first- and second-order poles in [CT15; CT17]. There is work in progress to generalize these results.[7]

In the case that we have a model with bound states the scattering function has poles within the physical strip $\mathbb{S}(0,\pi)$. They are expected to lie on the imaginary axis $i(0,\pi)$: For on-shell momenta, energy-momentum conservation can otherwise not be satisfied (see Remark A.5.1 below) and thus bound state poles lying off the imaginary axis have to be unstable. In a similar way resonances, i.e., unstable particles, are characterized by poles within $\mathbb{S}(-\pi,0)$; resonances will not have further relevance for this document, for details we refer to [CF05].

To begin with, let us explain what a bound state is: The formation of a bound state is specified by a process $ij \to k$ where two particles of type i and j form a

[7] K. Shedid Attifa, University of Leipzig.

single particle of type k. Let us denote the particle masses and momenta by $m_{i/j/k}$ and $p_{i/j/k}$, respectively. Energy-momentum conservation, $p_i + p_j = p_k$, necessitates the existence of two *fusion angles* $\theta_{ij}^k, \theta_{ji}^k \in (0, \pi)$ such that $\theta_{ij}^k + \theta_{ji}^k \in (0, \pi)$ and

$$p(\zeta + i\theta_{ij}^k; m_i) + p(\zeta - i\theta_{ji}^k; m_j) = p(\zeta; m_k) \quad \forall \zeta \in \mathbb{C}, \tag{A.71}$$

where the mass of the bound state m_k is fixed to be

$$m_k = \sqrt{m_i^2 + m_j^2 + 2m_i m_j \cos\theta_{(ij)}^k}, \quad \theta_{(ij)}^k := \theta_{ij}^k + \theta_{ji}^k, \tag{A.72}$$

and, similarly,

$$m_i = \sqrt{m_k^2 + m_j^2 - 2m_k m_j \cos\theta_{ji}^k}, \quad m_j = \sqrt{m_k^2 + m_i^2 - 2m_k m_i \cos\theta_{ij}^k}. \tag{A.73}$$

Remark A.5.1. The expressions above, in particular the generality of (A.71), can be argued as follows: For on-shell momenta we parameterize the conservation relation $p_i + p_j = p_k$ by complex rapidities $\zeta_{i/j/k} \in \mathbb{C}$ and denote their differences as $\zeta_{ij} := \zeta_i - \zeta_j$ and so on. We restrict to $|\operatorname{Im}\zeta_{ij}|, |\operatorname{Im}\zeta_{jk}|, |\operatorname{Im}\zeta_{ik}| < \pi$ since shifts in $i\pi$ yield relative signs, $p(\zeta) = -p(\zeta + i\pi)$, which change the interpretation of the labels i, j, and k (e.g., $p_i + p_j = p_k$ becomes $p_i - p_j = p_k \Leftrightarrow p_i = p_k + p_j$). Squaring the conservation relation yields

$$m_i^2 + m_j^2 + 2m_i m_j \operatorname{ch}\zeta_{ij} = m_k^2. \tag{A.74}$$

Since m_k^2 is real, it is required that $\operatorname{ch}\zeta_{ij}$ is real so that either $\operatorname{Re}\zeta_{ij} = 0$ or $\operatorname{Im}\zeta_{ij} = 0$. For $\operatorname{Im}\zeta_{ij} = 0$ one has $m_k \geq m_i + m_j$ which is inconsistent with the interpretation as a bound state ($m_k < m_i + m_j$) so that $\operatorname{Re}\zeta_{ij} = 0$ is the desired option. Analogously, one can square $p_i = p_k - p_j$ and $p_j = p_k - p_i$ to obtain that $\operatorname{Re}\zeta_{ik} = 0$ and $\operatorname{Re}\zeta_{jk} = 0$. We then define $\theta_{ij}^k := \operatorname{Im}\zeta_{ik}$, $\theta_{ji}^k := \operatorname{Im}\zeta_{kj}$, and $\theta_{(ij)}^k := \operatorname{Im}\zeta_{ij} = \theta_{ij}^k + \theta_{ji}^k$—all of them within $(-\pi, \pi)$ by assumption—so that (A.74) implies (A.72) and (A.73) holds analogously. For $\zeta_k = \zeta$ the conservation equation becomes (A.71). Taking the imaginary part of the 0-component of (A.71) at real ζ yields $(m_i \sin\theta_{ij}^k - m_j \sin\theta_{ji}^k) \operatorname{sh}\zeta = 0$ so that θ_{ij}^k and θ_{ji}^k must have the same sign; its choice corresponds to a relabeling $i \leftrightarrow j$ and is thus conventional (here we choose $+$).

The bound state k can be treated as a new (or existing) particle type with mass m_k as given above and some charge q_k and spin s_k which depend on the individual charges q_i, q_j and spins s_i, s_j, respectively. The bound state particle types $\mathcal{I}_B$ are treated on the same footing as elementary particle types $\mathfrak{I}$, i.e., $\mathcal{I}_B \subset \mathcal{I}$. It must be specified, though, which formation processes are possible. This role is played by the fusion rules $\mathfrak{F}$; recall here that these supplement the definition of a particle spectrum $(\mathfrak{I}, \mathcal{G}, \mathfrak{C}, \mathfrak{F})$ from Section 2.2. A fusion rule $f \in \mathfrak{F}$ is of the form $f = ij \to k$, $i, j, k \in \mathcal{I}$, and represents the fact that a fusion of particles of type i and j to a particle of type k is allowed. We demand that the set of fusion rules is closed under exchange of input particles, charge conjugation, and crossing symmetry: For each $ij \to k \in \mathfrak{F}$ also $ji \to k$, $\overline{i}\overline{j} \to \bar{k}$, $i\bar{k} \to \bar{j}$, $j\bar{k} \to \bar{i}$ are elements of $\mathfrak{F}$. The fusion angles are specified as secondary data by the construction from above:

Definition A.5.2. For each $(ij \to k) \in \mathcal{F}$ and particle masses m_i, m_j, m_k we define the *fusion angles* $\theta_{ij}^k, \theta_{ji}^k, \theta_{(ij)}^k \in (0, \pi)$ by the expressions given in (A.72) and (A.73).

As a consistency condition one may derive that

$$\theta_{(ij)}^k + \theta_{(jk)}^i + \theta_{(ki)}^j = 2\pi. \tag{A.75}$$

At the level of the scattering function, the existence of these fusion processes manifests by the appearance of poles within the physical strip and imposes additional constraints. As these concepts depend on the identification of particles, we base them on a one-particle little space $(\mathcal{K}, V, J, M)$ with a grading by a (here: finite) set of particle types $\mathfrak{I}$, i.e., $\mathcal{K} = \oplus_{i\in\mathfrak{I}}\mathcal{K}_i$ with orthogonal projections E_i corresponding to the closed subspaces $\mathcal{K}_i$ and similarly for V, J, and M. Then:

Definition A.5.3. (Intertwiner) Assume a little space $(\mathcal{K}, V, J, M)$ with a grading by particle types $\mathfrak{I}$. Then for $i, j, k \in \mathfrak{I}$ an operator $\varphi \in \mathcal{B}(\mathcal{K}^{\otimes 2}, \mathcal{K})$ is referred to as an *intertwiner* (with respect to $ij \to k$) iff it is complete and normalized,

$$\varphi^*\varphi = E_i \otimes E_j, \quad \varphi\varphi^* = E_k, \tag{A.76}$$

and equivariant/intertwining with respect to $V(g)$, J, and M, i.e.,

$$\forall g \in \mathcal{G} : \varphi V(g)^{\otimes 2} = V(g)\varphi, \quad \varphi J^{\otimes 2} = J\varphi, \quad \varphi M^{\otimes 2} = M\varphi. \tag{A.77}$$

In this case we often denote φ as φ_{ij}^k and its adjoint by φ_k^{ij}.

It is immediate that intertwiners are norm-preserving and that they are uniquely determined up to a phase factor. Also, using (A.76), Equation (A.77) simplifies to $\varphi(V_i(g) \otimes V_j(g)) = V_k(g)\varphi$ for all $g \in \mathcal{G}$, and analogously for M and J. Another simple consequence is that $d_{\mathcal{K}_k} = d_{\mathcal{K}_i} d_{\mathcal{K}_j}$.

In the presence of bound states we will always suppose that such intertwiners exist for each $ij \to k \in \mathfrak{F}$. In the scalar case, $d_{\mathcal{K}} = 1$, or without global symmetry, $\mathcal{G} = \{e\}$, finding the intertwiners is trivial: Define $\varphi(e_{i,\alpha} \otimes e_{j,\beta}) = e_{k,\gamma}$ for some ONBs $\{e_{i/j/k,\alpha}\}$ of $\mathcal{K}_{i/j/k}$ and some identification of (α, β) with γ.

In more general cases, finding an intertwiner corresponds to finding a Clebsh-Gordon type decomposition of the tensor product representation $V_i(g) \otimes V_j(g)$ into irreducible representations $V_l(g)$ which sum to $V_k(g)$.

As our last point, we explore the additional properties required for the scattering function. In order to do this we introduce

Definition A.5.4. (S-function residues) Assume a little space $(\mathcal{K}, V, J, M)$ with a particle type grading $\mathfrak{I}$ and a set of fusion rules $\mathfrak{F}$. The *S-function residues* $R_f \in \mathcal{B}(\mathcal{K}_k)$, $f \in \mathfrak{F}$, are defined by

$$-i \operatorname*{res}_{\zeta = i\theta} S(\zeta) = \sum_{\substack{ij \to k \in \mathfrak{F}, \\ \text{s.t. } \theta^k_{(ij)} = \theta}} \varphi^{ij}_k R_{ij\to k} \varphi^k_{ij} \tag{A.78}$$

for some choice of intertwiners φ^k_{ij} with respect to $ij \to k$. For usage in the main text, we define *bound state intertwiners* by

$$\Gamma^k_{ij} = \sqrt{|R_{ij\to k}|}\varphi^k_{ij}, \quad \Gamma^{ij}_k = \sqrt{|R_{ij\to k}|}\varphi^{ij}_k \tag{A.79}$$

as an element of $\mathcal{B}(\mathcal{K}^{\otimes 2}, \mathcal{K})$ and $\mathcal{B}(\mathcal{K}, \mathcal{K}^{\otimes 2})$, respectively.

Since φ^k_{ij} is uniquely defined up to a phase factor, $R_{ij\to k}$ is uniquely determined by (A.78). The bound state intertwiner, however, depends on this phase and in concrete examples care has to be taken to select a consistent choice for these phases [Que99]. For (A.78) we remark that in many typical examples there is only a single combination $ij \to k$ with $\theta = \theta^k_{(ij)}$ so that the sum over fusion rules collapses to a single term. We finally define the properties we expect for the S-function:

Definition A.5.5. Given a one-particle little space $(\mathcal{K}, V, J, M)$ with a particle type grading $\mathfrak{I}$ and a set of fusion rules $\mathfrak{F}$. We say that an S-function S is *compatible with the fusion rules* iff it satisfies the following further conditions:

(SB1) Pole structure: For each $f = ij \to k \in \mathcal{F}$, $S(\zeta)$ has a simple pole at $\zeta = i\theta^k_{(ij)}$.

(SB2) Pole positivity: For each $f \in \mathcal{F}$, $R_f \geq 0$.

(SB3) Bootstrap equation: For each $k \in \mathfrak{I}$,

$$\begin{aligned}(1 \otimes E_k)&S(\zeta)(E_k \otimes 1) \\ &= \sum_{ij\to k \in \mathfrak{F}} (1 \otimes \varphi^k_{ij} R^{-\frac{1}{2}}_{ij\to k}) S(\zeta - i\theta^k_{ji})_{23} S(\zeta + i\theta^k_{ij})_{13} (\varphi^{ij}_k R^{\frac{1}{2}}_{ij\to k} \otimes 1),\end{aligned}$$

for some choice of intertwiners φ^k_{ij}.

The restriction to simple poles in (SB1) is for simplicity. Higher-order poles are possible (e.g., in the $\mathbb{Z}(n)$-Ising model [BFK06]) but will not appear in examples presented in this document. (SB2) is motivated by the positivity of the scattering matrix residues in quantum mechanical potential scattering [Kar79b][8]. The bootstrap equation (SB3) is a consequence of treating bound states on the same footing as an elementary particle, in particular, supposing that they are also subject to factorizing scattering; the expression arises by taking the bound states residue $\zeta_{ij} = i\theta^k_{(ij)}$ of the three-particle scattering function $S^{(3)}(\zeta_i, \zeta_j, \zeta_l)$ with $\zeta = \zeta_{kl}$ [Kar79b].

A scattering function which is compatible with the fusion rules of the model has all the necessary poles to account for the bound states in the model and treats these bound states on the same footing as ordinary particles.

Definition A.5.6. In the case that a scattering function which is compatible with the fusion rules and has no more than the necessary poles and zeroes (resulting from properties (S1)–(S7) and (SB1)–(SB3)) it is referred to as *maximally analytic*.

A.6 Miscellaneous

We collect here proofs of various statements from the main text which can be easily proven but should not distract the reader there.

The first two results concern the identification of two-tensors $\mathcal{K}^{\otimes 2}$ with operators $\mathcal{B}(\mathcal{K})$ using the $\hat{\ }$-isomorphism defined in (2.4).

Lemma A.6.1. *For arbitrary $F \in \mathcal{K}^{\otimes 2}$ the following identities hold:*

$$\widehat{A^{\otimes 2}F} = A\hat{F}A^*, \quad A \in \mathcal{B}(\mathcal{K}),\ [A, J] = 0 \tag{A.80}$$

$$\widehat{J^{\otimes 2}F} = J\hat{F}J, \tag{A.81}$$

$$\widehat{\mathbb{F}F} = J\hat{F}^*J, \tag{A.82}$$

$$\widehat{I_{\otimes 2}} = \mathbb{1}_{\mathcal{K}}, \tag{A.83}$$

$$\|\hat{F}\|_{\mathcal{B}(\mathcal{K})} \leq \|F\|_{\mathcal{K}^{\otimes 2}}. \tag{A.84}$$

Proof. For all $u, v \in \mathcal{K}$ we find

$$\begin{aligned}
(u, \widehat{A^{\otimes 2}F}v) &= \left(u \otimes Jv, A^{\otimes 2}F\right)_{\mathcal{K}^{\otimes 2}} = (A^*u \otimes JA^*v, F)_{\mathcal{K}^{\otimes 2}} \\
&= (A^*u, \hat{F}A^*v) = (u, A\hat{F}A^*v). \\
(u, \widehat{J^{\otimes 2}F}v) &= \left(u \otimes Jv, J^{\otimes 2}F\right)_{\mathcal{K}^{\otimes 2}} = \overline{(Ju \otimes J^2v, F)}_{\mathcal{K}^{\otimes 2}} \\
&= \overline{(Ju, \hat{F}Jv)} = (u, J\hat{F}Jv).
\end{aligned}$$

[8] For comparison with the reference note the different sign convention for the Minkowski metric and that $\mathrm{res}_{(p_1+p_2)^2=-m_b^2} = -2i\sin\theta\ \mathrm{res}_{\theta_{12}=i\theta}$ where m_b is related to θ as m_k is related to $\theta^k_{(ij)}$ in (A.72). Thus positivity of the left side residue (Eq. (24) in the reference with $\eta_b = 1$) corresponds to $-i\,\mathrm{res}_{\zeta=i\theta}$ being positive.

$$
\begin{aligned}
(u, \widehat{\mathbb{F}F}v) &= (u \otimes Jv, \mathbb{F}F)_{\mathcal{K}^{\otimes 2}} = (Jv \otimes u, F)_{\mathcal{K}^{\otimes 2}} \\
&= \left(Jv \otimes J^2 u, F\right)_{\mathcal{K}^{\otimes 2}} = (Jv, \hat{F}Ju) = (u, J\hat{F}^* Jv). \\
(u, \widehat{I_{\otimes 2}}v) &= (u \otimes Jv, I_{\otimes 2})_{\mathcal{K}^{\otimes 2}} = (u, v) = (u, \mathbb{1}_{\mathcal{K}} v),
\end{aligned}
$$

implying (A.80)–(A.83). Lastly, we prove (A.84) by

$$
\begin{aligned}
\|\hat{F}\|_{\mathcal{B}(\mathcal{K})} &= \sup_{u \in \mathcal{K}, \|u\|=1} |(u, \hat{F}u)| = \sup_{u \in \mathcal{K}, \|u\|=1} |(F, u \otimes Ju)| \\
&\leq \sup_{u \in \mathcal{K}, \|u\|=1} \|F\|_{\mathcal{K}^{\otimes 2}} \|u\| \|Ju\| = \|F\|_{\mathcal{K}^{\otimes 2}}.
\end{aligned}
\tag{A.85}
$$

□

Lemma A.6.2. *$I_{\otimes 2}$ is invariant under the action of $\mathbb{F}$ and of $U^{\otimes 2}$ for any $U \in \mathcal{B}(\mathcal{K})$ with U unitary or anti-unitary and $[U, J] = 0$. Also, we have $\|I_{\otimes 2}\|_{\mathcal{K}} = \sqrt{d_{\mathcal{K}}}$.*

Proof. Let $u, v \in \mathcal{K}$. Then $(u \otimes v, I_{\otimes 2}) = (u, Jv)$. Invariance under $\mathbb{F}$ follows by

$$
(u \otimes v, \mathbb{F}I_{\otimes 2}) = (v \otimes u, I_{\otimes 2}) = (v, Ju) = (u, Jv) = (u \otimes v, I_{\otimes 2}). \tag{A.86}
$$

For unitary U, we have

$$
(u \otimes v, U^{\otimes 2} I_{\otimes 2}) = (U^{-1}u \otimes U^{-1}v, I_{\otimes 2}) = (U^{-1}u, \mathbb{1}_{\mathcal{K}} JU^{-1}v) = (u, UJU^{-1}v). \tag{A.87}
$$

For antiunitary U the intermediate expressions have a complex conjugation but the concluding identity holds unmodified. The r.h.s. of (A.87) evaluates to (u, Jv) iff $UJU^{-1} = J$, or equivalently $[U, J] = 0$.

The norm follows by (2.5) via

$$
\|I_{\otimes 2}\|_{\mathcal{K}}^2 = (I_{\otimes 2}, I_{\otimes 2}) = \sum_{\alpha, \beta} (e_\alpha \otimes Je_\alpha, e_\beta \otimes Je_\beta) = \sum_{\alpha\beta} \delta_\beta^\alpha \delta_\alpha^\beta = d_{\mathcal{K}}. \tag{A.88}
$$

□

Proposition A.6.3. *The set of states*

$$
\mathcal{P}_S(\varphi_1 \otimes ... \otimes \varphi_n) = \tfrac{1}{\sqrt{n!}} z_S^\dagger(\varphi_1)...z_S^\dagger(\varphi_n)\Omega, \quad \varphi_j \in \mathcal{H}_1,\ j = 1, .., n, \tag{A.89}
$$

forms a total subset of $\mathcal{H}_S$

Proof. Equality of the expressions in (A.89) is obtained by repetitive application of the definition of $z_S^\dagger$ (2.26)

$$
\begin{aligned}
\tfrac{1}{\sqrt{n!}} z_S^\dagger(\varphi_1)...z_S^\dagger(\varphi_n)\Omega &= \tfrac{1}{\sqrt{(n-1)!}} \mathcal{P}_S(\varphi_1 \otimes z_S^\dagger(\varphi_2)...z_S^\dagger(\varphi_n)\Omega) \\
&= ... \\
&= \mathcal{P}_S(\varphi_1 \otimes \mathcal{P}_S(\varphi_2 \otimes \mathcal{P}_S(...\varphi_n))),
\end{aligned}
\tag{A.90}
$$

where the inner $\mathcal{P}_S$ drop out: For any $n \in \mathbb{N}$,

$$\begin{aligned} \mathcal{P}_{S,n}(1 \otimes \mathcal{P}_{S,n-1}) &= \tfrac{1}{n!(n-1)!} \sum_{\tau \in \mathfrak{S}_n} \sum_{\rho \in \mathfrak{S}_{n-1}} D_n^\tau (1 \otimes D_{n-1}^\rho) \\ &= \tfrac{1}{n!(n-1)!} \sum_{\tau \in \mathfrak{S}_n} \sum_{\rho \in \mathfrak{S}_{n-1}} D_n^\tau D_n^{\mathrm{id}\otimes\rho} \\ &= \tfrac{1}{n!(n-1)!} \sum_{\tau \in \mathfrak{S}_n} \sum_{\rho \in \mathfrak{S}_{n-1}} D_n^{\tau\circ(\mathrm{id}\otimes\rho)} \\ &= \tfrac{1}{n!} \left(\frac{\sum_{\rho \in \mathfrak{S}_{n-1}}}{(n-1)!} \right) \sum_{\tau' = \tau\circ(\mathrm{id}\otimes\rho) \in \mathfrak{S}_n} D_n^{\tau'} \\ &= \tfrac{1}{n!} \sum_{\tau' \in \mathfrak{S}_n} D_n^{\tau'} = \mathcal{P}_{S,n} \,. \end{aligned} \tag{A.91}$$

Then it is evident that the linear span of (A.89) includes the subset of finite particle states $\mathcal{H}_S^{\mathrm{f}}$ which is dense in $\mathcal{H}_S$. □

Lemma A.6.4. *The properties (S3), (S6), and (S7) imply, respectively, that $U(j)$, $U_S(x, \lambda)$, and $V_S(g)$ commute with $\mathcal{P}_S$ at the two-particle level for all $(x, \lambda) \in \mathcal{P}_+$ and $g \in \mathcal{G}$.*

Proof. Since $\mathcal{P}_{S,2} = \frac{1}{2}(1 + S_\leftarrow)$ it suffices to prove commutativity with $S_\leftarrow$. The expressions in Proposition 2.4.1 and Equation (2.7) for $U(j)$, $U_S(x, \lambda)$, and $V_S(g)$ yield that, for $\psi \in \mathcal{H}_{S,2}$ and $\boldsymbol{\theta} \in \mathbb{R}^2$,

$$(U(j)\psi)(\boldsymbol{\theta}) = J^{\otimes 2}\mathbb{F}\psi(\overleftarrow{\boldsymbol{\theta}}), \tag{A.92}$$

$$(U_S(x, \lambda)\psi)(\boldsymbol{\theta}) = e^{iP(\boldsymbol{\theta}).x}\psi(\boldsymbol{\theta} - \lambda\mathbf{1}), \tag{A.93}$$

$$(V_S(g)\psi)(\boldsymbol{\theta}) = V(g)^{\otimes 2}\psi(\boldsymbol{\theta}), \tag{A.94}$$

where $\mathbf{1} = (1, 1)$. Thus, using (S3), we find

$$\begin{aligned} (U(j)S_\leftarrow\psi)(\boldsymbol{\theta}) &= J^{\otimes 2}\mathbb{F}(S_\leftarrow\psi)(\overleftarrow{\boldsymbol{\theta}}) \\ &= J^{\otimes 2}\mathbb{F}S(\theta_1 - \theta_2)\psi(\boldsymbol{\theta}) \\ &= S(\theta_2 - \theta_1)\mathbb{F}J^{\otimes 2}\psi(\boldsymbol{\theta}) \\ &= S(\theta_2 - \theta_1)(U(j)\psi)(\overleftarrow{\boldsymbol{\theta}}) \\ &= (S_\leftarrow U(j)\psi)(\boldsymbol{\theta}). \end{aligned} \tag{A.95}$$

Due to (S6) defining $M_1 = M \otimes 1$ and $M_2 = 1 \otimes M$ we have that $M_1 S(\zeta) = S(\zeta)M_2$ and $M_2 S(\zeta) = S(\zeta)M_1$ for all $\zeta \in \mathbb{C}$. As $P(\boldsymbol{\theta}) = p(\theta_1; M_1) + p(\theta_2, M_2)$, it follows that $P(\boldsymbol{\theta})S(\zeta) = S(\zeta)P(\overleftarrow{\boldsymbol{\theta}})$. As a result, we find

$$\begin{aligned}(U_S(x,\lambda)S_{\leftarrow}\psi)(\boldsymbol{\theta}) &= e^{iP(\boldsymbol{\theta}).x}(S_{\leftarrow}\psi)(\boldsymbol{\theta} - \lambda\mathbf{1}) \\ &= e^{iP(\boldsymbol{\theta}).x}S(\theta_2 - \theta_1)\psi(\overleftarrow{\boldsymbol{\theta}} - \lambda\mathbf{1}) \\ &= S(\theta_2 - \theta_1)e^{iP(\overleftarrow{\boldsymbol{\theta}}).x}\psi(\overleftarrow{\boldsymbol{\theta}} - \lambda\mathbf{1}) \\ &= S(\theta_2 - \theta_1)(U_S(x,\lambda)\psi)(\overleftarrow{\boldsymbol{\theta}} - \lambda\mathbf{1}) \\ &= (S_{\leftarrow}U_S(x,\lambda)\psi)(\boldsymbol{\theta}).\end{aligned} \tag{A.96}$$

Lastly, using (S7), it is also clear that $[V_S(g), S_{\leftarrow}] = 0$. □

Appendix B

Literature survey: Form factor conventions

There are various conventions for the form factors in the literature so that it appears useful to briefly compare them. Let us survey the conventions which are used in [Smi92], [Que99], [Bab+99], [BC15], and [AL17]. To be clear about the definitions, form factors $F_{m+n}^{[A]}$, $m, n \in \mathbb{N}_0$, of a given operator A are defined as extensions of the expression

$$(F_{m+n}^{[A]})_{\boldsymbol{\alpha\beta}}(\boldsymbol{\theta}, \boldsymbol{\eta}+i\boldsymbol{\pi}) = \sqrt{m!n!}\, C_{\boldsymbol{\alpha\alpha'}} \,\langle \boldsymbol{\theta}_{\boldsymbol{\alpha}'}|_S \, A \, |\overleftarrow{\boldsymbol{\eta}}_{\overleftarrow{\boldsymbol{\beta}}}\rangle_S\,, \qquad \theta_i \neq \eta_j,\, 1 \leq i \leq m, 1 \leq j \leq n, \tag{B.1}$$

which is defined in (3.2). Vice versa, given such form factors, the corresponding operator A is defined in (3.1) and (3.3).

Other conventions in the literature differ by constant prefactors, changes in the order of rapidities, or where $i\pi$-shifts are used. In the table below we list for each reference the equations or sections which are necessary to fix the convention for the form factors. Further, we give the expression corresponding to (B.1) in the notation employed by the reference. Note that due to the assumptions on $\boldsymbol{\theta}$ and $\boldsymbol{\eta}$ in (B.1) we hit no kinematic poles. Note also that the relevant equations in this document to fix the conventions are, apart from (B.1), the ZF algebra relations (2.30) or (2.31) and the definition of the rapidity eigenstates (2.42).

Reference	Relevant parts	$(F_{m+n}^{[A]})_{\boldsymbol{\alpha\beta}}(\boldsymbol{\theta}, \boldsymbol{\eta} + i\boldsymbol{\pi})$ in notation of ref.
[Smi92]	Chapter 1	$f_{\beta\boldsymbol{\alpha}}(\boldsymbol{\eta}, \boldsymbol{\theta} + i\boldsymbol{\pi})$
[Bab+99]	Eqs. (2.1, 2.2, 3.9)	$(4\pi)^{-\frac{n}{2}} f^{A}_{\overleftarrow{\boldsymbol{\beta}}}(\overleftarrow{\boldsymbol{\eta}} + i\boldsymbol{\pi})$ for $m = 0$
[Que99][1]	Secs. 1.1, 1.2, 2.2.2, Eq. (2.11)	$(4\pi)^{-\frac{m+n}{2}} f^{A}_{\boldsymbol{\alpha}\overleftarrow{\boldsymbol{\beta}}}(\boldsymbol{\theta} + i\boldsymbol{\pi}, \overleftarrow{\boldsymbol{\eta}})$
[BC15]	Eqs. (2.31, 2.35–2.40, 5.4–5.6)	$f^{[A]}_{m,n}(\boldsymbol{\theta}, \boldsymbol{\eta})$ for $d_{\mathcal{K}} = 1$
[AL17]	Eqs. (2.28c, 4.1)	$\sqrt{m!} C_{\boldsymbol{\alpha\alpha'}} (A\Omega)^{\boldsymbol{\alpha}'}_{m}(\boldsymbol{\theta})$ for $n = 0$

Appendix C

Stress-energy tensor

C.1 Stress-energy tensors for the free scalar field

In this section, we compute the canonical (Noetherian) and metrical (gravitational or Hilbert) versions of the stress-energy tensor for the free scalar field on Minkowski space, in classical and quantum field theory. Most of the expressions presented here are well known[1] and apply to arbitrary spacetime dimensions but for simplicity we restrict the presentation to 1+1d. What is not so well known is that there is a one-parameter family of valid metrical stress-energy tensors on Minkoski space (one of which agrees with the canonical one). The results on one-particle QEIs presented in Chapter 7 constrain the one-parameter family: For too large parameter the stress-energy tensor violates one-particle QEIs and is therefore not a physically reasonable stress-energy tensor. This was noticed already before in [BC16, Sec. 8] whereas the contact with the metric version of the stress-energy tensor was not presented therein.

To start with, consider a classical scalar free field ϕ of mass m on Minkowski space $\mathbb{M}$. Its Lagrangian is given as

$$\mathcal{L} = \tfrac{1}{2}\left(g_{\mu\nu}\partial^\mu\phi\partial^\nu\phi - m^2\phi^2\right), \tag{C.1}$$

where $g = \mathrm{diag}(+1, -1)$ denotes the Minkowski metric, and the corresponding equation of motion reads $\Box\phi + m^2\phi = 0$, where $\Box = g_{\mu\nu}\partial^\mu\partial^\nu$. The canonical stress-energy tensor is then given by

$$T^{\mu\nu}_{\text{can}} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial^\nu\phi - g^{\mu\nu}\mathcal{L} \tag{C.2}$$

which evaluates to

$$T^{\mu\nu}_{\text{can}} = \partial^\mu\phi\partial^\nu\phi - \tfrac{1}{2}g^{\mu\nu}\left(\partial_\rho\phi\partial^\rho\phi - m^2\phi^2\right). \tag{C.3}$$

On the other hand, the metrical stress-energy tensor is given by

$$T^{\mu\nu}_{\text{metr}} = \frac{-2}{\sqrt{-\det h}}\frac{\delta\mathcal{S}_{\text{mat}}}{\delta h^{\mu\nu}} \restriction_{h^{\mu\nu}=g^{\mu\nu}}, \tag{C.4}$$

[1] Cf., e.g., [Wal84, Secs. 4.2, 4.3, Appendix E.1] for a classical field theory treatment including a brief discussion of the canonical and metrical stress-energy tensor. An expression for the metrical stress-energy tensor with arbitrary curvature coupling is for instance given in [Few12, Sec. 2.1]. Note here that the references work in 1+3d and with opposite sign convention for the metric. Note also that we treat just flat spacetime at the end so transferring the expressions from the reference all curvature terms are set to zero. The quantization of a free field on Minkowski space is also standard and can be found in any standard book on QFT.

where $h^{\mu\nu}$ denotes a family of metrices perturbing the Minkowski metric $g^{\mu\nu}$ and the action for the matter part of the system is given by

$$\mathcal{S}_{\text{mat}} = \tfrac{1}{2}\int \left(h_{\mu\nu}\nabla^\mu\phi\nabla^\nu\phi - m^2\phi^2 - \xi R\phi\right)\text{vol}_h, \tag{C.5}$$

where ∇ and vol_h denote the covariant derivative and the volume form with respect to h and where R denotes the scalar curvature and ξ the curvature coupling. From the perspective of Minkowski space to which we restrict a posteriori and where $R = 0$, we can treat ξ as a free parameter. The metrical stress-energy tensor then evaluates to

$$T^{\mu\nu}_{\text{metr}} = T^{\mu\nu}_{\text{can}} + \xi\left(g^{\mu\nu}\Box - \partial^\mu\partial^\nu\right)\phi^2 \tag{C.6}$$

Note that all choices of ξ are physically equivalent: This is because the 00- and 01-component of the terms proportional to ξ in (C.6) are spatial derivatives[2]. As a result, provided that the field ϕ decays suitable fast towards spatial infinity, the integral of these terms over the whole space vanishes and the total energy-momentum operator

$$P^\mu = \int ds\, T^{\mu 0}_{\text{metr}}(0, s) \tag{C.7}$$

does not depend on the choice of ξ.

The corresponding quantum stress-energy tensors can be obtained by quantization. The classical field ϕ is then replaced by the analogous quantum field

$$\phi(x) = \mathcal{N}\int \left(a(\lambda)e^{-ip(\lambda).x} + a^\dagger(\lambda)e^{ip(\lambda).x}\right)d\lambda, \tag{C.8}$$

where $\mathcal{N}$ is a normalization constant which is later put to $\sqrt{4\pi}^{-1}$ and $\phi(x)$ is to be interpreted as an operator-valued distribution with $a^\sharp$ denoting the bosonic creation- and annihilation-operators (cf. Rem. 2.4.3 for $d_{\mathcal{K}} = 1$) satisfying

$$[a(\lambda), a(\lambda')] = [a^\dagger(\lambda), a^\dagger(\lambda')] = 0, \quad [a(\lambda), a^\dagger(\lambda')] = \delta(\lambda - \lambda'). \tag{C.9}$$

Such a field solves $\Box\phi + m^2\phi = 0$ (in the distributional sense) and is normalized such that

$$\langle\theta|\phi(x)|\Omega\rangle = \langle\Omega|a(\theta)\phi(x)|\Omega\rangle = \langle\Omega|\Omega\rangle\,\mathcal{N}e^{ip(\theta).x} = \mathcal{N}e^{ip(\theta).x}, \tag{C.10}$$

where $|\Omega\rangle$ denotes the vacuum and $|\theta\rangle = a^\dagger(\theta)\,|\Omega\rangle$ an (improper) rapidity eigen state (as introduced in Sec. 2.4 with $\mathcal{K} = \mathbb{C}$ and $S = 1$). The quantum stress tensor is obtained by normal ordering (also known as Wick ordering) of the classical expressions, i.e., all appearances of $a^\dagger$ are relocated to the left of a. Indicating the quantized stress-energy tensors by putting hats, we have

$$\hat{T}^{\mu\nu}_{\text{can}} = {:}T^{\mu\nu}_{\text{can}}{:}, \quad \text{and} \quad \hat{T}^{\mu\nu}_{\text{metr}} = {:}T^{\mu\nu}_{\text{metr}}{:} = \hat{T}^{\mu\nu}_{\text{can}} + \xi\left(g^{\mu\nu}\Box - \partial^\mu\partial^\nu\right){:}\phi^2{:}. \tag{C.11}$$

[2]They evaluate to $-\partial^1\partial^1\phi^2$ and $-\partial^0\partial^1\phi^2$, respectively.

For comparison with the general form of the stress-energy tensor at one-particle level, we compute one-particle expectation values. Recall, that we can identify the stress-energy tensor at one-particle level with its two-particle form factor (Eq. (5.9)) and specifically, $\langle\theta|T^{\mu\nu}(x)|\eta\rangle$ with $F_2^{\mu\nu}(\theta,\eta+i\pi;x)$ as[3] for all $\varphi,\chi\in L^2(\mathbb{R})$ and $x\in\mathbb{M}$

$$\langle\varphi,T^{\mu\nu}(x)\chi\rangle=\int d\theta d\eta\,\overline{\varphi(\theta)}\,\langle\theta|T^{\mu\nu}(x)|\eta\rangle\,\chi(\eta)=\int d\theta d\eta\,\overline{\varphi(\theta)}F_2^{\mu\nu}(\theta,\eta+i\pi;x)\chi(\eta). \tag{C.12}$$

The ingoing assumption $\langle\Omega|\hat{T}^{\mu\nu}(x)|\Omega\rangle=0$ is clearly satisfied[4].

To begin with, we compute the one-particle expectation values of $:\phi^2:$, $:\partial^\mu\phi\partial^\nu\phi:$, and $\partial^\mu\partial^\nu:\phi^2:$. By (C.8) and the normal ordering prescription we have that

$$\begin{aligned}:\phi^2:(x)&=\mathcal{N}^2\int d\lambda d\lambda'\Big(a(\lambda)a(\lambda')e^{-i(p(\lambda)+p(\lambda')).x}+a^\dagger(\lambda)a(\lambda')e^{i(p(\lambda)-p(\lambda')).x}\\&\qquad+a^\dagger(\lambda')a(\lambda)e^{i(-p(\lambda)+p(\lambda')).x}+a^\dagger(\lambda)a^\dagger(\lambda')e^{i(p(\lambda)+p(\lambda')).x}\Big)\\&=\mathcal{N}^2\int d\lambda d\lambda'\Big(a(\lambda)a(\lambda')e^{-i(p(\lambda)+p(\lambda')).x}+2a^\dagger(\lambda)a(\lambda')e^{i(p(\lambda)-p(\lambda')).x}\\&\qquad+a^\dagger(\lambda)a^\dagger(\lambda')e^{i(p(\lambda)+p(\lambda')).x}\Big).\end{aligned}\tag{C.13}$$

In one-particle expectation values, the terms with $a(\lambda)a(\lambda')$ and $a^\dagger(\lambda)a^\dagger(\lambda')$ drop out: Using the commutation relations (C.9), we have

$$a(\lambda)a(\lambda')\,|\theta\rangle=a(\lambda)a(\lambda')a^\dagger(\theta)\,|\Omega\rangle=\delta(\lambda'-\theta)a(\lambda)\,|\Omega\rangle=0$$

and analogously for the other term by using $a(\lambda)^*=a^\dagger(\lambda)$. Also due to (C.9), we have

$$\langle\theta|a^\dagger(\lambda)a(\lambda')|\eta\rangle=\delta(\theta-\lambda)\delta(\eta-\lambda') \tag{C.14}$$

so that

$$\begin{aligned}\langle\theta|:\phi^2:(x)|\eta\rangle&=2\mathcal{N}^2\int d\lambda d\lambda'\,e^{i(p(\lambda)-p(\lambda')).x}\,\langle\theta|a^\dagger(\lambda)a(\lambda')|\eta\rangle\\&=2\mathcal{N}^2\int d\lambda d\lambda'\,e^{i(p(\lambda)-p(\lambda')).x}\delta(\theta-\lambda)\delta(\eta-\lambda')\\&=2\mathcal{N}^2e^{i(p(\theta)-p(\eta)).x}.\end{aligned}\tag{C.15}$$

Analogously, we find that

$$\begin{aligned}&\langle\theta|:\partial^\mu\phi\partial^\nu\phi:(x)|\eta\rangle\\&=\mathcal{N}^2\int d\lambda d\lambda'\,(ip^\mu(\lambda)(-i)p^\nu(\lambda')+(-i)p^\nu(\lambda)ip^\mu(\lambda'))e^{i(p(\lambda)-p(\lambda')).x}\,\langle\theta|a^\dagger(\lambda)a(\lambda')|\eta\rangle\\&=\mathcal{N}^2(p^\mu(\theta)p^\nu(\eta)+p^\nu(\theta)p^\mu(\eta))e^{i(p(\theta)-p(\eta)).x}\end{aligned}\tag{C.16}$$

[3]Note that here $\mathcal{K}=\mathbb{C}$.

[4]It is satisfied by all normally ordered expresssions involving a creation or annihilation operator due to $a(\lambda)\,|\Omega\rangle=0$ and $\langle\Omega|\,a^\dagger(\lambda)=0$.

and

$$\begin{aligned}\langle\theta|\partial^\mu\partial^\nu{:}\phi^2{:}(x)|\eta\rangle &= \mathcal{N}^2\partial^\mu\partial^\nu\,\langle\theta|{:}\phi^2{:}(x)|\eta\rangle \\ &= -2\mathcal{N}^2(p(\theta)-p(\eta))^\mu(p(\theta)-p(\eta))^\nu e^{i(p(\theta)-p(\eta)).x}.\end{aligned} \tag{C.17}$$

Let $p = p(\theta)$ and $q = p(\eta)$, then combine (C.15)–(C.17) to find

$$\langle\theta|\hat{T}^{\mu\nu}_{\text{can}}(x)|\eta\rangle = e^{i(p-q).x}\mathcal{N}^2\left(p^\mu q^\nu + p^\nu q^\mu - g^{\mu\nu}(p.q-m^2)\right) \tag{C.18}$$

and

$$\langle\theta|\hat{T}^{\mu\nu}_{\text{metr}}(x)|\eta\rangle = \hat{T}^{\mu\nu}_{\text{can}}(x) + 2\xi e^{i(p-q).x}\mathcal{N}^2\left(g^{\mu\nu}(p-q).(p-q)-(p-q)^\mu(p-q)^\nu\right). \tag{C.19}$$

It is then easy to compute, using the standard trigonometric relations,

$$\operatorname{ch}(\theta+\eta) = \operatorname{ch}\theta\operatorname{ch}\eta + \operatorname{sh}\theta\operatorname{sh}\eta, \quad \text{and} \quad \operatorname{sh}(\theta+\eta) = \operatorname{sh}\theta\operatorname{ch}\eta + \operatorname{ch}\theta\operatorname{sh}\eta,$$

that

$$p^\mu q^\nu + p^\nu q^\mu - g^{\mu\nu}(p.q-m^2) = 2m^2\begin{pmatrix}\operatorname{ch}^2\frac{\theta+\eta}{2} & \frac{1}{2}\operatorname{sh}(\theta+\eta)\\ \frac{1}{2}\operatorname{sh}(\theta+\eta) & \operatorname{sh}^2\frac{\theta+\eta}{2}\end{pmatrix} = 4\pi G^{\mu\nu}_{\text{free}}(\tfrac{\theta+\eta}{2}), \tag{C.20}$$

with $G^{\mu\nu}_{\text{free}}$ as given in (5.42) for $M = m\mathbb{1}$. Analogously, using also

$$\operatorname{ch}\theta-\operatorname{ch}\eta = 2\operatorname{sh}\tfrac{\theta-\eta}{2}\operatorname{sh}\tfrac{\theta+\eta}{2}, \quad \operatorname{sh}\theta-\operatorname{sh}\eta = 2\operatorname{sh}\tfrac{\theta-\eta}{2}\operatorname{ch}\tfrac{\theta-\eta}{2}, \quad 2\operatorname{sh}^2\tfrac{\theta-\eta}{2} = \operatorname{ch}(\theta-\eta)-1,$$

one finds that

$$g^{\mu\nu}(p-q).(p-q)-(p-q)^\mu(p-q)^\nu = -2m^2(\operatorname{ch}(\theta-\eta)-1)\begin{pmatrix}\operatorname{ch}^2\frac{\theta+\eta}{2} & \frac{1}{2}\operatorname{sh}(\theta+\eta)\\ \frac{1}{2}\operatorname{sh}(\theta+\eta) & \operatorname{sh}^2\frac{\theta+\eta}{2}\end{pmatrix}. \tag{C.21}$$

With $F^{\mu\nu}_{2,\text{can/metr}}(\theta,\eta+i\pi;x) = \langle\theta|\hat{T}^{\mu\nu}_{\text{can/metr}}(x)|\eta\rangle$, $P^\mu(\theta,\eta+i\pi) = p^\mu(\theta)-p^\mu(\eta) = p-q$ and choosing $\mathcal{N}^2 = (4\pi)^{-1}$, we then find

$$F^{\mu\nu}_{2,\text{can}}(\theta,\eta+i\pi;x) = e^{iP(\theta,\eta+i\pi).x}G^{\mu\nu}_{\text{free}}(\tfrac{\theta+\eta}{2}), \tag{C.22}$$

$$F^{\mu\nu}_{2,\text{metr}}(\theta,\eta+i\pi;x) = F^{\mu\nu}_{2,\text{can}}(\theta,\eta+i\pi;x)(1-2\xi(\operatorname{ch}(\theta-\eta)-1)). \tag{C.23}$$

This agrees with the general form of the stress-energy tensor at one-particle level (Thm. 5.3.1, Cor. 5.3.3) for the specific case $S = 1$, $\mathcal{K} = \mathbb{C}$, $Jz = \bar{z}$, $M = m1_{\mathbb{C}}$, $\mathcal{G} = \mathbb{Z}_2$, and $V(\pm1) = \pm1_{\mathbb{C}}$. In particular, we find that $F(\theta+i\pi) = 1-2\xi(\operatorname{ch}\theta-1)$. Note that in view of Theorem 7.1.1 a QEI of the form (7.8) holds if $|\xi| < \frac{1}{4}$ and cannot hold if $|\xi| > \frac{1}{4}$ as outlined before in the discussion [BC16, Sec. 8].

C.2 A weaker notion for the density property

In this section, we first briefly introduce a weaker notion of the density property (t10) which was used in [Ver00; MPV22] and then specify how it relates to (t10).

As a preliminary, for arbitrary open finite regions $\mathcal{O} \subset \mathbb{M}$ we introduce $\mathcal{A}_\infty(\mathcal{O})$, a subalgebra of $\mathcal{A}(\mathcal{O})$ which is obtained by applying

$$A \mapsto u_f(A) = \int dt\, f(t) U(te_0, 0) A U(te_0, 0)^{-1}, \quad f \in \mathcal{S}(\mathbb{R})$$

such that $u_f(A)$ is still localized in $\mathcal{O}$. In that regard, we note that $\mathcal{A}_\infty(\mathcal{O})\Omega$ is dense in $C^\infty(P^0)$. The weaker density property then reads: For arbitrary $\mathcal{O}$ as above, $A \in \mathcal{A}_\infty(\mathcal{O})$ and $f \in \mathcal{D}(\mathbb{M})$ such that $f(t,x) = f_0(t)\underline{f}(x)$ with $\int f_0(t)dt = 1$ and $\underline{f} \restriction_{I_\mathcal{O}} = 1$, where $I_\mathcal{O} \subset \mathbb{R}$ is sufficiently big to contain all spatial slices of $\mathcal{O}''$ as subsets of $\mathbb{R}$, then

$$[P^\mu - T^{0\mu}(f), A]\Omega = 0 \tag{C.24}$$

Let $\mathcal{C}$ denote the class of vector states for which (t10) holds. Relying mostly on the discussion in [MPV22, Eqs. (3.12–3.16)] we can show that (t10) implies (C.24) if $\mathcal{C}$ is large enough and if we assume locality of $T^{\mu\nu}(f)$ relative to the observables (Rem. 3.1.4) which is stronger than (t2). Due to (t10), or more particular, due to (5.4), we have

$$\langle \varphi, [P^\mu, A]\Omega \rangle = \int \langle \varphi, [T^{\mu 0}(x), A]\Omega \rangle \, dx^1, \tag{C.25}$$

provided $\Omega, A\Omega, \varphi, A^*\varphi \in \mathcal{C}$. On the other hand, for arbitrary $A \in \mathcal{A}_\infty(\mathcal{O})$, $\varphi \in C^\infty(P^0)$, and f chosen as above with $t_0 \in \operatorname{supp} f_0$, it holds that

$$\langle \varphi, [T^{\mu 0}(f), A]\Omega \rangle = \int dtdx\, f_0(t)\underline{f}(x) \,\langle \varphi, [T^{\mu 0}(t,x), A]\Omega \rangle \tag{C.26}$$

$$= \int f_0(t)dt \cdot \int dx\, \underline{f}(x) \,\langle \varphi, [T^{\mu 0}(t_0,x), A]\Omega \rangle \tag{C.27}$$

$$= \int dx \,\langle \varphi, [T^{\mu 0}(t_0,x), A]\Omega \rangle \tag{C.28}$$

which takes the form of the r.h.s. of (C.25). Thus (t10) implies (C.25) provided that $\mathcal{C}$ is sufficiently large. The derivation steps are as follows: Equation (C.26) holds by Remark 5.1. Equation (C.27) follows since $\int dx\, \underline{f}(x) \,\langle \varphi, [T^{\mu 0}(t,x), A]\Omega \rangle$ is invariant in t by the continuity equation (t9) and the divergence theorem. In more detail, differentiating with respect to t reduces to a spatial derivative by the continuity equation. Note here that the commutator vanishes for all $(t,x) \in \mathcal{O}'$ (or equivalently outside $\mathcal{O}''$) as $A \in \mathcal{A}(\mathcal{O}) \subset \mathcal{A}(\mathcal{O}')'$ and by locality of $T^{\mu\nu}(f)$ relative to the observables; as a result, the support of the commutator in x for fixed t is contained in $I_{\mathcal{O},t} := \{x : (t,x) \in \mathcal{O}''\}$ so that boundary terms vanish and the spatial derivative acts on $\underline{f}$ by partial integration. Also, $\underline{f}$ is equal to 1 in that region by assumption ($I_{\mathcal{O},t} \subset I_{\mathcal{O}}$[5]) so that the spatial derivative vanishes and $\int dx\, \underline{f}(x) \langle \varphi, [T^{\mu 0}(t,x), A]\Omega\rangle$ is invariant in t. Thus we may pick $t = t_0$ and take the integration over f_0 out, yielding (C.27). Equation (C.28) follows by assumption, $\int f_0(t)dt = 1$ and $\underline{f} \restriction_{I_{\mathcal{O}}} = 1$.

C.3 Stress-energy tensor at one-particle level generating the boosts

For simplicity, we restrict to a single mass sector $m = m_\alpha = m_\beta$. Using the diagonal-in-mass expression for the stress-energy tensor at one-particle level (Thm. 5.3.1, Cor. 5.3.3) with $F(\theta + i\pi)$ replaced by $F(\theta)$, the r.h.s of (5.53) evaluates to

$$\langle\theta_\alpha| \int sT^{00}(0,s)ds\,|\eta_\beta\rangle = \int s\,\langle\theta_\alpha|T^{00}(0,s)|\eta_\beta\rangle\,ds \tag{C.29}$$

$$= \int s\,e^{i(p_1(\theta)-p_1(\eta))s}ds\;\langle\theta_\alpha|T^{00}(0)|\eta_\beta\rangle \tag{C.30}$$

$$= \int s\,e^{i(p_1(\theta)-p_1(\eta))s}ds\,F^{00}_{2\,\bar\alpha\beta}(\theta,\eta+i\pi) \tag{C.31}$$

$$= \int s\,e^{i(p_1(\theta)-p_1(\eta))s}ds\,\frac{m^2}{2\pi}\operatorname{ch}^2\tfrac{\theta+\eta}{2}F_{\bar\alpha\beta}(\eta-\theta). \tag{C.32}$$

Introducing $\rho = \frac{\theta+\eta}{2}$ and $\tau = \theta - \eta$, we obtain $p_1(\theta) - p_1(\eta) = 2m\operatorname{sh}\frac{\tau}{2}\operatorname{ch}\rho$ which for $\tau, \rho \in \mathbb{R}$ vanishes iff $\tau = 0$. Moreover, $\int se^{iks}ds = -i2\pi\delta'(k)$. Combining these two properties, we obtain

$$\langle\varphi| \int -sT^{00}(0,s)ds\,|\chi\rangle \tag{C.33}$$

$$= \int d\theta d\eta\,\overline{\varphi^\alpha(\theta)}\chi^\beta(\eta)\;\langle\theta_\alpha|\int -sT^{00}(0,s)ds\,|\eta_\beta\rangle \tag{C.34}$$

$$= im^2\int d\rho d\tau\,\overline{\varphi^\alpha(\rho+\tfrac{\tau}{2}})\chi^\beta(\rho-\tfrac{\tau}{2})F_{\bar\alpha\beta}(-\tau)\,\delta'(2m\operatorname{sh}\tfrac{\tau}{2}\operatorname{ch}\rho)\operatorname{ch}^2\rho \tag{C.35}$$

$$= -im^2\int d\rho d\tau\,\tfrac{\partial}{\partial\tau}\Big(\overline{\varphi^\alpha(\rho+\tfrac{\tau}{2})}\chi^\beta(\rho-\tfrac{\tau}{2})F_{\bar\alpha\beta}(-\tau)\Big)\,\delta(\tau)\Big(2m\operatorname{ch}\rho\frac{\partial\operatorname{sh}\frac{\tau}{2}}{\partial\tau}\Big)^{-2}\operatorname{ch}^2\rho \tag{C.36}$$

$$= -im^2\int d\rho d\tau\,\tfrac{\partial}{\partial\tau}\Big(\overline{\varphi^\alpha(\rho+\tfrac{\tau}{2})}\chi^\beta(\rho-\tfrac{\tau}{2})F_{\bar\alpha\beta}(-\tau)\Big)\,\delta(\tau)\Big(m\operatorname{ch}\rho\Big)^{-2}\operatorname{ch}^2\rho \tag{C.37}$$

$$= -i\int d\rho d\tau\,\tfrac{\partial}{\partial\tau}\Big(\overline{\varphi^\alpha(\rho+\tfrac{\tau}{2})}\chi^\beta(\rho-\tfrac{\tau}{2})F_{\bar\alpha\beta}(-\tau)\Big)\,\delta(\tau). \tag{C.38}$$

Assuming $F'_{\bar\alpha\beta}(0) = 0$ (see below), the τ-derivative acts only on φ and χ. After integration in τ, effectively setting $\tau = 0$, one then obtains:

$$= -i\int d\rho\,\tfrac{1}{2}\Big(\overline{\varphi'^\alpha(\rho)}\chi^\beta(\rho) - \overline{\varphi^\alpha(\rho)}\chi'^\beta(\rho)\Big)\,F_{\bar\alpha\beta}(0) \tag{C.39}$$

$$= i\int d\rho\,\overline{\varphi^\alpha(\rho)}\chi'^\beta(\rho)\delta_{\alpha,\beta} \tag{C.40}$$

[5]Note that it is sufficient here that $I_{\mathcal{O},t} \subset I_{\mathcal{O}}$ for $t \in \operatorname{supp} f_0$ so that we could have choosen $I_{\mathcal{O}}$ smaller at the cost of keeping a dependence on f_0. A possible choice would be $I_{\mathcal{O}} = \cup_{t\in\operatorname{supp} f_0} I_{\mathcal{O},t}$.

$$= -i \langle \varphi, \chi' \rangle \,, \tag{C.41}$$

using that $F_{\bar{\alpha}\beta}(0) = \delta_{\alpha,\beta}$ by Theorem 5.3.1(g), partial integration, and the vanishing of the boundary term due to the compact support of φ and χ. The assumption $F'(0) = 0$ is likely to hold since it decomposes (Prop. 5.3.4) into polynomial factors which have $\frac{d}{d\theta} Q(\operatorname{ch}\theta) \restriction_{\theta=0} = \operatorname{sh}\theta \, Q'(\operatorname{ch}\theta) \restriction_{\theta=0} = 0$ and into minimal solutions which satisfy $f'_{\min}(i\pi) = 0$ in case that their integral representation exists (Sec. 4.2).

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