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Four-qubit CHSH game

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Abstract
In this paper, the CHSH quantum game is extended to four players. This is achieved by exploring all possible 4-variable Boolean functions to identify those that yield a game scenario with a quantum advantage using a specific entangled state. Notably, two new four-player quantum games are presented. In one game, the optimal quantum strategy is achieved when players share a $|GHZ_4\rangle = \frac{1}{\sqrt{2}}(|0000\rangle + |1111\rangle)$ state, breaking the traditional 10% gain observed in 2- and 3-qubit CHSH games and achieving a 22.5% gap. In the other game, players gain a greater advantage using a $|W_4\rangle = \frac{1}{2}(|0001\rangle + |0010\rangle + |0100\rangle + |1000\rangle)$ state as their quantum resource. Quantum games with other four-qubit entangled states are also explored. To demonstrate the results, these game scenarios are implemented on an online quantum computer, and the advantage of the respective quantum resource for each game is experimentally verified.

Keywords CHSH · Quantum games · Entanglement · IBM quantum platform

1 Introduction

Quantum games form a framework in which players can take advantage of non-classical resources—such as entanglement or quantum contextuality—to design

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strategies that surpass what is achievable with purely classical means. In this setting, some of the counterintuitive aspects of quantum theory can be highlighted through simple game scenarios [1–3]. It is known that quantum strategies allows one to overperform on the classical Prisoner's Dilemma [4], and experimental demonstrations of this idea have been realized [5, 6]. Among others quantum games, the CHSH game [7], named after the CHSH inequality of Clauser, Horne, Shimony, and Holt [8], will be the subject of this paper.

The two-qubit CHSH game is defined as follows. A referee, denoted by Charlie (C), interacts with two players, Alice (A) and Bob (B), who are not allowed to communicate during the game. Charlie selects binary inputs x and y (with $x, y \in \mathbb{B} = \{0, 1\}$) and distributes them to Alice and Bob, respectively. Each player must provide a binary output: Alice answers with a and Bob with b. Their answers are considered successful if they satisfy the winning condition

$$x \cdot y = a \oplus b \; . \tag{1}$$

While the players may agree on a strategy beforehand, under classical resources and no communication,[1] their maximum winning probability is bounded by $p_{LR} \leq 0.75$. If, instead, they share the entangled EPR state $|EPR\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$, with one qubit given to each, they can adopt the following measurement strategy:

- For $x = 0$, Alice measures in the Z basis, reporting 0 if the outcome is $+1$ and 1 otherwise.
- For $x = 1$, Alice measures in the X basis and uses the same output rule.
- For $y = 0$, Bob measures in the $\frac{Z + X}{\sqrt{2}}$ basis, reporting 0 for $+1$ and 1 otherwise.
- For $y = 1$, Bob measures in the $\frac{Z - X}{\sqrt{2}}$ basis, with the same assignment.

This strategy achieves a success probability of $\cos(\pi/8)^2 \approx 0.85$, which is about 10% higher than the best classical value.

In [9], this concept was generalized to include three players and a referee by analyzing Boolean conditions of the form

$$f(a, b, c) = g(x, y, z) \bmod 2 \; . \tag{2}$$

A systematic exploration of all such cases revealed that distinct entangled states, notably the three-qubit GHZ and W states, $|GHZ_3\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$ and $|W_3\rangle = \frac{1}{\sqrt{3}}(|100\rangle + |010\rangle + |001\rangle)$, provide advantages for different families of games. The study also showed how this framework reduces consistently to the $n = 2$ CHSH scenario.

[1] These assumptions correspond to local realism (LR). By contrast, strategies exploiting quantum mechanics are denoted QM.

Moving to $n = 4$ players introduces two main difficulties. First, the number of Boolean relations to be tested grows rapidly (see Sect. 2). Second, in contrast with the tripartite case, the four-qubit setting admits infinitely many inequivalent classes of entangled states [10]. These challenges required us to refine the approach of [9], which in turn led to new insights. In particular, we identified a four-player game for which quantum resources yield more than a 22% advantage over classical strategies—a large increase compared with the 10% improvement available in the CHSH and three-qubit cases. This prediction was further validated by running experiments on an online quantum computer.

The remainder of the paper is structured as follows. Section 2 revisits the generalization procedure, which involves Boolean equations of the type $f(x_1, \ldots, x_n) = g(a_1, \ldots, a_n)$ over $2n$ binary variables: $(x_1, \ldots, x_n) \in \mathbb{B}^n$ denote the referee's questions, and $(a_1, \ldots, a_n) \in \mathbb{B}^n$ the players' answers. Given an entangled state shared among the players, one can then test whether a quantum advantage exists by exploring all possible measurement strategies. For the four-qubit case, we explain the selection methods used to reduce the computational complexity.

In Sect. 3, we focus on a four-player game in which the shared state $|GHZ_4\rangle = \frac{1}{\sqrt{2}}(|0000\rangle + |1111\rangle)$—the four-qubit extension of $|GHZ_3\rangle$—offers a significant advantage. The motivation for this choice comes from our previous study of the role of $|GHZ_3\rangle$ in the tripartite analysis of [9]. We also examine strategies based on the $|W_4\rangle$ state, the analogue of $|W_3\rangle$, and identify cases where $|W_4\rangle$ not only outperforms classical resources but also provides a better outcome than $|GHZ_4\rangle$. Additional classes of states, including maximally entangled four-qubit states and families from the classification of [10], are likewise considered.

Section 4 then describes the implementation of two representative games on IBM's Quantum Platform: one in which $|GHZ_4\rangle$ enables a quantum advantage, and another where $|W_4\rangle$ proves superior. In both cases, the experimental results exceed the classical bound and clearly distinguish between the two entangled resources.

Finally, Sect. 5 provides concluding remarks and outlines possible extensions of this work.

2 Generalization of the CHSH game

In this subsection, the CHSH game is extended to an n-party framework. For a fixed number of players n, we enumerate all possible binary equations and investigate the winning probabilities obtained under classical deterministic strategies compared with those arising from quantum strategies using shared entangled n-qubit states. We also outline the procedure employed to determine the optimal classical and quantum strategies for a generalized CHSH game.

2.1 Definition of the game

Following [9], the n participants are denoted $A_1, A_2, \ldots, A_n$. Each player $i \in [\![1, n]\!]$ receives a binary question $x_i \in \mathbb{B}$ from the referee and must return an answer $a_i \in \mathbb{B}$. The team wins the game whenever the following relation holds:

$$f(x_1, x_2, \ldots, x_n) = g(a_1, a_2, \ldots, a_n) \ . \tag{3}$$

Here, $f, g : \mathbb{B}^n \to \mathbb{B}$ are Boolean functions, and each input x_i and output a_i must occur at least once in f and g, respectively, so that all players actively contribute to the game. Observe that Eq. (2) corresponds to Eq. (3) specialized to the case $n = 3$.

A strategy implemented by the players is denoted h and defined as

$$h: \begin{cases} \mathbb{B}^n \to \mathbb{B}^n \\ (x_1, x_2, \ldots, x_n) \mapsto (a_1, a_2, \ldots, a_n) \end{cases}, \tag{4}$$

which prescribes the answers given depending on the questions asked.

Thus, the task is to identify a strategy h that maximizes the probability that the condition

$$f(x_1, x_2, \ldots, x_n) = g(h(x_1, x_2, \ldots, x_n)) \tag{5}$$

is satisfied.

2.2 Best classical strategy

In the classical deterministic case, each player predetermines their output for the two possible questions. Hence, the global strategy h can be factorized as

$$h(x_1, x_2, \ldots, x_n) = \big(h_1(x_1), h_2(x_2), \ldots, h_n(x_n)\big) \tag{6}$$

where h_i is the local strategy for player i. To fix h_i, one only needs to choose the values $h_i(0)$ and $h_i(1)$, requiring $2n$ bits to encode one global strategy. Therefore, the number of distinct classical strategies equals 2^{2n}. To determine the best-performing one, all strategies are enumerated, their success probabilities are computed, and the optimal candidates are identified.

2.3 Best quantum strategy

In a quantum setting, the players share an entangled n-qubit state $|\psi_n\rangle$. Entanglement is essential here; without it, the strategy is effectively classical with randomness. Each player can act locally on their qubit with a unitary operator depending on their received input. Specifically, if player i receives input x_i, they apply a unitary gate U_{i,x_i}. After all players apply their operations, they measure their qubits and use the measurement outcomes as their responses. The overall process is depicted in Fig. 1.

$$(x_1, x_2, \ldots, x_n) \xrightarrow{\text{apply strategy}} U_{1,x_1} \otimes \cdots \otimes U_{n,x_n} |\psi_n\rangle \xrightarrow{\text{measure}} (a_1, a_2, \ldots, a_n)$$

Fig. 1 Illustration of how an n-party quantum strategy is executed

As mentioned in [9], any single-qubit unitary U_{i,x_i} can be written in terms of three parameters $\theta_{i,x_i}, \phi_{i,x_i}, \lambda_{i,x_i}$ as

$$U_{i,x_i}(\theta_{i,x_i}, \phi_{i,x_i}, \lambda_{i,x_i}) = \begin{pmatrix} \cos\left(\frac{\theta_{i,x_i}}{2}\right) & -e^{j\lambda_{i,x_i}} \sin\left(\frac{\theta_{i,x_i}}{2}\right) \\ e^{j\phi_{i,x_i}} \sin\left(\frac{\theta_{i,x_i}}{2}\right) & e^{j(\phi_{i,x_i}+\lambda_{i,x_i})} \cos\left(\frac{\theta_{i,x_i}}{2}\right) \end{pmatrix} \tag{7}$$

where j denotes the imaginary unit and the ranges are given by $\theta_{i,x_i} \in [0, \pi]$, $\phi_{i,x_i}, \lambda_{i,x_i} \in [0, 2\pi]$.

2.4 Reduction of the problem

Since there are $2^{16} = 65,536$ Boolean functions from $\mathbb{B}^4$ to $\mathbb{B}$, an exhaustive comparison would involve checking $65,536^2$ possible winning conditions of the form $f(w, x, y, z) = g(a, b, c, d)$. Optimizing the parameters $\theta_{i,x_i}, \phi_{i,x_i}, \lambda_{i,x_i}$ for just one equation requires at least two seconds of computation on a typical laptop. To reduce computational load, algorithmic optimizations, parallel computation, and function selection heuristics are employed. A first simplification is to fix a choice of g, leaving only 65, 536 possible f functions.

2.4.1 Reduction of functions

The initial reduction halves the number of functions considered. Indeed, if one compares the games $\overline{f(w, x, y, z)} = g(a, b, c, d)$ and $f(w, x, y, z) = g(a, b, c, d)$, they yield the same optimal performance whenever g allows a player to flip the result by inverting their output. Note that $\overline{f(w, x, y, z)} = f(w, x, y, z \oplus 1)$. For instance, if $g(a, b, c, d) = a \oplus b \oplus c \oplus d$, then flipping the response of a single player transforms a winning strategy for f into a winning strategy for $\overline{f}$. Thus, restricting to this class of g reduces the number of relevant f to 32, 768.

A second reduction removes function variants that differ only by negating some of the input variables. For n variables, there are 2^n such equivalent versions. After accounting for this, the number of distinct functions for $n = 4$ decreases to 4, 014. A final filtering step requires that all inputs w, x, y, z and outputs a, b, c, d appear explicitly in the equation, leaving 3, 907 non-trivial functions to analyze.
After these reductions, the total time needed to evaluate all 3, 907 cases for a single fixed g is approximately 40 min.

2.4.2 Parallel implementation

To take advantage of multiple CPU cores, the optimization routines were parallelized using the `joblib` package. This allowed concurrent evaluations of distinct functions, yielding a substantial reduction in runtime.

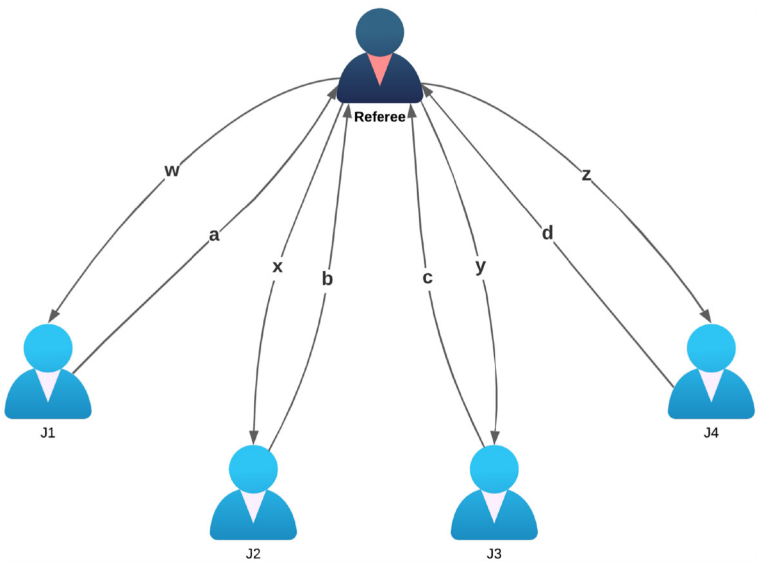

Fig. 2 CHSH game setup with 4 players: The referee sends a question w to $J1$, x to $J2$, y to $J3$, and z to $J4$. The players win the game if and only if their answers (a, b, c, d) satisfy the binary equation $f(w, x, y, z) = g(a, b, c, d)$ that defines the game

3 The four-qubit CHSH games

In this section, new and non-equivalent game versions of the CHSH games for four players are investigated. We first consider the games based on quantum strategies using as four-qubit entangled states either a $|GHZ_4\rangle$ or a $|W_4\rangle$ state. Recall that:

- $|GHZ_4\rangle = \frac{1}{\sqrt{2}}(|0000\rangle + |1111\rangle)$,
- $|W_4\rangle = \frac{1}{2}(|1000\rangle + |0100\rangle + |0010\rangle + |0001\rangle)$.

Then, we also discuss other examples of four-qubit genuine entangled states known as critical states and finally one considers the full four-qubit classification as described in [10].

3.1 Game's setup

The four participants are denoted $J1$, $J2$, $J3$, and $J4$. As discussed in the general framework in Sect. 2, they all take part in the same game, share a common four-qubit state, and aim to fulfill a single defining equation.

The referee assigns binary inputs w, x, y, and z to $J1$, $J2$, $J3$, and $J4$, respectively. Each player then applies their chosen strategy and produces outputs a, b, c, and d accordingly (see Fig. 2). The success is determined by whether these outputs satisfy

Table 1 Quantum strategies for four players game given by Eq. (9) where the maximum advantage is achieved with a $|GHZ_4\rangle$ state

Equation	$w = x = y = z = 0$	$w = x = y = z = 1$
$(xyz) + (xy\overline{w}) + (xz\overline{w}) + (yz\overline{w}) + (w\overline{x}\,\overline{y}\,\overline{z}) = a \oplus b \oplus c \oplus d$	$U_{1,0}(\frac{3\pi}{2}, \frac{13\pi}{15}, \frac{7\pi}{5})$	$U_{1,0}(\frac{\pi}{2}, \frac{11\pi}{7}, \frac{21\pi}{11})$
	$U_{2,0}(\frac{3\pi}{2}, \frac{6\pi}{5}, \frac{19\pi}{12})$	$U_{2,1}(\frac{3\pi}{2}, \frac{19\pi}{15}, \frac{\pi}{12})$
	$U_{3,0}(\frac{\pi}{2}, \frac{27\pi}{14}, \frac{8\pi}{7})$	$U_{3,1}(\frac{\pi}{2}, \frac{23\pi}{12}, \frac{23\pi}{14})$
	$U_{4,0}(\frac{3\pi}{2}, \frac{11\pi}{9}, \frac{8\pi}{13})$	$U_{4,1}(\frac{\pi}{2}, \frac{3\pi}{14}, \frac{\pi}{9})$

Angles are given in radians. With the quantum strategy, the players win with a probability ≈ 0.8535 compared to 0.6225 with classical strategies. This demonstrates a significant advantage for quantum strategies, with a gap of 22.5% between the quantum and classical success probabilities

the defining equation of the game, which is expressed as follows:

$$f(w, x, y, z) = g(a, b, c, d) \tag{8}$$

As recalled in Sect. 2.4.1, it is necessary to filter the useful equations from all the possible ones. We listed the 3907 possible functions from $\mathbb{B}^4$ to $\mathbb{B}$, after removing the non-relevant ones.

3.2 Gain with $|GHZ_4\rangle$

Concerning the choice of the g function, we first generalized the function $g(a, b) = a \oplus b$ used in the 2-qubit CHSH game (see Eq. (1)). For four players, it becomes:

$$g(a, b, c, d) = a \oplus b \oplus c \oplus d$$

The first interesting fact is that for four-qubit CHSH games it is possible to have a gap between quantum and classical strategies higher than 10% (the limit observed in the two and three-qubit CHSH game [9]), but indeed a gap of 22.5% is achieved for the game defined by:

$$(xyz) + (xy\overline{w}) + (xz\overline{w}) + (yz\overline{w}) + (w\overline{xyz}) = a \oplus b \oplus c \oplus d \tag{9}$$

This good performance can be attributed to the symmetric nature of the function $g(a, b, c, d) = a \oplus b \oplus c \oplus d$ and the strong correlations and entanglement properties inherent in the $|GHZ_4\rangle$ state. This symmetry aligns well with the requirements of the equation, ensuring that the outcomes of the quantum measurements match the desired function. An explicit quantum strategy is given in Table 1. Note that the quantum gain is of 85.35% which is similar to the quantum gains obtained for the analogue two and three-qubit CHSH games with the best strategy. But the gap between quantum and classical is much larger due to the impossibility to win the game classically with a probability higher than 0.6225 in the four players game. The larger number of variables (players) makes it more difficult to find a deterministic strategy.

Table 2 Quantum strategies for four players game where the maximum advantage is achieved with a $|W_4\rangle$ state

Equation	$w = x = y = z = 0$	$w = x = y = z = 1$
$(wx) + (wy) + (wz) + (xy) + (xz) + (yz) = (\overline{a}bcd) + (a\overline{b}cd) + (ab\overline{c}d) + (abc\overline{d})$	$U_{1,0}(\frac{3\pi}{4}, \frac{\pi}{2}, \frac{\pi}{15})$	$U_{1,1}(\frac{14\pi}{15}, \frac{14\pi}{11}, \frac{16\pi}{15})$
	$U_{2,0}(\frac{3\pi}{4}, \frac{13\pi}{12}, \frac{31\pi}{15})$	$U_{2,1}(\frac{14\pi}{15}, \frac{28\pi}{15}, \frac{16\pi}{15})$
	$U_{3,0}(\frac{3\pi}{4}, \frac{4\pi}{5}, \frac{\pi}{15})$	$U_{3,1}(\frac{14\pi}{15}, \frac{23\pi}{13}, \frac{16\pi}{15})$
	$U_{4,0}(\frac{3\pi}{4}, \frac{\pi}{4}, \frac{\pi}{15})$	$U_{4,1}(\frac{14\pi}{15}, \frac{19\pi}{9}, \frac{16\pi}{15})$

Angles are given in radian. With the quantum strategy, the players win with a probability ≈ 0.7499 compared to 0.6875 with classical strategies. This demonstrates a significant advantage for quantum strategies, with a gap of 6.25% between the quantum and classical success probabilities

3.3 Gain with $|W_4\rangle$

In order to find the equation where $|W_4\rangle$ has a better performance than $|GHZ_4\rangle$, we need to "break" the symmetries of Eq. (9). This is done again by generalizing to the four-qubit case an equation of [9]. It is thus possible to find a four-qubit game where using a $|W_4\rangle$ state gives a better advantage compared to a $|GHZ_4\rangle$ state :

$$(wx)+(wy)+(wz)+(xy)+(xz)+(yz) = (\overline{a}bcd)+(a\overline{b}cd)+(ab\overline{c}d)+(abc\overline{d}). \quad (10)$$

The best classical strategy allows one to win this new game with probability 0.6875, while the gain increases to probability approximately 0.7499 when using a $|W_4\rangle$ state for the quantum strategy (see Table 2). In comparison, with a $|GHZ_4\rangle$ state, the success probability only reaches 0.5727.

3.4 Other four-qubit entangled states: the critical states

In the four-qubit Hilbert space, there are an infinite number of different classes of entanglement (see Sect. 3.5) and the question of deciding what a maximally four-qubit entangled state is, is not fully answered [11]. In this subsection, one tests some alternative of four genuine entangled states known as critical states [12]. Critical states maximize some entanglement monotones defined by algebraic invariants. Here are the explicit forms of three four-qubit critical states:

- $|MP\rangle = \frac{1}{2}(|0000\rangle + |0011\rangle + |1100\rangle + |1111\rangle)$, known as the Mermin–Peres state.
- $|C_1\rangle = \frac{1}{2}(|0000\rangle + |0011\rangle + |1100\rangle - |1111\rangle)$, the cluster state.
- $|L\rangle = \frac{1}{4}[(1+\omega)(|0000\rangle + |1111\rangle) + (1-\omega)(|0011\rangle + |1100\rangle) + \omega^2(|0110\rangle + |1001\rangle + |1010\rangle + |0101\rangle)]$.

The $|MP\rangle$ state is used as a resource to play the quantum game known as the Mermin–Peres magic square but also variation of those games [13]. Cluster states are states that

Table 3 Best quantum gain when playing the four-qubit game of Eq. (9) with the four-qubit critical states, $|MP\rangle$, $|C_1\rangle$ and $|L\rangle$

Equation	State	Best quantum gain
$(xyz) + (xy\overline{w}) + (xz\overline{w}) + (yz\overline{w}) + (w\overline{xyz}) = a \oplus b \oplus c \oplus d$	$\lvert MP\rangle$	0.6767
	$\lvert C_1\rangle$	0.6767
	$\lvert L\rangle$	0.6767

One recalls that the best classical gain for Eq. (9) is 0.6225

Table 4 Best quantum gains when we analyze all possible strategy for all possible games of type $f(x, y, z, w) = a \oplus b \oplus c \oplus d$

State	Best quantum gain	Best classical
$\lvert MP\rangle$	0.7499	0.625
$\lvert C_1\rangle$	0.7499	0.625
$\lvert L\rangle$	0.6767	0.625

maximize the Renyi α-entropy [11, 14], by symmetry of the role of the players one only considers $|C_1\rangle$, the other cluster states being obtained by transposition. Finally, $|L\rangle$ is a state that maximizes the hyperdeterminant, an analogue of the 3-tangle for four-qubit states, see [14, 15].

The next two tables summarize our findings for those three critical states. Table 3 provides the score of the best quantum strategy to play the game given by Eq. (9). Then, Table 4 gives the best possible quantum gain and the corresponding best classical when we consider all games of type $f(x, y, z, x) = a \oplus b \oplus c \oplus d$.

3.5 The four-qubit classification

It is well known, as demonstrated by [10], that there exist nine types of entangled four-qubit states. More precisely, the number of classes of entanglement under the so-called SLOCC group is infinite but can be grouped in 9 families, 6 of them depending on parameters, 3 being parameters free. Let us recall with Table 5 the normal forms of those families (in its corrected version from [16]).

In this section, various four-qubit entangled states are tested, with each representing one of these nine types.

In Table 6, we report the score obtained for the best quantum strategy when playing the game defined by Eq. (9) with an instance of each of the nine-family. For the 6 families depending on parameters, we randomly choose a set of parameters (we repeated it four times) and collected the best gain and also the average quantum gain over the four sets of parameters. Once again with Eq. (9), we do not get any higher score than 0.8535.

If one now considers all games based on $f(x, y, z, w) = a \oplus b \oplus c \oplus d$, we do not get anything better than the difference between the best quantum strategy and the best classical ones that are reached for $|GHZ_4\rangle$.

Table 5 The four-qubit classification into 9 families of entanglement following [10] and [16]

Name	Normal form
G_{abcd}	$\frac{a+d}{2}(\|0000\rangle + \|1111\rangle) + \frac{a-d}{2}(\|0011\rangle + \|1100\rangle)$ $+\frac{b+c}{2}(\|0101\rangle + \|1010\rangle) + \frac{b-c}{2}(\|0110\rangle + \|1001\rangle)$
L_{abc_2}	$\frac{a+b}{2}(\|0000\rangle + \|1111\rangle) + \frac{a-b}{2}(\|0011\rangle + \|1100\rangle) + c(\|1010\rangle + \|0101\rangle) + \|0110\rangle$
$L_{a_2b_2}$	$a(\|0000\rangle + \|1111\rangle) + b(\|0101\rangle + \|1010\rangle) + \|0110\rangle + \|0011\rangle$
L_{ab_3}	$a(\|0000\rangle + \|1111\rangle) + \frac{a+b}{2}(\|0101\rangle + \|1010\rangle) + \frac{a-b}{2}(\|0110\rangle + \|1001\rangle)$ $+\frac{i}{\sqrt{2}}(\|0001\rangle + \|0010\rangle - \|1110\rangle - \|1101\rangle)$
L_{a_4}	$a(\|0000\rangle + \|0101\rangle + \|1010\rangle + \|1111\rangle) + i\|0001\rangle + \|0110\rangle - i\|1011\rangle$
$L_{a_2 0_{3\oplus 1}}$	$a(\|0000\rangle + \|1111\rangle) + \|0011\rangle + \|0101\rangle + \|0110\rangle$
$L_{0_{7\oplus \bar{1}}}$	$\|0000\rangle + \|1011\rangle + \|1101\rangle + \|1110\rangle$
$L_{0_{5\oplus \bar{3}}}$	$\|0000\rangle + \|0101\rangle + \|1000\rangle + \|1110\rangle$
$L_{0_{3\oplus \bar{1}} 0_{3\oplus \bar{1}}}$	$\|0000\rangle + \|0111\rangle$

Up to local changes of basis, i.e., SLOCC= $SL_2(\mathbb{C}) \times SL_2(\mathbb{C}) \times SL_2(\mathbb{C}) \times SL_2(\mathbb{C})$ transformations, all four-qubit quantum states can be transformed into one of those 9 families. Note that the first 6 depend on parameters

Table 6 Quantum gain for different parametric and nonparametric four-qubit entangled states when playing the four-player games defined by Eq. (9)

Equation	State	Best Quantum Gain	Average Gain
$(xyz) + (xy\overline{w}) + (xz\overline{w}) + (yz\overline{w}) + (w\overline{x}\overline{y}\overline{z}) = a \oplus b \oplus c \oplus d$	G_{abcd}	0.8535	0.7737
	L_{abc_2}	0.8532	0.7524
	$L_{a_2b_2}$	0.8534	0.7250
	L_{ab_3}	0.7357	0.7072
	L_{a_4}	0.6985	0.6774
	$L_{a_2 0_{3\oplus 1}}$	0.8535	0.7164
	$L_{0_{7\oplus \bar{1}}}$	0.6586	X
	$L_{0_{5\oplus \bar{3}}}$	0.6530	X
	$L_{0_{3\oplus \bar{1}} 0_{3\oplus \bar{1}}}$	0.7499	X

The first six families of the four-qubit classification depend on parameters, and we have computed the best quantum strategies for different random choices of parameters (we randomly repeated it four times). The first column provides the best quantum advantage obtained and the second one the average best gain over the different random choices of parameters. This does not apply to the last three classes that are parameter-free. Recall that the classical gain for Eq. (9) is 0.6225. With a $|GHZ_4\rangle$ states, one can win the game with probability 0.8535, and if one uses $|W_4\rangle$ as our entangled resource, the gain is of 0.69

Table 7 Game scores for different four-qubit entangled states and $g(a,b,c,d) = a \oplus b \oplus c \oplus d$

State	Game score	State	Game score
G_{abcd}	0.3452	$L_{a_2 0_{3 \oplus 1}}$	0.2554
L_{abc_2}	0.2622	$L_{0_{7 \oplus \bar{1}}}$	0.0031
$L_{a_2 b_2}$	0.2544	$L_{0_{5 \oplus \bar{3}}}$	0.0033
L_{ab_3}	0.2450	$L_{0_{3 \oplus \bar{1}} 0_{3 \oplus \bar{1}}}$	0.2804
L_{a_4}	0.2179		

3.6 Notion of game score

We define the notion of *game score* for a given four-qubit entangled state and a given function $g(a, b, c, d)$ as the proportion in percentage of functions $f(w, x, y, z)$ which have a quantum gain higher than 1% compared to the classical gain.

In the case where $|GHZ_4\rangle$ is used and $g(a, b, c, d) = a \oplus b \oplus c \oplus d$, a game score of 26.34% is obtained. In other words, on the 3907 functions f to test, 26.34% of them have a quantum gain higher than 1% compared to the classic one. Concerning the average gap between quantum and classic gain, in this case, it is equal to 5.49%. When we consider the $|W_4\rangle$ state, this game score drops to 24%

Table 7 collects the game score obtained for the nine types of entangled four-qubit states. For parametric states, one randomly chose the parameters and took the best game score after several trials. When a random G_{abcd} state is used, a game score of 34.52% is obtained, much better than with a $|GHZ_4\rangle$ state. But concerning the average gap, this one is equal to 3.73%, so lower than the previous one with $|GHZ_4\rangle$.

3.7 Retrieving gains of Table 6

We now explain why, in Table 6, one recovers for the families G_{abcd}, L_{abc_2} $L_{a_2b_2}$ and $L_{a_2 0_{3 \oplus 1}}$ the same best quantum gain as the one obtained with a $|GHZ_4\rangle$ state. We also discuss the quantum gain for L_{ab_3} and L_{a_4} and their relation to the critical states $|MP\rangle$.

3.7.1 Retrieving $|GHZ_4\rangle$ Gain

Regarding the four-qubit parametric entangled states, achieving the same gain as $|GHZ_4\rangle$ is possible when these parametric states can produce a $|GHZ_4\rangle$ state. It appears clearly that $|GHZ_4\rangle$ belongs to an orbit of the family G_{abcd}. For example, one can choose $a = b$ and $c = d = 0$ to recover the usual $|GHZ_4\rangle$ state.

If one considers the family L_{abc_2} with $c = 0$, one has

$$\lim_{a=b\to\infty} L_{abc_2} = |GHZ_4\rangle.$$

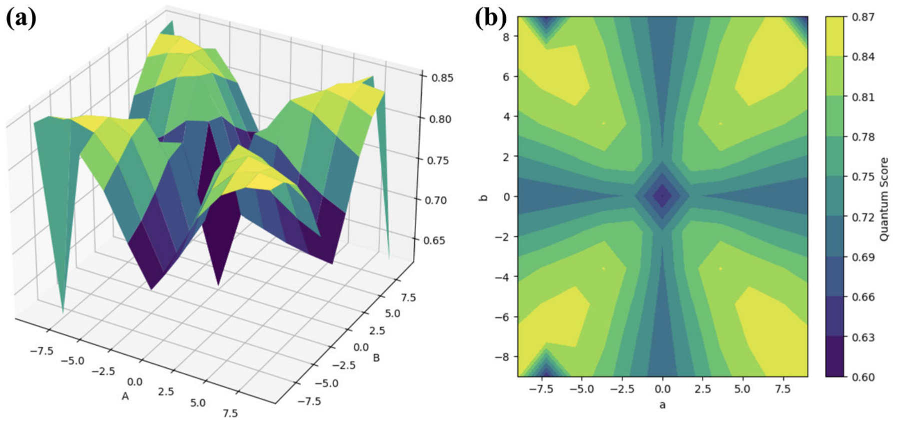

Fig. 3 Evolution of the quantum gain of L_{abc_2} for Eq. (9) with a and b between -9 and 9 with $c = 0$ in a 3D (**A**) and 2D (**B**) representations

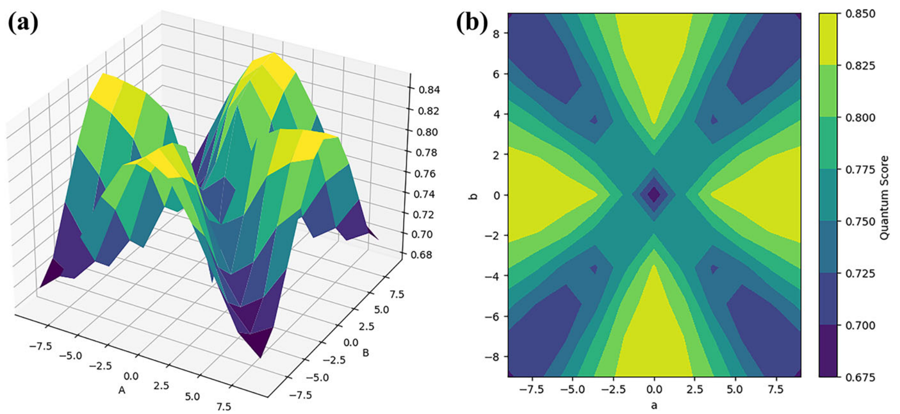

Fig. 4 Evolution of the quantum gain of $L_{a_2b_2}$ for Eq. (9) with a and b between -9 and 9 in a 3D (**A**) and 2D (**B**) representations

And the same holds if $a = \pm b \to \pm\infty$. Therefore:

$$\lim_{\substack{a=\pm b\to\pm\infty \\ c=0}} Gain(L_{abc_2}) = Gain(|GHZ_4\rangle) = 0.8535. \tag{11}$$

To illustrate this property, one plotted in Fig. 3a and b the gain for the game based on Eq. (9) with the quantum state L_{abc_2} for different values of a and b and $c = 0$. The limits of Eq. (11) can be recovered from Fig. 3.

With $L_{a_2b_2} = a(|0000\rangle + |1111\rangle) + b(|0101\rangle + |1010\rangle) + |0110\rangle + |0011\rangle$, two $|GHZ_4\rangle$ states can be observed: The first is defined by $a(|0000\rangle + |1111\rangle)$, and the second is defined by $b(|0101\rangle + |1010\rangle)$. So by studying the quantum gain of $L_{a_2b_2}$

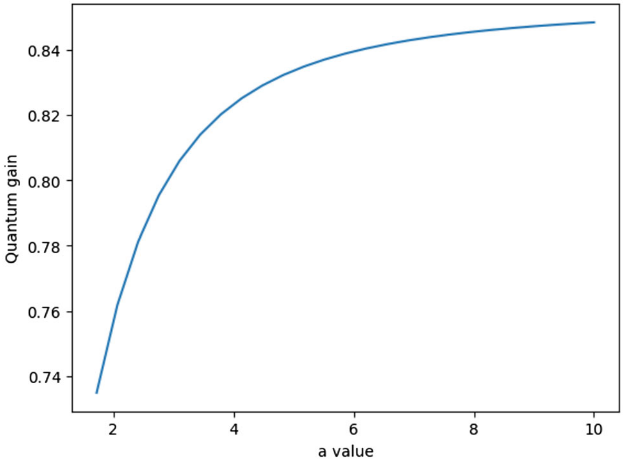

Fig. 5 Evolution of the quantum gain of $L_{a_2 0_{3\oplus 1}}$ with a increasing from 2 to 10 approaches $|GHZ_4\rangle$ state

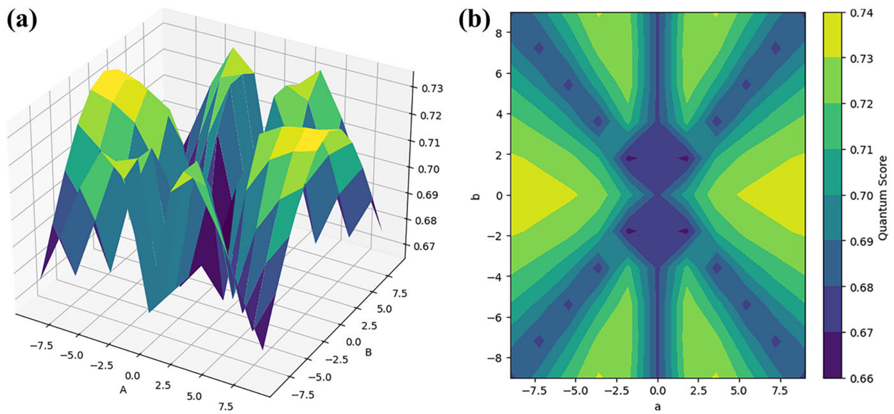

Fig. 6 Evolution of the quantum gain of $L_{a_2 b_2}$ for Eq. (9) with a and b between −9 and 9 in a 3D (**A**) and 2D (**B**) representations

when a goes to ∞ and b goes to 0, it is possible to deduce that :

$$\lim_{\substack{a\to\infty \\ b\to 0}} Gain(L_{a_2 b_2}) = \lim_{\substack{b\to\infty \\ a\to 0}} Gain(L_{a_2 b_2}) = Gain(|GHZ_4\rangle) = 0.8535$$

We can clearly see this in Fig. 4a and b, we can also observe that the quantum gain is minimum when $a = b$.

In the case of $L_{a_2 0_{3\oplus 1}} = a(|0000\rangle + |1111\rangle) + (|0011\rangle + |0101\rangle + |0110\rangle)$, the $|GHZ_4\rangle$ state is again clearly visible. Thus, it is possible to retrieve $|GHZ_4\rangle$ score, for $(xyz) + (xy\overline{w}) + (xz\overline{w}) + (yz\overline{w}) + (w\overline{xyz}) = a \oplus b \oplus c \oplus d$ by increasing the coefficient a. One observes that (Fig. 5):

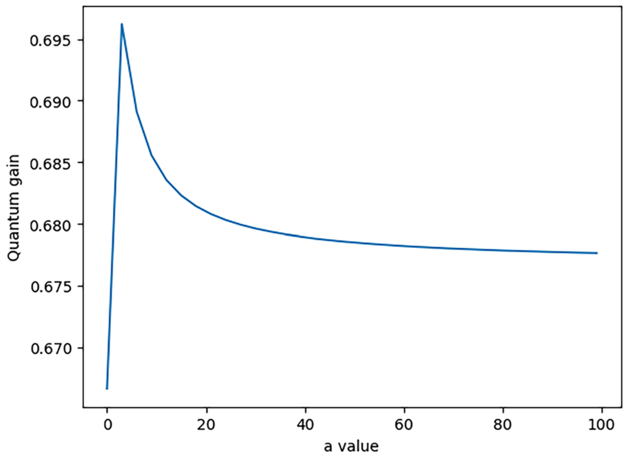

Fig. 7 Evolution of the quantum gain of L_{a_4} for Eq. (9) with a increasing from 0 to 100. For $a = 0$, the gain is similar to what one would have obtained for the state $i|0001\rangle + |0110\rangle - i|1011\rangle$ and as a increases, the gain approaches the gain obtained with the critical state $|MP\rangle$ which follows from the limit calculation. Interestingly, a pick is reached for $a = 2$

$$\lim_{a\to\infty} Gain(L_{a_2 0_{3\oplus 1}}) = Gain(|GHZ_4\rangle) = 0.8535$$

.

3.7.2 Retrieving $|MP\rangle$ Gain

Let us now consider the parametric states L_{ab_3} and L_{a_4} when used with Eq. (9). We first observe that $|MP\rangle$ belongs to the L_{ab_3} family for $a = b = 0$. Indeed, we have:

$$L_{ab_3} \underbrace{=}_{a=b=0} \frac{i}{\sqrt{2}}(|0001\rangle + |0010\rangle - |1110\rangle - |1101\rangle = \frac{i}{\sqrt{2}}((|00\rangle - |11\rangle)(|10\rangle + |01\rangle).$$

Therefore, as can be seen from Fig. 6a and b:

$$\lim_{\substack{a\to 0\\ b\to 0}} Gain(L_{ab_3}) = Gain(|MP\rangle) = 0.6767$$

Similarly, up to a permutation of the second and third qubits, one can see that

$$\lim_{a\to\infty} L_{a_4} = |MP\rangle$$

Figure 7 illustrates this fact as one sees the gain to converge to 0.6767 when a increases.

$$\lim_{a\to\infty} Gain(L_{a_4}) = Gain(|MP\rangle) = 0.67677.$$

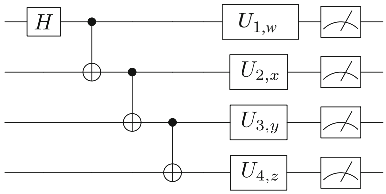

Fig. 8 Quantum circuit used for applying the quantum strategy with $|GHZ_4\rangle$. The optimal angles of the local unitary transformations depend on the question sent to the players and are reported in Table 1

4 Four-qubit CHSH games on a quantum computer

In this section, two different four-qubit CHSH games are played on real quantum devices. The first aim is to highlight the violation of classical bounds using our adapted CHSH games, and the second is to examine the behavior of the quantum protocol in the presence of noise as well as on a real quantum device. Two distinct games are selected: one where the $|GHZ\rangle$ state outperforms the classical gain, and another where the $|W\rangle$ state surpasses both the classical limit and the $|GHZ\rangle$ state in terms of score. Those are the games studied in Sect. 3. In both scenarios, the experimental advantage given by the quantum strategies surpasses the classical bound as predicted by the theory.

4.1 Experiment with $|GHZ_4\rangle$

We first test the game defined by Eq. (9). Recall that, in that case, one can outperform the best classical strategy (0.6225 of gain) with a quantum strategy given in Table 1 involving the $|GHZ_4\rangle$ state (0.8535 of gain).

To play the game, the $|GHZ_4\rangle$ state is first generated by applying a Hadamard gate on the first qubit followed by a series of CNOT gates. Subsequently, each player applies a unitary transformation on their qubit based on the received question, with angles provided in Table 1. Finally, each qubit is measured in the computational basis, and the results are returned to the referee. The circuit executed for this process is depicted in Fig. 8 and the results are provided on Fig. 9 where we included comparison to the classical strategy.

4.2 Experiment with $|W_4\rangle$

For the second experiment, one considers the game where the four players win whenever their answers $(a, b, c, d) \in \mathbb{B}^4$ satisfy Eq. (10) for a choice of question $(w, x, y, z) \in \mathbb{B}^4$.
In order to construct a $|W_4\rangle$ state with four qubits, we use a method inspired by [17], an adapted by [18] using rotations around the Y-axis, controlled-Z, controlled-NOT and Hadamard gates, see Fig. 10 (an alternative description and discussion on its physical implementation can be found in [19]):

We ran this game on IBM_Sherbrooke device. Figure 11 presents our results in the form of a histogram comparing the percentage of victory when playing the game with a $|W_4\rangle$ state and with the best classical strategy. On average, the percentage of wins is higher with $|W_4\rangle$ and beats the classical bound of 68.75%.

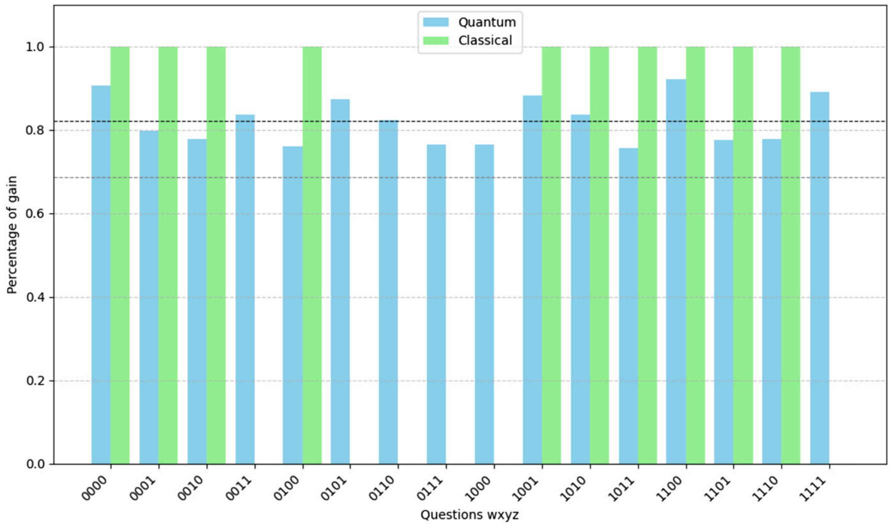

Fig. 9 Histogram of the percentage of victory as a function of the question sent to the four players in the game defined by Eq. (9). In green a classical strategy that achieves an average gain of 62.25% of victory. In blue, the quantum gain with an average score of 82.19%, the four players sharing a $|GHZ_4\rangle$ entangled state. The dashed line represents the average victory. The experiment was performed on IBM Sherbrooke with 10,000 shots on October 14, 2024

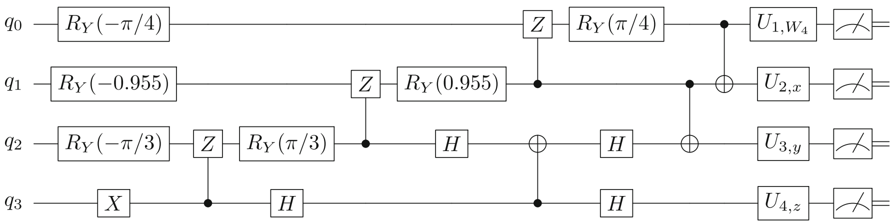

Fig. 10 Quantum circuit used for applying the quantum strategy with $|W_4\rangle$. The optimal angles of the local unitary transformations depend on the question sent to the players and are reported in Table 2

5 Conclusion

In this article, we explored the generalization of CHSH quantum games to four qubits, inspired by previous advances in two- and three-qubit games [8, 9]. Our investigation led to the discovery and analysis of new quantum games. We explicitly described a game using the four-qubit entangled state $|GHZ_4\rangle$ as the optimal resource and a game using the state $|W_4\rangle$ as the optimal resource. Significantly, we demonstrated that the usual quantum advantage of 10%, observed in two- and three-qubit games, was not only achieved but also doubled, reaching 22.5% in the case of the $|GHZ_4\rangle$ game. Additionally, we tested critical states that are candidates for maximally entangled states as well as generic states from the four-qubit classification [10], further expanding our understanding of the potential use of those different types of entanglement. Interestingly, these maximally entangled states are not the quantum resources that

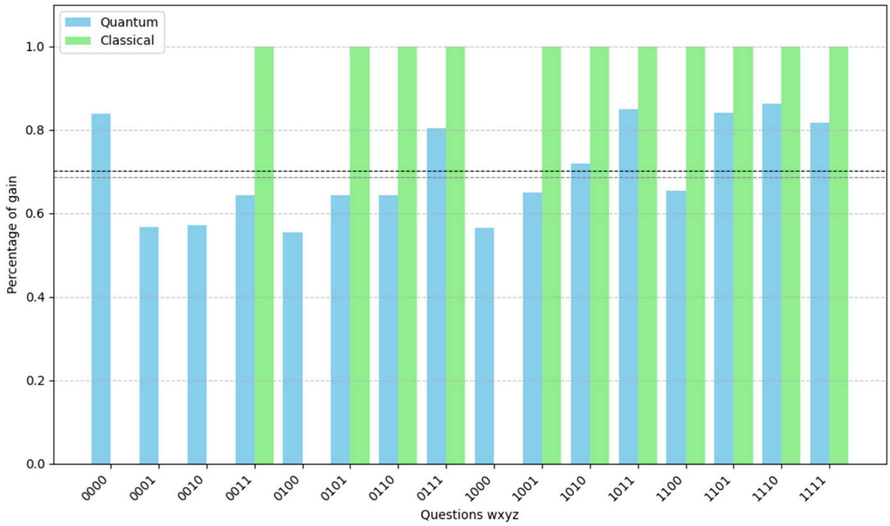

Fig. 11 Histogram of the percentage of victory as a function of the question sent to the four players for the game defined by Eq. (10). In green a classical strategy that achieves an average gain of 68.75% of victory. In blue, the quantum gain with an average score of 70.13%, the four players sharing a $|W_4\rangle$ entangled state. The dashed line represents the average victory. The experiment was performed on IBM Sherbrooke with 10,000 shots on October 14, 2024

provide the best quantum advantage in our four-qubit quantum game setting. It raises the question of what type and what amount of entanglement is the most efficient for those quantum protocols.

In our experiments with the IBM Quantum computers, one obtained a clear violation of the classical bounds for all tested scenarios. The experimental results are very close to the theoretical values and did not require a deeper analysis of the noise effects [20]. The recent successful implementation [21] of the Mermin pseudo-telepathy game on the new generation of IBM Quantum processors confirms the increasing reliability of those quantum processors. This advancement highlights not only the power of four-qubit entangled states to overcome classical limits but also the diversity of new possible games in this framework. It is important to note that, despite the numerous results obtained, we had to reduce the problem size due to computational time constraints, making an exhaustive search impractical. However, our work confirms that for games based on conditions of types

$$f(w, x, y, , z) = a \oplus b \oplus c \oplus d,$$

the best quantum advantage, over all possible f is obtained for the $|GHZ_4\rangle$ state. One expects this to remain true for a larger number of qubits. For practical applications, our framework offers a general method to decide if a four-player no-communication game, defined by a winning condition of type $f(w, x, y, z) = g(a, b, c, d)$, for f and g Boolean functions, can be won advantageously with quantum entanglement. We hope this will provide a near-future scenario for quantum protocols, such as self-testing, device-independent quantum computing, or quantum key distribution [22].

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Author Contributions All co-authors equally contributed to the research and the manuscript

Data availability All codes and results of our quantum experiments are available at: https://github.com/ColibrITD-SAS/publications_material/tree/main/four_qubit_CHSH_game.

Declarations

Conflict of interest The authors declare no Conflict of interest.

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