Quantum Separation of Variables in Higher Rank and Supersymmetric Integrable Models

Louis Vignoli

Numéro National de Thèse : 2021LYSEN052

THÈSE DE DOCTORAT DE L'UNIVERSITÉ DE LYON

opérée par

l'École Normale Supérieure de Lyon

École Doctorale N° 52
Physique et Astrophysique de Lyon (PHAST)

Discipline : Physique

Soutenue publiquement le 29/10/2021, par :

Louis VIGNOLI

Quantum Separation of Variables in Higher Rank and Supersymmetric Integrable Models

Séparation de variables quantiques pour les modèles intégrables de plus haut rangs et supersymétriques

---

Devant le jury composé de :

Didina SERBAN	Ingénieure CEA	CEA Saclay	Rapporteure
Véronique TERRAS	Directrice de recherche	U. Paris–Saclay	Rapporteure
Frank GÖHMANN	Professeur	U. de Wuppertal	Examinateur
Éric RAGOUCY	Directeur de recherche	U. de Savoie	Examinateur
Jean Michel MAILLET	Directeur de recherche	ENS de Lyon	Directeur de thèse
Giuliano NICCOLI	Directeur de recherche	ENS de Lyon	Codirecteur de thèse

Contents

Remerciements

À Jean Michel Maillet et Giuliano Niccoli, mes deux directeurs de thèse. Vous m'avez accordé votre confiance en m'offrant de travailler sous votre supervision sur ces sujets que vous chérissez. Merci pour votre encadrement bienveillant, vos conseils avisés et vos critiques toujours justes. Je vous dois autant scientifiquement que humainement ; j'espère devenir un aussi bon mentor que vous l'avez été pour moi.

À tous les membres de mon jury de thèse :

À Franck Göhmann, pour avoir amené son expertise sur les modèles supersymétriques et le modèle de Hubbard,

À Didina Serban et Véronique Terras pour leur relecture attentive du manuscrit et les commentaires pertinents qui ont permis son amélioration,

À Éric Ragoucy pour avoir présidé le jury avec expérience lors de la soutenance.

Au Laboratoire de Physique à l'École Normale Supérieure de Lyon, où que j'ai pu conduire mes travaux de recherche. Je souhaite remercier l'ensemble de ses membres pour le sérieux scientifique et l'enthousiasme général qui y règne. Ses deux directeurs que j'ai connu incarne particulièrement cet état d'esprit :

À Thierry Dauxois pour son accueil chaleureux et notre rivalité complice entre les deux Olympiques—bien qu'il n'y en qu'un seul,

À Jean-Christophe Géminard pour son accessibilité et sa volonté constante d'offrir à tous les meilleures conditions de travail malgré de difficiles conditions.

À l'École Doctorale PHAST 52 et son directeur Dany Davesne pour l'encadrement de mon diplôme et la formation prodiguée au cours de ce dernier.

À tous les collègues du laboratoire avec qui j'ai partagé le quotidien de la recherche : David, Sylvain, Éric, Antoine, Jérémy, Jérémy, Benjamin G., Benjamin C., Yannick, Louison, Jason, Christophe, Harriet, Salvish, Henning, Denis, François D., François G., Marc M., Marc G., Etera, Karol, Jérémie.

Au personnel du laboratoire qui s'est assuré de la bonne conduite de ma thèse avec une grande bienveillance et beaucoup de bonne humeur : Laurence, Fatiha, Nadine.

Au Département de Physique, pour m'avoir confié l'enseignement en Licence, Master et Préparation à l'Agrégation de nombreux cours et travaux dirigés. Savoir c'est d'abord apprendre et puis partager. Merci à Cendrine, Romain Volk et Christophe Winisdoerffer.

À mes colocataires successifs du 86 avenue Jean Jaurès : Théo, Lou-Alma, Marion, Sow, Salomé, Camille et Érica. Merci d'avoir contribué au quotidien chaleureux de cet appartement.

À Aleksandra, Gilles, Philip, Andrew, et à tous les participants de l'école d'été des Houches Integrability in Atomic and Condensed Matter Physics, ainsi qu'à tout ceux des différentes éditions du Student Workshop in Integrability. Qu'il est bon de faire parti d'une communauté de jeunes gens aussi dynamiques et accueillants.

À Alain Lhopital, Laurent Barbaza, Jean-Claude Martin, Pierre Grécias et Christian Garing ; vous m'avez forgé un esprit de science que j'exerce chaque jour avec fierté.

À Michel Gonin, Thomas Mueller, Benjamin Quillain et Laura Dominé ; c'est par vous que j'ai véritablement découvert la recherche scientifique et en ai fait ma vie.

À mes nombreux amis qui m'ont accompagné durant ces années, et plus particulièrement :

À Camille, pour le partage des nos bureaux qui m'a procuré tant de bonheur. Merci pour ces quatre années ensemble autour de notre goût pour la science, les origamis, la typographie bien faite et zoomée de près et les photos de chats. Il y a un peu de toi dans chacune des équations, chacun des symboles de ce manuscrit.

À Noémie D., sans qui cette heureuse cohabitation n'aurait eu que la saveur—un peu trop aride—des calculs abstraits. Merci pour toutes ces nécessaires pauses cappuccino que nous n'avions pas la présence d'esprit de prendre, le nez dans les équations, et cette joie de vivre qui a ensoleillé notre bureau si mal exposé.

À Augustin, d'être mon compagnon de science depuis nos caserts face à face et de rester sur le front de la recherche pour moi.

À Simon, pour sa curiosité constante et notre intérêt commun aux belles mathématiques.

À Jules ; je te dois les chapitres 3 et 4 de ce manuscrit grâce à notre séjour à Sanary et ton vélo si véloce.

À Lucas ; je te dois les chapitres 6 et 7 par ton séjour à Marseille et ton attachement inexpugnable aux mosaïques urbaines.

Merci pour vos amitiés aussi faciles que précieuses.

Au chat Piana, pour l'animation des laborieuses journées de confinement.

À ma soeur, mon père, ma mère, et toute ma famille ; votre soutien indéfectible est ma plus grande force.

Enfin, à Noémie. Je n'ai rien à écrire que tu ne sais pas déjà, et je te dois tant.

Chapter 1 Introduction

Conserved quantities are quantities that remain constant during the evolution of a physical system. They are central in this thesis. Conserved quantities form the defining feature of the systems studied, the integrable models, and are precisely used in the techniques we develop here to solve these models, the quantum separation of variables.

It has been a very long and difficult progress to understand there were invariants in the physical processes of Nature, and that these conserved quantities were essential tools in their description. Humans have proved to have a very developed ability to identify constant and recurring features in their changing surroundings. This immutable traits and periodic events allow for predictions, giving humans some control over nature, some sense of mastery or a first form of knowledge.

Astronomy was a universal practice among the antique civilizations. They identified that stars appear fixed on a celestial sphere, organized in everlasting constellations, slowly rotating during the year around the fixed Pole Star in Ursa Minor. Periodic astronomical events, from the sunset to solstices, played a key role in lives of ancient men, from agricultural planning to religious life. There has been strong motivation to find an explanation for those. At first, scientific and religious arguments mixed up in the proposed explanations. Gods of ancients Greeks and Romans manifest in the sky as the planets, who move freely in the cosmos because of their divine status. Yet, some are entitled to constant behaviors, such as Helios driving his chariot every day, or Atlas lifting the Earth still in the cosmos for eternity.

Later, the schism between science and religion became more and more inevitable. With the emergence of physics as the science of the behavior of material reality, the search for constant values and conservation laws in physical processes guided the experimental and theoretical approaches. We will illustrate this with the slow elaboration of the concepts of momentum, angular momentum and energy. That led to the foundation of modern physics.

There was already some intuition of a conserved quantity when studying the transmission of motion, from one objects to another, which later led to the concept of inertia, and the conservation of momentum. Galileo observed the velocity of a body do not vary in the absence of an exterior force [1, 2], and put an end to centuries of Aristotelian understating of motion, according to which constant velocity requires a continuous propelling. With some metaphysical justification, Descartes therefore wrote that God, through the law of natures he prescribed, "'conserve maintenant en l'Univers, par son concours ordinaire, autant de mouvement et de repos qu'il y en a mis en le créant. [...] une certaine quantité qui n'augmente ni ne diminue jamais, encore qu'il y en ait tantôt plus et tantôt moins en quelques-unes de ses parties.'" ([3], article 36) While disciples of Galileo formulated many statements on the idea of the conservation of movement [4], it is Newton who stated first the modern form of the *principle of inertia* in its *Philosophiæ Naturalis Principia Mathematica*: "Every body perseveres in its state, of rest or of uniform motion in a right line, unless it is compelled to change that state by forces impressed thereon" [5]. The conservation of the total vectorial momentum $\sum_i m_i \vec{\mathbf{v}}_i$ was then recognized by Huygens in isolated systems [6]: despite internal interactions in the system which reallocate momentum from parts to parts, the whole exhibits an invariant quantity, a conserved total momentum.

The study of the motion of planets in the solar systems also led very early to the identification of fundamental conservation laws of mechanics. Danish astronomer Tycho Brahe gathered a large amount of

precise observational data. They allowed his assistant Johannes Kepler to later state in the 1610s his three laws of planetary motion in the solar system [7, 8]. The second law notably states the steadiness of the areal velocity along the orbit of planets around the sun. This conservation of the areal velocity in planetary motion is an early understanding of conservation of the angular momentum, the rotational equivalent of (linear) momentum. If Bernoulli already talked of a "moment of rotational motion" [9] in 1744, the work of Poinsot may be considered as a real understanding of this object [10]; he represented rotations as a line segment perpendicular to the rotation place and introduced the concept of "conservation of moments" in his *Mémoires sur la composition des moments et des aires* [11].

During his experiments on elastic shocks, Huygens observed the conservation of the scalar quantity $\sum_i m_i \vec{\mathbf{v}}_i^2$ [6]. It enabled the concept of the conservation of *vis viva* (lively force) by Leibniz [12], aside of the inertia principle, which then led to the concept of *energy* in mechanics and its conservation. The first well-formed idea of energy emerged from the study of mechanical systems as the one of kinetic energy. Indeed, Lagrange proved in his *Mécanique analytique* [13] the *vis viva theorem*, which is better known as the theorem of kinetic energy today: during a non-dissipative process, the work received by each mass point of mass m and velocity $\vec{\mathbf{v}}$ is equal to half the increase in its lively force $m\vec{\mathbf{v}}^2$. Isolated systems therefore show constant energy over time, and energy joined momentum and angular momentum as a first class concept of mechanics. In its celebrated translation of Newton's *Principia Mathematica*, Émilie du Châtelet, with great knowledge of both Newton's and Leibniz's scientific works, also postulated the existence of a law of a conservation of a *total energy*, of which the kinetic one mv^2 is just a possible form [14]. It remained to understand that conservative forces are derived from potential functions of the position variables, and that their work identifies with the decrease of this function along the trajectory of the motion. This function was named *potential energy* by Rankine [15], and it was then possible to recognize the sum of the kinetic and potential energies as a conserved quantity called the *mechanical energy*. While mechanics was the first field to grasp the concept, the considerable developments of thermodynamic, started as the field of physics devoted to heat and its propagation, would put energy at the center of the stage.

Conversion of heat into mechanical work was made possible by the heat machines, invented by Newcomen and Savery and perfected by Watt [16]. This propelled humanity in a new paradigm, just like science. Careful studies by Carnot [17] on the engine cycles of these machines contributed greatly to the early developments of thermodynamics and most importantly to the discovery of its famous second law.

Thermodynamic highlighted the equivalence between the quantity of heat produced and the amount of work given to the machine. It allowed to consider these quantities as two forms of the same underlying entity, the energy. The experiments of Joule in the 1840s were decisive [18] in the proof of this correspondence. In these experiments, a falling mass induces the rotation of a paddle wheel immersed in the water of a calorimeter, whose temperature is observed to increase. It gave a clear demonstration of the conversion of potential energy in heat. This idea is also found in the works of von Mayer [19] on the human metabolism and photosynthesis, enlarging the circle of possible energy form to food and light.

These observations were encapsulated formally with the full statement of the first principle of thermodynamic, which came in 1850 from Clausius [20]: "In all cases in which work is produced by the agency of heat, a quantity of heat is consumed which is proportional to the work done; and conversely, by the expenditure of an equal quantity of work an equal quantity of heat is produced." [20]. He also introduces, necessarily, the internal energy function: "In a thermodynamic process involving a closed system, the increment in the internal energy is equal to the difference between the heat accumulated by the system and the work done by it." [20]

In parallel, the kinetic theory of gases developed and was formalized by Maxwell [21, 22] and Boltzmann [23]. This progress allowed to interpret the macroscopic internal energy of a material as the sum of the mechanical microscopic energies of its constituents, namely the sum of their kinetic energy and the potential energy of their interactions. It enabled the understanding that non-conservative mechanical

processes were in fact conservative when one accounts for the possibility of microscopic degrees of freedom to bear mechanical energy, whose macroscopic expression is heat. Energy is therefore not lost in non-conservative mechanical processes, but rather *dissipated* among the constituent of materials.

As we can see, energy has been a very difficult concept to grasp. Quite early, it has been identified as a universal scalar quantity that is invariant during the conservative evolution of the system at hand. But because it can take many forms, it has been very easy to lose track of some parts when computing the balance along the evolution of a system. All along the history of modern science, the energy has been found conserved, except in some experimental settings showing it was not in non-conservative settings, only to be found conserved again when accounting for hidden degrees of freedom, novel forms of energy and new ways to exchange it.

Feynman has, as always, a clever analogy to introduce energy to students in its Physics Lectures [24]; the one of a newborn playing with 28 indestructible blocks. No matter what he does with the blocks, his mother always finds a quantity to be computed that equals to 28: weighting a box, calculating the displacement of the level of water, or accounting for the presence of visitors bringing or grabbing some blocks. It is not the concept of blocks that matter here, as it's rather misleading for the classical meaning of energy (though it is more accurate in the quantum point of view). It is really the idea that a quantity remains equal to the same amount while being distributed in a system in very diverse ways, which a priori are very distinguishable objects, systems, concepts. In other words, a conserved quantity may be scattered in different forms and in different parts of a system, but its collected sum is a constant over time. This idea of a unified energy quantity was introduced in all generality by von Helmoltz in 1847 in his book *On the conservation of Force*[1] [25]. He postulated an underlying relationship between mechanics, heat, light, electricity and magnetism by treating them as manifestations of a single energy, unifying concepts from very diverse areas of physics.

As of today, modern science has identified energy under numerous forms: kinetic energy, potential energy deriving from a force field, electrical energy disposable from a difference of charge density, chemical energy bore by the various types of chemicals bonds, nuclear energy contained inside the nucleus and releasable by the fission and fusion processes, etc. Energy is exchanged between its different forms by various processes: contact mechanical interactions, thermal exchange, motion of massive or charged bodies trough space, radiation of light, excitation by electrical and magnetic fields, etc. Thanks to the modern developments on gravity, massive matter was found to also be a form of energy, which is encompassed by the celebrated $E = mc^2$ equation of Einstein's general relativity [26–28]. This equivalence is best illustrated by the disintegration of radioactive elements for example, where part of the mass "evaporates" in an electromagnetic radiation [29]. Numerous experiments have verified it, from the simple idea of "weighing photons" in a varying gravitational field [30], to modern and accurate measurements of atomic-mass difference compared to the wavelength of atoms spectra [31]. Or even more strikingly, but tragically, by explosions of atomic bombs. In the end, all these different form of energy and exchange processes have been unified as manifestations in different contexts and space scales of the four fundamental interactions that are the electromagnetic, weak, strong and gravitational interactions, which are mediated by their own gauge bosons [32, 33]. But depending on the context, some forms of energy and energy transfers are more suited to the description of the physical processes.

The conservation of energy for an isolated system is a foundational concept of physics that is shown satisfied in any experiments, from astrophysics to nuclear physics scale, in relativistic regimes, or quantum processes. In fact, one could say most of the research efforts in modern physics were made in the way of identifying quantities that can be conserved under certain conditions, the ways they change when they are not, and how they are exchanged between systems.

Conserved quantities are defining traits of a physical system and therefore are of great usefulness in

[1] Here the word "force" has to be understood as the proto concept of energy.

the study of their evolution. Indeed, conservation of a quantity during the system's evolution may be viewed as a constraint that has to be satisfied, limiting the size of the available phase space.

This can be illustrated with the Kepler problem. Consider two mass points in interaction by a central force deriving from a potential in $1/r$. It is well known this can be reduced to the study of an equivalent one-body problem, so that the system really as 6 dynamical variables, the components of the position and momentum vectors, with unknown dynamics one wants to compute. The conservation of the angular momentum (vector) enforces the flatness of the motion, reducing this number to 4 and already producing a great simplification in the description of the possible trajectories, and the possibility to use polar coordinates in place of spherical ones. Using the other conserved quantities, it is a classic exercise to compute the time evolution of the system in terms of quadrature by computing the *first integrals* [34]. All the parameters defining the orbit are expressed in terms of the conserved quantities of the Kepler problem, their exact values being determined by the initial conditions.

Note that one can exhibit additional conserved quantities, such as the Laplace–Runge–Lenz vector [34, 35], which can be leveraged in other methods of resolution to obtain information on the trajectory, or perform perturbative calculation around the $1/r$ potential as it is done for the computation of the precession of Mercure's perihelion [36].

The central role played by conserved quantities is better understood in the Hamiltonian formalism of classical mechanics [34, 37]. There, the configuration of a mechanical system is given by the knowledge of n *generalized coordinates* q_i, specifying a point on the n-dimensional real manifold *configuration space*. The additional knowledge of the *conjugated momenta* p_i complete the description of the mechanical system's state, and specifies a point on the *phase space* $\mathcal{M}$. The phase space is a $2n$-dimensional symplectic manifold, with canonical Poisson brackets between the canonical variables (q_i, p_i). The time evolution of the canonical variables is given by the flow of the *Hamiltonian* function $H : \mathcal{M} \to \mathbb{R}$, which gives the Hamilton equations of the motion

$$\forall i \in [\![1, n]\!], \quad \frac{\mathrm{d}q_i}{\mathrm{d}t} = \frac{\partial H}{\partial p_i}, \quad \frac{\mathrm{d}p_i}{\mathrm{d}} = -\frac{\partial H}{\partial q_i}.$$

In this picture, conserved quantities are functions F_j of the dynamical variables (q_i, p_i) whose image remains constant along the physical trajectory in the phase space. In general,

$$\frac{\mathrm{d}F_j}{\mathrm{d}t} = \frac{\partial F_j}{\partial t} + \{F_j, H\},$$

but looking among the functions that do not depend explicitly on time, conserved quantities are functions in involution with the Hamiltonian

$$\frac{\mathrm{d}F_j}{\mathrm{d}t} = \{F_j, H\} = 0.$$

The numerical fixed values of conserved quantities—fixed by the initial values of the constants of the motion—therefore defines the *level submanifold* of the phase space, i.e. the subspace fixed by the equations $F_j(m) = f_j = \text{cste} \in \mathbb{R}$ for $m \in \mathcal{M}$ on which the physical motion is constrained. Each additional independent conserved quantity identified thus simplifies further the resolution of the equations of the motion by narrowing down the portion of the phase space reachable during the motion. Great efforts are thus made to identify the conserved quantities systematically, and, if possible, from first principles.

The role of symmetries is central in this search. Any symmetries of the Hamiltonian is indeed associated to a conserved quantity. The most obvious ones are *cyclic coordinates*, namely coordinates that do not appear explicitly in the Hamiltonian—which is thus invariant under translations along these coordinates. The conjugated momenta of cyclic coordinates are seen conserved easily by the Hamilton equations, and are therefore conserved quantities. The powerful Noether's theorem clarify the picture in the case of continuous system: for every differentiable symmetry of a conservative system, there is a corresponding

conservation law [38]. From this point of view, invariance under space translations corresponds to conservation of momentum, invariance under space rotations corresponds to conservation of angular momentum, and time invariance is associated with the conservation energy.

Symmetries in physics are best understood using *groups and their representations* [39–44]. For differential symmetries involving Lie groups, it is often beneficial to consider the corresponding Lie algebra [45, 46]. The study of the underlying symmetry group or algebra, and their representations on the space of physical states, is a mandatory step to identify conserved quantities in a systematic way.

*
* *

The possibility to exactly solve analytically and in closed form the equations of motion of mechanical systems is very appealing. An obvious reason is that exact formulas of the evolution of a system should contain all the results a physicist is looking for. Moreover, while numerical calculations or computer simulations are more and more common, the computational cost remains prohibitive for many systems, despite their simplicity. Besides, a system may not be exactly solvable in general, but actually is for some specific values of its parameters. Perturbation theory can then be applied to obtained substantial results in the vicinity of these solvable points in the parameter space.

The importance of conserved quantities in exact methods has been acknowledged with the concept of *integrable system*. Integrability is the notion of total and exact solvability made possible by the presence of enough conserved quantities, and the different methods to obtain the closed form solutions. Concerning Hamiltonian systems, there exists a precise definition of this concept called *Liouville integrability* [47]. For a phase space of dimension $2n$, a system is Liouville integrable if there exists a set of n independent Poisson commuting conserved quantities. This property ensures that the system is solvable in closed form, thanks to the Liouville–Arnol'd theorem [48][2]. For Liouville integrable systems, it states the existence of canonical coordinates whose conjugated momenta are constants of the motion, so that the time evolution of the system in these canonical coordinates is obtained by trivial independent integrations. Rearranging the conserved quantities properly, the canonical *action–angle variables* are obtained from the canonical coordinates aforementioned, and in the compact case proved to be the adapted coordinates for the topology of the level submanifold of integrable systems: it is diffeomorphic to a n-dimensional torus [37, 48, 49]. This produces a foliation of the phase space by n-tori parametrized by the value of the conserved quantities.

The identification and construction of conserved quantities remains a difficult problem in general, as the symmetries of the Hamiltonian may be convoluted or not obvious. As we shall see, the Lax formalism [50] provides a framework to describe mechanical systems such that conserved quantities are easily obtained.

With great success in the study of mechanical systems with a finite number of degrees of freedom, the search for exact methods and an extension of the notion of integrability continued in close areas of physics. Two categories of system had a particularly lively developments of the integrability ideas: continuous systems, such as hydrodynamics or classical field theory, and statistical models on lattices.

In mechanics of continuous systems, the Korteweg–de Vries (KdV) equation introduced by Boussinesq [51, 52], which aims to describe waves in shallow waters, is a prototypical example of an exactly solvable model. It provided a satisfactory explanation of an intriguing phenomenon observed by Scott Russel [53]: the solitary wave. The KdV equation admits soliton solutions, which are waves with an invariant shape and constant velocity. The interaction of two solitons is reduced to a temporal shift in their propagation while shapes remain invariant. Infinitely many conserved quantities are associated to this conservation of the shape and velocity, and were identified in the works of Gardner, Green, Kruskal

[2] Some discussion over the topology of the phase space are also necessary.

and Miura [54] and Lax [50]. The Lax formalism proved especially useful in the study of integrable models. Its idea is to repack the equations of the motion in the form of a matrix commutator with the help of two matrices (L, V), the *Lax pair*, over an auxiliary space $\mathbb{V}$:

$$\frac{\mathrm{d}L}{\mathrm{d}t} = [L, V].$$

One of them, the Lax matrix L, then contains conserved quantities as its spectral invariants

$$\forall k \in [\![0, \dim \mathbb{V}]\!], \quad \frac{\mathrm{d}}{\mathrm{d}t} \operatorname{tr}\left(L^k\right) = 0.$$

The Lax pair framework was later extended to the continuous case by Zakharov and Shabat [55, 56] in the form of a zero-curvature equation between two differential operators generalizing the Lax pair of matrices. The possibility to study an auxiliary problem in the form of a linear system

$$\partial_\mu \psi = A_\mu \psi \quad \text{with} \quad F_{\mu\nu} = \left[\partial_\mu - A_\mu, \partial_\nu - A_\nu\right] = 0$$

rather than the equation of evolution of the field itself led to the development of the Classical Inverse Scattering Method (CISM). Faddeev and Zakharov gave a Hamiltonian interpretation of this scheme [57]. The solution $\psi(x, t)$ of the auxiliary problem on the real line $\mathbb{R}$, and its associated scattering data, have relatively simple evolution equations, which allows constructing from them classical field theory equivalents of the action–angle variables. Obtaining the initial fields from the scattering data is called the inverse problem, and its resolution solve the dynamics and give solutions to the equations of motion. Methods to do so were developed by Gel'fand, Levitan and Marchenko [58, 59], among other ideas such as the reduction to a Riemann–Hilbert problem [49, 60, 61] and algebraic factorization methods [49]. This enabled the study of other classical field theory systems as the classical sine-Gordon equation [60] or the non-linear Schrödinger equation [55] for example. In general, solitons solutions have been extensively studied in classical and then quantum field theories [62, 63].

Another category of models that played an important role in the development of modern integrability is the one of statistical models on lattices. These models aim to provide a microscopic explanation to statistical physics phenomena like phase transition, for example the ferromagnetic–paramagnetic transition of iron. A well known one is the Ising model, introduced by Lenz and solved exactly in the one-dimensional case by its student Ising [64, 65]. In one dimension, it does not exhibit any phase transition. This was a disappointing result, and motivated Heisenberg to propose a quantum version in one dimension, as we shall see soon. The two-dimensional classical Ising model on the rectangular lattice proved to be a much more difficult problem, and was exactly solved by Onsager and Kaufman in the 1940s [66–68], who showed this time there was a phase transition in the two-dimensional case. The resolution, using the Onsager and Clifford algebra, is prototypical to the current philosophy behind integrability: embedding the system in a rich algebraic structure that provides the tools for its resolution.

In the meantime, the quantum theory has been developed, and a quantum version of the one-dimensional Ising model was introduced by Heisenberg, with the hope to exhibit phase transition, contrary to the classical one-dimensional model: the Heisenberg XXX spin chain [69]. It consists in a chain of usual quantum $1/2$-spins, interacting with their two nearest neighbors in an isotropic manner. Bethe obtained the spectrum of the Hamiltonian for periodic boundary condition in 1931 [70], with a method that now bears his name: the coordinate Bethe Ansatz (CBA). The main idea of the CBA is to write an ansatz (a trial answer) for the eigenstates as a sum of planes waves. By enforcing that it is an eigenstate, the *Bethe equations* on the rapidities of the plane waves are obtained. The works of Orbach [71] and Walker [72] allowed to apply this ansatz to generalizations of the Heisenberg model, like the XXZ chain, where an anisotropy in the interaction between spins is introduced in one direction.

The energy of the fundamental state of this model was computed by Yang and Yang [73–75], which exhibited the link with the vertex models found in statistical physics imagined originally to describe the microscopic configurations of ice crystals [76–79].

The study of lattice models owes a lot to the works of Baxter. He introduced a new approach relying on the Q operator and a "Baxter equation" for 6-vertex and 8-vertex models [80]. This proved especially powerful for the 8-vertex models, leading to new results [81–83]. The 8-vertex model contains the 6-vertex model and the Ising model as particular case, so that this method was thought very general. Baxter also produced key exact results for the eight-vertex model [80, 81, 84], such as the computation of the partition function. He showed that the transfer matrices form a one-parameter commuting family, thanks to the existence of relations for the Boltzmann weights [80, 85]. These results were derived from the star-triangle relations obtained for the 8-vertex weights and their matrix rewriting. The same type of equations were obtained by Yang in the context of factorizable scattering processes [86]. These equations now goes under the celebrated name of the *Yang–Baxter equation*, as named by Faddeev and collaborators, and write

$$R_{12}(u,v)R_{13}(u,w)R_{23}(v,w) = R_{23}(v,w)R_{13}(u,w)R_{12}(u,v).$$

The R-matrix $R(u,v)$ is interpreted as the matrix gathering the Boltzmann weights of a 2D lattice vertex model, or, as found by C. N. Yang, are related to the two-body S-matrix for a factorizable scattering of quantum particles moving on the real line, with rapidities u and v [87, 88]. The associated Yang–Baxter algebra is generated by the elements $M_{ij}(u)$ of a monodromy matrix $M(u)$, with relations

$$R_{12}(u,v)M_1(u)M_2(v) = M_2(v)M_1(u)R_{12}(u,v),$$

where $M_1(u) = M(u)\otimes 1$, $M_2(u) = 1\otimes M(u)$. The *quantum R-matrix* intertwines two copies of the same *monodromy* operators. The trace of the monodromy matrix gives the *transfer matrix*

$$T(u) = \operatorname{tr} M(u).$$

The commutativity of the family of transfer matrices $(T(u))_u$ is given by the Yang–Baxter algebra relations, provided $R(u)$ is invertible. If the Hamiltonian is found to be a function of these matrices, then one gets a whole family of conserved quantities of the system at hand.

The knowledge of conserved charges is not sufficient to claim integrability. For a classical system, as we said above, it is also necessary to prove there are in sufficient number, independent and in involution to claim that the system is Liouville integrable. To do so, one need to compute systematically the Poisson brackets between the entries of the Lax matrix. Just like the R-matrix gave the commutators between the entries of the monodromy, is it possible to pack the Poisson brackets of the Lax matrix entries in a compact, similar form? Remarkably, these Poisson brackets can be written in a commutator with the help of a classical r-matrix [49, 89–91], for example

$$\{L_1(u), L_2(v)\} = [r_{12}(u,v), L_1(u) + L_2(v)].$$

Historically, this object was identified after its quantum counterpart, the R-matrix introduced above, and the classical limit of the quantum R-matrix indeed gives a classical r-matrix.

Using techniques derived from the CBA, Lieb and Liniger computed the energies of a one-dimensional gas of bosons in delta interaction, corresponding to a non-linear quantum Schrödinger equation [92, 93]. On the other hand, solutions of the non-linear (classical) Schrödinger equation were obtained, thanks the CISM [54]. This raised the question of the existence of a quantum version of the CISM method, which was elucidated with the developments of the *Quantum Inverse Scattering Method* (QISM) [90, 94–97]. The main idea is the construction of a monodromy matrix for the problem from quantum local Lax

operators, whose elements are auxiliary operators and quantum counterparts to the scattering data in the CISM. As described above, a R-matrix solution to the Yang–Baxter equation prescribes the commutators between the entries of the monodromy matrix, forming a Yang–Baxter algebra. Conserved quantities are then obtained from the family of the transfer matrices. Besides, the Yang–Baxter algebra can also be leverage to reconstruct the eigenstates of the conserved quantities. Indeed, the QISM serves as the privileged framework for the definition of an algebraic version of the CBA, the Algebraic Bethe Ansatz (ABA) [95, 98, 99]. For a review, see [100]. The core idea of the ABA is that the off-diagonal elements of the monodromy matrix somehow contain creation and annihilation operators for the eigenstates of the trace. The idea stem from the CISM, where these off-diagonal elements contain the action–angle variables. Hence, the repeated action of an operator $B(u)$ constructed from the off-diagonal elements over a well-chosen reference state $|0\rangle$ should therefore produce an eigenstate of the form

$$B(u_1)\dots B(u_m)\,|0\rangle\,,$$

under conditions on the values of the spectral parameters $u_1,\dots,u_m$, which are eventually written as the Bethe equations. The joint development of QISM and the ABA allowed numerous results in quantum integrable models, such as the quantum sine-Gordon model [95], the massive Thirring model [101] and the quantum Heisenberg chains [102, 103]. We shall focus heavily on the history of the later models, usually referred to as “spin chains”. The monograph [99] by Bogoliubov, Izergin and Korepin describe the formalism of QISM and ABA, as well as many results obtained by these techniques.

The computation of the monodromy and transfer matrices from local Lax matrices amounts to the “direct” problem of quantum integrability. The actual *inverse problem* is the computation of the dynamical variables, say the local spin operators for a spin chain, in terms of the monodromy matrix elements, namely the diffusion data. The effective resolution of the inverse problem was achieved only around the year 2000, when Kitanine, Maillet and Terras [104–106] obtained explicit expression of the local operators in terms of the monodromy matrix elements. Their results were extended by Göhmann and Korepin to the supersymmetric case [107].

These developments on the quantum side had great repercussions on the theory of classical integrable models. Sklyanin realized that the R-matrix, the Yang–Baxter equation and the Yang–Baxter algebras had classical counterparts, which could be obtained as a classical limits of the quantum objects [89]. This led to a program of classification of the classical integrable models by their Yang–Baxter algebra, using Lie algebra theory [108–110]. The monograph of Bernard, Babelon and Talon [49] is a modern account of the progress in this field during the last decades of the 20$^{\text{th}}$ century.

Similarly, great efforts were made to construct and classify the solutions of the quantum Yang–Baxter equation. The works of the Kulish and Reshetikhin [111–113], Jimbo [114] and Drinfel’d [115] led to the discovery of *quantum groups*, which may be introduced as deformed universal enveloping algebras of Lie algebras and Lie groups, replacing the role played by Lie algebras in the classical case. The R-matrices already identified were found to be representations of a universal object, the universal R-matrix, and quantum integrable models were shown to be representations of these quantum groups. This opened a new field of research in mathematics, the study of *quantum algebras* [116–118]. The tools of integrability also proved useful for the computation of invariants in knot theory and invariants of 3-dimensional manifolds [119–121]. This makes integrable models a nice example of the deep interleaving between mathematics and physics, one being inspired by the other and conversely.

The underlying symmetries of quantum one-dimensional spin chains have been identified, depending on the anisotropy of the coupling, to be prescribed by *Yangians* $\mathscr{Y}(\mathfrak{gl}(n))$, *quantum affine algebras* $U_q(\mathfrak{gl}(n))$, and related algebras [117]. As the classification of quantum integrable models developed, the study of *higher rank models* gained weight. The Heisenberg XXX chain is associated to the fundamental representation of the Yangian $\mathscr{Y}(\mathfrak{gl}(2))$ of the $\mathfrak{gl}(2)$ Lie algebra, in link with the simple Lie algebra

$A_1 = \mathfrak{sl}(2)$, of rank 1. Keeping the Lie algebra in the same series, *higher rank models* are associated to the Yangian $\mathscr{Y}(\mathfrak{gl}(\mathfrak{n}))$, with $\mathfrak{n} \geq 3$. The ABA proved to be useful for these models, leading to the development of a Nested version of the Algebraic Bethe Ansatz (NABA) [122, 123], where creation operators acting over a reference state have a non-trivial expression in terms of the off-diagonal elements of the monodromy matrix, and involve several levels of Bethe roots. The ABA techniques have also been extended to the supersymmetric case of $\mathscr{Y}(\mathfrak{gl}(\mathfrak{m}|\mathfrak{n}))$ models [124–129].

For quantum integrable model on the lattice, a longstanding goal—solved for some models by now—is the computation of the correlation functions between operators, which are quantities of the form

$$\frac{\mathrm{tr}_{\mathscr{H}}(\mathcal{O}e^{-H/k_BT})}{\mathrm{tr}_{\mathscr{H}}(e^{-H/k_BT})},$$

where $\mathcal{O}$ is an observable over the Hilbert space $\mathscr{H}$ of the system, H is the Hamiltonian, T the temperature and k_B the Boltzmann constant. In the limit of zero temperature, and assuming there exists only one fundamental state $|f\rangle$ of the lowest energy, correlation functions are reduced to a single matrix element, the normalized expectation value of $\mathcal{O}$ in the fundamental state

$$\frac{\langle f|\mathcal{O}|f\rangle}{\langle f|f\rangle}.$$

Usually, in lattice models, one would like to compute the above quantity for local operators, or products of local operators. A quantum integrable model on the lattice may be considered “solved” when it is possible to compute such correlation functions exactly.

This first requires to compute the fundamental state $|f\rangle$. In the QISM and ABA framework, it is written in the form of a Bethe state. If $\mathcal{O}$ is some monodromy matrix elements, its action on an on-shell Bethe state may be computed in the form of a linear combination of off-shell Bethe states, thanks to the Yang–Baxter relations and the choice of the reference state. This leaves the correlation functions as sums of on-shell/off-shell scalar products, which for some models have been computed as determinants in the works of Gaudin [130], Korepin [131] and Slavnov [132]. The action of a local operator on these states is a priori not known, but the resolution of the quantum inverse problem allows writing local operators in terms of the matrix elements of the monodromy. This enables the computation of correlation functions by ABA procedure and the repacking of the final formulas into determinants [105]. In collaboration with Slavnov, Kozlowski and Niccoli, many results stemmed from the first ones obtained on the XXZ chain by Kitanine, Maillet and Terras, up to the computation of the thermodynamic limit of multiple integral representations of correlation functions [105, 133–139]. The computation of the dynamical structure factor with the help of Caux [140, 141] allowed to successfully compare the results from integrability with the neutron scattering experiments [142, 143]. Important results were obtained in the temperature dependent case by Göhmann, Klümper and Suzuki [144–147], and also by Kozlowski, Maillet and Slavnov [148, 149]. For the higher rank case, many results have been obtained by Ragoucy, Slavnov, Beillard, Pakuliak and Lyashyk, allowing to express scalar products as determinants and computing some form factors [129, 150–156]. Some of these results were obtained in great generality, holding for supersymmetric models as well [127, 129, 157–162].

At this point it worths pointing out that the tools of integrability have been extensively used by the AdS/CFT correspondence and super Yang–Mills communities to derive exact results in quantum field theories using the lattice as a regularization [163–171]. This makes integrability a topic of formidable interaction between various branches of physics and mathematics.

However, and despite its considerable successes, the algebraic Bethe Ansatz suffers from some limitations. The first and most obvious one is that it is an ansatz; it is necessary to verify that all the eigenstate of the model are obtained by the ABA procedure. This counting is called the problem of the

completeness of the Bethe Ansatz, and is a non-trivial task. Various techniques have been employed [172–174], and the recent work of Chernyak, Leurent and Volin [175] seems to give a new understanding of the Bethe equations completeness problem.

Furthermore, the ABA needs a reference state to be performed. For some not-so-intricate systems, such as the antiperiodic XXZ chain [176] or the Toda chain [177], such a state simply does not exist, which makes the ABA fails from the very start.

Finally, the nesting procedure of the NABA in the higher rank case is quite heavy [129]. In particular, it would be desirable to have a representation of the eigenstates simpler than the nested one. The look for more compact representations found some success with the conjectures of Gromov, Levkovich–Maslyuk and Sizov on the expression of a single $\mathcal{B}(u)$ operator in higher rank models that led to compact representations of the eigenstates [178, 179].

Several variants of the ABA method were proposed to overcome these difficulties, such as the off-diagonal Bethe Ansatz [180] or the modified Bethe Ansatz [181]. But ultimately, there is room for another approach of integrable models, partly because of the aforementioned limitations of the Bethe ansatz techniques, but also because the definition of quantum integrability remains unsatisfactory. The subject of this thesis is the development of new techniques of separation of variables for higher rank and supersymmetric quantum integrable models, with the goal of finding more simple representations of the basic objects, like the transfer matrix spectrum characterization, representations of the eigenstates, scalar products, and ultimately make the first steps towards correlation functions.

*
* *

The first occurrence of separation of variables (SoV) may be attributed to d'Alembert in its *Traité de dynamique* in 1758 [182, 183], and is found in the part dedicated to the study of the wave equation that bears its name. It has been used extensively by Fourier during the 19$^{\text{th}}$ century to solve various differential equation, especially the heat equation [184], so that it also referred to as the "Fourier method" in english literature. Consider the d'Alembert wave equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0.$$

With the coordinates change

$$\begin{cases} p = x - ct \\ q = x + ct \end{cases} \quad \Leftrightarrow \quad \begin{cases} x = \dfrac{p+q}{2} \\ t = \dfrac{q-p}{2c}, \end{cases}$$

the wave equation is rewritten

$$\frac{\partial^2 u}{\partial q \partial p} = 0.$$

Clearly the coordinates (p, q) are more adapted to describe the motion: solutions to the above equations are easily computed to be of the form

$$u(x,t) = f\big(p(x,t)\big) + g\big(q(x,t)\big) = f(x-ct) + g(x+ct),$$

where f and g are C^2 functions of their arguments. They have a separate form, in two variables $p = x - ct$ and $q = x + ct$, with f depending only on p and g only on q. Initial conditions and conditions at the boundary of the space-time domain considered fix the solution. This example illustrate how separate variables can drastically simplify the resolution of the dynamic. Henceforth, it is very desirable to separate the variables whenever it is possible. This procedure decouples the equations of the motion of the system at hand in independent and separate problems of lower complexity.

Separation of variables was found very useful in classical mechanics in the resolution of the Hamilton–Jacobi equation for Hamiltonian systems [34]. In practice, Hamilton–Jacobi equations of separate forms are among the only cases where the resolution is tractable and the action–angle variables can be computed in closed form. In all generalities, the $2n$ canonical variables (x_i, z_i) of a $2n$-dimensional Hamiltonian system are separate if there exist n independent *separate relations* of the form

$$\mathcal{F}_i(q_i, p_i, F_1, \dots, F_n) = 0.$$

Note that $\mathcal{F}_i$ depends only on the corresponding q_i and p_i, and a priori also on all the conserved quantities F_i. Realizing p_i as $\partial/\partial q_i$, one sees clearly that the above equations gives independent ordinary differential equations for each coordinate q_i.

It is Sklyanin who laid the foundations of SoV in the inverse scattering framework for classical and quantum integrable models, with its seminal paper on the Toda chain [177] in which he acknowledges inspiration from Gutzwiller's results [185] and the help of Komarov. Sklyanin introduced SoV techniques in the CISM context [186–188], and contributions for the generic $\mathfrak{gl}(\mathfrak{n})$ case were made by Scott [189] and Gekhtman [190]. The idea relies on the existence of a pair of two functions $(\mathcal{A}(u), \mathcal{B}(u))$, such that the roots x_i of $\mathcal{B}$ and their image $z_i = \mathcal{A}(x_i)$ by $\mathcal{A}$ form conjugated canonical variables. The (x_i, z_i) are then shown to be separate variables for the dynamical problem, because of the form of the $\mathcal{A}(u)$ and $\mathcal{B}(u)$ functions.

Having developed SoV for classical spin chains models, Sklyanin was able to extend his construction to $\mathfrak{gl}(2)$ quantum spin chains [191–193], under the name of *functional Bethe Ansatz*[3]. The principles remain the same, with the $\mathcal{A}(u)$ and $\mathcal{B}(u)$ functions promoted to operators. This makes the definitions of the *operatorial roots* x_i of $\mathcal{B}(u)$ more subtle. The separate basis is the eigenbasis of the $\mathcal{B}(u)$ operator, which should be diagonalizable with simple spectrum. The conjugate momenta z_i to the x_i give shifts on the spectrum of the x_i operators, and therefore on the spectrum of $\mathcal{B}(u)$. In the separate basis, the wave functions of the eigenstates of the transfer matrix factorizes as a product of one-variable wave functions, each of them satisfying an independent finite-difference equation. Hence, separation of variables in the quantum case indeed reduces drastically the complexity of the multi-variable problem to several one-variable ones.

Since these seminal models, SoV has been developed for a wide range of other important models. SoV has been implemented for the rational Gaudin model by Sklyanin [191], and extended to the quantum elliptic case with the help of Takebe [194]. He also considered the infinite volume case for the non-linear Schrödinger model [195] and the sinh-Gordon model [196]. Furthermore, he studied the 3-particles quantum Calogero–Sutherland model with Kuznetsov [197], the A_2 Ruijsenaars model with Nijhoff [198, 199]. The non-compact XXX chain case was studied in the works of Derkachov, Korchemsky and Manashov [200–202], while the lattice sinh-Gordon model was tackled by Bytsko and Teschner [203]. Additional contributions to the XXX spin $1/2$ chain SoV were made by Frahm and collaborators [204, 205].

The SoV method for higher rank quantum integrable model was initiated by Sklyanin himself, by quantizing his classical construction. In [206], he constructed separate variables for the $\mathfrak{gl}(3)$ quantum spin chain by constructing corresponding $\mathcal{A}(u)$ and $\mathcal{B}(u)$ operators. He already acknowledged *en passant* that for the scheme to be well-defined, one needs to verify a non-intersection condition between the spectrum of the roots of $\mathcal{B}(u)$ and possible poles of $\mathcal{A}(u)$. This construction was discussed in the generic $\mathfrak{n} \geq 2$ case by Smirnov [207].

Later, Gromov, Levkovich-Maslyuk and Sizov used exact computer-aided computations to conjecture and verify for small length chain the form of the spectrum of $\mathcal{B}(u)$ operators that would yield separate

[3] In the considered models, the $\mathcal{A}(u)$ and $\mathcal{B}(u)$ functions were polynomials in u or $e^{u\eta}$ for some $\eta \in \mathbb{C}$, making all the manipulations rather explicit.

variables from its operatorial roots for higher rank (and supersymmetric) models [178, 179]. They have not considered the question of the operator $\mathcal{A}(u)$, though.

In the 2010s, Maillet and Niccoli, in collaboration with Grosjean and Teschner used the Sklyanin SoV method to get great results over various quantum integrable models [176, 208–216]. For the $\mathfrak{gl}(2)$ models in particular, the complete SoV characterization of the transfer matrix spectrum allowed to prove the completeness of the Bethe Ansatz characterization, and to compute the form factors of local operators thanks to the separate representation of the eigenstates of the transfer matrix. This work has been pursued in collaboration with Terras and Kitanine towards the computation of correlation functions [217–220], with some very recent results in this domain [221, 222]. The open case boundary case was recently studied with Pezelier [223, 224].

A new idea emerged recently from this line of research: the construction of separate bases from conserved quantities [225]. A most general form of such bases is

$$\langle L| \prod_{j=1}^{N} \prod_{k_j=1}^{h_j} T\left(y_j^{(k_j)}\right),$$

where the $y_j^{(k_j)}$ are complex numbers and $T(u)$ is the transfer matrix generating the conserved quantities of the model at hand [see 225, Definition 2.2]. For example, for the $\mathscr{Y}(\mathfrak{gl}(\mathfrak{n}))$ fundamental models, the basis constructed from the powers of the transfer matrix evaluated in the inhomogeneity parameters ξ_j of the models

$$\langle S| \prod_{j=1}^{N} T\left(\xi_j\right)^{h_j},$$

where the h_j are integers between 0 and $n-1$, is shown to be separate provided weak restrictions on the inhomogeneities of the model, and that the boundary conditions are given by a twist matrix with simple spectrum. The possibility to construct such bases originates from the existence of cyclic vectors for simple spectrum matrices, or non-derogatory matrices [226]. There are in general many possible choices for the covector $\langle S|$; a useful one for quantum integrable lattice models is a tensor product form made up from the cyclic covectors of the twist matrix at each lattice site.

This construction of separate bases from the transfer matrix itself has some advantages to Sklyanin's approach of SoV. First, it bypasses the need for the identification and construction of the $\mathcal{A}(u)$ and $\mathcal{B}(u)$ operators[4], and in place relies on a fundamental object of quantum integrability, the transfer matrix. This is desirable, for it minimizes the number of different objects and the complexity of the SoV procedure, but also because the construction of proper $\mathcal{A}(u)$ and $\mathcal{B}(u)$ operators has been identified as a blocking point for the proper generalization of Sklyanin's SoV to higher rank model. Furthermore, with this construction, the knowledge of an eigenvalue fixes completely its unique associated eigenvector by its wave functions in the separate basis—one proves the transfer matrix has simple spectrum, thanks to the twist matrix having this property.

In this new take on quantum separation of variables, the *fusion relations* verified by the fused transfer matrices play a key role [111, 112, 128, 227–231]. They originate from the decomposition in irreducible components of the tensor products representations. First, they allow to compute the action of the transfer matrix in the separate bases. But ultimately they also provide a characterization of the complete spectrum of the transfer matrix, which can be put in the form of a quantum spectral curve by exhibiting the necessary Baxter Q-functions.

This novel SoV procedure has been shown to be applicable in a wide range of cases. It was shown to be applicable for higher spin representations [232], where Q operators are used to construct the separate

[4] We still need to identify their spectrum, which are the separate variables.

basis. The open boundary case of $\mathfrak{gl}(2)$ models was considered in [233]. Separate bases of this form were also constructed in the general $\mathscr{Y}(\mathfrak{gl}(n))$ [234] and $U_q(\widehat{\mathfrak{gl}(n)})$ [235] cases, and used to provide a characterization of the complete spectrum of the transfer matrix.

Article [225] triggered general progress on a SoV procedure based on conserved quantities. Ryan and Volin [236] proved that the $\mathbb{B}(u)$ operator—conjectured by Gromov, Levkovich-Maslyuk, and Sizov in [178] to be the generalization of Sklyanin's $\mathcal{B}(u)$ operator in the general $\mathfrak{gl}(n)$ case—produces in its roots the same separated variables obtained in [225], and moreover can serve as the correct creation operator for a Bethe Ansatz description of the eigenstates. However, construction of the $\mathcal{A}(u)$ operator realizing the proper shifts in the $\mathcal{B}(u)$ spectrum remained unaddressed, as well as the completeness of the spectrum. These results were obtained for a general class of representations, and rely on the construction of the eigenbasis of the $\mathbb{B}(u)$ operators by deformation of the Gelfand–Tsetlin basis of $\mathscr{Y}(\mathfrak{gl}(n))$. This construction was later extended in [237] where the connection with Bäcklund flow was made, and a construction of the conjugate momentum variables of the separate (coordinate) variables was proposed, in the form of Wronksian of Q-functions. In collaboration with Cavaglià, Gromov and Levkovich-Maslyuk, some progress was also made in the direction of correlation functions, with results on the SoV measure for higher rank models [238–240].

*
* *

The work introduced in this thesis is two folds:

i) The characterization of the SoV measure in the $\mathscr{Y}(\mathfrak{gl}(3))$ fundamental models, making a first step towards the computation of correlation functions by SoV in higher rank models. This is the subject of chapter 7.

ii) The extension of this SoV procedure to the supersymmetric integrable models, and how to tackle the spectral problem of the transfer matrix in this framework. This is the subject of chapter 8.

We need to introduce several topics in details before discussing the original works of the thesis presented in chapters 7 and 8. This is the role of chapters 2 to 6, which are devoted to the exposition of the necessary techniques and ideas for the manuscript to be essentially self-contained.

We introduce classical and quantum integrable models and their description in the classical and quantum inverse scattering formalisms in chapters 2 and 3. The Algebraic Bethe Ansatz techniques and its nested versions are detailed in chapter 4, as well as the results towards correlation function obtained in the $\mathfrak{gl}(2)$ and higher rank case. Next, we describe the separation of variables procedure in general in chapter 5. The classical and quantum SoV principles are introduced, and we discuss the Sklyanin's SoV construction for classical and quantum integrable models. In chapter 5, the idea of separate bases from conserved quantities is given in details for $\mathscr{Y}(\mathfrak{gl}(n))$ models, laying the necessary concepts and notations for the discussion of results of the last two chapters.

Two articles are attached to the main manuscript:

[LV1] J. M. Maillet, G. Niccoli, and L. Vignoli, "Separation of variables bases for integrable $gl_{\mathcal{M}|\mathcal{N}}$ and Hubbard models," *SciPost Phys.*, vol. 9, p. 60, 4 2020. DOI: 10.21468/SciPostPhys.9.4.060

[LV2] J. M. Maillet, G. Niccoli, and L. Vignoli, "On Scalar Products in Higher Rank Quantum Separation of Variables," *SciPost Phys.*, vol. 9, p. 86, 6 2020. DOI: 10.21468/SciPostPhys.9.6.086

They are the author's peer-reviewed contributions to the field, and contain additional lengthy proofs and details of the results featured in chapters 7 and 8.

Chapter 2 Classical integrability

There are several frameworks in which we can describe the configurations of a mechanical system and study its dynamics. The Hamiltonian formulation [34, 37, 48] is proved most useful in the context of integrability, as integrable models and the role of conserved quantities are best described in it, but also because quantization of Hamiltonian systems is a customary procedure.

The role of conserved quantities in the resolution of Hamiltonian systems is emphasized, with the introduction of Liouville integrability [47] and the Liouville–Arnol'd theorem [48]. We also discussed the privileged role of action–angle variables in this setting [34, 49].

We introduce the Lax formalism [49, 50, 63, 241] in details for mechanical systems and classical field theory. The benefit of the Lax formulation is to possibility to produce conserved quantities systematically form the spectral invariants of the Lax matrix. Then, we introduce the (classical) inverse scattering method (CISM) [49, 54, 96], whose goal is to produce action–angle variables analogs for integrable models on continuous space or on the lattice. The direct problem is the construction of such variables from the diffusion data, while the inverse problem consists in expressing the original dynamical variables in terms of the diffusion data. The notion of classical r-matrix [89–91] and the associated integrable structures are then discussed, setting the course to the quantized version of these objects.

2.1 Classical integrability

Hamiltonian formulation of classical mechanics Consider a mechanical system with N degrees of freedom. A state of the system is described by N coordinates defining a point in the N-dimensional configuration space $\mathcal{C}$. These are the generalized coordinates $q_1, \dots, q_N$. The phase space of the system is the $2N$-dimensional cotangent manifold $\mathcal{M} = T^*\mathcal{C}$ with canonical conjugate coordinates $(q_1, \dots, q_N, p_1, \dots, p_N)$, that is with the following Poisson brackets

$$\{q_i, q_j\} = 0 = \{p_i, p_j\} \quad \text{and} \quad \{q_i, p_j\} = \delta_{ij}. \tag{2.1}$$

More precisely, to describe globally the phase space, a canonical atlas of local canonical charts is required [48]. In the following, it is implied that q_i, p_i are local coordinates.

$\mathcal{M}$ has a symplectic structure fixed by a closed and non-degenerate symplectic 2-form

$$\omega = -\mathrm{d}\alpha = \sum_{i=1}^{N} \mathrm{d}q_i \wedge \mathrm{d}p_i\,, \tag{2.2}$$

where $\alpha = \sum_{i=1}^{N} p_i\,\mathrm{d}q_i$ is the canonical 1-form. The Poisson brackets are defined over $\mathcal{M}$ by ω: for $f, g \in C^1(\mathcal{M})$ and X_f, X_g the vector fields associated to f and g by $\mathrm{d}f = \omega(X_f, \cdot)$ and $\mathrm{d}g = \omega(X_g, \cdot)$, we have

$$\{f, g\} := \omega(X_f, X_g) = X_f(g) = -X_g(f). \tag{2.3}$$

In local coordinates, they write

$$\{f,g\} = \sum_{i=1}^{N} \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q_i}\frac{\partial f}{\partial p_i}, \tag{2.4}$$

and

$$X_f = \sum_{i=1}^{N} \frac{\partial f}{\partial q_i}\frac{\partial}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial}{\partial q_i}, \tag{2.5}$$

and the same holds for X_g. The fact that the q_i, p_j are canonical conjugated coordinates is seen in their Poisson brackets: $\{q_i,p_j\} = \delta_{ij}$ and $\{q_i,q_j\} = 0 = \{p_i,p_j\}$, $\forall i,j \in [\![1,N]\!]$.

The dynamics of the system is prescribed by a Hamiltonian function $H(q,p,t) \in C^1(\mathcal{M})$ such that the equations of the motion read

$$\begin{aligned} \dot{q}_i &= \{q_i,H\} = \frac{\partial H}{\partial p_i}, \\ \dot{p}_i &= \{p_i,H\} = -\frac{\partial H}{\partial q_i}. \end{aligned} \tag{2.6}$$

This is the Hamiltonian formulation of classical mechanics [34, 37]. From now on, we restrict to time-independent Hamiltonian for simplicity.

Solving (2.6) is the main goal in the study of mechanical systems; it can be a challenging task. It is a system of $2N$ first-order coupled differential equations. A reformulation of the problem in a new, appropriate set of canonical coordinates proves useful in most of the cases.

Change of coordinates Other sets of canonical variables

$$Q_i(q,p,t),\ P_i(q,p,t), \tag{2.7}$$

have the property that if the equations of the motion for (q,p) are Hamilton's canonical equations (2.6) with some Hamiltonian $H(q,p)$, then there exists a new Hamiltonian $H'(Q,P) \in C^1(\mathcal{M})$ such that

$$\begin{aligned} \dot{Q}_i &= \{Q_i,H'\} = \frac{\partial H'}{\partial P_i}, \\ \dot{P}_i &= \{P_i,H'\} = -\frac{\partial H'}{\partial Q_i}, \end{aligned} \tag{2.8}$$

so that H' serves as the Hamiltonian function in the new coordinates. Canonical coordinate transformations are characterized by a *generating function* which makes the bridge between the two sets of canonical coordinates (q,p) and (Q,P) [34, 37]. For a "type 2" generating function of the form $F(q,P,t)$ that might depend explicitly on time, we have

$$p_i = \frac{\partial F}{\partial q_i}, \quad Q_i = \frac{\partial F}{\partial P_i}, \tag{2.9}$$

$$H' = H + \frac{\partial F}{\partial t}. \tag{2.10}$$

Under such transformation $(q,p) \to (Q,P)$, the 1-form α is transform in α', that differs from α by an exact differential, so that the 2-form ω is actually invariant.

Constants of the motion and integrability The knowledge of constants of the motion is crucial in the construction of canonical coordinates adapted to the dynamics of the system at hand. It is the key feature of *Liouville integrable systems* to ensure their solvability from the knowledge of their conserved quantities.

Liouville Integrability. *A mechanical system with N degrees of freedom is Liouville integrable if it possesses N independent conserved quantities F_i in involution*

$$\{F_i, F_j\} = 0. \tag{2.11}$$

The independence of the functions F_i is defined by the linear independence of the associated vector fields X_{F_i} at each point of $\mathcal{M}$.

Remark 1. There cannot be more that N independent quantities in involution otherwise the Poisson bracket would be degenerate, so the Hamiltonian H is necessarily a function of the F_i.

The integrability property is backed by the powerful

Liouville–Arnol'd theorem ([48]). *The solution of the equations of the motion of a Liouville integrable system is obtained by quadrature.*

More precisely, there exists a coordinate transformation to some canonical variables (ψ_i, F_i) in which the time evolution is linear. The generating function $S(q, F)$ of the canonical coordinate transformation is computed by quadrature. If the level manifold $\mathcal{M}_f$ specified by the fixed value of the constants of the motion $F_i = f_i$ is compact and connected, then it is diffeomorphic to the N-dimensional torus.

Proof. The system being isolated from external forces, its evolution takes place over the level manifold $\mathcal{M}_f$ defined by the scalar values f_i of the conserved quantities

$$\mathcal{M}_f = \left\{ m = m(q,p) \in \mathcal{M} \;\middle|\; \forall i = 1, \dots, N, \quad F_i(q,p) = f_i \right\}. \tag{2.12}$$

Suppose that we can solve for the momenta on $\mathcal{M}_f$ with $p_i = p_i(q, f)$. We construct the function $S(q, F)$ by integrating the canonical 1-form on $\mathcal{M}_f$ from the initial point m_0 to a generic point m

$$S(q,F) := \int_{m_0}^{m} \alpha = \int_{q_0}^{q} \sum_i p_i(q,f)\, \mathrm{d}q_i \,. \tag{2.13}$$

One should check this object is well-defined. Independence and involution of the F_i prove that

$$\mathrm{d}\alpha\,|_{\mathcal{M}_f} = \omega|_{\mathcal{M}_f} = 0, \tag{2.14}$$

making α a closed form [see 49, p. 9]. By Stokes theorem [242], this ensures that the value of the integral (2.13) is unchanged by continuous deformation of the path $m_0 \to m$. However, in the generic case the topology of $\mathcal{M}_f$ is such that there are non-trivial cycles, so the function $S(q, F)$ is a priori multivalued (depending on the number of cycles made by the integration path). Still, S exists and is well-defined. We will now show it is the generating function of a canonical coordinate transformation $(q, p) \to (\psi, F)$. Defining

$$\psi_i = -\frac{\partial S}{\partial F_i}, \tag{2.15}$$

we have

$$\mathrm{d}S = -\psi_i\, \mathrm{d}F_i + p_i\, \mathrm{d}q_i \quad \text{or} \quad p_i\, \mathrm{d}q_i = \psi_i\, \mathrm{d}F_i + \mathrm{d}S\,, \tag{2.16}$$

where the summation over i is assumed, so the canonical 1-form in the two sets of coordinates coincides up to a differential. With $\omega = -\mathrm{d}\alpha$, because $\mathrm{d}^2 = 0$, the symplectic 2-from ω coincides in the two coordinates systems

$$\omega = \sum_i \mathrm{d}q_i \wedge \mathrm{d}p_i = \sum_i \mathrm{d}\psi_i \wedge \mathrm{d}F_i \,. \tag{2.17}$$

So the variables (ψ_i, F_i) are canonical coordinates and relations (2.9), (2.10) are satisfied. Since $\partial S/\partial t =$

0 the new Hamiltonian is simply the old one written in terms of the new coordinates

$$H'(\psi, F) = H\big(q(\psi, F), p(\psi, F)\big). \tag{2.18}$$

By (2.8) it is clear that H' depends only on the constants of the motion F_i. The constant value of the Hamiltonian $H(q,p) = H'(F)$ over the level manifold $\mathcal{M}_f$ is the energy E of the system

$$H(q,p)\big|_{q_0 \to q \subset \mathcal{M}_f} = H'(f_1, \dots, f_n) = E. \tag{2.19}$$

H' has no explicit dependence in the ψ_i; they are tagged as *ignorable coordinates*[1]. Noting

$$\nu_i(F) := \dot{\psi}_i = \{\psi_i, H'\} = \frac{\partial H}{\partial F_i} = \text{const}, \tag{2.20}$$

we have the following linear time evolution for the system at hand when written in the (ψ, F) canonical coordinates

$$\begin{aligned} F_i(t) &= f_i, \\ \psi_i(t) &= \psi_i(0) + \nu_i\, t. \end{aligned} \tag{2.21}$$

To obtain this solution, we had to perform a single—but curvilinear over a N-dimensional submanifold—integration to compute S, and invert the coordinate transformation to get the ν_i from the explicit expression (2.18) of the Hamiltonian H in terms of the ψ_i and F_i. Hence, we solved it *by quadrature*, and some additional algebraic manipulations.

For the proof of the topology of $\mathcal{M}_f$ under the suitable conditions, see [48]. □

Action–Angle variables Since under suitable conditions $\mathcal{M}_f$ is a torus, we can choose an even more appropriate set of coordinates than (ψ, F) to describe the motion: the action–angle variables (θ, I).

Choosing the values f_i of the F_i fixes the level-manifold $\mathcal{M}_f$ and thus fixes a particular torus in the foliation in tori of $\mathcal{M}$. We may choose any N independent functions $I_j = I_j(F_1, \dots, F_N)$ such that once the values $\mathcal{I}_j$ of the I_j are known, then $\mathcal{M}_f$ is determined. But since the N-dimensional torus T_n is isomorphic to the product of N circles C_j, we define the *action variables* I_j as the integrals of the canonical 1-form over the closed cycles C_j

$$I_j = \frac{1}{2\pi} \oint_{C_j} \alpha = I_j(F_i). \tag{2.22}$$

Using the same generating function as before, but expressed in terms of the I_j

$$S(q, I) = \int_{m_0}^{m} \alpha, \tag{2.23}$$

the new canonical variables in the coordinate transformation characterized by S are the

$$I_j \quad \text{and} \quad \theta_k = \frac{\partial S}{\partial I_k}, \quad 1 \le j, k \le N. \tag{2.24}$$

[1] It remains useful to characterizes their dynamic to get the ones of the original coordinates (q, p).

By definition of θ_k,

$$\begin{aligned}\oint_{C_j} \mathrm{d}\theta_k &= \frac{\partial}{\partial I_k}\oint_{C_j} \mathrm{d}S = \frac{\partial}{\partial I_k}\oint_{C_j}\sum_i \frac{\partial S}{\partial q_i}\,\mathrm{d}q_i + \frac{\partial S}{\partial I_i}\,\mathrm{d}I_i \\ &= \frac{\partial}{\partial I_k}\oint_{C_j}\sum_i p_i\,\mathrm{d}q_i + 0 = \frac{\partial}{\partial I_k}\oint_{C_j}\alpha \\ &= 2\pi\frac{\partial I_j}{\partial I_k} = 2\pi\delta_{jk},\end{aligned} \tag{2.25}$$

since $\mathrm{d}I_j = 0$ on $\mathcal{M}_f \supset C_j$. So the θ_k are the *angle variables* along the cycle C_j conjugated to the action variable I_j.

Same as before, the evolution's equations in the action–angle variables write

$$\begin{aligned}I_j(t) &= \mathcal{I}_j, \\ \theta_k(t) &= \theta_k(0) + \omega_k t,\end{aligned} \tag{2.26}$$

where the $\omega_k = \omega_k(\mathcal{I}_j)$ are the frequencies of the periodic motion along each cycle C_k of the level manifold $\mathcal{M}_f$.

Action–Angle variables for the harmonic oscillator

Consider the one-dimensional harmonic oscillator of mass m and natural frequency $\omega = k/m$, k being the stiffness of the spring. It is a classical integrable system with a phase space of dimension 2 and canonical coordinates (q,p), respectively the position and momentum of the mass. Its Hamiltonian is

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2. \tag{2.27}$$

It is trivially Liouville integrable, since the energy E is a conserved quantity in this isolated system. Its value fixes a curve in the phase space on which the system evolves

$$E = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2 \quad \text{or} \quad p = \pm\sqrt{2mE - m^2\omega^2 q^2}, \tag{2.28}$$

so the level manifolds $\mathcal{M}_E$ are ellipses centered on $(0,0)$ and foliate the phase space for E in the range $[0,+\infty[$. From equations (2.13) and (2.22), the only action variable of the system is obtained by integrating the momentum over the cycle of the level manifold $\mathcal{M}_E$

$$J := \oint_{\mathcal{M}_E} p\,\mathrm{d}q\,. \tag{2.29}$$

With a polar parametrization of the integration path, the substitution

$$q = \sqrt{\frac{2E}{m\omega^2}}\sin\phi \tag{2.30}$$

reduces the integral to

$$J = \frac{2E}{\omega}\int_0^{2\pi}\cos^2\phi\,\mathrm{d}\phi = \frac{2\pi E}{\omega}. \tag{2.31}$$

Unsurprisingly, the action variable is the rescaled energy of the system, and the Hamiltonian may be

rewritten $H = \omega J/2\pi$. Eventually, the original coordinates take the form

$$q = \sqrt{\frac{J}{\pi m\omega}}\sin\theta, \tag{2.32}$$

$$p = \sqrt{\frac{mJ\omega}{\pi}}\cos\theta, \tag{2.33}$$

which effectively solves the dynamics of the system by providing the explicit time evolutions of the original coordinates. Also,

$$J = \frac{\pi}{m\omega}\left(p^2 + m^2\omega^2 q^2\right), \tag{2.34}$$

$$\theta = \arctan\left(m\omega\,\frac{q}{p}\right), \tag{2.35}$$

and one verifies $\{J,\theta\} = 2\pi$, so the action–angle (canonical) variables are $(J/2\pi\,,\theta)$ (a $1/2\pi$ normalization in the contour integral (2.29) would give the correct action straight away). The angle θ evolves linearly in times

$$\theta(t) = \theta_0 + \nu t \quad \text{given} \quad \nu = \frac{\partial H}{\partial \frac{J}{2\pi}} = \omega = \sqrt{\frac{k}{m}}. \tag{2.36}$$

One recovers the well-known formula for the pulsation of the oscillation. Here, the θ variable finds a direct physical interpretation as being the phase of the position, so it comes as no surprise that its frequency matches the physical one.

Computing the coordinate transformation to canonical coordinates made from constants of the motion, which are *pre-action–angle variables*, requires to compute a curvilinear integral over a curve lying on the level manifold. This is in general a non-trivial calculation. In practice, the integral (2.13) is tractable only when the variables are *separate*: it can then be reduced to a sum of one-variable independent integrals. This substantially alleviates the computational difficulty of the problem. Hence, developping tools to effectively separate the variables in integrable systems has been of extreme interest, and Quantum Separation of Variables (SoV) is the core subject of this thesis. We will describe such tools in chapters 5 and 6. It requires first to develop a more algebraic understanding of classical and quantum integrable models, that we will detail in the following sections. For now, we pursue the description of classical integrable models and the algebraic tools developed to study them systematically.

2.2 Classical Lax formalism and r-matrix

As we saw with the Liouville–Arnol'd theorem, conserved quantities are at the center of the description of integrable models. Any studies of integrable models should therefore be committed to identify clearly the constants of the motion at the earliest stage. Modern studies of integrable mechanical systems goes along with the Lax formalism, first developed for solving differential equations of continuous systems [50, 55, 56] and then extended to the general Hamiltonian description of classical mechanics [57, 63].

The seminal idea is to recast the equations of the motion into a commutator of two matrices, the Lax pair (L,V). By doing so, conserved quantities of the system are found in the spectral invariants of L, called the Lax matrix.

Lax pair. *Let $\mathcal{M}$ be the $2N$-dimensional phase space of a mechanical system whose dynamics is given by the Hamiltonian H independent of the time. Two $n \times n$ square matrices $L(m,u,t)$ and $V(m,u,t)$, with*

$(m,u) \in \mathcal{M} \times \mathbb{C}$, form a Lax pair if they rewrite the equations of the motion as

$$\frac{\mathrm{d}L}{\mathrm{d}t} = \{L, H\} = [V, L]. \tag{2.37}$$

$L(m,u,t)$ is the Lax matrix. The variable u is called the spectral parameter. *The space $\mathbb{V}_0 := \mathbb{C}^n$, $n \in \mathbb{Z}_{\leq 2}$, on which the matrix L and V are constructed is called the* auxiliary space. *We often omit the time and phase space dependency and simply write $L(u)$, $V(u)$.*

Note that for a given system, the Lax pair—if it exists—may not be unique. The dimension of the L and V matrices may even be different for different pairs associated to the same system. The addition of a spectral parameter is necessary to ensure the spectral invariants contains all the conserved quantities of the model, see the examples of tops in chapter 2 of [49]. In many integrable systems, the spectral parameter u is a complex number. However, it could be a more sophisticated object depending on the space-time symmetries of the system at hand. For example, it is a twistor in self-dual Yang–Mills theories [243, 244].

Set of commuting conserved quantities The interest of the Lax construction is the easy access to the conserved quantities of the system.

Indeed, the spectral invariants of the Lax matrix

$$\forall j \in [\![1, \dim \mathbb{V}_0]\!], \quad F_j(u) := \mathrm{tr}_{\mathbb{V}_0}\left(L_0(u)^j\right), \tag{2.38}$$

are integrals of the motion. The proof is done by direct computation of the flow of the F_j under the Hamiltonian. Omitting the spectral dependence in u, one has

$$\begin{aligned} \frac{\mathrm{d}F_j}{\mathrm{d}t} &= \left\{F_j, H\right\} = \mathrm{tr}_{\mathbb{V}_0}\left\{L_0^j, H\right\} \\ &= \mathrm{tr}_{\mathbb{V}_0}\left(\sum_{k=0}^{j-1} L_0^{j-1-k}\{L_0, H\}L_0^j\right) = j\,\mathrm{tr}_{\mathbb{V}_0}\left(L_0^{j-1}\{L_0, H\}\right) \\ &= j\,\mathrm{tr}_{\mathbb{V}_0}\left(L_0^{j-1}[V_0, L_0]\right) = 0, \end{aligned} \tag{2.39}$$

where we used the trace identify $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ over the auxiliary space.

Also, note that matrices of the form

$$L(u,t) = g(t)L(u,0)g(t)^{-1}, \tag{2.40}$$

where the invertible $g(t)$ matrix evolves by the equation

$$\frac{\mathrm{d}g}{\mathrm{d}t} = V(u,t)g(u,t) \quad \text{with} \quad g(u,0) = \mathrm{id}, \tag{2.41}$$

are solution to equation (2.37) Therefore, any function $F(L(u))$ invariant under the conjugation by $g(t)$ is a constant of the motion. Such functions are the eigenvalues of the Lax matrix $L(u)$, so the spectrum of L is preserved over time and L time evolution is said isospectral. The conserved charges F_j obtained above are symmetric functions of the eigenvalues of $L(u)$, so it is no surprise that they are conserved.

r-matrix The Lax pair is a convenient way to construct constants of the motion without referring to the Poisson structure of the phase space. But for Liouville integrability of the system, we need to have them independent and in involution. The discussion of the involution of the integrals of the motion F_j is tied to the existence of an r-matrix, which encodes a general form of Poisson bracket between the matrix elements of L that ensures the involution property of the conserved quantities [49, 62, 63, 89, 90, 241].

Theorem 1 (Babelon & Viallet, see [49, 241]). *The involution property of the spectral invariants of L is equivalent to the existence of a matrix $r_{12}(u,v|m)$ acting on $\mathbb{V}_0 \times \mathbb{V}_0$ whose matrix elements are function on $\mathcal{M}$, such that*

$$\{L_1(m,u), L_2(m,v)\} = [r_{12}(m|u,v), L_1(m,u)] - [r_{21}(m|v,u), L_2(m,v)]. \tag{2.42}$$

The indices 1 and 2 in L and r label two distinct copies of the auxiliary space $\mathbb{V}_0$.

The Poisson braket between the two Lax matrices has to be understood as the matrix whose coefficients are the Poisson brackets between the matrix elements of L_1 and L_2:

$$\{L_1(m,u), L_2(m,v)\} = \sum_{i,j,k,\ell} \big\{L_{ij}(m,u), L_{k\ell}(m,v)\big\} e_{ij} \otimes e_{k\ell}. \tag{2.43}$$

Most of the time, the dependence in $m \in \mathcal{M}$ is suppressed from the notation. The proof that the involution implies the existence of the r-matrix is done by direct construction of it, while for the converse way the Poisson brackets of the invariants of the Lax matrices write as trace over commutators, which vanish [49]. Integrability is then ensured once the independence of N spectral invariants of L is proven, $N \leq n = \dim \mathbb{V}_0$, which is now a model-dependent problem.

Lax pair and r-matrix of the harmonic oscillator

The Lax pair of the harmonic oscillator (2.27) with $m = 1$ are the 2×2 matrices

$$L = \begin{pmatrix} p & m\omega q \\ m\omega q & -p \end{pmatrix}, \quad V = \begin{pmatrix} 0 & -\omega/2 \\ \omega/2 & 0 \end{pmatrix}. \tag{2.44}$$

It is not necessary to introduce a spectral parameter here. One checks easily that the Lax equation $\mathrm{d}L/\mathrm{d}t = [V, L]$ is equivalent to the equations of the motion $\dot{q} = p/m$, $\dot{p} = -m\omega^2 q$. We also observe that the Hamiltonian can be written $\frac{1}{4m}\operatorname{tr}\big(L^2\big)$. The matrix

$$r = -\frac{\omega}{4E}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes L, \tag{2.45}$$

with $E = p^2/2m + m\omega^2 q^2/2$, is such that the relation (2.42) holds with the above L. For example, the equality of the $(1,2)$ matrix elements in equation (2.42) specialized to the above matrices computes from its left-hand side the Poisson bracket $\{p,q\}$, and the right-hand side is

$$-\frac{\omega}{4E}\big(2m^2\omega^2 q^2 + 2p^2\big) = -1, \tag{2.46}$$

which is the expected value of $\{p,q\}$ from the equations (2.1), (2.4). Note that the r-matrix is dynamical (it depends on the q and p variables).

Classical Yang–Baxter equation What are the solutions (L,r) of (2.42) such that the induced Poisson bracket indeed defines a symplectic structure on $\mathcal{M}$? This mainly amounts to the verification of the skew-symmetry property and the Jacobi identity for the Poisson bracket [49]. Skew-symmetry is obvious from (2.42), while the Jacobi identity may hold only after imposing some conditions on L and r. Constraints on L and r are decoupled when restricting to r-matrices that are constant on $\mathcal{M}$, i.e. which

have scalar entries. In particular, the Jacobi identity is satisfied if a constant r-matrix satisfies

$$[r_{12}(u,v),r_{13}(u,w)]+[r_{12}(u,v),r_{23}(v,w)]+[r_{32}(w,v),r_{13}(u,w)]=0, \tag{2.47}$$

When r is antisymmetric as $r_{12}(u,v)=-r_{21}(v,u)$, this is called the *classical Yang–Baxter equation*.

Unitary antisymmetric solutions of difference type to the classical YBE are constant r-matrices verifying

$$\begin{aligned} r_{12}(u,v)&=-r_{21}(v,u),\\ r_{12}(u,v)&=r_{12}(u-v). \end{aligned} \tag{2.48}$$

They are associated to the Poisson algebra with a linear in L right-hand side

$$\{L_1(u),L_2(v)\}=[r_{12}(u-v),L_1(u)+L_2(v)], \tag{2.49}$$

but also to another symplectic structure called the Skylanin's quadratic Poisson algebra

$$\{L_1(u),L_2(v)\}=[r_{12}(u-v),L_1(u)L_2(v)], \tag{2.50}$$

which is quadratic in L right-hand side, leading to a different class of Lax matrices and integrable models[2].

The classical numeric r-matrices satisfying (2.47) have been extensively studied and classified in families labelled by Lie algebras representation theory [112, 245]. A notable solution is the one associated to the $\mathfrak{gl}(\mathfrak{n})$ algebra, $\mathfrak{n}\geq 2$, with its fundamental vector representation taken for the spaces $\mathbb{V}_{1,2}$

$$r_{12}(u)=\frac{c}{u}\mathcal{P}_{12},\quad c\in\mathbb{C}, \tag{2.51}$$

where $\mathcal{P}$ is the permutation operator on $\mathbb{V}\otimes\mathbb{V}$; with e_{ij} the $\mathfrak{n}\times\mathfrak{n}$ coordinate matrices, it writes

$$\mathcal{P}=\sum_{i,j=1}^{\mathfrak{n}} e_{ij}\otimes e_{ji}. \tag{2.52}$$

Classical Inverse Scattering This discussion extends to classical field theory with a space-dependent Lax pair $(L(u,x,t),V(u,x,t))$, where x is the space coordinate. It rationalizes the Classical Inverse Scattering Method (CISM) originally designed by Gardner, Greene, Kruksal and Miura [54] under an L and r-matrix construction.

Considering a non-linear partial differential equation for a classical field $\psi(x,t)$ on the real line of the form

$$K\left(\psi,\frac{\partial\psi}{\partial t},\frac{\partial\psi}{\partial x}\right)=0, \tag{2.53}$$

we suppose there exists a Lax pair such that it can be recast in

$$[\partial_x-L,\partial_t-V]=0, \tag{2.54}$$

which is the continuous equivalent of (2.37). This allows to interpret the equations of the motion written in the form (2.54) as the compatibility condition—or zero-curvature equation—of the linear auxiliary system

$$\begin{cases} \partial_x\chi=L\chi,\\ \partial_t\chi=V\chi. \end{cases} \tag{2.55}$$

[2]It is possible to rewrite this quadratic algebra in a linear form and recover Poisson bracket of the form (2.42), but the r-matrix is then, obviously, a different one. In particular, it is non-constant.

The study of the partial differential equation (2.53) then reduces [90, 95] to the study of the spectral characteristics of the spatial part of the Lax pair, the linear differential operator $\partial_x - L(x,u)$.

Let us show in more details how this is done. Consider two points A and B of the two-dimensional space-time (x,t). Let $\mathcal{A}_x = L$ and $\mathcal{A}_t = V$. The Wilson line element between points A and B is the ordered exponential

$$W_{BA} = \overleftarrow{\exp}\int_A^B \mathcal{A}_\mu \,\mathrm{d}x^\mu\,, \tag{2.56}$$

with $\mathcal{A}_\mu = \mathcal{A}_\mu(x,t,u)$. For a spacetime with trivial topology, this Wilson line is independent of the integration path chosen between A and B, so that (2.54) is a zero-curvature equation for the $\mathcal{A}_\mu$. Consider now $A=(x_A,t)$ and $B=(x_B,t)$ two points on the same surface $t=$ cste of the space-time. Wilson line elements are multiplicative along the integration path so for two other points $A'=(x_{A'},t')$, $B'=(x_{B'},t')$ on another constant time surface, $t'\neq t$, one has

$$W_{BA}(t,u) = W_{BB'}W_{B'A'}(t',u)W_{A'A}. \tag{2.57}$$

For a system periodic in the box $[0,y]$, i.e. $\mathcal{A}_\mu(y,t,u) = \mathcal{A}_\mu(0,t,u)$ for $\mu = x,t$, the space-time has the geometry of a cylinder and the closed Wilson line between points $A=(0,t)$ and $B=(y,t)\equiv A$ is called the *monodromy* matrix[3]

$$M(u,t) := W_{AA}^{\circlearrowleft} = \overleftarrow{\exp}\oint_A^A L(x,t,u)\,\mathrm{d}x\,. \tag{2.58}$$

Then $W_{A'A} = W_{BB'}^{-1}$ due to spatial periodic boundary conditions and

$$W_{BA}(t,u) = W_{BB'}W_{B'A'}(t',u)W_{BB'}^{-1}. \tag{2.59}$$

With the partial differentials

$$\frac{\partial}{\partial x_B}W_{BA} = L(x_B,t_B,u)W_{BA}, \qquad \frac{\partial}{\partial x_A}W_{BA} = -W_{BA}L(x_A,t_A,u), \tag{2.60}$$

$$\frac{\partial}{\partial t_B}W_{BA} = V(x_B,t_B,u)W_{BA}, \qquad \frac{\partial}{\partial t_A}W_{BA} = -W_{BA}V(x_A,t_A,u), \tag{2.61}$$

and $t' = t+\varepsilon$, $\varepsilon\to 0$, from (2.59) one computes

$$\frac{\mathrm{d}M_{BA}}{\mathrm{d}t} = [V_A(0,t,u), M_{BA}(t,u)], \tag{2.62}$$

which is the Lax equation (2.37) for the monodromy matrix. Then, by a computation similar to (2.39)

$$\forall k\in\mathbb{N}, \quad \frac{\mathrm{d}}{\mathrm{d}t}\operatorname{tr}\big(M(t,u)^k\big) = 0, \tag{2.63}$$

that is the spectral invariants of the monodromy matrix are constants of the motion. In the case of twisted boundary conditions

$$\mathcal{A}_\mu(y,t,u) = K(u)^{-1}\mathcal{A}_\mu(0,t,u)K(u), \tag{2.64}$$

it is the *twisted monodromy* $M^{(K)}(u) = K(u)M(u)$ which gets a derivative in the form of a commutator

$$\frac{\mathrm{d}}{\mathrm{d}t}(K(u)M(t,u)) = [V(0,t,u), K(u)M(t,u)], \tag{2.65}$$

[3] A monodromy studies how objects behave as they run around a singularity.

and

$$\forall k \in \mathbb{N}, \quad \frac{\mathrm{d}}{\mathrm{d}t} \operatorname{tr}\big((K(u)M(t,u))^k\big) = 0. \tag{2.66}$$

Just as for point-like mechanical systems, it is possible to introduce a r-matrix to pack efficiently the Poisson brackets of the monodromy entries in a commutator form. Linear Poisson brackets for L reads

$$\{L_1(u,x,t), L_2(y,v,t)\} = \delta(x-y)\left[r_{12}(u,v), L_1(u,x,t) + L_2(y,t',v)\right], \tag{2.67}$$

and leads [90] to quadratic Poisson brackets for M

$$\{M_1(t,u), M_2(t,v)\} = [r_{12}(u,v), M_1(t,u)M_2(t,v)], \tag{2.68}$$

with $r(u,v)$ antisymmetric and satisfying the Yang–Baxter equation (2.47).

Then the transfer matrix

$$T(u) := \operatorname{tr} M(u) \tag{2.69}$$

defines a family of conserved quantities. Indeed, thanks to (2.68), it is trivial to show that the transfer matrix commutes with itself and generates a family of conserved quantities in involution as the coefficients in its expansion in u. One has,

$$\{T(u), T(v)\} = \operatorname{tr}_{12}\{M_1(u), M_2(v)\} = \operatorname{tr}_{12}[r_{12}(u,v), M_1(u)M_2(v)] = 0. \tag{2.70}$$

This computation extends to the trace of higher powers of M they are all in involution. If this generates enough conserved quantities, the system is integrable.

Consider now the 2×2 case and suppose V is diagonal. Noting

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad V = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}, \tag{2.71}$$

then (2.62) gives

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}a = 0, && \frac{\mathrm{d}}{\mathrm{d}t}d = 0 \\ &\frac{\mathrm{d}}{\mathrm{d}t}b = (\alpha-\beta)b, && \frac{\mathrm{d}}{\mathrm{d}t}c = (\beta-\alpha)c. \end{aligned} \tag{2.72}$$

Hence, a and d are constants of the motion, while $\ln(b)$ and $\ln(c)$ have linear time evolution: they open the way to obtain action–angle variables. More generally, if V is a constant matrix, the equations (2.72) give a linear time evolution for the monodromy matrix M, which are easily solved. For example, for systems over the real line $\mathbb{R}$ with specific boundary conditions at $\pm\infty$ such that $V \to$ cste, the computation of the scattering data by the monodromy matrix M is a direct transformation towards action–angle variables. Now, the main question is to obtain the inverse transformation, that is to solve explicitly the inverse problem by computing the dynamical variables in terms of the scattering data (the monodromy matrix entries) to obtain the solution of the original PDE (2.53). This is a non-trivial computation. It can be obtained by solving the Gelfand–Levitan–Marchenko linear integral equation, which requires the knowledge of the scattering data [58, 59]. The inverse scattering problem is also reducible to a Riemann–Hilbert factorization problem [49, 60, 246, 247].

Systems on the lattice The CISM works similarly for discrete theories on the 1-dimensional lattice with N sites. The discrete analogue of the Lax matrix now depends on the site index $n \in [\![1, N]\!]$ rather than the continuous space coordinate x. We note $L_{ab}(u)$ the Lax matrix of auxiliary space $\mathbb{V}_a$ with its entries acting over $\mathbb{V}_b$. It is instructive to see the Lax matrix $L_{0n}(u,t)$ as analogous to a monodromy matrix (2.58) calculated over a segment $[x_n, x_n + \Delta]$ where Δ is the lattice site spacing. Consequently,

the Poisson brackets for L at site n are quadratic [90, 248]

$$\big\{L_{1n}(n,u,t),L_{2n}(m,v,t')\big\}=\delta_{nm}\,[r_{12}(u-v),L_{1n}(n,u,t)L_{2n}(m,u,t)]. \tag{2.73}$$

Note that more general quadratic brackets can be introduced [248, 249]. The monodromy is now the ordered product along the lattice sites

$$M_0(u,t)=L_{0N}(u,t)\dots L_{01}(u,t), \tag{2.74}$$

and has quadratic Poisson brackets as well

$$\big\{M_1(u,t),M_2(v,t')\big\}=\,[r_{12}(u-v),M_1(u,t)M_2(u,t)]. \tag{2.75}$$

Similarly, the transfer matrix construct a family of quantities in involution, which are constants of the motion provided the Hamiltonian writes as a function of them. Should the Lax construction provides "enough" independent conserved quantities in involution, the system will be integrable.

As we will see soon with quantum spins chains, the structure for quantum integrable lattice models is very similar.

The classical $\mathfrak{sl}(2)$ spin-s chain [63, 187, 250, 251]

Consider classical $\mathfrak{sl}(2)$-spins at each site of a one-dimensional periodic lattice with N sites. Their angular momenta are represented by the vectors

$$\vec{\mathbf{S}}_n=\begin{pmatrix}S_n^x\\S_n^y\\S_n^z\end{pmatrix}\in R^3,\quad \big\|\vec{\mathbf{S}}_n\big\|=(S_n^x)^2+(S_n^y)^2+(S_n^z)^2=s^2. \tag{2.76}$$

The Poisson brackets between the components are

$$\big\{S_n^\alpha,S_m^\beta\big\}=\delta_{n,m}\varepsilon_{\alpha\beta\gamma}S^\gamma, \tag{2.77}$$

where $\varepsilon_{\alpha\beta\gamma}$ is the Levi–Civita symbol. For the Hamiltonian

$$H=-2\sum_{n=1}^{N}\ln\left(\frac{s^2+\vec{\mathbf{S}}_n\cdot\vec{\mathbf{S}}_{n+1}}{2s^2}\right), \tag{2.78}$$

the equations of the motion are

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{\mathbf{S}}_n=\big\{\vec{\mathbf{S}}_n,H\big\}=\frac{2\,\vec{\mathbf{S}}_n\wedge\vec{\mathbf{S}}_{n+1}}{s^2+\vec{\mathbf{S}}_n\cdot\vec{\mathbf{S}}_{n+1}}+\frac{2\,\vec{\mathbf{S}}_n\wedge\vec{\mathbf{S}}_{n-1}}{s^2+\vec{\mathbf{S}}_n\cdot\vec{\mathbf{S}}_{n-1}}. \tag{2.79}$$

They rewrite as the compatibility condition of the site n Lax pair

$$L_{0n}(u)=u+c\mathsf{S}_{0n}, \tag{2.80}$$

$$V_{0n}(u)=\frac{v_{0n}^+}{s+\frac{u}{c}}-\frac{v_{0n}^-}{s-\frac{u}{c}}, \tag{2.81}$$

where the S_{0n} matrix packs the spins components at site n in a $\mathfrak{sl}(2)$ matrix on the auxiliary space $\mathbb{V}_0=\mathbb{C}^2$

of the fundamental representation of $\mathfrak{sl}(2)$

$$\mathsf{S}_{0n} = \vec{\mathbf{S}}_n \cdot \vec{\sigma}_0 = \sum_{\alpha=x,y,z} S_n^\alpha \sigma_0^\alpha = \begin{pmatrix} S_n^z & S_n^x - iS_n^y \\ S_n^x + iS_n^y & -S_n^z \end{pmatrix}_0 = \begin{pmatrix} S_n^z & S_n^- \\ S_n^+ & -S_n^z \end{pmatrix}_0, \tag{2.82}$$

and the coefficients appearing in the V-matrix

$$v_{0n}^{\pm} = \frac{is^2}{s^2 + \vec{\mathbf{S}}_n \cdot \vec{\mathbf{S}}_n} \left(1 \pm \frac{\mathsf{S}_{0n}}{s}\right)\left(1 \pm \frac{\mathsf{S}_{0n-1}}{s}\right), \tag{2.83}$$

are such that the residues at the poles $\pm c/s$ cancel. Let us carry the derivation explicitly: from the compatibility condition

$$\frac{\mathrm{d}L_{0n}}{\mathrm{d}t} = V_{0n+1}(u)L_{0n}(u) - L_{0n}(u)V_{0n}(u), \tag{2.84}$$

which is the discrete analog of (2.62), the differential equation for the time evolution of S_{0n} is obtained at $u = 0$

$$c\frac{\mathrm{d}\mathsf{S}_{0n}}{\mathrm{d}t} = \frac{c}{s}\big(v_{0n+1}^+ - v_{0n+1}^-\big)\mathsf{S}_{0n} - \frac{c}{s}\mathsf{S}_{0n}\big(v_{0n}^+ - v_{0n}^-\big). \tag{2.85}$$

Since the product of S_{0k} matrices along the auxiliary space is given by

$$\mathsf{S}_{0k}\mathsf{S}_{0\ell} = \vec{\mathbf{S}}_k \cdot \vec{\mathbf{S}}_\ell + i\big(\vec{\mathbf{S}}_k \wedge \vec{\mathbf{S}}_\ell\big) \cdot \vec{\sigma}_0 \tag{2.86}$$

it gives

$$\begin{aligned}\frac{\mathrm{d}\mathsf{S}_{0n}}{\mathrm{d}t} = {} & \frac{2i}{s^2 + \vec{\mathbf{S}}_n \cdot \vec{\mathbf{S}}_{n+1}}\Big(s^2 + \vec{\mathbf{S}}_n \cdot \vec{\mathbf{S}}_{n+1} + i\big(\vec{\mathbf{S}}_{n+1} \wedge \vec{\mathbf{S}}_n\big) \cdot \vec{\sigma}_0\Big) \\ & - \frac{2i}{s^2 + \vec{\mathbf{S}}_n \cdot \vec{\mathbf{S}}_{n-1}}\Big(s^2 + \vec{\mathbf{S}}_n \cdot \vec{\mathbf{S}}_{n-1} + i\big(\vec{\mathbf{S}}_n \wedge \vec{\mathbf{S}}_{n-1}\big) \cdot \vec{\sigma}_0\Big),\end{aligned} \tag{2.87}$$

or

$$\frac{\mathrm{d}\vec{\mathbf{S}}_n}{\mathrm{d}t} \cdot \vec{\sigma}_0 = \left(\frac{2\,\vec{\mathbf{S}}_n \wedge \vec{\mathbf{S}}_{n+1}}{s^2 + \vec{\mathbf{S}}_n \cdot \vec{\mathbf{S}}_{n+1}} + \frac{2\,\vec{\mathbf{S}}_n \wedge \vec{\mathbf{S}}_{n-1}}{s^2 + \vec{\mathbf{S}}_n \cdot \vec{\mathbf{S}}_{n-1}}\right) \cdot \vec{\sigma}_0, \tag{2.88}$$

which are exactly the equations of the motion (2.79), once one identifies the coefficients of the auxiliary space $\mathfrak{sl}(2)$ generators $\sigma_0^{x,y,z}$.

Having a Lax pair, we now describe how to derive the constants of the motion from it. The Poisson brackets between the S_{0n} real coefficients $\mathsf{S}_{ij}^{(n)}$, $\mathsf{S}_{0n} = \mathsf{S}_{ij}^{(n)} e_{ij}^{(0)}$, are computed to be

$$\big\{\mathsf{S}_{ij}^{(n)}, \mathsf{S}_{k\ell}^{(m)}\big\} = i\delta_{n,m}\big(\mathsf{S}_{i\ell}^{(n)}\delta_{jk} - \mathsf{S}_{jk}^{(n)}\delta_{i\ell}\big), \tag{2.89}$$

so that the Poisson brackets between the Lax pairs coefficients (of the same site Lax matrix, the other vanish) admits the quadratic form

$$\big\{L_{ij}(u), L_{k\ell}(v)\big\} = \frac{ic}{u-v}\big(L_{kj}(u)L_{i\ell}(v) - L_{kj}(v)L_{i\ell}(u)\big). \tag{2.90}$$

One can put this formula into the compact form

$$\{L_{an}(u), L_{bn}(v)\} = [r_{ab}(u-v), L_{an}(u)L_{bn}(v)], \tag{2.91}$$

with the r-matrix

$$r_{ab}(u-v) = \frac{ic}{u-v}\mathcal{P}_{ab}. \tag{2.92}$$

Constructing the monodromy

$$M_0(u) = L_{0N}(u)\dots L_{01}(u), \tag{2.93}$$

the Poisson brackets (2.90) replicates themselves for $M_0(u)$ and one has

$$\{M_a(u), M_b(u)\} = [r_{ab}(u-v), M_a(u)M_b(v)]. \tag{2.94}$$

The transfer matrix

$$T(u) = \mathrm{tr}_0\left(M_0(u)\right), \tag{2.95}$$

is in involution with itself

$$\{T(u), T(v)\} = 0, \tag{2.96}$$

which is seen easily from the r-matrix form of the brackets (2.94). Thus its N subdominant coefficients in its u polynomial expansion forms a family in involution. The Hamiltonian H is some function of the J_k. Because

$$\frac{\mathrm{d}M(u)}{\mathrm{d}t} = [V_{01}(u), M(u)], \tag{2.97}$$

it is easy to check that the

$$J_k := \mathrm{tr}_0\left(M_0(u)^k\right), \quad 1 \le k \le N \tag{2.98}$$

are N constants of the motion—in particular, $T(u)$ is.

Generating integrable systems In the above discussion, we started from an integrable mechanical system and formalized the description of its integrable features thanks to the idea of the Lax pair. We constructed a Lax matrix that embed the conserved quantities and a r-matrix specifying the Poisson brackets of its entries. The natural question is now: *can we walk the converse path?* More precisely, having classified the r-matrices, can we generate integrable models upon request?

The answer is yes. Somehow, the r matrix contains the structure constants of the Poisson bracket algebra for the monodromy matrix elements, with the Yang–Baxter equation for r ensuring the Jacobi identify. Then finding a solution M (or L) of (2.75) is a problem of representation theory of the algebra defined by (2.75). See [248, 252] for additional discussion about this.

In short, from the knowledge of a r-matrix and an associated monodromy matrix M, it is sufficient to pick the Hamiltonian as some function of the spectral invariants, which have been shown to form a commuting family in involution. Besides, the equations of the motion have a Lax form, where the V matrix is constructed from r and M.

Let us be more specific. Consider $r(u)$ be an antisymmetric solution of difference form of the classical Yang–Baxter equation, and $M(u)$ a $n \times n$ monodromy matrix solution to (2.68) on a given $2n$-dimensional phase space $\mathcal{M}$ and n-dimension auxiliary space $\mathbb{V}_0$. We can define the Hamiltonians $F_k(u)$ defined form the spectral invariants of the monodromy matrix

$$F_k(u) := \mathrm{tr}\left(M(u)^k\right) \quad \text{for all} \quad k \in [\![1, n]\!]. \tag{2.99}$$

They are all in involution

$$\forall k, \ell \in [\![1, n]\!], \quad \{F_k(u), F_\ell(v)\} = 0. \tag{2.100}$$

The associated equations of the motion of the monodromy under the flow of these Hamiltonians admit a

Lax form

$$\frac{\mathrm{d}M(u)}{\mathrm{d}t_k} = \{M(u), F_k(v)\} = \big[V^{(k)}(u,v), M(u)\big], \tag{2.101}$$

where

$$V^{(k)}(u,v) = -k\,\mathrm{tr}_{\mathbb{V}_0}\big(M_0(u)^{k-1} r_{01}(u-v)\big). \tag{2.102}$$

Any C^1 function of its arguments $\mathcal{F}$ then defines the Hamiltonian

$$H(u) = \mathcal{F}\big(F_1(u), \dots, F_n(u)\big) \tag{2.103}$$

of an integrable system with n conserved quantities F_k, $k \in [\![1, n]\!]$, in involution, whose equations of the motion have a Lax form

$$\frac{\mathrm{d}M(u)}{\mathrm{d}t} = \{M(u), H(v)\} = [V(u,v), M(u)], \tag{2.104}$$

where

$$V(u,v) = \sum_{k=1}^{n} \frac{\partial \mathcal{F}}{\partial F_k}\Big|_{(F_1(v),\dots,F_n(v))} V^{(k)}(u,v). \tag{2.105}$$

Starting from a r-matrix, we can solve for L defined over $\mathcal{M}$ in (2.49) or (2.50). The couple (r, L) is then sufficient to construct an integrable mechanical system. Let $H = \sum_j c_j F_j$ be the Hamiltonian of our choice on $\mathcal{M}$, with the F_j of (2.38). Then the V-matrix of the Lax pair is given by the linear combination

$$V_0(u) = \sum_j c_j \Big(-j\,\mathrm{tr}_{\tilde{0}}\big(L_{\tilde{0}}^{j-1} r_{0\tilde{0}(u)}\big)\Big), \tag{2.106}$$

and the equations of the motion are given by (2.37). The integrals of the motion are the spectral invariants of L, and their independence should be checked regarding the model at hand.

In summary, the privileged objects in the theory of the classical integrable models are the r-matrix, the Lax matrix and the Yang–Baxter algebra. Together they pack the conserved quantities and the Poisson brackets of the system by the Yang–Baxter equation, paving the way for a systematic resolution of Liouville integrable models. For continuous or lattice systems, the explicit resolution is achieved by the CISM. In the following section 3.2, we will investigate the quantum counterparts of the Lax formalism, as well as a quantum version of the Inverse Scattering Method for lattice quantum systems.

Chapter 3 Quantum integrability

We begin this chapter with a broad discussion on the nature of quantum integrability, which, contrary to its classical counterpart, is not a definitive and established concept [see 253, 254, for example].

After that, we present the quantum version of the inverse scattering method (QISM), with the notion of quantum R-matrix, and Lax, monodromy and transfer matrices [90, 97, 99, 100, 103, 193].

We then introduce in details the model that are studied in this thesis: quantum spin chains, which are one-dimensional integrable models on the lattice first introduce by Heisenberg and heavily studied and generalized along the 20$^{\text{th}}$ century and until today [62, 69, 97, 99, 143, 193, 255, 256]. The resolution of the quantum inverse problem for $\mathscr{Y}(\mathfrak{gl}(\mathfrak{n}))$ spin chains [104–107, 257] is discussed at the end of the chapter.

3.1 Approaching quantum integrability

In a moment we will describe the quantization of the Lax formalism and how the Quantum Inverse Scattering Method (QISM) is a preferred framework to describe 1+1 quantum integrable models. But first, let us linger on what quantum integrability should be.

Like in the classical case, we expect to exhibit enough independent conserved charges Q_i containing or generating the Hamiltonian that are in involution. They would play the role the F_i had in the classical case. This should be sufficient to ensure the solvability of the model by a generic method that would be a quantum version of the Liouville–Arnol'd theorem, specific to integrable systems. The following words strike in the these sentences

1. *enough*: how many conserved quantities will ensure integrability?
2. *independent*: what is independence for observables?
3. *in involution*: what does it mean in the quantum context?
4. *solvability*: what are we solving for exactly?

The third observation is the easiest to answer to: quantization of Poisson brackets between functions gives commutators between observables [258]. We are thus looking for a *commutative* family of observables Q_i

$$\forall i,j \in \mathcal{I} \subset N^*, \quad \big[Q_i, Q_j\big] = 0, \quad i,j = 1,2,\dots. \tag{3.1}$$

Questions 1 and 2 are imbricated: adding additional observables that are not independent of the former should not add any information to the integrability—though "independent" is yet to be properly defined. So the answer to question 1 depends mainly on how we define independence between observables. At least, we ask for the set $\{Q_i\}$ to be simultaneously diagonalizable with simple spectrum, so that the Q_i label their common eigenstates uniquely, such that for two different eigenstates $|\lambda\rangle$ and $|\mu\rangle$ we have

$$\forall i, \quad Q_i|\lambda\rangle = q_i^\lambda |\lambda\rangle, \quad Q_i|\mu\rangle = q_i^\mu |\mu\rangle \quad \text{with} \quad q_j^\lambda \neq q_j^\mu \quad \text{for a least one index } j. \tag{3.2}$$

But the answer turns out to be more subtle than expected. Indeed, the notion of independence of the conserved quantities fails to find a direct quantum equivalent. Weigert discuss in details in [253] how additional conserved quantities may be constructed and joined to an existing set of commuting operators

with simple spectrum, such that the notion of functional independence of the constants of motion has to be dropped when transitioning to the quantum. This is due to a fundamental theorem worked by von Neumann [259]. This is a great annoyance in our search for a consistent definition of quantum integrability.

Lastly, we have to define when a quantum model is said to be solved. In the classical case, this is claimed when the whole dynamics is characterized as function in time depending on the constants of the motion and the initial conditions. For integrable models, this is obtained by finding the coordinate transformation to the action–angle variables, and it should require the computation of a sole integral, as the Liouville–Arnol'd theorem specifies by "in quadrature". From there we get $x^i(t) = \ldots$, $p_i(t) = \ldots$ and the configuration of the system is known at any time. This change of coordinate was achieved in the direct way $(q,p) \to M(u)$ by the construction of the monodromy operator, and in the inverse way $M(u) \to (q,p)$ by the reconstruction of the initial dynamical variables in terms of the action–angles ones, the diffusion data contained in the monodromy matrix elements. In the quantum case, we solve the dynamics by diagonalizing the Hamiltonian: eigenstates $|\varphi_n\rangle$ of H evolve over time only by a phase fixed by their energy

$$|\varphi_n(t)\rangle = e^{-i\hbar E_n t} |\varphi_n(0)\rangle , \tag{3.3}$$

so the evolution of generic states is obtained by decomposing them on the eigenbasis of the Hamiltonian and let each vector of the latter evolve

$$|\Psi(t)\rangle = \sum_n c_n |\varphi_n(t)\rangle = \sum_n c_n e^{-i\hbar E_n t} |\varphi_n(0)\rangle . \tag{3.4}$$

A mandatory step to call the system solved is therefore the diagonalization of the Hamiltonian. But it is not sufficient. Contrary to the configurations of a classical system, quantum states are difficult to access directly. Most likely, we would like to access to macroscopic quantities from the microscopic, quantum description of a material, such as critical exponents, magnetic susceptibilities, scattering cross-sections and angles, dynamical structure factors.... These quantities are way more accessible experimentally, and computing them allows to confront the theoretical predictions coming from a theory of quantum integrability to experimental results.

Hence, we have to add many more objects to the collection of the desirable quantities, rather than limiting us to the spectrum of H. In summary, we would like to compute

- the spectrum of the Hamiltonian
- its eigenstates
- the norm of the constructed eigenstates
- scalar products of generic state with eigenstates
- action of "physical" operators on the left and right eigenbasis
- their matrix elements in the basis, i.e. the form factors
- correlation functions
- and higher-level quantities depending on the previous quantities

The first points are directly related to (3.4). Correlation functions for an observable $\mathcal{O} \in \operatorname{End} \mathscr{H}$ are quantities of the form

$$\frac{\operatorname{tr}_{\mathscr{H}} \left(\mathcal{O} e^{-H/k_B T} \right)}{\operatorname{tr}_{\mathscr{H}} \left(e^{-H/k_B T} \right)}, \tag{3.5}$$

where T is the temperature and k_B the Boltzmann constant. The operator $\mathcal{O}$ is usually a product of local one-site operators, but may be a more intricate object. In the limit of null temperature, the contribution of the ground state $|\psi_g\rangle$ of the Hamiltonian is the only one to survive — because it has the smallest eigenvalue under the action of H[1]. So correlation functions are reduced to the expectation value of $\mathcal{O}$ in

[1] If ground state is not unique, i.e. if they are several state with the lowest energy, there are multiple contributions of the

the ground state

$$\frac{\langle\psi_g|\mathcal{O}|\psi_g\rangle}{\langle\psi_g|\psi_g\rangle}. \tag{3.6}$$

From all the above discussion, let us throw a tentative definition of the quantum version of Liouville integrability

Tentative quantum LI. *A quantum system with Hilbert space $\mathscr{H}$ and Hamiltonian H is said* integrable *if there exist a family of mutually commuting observables $\{Q_i\}_{i\in\mathcal{I}} \in \mathrm{End}(\mathscr{H})$, $\mathcal{I} \subset \mathbb{N}$, that are diagonalizable with simple spectrum and generate the Hamiltonian $H = f(Q_i)$.*

Similarly to the classical case, a quantum version of the inverse scattering method has been developed: the quantum inverse scattering method (QISM). In the classical case, the Lax matrix $L(u)$ gathered the constants of the motion in its spectral invariants; we may find a similar situation in the quantum case. The monodromy $M(u)$ is then construct from the local Lax matrices, and its spectral invariants, such as its trace, the transfer matrix $T(u)$, generate conserved quantities. Anticipating on the notation, we incorporate the spectral parameter by a rephrasing of the above definition around the transfer matrix.

Tentative quantum LI (one-parameter family version)**.** *A quantum system with Hilbert space $\mathscr{H}$ and Hamiltonian H is said* integrable *if there is a one-parameter family of operators $T(u) \in \mathrm{End}(\mathscr{H})$, $u \in \mathbb{C}$, such that*

1. $[T(u), T(v)] = 0$
2. *for some complex number u_0, $H = F(T(u))|_{u=u_0}$*
3. *$T(u)$ is diagonalizable and has simple spectrum: $\forall \lambda(u), \mu(u) \in \mathrm{Sp}(T(u))$, $\lambda(u) \neq \mu(u)$.*

As anticipated, these definitions cannot be seen as completely satisfactory. The crucial notion of independence is lacking and does not constrain enough the choice of the conserved charges that would claim the integrability of the model. Moreover, the definition of classical Liouville integrability is backed by the Liouville–Arnol'd theorem. It ensures that the conserved quantities can be recast in action variables, and that the coordinate transformation to the action–angle variables is obtained by quadrature. The above definitions are orphan of such a theorem, making these definitions of quantum integrability a purely *descriptive* feature.

Somehow, what is missing is a property leading to solvability, i.e. a property, an object or a tool that allows the computation of the spectrum by means of quadratures or some analogous procedure. As we will see, the Yang–Baxter algebra provides at least for a class of models a framework in which a such procedure can be performed.

It is puzzling that no definitive definition of *what a quantum integrable model is* exists to this day. Still, the study of quantum integrable models is vivid and plethoric, and in fact not really embarrassed by the absence of a precise definition. For recent reviews of what quantum integrability should be, see for example [79], and [254] with citations 1–4 therein.

However, there is a clearer picture of quantum integrability in the context of separation of variables. We will describe in great details the classical and quantum SoV method in chapter 5 and 6.

Being fair with History

As stated in the introduction of this thesis, the quantum Yang–Baxter structures have been found and studied before their classical counterparts.The classical construction has been linked naturally to the

form (3.5).

existing framework of classical integrable models, while the quantum one has not found at the time an already existing extensive groundwork to rely on.

That is why introductions to quantum integrable systems often start from the R matrix and the YBE, and go down the algebraic ladder until reaching an actual physical system. In the next section, we describe this process and the algebraic framework in which quantum integrable systems are best understood: the quantum inverse scattering method.

3.2 Quantum inverse scattering method

Quantum Yang–Baxter equation Since there are no quantum version available of the Liouville integrability property and its Liouville–Arnol'd sidekick, a most common way to define quantum integrable models is to construct systems with a quantum version of the Lax formalism and Yang–Baxter structures.

$$\begin{array}{ccccc} H_{\mathrm{cl}} & \longleftarrow & L_{\mathrm{cl}} & \longleftarrow & r \\ & & & & \uparrow \quad \text{classical limit} \\ H_{\mathrm{q}} & \longleftarrow & L_{\mathrm{q}} & \longleftarrow & R \end{array}$$

Starting from R matrices satisfying a *quantum* version of the YBE, it is easy to produce non-trivial quantum systems, with "enough" conserved charges, for which we can solve in closed form numerous quantities. The Quantum Inverse Scattering Method framework gives a generic way to produce such quantities. It all start with a quantum equivalent of the classical Yang–Baxter equation for a quantum R matrix, and a quantum Lax matrix. Let us introduce for the quantum case the "$R \to L \to M \to T \to H$ and conserved quantities" chain that we have already described for the classical case.

Being simplistic, quantization mainly amounts to the transformation of functions over the phase space $\mathcal{M}$ in linear operators over a Hilbert space $\mathscr{H}$, mapping the Poisson brackets to the commutator $\{\,,\} \to i\hbar[\,,]$.

The (quantum) Yang–Baxter equation [100, 260] writes

$$R_{12}(u,v)R_{13}(u,w)R_{23}(v,w) = R_{23}(v,w)R_{13}(u,w)R_{12}(u,v), \tag{3.7}$$

where $R_{ab}(u,v)$ is an $n \times n$ square matrix acting non trivially over $\mathbb{V}_a \otimes \mathbb{V}_b$ only, with $\mathbb{V}_{1,2,3} \simeq \mathbb{V} \simeq \mathbb{C}^{\mathfrak{n}}$. For a R matrix of difference form, one simplifies the notation by writing $R(u) := R(u,0)$ and the YBE becomes

$$R_{12}(u-v)R_{23}(v)R_{23}(v) = R_{23}(v)R_{23}(v)R_{12}(u-v). \tag{3.8}$$

From the semi-classical expansion

$$R(u) = 1 + i\hbar r(u) + \mathcal{O}(\hbar^2), \tag{3.9}$$

the classical Yang–Baxter equation (2.47) for r is recovered from (3.7). Note that the Yang–Baxter equation is homogeneous, so that it still holds after $R(u) \to f(u)R(u)$ where f is any function of the spectral parameter.

In the $\mathfrak{gl}(\mathfrak{n})$-invariant case (2.51), the quantum R-matrix is

$$R(u) = 1 + \frac{\eta}{u}\mathcal{P}, \quad \eta = i\hbar c, \tag{3.10}$$

where $\mathcal{P}$ is the permutation operator. One indeed verifies that $[R(u), A \otimes 1 + 1 \otimes A] = 0$ for any $A \in \mathfrak{gl}(\mathfrak{n})$.

It is common to multiply it by u and use the non-singular solution

$$R(u) = u + \eta\mathcal{P}. \tag{3.11}$$

The classical r matrix encoded the behavior under the Poisson bracket of the Lax matrix. Similarly, the quantum R matrix characterized the commutator between the coefficient of the Lax matrix.

Quantum Lax matrix and the monodromy matrix [90, 100] For a fixed R-matrix and a given Hilbert space $\mathscr{H}$, one can look for solutions L of the quantum counterpart of the quadratic algebra (2.50)

$$R_{12}(u,v)L_{1Q}(u)L_{2Q}(v) = L_{2Q}(v)L_{1Q}(u)R_{12}(u,v). \tag{3.12}$$

$L_{0Q}(u)$ is the quantum Lax matrix, it is a $n \times n$ square matrix in the auxiliary space $\mathbb{V}_0 \simeq \mathbb{C}^n$ whose entries are operators on the Hilbert space $\mathscr{H}$, symbolized here by the index Q. The indices 1 and 2 label two copies of the auxiliary matrix spaces $\mathbb{V}_0 \simeq \mathbb{C}^n$. Identifying the matrix entries from the left and right-hand sides of (3.12), one gets n^2 algebraic relations between the Lax matrix entries often referred to as the Fundamental Commutation Relations (FCR). This forms a *Yang–Baxter algebra* associated to the R-matrix. The generators are the Lax matrix entries, which are represented as operators over the space $\mathscr{H}$, and the structure constants are given in terms of the R-matrix coefficients. We describe in greater details the mathematical structure of these algebras, and especially the Yangians of classical Lie algebras, in appendix A. It is worth noting that the classical limit of (3.12) gives the quadratic Sklyanin bracket (2.50).

Quantum inverse scattering method (QISM) Let us now consider a quantum system on a one-dimensional lattice of length N with Hilbert space $\mathscr{H} = \otimes_{k=1}^{N} \mathbb{V}_k$. A quantum counterpart to the classical inverse scattering method exists and is unsurprisingly called the quantum inverse scattering method. Let $\mathscr{H}_k$ and $L_{0k}(u)$ be respectively the Hilbert space and the Lax matrix at sites $k \in [\![1,N]\!]$, where the Lax matrices share the same auxiliary space $\mathbb{V}_0 \simeq \mathbb{C}^n$ and verify (3.7) with the same fixed numeric R-matrix $R(u)$. A nice feature of (3.7) is that it behaves well under tensorization of multiples copies of L. It is similar to the property of composition satisfied by Wilson lines in the continuous case. Indeed, as operators acting on $\mathbb{V}_0 \otimes \mathbb{V}_{\bar{0}} \otimes \mathbb{V}_k \otimes \mathbb{V}_\ell$

$$\begin{aligned} &R_{0\bar{0}}(u,v)L_{0k}(u)L_{0\ell}(u)L_{\bar{0}k}(v)L_{\bar{0}\ell}(v) = R_{0\bar{0}}(u,v)L_{0k}(u)L_{\bar{0}k}(v)L_{0\ell}(u)L_{\bar{0}\ell}(v) \\ &= L_{\bar{0}k}(v)L_{0k}(u)R_{0\bar{0}}(u,v)L_{0\ell}(u)L_{\bar{0}\ell}(v) = L_{\bar{0}k}(v)L_{0k}(u)L_{\bar{0}\ell}(v)L_{0\ell}(u)R_{0\bar{0}}(u,v) \\ &= L_{\bar{0}k}(v)L_{\bar{0}\ell}(v)L_{0k}(u)L_{0\ell}(u)R_{0\bar{0}}(u,v). \end{aligned} \tag{3.13}$$

This can be shown as a consequence of the Hopf algebra structure of the Yang–Baxter algebra. So from a solution $L_{0j}(u) \in \mathrm{End}(\mathbb{V}_0 \otimes \mathbb{V}_j)$ of (3.12), one can construct the monodromy operator

$$M_0(u) = L_{0N}(u)\dots L_{01}(u) \in \mathrm{End}(\mathbb{V}_0 \otimes \mathscr{H}), \tag{3.14}$$

in a way similar to the classical construction (2.74), which verifies

$$R_{12}(u-v)M_1(u)M_2(v) = M_2(v)M_1(u)R_{12}(u-v). \tag{3.15}$$

The $\mathfrak{n}^2$ relations defining the algebra between the monodromy entries $M_{ij}(u)$ are obtained by identifying the left and right-hand sides of (3.15). For the usual $\mathfrak{gl}(\mathfrak{n})$-invariant R-matrix, these fundamental commutation relations write

$$\forall i,j,k,\ell \in [\![1,\mathfrak{n}]\!], \quad \big[M_{ij}(u), M_{k\ell}(v)\big] = \frac{\eta}{u-v}\big(M_{kj}(v)M_{i\ell}(u) - M_{kj}(u)M_{i\ell}(v)\big). \tag{3.16}$$

Conserved quantities are constructed from the quantum Lax matrix through the transfer matrix, which is the trace of (3.14) on the auxiliary space

$$T(u) = \operatorname{tr}_0 M_0(u) \in \operatorname{End} \mathscr{H}, \tag{3.17}$$

with $\mathscr{H} = \otimes_{k=1}^{N} \mathscr{H}_k$. Thanks to the Yang–Baxter equation, with R invertible, the one-parameter family of transfer matrices $\{T(u)\}|_{u\in\mathbb{C}}$ is commutative [80, 85, 100, 102, 103]

$$[T(u), T(v)] = 0, \tag{3.18}$$

so that transfer matrix coefficients in its u expansion are a set of mutually commuting operators. When the Hamiltonian is a function of them, this defines a family of conserved charges, which can be leveraged to diagonalize the Hamiltonian: they are all diagonalizable in the same basis.

The interest of this construction truly reveals in chapter 4, where the monodromy entries and theirs algebraic relations are used to generate the eigenvectors of the $T(u)$ family. This constitutes the direct problem part of the QISM, namely the construction of commuting integrals of motion and generators of the Yang–Baxter algebra that surround them. To solve the dynamics, one also has to deal with the resolution of the quantum inverse problem, namely the reconstruction of local quantum operators in terms of the monodromy matrix entries $M_{ij}(u)$, i.e. the scattering data. The explicit resolution for $\mathfrak{gl}(\mathfrak{n})$ models will be given in section 3.3.3 below. Before this, we consider fundamental models that can be described in the QISM framework and are at the center of this thesis.

3.3 Spin chains

Spin chains are 1+1 discrete quantum models on the lattice. Their studies have thrived in the integrable world, and they are at the center of this thesis. Note that we already presented a chain of classical spins in 2.2. The quantum XXX $1/2$ spin chain is a simple yet non-trivial 2^N-dimensional quantum model that is fundamental in the study of integrable models.

3.3.1 The XXX Heisenberg $1/2$ spin chain

Most simple examples of non-trivial spin chains are the $\mathfrak{sl}(2)$-invariant ones, with usual $1/2$ spins at each site that are linear combination of up $|\uparrow\rangle$ and down $|\downarrow\rangle$ states. They have been introduced by Heisenberg in [69], and are nowadays called after his name.

The model and its Hamiltonian Consider a one dimensional lattice with N sites equipped with $1/2$ spins. We represent them over $\mathbb{C}^2$ with $|\uparrow\rangle = \binom{1}{0}$ and $|\downarrow\rangle = \binom{0}{1}$, and note $S_k^\alpha = 1/2\,\sigma_k^\alpha$, with $\alpha \in \{x, y, z\}$, the spin operators at site k. The operators $\sigma_k^\alpha \in \operatorname{End}(\mathbb{C}^2)$ are the usual Pauli matrices

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.19}$$

The Hilbert space $\mathbb{V}_k$ at each site is the $\mathfrak{sl}(2)$ fundamental representation space $\mathbb{V}_k = \mathbb{C}^2$, so that the total Hilbert space is $\mathscr{H} = \otimes_{k=1}^{N} \mathbb{V}_k = (\mathbb{C}^2)^{\otimes N}$ of complex dimension 2^N.

Each spin interacts with its two nearest neighbors by a spin–spin interaction, and the chain closed on itself with periodic boundary conditions by identifying site $N+1$ with site 1 such that $S_{N+m}^\alpha = S_m^\alpha$. The Hamiltonian has therefore the following form

$$H = C + J \sum_{k=1}^{N} \vec{\mathbf{S}}_k \cdot \vec{\mathbf{S}}_{k+1} = C + J \sum_{k=1}^{N} \sum_{\alpha=x,y,z} S_k^\alpha S_{k+1}^\alpha. \tag{3.20}$$

The constant C shifts the overall spectrum and may be chosen at convenience. The constant J quantifies the spin–spin interaction. Whether it is positive or negative, it favors anti-aligned or aligned adjacent spins, changing the energy ordering of the eigenstates of H and its fundamental statement. All the coefficients in front of the $S_k^\alpha S_{k+1}^\alpha$ are equal (to J), hence the name XXX for this model. The total spin operator is defined in component as

$$\forall \alpha = x, y, z, \quad S^\alpha = \sum_{k=1}^{N} S_k^\alpha. \tag{3.21}$$

The components S^α are representations of the Pauli generators of the $\mathfrak{sl}(2)$ Lie algebra over the Hilbert space $\mathscr{H} = (\mathbb{C}^2)^{\otimes N}$, up to a $1/2$ coefficient. The model is $\mathfrak{sl}(2)$-invariant since

$$\forall \alpha = x, y, z, \quad [H, S^\alpha] = 0. \tag{3.22}$$

QISM description Let $\mathbb{V}_0 = \mathbb{C}^2$ be the auxiliary space, the same as the one on each site. The Lax matrix at site k is

$$L_{0k}(u) = u + \eta \mathcal{P}_{0k}, \tag{3.23}$$

where $\mathcal{P}_{0k} = \sum_{i,j=1}^{2} e_{ij}^{(0)} \otimes e_{ji}^{(k)}$ is the permutation operator on $\mathbb{V}_0 \otimes \mathbb{V}_k$. Note that since

$$\mathcal{P} = \frac{1}{2}\left(1 + \sum_{\alpha=x,y,z} \sigma^\alpha \otimes \sigma^\alpha\right), \tag{3.24}$$

one has

$$L_{0k}(u) = \left(u + \frac{\eta}{2}\right) + \eta \sum_{\alpha=x,y,z} S_k^\alpha \sigma_0^\alpha = \begin{pmatrix} u + \frac{\eta}{2} + S_k^z & \eta S_k^- \\ \eta S_k^+ & u + \frac{\eta}{2} + S_k^z \end{pmatrix}_0, \tag{3.25}$$

where $S_k^\pm = S_k^x \pm i S_k^y = \frac{1}{2}\left(\sigma_k^x \pm \sigma_k^y\right) \equiv \sigma_k^\pm$. This is the basic building block of the integrable structure of the XXX chain.

Writing

$$L_{0k} = \mathcal{L}_{ij}^{(k)} e_{ij}^{(0)}, \tag{3.26}$$

with $e_{ij}^{(0)}$ the elementary matrices of $\mathbb{V}_0$, the commutators between the Lax matrix entries $\mathcal{L}_{ij}^{(k)}$ are

$$\left[\mathcal{L}_{ij}^{(k)}(u), \mathcal{L}_{kl}^{(k)}(v)\right] = \frac{\eta}{u-v}\left(\mathcal{L}_{kj}^{(k)}(v)\mathcal{L}_{il}^{(k)}(u) - \mathcal{L}_{kj}^{(k)}(u)\mathcal{L}_{il}^{(k)}(v)\right), \tag{3.27}$$

while $\left[\mathcal{L}_{ij}^{(n)}, \mathcal{L}_{k\ell}^{(m)}\right] = 0$ for $n \neq m$. They are written in the compact Yang–Baxter form

$$R_{ab}(u-v) L_{an}(u) L_{bn}(v) = L_{bn}(v) L_{an}(u) R_{ab}(u-v), \tag{3.28}$$

using the $\mathfrak{gl}(2)$-invariant quantum R-matrix

$$R(u) = u + \eta \mathcal{P}. \tag{3.29}$$

The matrix representation of R in $\mathbb{V} \otimes \mathbb{V} \simeq \mathbb{C}^4$ is

$$R(u) = \begin{pmatrix} u+\eta & 0 & 0 & 0 \\ 0 & u & \eta & 0 \\ 0 & \eta & u & 0 \\ 0 & 0 & 0 & u+\eta \end{pmatrix}. \tag{3.30}$$

Immediately, one remarks that $L_{0n}(u) = R_{0n}(u)$, so that (3.28) is nothing but a rewriting of the Yang–

Baxter equation (3.8) verified by the R-matrix.

The monodromy of the periodic N-site XXX chain is constructed by taking the ordered product along the auxiliary space of the Lax matrices

$$M(u) = L_{0N}(u)\dots L_{01}(u). \tag{3.31}$$

The Yang–Baxter equation (3.28) replicates to the monodromy, and one has

$$R_{ab}(u-v)M_a(u)M_b(v) = M_b(v)M_a(u)R_{ab}(u-v), \tag{3.32}$$

with $M_a(u) = M(u)\otimes 1$ and $M_b(u) = 1\otimes M(u)$ in $\mathbb{V}_a\otimes\mathbb{V}_b \simeq \mathbb{V}_0\otimes\mathbb{V}_0 \simeq \mathbb{C}^4$, or in components

$$\big[M_{ij}(u), M_{k\ell(v)}\big] = \frac{\eta}{u-v}\big(M_{kj}(v)M_{i\ell}(u) - M_{kj}(u)M_{i\ell}(v)\big), \tag{3.33}$$

with the decomposition on the auxiliary space

$$M(u) = M_{ij}(u)\, e_{ij}^{(0)}. \tag{3.34}$$

Since the monodromy is a 2-by-2 matrix in the auxiliary space, it is customary and useful to write it as the following matrix in the auxiliary space

$$M(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}_0. \tag{3.35}$$

The transfer matrix

$$T(u) = \operatorname{tr}_0 M(u) = A(u) + D(u) \tag{3.36}$$

commutes with itself

$$[T(u), T(v)] = 0, \tag{3.37}$$

by virtue of (3.32). Expanding T in u by

$$T(u) = F_1 + F_2 u + \dots + F_N u^{N-1} + u^N, \tag{3.38}$$

the F_i forms a commutative family of N operators.

Eventually, we should find the XXX Hamiltonian as a function of the F_i to justify this whole construction. It appears to be constructed by F_2, but let us first compute $F_1 = T(0)$. One has

$$T(0) = \eta^N \mathcal{P}_{12}\dots\mathcal{P}_{N-1N} =: \eta^N U \quad \text{with} \quad U S_k^\alpha = S_{k+1}^\alpha U, \tag{3.39}$$

so $\eta^{-N}\mathcal{Q}_1 = U \in \operatorname{End}(\mathscr{H})$ is the right-shift operator in the number of sites. By definition, the momentum operator P produces an infinitesimal shift along the chain, therefore

$$U = e^{iP}, \tag{3.40}$$

and

$$P = -i\ln\big(\eta^{-N}T(0)\big). \tag{3.41}$$

Now,

$$F_2 = \frac{\mathrm{d}}{\mathrm{d}u}T(u) = \eta^{N-1}\sum_{k=1}^{N}\mathcal{P}_{12}\dots\mathcal{P}_{k-1k+1}\dots\mathcal{P}_{N-1N} = \eta^{-1}T(0)\sum_{k=1}^{N}\mathcal{P}_{kk+1}, \tag{3.42}$$

where by the use of (3.24), we identify the Hamiltonian and write

$$F_2 = \frac{T(0)}{2\eta}\big(N + 4J^{-1}(H - C)\big). \tag{3.43}$$

One finally expresses the Hamiltonian (3.20) in terms of the transfer matrix with

$$J^{-1}(H - C) = \frac{\eta}{2}\frac{\mathrm{d}}{\mathrm{d}u}\ln T(u)\big|_{u=0} - \frac{N}{4}. \tag{3.44}$$

Using the QISM formalism, one has constructed N commutative conserved charges of the XXX model, collected in one operator, the transfer matrix (3.36), which profit of a rich surrounding Yang–Baxter algebraic structure given by (3.32)–(3.33) that can be leveraged to solve the system. In chapter 4, we will diagonalize the $T(u)$ family (3.36) using the Algebraic Bethe Ansatz, obtaining the spectrum and eigenstates of the XXX model.

Physical realization of the Heisenberg spin chain

The $KCuF_3$ crystal has a fully three-dimensional structure, but displays magnetic properties characteristic to one-dimensional antiferromagnetic magnets. It appears that the Cu^{2+} octahedral orbitals are deformed by the Jahn–Teller effect, resulting in strong magnetic interactions along one axis of the crystal, while magnetic interactions in the perpendicular directions are very weak due to a poor overlap of the orbitals. The magnetic behavior of the $KCuF_3$ crystal is therefore effectively one-dimensional: the ratio between the longitudinal and orthogonal interaction constants is close to 0.027 [261]. Moreover, the Cu^{2+} ions are effective $1/2$ spins, in interaction mostly with their nearest neighbors along the chain axis.

Hence, the description of the magnetic property of the $KCuF_3$ compound is well achieved by a long Heisenberg XXZ chain embedded in a uniform external magnetic field $\vec{h} = h\vec{e}_z$ with Hamiltonian

$$H^{\mathrm{KCuF_3}}_{\mathrm{effective}} = -hS_z + \sum_{k=1}^{N}\big(\sigma^x_k\sigma^x_{k+1} + \sigma^y_k\sigma^y_{k+1} + \Delta\big(\sigma^z_k\sigma^z_{k+1} - 1\big)\big), \tag{3.45}$$

where Δ quantifies the anisotropy of the spin coupling in the z direction. It is very close from 1, so the XXX Hamiltonian (3.20) is a very good approximation of (3.45).

Ideally, studying the finite chain and then taking the large chain limit $N \to +\infty$ should extract the physical magnetic properties of $KCuF_3$ from a microscopic model [143]. It is sufficient to take a very long but finite chain—for example $N \sim 500$—to get a good approximation converging in $1/N$ of the dynamical structure factor.

The $\mathscr{Y}(\mathfrak{gl}(2))$ As we see from the above construction, the QISM promotes the use of the monodromy entries over the local spin operators for the study of the XXX chain, so it is worth describing their algebra in greater details.

Here we deal with a representation of this algebra over $\mathscr{H}$, described by the 4 generators $M_{ij}(u) \in \mathrm{End}(\mathscr{H})$ and the 16 relations (3.33) obtained by identifying matrix elements by matrix elements the two sides of the Yang–Baxter equation (3.32). We identified this to be a representation of a Yangian algebra [117], and more precisely the Yangian subalgebra $\mathscr{Y}(\mathfrak{sl}(2)) \subset \mathscr{Y}(\mathfrak{gl}(2))$ associated to the $\mathfrak{gl}(2)$ Lie algebra. Appendix A provides a summary of Yangians and their properties.

Note that the one-site Lax matrix $L_{0j}(u)$ already defines a representation of $\mathscr{Y}(\mathfrak{gl}(2))$—over $\mathbb{V}_j \simeq \mathbb{C}^2$—thanks to equation (3.28). In fact, it is only because the Lax operator (3.23) verifies the YBE that the monodromy satisfies it as well. This is well explained by the Hopf algebra structure of $\mathscr{Y}(\mathfrak{gl}(2))$: the

comultiplication $\Delta : \mathscr{Y}(\mathfrak{gl}(2)) \to \mathscr{Y}(\mathfrak{gl}(2)) \otimes \mathscr{Y}(\mathfrak{gl}(2))$ is an algebra automorphism. Using the fundamental evaluation representation $\rho_u^{\mathbb{C}^2}$ over $\mathbb{C}^2$ and starting from the one-site YBE (3.28), $\left(\rho_{u_{j-1}}^{\mathbb{C}^2} \otimes \rho_{u_j}^{\mathbb{C}^2}\right) \circ \Delta$ applied on the generators of $\mathscr{Y}(\mathfrak{gl}(2))$ implies that (3.28) also holds with the substitution $L_{0j}(u) \to L_{0j}(u_j) L_{0j-1}(u_{j-1})$. This construction repeats until one obtains the monodromy corresponding to the chain of length N, and with $u_j = u - \xi_j$ we recover the inhomogeneous system.

The Yang–Baxter equation is ultimately what provides quantum integrability to the system: it allows the construction of a commuting family of operators. Now that we have identified the algebra behind it, we can take a constructive point of view. Starting from Yangian algebras $\mathscr{Y}(\mathcal{A})$, with $\mathcal{A}$ a given algebra, or other similar algebras whose relations between generators packs in a Yang–Baxter form, a representation of $\mathscr{Y}(\mathfrak{gl}(\mathcal{A}))$ over a Hilbert space $\mathscr{H}$ shall grant a quantum integrable model. We elaborate on this in a forthcoming section, with the $\mathfrak{gl}(2)$ chain with spin $s \geq 1$ and $\mathfrak{gl}(n)$ spin chains.

XXZ and XYZ models A generalization of (3.20) is achieved by allowing different coupling constant J_α along the $\alpha = x, y, z$ directions of the spins

$$H = C + \sum_{k=1}^{N} \sum_{\alpha=x,y,z} J_\alpha \, S_k^\alpha S_{k+1}^\alpha. \tag{3.46}$$

The generic case $J_x \neq J_y \neq J_z$ is the XYZ model, and two equal constants $J_x = J_y \neq J_z$ give the XXZ model.

The R and Lax matrix associated to these models are of $\mathfrak{gl}(2)$-type, but differ a bit. The 6-vertex R-matrix solution of the YBE is associated to the XXX and XXZ models and is of the form

$$R(u) = \begin{pmatrix} \varphi(u+\eta) & 0 & 0 & 0 \\ 0 & \varphi(u) & \varphi(\eta) & 0 \\ 0 & \varphi(\eta) & \varphi(u) & 0 \\ 0 & 0 & 0 & \varphi(u+\eta) \end{pmatrix}, \tag{3.47}$$

the φ function being

$$\varphi(u) = \begin{cases} u & \text{for the XXX model,} \\ \sinh(u) & \text{for the XXZ model,} \end{cases} \tag{3.48}$$

where $\sinh(u)$ is the hyperbolic sinus function. The XXX and XXZ R-matrices are respectively of rational and trigonometric form. Identifying the spin $1/2$ operators in each 2×2 block of the R-matrix, the one-site Lax matrix of the fundamental model is

$$L_{0k}(u) = \begin{pmatrix} \varphi\left(u + \eta/2 + \eta S_k^z\right) & \varphi(\eta) S_k^- \\ \varphi(\eta) S_k^+ & \varphi\left(u + \eta/2 - \eta S_k^z\right) \end{pmatrix}. \tag{3.49}$$

The 8-vertex R-matrix associated to the XYZ model has the form

$$R(u) = \begin{pmatrix} \mathfrak{a}(u) & 0 & 0 & \mathfrak{d}(u) \\ 0 & \mathfrak{b}(u) & \mathfrak{c}(u) & 0 \\ 0 & \mathfrak{c}(u) & \mathfrak{b}(u) & 0 \\ \mathfrak{d}(u) & 0 & 0 & \mathfrak{a}(u) \end{pmatrix}, \tag{3.50}$$

where the $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{d}$ are rational expressions of Jacobi theta functions $\theta_j(u|k\omega)$, $j \in [\![1,4]\!]$, $k = 1, 2$ with

quasi-periods π and $k\pi\omega$ [262]

$$\begin{aligned}\mathfrak{a}(u) &= \frac{2\theta_4(\eta|2\omega)\theta_1(u+\eta|2\omega)\theta_4(u|2\omega)}{\theta_2(0|\omega)\theta_4(0|2\omega)}, \quad \mathfrak{b}(u) = \frac{2\theta_4(\eta|2\omega)\theta_1(u|2\omega)\theta_4(u+\eta|2\omega)}{\theta_2(0|\omega)\theta_4(0|2\omega)},\\ \mathfrak{c}(u) &= \frac{2\theta_1(\eta|2\omega)\theta_4(u|2\omega)\theta_4(u+\eta|2\omega)}{\theta_2(0|\omega)\theta_4(0|2\omega)}, \quad \mathfrak{d}(u) = \frac{2\theta_1(\eta|2\omega)\theta_1(u+\eta|2\omega)\theta_1(u|2\omega)}{\theta_2(0|\omega)\theta_4(0|2\omega)}.\end{aligned} \tag{3.51}$$

It is called an elliptic model.

From a quantum group point of view, the XXX model is associated to the Yangian algebra $\mathscr{Y}(\mathfrak{gl}(2))$, the XXZ model is associated to the quantum affine algebra $U_q(\widehat{\mathfrak{gl}(2)})$, and the XYZ model to the elliptic quantum algebra $U_{q,p}(\widehat{\mathfrak{gl}(2)})$ [263, 264].

3.3.2 Inhomogeneous twisted $\mathfrak{gl}(\mathfrak{n})$ spin chains

As we see from the QISM illustrated on the common XXX spin chain, the fundamental objects of quantum integrable models are the R-matrix and the monodromy matrix who verify the Yang–Baxter equation, and the underlying algebra structure is a Yangian one.

The construction can be extended to a wide class of R-matrix and monodromy matrix, while keeping the Yang–Baxter equation verified, ensuring the integrability of the one-dimensional quantum system defined by the transfer matrix.

Starting from the XXX model, this is done in several ways, such as *i)* adding inhomogeneities at the sites of the chain *ii)* imposing other boundary conditions at the edges of the chain *iii)* having different quantum objects at each site along the chain *iv)* allowing other symmetry algebras than $\mathscr{Y}(\mathfrak{gl}(2))$.

i) Inhomogeneities The Yang–Baxter equation (3.8) with site-dependent inhomogeneities along the chain. Indeed, the following equation holds

$$R_{12}(u,v)R_{1j}(u,\xi_j)R_{2j}(v,\xi_j) = R_{2j}(v,\xi_j)R_{1j}(u,\xi_j)R_{12}(u,v), \tag{3.52}$$

at any site $j \in [\![1,N]\!]$. Therefore, one can choose a different ξ_j in the Lax matrix at each site,

$$L_{0k}(u) = R_{0k}(u,\xi_k), \quad \xi_k \in \mathbb{C}, \tag{3.53}$$

(which manifest as a shift in the spectral parameter for R-matrix of difference form) while keeping (3.28) verified, so the inhomogenous monodromy

$$M(u) = L_{0N}(u)\dots L_{01}(u) \tag{3.54}$$

still verifies (3.15), and the transfer matrix still defines an integrable model.

The ξ_j are called the inhomogeneities parameters; they may be kept in generic position. In the $\eta \to 0$ limit, the inhomogeneous XXX spin chain gives the Gaudin model [130, 265, 266], where the inhomogeneities ξ_k are interpreted as the particles' positions [191]. The possibility to add an extra degree of freedom at each site may prove useful in some calculation, where the analytic dependence in the ξ_j provides insights in the mathematical structure of the quantity at hand. This comes at the price of an additional step to recover a meaningful physical model in the end: performing the homogeneous limit $\xi_1,\dots,\xi_N \to \xi$, with $\xi = 0$ for example, in the final formulas.

ii) Boundary conditions For the above XXX chain, periodic boundary conditions have been imposed by identifying the quantum space at site $N+1$ with the one at site 1, but other boundary conditions that keep the integrability may be specified. Rather that making the plain identification, one may identify the

spaces $\mathbb{V}_{N+1}$ and $\mathbb{V}_1$ up to an isomorphism $\mathbb{V}_1 \simeq \mathbb{V}_{N+1}$ making the chain quasi-periodic. This is encoded by a twist matrix $K \in \mathrm{End}(\mathbb{V})$, $\det K \neq 0$, and the Hamiltonian then becomes

$$H = C + J\sum_{k=1}^{N-1} \vec{\mathbf{S}}_k \cdot \vec{\mathbf{S}}_{k+1} + J\vec{\mathbf{S}}_N \cdot \vec{\mathbf{S}}_{N+1} = C + J\sum_{k=1}^{N-1} \vec{\mathbf{S}}_k \cdot \vec{\mathbf{S}}_{k+1} + J\vec{\mathbf{S}}_N \cdot \left(K_1 \vec{\mathbf{S}}_1 K_1^{-1}\right). \tag{3.55}$$

The monodromy is twisted as well by

$$M^K(u) = K_0 L_{0N}(u) \dots L_{01}(u), \tag{3.56}$$

so that the transfer matrix $T^K(u) = \mathrm{tr}_0\, M^K(u)$ gives the proper Hamiltonian (3.55) by (3.44). Note that since $K \in \mathtt{GL}(2)$,

$$[R(u), K \otimes K] = 0 \quad \text{or written differently} \quad R_{12}(u) K_1 K_2 = K_2 K_1 R_{12}(u), \tag{3.57}$$

so the Yang–Baxter equation (3.32) indeed holds for $M^K(u)$ as well, and the whole scheme is preserved for the twisted XXX chain. K is interpreted as a Lax matrix with a trivial quantum representation, or simply as an element of the group under which the R-matrix is invariant, here $\mathtt{GL}(2)$.

In this thesis, the discussion focuses solely on quasi-periodic models, but other boundary conditions may be imposed while keeping the system integrable. The reflection algebra [267] allows for instance the construction of open spin chains—spin chains that are not closed on themselves and have non-trivial magnetic fields applied on the boundary sites. See for example the thesis [268] as a resource for SoV techniques in quantum integrable models with open boundary conditions.

iii) Representations on the quantum sites Let the representations $(\rho, \mathbb{V}_k)$ at each site be the $\mathfrak{sl}(2)$ spin s_k ones, with $s_k \in \mathbb{Z}_{>0}/2$, and $\mathbb{V}_k$ a space of dimension $2s_k + 1$. The s_k may be different from each others. The $\mathfrak{sl}(2)$ generators

$$\left[h, x^{\pm}\right] = \pm x^{\pm}, \quad \left[x^+, x^-\right] = 2h, \tag{3.58}$$

are represented over $\mathbb{V} = \mathbb{C}^{2s_k+1}$ explicitly as

$$\rho_k(h) = \begin{pmatrix} s_k & & & \\ & s_k - 1 & & \\ & & \ddots & \\ & & & -s_k \end{pmatrix}, \quad \rho_k(x^+) = \rho_k(x^-)^t = \begin{pmatrix} 0 & j_n(1) & & \\ & \ddots & \ddots & \\ & & \ddots & j_n(2s_n) \\ & & & 0 \end{pmatrix}, \tag{3.59}$$

where $j_n(a) = \sqrt{a(2s_n + 1 - a)}$. Keeping the auxiliary space $\mathbb{V}_0$ to $\mathbb{C}^2$, the Lax operators is then written

$$L_{0k}(u) = u + \eta \sum_{i,j=1}^{2} e_{ij}^{(0)} \otimes \rho_k(e_{ji}) = \begin{pmatrix} u + \frac{\eta}{2} + \rho_k(h) & \eta\rho_k(x^-) \\ \eta\rho_k(x^+) & u + \frac{\eta}{2} - \rho_k(h) \end{pmatrix}_0, \tag{3.60}$$

and verifies (3.28) with the 4-by-4 R-matrix (3.30). The construction of the monodromy and transfer matrix is the same as above, exhibiting form $T(u)$ a commuting family of N operators acting on $\mathscr{H} = \otimes_{k=1}^N \mathbb{V}_k \simeq \otimes_{k=1}^N \mathbb{C}^{2s_k+1}$. Any Hamiltonian chosen as a function of these N quantities makes the model integrable.

Remark that the auxiliary space has been hold to the fundamental representation $\mathbb{C}^2$ here. It is possible to make this construction with a different representation $(\rho_0, \mathbb{V}_0)$ of $\mathfrak{sl}(2)$ for the auxiliary space, because

the Yang–Baxter equation $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$ holds at the virtual algebra level for

$$R(u) = u1 \otimes 1 + \eta \sum_{i,j=1}^{2} e_{ij} \otimes e_{ji}, \tag{3.61}$$

where the e_{ij} are the unrepresented elementary generators of the $\mathfrak{gl}(2)$ algebra.

When all the quantum spaces are equipped with the same representation $(\rho, \mathbb{V})$ with $\dim \mathbb{V} = 2s+1$, one gets the spin s chain. Taking the same representation on the auxiliary space is then beneficial, since we recover the desirable feature that Lax matrices degenerate to the permutation operator on $\mathbb{V} \otimes \mathbb{V}$

$$L_{ab}(\xi_j) = R_{ab}(0) = \eta \sum_{i,j=1}^{\dim \mathbb{V}} \rho(e_{ij}) \otimes \rho(e_{ji}) = \eta \mathcal{P}^{\mathbb{V}\otimes\mathbb{V}}. \tag{3.62}$$

It is also possible to put representations of different spin values at each site while keeping the model integrable.

iv) Other symmetry algebra Two equations are at the core of the description of quantum integrable system by the quantum inverse scattering method

$$R_{12}M_1M_2 = M_2M_1R_{12}, \tag{3.63}$$

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}. \tag{3.64}$$

On page 45, we identified that the XXX spin chain of length N is obtained as the N-fold tensor product of fundamental evaluation representations of the Yangian $\mathscr{Y}(\mathfrak{gl}(2))$. The two RRR and RMM equations originated naturally from this underlying algebraic structure.

Starting from similar algebras, different integrable models are constructed. In appendix A, we review the mathematical structure of Yangian algebras $\mathscr{Y}(\mathfrak{gl}(n))$, and show how spin chain models are obtained as representations of these algebras.

3.3.3 Reconstruction of local spin operators

The QISM formalism puts the monodromy entries at the center of the stage with their rich algebraic structure. But they are highly non-local quantities, being a sum of products of the local spin operators along all the sites of the chain. A quick induction shows that they are a sum of 2^{N-1} terms, where each term is a product of N local one-site operators.

Can one express the local one-site spin operators $S_j^{\pm}$, S_j^z, in terms of the monodromy entries A, B, C, D? This is actual the *inverse problem*. In the classical case, this solves the dynamics by expressing the initial dynamical variables in terms of the action–angle variables of the system.

The solution was obtained by Maillet, Kitanine and Terras, first using the Drindfel'd twist and the F-basis [104] for the XXX and XXZ chains. This was extended to a large class of chains in [106], where formulas are obtained in a more direct way which bypass the use of the F-basis. These results have been generalized to the supersymmetric case in [107].

The reconstruction of local operators of fundamental $\mathfrak{gl}(n)$ chains is surprinsigly simple: local spin operators at site n are equal to a corresponding monodromy entry dressed by a product of transfer matrices. Specializing in the $\mathfrak{gl}(2)$ case, the proof is rather short. Let us consider the operator $S_1^- = \sigma_1^-$. Then, by (3.31),

$$\mathrm{tr}_0\left(e_{21}^{(0)} M(0)\right) = e_{21}^{(1)} \eta^N U = \sigma_1^- T(0), \tag{3.65}$$

while using the matrix representation (3.35) the trace gives

$$\mathrm{tr}_0\left(e_{21}^{(0)}M(0)\right)=\mathrm{tr}_0\begin{pmatrix}0 & *\\ * & B(0)\end{pmatrix}=B(0). \tag{3.66}$$

The spin operator on the first site of the chain is therefore simply

$$S_1^-=B(0)T(0)^{-1}. \tag{3.67}$$

The expression at an arbitrary site n is obtained by shifting the chain using the U operator constructed in (3.39)

$$S_n^-=T(0)^{n-1}B(0)T(0)^{-n}. \tag{3.68}$$

This computation extends to the XXZ and XYZ chains and also to twisted inhomogeneous $\mathfrak{gl}(\mathfrak{n})$ models with $n\geq 2$, making this powerful result very general

$$e_{ij}^{(n)}=\left(\prod_{k=1}^{n-1}T^K(\xi_k)\right)\mathrm{tr}_0\left(e_{ji}^{(0)}M^K(\xi_n)\right)\left(\prod_{k=1}^{n}T^K(\xi_k)\right)^{-1}. \tag{3.69}$$

Formulas for $\mathfrak{gl}(\mathfrak{n})$ models, higher spin chains and $\mathfrak{gl}(m|n)$ supersymmetric models are similar [106, 107], since this computation relies mostly on the existence of a point u_0 such that $L_{ab}(u_0)\propto\mathcal{P}_{ab}$.

Like in the classical case, the quantum inverse problem allows for computing the dynamics, namely form factors and correlation functions of these spin chains [257]. We will review the results enabled by its resolution in the next chapter.

Chapter

4 Algebraic Bethe ansatz techniques

In this chapter, we describe the techniques of the Algebraic Bethe Ansatz (ABA) for the $\mathfrak{gl}(2)$ models [100], as well as its Nested version (NABA) for higher rank models [129]. The ABA idea is to extract creation and annihilation operators from the off-diagonal elements of the monodromy matrix, and create the eigenstates of the transfer matrix by action of these creation operators on a pseudo-vacuum reference state. This gives an ansatz form for the eigenstates, namely we search for eigenstates of the form of Bethe vectors. The action of the transfer matrix on Bethe vectors can be computed by using the Yang–Baxter algebra of the monodromy entries [99]. Unwanted terms arise, and are cancelled by requiring the spectral parameters in which the operators are evaluated satisfy a set of tightly coupled equations, the Bethe equations. This line of research has proven extremely successful and rich of results in the past decades. We close the chapter with a review of the ABA results towards the computation of correlation functions, but also highlight the limitations of this framework justifying the developments of other techniques for the study of quantum integrable models.

4.1 The $\mathfrak{gl}(2)$ model: from the spectrum...

We make use of the notations introduced in Section 3.3.1.

Fundamental commutation relations First, let us explicit the FCR (3.33) of the $\mathfrak{gl}(2)$ Yang–Baxter algebra in terms of the monodromy matrix elements A, B, C, D introduced in (3.35).

$$[A(u),A(v)] = 0, \tag{Y.1}$$
$$[B(u),B(v)] = 0, \tag{Y.2}$$
$$[C(u),C(v)] = 0, \tag{Y.3}$$
$$[D(u),D(v)] = 0, \tag{Y.4}$$

$$A(u)B(v) = f(v,u)\,B(v)A(u) + g(u,v)\,B(u)A(v), \tag{Y.5}$$
$$A(u)C(v) = f(u,v)\,C(v)A(u) + g(v,u)\,C(u)A(v), \tag{Y.6}$$

$$B(u)A(v) = f(v,u)\,A(v)B(u) + g(u,v)\,A(u)B(v), \tag{Y.7}$$
$$B(u)D(v) = f(u,v)\,D(v)B(u) + g(v,u)\,D(u)B(v), \tag{Y.8}$$

$$C(u)A(v) = f(u,v)\,A(v)C(u) + g(v,u)\,A(u)C(v), \tag{Y.9}$$
$$C(u)D(v) = f(v,u)\,D(v)C(u) + g(u,v)\,D(u)C(v), \tag{Y.10}$$

$$D(u)B(v) = f(u,v)\,B(v)D(u) + g(v,u)\,B(u)D(v), \tag{Y.11}$$
$$D(u)C(v) = f(v,u)\,C(v)D(u) + g(u,v)\,C(u)D(v), \tag{Y.12}$$

$$[A(u),D(v)] = g(u,v)\big(C(v)B(u) - C(u)B(v)\big), \tag{Y.13}$$
$$[B(u),C(v)] = g(u,v)\big(D(v)A(u) - D(u)A(v)\big), \tag{Y.14}$$
$$[C(u),B(v)] = g(u,v)\big(A(v)D(u) - A(u)D(v)\big), \tag{Y.15}$$
$$[D(u),A(v)] = g(u,v)\big(B(v)C(u) - B(u)C(v)\big). \tag{Y.16}$$

with

$$f(u,v) = \frac{\varphi(u-v+\eta)}{\varphi(u-v)}, \quad g(u,v) = \frac{\varphi(\eta)}{\varphi(u-v)}. \tag{4.1}$$

The φ function depends on the model at hand as in (3.48).

The main idea of the Bethe Ansatz in its algebraic form is to generate the eigenstates of T as the descendants of a reference state $|0\rangle$ by the successive action of some creation operators, yet to be defined, but available somewhere in the Yang–Baxter algebra. In a way, this is a generalization of the harmonic oscillator treatment where the eigenstates $a^\dagger \dots a^\dagger |0\rangle$ of the particle number operator $N = a^\dagger a$ are created by the successive action of creation operators on a pseudo-vacuum state $|0\rangle$, with the algebra

$$\left[a, a^\dagger\right] = 1, \quad [N,a] = -a, \quad \left[N, a^\dagger\right] = a^\dagger, \tag{4.2}$$

and

$$a|0\rangle = 0, \quad a^\dagger|0\rangle \neq 0. \tag{4.3}$$

Since the underlying Yang–Baxter algebra of the model has a richer structure, the setting is more involved, and the actual computation of the eigenstates require more work.

A toy example

Let (J_i, θ_k) be the action–angle variables of a classical integrable mechanical system. We have $\{J_i, \theta_k\} = \delta_{ik}$. Under quantization, the Poisson bracket becomes a commutator

$$[J_i, \theta_k] = i\hbar\delta_{ik}. \tag{4.4}$$

Let us note $A_k := J_k$ and $B_k := e^{ib_k\theta_k}$, $b_k \in \mathbb{C}$, for a fixed k. Since $[J_k, \theta_k^a] = ai\hbar\theta_k^{a-1}$, we have

$$\begin{aligned}[A_k, B_k] &= \sum_{a=0}^{+\infty} \frac{i^a}{a!}\left[J_k, \theta_k^a\right] \\ &= -b_k\hbar \sum_{a=1}^{+\infty} \frac{i^{a-1}}{(a-1)!}\theta_k^{a-1} \\ &= -b_k\hbar B_k.\end{aligned} \tag{4.5}$$

Let $|\varphi\rangle$ be an eigenstate of A_k of eigenvalue λ. The action of B_k on $|\varphi\rangle$ generates a new eigenstate of A_k

$$A_kB_k|\varphi\rangle = B_kA_k|\varphi\rangle - \hbar b_kB_k|\varphi\rangle = (\lambda - b_k\hbar)\, B_k|\varphi\rangle, \tag{4.6}$$

of shifted eigenvalue $\lambda - b_k\hbar$, provided that $B_k|\varphi\rangle \neq 0$. B_k thus realizes a shift operation on the A_k spectrum, meaning that the choice of the parameter b_k defining B_k is a rather subtle issue.

Consider now two operators $\mathcal{A}(u)$ and $\mathcal{B}(u)$ satisfying an exchange algebra

$$\mathcal{A}(u)\mathcal{B}(v) = \beta(u,v)\mathcal{B}(v)\mathcal{A}(u), \tag{4.7}$$

for some u, v. Then for $|\varphi\rangle$ an eigenstate of $\mathcal{A}$ of eigenvalue $\lambda(u)$, the action of $\mathcal{B}$ generates a new eigenstate

$$\mathcal{A}(u)\mathcal{B}(v)|\varphi\rangle = \beta(u,v)\lambda(u)\mathcal{B}(v)|\varphi\rangle\,, \tag{4.8}$$

whose eigenvalue is obtained from the original $\lambda(u)$ one by a multiplicative coefficient $\beta(u,v)$. One should check that (4.8) is a non-zero vector, though.

This motivates the use of the off-diagonal elements of the monodromy matrix to generate the eigenvectors of the transfer matrix from a first known eigenstate $|0\rangle$. Indeed, we know from the quantization procedure that they should contain in some form the quantum counterparts to the angle variables, which in classical $\mathfrak{gl}(n)$ models are generated by off-diagonal elements of the monodromy matrix, see page 31.

Bethe states and ABA calculation Let us go back to the $\mathfrak{gl}(2)$ model with periodic boundary conditions. Let $|0\rangle \in \mathscr{H}$ be a reference state verifying

$$\begin{aligned} &A(u)|0\rangle = a(u)|0\rangle\,, \qquad && D(u)|0\rangle = d(u)|0\rangle\,, \\ &B(u)|0\rangle \neq 0, && C(u)|0\rangle = 0, \end{aligned} \tag{4.9}$$

and consider a *Bethe state* of level m

$$B(u_1)\dots B(u_m)|0\rangle\,, \tag{4.10}$$

where $u_1,\dots,u_m$ are m complex numbers. The state $|0\rangle$ is therefore an eigenstate of $T(u) = A(u) + D(u)$ and shall play the role of the *highest weight state*, such that its descendants (4.10) by the creation operators $B(u_k)$ are eigenstates.

The action of $A(u)$ over this state is easily computed thanks to the above FCR's. Shifting $A(u)$ to the right by making it pass through all the B's to finally act on its eigenvector $|0\rangle$ produces 2^m terms, that re-sum in only $m+1$ different contribution thanks to the commutation of the B's

$$A(u)B(u_1)\dots B(u_m)|0\rangle = \Lambda_0\, B(u_1)\dots B(u_m)|0\rangle + \sum_{k=1}^{m} \Lambda_k\, B(u)B(u_1)\dots\widehat{B(u_k)}\dots B(u_m)|0\rangle\,, \tag{4.11}$$

where the hat $\widehat{\ }$ means the concerned term is omitted. The Λ_0 coefficients is given by following only the direct term in the FCR (Y.5) until $A(u)$ acts on $|0\rangle$, and therefore is

$$\Lambda_0 = a(u)\prod_{\ell=1}^{m} f(u_\ell, u). \tag{4.12}$$

Λ_1 is obtained by using the first indirect term once and then only the direct terms to push $A(u_1)$ to the right and make it act on $|0\rangle$, so

$$\Lambda_1 = a(u_1)g(u,u_1)\prod_{\ell=2}^{m} f(u_\ell, u_1). \tag{4.13}$$

Because the B's commute, the Λ_k coefficients are obtained simply by substituting $u_1 \to u_k$, and eventually

$$\begin{aligned} A(u)\,B(u_1)\dots B(u_m)|0\rangle = a(u)\prod_{\ell=1}^{m} f(u_\ell,u)\,B(u_1)\dots B(u_m)|0\rangle \\ + \sum_{k=1}^{m} a(u_k)g(u,u_k)\prod_{\substack{\ell=1\\ \ell\neq k}}^{m} f(u_\ell,u_k)\,B(u)B(u_1)\dots\widehat{B(u_k)}\dots B(u_m)|0\rangle\,. \end{aligned} \tag{4.14}$$

The action of $D(u)$ is computed similarly

$$\begin{aligned} D(u)B(u_1)\dots B(u_m)|0\rangle &= d(u)\prod_{\ell=1}^{m} f(u,u_\ell)B(u_1)\dots B(u_m)|0\rangle \\ &+ \sum_{k=1}^{m} d(u_k)g(u_k,u)\prod_{\substack{\ell=1\\ \ell\neq k}}^{m} f(u_k,u_\ell)B(u)B(u_1)\dots\widehat{B(u_k)}\dots B(u_m)|0\rangle\,, \end{aligned} \tag{4.15}$$

and the action of $T(u)$ on (4.10) is therefore

$$\begin{aligned} T(u)B(u_1)\dots B(u_m)|0\rangle &= \left(a(u)\prod_{\ell=1}^{m} f(u,u_\ell) + d(u)\prod_{\ell=1}^{m} f(u,u_\ell)\right)B(u_1)\dots B(u_m)|0\rangle \\ &+ \sum_{k=1}^{m}\left(a(u_k)g(u,u_k)\prod_{\substack{\ell=1\\ \ell\neq k}}^{m} f(u_\ell,u_k) + d(u_k)g(u_k,u)\prod_{\substack{\ell=1\\ \ell\neq k}}^{m} f(u_k,u_\ell)\right) \\ &\times B(u)B(u_1)\dots\widehat{B(u_k)}\dots B(u_m)|0\rangle\,. \end{aligned} \tag{4.16}$$

It is clear that if (4.10) is a non-zero vector, it is an eigenstate of $T(u)$ of eigenvalue

$$t(u) = a(u)\prod_{\ell=1}^{m} f(u_\ell,u) + d(u)\prod_{\ell=1}^{m} f(u,u_\ell), \tag{4.17}$$

as soon as the m unwanted contributions in (4.16) vanish, which is the case if the $u_1,\dots,u_m$ verify the m Bethe equations

$$\frac{a(u_k)}{d(u_k)} = \prod_{\substack{\ell=1\\ \ell\neq k}}^{m}\frac{f(u_\ell,u_k)}{f(u_k,u_\ell)}. \tag{4.18}$$

Note that these condition for (4.10) to be an eigenstate of $T(u)$, $u\in\mathbb{C}$, are independent of u (whenever $g(u,v)=-g(v,u)$). This is good agreement with the fact that the $(T(u))_{u\in\mathbb{C}}$ family is commutative, and thus diagonalizable in the same basis. Bethe vectors (4.10) whose parameters satisfy the Bethe equations are called *on-shell*, in opposition with *off-shell* Bethe vectors if the u's are arbitrary complex numbers. Specializing to the inhomogeneous model, the Bethe equations and the eigenvalue associated to the Bethe state (4.10) write explicitly

$$\prod_{j=1}^{N}\frac{\varphi(u_k-\xi_j+\eta)}{\varphi(u_k-\xi_j)} = \prod_{\substack{\ell=1\\ \ell\neq k}}^{m}\frac{\varphi(u_k-u_\ell+\eta)}{\varphi(u_k-u_\ell-\eta)}, \tag{4.19}$$

$$t(u) = \prod_{j=1}^{N}\varphi(u-\xi_j+\eta)\prod_{\ell=1}^{m}\frac{\varphi(u-u_\ell-\eta)}{\varphi(u-u_\ell)} + \prod_{j=1}^{N}\varphi(u-\xi_j)\prod_{\ell=1}^{m}\frac{\varphi(u-u_\ell+\eta)}{\varphi(u-u_\ell)}. \tag{4.20}$$

Note that (4.17) is then an analytic function in u, since the m apparent poles at the u_ℓ are cancelled out by the Bethe equations. An analytic way to recover the Bethe equations is thus to require the analiticity of the eigenvalue of generic Bethe states on the complex plane.

For Bethe state of level 1, there is only one Bethe root u_1 constrained only by a unique Bethe equation (4.19). For Bethe state of level 2, the two Bethe roots u_1 and u_2 have to satisfy the coupled system of two equations (4.19), and already the resolution is more intricate. One may try the naive solution $u_1=\xi_j-\eta$ for some j between 1 and N, cancelling the LHS of the first Bethe equation, but this imposes that $\varphi(u_2-u_1+\eta)=0$ or $u_2=\xi_j$. The corresponding Bethe state $B(\xi_j)B(\xi_j-\eta)|0\rangle$ vanishes,

since $B(\xi_j)B(\xi_j-\eta)=0$ in the $\mathfrak{gl}(2)$ case[1], so this solution of the Bethe equations is not admissible, in the sense that it does not correspond to an eigenvector.

Choice of the reference state The reference state $|0\rangle$ may be constructed in a tensor product form one-site similar states. At site k, the action of the Lax matrix on the vector $|0\rangle_k = \binom{1}{0}$ becomes upper-triangular in the auxiliary space

$$L_{0k}(u)\,|0\rangle_k = \begin{pmatrix} \varphi(u+\eta) & * \\ 0 & \varphi(u) \end{pmatrix}_0 |0\rangle_k\,. \tag{4.21}$$

Hence, the action of the monodromy matrix on

$$|0\rangle = \otimes_{k=1}^N |0\rangle_n = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{4.22}$$

is easily computed by multiplication of scalar upper-triangular matrices

$$M(u)\,|0\rangle = \begin{pmatrix} \prod_{k=1}^N \varphi(u-\xi_k+\eta) & * \\ 0 & \prod_{k=1}^N \varphi(u-\xi_k) \end{pmatrix}_0 |0\rangle \equiv \begin{pmatrix} A(u)\,|0\rangle & B(u)\,|0\rangle \\ C(u)\,|0\rangle & D(u)\,|0\rangle \end{pmatrix}_0 . \tag{4.23}$$

The state $|0\rangle$ indeed verifies (4.9) by identification.

Left Bethe covectors Performing the same calculation with Bethe covectors

$$\langle 0|\,C(v_1)\dots C(v_m), \tag{4.24}$$

with $\langle 0|$ such that

$$\langle 0|\,C(u) \neq 0 \quad \text{and} \quad \langle 0|\,B(u) = 0, \tag{4.25}$$

one proves they are eigenvectors of $T(u)$ if the v_j verify the same Bethe equations (4.19). The computation of the scalar products between Bethe co-vectors and vectors

$$\langle 0| \prod_{j=1}^m C(v_j) \prod_{k=1}^{m'} B(u_k)\,|0\rangle\,, \tag{4.26}$$

is of prime interest for the computation of correlation functions. We elaborate on this in the next section.

Energy and momentum of the eigenstates The physical characteristics of the eigenstates are deduced from equations (3.41) and (3.44) in the homogeneous limit $\xi_i \to 0$. Noting

$$\mathbb{B}(\bar{u}) := B(u_1)\dots B(u_m)\,|0\rangle\,, \tag{4.27}$$

we have

$$P\mathbb{B}(\bar{u}) = \sum_{k=1}^m p_0(u_k)\,\mathbb{B}(\bar{u}), \qquad p_0(u) = -i\ln\frac{\varphi(u+\eta)}{\varphi(u)}, \tag{4.28}$$

$$J^{-1}(H-C)\mathbb{B}(\bar{u}) = -N\frac{\varphi'(\eta)}{2}\mathbb{B}(\bar{u}) + \sum_{k=1}^m \varepsilon_0(u_k)\,\mathbb{B}(\bar{u}), \qquad \varepsilon_0(u) = \frac{\varphi(\eta)^2}{2\varphi(u)\varphi(u+\eta)}. \tag{4.29}$$

[1] This can be proved by using the fact that $(\sigma_j^-)^2 = 0$, and using the reconstruction of σ_j^- by the quantum inverse problem which can be done in two ways: from $B(\xi_j)$ and from $B(\xi_j-\eta)$ [104].

Hence the Bethe roots sums independently in individual contributions to the momentum and energy of the eigenstate, justifying a quasiparticle interpretation: each $B(u_k)$ is a creation operator spawning a quasiparticle of momentum $p_0(u_k)$ and energy $\varepsilon_0(u_k)$ out of the pseudo-vacuum $|0\rangle$. Energy and momentum are linked through

$$\varepsilon_0(u) = i\frac{\varphi(\eta)}{2}\frac{\mathrm{d}p_0}{\mathrm{d}u}(u), \tag{4.30}$$

where the spectral parameter u plays the role of the quasiparticle rapidity[2].

The fundamental state $|\psi\rangle_g$ is the lowest energy state. It is not necessarily the reference state $|0\rangle$ as one can see from (4.29).

Limitations of the ABA The Bethe equations are a system of m fully coupled polynomial equations in the u_k; their resolution is far from trivial. Counting the solutions of the Bethe equations which provide non-zero Bethe eigenstate of the transfer matrix is a difficult task. Moreover, the completeness of the Bethe equations has to be verified: since we impose the form of the eigenstates by ansatz, there is no guarantee we obtain all the eigenstates of T in this way. Completeness of the Bethe equations was investigated in [172–174, 212, 217, 269], and in particular was proved for the XXX chain in [174]. It remains an open problem in the general case, though recent work from Chernyak, Leurent and Volin seems to provide new insights [175]. Another problem is the existence of a reference state $|0\rangle$ from which the eigenstates are constructed. For not so complicated models such as the antiperiodic XXZ chain, such a state does not exist, so the ABA fails straight from the beginning.

4.2 . . . to the correlations functions

The Bethe Ansatz has given access to the spectrum and the left and right eigenstates of the model. But we would like to extract more information and compute all meaningful physical quantities from the integrable structure of the model, namely compute the form factors and correlation functions of the local spin operators S_k^α, $\alpha \in \{x,y,z\}$, $k \in \{1,\dots,N\}$.

Eventually, the building blocks of these quantities are scalar products between off-shell and on-shell Bethe states. We will describe their computation first.

Scalar products (Slavnov's overlaps) Scalar products between on-shell/off-shell Bethe states have been characterized extensively in the literature. Let

$$S_m(\{\mu_j\},\{\lambda_k\}) = \langle 0|\prod_{j=1}^{m} C(\mu_j)\prod_{k=1}^{m} B(\lambda_j)\,|0\rangle\,, \tag{4.31}$$

with $\{\lambda_k\}$ solutions to the Bethe equations. For the $\mathfrak{gl}(2)$ case, the computation (4.31) amounts to make the C operators from the Bethe covector pass through the B operators of the Bethe vector until they hit the reference state $|0\rangle$, on which their action vanishes. Many additional terms arise in this process because of the form (Y.15) of the commutator $[C(u),B(v)]$. It appears that the final result is expressed concisely using determinants. A first step was obtained by Slavnov [132] who derived the following determinant formula

$$S_m(\{\mu_j\},\{\lambda_k\}) = \frac{\det H(\{\mu_j\},\{\lambda_k\})}{\prod_{j>k}\varphi(\mu_k-\mu_j)\prod_{\alpha<\beta}\varphi(\lambda_\beta-\lambda_\alpha)}, \tag{4.32}$$

[2] For the XXX model with $\eta = i$, one gets $p_0(u) = -i\ln(u+\eta)/u = \dots$.

where H is the $m \times m$ matrix of elements

$$H_{ab} = \frac{\varphi(\eta)}{\varphi(\lambda_a - \mu_b)} \left(a(\mu_b) \prod_{k \neq a} \varphi(\lambda_k - \mu_b + \eta) - d(\mu_b) \prod_{k \neq a} \varphi(\lambda_k - \mu_b - \eta) \right). \tag{4.33}$$

His demonstration also proved the symmetry in the Bethe roots of the scalar products: we have $S_m(\{\mu_j\},\{\lambda_k\}) = S_m(\{\lambda_k\},\{\mu_j\})$, where the λ_k verifies the Bethe equations.

This result had been refined ten years later by Maillet, Kitanine and Terras [104, 105], when they obtained (4.32) in another manner, and identified that it could be rewritten in a ratio of two determinants

$$S_m(\{\mu_j\},\{\lambda_k\}) = \frac{\det \mathcal{T}(\{\mu_j\},\{\lambda_k\})}{\det V(\{\mu_j\},\{\lambda_k\})}, \tag{4.34}$$

where $\mathcal{T}$ and V are the $m \times m$ matrices

$$\mathcal{T}_{ab} = \frac{\partial}{\partial \lambda_a} t(\mu_b,\{\lambda_k\}) \quad \text{and} \quad V_{ab} = \frac{1}{\varphi(\mu_b - \lambda_a)}. \tag{4.35}$$

The function $t(u,\{\lambda_k\})$ is the eigenvalue of the transfer matrix $T(u)$ on the on-shell Bethe state $\prod_{k=1}^m B(\lambda_j)|0\rangle$. Remark that on-shell Bethe states with a different number of excitations, i.e. a different number of Bethe roots, are orthogonal (for periodic boundary conditions).

A key ingredient in their derivation was the use of the F-basis [270]. In the F-basis, the monodromy entries $B(u)$ and $C(u)$ write as a sum of only N local, one-site terms, dressed by a tensor product of diagonal operators acting on the other sites [104]. This is a huge combinatorial simplification to the 2^N-term expression of $B(u)$ and $C(u)$ in the original spin basis $|\uparrow\uparrow \ldots \uparrow\rangle, \ldots, |\downarrow\downarrow \ldots \downarrow\rangle$, where each term is a product of N spin operators along the chain.

Note that the formula (4.34) is nothing but the product of a Jacobian of the transfer matrix eigenvalue and a normalization factor. The recent article [271] gives some understanding of *why* scalar products have this determinant form. It shows that the scalar products satisfy a homogeneous system of linear equations, hence the determinant form, while the Jacobian arises from a particular property of the eigenvalue $t(u)$ — namely its linear and symmetric dependence in the Bethe roots, up to a symmetric normalization factor. Still, it would be of great interest to understand why it is a Jacobian.

Norm of on-shell Bethe vectors — the Gaudin formula When the λ_k verify the Bethe equations, $S_m(\{\lambda_k\}) := S_m(\{\lambda_k\},\{\lambda_k\})$ is simply the squared norm of the Bethe vector (4.27). The formula (4.34) is then reducible and can be written as a unique determinant, already conjectured by Gaudin [272] and later argued by Korepin [131]

$$S_m(\{\lambda_k\}) = \varphi(\eta)^m \left(\prod_{k=1}^m a(\lambda_k) d(\lambda_k) \right) \left(\prod_{p \neq q} \frac{1}{f(\lambda_p,\lambda_q)} \right) \det G(\{\lambda_k\}), \tag{4.36}$$

where G is the $m \times m$ Gaudin matrix of elements

$$G_{ab} = -\frac{\partial}{\partial \lambda_b} \ln \left(\frac{a(\lambda_a)}{d(\lambda_a)} \prod_{\substack{k=1 \\ k \neq a}}^m \frac{f(\lambda_a,\lambda_k)}{f(\lambda_k,\lambda_a)} \right). \tag{4.37}$$

Equation (4.34) gives a direct proof of this result by taking the limits $\mu_j \to \lambda_j$ for $j = 1,\ldots,m$.

Form factors The $\mathfrak{gl}(2)$ Yang–Baxter algebra (Y.1)–(Y.16) allows to determine the actions of the A, B, C, D operators on the left and right Bethe states. On the other hand, the action of the local operators

S_k^α on the Bethe state is not immediate. Thankfully, they are reconstructed rather easily from the A, B, C, D, see page 49.

A tedious but straightforward induction computation give the actions of A, B and D on generic Bethe covectors [99, 257]

$$\langle 0|\prod_{j=1}^{m} C(u_j)\, A(u_{m+1}) = \sum_{k=1}^{m+1} a(u_k) \frac{\prod_{l=1}^{m} \varphi(u_l - u_k + \eta)}{\prod_{\substack{l=1\\ l\neq k}}^{m+1} \varphi(u_l - u_k)} \langle 0|\prod_{\substack{j=1\\ j\neq k}}^{m+1} C(u_j), \tag{4.38}$$

$$\langle 0|\prod_{j=1}^{m} C(u_j)\, D(u_{m+1}) = \sum_{k=1}^{m+1} d(u_k) \frac{\prod_{l=1}^{m} \varphi(u_k - u_l + \eta)}{\prod_{\substack{l=1\\ l\neq k}}^{m+1} \varphi(u_k - u_l)} \langle 0|\prod_{\substack{j=1\\ j\neq k}}^{m+1} C(u_j), \tag{4.39}$$

$$\langle 0|\prod_{j=1}^{m} C(u_j)\, B(u_{m+1}) = \sum_{k=1}^{m+1} d(u_k) \frac{\prod_{l=1}^{m} \varphi(u_k - u_l + \eta)}{\prod_{\substack{l=1\\ l\neq k}}^{m+1} \varphi(u_k - u_l)} \tag{4.40}$$

$$\times \sum_{\substack{k'=1\\ k'\neq k}}^{m+1} \frac{a(u_k)}{u_{m+1} - u_k + \eta} \frac{\prod_{\substack{p=1\\ p\neq k'}}^{m+1} \varphi\big(u_p - u_k + \eta\big)}{\prod_{\substack{p=1\\ p\neq k,k'}}^{m+1} \varphi\big(u_p - u_k\big)} \langle 0|\prod_{\substack{j=1\\ j\neq k,k'}}^{m+1} C(u_j). \tag{4.41}$$

The action of $C(u_{m+1})$ is simply absorbed in the Bethe vector and increase its rank by one. Similar relations are obtained for the action of A, C and D on the Bethe vectors, so the computation of form factors may be done by keeping on-shell states at the left or the right of the scalar products and using the scalar product formula (4.32).

The action of the monodromy entries over on-shell Bethe states gives a sum of *off-shell* Bethe states. Therefore, the computation of form factors reduces to a sum of scalar products between on-shell and off-shell Bethe states. While the formula (4.32) for scalar products have been obtained decades ago, it was the resolution of the quantum inverse problem, namely reconstruction of local operators in terms of the monodromy matrix elements obtained in [104] and [106], that enabled the systematic computation of the form factors. For periodic boundary conditions, since scalar products of states with a different number of Bethe roots vanish, only the following form factors are non-zero

$$\begin{gathered}\langle 0|\prod_{j=1}^{m} C(\mu_j)\, S_n^+ \prod_{k=1}^{m+1} B(\lambda_k)\,|0\rangle\,, \quad \langle 0|\prod_{j=1}^{m+1} C(\mu_j)\, S_n^- \prod_{k=1}^{m} B(\lambda_k)\,|0\rangle\,, \\ \langle 0|\prod_{j=1}^{m} C(\mu_j)\, S_n^z \prod_{k=1}^{m} B(\lambda_k)\,|0\rangle\,.\end{gathered} \tag{4.42}$$

For example, explicitly,

$$\begin{aligned}\langle 0|\prod_{j=1}^{m+1} C(\mu_j)\, S_n^- \prod_{k=1}^{m} B(\lambda_k)\,|0\rangle &= \langle 0|\prod_{j=1}^{m+1} C(\mu_j) \prod_{a=1}^{n-1} T(\xi_a)\, B(\xi_n) \prod_{a=1}^{n} T(\xi_a)^{-1} \prod_{k=1}^{m} B(\lambda_k)\,|0\rangle \\ &= \frac{\phi_{n-1}(\{\lambda_k\})}{\phi_{n-1}(\{\mu_j\})} \frac{\prod_{k=1}^m \varphi(\lambda_k-\xi_n+\eta)}{\prod_{k=1}^m \varphi(\mu_j-\xi_n+\eta)} \\ &\quad \times \frac{\det H^-(n,\{\mu_j\},\{\lambda_k\})}{\prod_{j>k}\varphi(\mu_k-\mu_j)\prod_{\beta>\alpha}\varphi(\lambda_\beta-\lambda_\alpha)},\end{aligned} \tag{4.43}$$

with $\phi_n(\{\lambda_k\}) = \prod_{j=1}^n \prod_{k=1}^m f(\lambda_k,\xi_j)$ and H^- is the modified $m\times m$ matrix H from (4.35)

$$H^-_{ab}(n,\{\mu_j\},\{\lambda_k\}) = H_{ab}(\{\mu_j\},\{\lambda_k\}) \qquad \text{for} \quad b<m, \tag{4.44}$$

$$H^-_{am}(n,\{\mu_j\},\{\lambda_k\}) = \frac{\varphi(\eta)}{\varphi(\lambda_a-\xi_n+\eta)\varphi(\lambda_a-\xi_n)} \qquad \text{otherwise.} \tag{4.45}$$

For the detail computations and the exact expressions of the form factors, see Proposition 5.1 and 5.2 of [104].

Correlation functions Let $e^{(k)}_{\alpha_k,\beta_k}$ be the elementary quantum operators acting at the quantum site k, with $\alpha_k,\beta_k = 1,2$. Recalling from Section 3.1, we are interested in the computation of correlation function at zero temperature of the form, the elementary blocks

$$\mathcal{F}_n\big(\{\alpha_k,\beta_k\};\psi_g\big) = \frac{\big\langle\psi_g\big|\prod_{k=1}^m e^{(k)}_{\alpha_k,\beta_k}\big|\psi_g\big\rangle}{\big\langle\psi_g\big|\psi_g\big\rangle}. \tag{4.46}$$

Thanks to (3.69), the local operators are reconstructed as

$$e^{(k)}_{\alpha_k,\beta_k} = \left(\prod_{a=1}^{k-1} T(\xi_a)\right) M_{\beta_k,\alpha_k}(\xi_k) \left(\prod_{a=1}^{k} T(\xi_a)\right)^{-1}, \tag{4.47}$$

while the fundamental state writes as an on-shell Bethe vector

$$\big|\psi_g\big\rangle = \prod_{j=1}^m B(u_j)\,|0\rangle \quad \text{and} \quad \big\langle\psi_g\big| = \langle 0|\prod_{j=1}^m C(u_j), \tag{4.48}$$

where the u_j are solutions to the Bethe equations (4.19). So the following n-point correlation function of n adjacent local operators

$$\mathcal{F}_n\big(\{\alpha_k,\beta_k\};\{u_j\}\big) = \frac{\langle 0|\prod_{j=1}^m C(u_j)\ \prod_{k=1}^n e^{(k)}_{\alpha_k,\beta_k}\ \prod_{j=1}^m B(u_j)\,|0\rangle}{\langle 0|\prod_{j=1}^m C(u_j)\prod_{j=1}^m B(u_j)\,|0\rangle}, \tag{4.49}$$

is the expectation value of a product of monodromy entries in the ground state $\prod_{j=1}^m B(u_j)\,|0\rangle$

$$\mathcal{F}_n\big(\{\alpha_k,\beta_k\};\{u_j\}\big) = \psi_n(\{u_j\}) \frac{\langle 0|\prod_{j=1}^m C(u_j)\ \prod_{k=1}^n M_{\beta_k,\alpha_k}(\xi_k)\ \prod_{j=1}^m B(u_j)\,|0\rangle}{\langle 0|\prod_{j=1}^m C(u_j)\prod_{j=1}^m B(u_j)\,|0\rangle}, \tag{4.50}$$

where

$$\psi_n(\{u_j\}) = \prod_{\alpha=1}^n \left(\frac{1}{a(\xi_\alpha)} \prod_{j=1}^m \frac{1}{f(u_j,\xi_\alpha)}\right) \tag{4.51}$$

is a scalar coefficient given by the action of the dressing product of transfer matrices in (4.47), namely the eigenvalue of the product $\left(\prod_{\alpha=1}^{n} T(\xi_\alpha)\right)^{-1}$ corresponding to the Bethe eigenstate $\prod_{j=1}^{m} B(u_j)|0\rangle$.

From there, formula (4.50) may be computed explicitly in terms of the Bethe roots [105]: one makes the monodromy entries act on the left Bethe state and produce a sum of off-shell/on-shell scalar products of the form (4.31). This produces n sums over the Bethe parameters of the left Bethe state. In the thermodynamic limit $N \to +\infty$, these n sums become n integrals over Bethe roots that are continuous parameters, over fixed ranges and with certain densities for the Bethe roots. The reader can refer to [105] for explicit formula of the emptiness formation probability and more general correlation functions. Further important objects are the physical 2-point functions $\langle \sigma_1^z \sigma_{m+1}^z \rangle$ and their arguments for $m \to +\infty$ with $m/N \to 0$, see [105, 133, 134, 273–276].

This closes the study of the $\mathfrak{gl}(2)$ model from the algebraic Bethe ansatz view-point. We will now describe how it extends to higher rank models

4.3 Higher rank models

The Algebraic Bethe Ansatz method generalizes to higher rank $\mathfrak{gl}(\mathfrak{n})$ models where $n \geq 3$. As usual the first step is the most difficult: most of the results have been obtained first by adapting the proofs to the $\mathfrak{gl}(3)$ model, before being extended to the general case.

As we will see, with the ABA for higher rank spin chains, the eigenstates are given by cumbersome formulas. This makes without surprises the results on scalar products and correlation functions just as intricate, despite considerable recent progress [129, 271].

The nested ABA The above Bethe Ansatz scheme generalizes to the higher rank model: a reference state $|0\rangle$ serves as a starter to create other eigenstates using creation operators derived from the monodromy $M(u)$.

While in the $\mathfrak{gl}(2)$ model there was only one raising operator $B(u) \equiv M_{12}(u)$ for the Bethe states, there are now $\mathfrak{n}(\mathfrak{n}-1)/2$ operators $M_{ij}(u)$, $1 \leq i < j \leq n$, that can create excitations from a pseudo-vacuum state $|0\rangle$, and $\mathfrak{n}-1$ sets of Bethe roots $\{u_1^{(j)}, \dots, u_{a_j}^{(j)}\}$ where $j = 1, 2, \dots \mathfrak{n}-1$ and $a_j \in \mathbb{Z}_{\geq 0}$.

The bar notations introduced in [158] are useful to write formulas of the NABA concisely (see also [129]). They are as follows:

- the overhead bar denotes a set of elements: $\bar{u} := \{u_1, \dots, u_m\}$
- individual elements of a set have a latin subscript
- cardinality is noted by a hash: $\#\bar{u} = m$
- subsets of elements are denoted by roman indices: $\bar{u}_{\mathrm{I}}$, $\bar{u}_{\mathrm{II}}$
- a overhead bar with a single latin subscript denotes a subset of cardinality $\#\bar{u} - 1$, for example $\bar{u}_k := \bar{u} \setminus u_k$
- therefore the set of level j Bethe roots is $\bar{u}^{(j)} = \{u_1^{(j)}, \dots, u_{m_j}^{(j)}\}$, and the set of all Bethe roots is $\bar{u} = \{\bar{u}^{(1)}, \dots, \bar{u}^{(\mathfrak{n}-1)}\}$
- when a quantity takes a set as an argument, the product over the elements of the set is implied:

$$f(z, \bar{u}^{(j)}) := \prod_{k=1}^{m_j} f(z, u_k^{(j)}), \quad f(z, \bar{u}) := \prod_{j=1}^{\mathfrak{n}-1} \prod_{k=1}^{m_j} f(z, u_k^{(j)}),$$
$$f(\bar{u}_{\mathrm{I}}, \bar{u}_{\mathrm{II}}) := \prod_{w \in \bar{u}_{\mathrm{I}}} \prod_{w' \in \bar{u}_{\mathrm{II}}} f(w, w'). \tag{4.52}$$

Generic Bethe vectors and covectors, associated to the Bethe roots $\bar{u}$, are also denoted with the bar notation as $\mathbb{B}(\bar{u})$ and $\mathbb{C}(\bar{u})$ respectively[3]. Their form is now non-trivial, since the monodromy entries

above the diagonal are not independent creation operators for the Bethe states. Moreover, the $M_{ij}(u)$, with $i<j$, do not commute together, so in order to properly count the different Bethe states, one should agree on an ordering in the M_{ij} (which is not specified in the $\mathbb{B}(\bar{u})$ notation).

Let us be specific with the $\mathfrak{gl}(3)$ fundamental model, the first higher rank spin chain. Noting $\bar{u} := \{u_1^{(1)},\dots,u_{a_1}^{(1)}\}$ and $\bar{v} := \{u_1^{(2)},\dots,u_{a_2}^{(2)}\}$ the Bethe roots of order 1 and 2, respectively, Bethe vectors write

$$\mathbb{B}(\bar{u},\bar{v}) = \sum_{\substack{\{\bar{u}_{\mathrm{I}},\bar{u}_{\mathrm{II}}\}\\ \{\bar{v}_{\mathrm{I}},\bar{v}_{\mathrm{II}}\}}} C(\bar{u}_{\mathrm{I}},\bar{u}_{\mathrm{II}},\bar{v}_{\mathrm{I}},\bar{v}_{\mathrm{II}}) M_{12}(\bar{u}_{\mathrm{II}}) M_{13}(\bar{u}_{\mathrm{I}}) M_{23}(\bar{v}_{\mathrm{II}})\,|0\rangle\,, \tag{4.53}$$

where the sum is taken over 2-partitions of the sets of Bethe roots, with $\bar{u}_{\mathrm{I}} \sqcup \bar{u}_{\mathrm{II}} = \bar{u}$ and $\bar{v}_{\mathrm{I}} \sqcup \bar{v}_{\mathrm{II}} = \bar{v}$. $C(\bar{u}_{\mathrm{I,II}},\bar{v}_{\mathrm{I,II}})$ is a convoluted coefficient in the Bethe roots [see 129].

Like in the $\mathfrak{gl}(2)$ case, acting with the transfer matrix on the state $\mathbb{B}(\bar{u})$ produces many unwanted terms. These contributions cancel each other for certain values of the $\bar{u}$ parameters. Returning to the general $\mathscr{Y}(\mathfrak{gl}(n))$ case, a Bethe vector $\mathbb{B}(\bar{u})$ is an eigenvector of $T(z)$

$$T(z)\mathbb{B}(\bar{u}) = t(z;\bar{u})\mathbb{B}(\bar{u}) \tag{4.54}$$

of eigenvalue

$$t(z;\bar{u}) = \sum_{i=1}^{\mathfrak{n}} \alpha_i(z) f(z,\bar{u}^{(i-1)}) f(\bar{u}^{(i)},z) \tag{4.55}$$

provided the Bethe roots $\bar{u}$ satisfy the coupled system of $\mathfrak{n}-1$ Bethe equations

$$\forall i = 1,\dots,\mathfrak{n}-1, \quad \frac{\alpha_i(\bar{u}_{\mathrm{I}}^{(i)})}{\alpha_{i+1}(\bar{u}_{\mathrm{I}}^{(i)})} = \frac{f(\bar{u}_{\mathrm{I}}^{(i)},\bar{u}_{\mathrm{II}}^{(i)})}{f(\bar{u}_{\mathrm{II}}^{(i)},\bar{u}_{\mathrm{I}}^{(i)})} \frac{f(\bar{u}^{(i+1)},\bar{u}_{\mathrm{I}}^{(i)})}{(\bar{u}_{\mathrm{I}}^{(i)},\bar{u}^{(i-1)})}, \tag{4.56}$$

with the convention $\bar{u}^{(0)} = \emptyset = \bar{u}^{(\mathfrak{n})}$. Bethe covectors $\mathbb{C}(\bar{u},\bar{v})$ have a similar expression [129] and are eigencovectors of same eigenvalue (4.55) when on-shell.

Equation (4.53) is an explicit formula among the known characterizations of the $\mathfrak{gl}(\mathfrak{n})$ Bethe vectors:

- the trace formula — Bethe vectors are obtained as a trace over a big auxiliary space, whose dimension is the number of Bethe roots
- the recursion formula — Bethe vectors are expressed in terms of the ones with a smaller number of Bethe roots
- the current presentation formula — from the current realization of the algebra, Bethe vectors are obtained by a projection method
- the explicit formula — Bethe vectors write explicitly as a linear combination of the different possible ordered combinations of the M_{ij}, $1 \le i < j \le n$.

These are all equivalent presentations of generic Bethe vectors [129]. Some are more convenient to manipulate depending on what one wants to achieve, but they all account for the underlying complexity inherent to generic Bethe vectors — that is the correctly weighted combination of the M_{ij}, $1 \le i < j \le N$ supplemented by the deep coupling between Bethe roots at all level. We have not discussed the normalization of Bethe vectors here, but note that imposing a particular coefficient for the "main term" in the Bethe vectors is a key ingredient in the derivations of above and upcoming formulas.

Scalar products Because of the form of the Bethe vectors, computation of off-shell/on-shell scalar products is already a really difficult task. An analog of the determinant formula (4.31) for higher rank

[3] We deviate from the notation $\mathbb{B}_{\bar{m}}(\bar{u})$ of [129] to ease the formulas. Since the cardinality of the $\bar{u}^{(j)}$ sets is fixed to m_j at definition, it is not crucial to remind it in the Bethe vector notation — at least for our usage of them.

algebras is not known as today, though it has been obtained in some particular cases [129, 150, 151, 277].

However, there exists a sum formula, first found for $\mathfrak{gl}(2)$ in [131] and then generalized to the higher rank case in [160–162]. The scalar product of two Bethe vectors is written

$$\mathbb{C}(\bar{s})\mathbb{B}(\bar{t}) = \sum_{\mathrm{I},\mathrm{II}} c(\bar{s}_{\mathrm{I}}, \bar{t}_{\mathrm{II}}) W_{\mathrm{part}}(\bar{s}_{\mathrm{I}}, \bar{s}_{\mathrm{II}} | \bar{t}_{\mathrm{I}}, \bar{t}_{\mathrm{II}}), \tag{4.57}$$

where $c(\bar{s}_{\mathrm{I}}, \bar{t}_{\mathrm{II}})$ is a scalar coefficient depending on the representation, and W_{part} is computed from the highest coefficients of the Bethe vectors.

The squared norm of on-shell Bethe vectors enjoys a Gaudin formula similar to (4.36), where the Gaudin matrix is a block matrix consisting of $\mathfrak{n}-1$ square blocks, each of them being of size $m_j \times m_j$ the number of Bethe roots at level j [129].

Form factors and correlation functions Form factors

$$\mathbb{C}(\bar{u}) M_{ij}(z) \mathbb{B}(\bar{v}) \tag{4.58}$$

and correlation functions

$$\mathbb{C}(\bar{u}) \prod_{k=1}^{n} e^{(k)}_{\alpha_k,\beta_k} \mathbb{B}(\bar{v}) \tag{4.59}$$

may be computed in the same fashion as in the $\mathfrak{gl}(2)$ model. Because of the form of Bethe vectors and scalar products, this turns out to be a tremendous task which requires the use of tricks and subtler approaches than plain, brute force calculations. We will not linger on these and refer to article [129] which presents a detailed summary of these techniques.

However, let us note that it could be sufficient to compute just one form factors, as the other ones can be obtained from the form factor of a diagonal matrix element in special limits of the Bethe parameters by the zero mode method [278].

4.4 Summary

The Algebraic Bethe Ansatz technique is a powerful method to generate eigenstates of a quantum integrable model as Bethe vectors, using creation operators on a pseudo-vacuum state called the reference state. While the family of conserved quantities is constructed from the diagonal elements of the monodromy, equivalent of creation/annihilation operators are constructed from the off-diagonal ones at special points. These points, the Bethe roots, are tightly constrained all together by the Bethe equations, which arise from the cancellation of the unwanted terms in the action of the transfer matrix over Bethe vectors, as in (4.16). From the Bethe vector description of the eigenstate, important results for scalar products, form factors and correlation functions of $\mathfrak{gl}(\mathfrak{n})$ models have been obtained.

However, the above review pointed out the fundamental requirements necessary to implement the ABA procedure. The first one is the necessity of a reference state, whose existence is not guaranteed as we already highlighted. This is a great restriction that can make the ABA fail from the very start. Moreover, being an ansatz, one should ensure that all the eigenvectors of the transfer matrix are indeed obtained by this procedure. In particular, this requires to check that the constructed vector are non-zero. Finally, one can mention the intricacy of the nested procedure for higher rank models, which impact the calculations that rely on them this description of the eigenstates. Scalar products and form factors are therefore difficult to handle in their NABA form, and this as early as the first higher rank case of the $\mathfrak{gl}(3)$ models. This motivates alternative approaches, such as the quantum separation of variables which is at the center of this thesis.

Why separation of variables seems such a promising tool for quantum integrable models? A first strong argument for the SoV method is that it is already a canonical method for treating classical integrable models. Besides, SoV already proved most useful at the quantum level for $\mathfrak{gl}(2)$ models, in particular when tackling non-trivial boundary conditions, all the more so as the ABA does not work in some of these settings—see references [176, 214, 215] for antiperiodic $\mathfrak{gl}(2)$ chains for example—even though there are some variants such as the modified ABA that allow to tackle these models [181, 279]. Moreover, observations based mainly on numerics for small chains [178, 280] indicated that more compact descriptions of eigenstates (than with ABA) could be obtained by SoV in $\mathfrak{gl}(n)$ models, $n \geq 3$. [4] In the next chapters, we present quantum SoV in details, from the classical case to the quantum higher rank and supersymmetric chains.

[4] As we will see, this is indeed achieved by the SoV framework developed by Maillet and Niccoli in [225], which give the whole spectrum. Ryan and Volin then proved that this SoV construction is equivalent to the one relying on the eigenbasis the $\mathcal{B}(u)$ operator [236, 237], so that the transfer matrix eigenstate have a non-nested and compact form using a single $\mathcal{B}(u)$ operator.

Chapter 5 Fundamentals of separation of variables

In this chapter, we introduce the separation of variables procedure in generality for classical [34] and quantum Hamiltonian systems [281]. We begin by a description of separation of variables as a tool to solve the differential equations that characterize the motion in classical and quantum Hamiltonian systems. Then, we focus on Sklyanin construction of separate variables in classical integrable models in details [186–190], and show how to quantize this construction to produce a quantum SoV procedure [177, 188, 192, 193, 206]. Some limitations of the quantization procedure for higher rank $\mathscr{U}(\mathfrak{gl}(\mathfrak{n}))$ quantum integrable models are discussed [206], and a review of SoV results of the last decades closes the chapter.

5.1 SoV in classical systems

5.1.1 Classical SoV

For a classical Liouville integrable system, we outlined in Chapter 2 that the Liouville–Arnol'd theorem ensures the existence of desirable canonical coordinate transformation[1] $(q,p)\to(\psi,F)$ computable by quadrature. "By quadrature" simply means there is a sole integral to perform to compute the generating function $S(q,F)$ of the canonical coordinate transformation, which is the curvilinear integral (2.13)

$$S(q,F)=\int_{m_0}^{m}\alpha=\int_{q_0}^{q}\sum_i p_i(q,f)\,\mathrm{d}q_i \tag{5.1}$$

on the curve $m_0(q_0,p_0)\to m(q,p)$ laying on the level submanifold $\mathcal{M}_f$ specified by the values f_i of the constants of the motion F_i. To compute S, one should first get the actual parametrization $p_i(q,f)$ of the canonical momenta by the q_i's over $\mathcal{M}_f$. Since S defines a canonical transformation, these are simply the partial differential of S with respect to the q_i's we already stated in (2.9)

$$p_i(q,f)=\frac{\partial S}{\partial q_i}. \tag{5.2}$$

Plugging these in (2.19) it gives

$$H\left(q_1,\dots,q_n,\frac{\partial S}{\partial q_1},\dots,\frac{\partial S}{\partial q_n}\right)=E. \tag{5.3}$$

This is known as the Hamilton–Jacobi equation — and more precisely as the *restricted* Hamilton–Jacobi equation, for systems where the Hamiltonian does not involve time explicitly. It is a partial differential equation of order 1 in n variables $q_1,\dots,q_n$ for the generating function $S(q_1,\dots,q_n,F_1,\dots,F_n)$. Solving it for S provides the desired coordinate transformation and solves the mechanical system, but it is a priori a hard task. In practice, this equation is tractable only *when the variables are separable*.

Separable coordinate in the H–J equation

[1] That is to canonical variables where the time evolution is linear.

Definition. *A coordinate q_1 is said to be* separable in the H–J equation *if the generating function $S(q,F)$ can be split in two additive terms $S_1(q_1,F)$ and $S^\bullet(q_2,\dots,q_n,F)$—one depending* only *on q_1 and one independent of q_1*—and *the H–J equation can be split in two independent H–J equations: one for S_1 and one for $S^\bullet$.*

Motivation. Say q_1 is a separable variable as explained above. Then

$$S = S_1(q_1,F_1,\dots,F_n) + S^\bullet(q_2,\dots,q_n,F_1,\dots,F_n), \tag{5.4}$$

so (5.3) becomes

$$H\left(q_1, \frac{\mathrm{d}S_1}{\mathrm{d}q_1}, q_2, \dots, q_n, \frac{\partial S^\bullet}{\partial q_2}, \dots, \frac{\partial S^\bullet}{\partial q_n}\right) = E. \tag{5.5}$$

In practice the H–J equation is separable in two when the contribution of q_1 and $p_1 \equiv \partial S/\partial q_1$ can be "segregated" in the Hamiltonian[2], such that the above equation can be rewritten

$$H\left(q_1, \frac{\mathrm{d}S_1}{\mathrm{d}q_1}, f\left(q_2, \dots, q_n, \frac{\partial S^\bullet}{\partial q_2}, \dots, \frac{\partial S^\bullet}{\partial q_n}\right)\right) = E. \tag{5.6}$$

The function f is some helper function that repacks the dependence of H in the $2n$ variables q_i, p_i in a dependence in *three* variables q_1, p_1 and the function f of the remaining canonical variables $q_{i\geq 2}$, $p_{i\geq 2}$. Note that $f(q_2,\dots,q_n,p_2,\dots,p_n)$ is then necessarily a constant of the motion:

$$\begin{aligned}
\{H,f\} &= \sum_{i=1}^{n} \frac{\partial H}{\partial p_i}\frac{\partial f}{\partial q_i} - \frac{\partial f}{\partial p_i}\frac{\partial H}{\partial q_i} \\
&= \frac{\partial H}{\partial p_1}\frac{\partial f}{\partial q_1} - \frac{\partial f}{\partial p_1}\frac{\partial H}{\partial q_1} + \sum_{i=2}^{n} \frac{\partial H}{\partial p_i}\frac{\partial f}{\partial q_i} - \frac{\partial f}{\partial p_i}\frac{\partial H}{\partial q_i} = 0 \\
&= 0 + \frac{\partial H}{\partial f}\{f,f\} = 0.
\end{aligned} \tag{5.7}$$

In principle (5.6) can be inverted by the implicit function theorem: there exists a function g such that

$$f\left(q_2, \dots, q_n, \frac{\partial S^\bullet}{\partial q_2}, \dots, \frac{\partial S^\bullet}{\partial q_n}\right) = g\left(q_1, \frac{\mathrm{d}S_1}{\mathrm{d}q_1}, E\right). \tag{5.8}$$

The two sides of this formula are equal but depends on independent variables, respectively $q_{i\geq 2}$ and q_1. They have to be equal to the same constant, the separation constant α_1, which is the value of the constant function f and is some function of the f_i. With the notation $H^\bullet := f$ and $H_1 := g$, we have

$$H_1\left(q_1, \frac{\mathrm{d}S_1}{\mathrm{d}q_1}, E\right) = \alpha_1, \tag{5.9}$$

$$H^\bullet\left(q_2, \dots, q_n, \frac{\partial S^\bullet}{\partial q_1}, \dots, \frac{\partial S^\bullet}{\partial q_n}\right) = \alpha_1, \tag{5.10}$$

and it is clear that the H–J equation has been separate in two independent H–J equations. □

If all the coordinates q_i are separable, the system is said completely separable. By iterating the above procedure on the residual term $H^\bullet$, the generating function is fully separate as a sum of n terms with a mutually exclusive dependence in one q_i, while the Hamilton–Jacobi equation is split in n differential

[2] Quote from [34].

equations of order one *in one variable* for the function S

$$S(q_1,\dots,q_n,F) = S_1(q_1,F) + \cdots + S_n(q_n,F), \tag{5.11}$$

$$H_i\left(q_i, \frac{\partial S_i}{\partial q_i}, F\right) = \alpha_i \quad \text{for} \quad i = 1,\dots,n. \tag{5.12}$$

Note that the separation constants α_i are some function of the values f_i of the conserved quantities F_i on the level manifold $\mathcal{M}_f$, so we can write

$$\tilde{H}_i\left(q_i, \frac{\partial S_i}{\partial q_i}, F_1,\dots,F_n\right) = 0 \tag{5.13}$$

in place of (5.12), where $\tilde{H}_i = H_i - \alpha_i$.

The generating function $S(q,F)$ is now constrained by its n independent partial derivatives, for which we do have closed equations. Solving these n differential equations in one variable gives (by (2.9)) the parametrization $p_i(q_i,F)$ over $\mathcal{M}_f$ of the original canonical momenta — which in fact depend solely on their associated coordinate q_i, a feature of the separate variables. The generating function S of the coordinate transformation $(q,p) \to (\psi,F)$ that linearizes the dynamics is then constructed by n independent quadratures in an additive separate form (5.11) which explicitly is

$$S(q_1,\dots,q_n,F_1,\dots,F_n) = \sum_{i=1}^{n} \int_{q_i(m_0)}^{q_i(m)} p_i(q_i,F_1,\dots,F_n)\,\mathrm{d}q_i\,. \tag{5.14}$$

Remark 1 (Ignorable coordinates). Coordinates that do not appear explicitly in the Hamiltonian are called *ignorable* or *cyclic*. They are easily separable. The conjugate momentum p_1 of a cyclic variable q_1 is constant since $\dot{p}_1 = \{p_1,H\} = -\partial H/\partial q_1 = 0$. Coordinate q_1 is also immediately separable: with S in the form (5.4), equation (5.5) involves only $S^\bullet$ and S_1 is solution of the equation

$$p_1 = \alpha_1 = \frac{\partial S_1}{\partial q_1}, \tag{5.15}$$

where α_1 is some (separation) constant.

Separate variables in Liouville integrable systems Essentially, what is done on the H–J equation is merely the extraction of a conserved quantity from the expression of the Hamiltonian. When working with a Liouville integrable system, we already have some knowledge of conserved quantities. Therefore, we can define separate canonical variables in the following way.

Definition (Classical separate variables)**.** *Consider a Liouville integrable system with a phase space $\mathcal{M}$ and Hamiltonian H, and N independent constants of the motion $F_1,\dots,F_N$ in involution*

$$\forall i,j = 1,\dots,N, \quad \{F_i,F_j\} = 0 \quad \textit{and} \quad \{H,F_j\} = 0. \tag{5.16}$$

Let (x_i,z_i) be $2N$ canonical variables

$$\{x_i,x_j\} = \{z_i,z_j\} = 0, \quad \{x_i,z_j\} = \delta_{ij}. \tag{5.17}$$

The x_i are separate coordinates if there exists independent separate relations

$$\mathcal{F}_i\left(x_i,z_i,F_1,\dots,F_n\right) = 0, \tag{5.18}$$

where the $\mathcal{F}_i$ are C^1 functions and $\partial F_i/\partial z_i \neq 0$. The system is thus fully separate.

Change of variables Usually the original canonical coordinates (q_i, p_i) do not enable the separation of the system, and it is necessary to perform a first coordinate transformation to new canonical variables, say (x_i, z_i), that are separable

$$(q,p) \xrightarrow[\text{separate variables}]{1^{\text{st}}\text{ CT to}} (x,z) \xrightarrow[\text{pre-action–angle variables}]{2^{\text{nd}}\text{ CT to}} (\psi,F). \tag{5.19}$$

It is now the generating function $S(x,F)$ that we solve for. Usually, the coordinate transformation to the separate variables is hinted by the symmetries and invariances of the system and other physically meaningful statement made on it.

Central force problem by separation of variables

Let $\vec{\mathbf{r}}$ be the position of a particle of mass m, with Cartesian coordinates $\vec{\mathbf{r}} = (x,y,z)$ in its configuration space $\mathbb{R}^3$. The phase space of the system is the 6-dimensional Poisson manifold with original canonical variables (x,y,z,p_x,p_y,p_z), where the p_i's are the conjugated canonical momenta to the x,y,z coordinates. The particle evolves in the potential $V(r)$ of some central force $\vec{\mathbf{F}}(\vec{\mathbf{r}})$ parallel to the position vector $\vec{\mathbf{r}}$ from the origin, $r = \sqrt{x^2+y^2+z^2}$; the dynamics is thus prescribed by a Hamiltonian of the form

$$H = \frac{\|\vec{\mathbf{p}}\|^2}{2m} + V(r) \quad \text{using} \quad \vec{\mathbf{p}} = \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix}. \tag{5.20}$$

It is well-known that the motion of this isolated system is planar and has constant areal velocity, reducing the number of effective degrees of freedom of the particle. Precisely,

- because the force derives from a potential, it is conservative and the energy E is conserved,
- $\mathrm{d}\vec{\mathbf{L}}/\mathrm{d}t = \vec{\mathbf{r}} \times \vec{\mathbf{F}} = \vec{\mathbf{0}}$ given that $\vec{\mathbf{F}} \parallel \vec{\mathbf{r}}$, so the norm and direction of the angular momentum $\vec{\mathbf{L}}$ are conserved, hence $\left\|\vec{\mathbf{L}}\right\|$ and the projection $L_\uparrow$ of the angular momentum on some axis $O_\uparrow$ orthogonal to the plane of the motion (usually set to be O_z) are constants of the motion.

This gives three constants of the motion $F_1 = E$, $F_2 = \left\|\vec{\mathbf{L}}\right\|$ and $F_3 = L_\uparrow$. One can prove they are independent and in involution with additional calculations, making this system Liouville integrable. This knowledge acquired from the conserved quantities can be leveraged to simplify the description of problem (like using polar coordinates in the plane of motion), but let us look for separate variables straight away. Because of the spherical symmetry of $V(r)$, the spherical coordinates (r,θ,φ) are the adapted way to describe this system. With $\vec{\nabla} = \left(\partial/\partial r\,, r^{-1}\,\partial/\partial\theta\,, (r\sin\theta)^{-1}\,\partial/\partial\varphi\right)$ the gradient in spherical coordinates, the Hamiltonian becomes

$$H = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\varphi^2}{r^2\sin^2\theta}\right) + V(r). \tag{5.21}$$

This system is completely separable in these canonical variables as we shall see. Let us pick the generating function $S(\vec{\mathbf{r}},\vec{\mathbf{F}})$ of the coordinate transformation $(\vec{\mathbf{r}},\vec{\mathbf{p}}) \to (\vec{\psi},\vec{\mathbf{F}})$ in a separate form in the spherical coordinates

$$S(\vec{\mathbf{r}},\vec{\mathbf{F}}) = S_r(r,\vec{\mathbf{F}}) + S_\theta(\theta,\vec{\mathbf{F}}) + S_\varphi(\varphi,\vec{\mathbf{F}}). \tag{5.22}$$

The coordinate φ is ignorable, so we can introduce the constant α_φ with $\partial S_\varphi/\partial\varphi = \alpha_\varphi$. The H–J

equation (5.3) is written

$$\left(\frac{\partial S_r}{\partial r}\right)^2 + \frac{1}{r^2}\left[\left(\frac{\partial S_\theta}{\partial \theta}\right)^2 + \left(\frac{\alpha_\varphi}{\sin\theta}\right)^2\right] + 2mV(r) = 2mE. \tag{5.23}$$

The θ-dependent part is "segregated", so following (5.6) this is separate in two equations with the introduction of the constant α_θ

$$\left(\frac{\partial S_\theta}{\partial \theta}\right)^2 + \frac{\alpha_\varphi^2}{\sin^2\theta} = \alpha_\theta^2, \tag{5.24}$$

$$\left(\frac{\partial S_r}{\partial r}\right)^2 + \frac{\alpha_\theta^2}{r^2} + 2mV(r) = 2mE. \tag{5.25}$$

Together with

$$\frac{\partial S_\varphi}{\partial \varphi} = \alpha_\varphi, \tag{5.26}$$

this makes the system entirely separate, and integrating these three equations provide a formal solution to the H–J equations of the system, solving its dynamics. Obviously the resolution is a bit intricate, because we have not accounted for the planar motion in the description of the system, so both the angle θ and φ parametrize the trajectory[3]. Nevertheless, the separation constants have a clear physical interpretation:

- $\alpha_0 \equiv E$ is the energy of the system,
- $\alpha_\varphi = p_\varphi$ is the constant value of the projection of the angular momentum $\vec{\mathbf{L}} = \vec{\mathbf{r}} \times \vec{\mathbf{p}}$ along the polar axis O_φ[4],
- $\alpha_\theta = \sqrt{p_\theta^2 + \frac{p_\varphi^2}{\sin^2\theta}}$ is the constant norm $\left\|\vec{\mathbf{L}}\right\|$ of the angular momentum.

We identify easily $F_1 = E \equiv \alpha_0$ and $F_3 = \left\|\vec{\mathbf{L}}\right\| \equiv \alpha_\theta$, while α_φ is a projection of the $\vec{\mathbf{L}}$ vector, also constant since $\partial\vec{\mathbf{L}}/\partial t = \vec{\mathbf{0}}$, but a priori O_φ is not the axis orthogonal to the plane of the motion. Nonetheless, it is uniquely defined by the constants F_i by some function $\alpha_\varphi(F_1, F_2, F_3)$.

In summary, separation of variables (SoV) in a Liouville integrable classical system is a procedure that identifies or constructs canonical variables in which the Hamilton–Jacobi equation becomes tractable, thanks to the newly obtained separate form. In essence, it performs a reduction from a coupled n variable problems to n independent problems in one variable. It is a primer to the full resolution of the system: solving the Hamilton–Jacobi equation characterizes a coordinate transformation to momenta that are constants of the motions, effectively solving the dynamics.

In the context of the inverse scattering method, Sklyanin has introduced an approach to define separation of variables for integrable models. We expose the classical and quantum cases in the next sections.

5.1.2 Classical SoV for integrable systems in the CISM framework

The classical spin 1/2 chain of length N is a Liouville integrable system: thanks to the CISM, one can construct N independent constants of the motion in involution. Characterizing the coordinate transform

[3] The description of the system in polar coordinates is recovered by taking the initial condition $\theta(t=0) = \pi/2$ and $\dot{\varphi}(t=0) = 0$, fixing $\theta = \pi/2$ so that the motion is described only with the (polar) coordinates (r, φ).

[4] p_θ is the canonical momentum conjugated to the coordinate θ, and hence the component of $\vec{\mathbf{L}}$ along the O_φ axis. It is *not* the component of the momentum vector $\vec{\mathbf{p}} = m\dot{\vec{\mathbf{r}}}$ along the $\hat{\mathbf{e}}_\varphi$ unit vector of the local spherical frame.

to the constants of the motion solves the system. Can we use separation of variables to do so? Is the system fully separable? What are the separate variables? In the framework of the CISM, these questions about SoV were pioneered by Sklyanin see [186, 187] and the review [188]. We need to introduce some objects in order to describe the SoV method.

Spectral curve & Baker–Akhiezer function Consider the classical $\mathfrak{gl}(\mathfrak{n})$ spin chains introduced in Chapter 2. Let $\chi_M(u,z)$ be the characteristic polynomial of the $\mathfrak{n}\times\mathfrak{n}$ monodromy matrix $M(u)$

$$\chi_M(u,z) := \det\big(z1 - M(u)\big), \tag{5.27}$$

where the determinant is taken on the auxiliary space $\mathbb{V}_0$. The spectral curve is the locus of the eigenvalues $z(u)$ of the monodromy matrix at point u for the spectral parameter. It is an algebraic curve in $\mathbb{C}^2$ defined by the equation

$$\chi_M(u,z) = 0. \tag{5.28}$$

It implicitly defines the eigenvalue $z(u)$ of $M(u)$ as a $\mathfrak{n}$-multivalued function on $\mathbb{C}$. The eigenfunction $\Omega(u)$ associated to an eigenvalue $z(u)$

$$M(u)\Omega(u) = z(u)\Omega(u) \tag{5.29}$$

is called the Baker–Akhiezer function [188, 282, 283]. The normalization of the $\Omega(u)$ vector will play a specific role, so we fix it by the use of some coefficients $\alpha_i(u)$ and the linear constraint on the components of $\Omega(u)$

$$\sum_{i=1}^{\mathfrak{n}} \alpha_i(u)\Omega_i(u) = 1. \tag{5.30}$$

Relation between the spectral invariants The determinant in (5.27) may be expanded as

$$\chi_M(u,z) = \sum_{k=0}^{\mathfrak{n}} (-1)^k z^{\mathfrak{n}-k} T_k(u), \tag{5.31}$$

where the spectral invariants $T_k(u)$ are defined as

$$T_k(u) = \operatorname{tr} \bigwedge^k M(u). \tag{5.32}$$

In particular, $T_0(u) = 1$ and $T_n(u) = \det M(u)$. For a point $(a,b)\in\mathbb{C}^2$ on the spectral curve, (5.31) gives a non-trivial relation between the spectral invariants of $M(u)$.

Sklyanin "magic recipe" From the B–A function, Sklyanin gives a way to construct separate variables

> "*Take the poles of the properly normalized Baker–Akhiezer function and the corresponding eigenvalues of the monodromy [Lax] operator and you obtain a* SoV." — Sklyanin in [188, p. 41].

The exact statement is the following

Theorem 2 (SoV in the CISM framework)**.** *Consider the classical $\mathfrak{gl}(n)$ spin chain of length N. There exist two functions $\mathcal{A}(u)$ and $\mathcal{B}(u)$ such that*

- *$\mathcal{A}(u)$ is a rational function of the monodromy entries $M_{ij}(u)$*
- *$\mathcal{B}(u)$ is a polynomial of degree $N\ \mathfrak{n}(\mathfrak{n}-1)/2$ in the $M_{ij}(u)$*

Then the variables x_i and z_i defined by

$$\mathcal{B}(x_i) = 0 \quad \textit{and} \quad z_i := \mathcal{A}(x_i), \tag{5.33}$$

have Poisson brackets

$$\{x_i, x_j\} = 0 = \{z_i, z_j\}, \quad \{z_i, x_j\} = \delta_{ij} z_i, \tag{5.34}$$

and verify by the separate relations

$$\det(z_i - M(x_i)) = 0. \tag{5.35}$$

If there are $\mathfrak{n}(\mathfrak{n}-1)/2$ such pairs, the $(x_i, \eta^{-1}\ln z_i)$ form canonical separate variables (for the quadratic Sklyanin's Poisson brackets (2.68)).

This was investigated by Sklyanin himself for the $\mathfrak{gl}(2)$ and $\mathfrak{gl}(3)$ case [187, 192, 193], and later expanded for a wider range of $\mathfrak{gl}(\mathfrak{n})$ classical systems by Scott [189] and Gekhtman [190], see also for a synthesis [188].

The above theorem does not explicitly mention the "poles of the Baker–Akhiezer function", but is simple to feel why they can be separate coordinates, and motivate that it is the primer of the $\mathcal{A}$ and $\mathcal{B}$ functions.

Let x_i be a pole of $\Omega(u)$ and $z_i := z(x_i)$. The point $(x_i, z_i) \in \mathbb{C}^2$ lies on the spectral curve, therefore

$$\chi_M(z_i, x_i) = 0, \quad \text{which rewrite} \tag{5.36}$$

$$\sum_{k=0}^{\mathfrak{n}} (-1)^k z_i^{\mathfrak{n}-k} T_k(x_i) = 0. \tag{5.37}$$

This equation is of the form (5.18)

$$\mathcal{F}(x_i, z_i, F_1, \dots, F_N) = 0, \tag{5.38}$$

therefore the couples $(x_i, \eta^{-1}\ln z_i)$ are separate variables for the chain once it is verified that there are N independent such couples which are conjugated canonical variables, that is

$$\forall i, j \in [\![1, N]\!], \quad \{x_i, x_j\} = \{z_i, z_j\} = 0 \quad \text{and} \quad \{x_i, z_j\} = \delta_{ij}. \tag{5.39}$$

Let $\Omega^{(i)}(u) := \mathrm{Res}(\Omega(u); x_i)$. Equations (5.29) and (5.30) evaluated in each couple x_i, z_i give

$$M(x_i)\Omega^{(i)} = z_i \Omega^{(i)}, \quad \sum_{k=1}^{\mathfrak{n}} \alpha_k(x_i)\Omega_k^{(i)} = 0. \tag{5.40}$$

This is a $\mathfrak{n}$-dimensional linear problem for the (residue) vector $\Omega^{(i)}$. There exists a non-zero solution if $\mathrm{rank}\,\mathbb{M} = \mathfrak{n} - 1$, where $\mathbb{M}$ is the $(\mathfrak{n}+1) \times \mathfrak{n}$ matrix

$$\mathbb{M} := \begin{pmatrix} \vec{\alpha}(x_i) \\ M(x_i) - z_i 1 \end{pmatrix} = \begin{pmatrix} \alpha_1(x_i) & \dots & \alpha_{\mathfrak{n}}(x_i) \\ M_{11}(x_i) - z_i & \dots & M_{1\mathfrak{n}}(x_i) \\ \vdots & \ddots & \vdots \\ M_{\mathfrak{n}1}(x_i) & \dots & M_{\mathfrak{n}\mathfrak{n}}(x_i) - z_i \end{pmatrix}. \tag{5.41}$$

If it is the case, any two minors of order $\mathfrak{n}$ of $\mathbb{M}$ vanish and form a system with roots (x_i, z_i), which allow in principles to count the number of pairs (x_i, z_i) and compute their Poisson brackets. The choice of the normalization of the Baker–Akhiezer function reveals crucial in these discussions. For the $\mathfrak{gl}(\mathfrak{n})$ case, it

turns out that any constant numeric normalization vector $\vec{\alpha}(u) \in \mathbb{C}^n$ produces SoV. A simplest choice[5] is

$$\alpha_1(u) = 1 \quad \text{and} \quad \alpha_2(u) = \cdots = \alpha_n(u) = 0. \tag{5.42}$$

$\mathfrak{gl}(2)$ case Specializing to the $\mathfrak{gl}(2)$ case, we can take the vanishing minors of order $n = 2$ to be

$$\begin{vmatrix} 1 & 0 \\ M_{11}(x_i) - z_i & M_{12}(x_i) \end{vmatrix} = 0, \quad \begin{vmatrix} 1 & 0 \\ M_{21}(x_i) & M_{22}(x_i) - z_i \end{vmatrix} = 0. \tag{5.43}$$

Using the A, B, C, D notation, this gives immediately

$$B(x_i) = 0, \tag{5.44}$$
$$D(x_i) = z_i. \tag{5.45}$$

In the end, the separate variables are simply the zeroes of the $B(u)$ polynomial, and the z_i's are defined equally simply as the images by D of the x_i's[6]. We identify immediately $\mathcal{A}(u) = D(u)$ and $\mathcal{B}(u) = B(u)$. In $u = x_i$ the monodromy becomes triangular

$$M(x_i) = \begin{pmatrix} A(x_i) & 0 \\ C(x_i) & z_i \end{pmatrix}, \tag{5.46}$$

so it is no surprise its eigenvalue $z_i = z(x_i)$ is given by the diagonal element $D(x_i)$. The explicit separate relations for the (x_i, z_i) are

$$\det M(x_i) - z_i T(x_i) + z_i^2 = 0. \tag{5.47}$$

It remains to compute the Poisson brackets between the x_i's and z_i's. This can be achieved by the knowledge of the quadratic Poisson brackets between the monodromy matrix elements (2.75). The r-matrix that gives the Poisson bracket is $r(u) = \frac{c}{u}\mathcal{P}$, where $\mathcal{P}$ is the permutation operator. Taking particular indices in (2.75), one obtains

$$\{B(u), B(v)\} = 0, \tag{5.48}$$
$$\{D(u), D(v)\} = 0, \tag{5.49}$$
$$\{D(u), B(v)\} = \frac{c}{u - v}(B(u)D(v) - B(v)D(u)). \tag{5.50}$$

The involution of the B's entrain immediately that $\{x_i, x_j\} = 0$, for all i, j. Now, for any function F over the phase space,

$$0 = \{F, B(x_i)\} = \{F, B(u)\}_{u=x_i} + \frac{\mathrm{d}B}{\mathrm{d}u}\Big|_{u=x_i} \{F, x_i\}. \tag{5.51}$$

[5] Note that historic literature [188, for example] usually takes $\alpha_1 = \cdots = \alpha_{n-1} = 0$ and $\alpha_n = 1$, which produces an equivalent discussion.

[6] Sklyanin's is used to take $\mathcal{A}(u) = A(u)$ [187], which is essentially the same thing.

Therefore,

$$\begin{aligned}\{z_i, x_j\} &= \{D(u), x_j\}_{u=x_j} + \frac{\mathrm{d}D}{\mathrm{d}u}\Big|_{u=x_i}\{x_i, x_j\} \\ &= -\frac{1}{\frac{\mathrm{d}B}{\mathrm{d}v}\Big|_{x_j}}\{D(u), B(v)\}_{\substack{u=x_i \\ v=x_j}} + 0 \\ &= -\frac{c}{(x_i - x_j)\frac{\mathrm{d}B}{\mathrm{d}v}\Big|_{x_j}}\big(B(x_i)D(x_j) - B(x_j)D(x_i)\big).\end{aligned} \tag{5.52}$$

This indeed gives 0 for $i \neq j$, while for the case $i = j$ the L'Hôpital rule gives $\{z_i, x_i\} \propto \delta_{ij}$. A similar calculation gives the Poisson brackets $\{z_i, z_j\}$, which are computed to be zero, and the (x_i, z_j) are conjugate canonical variables.

Higher rank case In the $\mathfrak{gl}(\mathfrak{n})$ case, the functions $\mathcal{A}(u)$ and $\mathcal{B}(u)$ may be constructed similarly by eliminating x_i and z_i (respectively) from a system of two vanishing minors of order $\mathfrak{n}$ — but their expressions are a bit more intricate [187, 188, 193]. For $\mathfrak{n} = 3$, eventually one computes

$$\mathcal{B}(u) = M_{12}(u)\begin{vmatrix} M_{12}(u) & M_{13}(u) \\ M_{22}(u) & M_{23}(u)\end{vmatrix} + M_{13}(u)\begin{vmatrix} M_{12}(u) & M_{13}(u) \\ M_{32}(u) & M_{33}(u)\end{vmatrix}, \tag{5.53}$$

and there are two choices for $\mathcal{A}(u)$

$$\mathcal{A}(u) = M_{12}(u)^{-1}\begin{vmatrix} M_{12}(u) & M_{13}(u) \\ M_{32}(u) & M_{33}(u)\end{vmatrix} \quad \text{or} \quad \mathcal{A}(u) = -M_{13}(u)^{-1}\begin{vmatrix} M_{12}(u) & M_{13}(u) \\ M_{22}(u) & M_{23}(u)\end{vmatrix}. \tag{5.54}$$

From there, one can do the same program as above and get separate relations similar to (5.47) [187, 188, 193].

5.2 Quantum SoV

As we observed in Section (3.1), the notion of integrability could not be enlarged to the quantum world as it is, because of the ambiguity in defining the number of independent conserved charges. Fortunately the situation is better for the separation of variables.

Quantum separate variables

Definition (Quantum separate variables). *Consider a quantum system over a Hilbert space $\mathscr{H}$ with some Hamiltonian H, and N conserved quantities H_j such that*

$$\forall j, k = 1, \dots, N, \quad \big[H_j, H_k\big] = 0 \quad \textit{and} \quad \big[H, H_j\big] = 0. \tag{5.55}$$

Let (X_j, Z_j) be N couples of operators over some Hilbert space $\mathscr{H}$, with canonical commutation relations

$$\big[X_j, X_k\big] = \big[Z_j, Z_k\big] = 0, \tag{5.56}$$

$$\big[Z_j, X_k\big] = -\delta_{ik} i\hbar, \tag{5.57}$$

and X_j's are diagonalizable with simple spectrum. The X_j's are quantum separate variables for the spectral problem of H if there exists N separate relations of the form

$$\mathcal{F}_j\big(X_j, Z_j, H_1, \dots, H_N\big) = 0 \tag{5.58}$$

for $j = 1, \ldots, N$, where some order between the non-commutative operators X_i, Z_j and the commutative $\{H_j\}$ family has to be prescribed. The basis of common eigenvectors of the X_j's is the separate basis. Each vector is labelled uniquely by its eigenvalues x_j under the action of the X_j

$$|\vec{\mathbf{x}}\rangle = |x_1, \ldots, x_N\rangle \quad \textit{and} \quad X_j\,|\vec{\mathbf{x}}\rangle = x_j\,|\vec{\mathbf{x}}\rangle\,. \tag{5.59}$$

Let

$$\psi(x_1, \ldots, x_N) := \langle x_1, \ldots, x_N|\psi\rangle \tag{5.60}$$

be the wavefunction of a stationary state $|\psi\rangle$ of the system, where $\langle\vec{\mathbf{x}}| = \langle x_1, \ldots, x_N|$ is the (separate) Hilbert basis of eigencovectors associated to the separate coordinates X_j. Then the relations (5.58) suggests a factorized form of the wave function

$$\psi(x_1, \ldots, x_N) = \prod_{j=1}^{N} \psi_j(x_j). \tag{5.61}$$

For a continuous system with an infinite-dimensional Hilbert space, it is handy to realize quantum states as functions and work in the x-representation of the operators to picture the implications of the above definition. Operators X_j are the multiplication operators by the j-th coordinate x_j, and $Z_j = \partial\big/\partial X_j$. Then, acting on $\psi(\vec{\mathbf{x}})$ with the j-th relation (5.58), one can restrict the equation to the $\psi_j(x_j)$ term

$$\mathcal{F}_j\left(x_j, \frac{\partial}{\partial x_j}, h_1, \ldots, h_N\right)\psi_j(x_j) = 0, \tag{5.62}$$

where the h_i are the scalar value of the conserved quantities operators on the $\psi(\vec{\mathbf{x}})$ eigenstate. These are separate differential equations in one variable; the original spectral problem has been decoupled in independent simpler ones.

For a finite-dimensional Hilbert space, the "wavefunctions" (5.60) we consider are simply the coefficients of the linear decomposition of an eigenstate in the separate basis. The Z_j dependence in the separate relations (5.58) should be realized as a shift in the spectrum of X_j. Noting

$$\mathcal{S}_j := \{\, x_j^{(h_j)} \mid h_j = 1, \ldots, d_j \,\} \tag{5.63}$$

the d_j eigenvalues of X_j, this yields independent separate finite difference equations for the $\psi_j(x_j)$ factor of the form

$$\mathcal{F}_j\left(x_j^{(h_j)}, D_j^{\pm}, h_1, \ldots, h_N\right)\psi_j\left(x_j^{(h_j)}\right) = 0, \tag{5.64}$$

where $D_j^{\pm}$ are the shift operators in the spectrum of X_j

$$D_j^{\pm}\psi_j\left(x_j^{(h_j)}\right) = \psi_j\left(x_j^{(h_j \pm 1)}\right). \tag{5.65}$$

This shows how the original spectral problem separates in N independent ones of smaller dimension d_i on the Hilbert space $\mathscr{H}_j$, with

$$\mathscr{H} = \bigotimes_{j=1}^{N} \mathscr{H}_j \quad \text{and} \quad \prod_{j=1}^{N} d_j = \dim \mathscr{H}. \tag{5.66}$$

Link with classical SoV We can illustrate the classical limit with the simple case of the particle in N dimensions in some potential $V(\vec{\mathbf{x}})$. The Hamiltonian of this continuous system is of the form

$$H = -\frac{\hbar^2}{2m}\Delta + V(\vec{\mathbf{x}}). \tag{5.67}$$

Writing the spatial wavefunction of the state $|\psi\rangle$ in the form of a stationary phase

$$\psi(\vec{\mathbf{x}}) = \psi_0\, e^{\,iS(\vec{\mathbf{x}})/\hbar} \quad \big(\text{and} \quad \psi(\vec{\mathbf{x}},t) = \psi(\vec{\mathbf{x}})e^{-iEt/\hbar}\big), \tag{5.68}$$

and taking the classical limit in the (time independent) Schrödinger equation, we obtain

$$E = \frac{(\nabla S(\vec{\mathbf{x}}))^2}{2m} - \frac{i\hbar}{2m}\Delta S(\vec{\mathbf{x}}) + V(\vec{\mathbf{x}}) \xrightarrow[\hbar\to 0]{} \frac{(\nabla S(\vec{\mathbf{x}}))^2}{2m} + V(\vec{\mathbf{x}}), \tag{5.69}$$

which is the classical reduced Hamilton–Jacobi equation with a given energy E and Hamilton characteristic function $S(x)$ for the corresponding classical system — see equations (5.1) and (5.3) with $H = \vec{\mathbf{p}}^2/2m + V(\vec{\mathbf{x}})$. Now for a separate problem with a factorized wavefunction

$$\psi(\vec{\mathbf{x}}) = \psi_0 \prod_{j=1}^{N} e^{-i\hbar S_j(x_j)}, \tag{5.70}$$

one recovers the equations (5.11) and (5.12).

In the classical case, the separate relations are used to solve the equations of the motion, constructing by additive separate quadratures the generating function of the coordinate change to the constants of the motion which linearize the time evolution. In the quantum case, the separate relations are used to solve the spectral problem for the Hamiltonian, constructing the wavefunction of eigenstates by multiplicative separate wavefunctions responding to their own spectral problems of smaller dimension.

The hydrogen atom

Consider the Hamiltonian describing the motion of an electron in the potential of proton at the origin,

$$H = \frac{\vec{\mathbf{p}}^2}{2m} + V(\vec{\mathbf{r}}) \quad \text{with} \quad V(\vec{\mathbf{r}}) = -\frac{e^2}{4\pi\varepsilon_0 r}, \tag{5.71}$$

where $\vec{\mathbf{r}}$ is the position vector, $r = \|\vec{\mathbf{r}}\|$, and $\vec{\mathbf{p}}$ the momentum. The electron state is described by vectors $|\Psi\rangle$ of the Hilbert space $\mathscr{H} \simeq L^2(\mathbb{R}^3)$. The time-independent Schrödinger equation for the wave function $\Psi(\vec{\mathbf{r}}) = \langle\vec{\mathbf{r}}|\Psi\rangle$ is, with $\vec{\mathbf{p}} = -i\hbar\vec{\nabla}$,

$$-\frac{\hbar^2}{2m}\Delta\Psi + V(\vec{\mathbf{r}})\Psi = E, \quad \text{with} \quad V(\vec{\mathbf{r}}) = -\frac{e^2}{4\pi\varepsilon_0 r}, \tag{5.72}$$

for some energy $E \in \mathbb{R}$. Because of the central nature of the force—namely the potential $V(r)$ depends only on the radial distance r—the three components L_x, L_y and L_z of the orbital angular momentum $\vec{\mathbf{L}} = -i\hbar\,\vec{\mathbf{r}}\times\vec{\nabla}$ are operators that commute with the Hamiltonian. This can be shown using the fundamental commutation relations $[r_i, p_j] = i\hbar\delta_{ij}$. Therefore,

$$\big[H, \vec{\mathbf{L}}^2\big] = [H, L_z] = \big[\vec{\mathbf{L}}^2, L_z\big] = 0. \tag{5.73}$$

Any eigenstate of the Hamiltonian is thus also a shared eigenstate of the $\vec{\mathbf{L}}^2$ and L_z operators, so one can

search for the eigenstate of H in the set of common eigenstates of the commuting family $(H, \vec{\mathbf{L}}^2, L_z)$.

Because of the rotational invariance of the system, the spherical coordinates (r, θ, φ) are well suited to describe it. The wavefunction of the state $|\Psi\rangle$

$$\Psi(r, \theta, \varphi) = \langle r, \theta, \varphi | \Psi \rangle \tag{5.74}$$

are function of $L^2(\mathbb{R}_+ \times [0, \pi[\times [0, 2\pi[, r^2 \sin\varphi \, \mathrm{d}r \, \mathrm{d}\theta \, \mathrm{d}\varphi)$, and the H, $\vec{\mathbf{L}}^2$, L_z operators are represented as

$$\Delta = \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2}{\partial \varphi^2}, \tag{5.75}$$

$$\vec{\mathbf{L}}^2 = -\hbar^2 \left(\frac{1}{\sin\theta} \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial}{\partial \theta}\right) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial \varphi^2} \right), \tag{5.76}$$

$$L_z = -i\hbar \frac{\partial}{\partial \varphi}. \tag{5.77}$$

Defining the three operators

$$F_1\left(r, \frac{\partial}{\partial r}, H, \vec{\mathbf{L}}^2\right) := -\frac{\hbar^2}{2mr^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) + \frac{1}{2mr^2} \vec{\mathbf{L}}^2 - \frac{e^2}{4\pi\varepsilon_0 r} - H, \tag{5.78}$$

$$F_2\left(\theta, \frac{\partial}{\partial \theta}, \vec{\mathbf{L}}^2, L_z\right) := -\frac{1}{\sin\theta} \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial}{\partial \theta}\right) - \frac{L_z^2}{\hbar^2 \sin^2(\theta)} + \hbar^{-2} \vec{\mathbf{L}}^2, \tag{5.79}$$

$$F_3\left(\varphi, \frac{\partial}{\partial \varphi}, L_z\right) := -i\hbar \frac{\partial}{\partial \varphi} - L_z, \tag{5.80}$$

one can verify that their action vanishes on wavefunctions $\Psi()$ of eigenstate of the commuting family $(H, \vec{\mathbf{L}}^2, L_z)$

$$\forall a \in [\![1, 3]\!], \quad F_a \Psi(r, \theta, \varphi) = 0. \tag{5.81}$$

This gives relations of the form (5.58), so the spherical coordinates (r, θ, φ) are thus separate coordinates for this system.

Consider $\Psi(r, \theta, \varphi)$ an eigenfunction of the commuting family $(H, \vec{\mathbf{L}}, L_z)$, i.e.

$$H\Psi = E\Psi, \tag{5.82}$$

$$\vec{\mathbf{L}}^2 \Psi = h_2 \Psi, \tag{5.83}$$

$$L_z \Psi = h_3 \Psi, \tag{5.84}$$

with $E, h_2, h_3 \in \mathbb{R}$. The vanishing action of F_3 of Ψ gives the simple enough differential equation

$$\frac{\partial}{\partial \varphi} + \frac{h_3}{i\hbar} \Psi(r, \theta, \varphi) = 0, \tag{5.85}$$

with only derivatives in φ, so that we can solve it, giving

$$\Psi(r, \theta, \varphi) = A(r, \theta) e^{i\hbar^{-1} h_3 \varphi}, \tag{5.86}$$

with $A(r, \theta)$ some function of the r and θ variables. Acting further with F_2 on Ψ, one gets a differential equation for $A(r, \theta)$ with only derivatives in θ, so that it further separates the resolution in r and θ.

Eventually, the wavefunction can can be written in a factorized form

$$\Psi(r,\theta,\varphi) = R(r)\Theta(\theta)\Phi(\varphi), \tag{5.87}$$

where each factor verifies independent and separate differential equations.

The definitive form of the solution (5.87) is obtained by multiplying solutions of the three differential equations

$$F_1 R = 0, \quad F_2\Theta = 0, \quad F_3\Phi = 0, \tag{5.88}$$

and selecting the $\Psi = R\Theta\Phi$ that indeed verify the Schrödinger equation (5.72) on the whole space $\mathbb{R}$ and have a finite norm. The result is well-known: the discrete spectrum of the Hamiltonian is enumerated by the quantum numbers (n,ℓ,m)

$$E = E_n = -\frac{\mathcal{E}}{n^2} \quad \text{with} \quad n \in \mathbb{N}^* \quad \text{and} \quad \mathcal{E} = \frac{me^2}{2(4\pi\varepsilon_0)^2\hbar^2} \approx 13.6\,\text{eV}, \tag{5.89}$$

$$h_2 = \ell(\ell+1)\hbar \quad \text{with} \quad \ell \in [\![0, n-1]\!], \tag{5.90}$$

$$h_3 = m\hbar \quad \text{with} \quad m \in [\![-\ell, \ell]\!], \tag{5.91}$$

and the wavefunctions are of the form

$$\Psi_{n,\ell,m}(r,\theta,\varphi) = R_{n,\ell}(r)F_{\ell,m}(\theta)e^{im\varphi}, \tag{5.92}$$

where the angular part are the so-called spherical harmonics [284]

$$Y_{\ell,m}(\theta,\varphi), \tag{5.93}$$

and the radial part are generalized Laguerre polynomials [see 281, appendix B]

$$R_{n,\ell}(r) = L^{2\ell+1}_{n-\ell-1}(r). \tag{5.94}$$

Note that solutions with infinite norm are obtained for $E > 0$ and describe non-bound states corresponding to the continuous spectrum of the Hydrogen atom.

5.3 Quantum SoV for integrable systems in the QISM framework

5.3.1 SoV in the QISM description of quantum integrable models

For a quantum integrable model generated by some transfer matrix $T(u)$, the above definition of separate variables holds similarly, except that it is more convenient to express the separate relations (5.18) using the transfer matrix which packs the conserved quantities

$$\mathcal{F}\big(X_j, Z_j, T(X_j)\big) = 0 \tag{5.95}$$

In the QISM context, the core idea of Sklyanin's "magic recipe" is still the same: there should exist an operator $\mathcal{B}(u)$ whose properly defined operatorial roots are separate variables. Here, the roots X_j are operators over the Hilbert space, and they should commute together in order to write unambiguously

$$\mathcal{B}(u) = \mathcal{B}_0 \prod_{j=1}^{N} \big(u - X_j\big). \tag{5.96}$$

Like in the classical case, conjugated variables Z_j are obtained by evaluating an operator $\mathcal{A}(u)$ in the operatorial roots X_j, see (5.33). It is necessary to define how one substitutes an operator in place of the scalar parameter in $\mathcal{A}(u)$. We can choose the left ordering of the argument in the polynomial expansion of $\mathcal{A}$

$$\mathcal{A}(X_j) = \sum_k X_j^k \mathcal{A}_k \quad \text{given} \quad \mathcal{A}(u) = \sum_k u^k \mathcal{A}_k \quad \text{for all} \quad u \in \mathbb{C}. \tag{5.97}$$

In fact, a former name of the quantum separation of variables was the *Functional Bethe Ansatz* (FBA) [192, 193, 204, 205], whose name is a syncretism of the known way to solve quantum integrable systems and the necessity to introduce spaces of functions to rigorously describe the zeroes of operator-valued polynomials. One should also ensure that the spectrum of the X_j and the poles of $\mathcal{A}(u)$ do not intersect.

The separate relations shall be provided by some quantized version of (5.28), the quantum spectral curve, that we do not explicit for the moment. The common eigenbasis $|\vec{\mathbf{x}}\rangle = |x_1, \dots, x_n\rangle$ of the X_j operators, labelled by their eigenvalues, is also the eigenbasis of the $\mathcal{B}(u)$ operator. Hence, the separate basis may be constructed by the direct diagonalization of the $\mathcal{B}(u)$ operator.

5.3.2 The $\mathfrak{gl}(2)$ case

Let us show how to obtain separate variables *à la Sklyanin* for $\mathfrak{gl}(2)$ models associated to the 6-vertex algebra. The classical case hints us to look at $B(u) = M_{12}(u)$ and $D(u) = M_{22}(u)$ for the $\mathcal{B}(u)$ and $\mathcal{A}(u)$ operators, respectively. Let us consider a twisted monodromy $M(u) = M^K(u)$, and suppose the twist matrix K is such that $M_{12}(u)$ is diagonalizable with simple spectrum (note that for $K = \mathrm{Id}$, this is not the case). Having

$$[B(u), B(v)] = 0 \tag{5.98}$$

is already good news: we can define the commuting root operators X_j and write (5.96) as

$$B(u) = B_0 \prod_{j=1}^{N} \varphi\big(u - X_j\big). \tag{5.99}$$

Since $\mathcal{B}(u) = B(u)$ is diagonalizable with simple spectrum, then its operatorial roots X_j are simultaneously diagonalizable with simple spectrum: following (5.63) we note

$$x_j^{(h_j)} \in \mathbb{C} \quad \text{for} \quad 1 \leq h_j \leq d_j, \tag{5.100}$$

the d_j eigenvalues of the X_j operator. We have

$$\prod_{j=1}^{N} d_j \equiv \dim \mathscr{H}, \tag{5.101}$$

and the eigenvectors of $B(u)$ consists in the $d_1 \dots d_N$ vectors

$$\left|\vec{\mathbf{x}}^{(\vec{\mathbf{h}})}\right\rangle := \left|x_1^{(h_1)}, \dots, x_N^{(h_N)}\right\rangle \tag{5.102}$$

of eigenvalue

$$b_{\vec{\mathbf{h}}}(u) = B_0 \prod_{j=1}^{N} \varphi\Big(u - x_j^{(h_j)}\Big). \tag{5.103}$$

To ease the heavy notation (5.102), we will make use of the shorthand notation $|\vec{\mathbf{x}}\rangle = |x_1, \dots, x_N\rangle$ in the following to denote a generic vector of the separate basis.

What is the action of the conjugated $D(x_j)$ operators on the separate basis? By equation (Y.8) one

computes

$$B(u)D(x_j)\,|\vec{\mathbf{x}}\rangle = \frac{\varphi\big(u-x_j+\eta\big)}{\varphi\big(u-x_j\big)}\,b_{\vec{\mathbf{h}}}(u)D(x_j)\,|\vec{\mathbf{x}}\rangle\,. \tag{5.104}$$

Hence, unless $D(x_j)\,|\vec{\mathbf{x}}\rangle$ is zero, $D(x_j)\,|\vec{\mathbf{x}}\rangle$ is an eigenvector of eigenvalue

$$B_0\,\varphi\big(u-(x_j-\eta)\big)\prod_{k\neq j}^{N}\varphi\Big(u-x_k^{(h_k)}\Big). \tag{5.105}$$

Therefore, $D(x_j)$ is the *shift operator by* $-\eta$ in the spectrum $\mathcal{S}_j$ of X_j, and the eigenvalues can be labelled and ordered by successive shifts of the η parameter

$$\mathcal{S}_j = \Big(x_j^{(1)},\dots,x_j^{(d_j)}\Big) = \Big(x_j^{(0)},x_j^{(0)}-\eta,\dots,x_j^{(0)}-\big(d_j-1\big)\eta\Big), \tag{5.106}$$

for some $x_j^{(0)}\in\mathbb{C}$, such that

$$x_j^{(h_j)} = x_j^{(0)} + (h_j-1)\eta. \tag{5.107}$$

Similarly, the action of $A(x_j)=M_{11}(x_j)$ performs the opposite shift $x_j\to x_j+\eta$ on the eigenvectors of the separate basis, so we have

$$A(x_j)\,|\vec{\mathbf{x}}\rangle \propto \big|x_1,\dots,x_j+\eta,\dots,x_N\big\rangle, \tag{5.108}$$
$$D(x_j)\,|\vec{\mathbf{x}}\rangle \propto \big|x_1,\dots,x_j-\eta,\dots,x_N\big\rangle. \tag{5.109}$$

Using the notation $\vec{\mathbf{x}}+a\hat{\mathbf{e}}_j := (x_1,\dots,x_j+a,\dots,x_N)$, this gives

$$\begin{aligned} A(x_j)\,|\vec{\mathbf{x}}\rangle &\propto \big|\vec{\mathbf{x}}+\eta\hat{\mathbf{e}}_j\big\rangle,\\ D(x_j)\,|\vec{\mathbf{x}}\rangle &\propto \big|\vec{\mathbf{x}}-\eta\hat{\mathbf{e}}_j\big\rangle. \end{aligned} \tag{5.110}$$

On covectors of the left separate basis, $A(x_j)$ and $D(x_j)$ shift the spectrum in the opposite way:

$$\begin{aligned} \langle\vec{\mathbf{x}}|A(x_j) &\propto \big\langle\vec{\mathbf{x}}-\eta\hat{\mathbf{e}}_j\big|,\\ \langle\vec{\mathbf{x}}|\,D(x_j) &\propto \big\langle\vec{\mathbf{x}}+\eta\hat{\mathbf{e}}_j\big|. \end{aligned} \tag{5.111}$$

Using the quantum determinant (A.17), whose explicit expression in this case is

$$\text{q-det}\,M(u) = \det K a(u)d(u-\eta) = \det K\prod_{j=1}^{N}(u-\xi_j+\eta)(u-\xi_j-\eta), \tag{5.112}$$

where $d(u)=\prod_{j=1}^{N}(u-\xi_j)=a(u-\eta)$ are the eigenvalue of the untwisted diagonal matrix elements of the untwisted monodromy on the highest weight state, we compute

$$\langle\vec{\mathbf{x}}|A(x_j)D(x_j-\eta) = \text{q-det}\,M(x_j)\,\langle\vec{\mathbf{x}}|\,. \tag{5.113}$$

Hence, with a proper choice in the normalization, we may put

$$\begin{aligned} \langle\vec{\mathbf{x}}|A(x_j) &= a(x_j)\big\langle\vec{\mathbf{x}}-\eta\hat{\mathbf{e}}_j\big|,\\ \langle\vec{\mathbf{x}}|\,D(x_j) &= d(x_j)\big\langle\vec{\mathbf{x}}+\eta\hat{\mathbf{e}}_j\big|. \end{aligned} \tag{5.114}$$

Spectral problem We can now describe the spectral problem of the model using the separate basis constructed above. Let $|t\rangle$ be an eigenvector of eigenvalue $t(u)$ for the transfer matrix $T(u)=A(u)+D(u)$.

Ultimately we want to solve the spectral equation $T(u)|t\rangle = t(u)|t\rangle$. To do so, we can make good use of the above left separate basis by computing

$$\langle \vec{\mathbf{x}}| \, T(x_j) \, |t\rangle \tag{5.115}$$

in two different ways: *i)* by making T acts to the right on its eigenvector and *ii)* by making $T = A + D$ acts to the left on the separate basis. It yields the following equation

$$\forall \vec{\mathbf{x}} \in \mathcal{S}_1 \times \cdots \times \mathcal{S}_N, \quad t(x_j) \, \langle \vec{\mathbf{x}}|t\rangle = a(x_j) \left\langle \vec{\mathbf{x}} - \eta \hat{\mathbf{e}}_j \middle| t \right\rangle + d(x_j) \left\langle \vec{\mathbf{x}} + \eta \hat{\mathbf{e}}_j \middle| t \right\rangle . \tag{5.116}$$

Reintroducing the exact notation for the spectrum, the wavefunctions of eigenstate $|t\rangle$

$$\psi_t\left(x_1^{(h_1)}, \dots, x_N^{(h_N)}\right) = \left\langle x_1^{(h_1)}, \dots, x_N^{(h_N)} \middle| t \right\rangle \quad \text{for} \quad (x_1^{h_1}, \dots, x_N^{(h_N)}) \in \mathcal{S}_1 \times \cdots \times \mathcal{S}_N, \tag{5.117}$$

which are the 2^N components of the eigenstate in the separate basis, obey the following second order finite difference equation

$$\begin{aligned} t\left(x_j^{(h_j)}\right)\psi_t\left(x_1^{(h_1)}, \dots, x_N^{(h_N)}\right) = a\left(x_j^{(h_j)}\right)\psi_t\left(x_1^{(h_1)}, \dots, x_j^{(h_j+1)}, \dots, x_N^{(h_N)}\right) \\ + d\left(x_j^{(h_j)}\right)\psi_t\left(x_1^{(h_1)}, \dots, x_j^{(h_j-1)}, \dots, x_N^{(h_N)}\right). \end{aligned} \tag{5.118}$$

Using a factorized form (5.61) of the wavefunction ψ_t, these form N separate equations of the form (5.58). They manifest as finite difference equations, because we deal with a Hilbert space of finite dimension. It is enough to solve this system of finite-difference equation to solve the spectral problem of the twisted XXX chain, offering an alternate picture from the one of the ABA.

How does the quantum spectral curve arise in this discussion? In the classical case, it was the object which provides the separate relations from the help of an eigenvector of the monodromy matrix, the Baker–Akhiezer function. In the above discussion, we obtained the separate relations straight from the ad-hoc constructed separate basis without referring to the spectral curve once. But there is a way to recover it from the quantum determinant: from the expression $\text{q-det}\, M(u) = A(u)D(u-\eta) - B(u)C(u-\eta)$, the following equation holds

$$A(u)A(u-\eta) - A(u)T(u-\eta) + \text{q-det}\, M(u) + B(u)C(u-\eta) = 0. \tag{5.119}$$

Sandwiching this equation by a covector of the separate basis at the left and an eigenvector of the transfer matrix at the right and putting $u \leftarrow x_j^{(h_j)}$, it produces the quantum analog of the spectral curve (5.31)

$$\begin{aligned} a\left(x_j^{(h_j)}\right)a\left(x_j^{(h_j+1)}\right)\psi_t\left(x_1^{(h_1)}, \dots, x_j^{(h_j+2)}, \dots, x_N^{(h_N)}\right) \\ - a\left(x_j^{(h_j)}\right)t\left(x_j^{(h_j+1)}\right)\psi_t\left(x_1^{(h_1)}, \dots, x_j^{(h_j+1)}, \dots, x_N^{(h_N)}\right) \\ - \text{q-det}\, M(x_j)\psi_t\left(x_1^{(h_1)}, \dots, x_j^{(h_j)}, \dots, x_N^{(h_N)}\right) = 0, \end{aligned} \tag{5.120}$$

which is exactly (5.118) after one factors out the common $a\left(x_j^{(h_j)}\right)$ factor, and rescale h_j in $h_j - 1$.

Necessity of the twist The attentive reader should have notice that for an untwisted $\mathfrak{gl}(2)$ model, say the XXX $\mathfrak{gl}(2)$ fundamental models with periodic boundary conditions, the $B(u) = M_{12}(u)$ operator is nilpotent, which is incompatible with the above construction. However, it is possible circumvent to this

issue by adding a non-trivial twist to the chain. For the XXX chain, taking $K = \left(\begin{smallmatrix} 1 & \alpha \\ 0 & 1 \end{smallmatrix}\right)$, we have

$$B^K(u) = B(u) + \alpha D(u), \tag{5.121}$$

in terms of the elements of the untwisted monodromy, which is now a diagonalizable operator, being a sum of a nilpotent and a diagonalizable one. The simplicity of the B^K spectrum is then ensured by taking an inhomogeneous chain up to some constraints. For the XXZ chain, non-trivial linear combinations $K = \alpha\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) + \beta\left(\begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix}\right)$ of the σ^x and σ^y Pauli matrices give a diagonalizable $B^K(u)$ operator. We can therefore state the following theorem for the 6-vertex Yang–Baxter algebra:

Theorem 3. *Consider the twisted inhomogeneous* XXX *chain of length N, with a twist matrix K that is not proportional to the identity, so that it can always be brought to the form*

$$K = \begin{pmatrix} k_1 & \alpha \\ 0 & k_2 \end{pmatrix} \quad \textit{with} \quad \alpha \neq 0 \tag{5.122}$$

by an isomorphism[7]*, and monodromy*

$$M^K(u) = K_0 R_{0N}(u - \xi_N) \dots R_{01}(u - \xi), \tag{5.123}$$

with the inhomogeneities ξ_j satisfying

$$\forall a, b \in [\![1, N]\!], a \neq b, \quad \forall r = -1, 0, 1, \quad \xi_a \neq \xi_b + r\eta. \tag{5.124}$$

The left separate basis is the basis of eigenvectors of the 2^N covectors $B^K(u) = B_0^K \prod_{j=1}^N \varphi\big(u - X_j\big)$ operators, which are the

$$\langle \vec{\mathbf{h}}| := \langle 0| \prod_{j=1}^{N} \big(A^K(\xi_j)\big)^{h_j} \quad \textit{for} \quad h_j \in \{0, 1\}, \tag{5.125}$$

of eigenvalue

$$b_{\vec{\mathbf{h}}}(u) = \alpha \prod_{j=1}^{N} \varphi\big(u - (\xi_j - h_j \eta)\big). \tag{5.126}$$

The spectrum of the operatorial roots X_j is therefore

$$\mathcal{S}_j = \{\xi_j, \xi_j - \eta\} \tag{5.127}$$

which is indeed of the form (5.106) *and is simple thanks to conditions* (5.124)*. The above separate basis allows to re-express the spectral problem for the transfer matrix $T^K(u) = \operatorname{tr} M^K(u)$ as a system of discrete equations* (5.120) *with*

$$\text{q-det}\, M(u) = \det(K) \prod_{j=1}^{N} \varphi\big(u - \xi_j + \eta\big)\varphi\big(u - \xi_j + \eta\big) \tag{5.128}$$

and $a(u)$ being replaced by

$$a^K(u) = k_1 a(u) = k_1 d(u + \eta) = k_1 \prod_{j=1}^{N} (u - \xi_j + \eta). \tag{5.129}$$

Proof. These results are obtained by specializing the above discussions on the eigenstate of $\mathcal{B}(u) = B^K(u)$ and the spectral problem of the transfer matrix to the fundamental representation of $\mathfrak{gl}(2)$, and not

[7] See [225], section 3.2.

forgetting to take the twist into account.

The operator $B^K(u) = k_1 B(u) + \alpha D(u)$ is diagonalizable since $D(u)$ is and $\alpha \neq 0$. The family (5.125) is proved to be the eigenbasis of $B^K(u)$ by induction.

Any eigenstate $|t\rangle$ of $T^K(u)$ has its eigenvalue $t(u)$ and wavefunctions in the separate basis $\psi_t(\vec{\mathbf{x}})$ satisfy the system of equations (5.120), with the correct replacement for $a(u)$ and q-det $M(u)$. □

The above computations generalize to many $\mathfrak{gl}(2)$ models. A most interesting case is the one of the antiperiodic XXZ chain, because the ABA cannot be applied in this case for the lack of a reference state.

Example: SoV for the XXZ antiperiodic chain [176]

Consider the inhomogeneous XXZ spin 1/2 model of length N with the antiperiodic twist $K = \sigma^x = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. The explicitly twisted monodromy of this system is

$$M^K(u) = K R_{0N}(u-\xi_N)\dots R_{01}(u-\xi_1) = \begin{pmatrix} C(u) & D(u) \\ A(u) & B(u) \end{pmatrix}. \tag{5.130}$$

From the above analysis, we define

$$\mathcal{A}(u) := M^K_{22}(u) = B(u), \tag{5.131}$$
$$\mathcal{B}(u) := M^K_{12}(u) = D(u). \tag{5.132}$$

The $\mathcal{B}(u) = D(u)$ operator is diagonalizable with simple spectrum, as desired. For generic inhomogeneities satisfying the condition (5.124), the left and right eigenbasis of $D(u)$ are the covectors and vectors

$$\left\langle \vec{\mathbf{h}} \right| := \langle 0 | \prod_{j=1}^{N} \left(\frac{C(\xi_j)}{d(\xi_j - \eta)} \right)^{h_j}, \tag{5.133}$$

$$\left| \vec{\mathbf{h}} \right\rangle := \prod_{j=1}^{N} \left(\frac{B(\xi_j)}{a(\xi_j)} \right)^{h_j} |0\rangle, \tag{5.134}$$

labelled by vectors $\vec{\mathbf{h}} = (h_1, \dots, h_N) \in \{0,1\}^N$, with $|0\rangle = \otimes_{j=1}^N \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ and $\langle 0| = \otimes_{j=1}^N (1,0)$[8]. The eigenvalue associated to $\left|\vec{\mathbf{h}}\right\rangle$ and $\left\langle \vec{\mathbf{h}} \right|$ under the left and right action of $D(u)$, respectively, is

$$d_{\vec{\mathbf{h}}}(u) = \prod_{j=1}^{N} \sinh\left(u - \xi_j - h_j \eta\right). \tag{5.135}$$

These are clearly of the form (5.125) and (5.126). Proof is made by direct calculation using the FCR of the trigonometric 6-vertex algebra. Similar calculations show that the action of the $B(u)$ and $C(u)$ operators over (5.133) and (5.133) is a non-local shift in the h_i—but the action of the $B(\xi_j)$ and $C(\xi_j)$ are. The SoV measure is diagonal with

$$\left\langle \vec{\mathbf{h}} \middle| \vec{\mathbf{k}} \right\rangle \propto \delta_{\vec{\mathbf{h}},\vec{\mathbf{k}}}. \tag{5.136}$$

From there, one may prove that the spectrum of the transfer matrix $T^K(u) = \operatorname{tr} M^K(u) = B(u) + C(u)$ are the functions $t(u)$ solutions of the N discrete equations

$$t(\xi_j)t(\xi_j - \eta) = \text{q-det}\, M^k(\xi_j) = -a(\xi_j)d(\xi_j - \eta), \tag{5.137}$$

within a certain set of Laurent polynomials of degree $N-1$ constrained by the length of the chain. This is

good agreement with (5.128). The eigenstates have a separate form

$$|t\rangle = \sum_{\vec{\mathbf{h}}} \prod_{j=1}^{N} Q_t(\xi_j - h_j\eta) \left|\vec{\mathbf{h}}\right\rangle, \tag{5.138}$$

and similarly for the left eigenstates, so the wavefunctions $\left\langle \vec{\mathbf{h}} \middle| t \right\rangle$ are indeed factorized. Details can be found in [176], where the analysis is pursued until form factors of local operators.

5.3.3 SoV for the $\mathfrak{gl}(3)$ model

The SoV description for the quantum $\mathfrak{gl}(3)$ chain was first made by Sklyanin himself [206], based on his work for the classical $\mathfrak{gl}(3)$ case [187][9]. Recall that the quantum-determinant is a central element of the Yangian algebra, which here for $\mathscr{Y}(\mathfrak{gl}(3))$ is explicitly written as

$$\text{q-det}\, M(u) = \text{tr}_{123}\left(P^-_{123} M_1(u) M_2(u+\eta) M_3(u+2\eta)\right), \tag{5.139}$$

where $P^-_{123} = 1/6 \sum_{\sigma\in\Sigma_3} \text{sign}(\sigma)\, P_\sigma$ is the antisymmetrizer over three copies of the auxiliary space. The matrix

$$U(u) := \left(3\,\text{tr}_{23}\, P^-_{123} M_2(u) M_3(u+\eta)\right)^{\intercal_1}, \tag{5.140}$$

is a quasi-inverse of the monodromy matrix, as by direct computation one proves

$$\text{q-det}(M(u)) = (M(u)^{\intercal}) U(u+\eta), \tag{5.141}$$

where $^{\intercal}$ denotes the matrix transposition. Note that $T_2(u) = \text{tr}\, U(u)$. Let

$$\mathcal{B}(u) := M_{23}(u) U_{31}(u-\eta) - M_{13}(u) U_{32}(u-\eta), \tag{5.142}$$

which mirrors the expression (5.53) of the classical case, up to the η shifts. Direct calculation shows that $[\mathcal{B}(u), \mathcal{B}(v)] = 0$, so it is meaningful to write

$$\mathcal{B}(u) = B_0 \prod_{k=1}^{3N} (u - X_k), \tag{5.143}$$

and $[X_j, X_k] = 0$. Similarly, the operator $\mathcal{A}(u)$ is defined as a deformation of the classical case as

$$\mathcal{A}(u) := -M_{23}(u-\eta)^{-1} U_{32}(u-\eta) = -U_{32}(u-\eta) M_{23}(u-\eta)^{-1}, \tag{5.144}$$

with $[\mathcal{A}(u), \mathcal{A}(v)] = 0$. Sklyanin proved [206] that

$$(u-v)\mathcal{A}(u)\mathcal{B}(v) = (u-v-\eta)\mathcal{B}(v)\mathcal{A}(u) + \mathcal{B}(u)\mathcal{E}(u,v), \tag{5.145}$$

where

$$\mathcal{E}(u,v) = \mathcal{A}(u)\big(M_{23}(u-\eta) M_{23}(u)\big)^{-1} M_{23}(v-\eta) M_{32}(v). \tag{5.146}$$

It is then tempting to define $Z_j := \mathcal{A}(u)|_{u=X_j}$ as the conjugated variable, as it seems to produce the desired shift in the spectrum of X_j. However,

[8] The notation in [176] is multiplicative, but it is easy to make the connection with $\lambda = \exp(u)$, $q = \exp(\eta)$ and $\eta_j = \exp(\xi_j)$, and a factor 2 in the R-matrix.

[9] Extension to the generic $\mathfrak{gl}(n)$ case was also discussed in [207].

"the condition that

$$\left(\cup_{j=1}^{3N} \mathrm{Sp}(X_j)\right) \cap \mathrm{poles}(\mathcal{A}(u)) = \emptyset \tag{5.147}$$

should be checked for any specific representation of the [$\mathfrak{gl}(3)$ Yang-Baxter] algebra,"

as Sklyanin points out very appropriately in [206], before saying that "here[for the unspecified representation at hand] we assume it to be satisfied". Indeed, in case the intersection (5.147) is non-empty, the right-hand side of (5.145) do not vanish for all $u = x_j \in \mathrm{Sp}(X_j)$. However, it can be checked that for fundamental $\mathfrak{gl}(3)$ chains with generic twist of length $N \leq 3$, the condition (5.147) do not hold: some zeros of $\mathcal{B}(u)$ ares poles of $\mathcal{A}(u)$ and thus $\mathcal{E}(u,v)$ is indeterminate of the form $0/0$ when the left-hand side vanishes. This jeopardizes the SoV program as described above, since the $Z_j := \mathcal{A}(X_j)$ operator fails to deliver the desired shift

$$Z_j X_k = \left(X_k - \eta \delta_{jk}\right) Z_j, \tag{5.148}$$

in the spectrum of X_j. Details and proofs of such an argument are found in appendix A of [225]. It is a natural conjecture to expect the issue does not solve by itself for longer chains. While other representations might not have this issue, the case of the fundamental one is already bothering. This forms a strong obstacle for Sklyanin quantum SoV method in higher rank algebra.

This objection has been made only fairly recently by Maillet and Niccoli—publication [225] appeared in 2018—together with a new SoV approach bypassing this problem. Let us add that for specific boundary conditions of the $\mathfrak{gl}(2)$ model, the $\mathcal{B}(u)$ operator already fails to provide a separation of variables. These are enough argument to look for other method to produce separate variables in quantum integrable models. Chapter 6 is entirely devoted to the description of the novel SoV method introduced in [225] which do not rely on Sklyanin's $\mathcal{B}(u)$ and $\mathcal{A}(u)$ operators.

5.4 Recent developments

Still, there has been many developments of Sklyanin's SoV in the last decades, mostly in the $\mathfrak{gl}(2)$ case though. The next section is devoted to a brief review of the recent literature and its results.

5.4.1 SoV for $\mathfrak{gl}(2)$ models

Sklyanin's SoV allows to tackle the spectral problem associated to the 6-vertex Yang–Baxter algebra in a different way from the one of the ABA. It is natural to either *i)* compute scalar products, form factors and correlation functions using SoV and try to reproduce some known results of the ABA, which was done fairly quickly [188, 193, 206], or *ii)* use SoV to compute the spectrum in $\mathfrak{gl}(2)$ models known for the failure of the ABA. We showed an example of the latter with the antiperiodic XXZ chain on page 82.

The models associated to the 6-vertex algebra with non-trivial periodic boundary conditions are natural candidates for SoV. In the case of the inhomogeneous antiperiodic XXX and XXZ chains, for which no reference vector is available for the ABA, the spectrum, eigenstates and form factors were computed for spin $1/2$ representations [176, 215, 285]. The results were extended to higher spin representations by [215, 222]. A rewriting of the formulas for the XXX spin $1/2$ chain was achieved in [218] and made them suitable for the homogeneous and thermodynamic limits. Correlation functions at zero temperature were later obtained in [221]. The spectrum, eigenvectors and the SoV measure of the antiperiodic dynamical 6-vertex Yang–Baxter algebra were characterized in [214], and form factors were studied in [286]. Similar results have been obtained in the open case for XXX and XXZ spin $1/2$ chain [211, 217, 219, 220, 287] and for the cyclic representations of the 6-vertex algebra [223, 224].

5.4.2 SoV for $\mathfrak{gl}(\mathfrak{n})$ models

Relying on the pioneer work of Sklyanin, the form of the $\mathcal{B}(u)$ operators for the SoV of $\mathfrak{gl}(\mathfrak{n})$, $\mathfrak{n} \geq 3$, models was proposed by Gromov, Levkovich-Maslyuk, and Sizov in [178]—and [288] for supersymmetric chains. The $\mathcal{B}(u)$ diagonalizability and its spectrum were conjectured and verified numerically for small length chain, and it was conjectured that the $\mathcal{B}(u)$ operator was able to produce eigenstate in an ABA-like form $\mathcal{B}(u_1)\dots\mathcal{B}(u_m)\,|0\rangle$. This was proved by Slavnov and Liashyk for the $\mathfrak{gl}(3)$ symmetric representations case [289]—see also [290].

From there, progresses on the $\mathfrak{gl}(\mathfrak{n})$ case benefited from the novel SoV construction from conserved quantities developed by Maillet and Niccoli in [225] that we will describe in the next chapters. Thanks to it, Ryan and Volin proved the diagonalizability and compute the spectrum of the $\mathcal{B}(u)$ [236]. Once it is shown that the new SoV basis [225] coincides with the eigenbasis of the $\mathcal{B}(u)$ operator, the ABA form in $\mathcal{B}(u)$ of the transfer matrix eigenvectors follows. We address the recent bibliography on this subject in more details in the next chapter.

Chapter 6 Separation of variables from transfer matrices

The seminal ideas of SoV are found in Sklyanin's methods, but we already discussed how the extension to the $\mathfrak{gl}(3)$ case does not work out of the box: the identification and construction of the proper $\mathcal{A}(u)$ and $\mathcal{B}(u)$ operators is not easy, i.e. the quantization of the classical counterparts is not immediate.

The central idea of this chapter, which originates from the seminal publication [225] from J. M. Maillet and G. Niccoli, is to construct a separate basis from conserved charges themselves, by acting on some reference vector. Conserved quantities are the core features of integrable models, and it is natural to leverage them to solve their own spectral problem. Besides, the very form of the basis presented here allows a separation of variables straightforwardly. The key computation relies on algebraic property specific to non-derogatory matrices, thus we begin by some linear algebra results on this family of matrices that will serve the following discussion. Then, we show for $\mathscr{V}(\mathfrak{gl}(n))$ rational fundamental models how the repeated action of transfer matrices evaluated in the inhomogeneities on certain seed states $\langle S|$ generates a separate basis as soon as the twist matrix K is non-derogatory. This SoV basis is exploited to solve the spectral problem for the $\mathfrak{gl}(2)$ case in details, while elements of the proofs for higher representations, higher rank, other boundary conditions or different quantum groups are summarized from the related references [225, 232–235]. We will also show the precise link with Sklyanin's original SoV for the $\mathfrak{gl}(2)$ case.

The ideas behind this new construction of a separate basis unlatched recent progress on quantum SoV, allowing the construction of the eigenbasis of the $\mathcal{B}(u)$ operators in higher rank models [236], which effectively factorizes the wavefunctions of eigenstates. The results were refined in [237], where the authors propose an explicit realization as Wronskians of the conjugate momenta that realize the shift in the separate basis.

6.1 Foreword: non-derogatory matrices

Definition. *A square matrix is* non-derogatory *if its characteristic polynomial and minimal polynomial coincide.*

This is equivalent to say the matrix has simple spectrum, i.e. each distinct eigenvalue has only one eigenvector: for a non-derogatory matrix $X \in \mathrm{Mat}(\mathbb{C}^d)$,

$$\forall \lambda \in \mathrm{Sp}(X), \quad \exists! v_\lambda \in \mathbb{C}^d, \quad X v_\lambda = \lambda v_\lambda. \tag{6.1}$$

It means that each eigenvalue has geometric multiplicity one, despite possibly being a multiple root in the characteristic polynomial. In terms of Jordan block, every Jordan block has an eigenvalue different from the other ones, see proposition 2. Such matrices are also called cyclic, see next paragraph.

Non-derogatory matrices have the property to be similar to the companion matrix of their characteristic polynomial. Let X be such a matrix and

$$\chi_X(u) = \det(u\,\mathrm{id} - X) = u^d + \sum_{k=0}^{d-1} a_k u^k \tag{6.2}$$

be the characteristic polynomial of X. There exists an invertible matrix V_X such that

$$C_X := \begin{pmatrix} 0 & 0 & \dots & 0 & -a_0 \\ 1 & 0 & \dots & 0 & -a_1 \\ 0 & 1 & \dots & 0 & -a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & -a_{d-1} \end{pmatrix} = V_X X V_X^{-1}. \tag{6.3}$$

One can also bring X to the transpose of the above companion matrix $C_X^\intercal$ with an invertible matrix V_X'.

6.1.1 Cyclic vectors and basis

Proposition 1. *Let $X \in \mathrm{Mat}(\mathbb{C}^d)$ be a non-derogatory matrix. There exists a vector $s \in \mathbb{C}^d$ such that*

$$\left(s, Xs, X^2 s, \dots, X^{d-1} s\right) \tag{6.4}$$

forms a basis of $\mathbb{C}^d$. The vector s is a cyclic vector associated to X. Similarly, there exist a covector $s' \in (\mathbb{C}^d)^* \simeq \mathbb{C}^d$ such that

$$\left(s', s'X, s'X^2, \dots, s'X^{d-1}\right) \tag{6.5}$$

is a basis of covector of $\mathbb{C}^d$.

Proof. Let $\sigma \in \mathbb{C}^d$ such that $\sigma_1 \neq 0$ and $\sigma_d = 0$. Then, for C_X from (6.3), noting e_i the canonical vectors of $\mathbb{C}^d$,

$$\forall n \in [\![0, d-1]\!], \quad C_X^n \sigma = \sigma_1 e_{n+1} + \alpha e_d, \quad \text{where} \quad \alpha \in \mathbb{C}. \tag{6.6}$$

It is clear that $\{\sigma, C_X\sigma, \dots C_X^{d-1}\sigma\}$ forms a basis of $\mathbb{C}^d$. Its image by the isomorphism V_X^{-1} is a basis, so with $s := V_X\sigma$ the family (6.4) is a basis.

It is possible to put X in the transpose companion form $C_X^\intercal$ through another isomorphism

$$C_X^\intercal = W_X X W_X^{-1}. \tag{6.7}$$

With a similar computation, a covector s' with $s'_1 \neq 0$ and $s'_d = 0$ is cyclic for X and (6.5) forms a basis. □

In fact, lots of vectors are cyclic for a given non-derogatory matrix. An alternate proof that (6.4) and (6.5) are bases using the Jordan canonical form highlight this.

Proposition 2. *Let $X \in \mathrm{Mat}(\mathbb{C}^d)$ be a non-derogatory matrix, with therefore simple spectrum $\mathrm{Sp}X = \{\lambda_1, \dots, \lambda_a\}$, $a \leq d$, so that*

$$\forall i, j \in [\![1, a]\!], \quad i \neq j, \quad \lambda_i \neq \lambda_j. \tag{6.8}$$

There exists an invertible matrix $\mathcal{V}_X$ that puts X in its Jordan normal form

$$J_X := \begin{pmatrix} J(\lambda_1, m_1) & & & \\ & J(\lambda_2, m_2) & & \\ & & \ddots & \\ & & & J_a(\lambda_a, m_a) \end{pmatrix} = \mathcal{V}_X X \mathcal{V}_X^{-1}, \tag{6.9}$$

where m_i is the algebraic multiplicity of the eigenvalue λ_i, $\sum_{i=1}^a m_i = d$, and $J(\lambda, m)$ is the $m \times m$ Jordan

block

$$J(\lambda,m)=\begin{pmatrix}\lambda & 1 & 0 & \cdots & 0\\ 0 & \lambda & 1 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \lambda & 1\\ 0 & 0 & 0 & 0 & \lambda\end{pmatrix}. \tag{6.10}$$

Let S be a covector in $(\mathbb{C}^d)^*$ and note

$$S\mathcal{V}_X=\left(s_1^{(1)},\dots,s_{m_1}^{(1)},\dots,s_1^{(a)},\dots,s_{m_a}^{(a)}\right). \tag{6.11}$$

If

$$\prod_{i=1}^{a}s_1^{(i)}\neq 0, \tag{6.12}$$

then the family $\left(S,SX,\dots,SX^{d-1}\right)$ is a basis of covectors.

Proof. Noting $\mathcal{E}_i:=\mathcal{V}_X^{-1}e_i$ the image of the canonical basis by $\mathcal{V}_X^{-1}$, the family $\left(S,SX,\dots,SX^{d-1}\right)$ is a basis if there exists an invertible bijective maping with another basis, for example the canonical one. This is the case if the determinant of the $d\times d$ matrix M, defined by its coefficients

$$\mathsf{M}_{ij}=SX^{i-1}\mathcal{E}_j, \tag{6.13}$$

is non-zero. Thanks to the block diagonal form, this determinant is the product of the determinant in each Jordan block $J(\lambda_i,m_i)$. Using standard linear algebra techniques [see 225, proposition 2.2], one eventually computes

$$\det\mathsf{M}=\prod_{i=1}^{a}\left(s_1^{(i)}\right)^{m_i}\prod_{1\leq i<j\leq a}(\lambda_j-\lambda_i)^{m_im_j}, \tag{6.14}$$

proving the desired result. □

The above proposition also holds for cyclic bases of vectors, but we anticipate on the integrable model case, where we will mainly use a basis of covectors similar to this one.

Remark 1 (Centralizer of X). Let

$$\mathcal{C}_X=\{M\in\mathrm{Mat}(\mathbb{C}^d)\mid[M,X]=0\} \tag{6.15}$$

be the centralizer of a non-derogatory matrix X. Because the minimal polynomial of X coincides with its characteristic polynomial, $\mathcal{C}_X$ has dimension d and consists in the set of matrices which are polynomials in X with coefficients in $\mathbb{C}$ [226]. Hence, the family $(1,X,\dots,X^{d-1})$ is a basis of the vector space $\mathcal{C}_X$. Products of two powers of X re-decompose as a linear combination of powers smaller than $d-1$. For $X\cdot X^{d-1}=X^d$, this is computed explicitly from the minimal/characteristic polynomial, see (6.21).

6.1.2 Spectrum and eigenvectors

The cyclic basis is useful in the study of the eigenvectors of X: for a given eigenvalue, the unique associated eigenvector is fully determined by its coefficients in the cyclic basis, which are easily computed to be powers of the eigenvalue. From now on we will use braket notation.

Proposition 3. *Let $X\in\mathrm{Mat}(\mathbb{C}^d)$ be a non-derogatory matrix. Let $\langle S|:=s'$ and $\langle f_n|:=\langle S|X^{n-1}$. A vector $|\lambda\rangle\in\mathbb{C}^n$ is an eigenvector of X of eigenvalue λ if and only if*

$$\forall n\in[\![1,d]\!],\quad \langle f_n|\lambda\rangle=\lambda^{n-1}\langle S|\lambda\rangle. \tag{6.16}$$

Proof. Suppose $|\lambda\rangle \neq 0$ is an eigenvector of X with eigenvalue λ. Then (6.16) is verified immediately, but one should check that $\langle S|\lambda\rangle \neq 0$. Because the $\langle f_n|$ form a basis of $\mathbb{C}^n$, the dual vectors

$$|f_n\rangle := \langle f_n|^* = \left(\bar{X}^\intercal\right)^{n-1}(\bar{s'})^\intercal \tag{6.17}$$

form a basis of $\mathbb{C}^n$. Since $|\lambda\rangle \neq 0$ and

$$|\lambda\rangle = \sum_{n=1}^{d} \frac{\langle f_n|\lambda\rangle}{\langle f_n|f_n\rangle}\,|f_n\rangle = \langle S|\lambda\rangle \sum_{n=1}^{d} \frac{\lambda^{n-1}}{\langle f_n|f_n\rangle}\,|f_n\rangle\,, \tag{6.18}$$

necessarily $\langle S|\lambda\rangle \neq 0$ and (6.16) indeed holds.

Consider now a vector $|v\rangle \in \mathbb{C}^d$ such that $\langle S|v\rangle \neq 0$ and for an eigenvalue $\lambda \in \mathrm{Sp}(X)$, it holds

$$\forall n \in [\![1,d]\!]\,, \quad \langle f_n|v\rangle = \lambda^{n-1}\,\langle S|v\rangle\,. \tag{6.19}$$

Is $|v\rangle$ the eigenvector of X of eigenvalue λ? For $n < d$,

$$\langle f_n|X|v\rangle = \langle S|X^n\,|v\rangle = \langle f_{n+1}|v\rangle = \lambda\,\langle f_n|v\rangle\,, \tag{6.20}$$

which puts us on good track. For $n = d$, the vector $\langle f_d|X = \langle S|X^d$ is no longer a vector of the $\langle f_n|$ basis. However, the power X^d can be rewritten thanks to the characteristic polynomial of X: by the Cayley–Hamilton theorem, X satisfies its characteristic equation

$$\chi_X(X) = 0 = X^d + a_{d-1}X^{d-1} + \cdots + a_1 X + a_0. \tag{6.21}$$

This effectively decomposes the right action of X on $\langle f_d|$ over the $\langle f_n|$ basis

$$\langle f_d|X|v\rangle = -\sum_{n=0}^{d-1} a_n\,\langle S|X^n|v\rangle = -\sum_{n=0}^{d-1} a_n\,\langle f_{n+1}|v\rangle = -\langle S|v\rangle \sum_{n=0}^{d-1} a_n\lambda^n. \tag{6.22}$$

Because $\lambda \in \mathrm{Sp}(X)$, $\chi_X(\lambda) = 0$ so we reconstruct λ^d in the above equation. Therefore,

$$\langle f_d|X|v\rangle = \lambda^d\,\langle S|v\rangle = \lambda\,\langle f_d|v\rangle\,. \tag{6.23}$$

Eventually, this proves that $|v\rangle$ is the eigenvector of X of eigenvalue λ — and may be noted $|\lambda\rangle := |v\rangle$. □

This discussion on non-derogatory matrices shows that a basis may be constructed by repeated action on a cyclic vector. This basis is especially useful for their spectral problem, because the eigenvectors' decomposition in the $|f_n\rangle = \langle f_n|^*$ basis is entirely determined by powers of corresponding eigenvalue, up to an overall normalization fixed by $\langle S|\lambda\rangle$. Hence, in such a basis, we can construct explicitly the unique eigenvector (up to a non-zero normalization) as soon as we know the eigenvalue. This is a very handy result, as it is often possible, and easier, to compute only the eigenvalues.

"Constructing a basis that facilitates the spectral problem" is clearly of prime interest for quantum integrable models. In the context of quantum integrability, can one construct a separate basis with similar properties as the ones of non-derogatory matrices? We investigate this in the next section.

6.2 Separate basis from conserved quantities

6.2.1 Basis from conserved quantities

While Sklyanin's SoV sounds a promising tool which could supersede ABA techniques, the identification of the $\mathcal{A}(u)$ and $\mathcal{B}(u)$ operators are in general not trivial. All the more so as it is also required to prove non-immediate properties, such as the diagonalizability of $\mathcal{B}(u)$, then characterize its spectrum and prove its simplicity, and ensure the $\mathcal{A}(u)$ operator evaluated in the spectrum of $\mathcal{B}(u)$ indeed perform shifts on the $\mathcal{B}(u)$ spectrum.

Even for the rank one $\mathfrak{gl}(2)$ case, there are models for which such a program cannot be completed, for example the XXZ chain with a diagonal twist. Working with a greater algebra make the picture more involved. In the $\mathfrak{gl}(3)$ case, we already investigated how the expression of $\mathcal{A}(u)$ given by Sklyanin cannot perform the desired shift over the full spectrum of $\mathcal{B}(u)$. This is essentially because it is not anymore a polynomial in the monodromy matrix entries.

These observations provide strong motivation to look for a more universal construction of SoV in quantum integrable models. Integrability leverages the conserved quantities to ensure solvability. Could they be used to developing a separation of variables as well?

It appears they can: repeated action of the transfer matrix evaluated in the inhomogeneities on a certain covector produce a basis, which is immediately separate for the spectral problem of the transfer matrix itself. The key point in this construction is that it relies on the very object we ought to diagonalize. This overcomes entirely the identification of the $\mathcal{A}(u)$ and $\mathcal{B}(u)$ operator families and the proofs of their required properties. Besides, this SoV construction applies for a larger class of models. It generalizes to higher rank rather trivially, though the details of the use of the basis to solve the spectral problem are much more tedious than in the $\mathfrak{gl}(2)$. We discuss them in section 6.2.5.

Consider the $\mathscr{Y}(\mathfrak{gl}(\mathfrak{n}))$ chain of length N with the commuting family of transfer matrices $(T(u))_{u\in\mathbb{C}}$, $\forall u, v \in \mathbb{C}$, $[T(u), T(v)] = 0$, and Hilbert space $\mathscr{H}$. Suppose one can construct $\dim \mathscr{H}$ matrices from $T(u)$ gathered in N sets

$$\{T_0^{(a)}, \dots, T_{d_a-1}^{(a)}\}, \tag{6.24}$$

with $a \in [\![1, N]\!]$, and $\prod_{a=1}^{N} d_a = \dim \mathscr{H}$. Some functional $f_{h_a}^{(a)}$ of the $(T(u))_{u\in\mathbb{C}}$ family are such that these matrices are obtained from the transfer matrix $T : \mathbb{C} \to \mathrm{End}(\mathscr{H})$

$$T_{h_a}^{(a)} = f_{h_a}^{(a)}\big[T\big]. \tag{6.25}$$

For any N-tuple in $\mathcal{I} = [\![0, d_1 - 1]\!] \times \cdots \times [\![0, d_N - 1]\!]$, we note

$$T_{\vec{\mathbf{h}}} := \prod_{a=1}^{N} T_{h_a}^{(a)} \tag{6.26}$$

the $\dim \mathscr{H}$ possible products made by picking one operator in each set. Because the $T_{h_a}^{(a)}$ are functions of the $(T(u))_{u\in\mathbb{C}}$ family, they commute together and with $T(u)$ for any $u \in \mathbb{C}$, and so do the $T_{\vec{\mathbf{h}}}$. We note

$$C_T := \{X \in \mathrm{End}(\mathscr{H}) \mid \forall u \in \mathbb{C}, \quad [X, T(u)] = 0\}. \tag{6.27}$$

Suppose there exists $\langle S| \in \mathscr{H}^*$ such that the $\dim \mathscr{H}$ covectors

$$\langle \vec{\mathbf{h}}| := \langle S|\, T_{\vec{\mathbf{h}}} \tag{6.28}$$

form a basis of $\mathscr{H}^*$. Let

$$\mathcal{G}(\langle S|) := \left\{ \langle S| T_{\vec{\mathbf{h}}} \;\middle|\; \vec{\mathbf{h}} \in \mathcal{I} \right\} \tag{6.29}$$

Proposition 4. *If $\mathcal{G}(\langle S|)$ is a basis of $\mathscr{H}^*$, then the family $T(u)$ has simple spectrum. Moreover, the family $(T_{\vec{\mathbf{h}}})_{\vec{\mathbf{h}}\in\mathcal{I}}$ is a complete set of commuting matrices, i.e. it has simple common spectrum.*

Proof. Let $|t\rangle, |t'\rangle \in \mathscr{H}$ be two non-zero eigenvectors of the $T(u)$, $u \in \mathbb{C}$, associated to the same eigenvalue $t(u) \in \mathrm{Fun}(\mathbb{C},\mathbb{C})$. Because $T_{\vec{\mathbf{h}}} \in C_T$, $|t\rangle$ and $|t'\rangle$ are also eigenvectors of $T_{\vec{\mathbf{h}}}$. We note $t_{\vec{\mathbf{h}}}$ the associated eigenvalue, which is the same for $|t\rangle$ and $|t'\rangle$ and is determined from the eigenvalue $t : u \mapsto t(u)$ by the knowledge of the functionals (6.25):

$$t_{\vec{\mathbf{h}}} = \prod_{a=1}^{N} t_{h_a}^{(a)} = \prod_{a=1}^{N} f_{\vec{\mathbf{h}}_a}^{(a)}[t] \quad \text{for all} \quad \vec{\mathbf{h}} \in \mathcal{I}. \tag{6.30}$$

We abuse the notation $t_{\vec{\mathbf{h}}_a}^{(a)}$ to also refer to the functionals from $\mathrm{Fun}(\mathbb{C},\mathbb{C})$ to $\mathbb{C}$ derived from (6.25). The wavefunctions of $|t\rangle$ in the basis $\mathcal{G}(\langle S|)$ are

$$\langle \vec{\mathbf{h}}|t\rangle = \langle S|t\rangle\, t_{\vec{\mathbf{h}}} \quad \text{and} \quad \langle \vec{\mathbf{h}}|t'\rangle = \langle S|t'\rangle\, t_{\vec{\mathbf{h}}} \tag{6.31}$$

Because $|t\rangle \neq 0$, $\langle S|t\rangle \neq 0$, and the same holds for $|t'\rangle$. With a renormalization of $|t'\rangle$, one can put $\langle S|t'\rangle = \langle S|t\rangle$ in all generality. Then,

$$\forall \vec{\mathbf{h}} \in \mathcal{I}, \quad \langle \vec{\mathbf{h}}| \big(|t\rangle - |t'\rangle\big) = \langle \vec{\mathbf{h}}|t\rangle - \langle \vec{\mathbf{h}}|t'\rangle = \langle S|t\rangle \big(t_{\vec{\mathbf{h}}} - t_{\vec{\mathbf{h}}}\big) = 0. \tag{6.32}$$

The image of $|t\rangle - |t'\rangle$ under the basis of linear forms $\langle \vec{\mathbf{h}}|$ is zero, so this vector is zero. Henceforth, the common spectrum of $(T(u))_{u\in\mathbb{C}}$ family is simple

Moreover, the $T_{\vec{\mathbf{h}}}$ commute together, as functions of $T(u)$, and share the same eigenvectors with $T(u)$. For $|t\rangle, |t'\rangle \in \mathscr{H}$ two eigenvectors of the family $(T_{\vec{\mathbf{h}}})_{\vec{\mathbf{h}}\in\mathcal{I}}$ with the same eigenvalues $t_{\vec{\mathbf{h}}}$, the above discussion holds starting from (6.31), so that $|t\rangle = |t'\rangle$, and $|t\rangle$ is defined by its wavefunctions in $\mathcal{G}(\langle S|)$. Therefore, the family $(T_{\vec{\mathbf{h}}})_{\vec{\mathbf{h}}\in\mathcal{I}}$ is a complete set of commuting matrices, namely it has simple common spectrum. □

Proposition 5. *If $\mathcal{G}(\langle S|)$ is a basis of $\mathscr{H}^*$, then the $T_{\vec{\mathbf{h}}}$ form a basis of the centralizer C_T of $(T(u))_{u\in\mathbb{C}}$.*

Proof. Suppose there exist a linear relation between the $T_{\vec{\mathbf{h}}}$, i.e. there exist coefficients $\lambda_{\vec{\mathbf{h}}} \in \mathbb{C}$, not all zero, such that

$$\sum_{\vec{\mathbf{h}}\in\mathcal{I}} \lambda_{\vec{\mathbf{h}}} T_{\vec{\mathbf{h}}} = 0. \tag{6.33}$$

Then, acting on $\langle S|$ by the right,

$$\sum_{\vec{\mathbf{h}}\in\mathcal{I}} \lambda_{\vec{\mathbf{h}}} \langle \vec{\mathbf{h}}| = 0, \tag{6.34}$$

so this would imply that $\mathcal{G}(\langle S|)$ is not a basis. By contraposition, if $\mathcal{G}(\langle S|)$ is a basis of $\mathscr{H}^*$, the $T_{\vec{\mathbf{h}}}$ are necessarily a free family. Besides, the family $(T(u))_{u\in\mathbb{C}}$ has simple spectrum by proposition 4, so its minimal polynomial coincide with its characteristic polynomial and its centralizer C_T has dimension $\dim \mathscr{H}$. The family $(T_{\vec{\mathbf{h}}})_{\vec{\mathbf{h}}\in\mathcal{I}}$ is a free family in C_T of cardinal $\dim \mathscr{H}$, and therefore is a basis of the matrices that commute with the $(T(u))_{u\in\mathbb{C}}$ family. □

By proposition 5, for $T(u) \in C_T$ at generic value $u \in \mathbb{C}$, it can be decomposed as a linear combination on the $T_{\vec{\mathbf{h}}}$ family, which is a basis of C_T. Therefore, there exists functions $C_{\vec{\mathbf{h}}}(u)$ such that

$$T(u) = \sum_{\vec{\mathbf{h}}\in\mathcal{I}} C_{\vec{\mathbf{h}}}(u) T_{\vec{\mathbf{h}}} \quad \text{for all} \quad u \in \mathbb{C}. \tag{6.35}$$

The product of two matrices $T_{\vec{\mathbf{h}}}$, $T_{\vec{\mathbf{k}}}$ is in C_T as well, so the $T_{\vec{\mathbf{h}}}$ are $\dim(\mathscr{H})$ generators of an associative abelian algebra with relations

$$T_{\vec{\mathbf{h}}} \cdot T_{\vec{\mathbf{k}}} = \sum_{\vec{\ell} \in \mathcal{I}} C^{\vec{\ell}}_{\vec{\mathbf{h}},\vec{\mathbf{k}}} T_{\vec{\ell}}. \tag{6.36}$$

The $C^{\vec{\ell}}_{\vec{\mathbf{h}},\vec{\mathbf{k}}}$ are numerical coefficients that satisfy the associativity and commutativity constraints, namely

$$C^{\vec{\ell}}_{\vec{\mathbf{h}},\vec{\mathbf{k}}} = C^{\vec{\ell}}_{\vec{\mathbf{k}},\vec{\mathbf{h}}}, \tag{6.37}$$

$$\sum_{\vec{\mathbf{p}} \in \mathcal{I}} C^{\vec{\mathbf{p}}}_{\vec{\mathbf{h}},\vec{\mathbf{k}}} C^{\vec{\mathbf{q}}}_{\vec{\mathbf{p}},\vec{\ell}} = \sum_{\vec{\mathbf{p}} \in \mathcal{I}} C^{\vec{\mathbf{p}}}_{\vec{\mathbf{k}},\vec{\ell}} C^{\vec{\mathbf{q}}}_{\vec{\mathbf{h}},\vec{\mathbf{p}}}. \tag{6.38}$$

Relations (6.36) also holds at the eigenvalue level

$$t_{\vec{\mathbf{h}}} \cdot t_{\vec{\mathbf{k}}} = \sum_{\vec{\ell} \in \mathcal{I}} C^{\vec{\ell}}_{\vec{\mathbf{h}},\vec{\mathbf{k}}} t_{\vec{\ell}}. \tag{6.39}$$

The functions $C_{\vec{\mathbf{h}}}(u)$ and the constant numbers $C^{\vec{\ell}}_{\vec{\mathbf{h}},\vec{\mathbf{k}}}$ are fixed by the model at hand and the choice of the $T^{(a)}_{h_a}$ operators. With the decomposition of the transfer matrix over the $T_{\vec{\mathbf{h}}}$, one also has

$$T(u) \cdot T_{\vec{\mathbf{k}}} = \sum_{\vec{\mathbf{h}} \in \mathcal{I}} C_{\vec{\mathbf{h}}}(u)\, T_{\vec{\mathbf{h}}} \cdot T_{\vec{\mathbf{k}}} = \sum_{\vec{\mathbf{h}},\vec{\ell} \in \mathcal{I}} C_{\vec{\mathbf{h}}}(u)\, C^{\vec{\ell}}_{\vec{\mathbf{h}},\vec{\mathbf{k}}}\, T_{\vec{\ell}}. \tag{6.40}$$

The relations (6.39) can serve as a characterization of the transfer matrix spectrum. Let

$$\Sigma := \left\{ \left(x_{\vec{\mathbf{h}}}\right)_{\vec{\mathbf{h}} \in \mathcal{I}} \;\middle|\; \forall \vec{\mathbf{h}}, \vec{\mathbf{k}} \in \mathcal{I}, \quad x_{\vec{\mathbf{h}}} \cdot x_{\vec{\mathbf{k}}} = \sum_{\vec{\ell} \in \mathcal{I}} C^{\vec{\ell}}_{\vec{\mathbf{h}},\vec{\mathbf{k}}} x_{\vec{\ell}} \quad \text{and} \quad \exists \vec{\mathbf{h}} \in \mathcal{I}, \quad x_{\vec{\mathbf{h}}} \neq 0 \right\} \tag{6.41}$$

be the set of $\dim \mathscr{H}$-tuples of complex numbers solutions to the $(\dim \mathscr{H})(\dim \mathscr{H} + 1)/2$ quadratic equations (6.39).

Theorem 4. *If $\mathcal{G}(\langle S|)$ is a basis of $\mathscr{H}^*$, then every eigenvalue in $\mathrm{Sp}(T(u))$ is characterized uniquely by a tuple of Σ. Namely, there is a bijection between $\mathrm{Sp}(T(u))$ and Σ.*

Proof. Suppose $\mathcal{G}(\langle S|)$ is a basis. Then the transfer matrix has simple spectrum, and the unique eigenvector $|t\rangle$ associated to $t(u) \in \mathrm{Sp}(T(u))$ is an eigenvector of the $T_{\vec{\mathbf{h}}}$ as well, with eigenvalues $t_{\vec{\mathbf{h}}}$ defined in (6.30). Obviously the tuple $\left(t_{\vec{\mathbf{h}}}\right)_{\vec{\mathbf{h}} \in \mathcal{I}}$ is in Σ—this is seen easily by computing the action of the $T_{\vec{\mathbf{h}}} \cdot T_{\vec{\mathbf{k}}}$ over $|t\rangle$.

Consider now $\left(x_{\vec{\mathbf{h}}}\right)_{\vec{\mathbf{h}} \in \mathcal{I}} \in \Sigma$ different from the zero tuple. We construct the non-zero vector $|x\rangle$ by fixing its images under the basis of linear forms $\langle \vec{\mathbf{h}}|$

$$\forall \vec{\mathbf{h}} \in \mathcal{I}, \quad \langle \vec{\mathbf{h}}|x\rangle = x_{\vec{\mathbf{h}}}. \tag{6.42}$$

One has

$$\begin{aligned}
\langle \vec{\mathbf{h}}|T(u)|x\rangle &= \langle S|T_{\vec{\mathbf{h}}} \cdot T(u)|x\rangle = \langle S|T_{\vec{\mathbf{h}}} \sum_{\vec{\mathbf{k}} \in \mathcal{I}} C_{\vec{\mathbf{k}}}(u) T_{\vec{\mathbf{k}}}|x\rangle \\
&= \sum_{\vec{\mathbf{k}},\vec{\ell} \in \mathcal{I}} C_{\vec{\mathbf{k}}}(u) C^{\vec{\ell}}_{\vec{\mathbf{h}},\vec{\mathbf{k}}} \langle S|T_{\vec{\ell}}|x\rangle = \sum_{\vec{\mathbf{k}},\vec{\ell} \in \mathcal{I}} C_{\vec{\mathbf{k}}}(u) C^{\vec{\ell}}_{\vec{\mathbf{h}},\vec{\mathbf{k}}} x_{\vec{\ell}} \quad \text{(identifying } \langle \vec{\ell}|x\rangle\text{)} \\
&= x_{\vec{\mathbf{h}}} \cdot \sum_{\vec{\mathbf{k}} \in \mathcal{I}} C_{\vec{\mathbf{k}}}(u) x_{\vec{\mathbf{k}}} \quad \text{(using relations (6.41)).}
\end{aligned} \tag{6.43}$$

Hence, defining the function

$$\tau_x(u) := \sum_{\vec{\mathbf{k}}\in\mathcal{I}} C_{\vec{\mathbf{k}}}(u) x_{\vec{\mathbf{k}}}, \tag{6.44}$$

one has that

$$\forall \vec{\mathbf{h}} \in \mathcal{I}, \quad \langle \vec{\mathbf{h}}|T(u)|x\rangle = \langle \vec{\mathbf{h}}|x\rangle\, \tau_x(u). \tag{6.45}$$

Therefore, $|x\rangle$ is an eigenvector of $T(u)$ with eigenvalue $\tau_x(u)$. Two different tuples of Σ cannot yield the same eigenvalue: let $(x_{\vec{\mathbf{h}}})_{\vec{\mathbf{h}}}, (y_{\vec{\mathbf{h}}})_{\vec{\mathbf{h}}} \in \Sigma$, $(x_{\vec{\mathbf{h}}})_{\vec{\mathbf{h}}} \neq (y_{\vec{\mathbf{h}}})_{\vec{\mathbf{h}}}$ such that $\tau_x(u) = \tau_y(u)$. Then the two states $|x\rangle$ and $|y\rangle$, having different images under the basis of linear form $\langle \vec{\mathbf{h}}|$, are different eigenvectors of $T(u)$, yet with the same eigenvalue. This is not possible, since $T(u)$ has simple spectrum.

Hence, every eigenvalue $t(u)$ of $T(u)$ is associated to a unique tuple $\left(t_{\vec{\mathbf{h}}}\right)_{\vec{\mathbf{h}}\in\mathcal{I}}$ in Σ, and any tuple of Σ defines an eigenvalue of $T(u)$. Therefore, $\#\Sigma = \#\,\mathrm{Sp}(T(u))$, and the two sets are in bijection. One way is realized by the $f^{(a)}_{h_a}$ functions, and the other by the linear combination (6.44) with the coefficients $C_{\vec{\mathbf{h}}}(u)$. □

Hence, whenever a basis of the form $\mathcal{G}(\langle S|)$ can be constructed, the spectrum of the transfer matrix has a characterization in terms of solutions of a system of quadratic equations given by the relations between the generators of the centralizer of $T(u)$ that constructs the basis.

However, the number of relations (6.39) is a priori $\dim\mathscr{H}(\dim\mathscr{H}+1)/2$, which in case of quantum lattice models is exponential in the number of sites, since $\dim\mathscr{H} = \prod_{j=1}^{N} m_j$, where m_j is the dimension of the representation of $\mathfrak{gl}(\mathfrak{n})$ at site $j \in [\![1, N]\!]$. For quantum models constructed from a Yang–Baxter algebra, one observes a drastic reduction of the number of relations necessary to characterize the spectrum. This is the hallmark of integrability. Thanks to the fusion relations holding between the transfer matrices of these models, the number of relations implying the full relations (6.39) becomes in such case polynomial in the number of sites. Such a reduction from *an exponential to a polynomial dependency* in the number of sites N happens due to the possible careful choice of the set of the $T_{\vec{\mathbf{h}}}$ in integrable models. We will give several examples of such reduction in the remaining of this thesis. For now, let us give here a toy example of this phenomenon.

A toy situation is the one where the relations remain local in the indices of the N-tuples $\vec{\mathbf{h}}$. Within our notation, the wavefunctions of eigenstates are already factorized

$$\langle \vec{\mathbf{h}}|t\rangle = \langle S|t\rangle \prod_{a=1}^{N} t^{(a)}_{h_a}. \tag{6.46}$$

Suppose the actions are moreover local in the indices h_a of the N-tuples of $\mathcal{I}$, that is

$$T^{(a)}_i \cdot T^{(a)}_j = \sum_{k=1}^{d_a} c^{(a),k}_{ij} T^{(a)}_k, \tag{6.47}$$

with some fixed numerical coefficients c^k_{ij}. The knowledge of the c^k_{ij} coefficients completely fixes the relations (6.36), by

$$T_{\vec{\mathbf{h}}} \cdot T_{\vec{\mathbf{k}}} = \prod_{a=1}^{N} T^{(a)}_{h_a} \cdot T^{(a)}_{k_a} = \prod_{a=1}^{N} \sum_{\ell_a=1}^{d_a} c^{(a),\ell_a}_{h_a k_a} T^{(a)}_{\ell_a} \equiv \sum_{\vec{\ell}\in\mathcal{I}} C^{\vec{\ell}}_{\vec{\mathbf{h}},\vec{\mathbf{k}}} T_{\vec{\ell}}. \tag{6.48}$$

Rather than the system of equations given in (6.41), it is now simpler to consider N systems. For all

$a \in [\![1,N]\!]$, let

$$\Sigma_a := \left\{ \left(x_1^{(a)},\dots,x_{d_a}^{(a)}\right) \,\middle|\, \forall i,j \in [\![1,d_a]\!], \quad x_i^{(a)}x_j^{(a)} = \sum_{k=1}^{d_a} c_{ij}^{(a),k} x_k^{(a)} \quad \text{and} \quad \exists j \in [\![1,d_a]\!], \quad x_j^{(a)} \neq 0 \right\} \tag{6.49}$$

be the set of d_a-tuples of complex numbers solutions to the $d_a(d_a+1)/2$ quadratic equations (6.47).

Proposition 6. *If $\mathcal{G}(\langle S|)$ is a basis of $\mathscr{H}^*$, and if the $T_i^{(a)}$ multiply locally as in (6.47), then every eigenvalue in $\mathrm{Sp}(T(u))$ is characterized uniquely by N tuples picked from $\Sigma_1,\dots,\Sigma_N$. Namely, there is a bijection between $\mathrm{Sp}(T(u))$ and $\Sigma_1 \times \cdots \times \Sigma_N$.*

Proof. Let $|t\rangle$ be an eigenvector of $T(u)$ of eigenvalue $t(u)$. It is also an eigenvector of the $T_{h_a}^{(a)}$, with eigenvalues $t_{h_a}^{(a)}$. It is clear that the $(t_1^{(a)},\dots,t_{d_a}^{(a)})$ are d_a-tuples of complex numbers in Σ_a^*, and this for every $a \in [\![1,N]\!]$.

Consider now $(x_1^{(a)},\dots,x_{d_1}^{(a)}) \in \Sigma_a$, for every $a \in [\![1,N]\!]$. We construct the vector $|x\rangle \in \mathscr{H}$ from its image by the linear form $\langle \vec{\mathbf{h}}|$:

$$\forall \vec{\mathbf{h}} \in \mathcal{I}, \quad \langle \vec{\mathbf{h}}|x\rangle = \prod_{a=1}^{N} x_{h_a}^{(a)}. \tag{6.50}$$

Then, by a proof similar to the one of Theorem 4, one proves that $|x\rangle$ is an eigenvector of $T(u)$ of eigenvalue

$$\tau_x(u) = \sum_{\vec{\mathbf{k}} \in \mathcal{I}} C_{\vec{\mathbf{k}}}(u) \prod_{a=1}^{N} x_{h_a}^{(a)}. \tag{6.51}$$

□

With such local actions for the $T_{h_a}^{(a)}$, the discrete characterization of the spectrum by the set of solutions Σ is separate in a product of N sets $\Sigma_1,\dots,\Sigma_N$. The system of equations (6.41) has been decoupled in the N independent systems (6.49), each of rank $d_a(d_a+1)/2$. This is a great simplification of the spectrum characterization. Moreover, eigenvectors are obtained from their corresponding eigenvalue with the knowledge of their image under the basis of linear forms $\langle \vec{\mathbf{h}}|$, which now has a factorized form.

In the following, we will give several example of quantum integrable models where a similar simplification can be made, although they are more intricate than the above toy model. In particular, in the next section where we take

$$T_{h_a}^{(a)} = T(\xi_a)^{h_a}, \quad \text{with} \quad h_a \in [\![0,\mathfrak{n}-1]\!], \tag{6.52}$$

we will see that we observe a drastic simplification of most of the above relations (6.47) and (6.48). The different cases have been treated in a series of articles, covering the $\mathfrak{gl}(2)$ case with fundamental [225] and higher spin representations [232], $\mathfrak{gl}(3)$ and $\mathfrak{gl}(\mathfrak{n})$ models [234, 235], and supersymmetric $\mathfrak{gl}(\mathfrak{n}|\mathfrak{m})$ models [LV1].

6.2.2 Bases from the powers of the transfer matrix

Following the case of non-derogatory matrices, we can define a family of vectors of the form (6.28) with powers of the transfer matrix evaluated in the inhomogeneities, and they will form a basis under some weak conditions on the twist and the inhomogeneities.

Theorem 5 ([225]). *Consider the Yangian $\mathscr{Y}\big(\mathfrak{gl}(\mathfrak{n})\big)$ R-matrix $R(u) = \varphi(u)\mathrm{id} + \varphi(\eta)\mathcal{P} \in \mathrm{End}(\mathbb{C}^{\mathfrak{n}} \otimes \mathbb{C}^{\mathfrak{n}})$. The fundamental inhomogeneous twisted chain of length N, with Hilbert space $\mathscr{H} = \otimes_{j=1}^{N} \mathbb{C}^{\mathfrak{n}}$, corresponds to the monodromy matrix*

$$M(u) = K_0 R_{0N}(u-\xi_N)\dots R_{01}(u-\xi_1), \tag{6.53}$$

and its transfer matrix $T(u) = \mathrm{tr}_0 M(u)$. If the twist matrix K is non-derogatory, the $\mathfrak{n}^N$ covectors

$$\langle \vec{\mathbf{h}}| := \langle S| \prod_{j=1}^{N} T(\xi_j)^{h_j}, \tag{6.54}$$

labelled by the N-tuples $\vec{\mathbf{h}} \in [\![1, \mathfrak{n}]\!]^N$, form a basis of $\mathscr{H}^*$ for generic inhomogeneities ξ_j when $\langle S|$ is of the form

$$\langle S| = \langle s| \otimes \cdots \otimes \langle s|, \tag{6.55}$$

where $\langle s|$ is a cyclic vector of the twist matrix K.

Proof. Let $\big|e_j\big\rangle$, $j \in [\![1, \mathfrak{n}^N]\!]$, be the canonical basis of $\mathscr{H}$, constructed by tensorization of vectors the local canonical bases at each site. We define the square matrix $\mathtt{M}$ of size $\mathfrak{n}^N$ by its elements

$$\mathtt{M}_{ij} = \big\langle \vec{\mathbf{h}}(i) \big| e_j \big\rangle, \tag{6.56}$$

where $\vec{\mathbf{h}}(i)$ is uniquely defined by

$$i = 1 + \sum_{a=1}^{N} h_a(i) a^{i-1}. \tag{6.57}$$

The $\langle \vec{\mathbf{h}}|$ form a basis of $\mathscr{H}^*$ if

$$\det(\mathtt{M}) \neq 0. \tag{6.58}$$

The above determinant is a polynomial in the twist matrix entries, the inhomogeneities ξ_j and the coefficients $\big\langle S \big| e_j \big\rangle$. It is sufficient to prove (6.58) in some limit in this parameter to prove it for almost all values of them. Let us impose inhomogeneities that are scaled from a single value relatively to the site they are attached

$$\exists \xi \in \mathbb{C}, \quad \forall j \in [\![1, N]\!], \quad \xi_j = j\xi. \tag{6.59}$$

Because

$$T(\xi_a) = R_{aa-1}(\xi_a - \xi_{a-1}) \ldots R_{a1}(\xi_a - \xi_1) K_a R_{aN}(\xi_a - \xi_N) \ldots R_{aa+1}(\xi_a - \xi_{a+1}), \tag{6.60}$$

$T(\xi_a)$ is a polynomial in ξ of degree $N-1$ and is of the form

$$T(\xi_a) = c_a \xi^{N-1} K_a + \mathcal{O}\left(\xi^{N-2}\right) \quad \text{given} \quad c_a = (-1)^{N-a}(a-1)!(N-a)!. \tag{6.61}$$

Therefore,

$$\big\langle \vec{\mathbf{h}} \big| = \xi^{(N-1)\sum_{j=1}^{N} h_j} \times \left(\prod_{j=1}^{N} c_a^{h_j} \right) \times \langle S| \prod_{j=1}^{N} K_j^{h_j} + \mathcal{O}\left(\xi^{-1+(N-1)\sum_{j=1}^{N} h_j} \right). \tag{6.62}$$

The determinant of $\mathtt{M}$ is now a polynomial in ξ as well, of degree $(N-1)\sum_{i=1}^{\mathfrak{n}^N} \sum_{j=1}^{N} h_j(i)$ and dominant coefficient

$$\prod_{i=1}^{\mathfrak{n}^N} \prod_{j=1}^{N} c_j^{h_j(i)} \det\big(\hat{\mathtt{M}}\big), \tag{6.63}$$

where $\hat{\mathtt{M}}$ is the matrix with coefficients

$$\hat{\mathtt{M}}_{ij} = \Big\langle S \Big| \prod_{a=1}^{N} K_a^{h_a(i)} \Big| e_j \Big\rangle. \tag{6.64}$$

Taking the covector $\langle S|$ in a tensor form of covector local in the site

$$\langle S| = \otimes_{j=1}^{N} \langle S, j| \quad \text{where} \quad \langle S, j| \in \mathbb{V}_j^*, \tag{6.65}$$

the above determinant is factorized as a product of N determinants over square matrices of size n

$$\det\big(\hat{\mathtt{M}}\big) = \prod_{a=1}^{N} \det(\hat{\mathtt{m}}_a), \tag{6.66}$$

where, using $|e_j\rangle = \otimes_{a=1}^{N} |e_j(a)\rangle$,

$$(\hat{\mathtt{m}}_a)_{ij} = \big\langle S, a\big|K_a^{i-1}\big|e_j(a)\big\rangle. \tag{6.67}$$

Because K is a non-derogatory matrix, there exists a covector $\langle S, a| \in \mathbb{V}_a$ such that $\det(\hat{\mathtt{m}}_a) \neq 0$, thanks to proposition 1. Even better, proposition 2 gives explicit form of cyclic covector $\langle S, a|$ such that (6.54) forms a basis. Hence, there exists $\langle S| \in \mathscr{H}^*$ such that $\det(\mathtt{M}) \neq 0$ for almost all values of the parameters it depends on. □

The above theorem is a specialization of the discussion of the previous section, with

$$T_{\vec{\mathbf{h}}} = \prod_{j=1}^{N} T(\xi_j)^{h_i} \quad \text{with} \quad \vec{\mathbf{h}} \in [\![0, N-1]\!]^N. \tag{6.68}$$

These operators are now proven to form a basis of the Bethe algebra of the model. Thanks to the power form of the basis, the relations (6.36) are not too complicated for many generators. If

$$\forall j \in [\![1, N]\!], \quad h_j + k_j \leq N - 1, \tag{6.69}$$

then we simply have

$$T_{\vec{\mathbf{h}}} \cdot T_{\vec{\mathbf{k}}} = T_{\vec{\mathbf{h}}+\vec{\mathbf{k}}}. \tag{6.70}$$

If there is only one $j \in [\![1, N]\!]$ such that $h_j + k_j \geq N$, then it is necessary to compute the linear decomposition of a higher power on the ones smaller than $N-1$. The characteristic polynomial is here of no help, since it is a polynomial of degree $\mathfrak{n}^N$. However, the fusion relations can be used to compute this decomposition. Indeed, evaluated in the inhomogeneities, the fusion relations links products of transfer matrices to fused transfer matrix. The interesting point is that the one corresponding to the totally antisymmetric representation, namely $T_{\mathfrak{n}}^{(1)}(u)$, coincide with the quantum determinant which is a central element. This is seen easily in the $\mathfrak{gl}(2)$ case: the relevant fusion relation is written in the inhomogeneities as

$$T(\xi_j)T(\xi_j - \eta) = \text{q-det}\, M(\xi_j). \tag{6.71}$$

While we do not have directly $T(\xi_j)^2 = \text{q-det}\, M(\xi_j)$, it is clear that this relation provides a reduction in the power of the transfer matrix, which is 2 in the left-hand side and 0 in the right-hand side. To resolve the inconvenience of having one transfer matrix shifted by η, it is possible to interpolate $T(u)$ in the correct points to make use of the fusion relation to compute the decomposition of $T_{\vec{\mathbf{h}}} \cdot T(u)$ over the $T_{\vec{\mathbf{h}}}$. This is done in the proof of proposition 7.

The $\mathfrak{gl}(\mathfrak{n})$ case is more involved, because the successive use of the tower of $\mathfrak{n}-1$ fusion relations to requires to interpolate once again between each step. We explain the procedure to compute $T_{\vec{\mathbf{h}}} \cdot T(u)$ in this manner in section 6.2.5, where we give elements of the proof of the spectral problem of $\mathfrak{gl}(\mathfrak{n})$.

Note that the possibility to construct the basis (6.54) ensures that the transfer matrix $T(u)$ itself is non-derogatory. In fact, we have the following stronger result.

Lemma 1. *If the matrix K has simple spectrum, then the transfer matrix has simple spectrum. Moreover, if K is diagonalizable with simple spectrum, then $T(u)$ itself is diagonalizable with simple spectrum.*

Proof. Suppose K has simple spectrum, and let $t(u)$ be an eigenvalue of $T(u)$ and $|t\rangle$ an eigenvector associated to $t(u)$. Then,

$$\langle \vec{\mathbf{h}}|t\rangle = \prod_{j=1}^{N} t(\xi_j)^{h_j} \langle S|t\rangle \tag{6.72}$$

so $|t\rangle$ is uniquely defined, up to an overall normalization, by the eigenvalue $t(u)$. All eigenvalues have geometric multiplicity of one, so $T(u)$ is non-derogatory.

Let us now restrict to the case of K diagonalizable with simple spectrum. Noting

$$\mathrm{Sp}(K) = \{k_1, \dots, k_{\mathfrak{n}}\}, \tag{6.73}$$

we label the eigenvectors by their corresponding eigenvalue

$$\langle k_j| \quad \text{and} \quad |k_j\rangle\,, \quad j \in [\![1, \mathfrak{n}]\!]\,. \tag{6.74}$$

The left and right eigenvectors of K form a basis of $(\mathbb{C}^n)^*$ and $\mathbb{C}^{\mathfrak{n}}$ respectively. Tensoring N-times, we construct bases for $\mathscr{H}^*$ and $\mathscr{H}$

$$\langle K, \vec{\mathbf{h}}| := \otimes_{j=1}^{N} \langle k_{h_j}|\,, \quad |K, \vec{\mathbf{h}}\rangle := \otimes_{j=1}^{N} |k_{h_j}\rangle\,. \tag{6.75}$$

Taking the inhomogeneities as in (6.59), the above bases are eigenbases of the leading coefficient of the transfer matrix polynomial expansion in ξ (6.61) of the $T(\xi_j)$. Thus, for some left eigenvector $\langle t|$ of $T(u)$, there exists a unique N-tuple $\vec{\mathbf{h}} \in [\![1, \mathfrak{n}]\!]^N$ such that

$$\lim_{|\xi|\to+\infty} \frac{1}{\mathsf{N}_t^{\leftarrow}(\xi)\xi^{N-1}} \langle t|\, T(\xi_j, \xi) = c_j k_{h_j} \langle K, \vec{\mathbf{h}}|\,, \tag{6.76}$$

for some proper normalization $\mathsf{N}_t^{\leftarrow}(\xi)$. Reconstructing $T(u)$ by Lagrange interpolation in the $\xi_j = j\xi$ and subtracting the dominant term, one can find that

$$\lim_{|\xi|\to+\infty} \frac{1}{\xi^{N-1}} \langle t|\big(T(u) - \xi^N (-1)^N N!\,\mathrm{tr}\,K\big) = (-1)^{N-1} \sum_{a=1}^{N} c_a k_{h_a} \prod_{\substack{b\neq a^N \\ b=1}} \frac{b}{a-b} \langle K, \vec{\mathbf{h}}|\,. \tag{6.77}$$

The left-hand side of the above limit, once rewritten using Lagrange interpolation, has the same form as the right-hand side. By the above limits, it has to hold

$$\lim_{|\xi|\to+\infty} \frac{1}{\mathsf{N}_t^{\leftarrow}(\xi)} \langle t| = \langle K, \vec{\mathbf{h}}|\,, \quad \lim_{|\xi|\to+\infty} \frac{1}{\mathsf{N}_t^{\rightarrow}(\xi)} |t\rangle = |K, \vec{\mathbf{h}}\rangle \tag{6.78}$$

with the proper normalizations $\mathsf{N}_t^{\leftarrow}(\xi)$ and $\mathsf{N}_t^{\rightarrow}(\xi)$ are fixed. Eventually, this implies that

$$\lim_{|\xi|\to+\infty} = \frac{\langle t|t\rangle}{\mathsf{N}_t^{\leftarrow}(\xi)\mathsf{N}_t^{\rightarrow}(\xi)} = \langle K, \vec{\mathbf{h}}|K, \vec{\mathbf{h}}\rangle = 1. \tag{6.79}$$

In a non-trivial Jordan block, left and right eigenvectors are orthogonal. Here, $\langle t|t\rangle \neq 0$, so the Jordan block associated to the eigenvalue $t(u)$ must be trivial, i.e. of size 1. Eventually, this makes the transfer matrix diagonalizable with simple spectrum. □

6.2.3 Separate property

The basis (6.54) is immediately seen to have the first features of a separate basis. Let $|t\rangle$ be an eigenvector of $T(u)$ of eigenvalue $t(u)$. Then, the wavefunction in the basis (6.54)

$$\Psi(\vec{\mathbf{h}}) = \langle \vec{\mathbf{h}}|t\rangle = \prod_{j=1}^{N} t(\xi_j)^{h_j} \langle S|t\rangle\,, \tag{6.80}$$

is factorized, with each factor depending on only one coordinate h_j.

Can we find some separate relations of the form (5.58)? The fusion relation (A.61) for $\mathfrak{gl}(2)$, specialized in the inhomogeneities, are

$$T(\xi_j)T(\xi_j-\eta) = \text{q-det}\, M(\xi_j). \tag{6.81}$$

This clearly has the desired form; in the next section we show how it is used in the resolution of the spectral problem for the $\mathfrak{gl}(2)$ case. For higher rank cases, the fusion relations shall provide separate relations as well, but in a more intricate way, see section 6.2.5.

6.2.4 Spectral problem for $\mathfrak{gl}(2)$ models

Consider the inhomogeneous XXX spin $1/2$ chain with twisted boundary conditions. As anticipated, the separate relations constraining the spectrum in this case are the fusion relations particularized in the inhomogeneities. These N relations form a system which select eigenvalues among a certain family of functions determined by the analytic properties of the transfer matrix.

Proposition 7. *The eigenvalues of $T(u)$ are the polynomials $t(u)$ in u of degree N and dominant coefficient $\operatorname{tr} K$, verifying the N relations*

$$\forall j \in [\![1,N]\!]\,, \quad t(\xi_j)t(\xi_j-\eta) = \text{q-det}\, M(\xi_j), \tag{6.82}$$

where the quantum determinant is the scalar

$$\text{q-det}\, M(u) = \det K \prod_{j=1}^{N} (u-\xi_j+\eta)(u-\xi_j-eta), \tag{6.83}$$

which fix the N remaining coefficients. Explicitly, they are the polynomials

$$t(u) = \operatorname{tr} K \prod_{j=1}^{N} (u-\xi_j) + \sum_{a=1}^{N} \prod_{\substack{b=1\\ b\neq a}}^{N} \frac{u-\xi_b}{\xi_a-\xi_b} x_a, \tag{6.84}$$

where the N-tuple $\{x_1,\dots,x_N\}$ is a solution to the linear system of N quadratic equations

$$x_n \left(\operatorname{tr} K \prod_{j=1}^{N} (x_n-\eta-\xi_j) + \sum_{a=1}^{N} \prod_{\substack{b=1\\ b\neq a}}^{N} \frac{x_n-\eta-\xi_b}{\xi_a-\xi_b} x_a \right) = \text{q-det}\, M(\xi_n) \quad \textit{for} \quad n \in [\![1,N]\!]\,. \tag{6.85}$$

The unique eigenvector associated to the above eigenvalue is characterized by the factorized wavefunction in the left SoV *Basis*

$$\langle \vec{\mathbf{h}}|t\rangle = \prod_{j=1}^{N} t(\xi_j)^{h_j}, \tag{6.86}$$

with the normalization $\langle S|t\rangle = 1$.

Proof. It is immediate that any eigenvalue $t(u)$ and associated unique eigenvector are of the above form, so one just needs to check the converse way. Consider the polynomial $f(u)$ of degree N defined by its values in the inhomogeneities $x_a := f(\xi_a)$, such that

$$\forall a \in [\![1,N]\!], \quad f(\xi_a) f(\xi_a - \eta) = \text{q-det}\, M(\xi_a). \tag{6.87}$$

By Lagrange interpolation, $f(u)$ can be written

$$f(u) = \text{tr}(K)\, u^N + \sum_{a=1}^{N} \prod_{\substack{b=1 \\ b\neq a}}^{N} \frac{u-\xi_b}{\xi_a - \xi_b} x_a, \tag{6.88}$$

and then (6.87) is explicitly a system of N equations in the x_a. We define the vector $|v\rangle$ by imposing its coordinates in the left SoV basis

$$\langle \vec{\mathbf{h}} | v \rangle = \prod_{j=1}^{N} x_j^{h_j}. \tag{6.89}$$

Consider $\langle \vec{\mathbf{h}}|$ such that $h_a = 0$ for some $a \in [\![1,N]\!]$. Then,

$$\langle \vec{\mathbf{h}} | T(\xi_a) | v \rangle = \langle h_1, \dots, 1, \dots, h_N | t \rangle = f(\xi_a) \prod_{\substack{j=1 \\ j\neq a}}^{N} f(\xi_j)^{h_j} = f(\xi_a) \langle \vec{\mathbf{h}} | v \rangle . \tag{6.90}$$

Now, if $h_a = 1$,

$$\begin{aligned} \langle \vec{\mathbf{h}} | T(\xi_a - \eta) | v \rangle = \text{q-det}\, M(\xi_a) \langle h_1, \dots, 0, \dots, h_N | t \rangle &= f(\xi_a) f(\xi_a - \eta) \prod_{\substack{j=1 \\ j\neq a}}^{N} f(\xi_a)^{h_j} \\ &= f(\xi_a - \eta) \prod_{j=1}^{N} f(x_j)^{h_j} = f(\xi_a - \eta) \langle \vec{\mathbf{h}} | v \rangle , \end{aligned} \tag{6.91}$$

thanks to (6.87). Given a fixed N-tuple $\vec{\mathbf{h}}$, and noting $\xi^{(h_i)} = \xi - h_i \eta$, the transfer matrix and the function $f(u)$ are written by Lagrange interpolation similarly

$$T(u) = \text{tr}(K)\, u^N + \sum_{a=1}^{N} \prod_{\substack{b=1 \\ b\neq a}}^{N} \frac{u - \xi_b^{(h_b)}}{\xi_a^{(h_a)} - \xi_b^{(h_b)}} T(\xi_a^{(h_a)}) \tag{6.92}$$

$$f(u) = \text{tr}(K)\, u^N + \sum_{a=1}^{N} \prod_{\substack{b=1 \\ b\neq a}}^{N} \frac{u - \xi_b^{(h_b)}}{\xi_a^{(h_a)} - \xi_b^{(h_b)}} f(\xi_a^{h_a}), \tag{6.93}$$

therefore

$$\forall u \in \mathbb{C}, \quad \langle \vec{\mathbf{h}} | T(u) | v \rangle = f(u) \langle \vec{\mathbf{h}} | v \rangle . \tag{6.94}$$

This holds for any $\langle \vec{\mathbf{h}}|$ of (6.54). Because it is a basis,

$$\forall u \in \mathbb{C}, \quad T(u) |v\rangle = f(u) |v\rangle , \tag{6.95}$$

so $|v\rangle$ is an eigenvector with eigenvalue $f(u)$ of the transfer matrix. It is unique since $T(u)$ is non-derogatory.

Note that for any eigenstates $|t\rangle$ one has $\langle S|t\rangle \neq 0$ since $|t\rangle \neq 0$, otherwise the right-hand side

in (6.89) always vanishes, which would contradict the fact that the $\langle\vec{\mathbf{h}}|$ are a basis. □

While the above proposition 7 gives a characterization of the spectrum through a quadratic system, it is possible to give a functional characterization as well by what is called the quantum spectral curve.

Proposition 8. *Under the same assumptions as above, a polynomial $t(u)$ is an eigenvalue of $T(u)$ if and only if there exists a unique polynomial $Q_t(u)$ of degree $M \leq N$ with no ξ_j for its roots such that the couple $(t(u), Q_t(u))$ lies on the quantum spectral curve*

$$k_1^2 a(u)d(u)Q_t(u-2\eta) - k_1 a(u)t(u-\eta)Q_t(u-\eta) + \text{q-det}\, M(u)Q_t(u) = 0. \tag{6.96}$$

Furthermore, the unique eigenvector $|t\rangle$ associated to $t(u)$ has the following wavefunctions in the SoV *basis*

$$\langle\vec{\mathbf{h}}|t\rangle = k_1^{\sum_{j=1}^N h_j} \prod_{j=1}^N Q_t(\xi_j). \tag{6.97}$$

Proof. Let $(t(u), Q_t(u))$ be a couple lying on the spectral curve. We note k_2 the second eigenvalue of K — which might be equal to k_1. Because $\forall j \in [\![1,N]\!]$, $\text{q-det}\, M(u)(\xi_j + \eta) = 0$, equation (6.96) evaluated in points $u = \xi_j$ and $u = \xi_j + \eta$ produces the $2N$ equations for $1 \leq j \leq N$

$$\begin{aligned} t(\xi_j - \eta)Q_t(\xi_j - \eta) - k_2 d(\xi_j - \eta)Q_t(\xi_j) = 0, \\ k_1 a(\xi_j)Q_t(\xi_j - \eta) - t(\xi_j)Q_t(\xi_j) = 0. \end{aligned} \tag{6.98}$$

Since $Q_t(\xi_j) \neq 0$ by hypothesis, multiplying the two equations gives

$$\forall j \in [\![1,N]\!], \quad t(\xi_j)t(\xi_j - \eta) = \det(K)\; a(\xi_j)d(\xi_j - \eta) \equiv \text{q-det}\, M(\xi_j), \tag{6.99}$$

where one identify the quantum determinant in the right-hand side. This is exactly the spectrum characterization obtained in proposition 7, so $t(u)$ is an eigenvalue of the transfer matrix. The wavefunctions are recovered from (6.86) after an overall normalization.

Conversely, let $t(u)$ be an eigenvalue of $T(u)$. Let $Q_t(u)$ be a polynomial of degree $M \leq N$, and consider

$$P(u) = k_1^2 a(u)d(u)Q_t(u-2\eta) - k_1 a(u)t(u-\eta)Q_t(u-\eta) + \text{q-det}\, M(u)Q_t(u). \tag{6.100}$$

The polynomial $P(u)$ is easily proved to be of maximum degree $3N-1$, as the leading contribution in u^{2N+M} vanishes. It is zero for all $\xi_j - \eta$, because the scalar coefficients of the three terms are zero in these points. The condition $P(\xi_j) = 0 = P(\xi_j + \eta)$ is equivalent to the systems (6.98) which reduces to the N following equations thanks to the fusion relations (6.82)

$$\forall j \in [\![1,N]\!], \quad t(\xi_j)Q_t(\xi_j) = k_1 a(\xi_j)Q_t(\xi_j - \eta). \tag{6.101}$$

On can prove such a polynomial $Q_t(u)$ satisfying these conditions exists and is unique, as it is done in [218][1]. The form of the unique eigenvector associated to $t(u)$ then follows from (6.86) and the above equation. □

6.2.5 Spectral problem for fundamental $\mathfrak{gl}(n)$ models

Like in the $\mathfrak{gl}(2)$ case, the characterization of the spectrum using the SoV basis (6.54) requires a subset of the fusion equations. Noting $T_a(u)$ the fused transfer matrices obtained from the antisymmetric repre-

[1] The form of the transfer matrix eigenvalues in terms of Q-functions is already accessible from the ABA.

sentations with $a \in [\![1,\mathfrak{n}]\!]$, we recall that the relevant fusion relations are written in the inhomogeneities as

$$\forall a \in [\![1,\mathfrak{n}-1]\!], \quad T(\xi_j)T_a(\xi_j-\eta) = T_{a+1}(\xi_j), \tag{6.102}$$

with the identifications $T_1(u) \equiv T(u)$ and $T_{\mathfrak{n}}(u) = \text{q-det}\, M(u)$.

Let us defined a handler quantity which will be useful to write interpolation formula concisely. For all integers $a \in [\![1,\mathfrak{n}]\!]$ and $p \in [\![1,N]\!]$, we define

$$g_p^{(a)}(u) = \prod_{\substack{q=1\\q\neq p}}^{N} \frac{u-\xi_q}{\xi_p-\xi_q} \prod_{j=1}^{N}\prod_{k=1}^{a-1} \frac{1}{\xi_p-\xi_j^{(-k)}}, \tag{6.103}$$

with $\xi_j^{(k)} = \xi_j - k\eta$. Then the fused transfer matrix $T_a(u)$ can be written by Lagrange interpolation in the inhomogeneities as

$$T_a(u) = \prod_{j=1}^{N}\prod_{k=1}^{a-1}\big(u-\xi_j-k\eta\big)\left[\sigma_a(K)\prod_{j=1}^{\mathfrak{n}}(u-\xi_j) + \sum_{p=1}^{N} g_p^{(a)}(u)T(\xi_p)T_{a-1}(\xi_p-\eta)\right], \tag{6.104}$$

where $\sigma_a(K) = \text{tr}_{1,\dots,a}(P^-_{1,\dots,a}K_1\dots K_a)$ is the a-th elementary symmetric polynomial in the eigenvalues of the twist K.

The following results was obtained in [234].

Theorem 6. *Suppose the family (6.54) is a basis of covectors of $\mathscr{H}^*$ under some conditions we suppose verified, as stated by Theorem 5. Let $\{x_1,\dots,x_N\} \in \mathbb{C}^N$ define the following polynomial by Lagrange interpolation in the inhomogeneities*

$$t(u) = \text{tr}\, K\prod_{j=1}^{N}(u-\xi_j) + \sum_{a=1}^{N} g_a^{(1)}(u)x_a. \tag{6.105}$$

We can further define the $\mathfrak{n}-2$ polynomials of degree aN for $2 \le a \le \mathfrak{n}-1$

$$t_a(u) = \prod_{j=1}^{N}\prod_{k=1}^{a-1}\big(u-\xi_j-k\eta\big)\left[\sigma_a(K)\prod_{j=1}^{\mathfrak{n}}(u-\xi_j) + \sum_{p=1}^{N} g_p^{(a)}(u)t(\xi_p)t_{a-1}(\xi_p-\eta)\right]. \tag{6.106}$$

If $\{x_1,\dots,x_N\}$ is a solution of the system of N equations of order $\mathfrak{n}$ in the x_j

$$x_j t_{\mathfrak{n}-1}(\xi_j-\eta) = \text{q-det}\, M(\xi_j) \quad \textit{for} \quad j \in [\![1,N]\!], \tag{6.107}$$

then the polynomial $t(u)$ defined above is an eigenvalue of the transfer matrix $T(u)$ — and the $t_a(u)$ are eigenvalues of the $T_a(u)$, respectively. Furthermore, the spectrum of $T(u)$ is simple: the eigenvector $|t\rangle$ associated to a given eigenvalue $t(u)$ is unique and is defined by its wavefunctions in the separate basis up to an overall normalization as

$$\forall \vec{\mathbf{h}} \in [\![0,\mathfrak{n}-1]\!]^N, \quad \langle \vec{\mathbf{h}}|t\rangle = \prod_{j=1}^{N} t(\xi_j)^{h_j}. \tag{6.108}$$

Proof. The proof follows the same line as the one for the $\mathfrak{gl}(2)$ case. Since the successive functions $t_a(u)$ are constructed by the interpolation formulas originated from the fusion relations between the transfer matrices, it is immediate that any eigenvalue of $T(u)$ has its values in the inhomogeneities $x_j = t(\xi_j)$ verify the system (6.107).

Conversely, let $\{x_1,\dots,x_N\}$ be a solution of (6.107) and $f(u)$, $f_2(u),\dots,f_n(u)$ be the polynomials defined recursively by Lagrange interpolations as in the statement of the theorem. Let $|v\rangle \in \mathscr{H}$ be the

vector with wavefunctions

$$\forall \vec{\mathbf{h}} \in [\![0, \mathfrak{n}-1]\!]^N, \quad \langle \vec{\mathbf{h}} | v \rangle = \prod_{j=1}^{} N f(u)^{h_j}. \tag{6.109}$$

Then one can show after lengthy calculation leveraging the fusion relations for the transfer matrices and the one verified by the f polynomials that

$$\forall \vec{\mathbf{h}} \in [\![0, \mathfrak{n}-1]\!]^N, \quad , \left\langle \vec{\mathbf{h}} \middle| T(u) \middle| v \right\rangle = f(u) \langle \vec{\mathbf{h}} | v \rangle . \tag{6.110}$$

Hence, $|v\rangle$ is an eigenstate of eigenvalue $f(u)$, and we may note $|t\rangle := |v\rangle$ and $t(u) := f(u)$. The exact computations of (6.110) in the case of a N-tuple $\vec{\mathbf{h}}$ with at least one entry equal to $\mathfrak{n}-1$ follows the lines given in (6.43), where the coefficients $C_{\vec{\mathbf{h}}}^{\vec{\mathbf{k}}}$ are obtained indirectly by the successive use of the fusion relations. Subtleties of the calculations are found in appendix A of reference [234]. □

A functional characterization is also known for $\mathfrak{gl}(\mathfrak{n})$ thanks to the spectral curve.

Proposition 9. *Let $t(u)$ be a polynomial of degree N. We construct the polynomials $t_a(u)$, $2 \leq a \leq \mathfrak{n}-1$ from $t(u)$ as in (6.106). Under the same assumptions as above, $t(u)$ is an eigenvalue of $T(u)$ if and only if there exists a unique polynomial $Q_t(u)$ of degree $M \leq N$ with no ξ_j for its roots such that the couple $(t(u), Q_t(u))$ lies on the quantum spectral curve of equation*

$$\sum_{a=0}^{\mathfrak{n}} \alpha_a(u) Q_t(u - a\eta) t_{\mathfrak{n}-a}(u - a\eta) = 0. \tag{6.111}$$

We make the identifications

$$t_0(u) \equiv 1, \quad t_1(u) \equiv t(u), \quad t_{\mathfrak{n}}(u) \equiv \text{q-det}\, M(u), \tag{6.112}$$

and the coefficients $\alpha_a(u)$ are defined as

$$\forall a \in [\![0, \mathfrak{n}]\!], \quad \alpha_a(u) = (-1)^{a+1} k_1^a \prod_{b=0}^{a-1} d(u - b\eta), \tag{6.113}$$

where k_1 is an eigenvalue of the twist matrix K. Moreover, the unique eigenvector $|t\rangle$ associated to $t(u)$ is defined by its wavefunctions in the SoV *basis up to an overall normalization as*

$$\forall \vec{\mathbf{h}} \in [\![0, \mathfrak{n}-1]\!]^N, \quad \langle \vec{\mathbf{h}} | t \rangle = \prod_{j=1}^{N} \left(\alpha_1(\xi_j) Q_t(\xi_j - \eta) \right)^{h_a} Q_t(\xi_j)^{\mathfrak{n}-1-h_j} \tag{6.114}$$

Proof. The proof is very similar to the $\mathfrak{gl}(2)$ case.

For a couple $(t(u), Q_t(u))$, evaluating (6.111) in the $(\mathfrak{n}-1)N$ points $\xi_j + k\eta$, $k \in [\![1, \mathfrak{n}-1]\!]$, one can prove that it holds

$$\forall j \in [\![1, N]\!], \quad \forall a \in [\![1, \mathfrak{n}-1]\!], \quad t(\xi_j) t_a(\xi_j - \eta) = t_{a+1}(\xi_j), \tag{6.115}$$

such that by Theorem 6 $t(u)$ is an eigenvalue of $T(u)$, and the form of the wavefunction of $|t\rangle$ follows.

Conversely, if $t(u)$ is an eigenvalue of $T(u)$ and $Q_t(u)$ is some polynomial of degree $M \leq N$. Constraining $Q_t(u)$ by imposing

$$\forall j \in [\![1, N]\!], \quad , \alpha_1(\xi_j) Q_t(\xi_j - \eta) = t(\xi_j) Q_t(\xi_j), \tag{6.116}$$

which is equivalent to place $(t(u), Q_t(u))$ on the quantum spectral curve, one can prove $Q_t(u)$ exists and is unique. Details can be found in the appendix B of reference [234]. The form of the wavefunctions of the eigenvector $|t\rangle$ follows easily. □

6.3 Link with Sklyanin SoV in the $\mathfrak{gl}(2)$ case

In the $\mathfrak{gl}(2)$ case, for a certain choice of the covector $\langle S|$, the basis (5.59) made from the power of the transfer matrix coincide with the Sklyanin's original SoV basis, namely the eigenbasis of the $\mathcal{B}(u) = M_{12}(u)$ entry of the monodromy.

Proposition 10. *Let*

$$\langle \vec{\mathbf{h}}|^{\mathrm{Skl}} := \langle 0| \prod_{j=1}^{N} \left(\frac{A^K(\xi_j)}{a(\xi_j)} \right)^{h_j}, \quad \textit{with} \quad \vec{\mathbf{h}} \in \{0,1\}^N, \tag{6.117}$$

which are the vectors defined in (5.125) *up to a scalar factor. For* $\langle S| = \langle 0| = \otimes_{j=1}^{N} \big(1, 0\big)$,

$$\langle \vec{\mathbf{h}}| = \langle \vec{\mathbf{h}}|^{\mathrm{Skl}}. \tag{6.118}$$

Proof. The proof is made by induction. For $\vec{\mathbf{h}} = (0, \dots, 0)$, then obviously

$$\langle \vec{\mathbf{0}}| = \langle S| = \langle 0| = \langle \vec{\mathbf{0}}|^{\mathrm{Skl}}, \tag{6.119}$$

and (6.118) is verified.

Take $\ell \in [\![1, N]\!]$ and suppose that for any $\vec{\mathbf{h}} \in \{0,1\}^N$ such that $\sum_{j=1}^{N} h_j = \ell$, equality (6.118) is verified. Let $\vec{\mathbf{h}} \in \{0,1\}^N$ be the specific N-tuple with

$$h_j = \begin{cases} 1 & \text{for } 1 \leq j \leq \ell, \\ 0 & \text{otherwise.} \end{cases} \tag{6.120}$$

We have

$$\langle \vec{\mathbf{h}}|\, D(\xi_{\ell+1}) = \langle \vec{\mathbf{h}}|^{\mathrm{Skl}}\, D(\xi_{\ell+1}) = \langle 0| \left(\prod_{j=1}^{\ell} \frac{A(\xi_j)}{a(\xi_j)} \right) D(\xi_{\ell+1}), \tag{6.121}$$

and using (Y.16) one time, it gives

$$\langle \vec{\mathbf{h}}|\, D(\xi_{\ell+1}) = \langle 0| \left(\prod_{j=1}^{\ell-1} \frac{A(\xi_j)}{a(\xi_j)} \right) \Big[D(\xi_{\ell+1}) A(\xi_\ell) - g(\xi_{\ell+1}, \xi_\ell) \big(B(\xi_\ell) C(\xi_{\ell+1}) - B(\xi_{\ell+1}) C(\xi_\ell) \big) \Big]. \tag{6.122}$$

The covector $\langle 0| \left(\prod_{j=1}^{\ell-1} A(\xi_j) \big/ a(\xi_j) \right)$ is an eigencovector of $B(u)$, whose eigenvalue vanishes in the ξ_j, $j \leq \ell$ (see (5.126)). Therefore, the two contributions in B are zeros, and everything happens as the A and D have commuted together. Iterating on this, one can bring $D(\xi_{\ell+1})$ all to the left of the product of $A(\xi_j)$ operators. Since $\langle 0|\, D(\xi_j) = \langle 0|\, d(\xi_j) = 0$, it means that the action (6.121) is zero. Hence, noting $\hat{\mathbf{e}}_j$ the N-tuple with 1 at the j-th position and 0 everywhere else,

$$\langle \vec{\mathbf{h}} + \hat{\mathbf{e}}_{\ell+1}| = \langle \vec{\mathbf{h}}|\, T(\xi_{\ell+1}) = \langle \vec{\mathbf{h}}|^{\mathrm{Skl}} A(\xi_{\ell+1}) = \langle \vec{\mathbf{h}} + \hat{\mathbf{e}}_{\ell+1}|^{\mathrm{Skl}}. \tag{6.123}$$

From every N-tuple of sum ℓ, we can rearrange the indices by a permutation and perform the above computation. Hence, we have just proven equality (6.118) for N-tuples of sum $\ell + 1$.

Eventually, this proves (6.118) by induction over ℓ. □

In $\mathfrak{gl}(\mathfrak{n})$ models, it is more tedious to verify that the basis (5.59) is the eigenbasis of the $\mathcal{B}(u)$ operator. Some results for $\mathfrak{gl}(3)$ chains of small length were presented in the Appendix A of [225] — along comments on the limitations of this SoV picture. The full proof has been obtained later in reference [236].

It is also possible to make the link with the ABA description of the eigenstates, see section IV.D of [225] for example.

6.4 Recent results on SoV

The results presented here are mostly a concise version of [225]. A series of publications from the same authors extended the methods to higher rank algebra, other representations and different boundary conditions. The higher rank spectral problem for $\mathscr{Y}(\mathfrak{gl}(\mathfrak{n}))$ was solved by SoV method in [234], while the trigonometric $\mathcal{U}_q(\mathfrak{gl}(\mathfrak{n}))$ models were treated in [235]. Higher spin representations we tackled by [232] in the $\mathfrak{gl}(2)$ case. In the open case, a SoV basis for $\mathscr{Y}(\mathfrak{gl}(n))$ models is constructed, and used to solve the spectral problem for $\mathfrak{n} = 2, 3$ in [233]. Extension to the supersymmetric $\mathfrak{gl}(\mathfrak{m}|\mathfrak{n})$ spin chains was considered in [IV1]; this will be developed in chapter 8.

Thanks to the insights provided by [225], the $\mathcal{B}(u)$ operator proposed in [178, 179] was diagonalized for $\mathfrak{gl}(\mathfrak{n})$ rectangular representations [236], with results refined in [237]. The authors obtain a basis as a deformation of the Gelfand–Tsetlin one, and prove that it is the eigenbasis of some $\mathcal{B}(u)$ operator which produces separates variables.

Efforts have been made towards correlation functions with progresses in the computation of scalar products of separate states. Publications [238, 291] argue about the characterization of the SoV measure in higher rank models, with article [240] being a great review of these results. We have contributed ourselves to the determination of the SoV measure in the $\mathfrak{gl}(3)$ case with article [IV2], which is the subject of the next chapter.

Chapter

7 Towards the dynamics in higher rank: scalar products for $\mathfrak{gl}(3)$ in SoV

This chapter is devoted to the results obtained in the article [LV2]. As we have shown in the previous chapter, one can construct left and right separate bases $\langle\vec{\mathbf{h}}|$, $|\vec{\mathbf{h}}\rangle$ for the fundamental $\mathscr{Y}(\mathfrak{gl}(\mathfrak{n}))$ model in particular, using powers of the transfer matrix in the inhomogeneities for example. Eigenstates are then characterized by their factorized wavefunctions in the left separate basis. One would like to compute scalar products of these eigenstates, paving the way towards form factors and correlation functions.

In the first section of this chapter, we introduce the bases ${}_p\langle\vec{\mathbf{h}}|$ and $|\vec{\mathbf{h}}\rangle_p$ orthogonal to the left and right separate bases, and the SoV measure $\mathcal{M}$ that computes scalar product of separate states.

We then consider the $\mathfrak{gl}(2)$ fundamental chain case, and show that the left and right separate bases obtained earlier for these models are orthogonal with respect to the canonical scalar product of the Hilbert space. More precisely, $|\vec{\mathbf{h}}\rangle_p = \left|\vec{\mathbf{h}}\right\rangle$ and ${}_p\langle\vec{\mathbf{h}}| = \langle\vec{\mathbf{h}}|$. This makes the computation of the SoV measure relatively easy, and we compute it exactly.

The higher rank case of the $\mathfrak{gl}(3)$ fundamental chain is more intricate. Using bases of the form (6.28), we have been are able to construct only pseudo-orthogonal left and right SoV bases at best. That is, though one has equality for a certain amount of tuples $\vec{\mathbf{h}}$, one has $|\vec{\mathbf{h}}\rangle_p \neq |\vec{\mathbf{h}}\rangle$ in general. In the third section, we give pseudo-orthogonal separate bases for the $\mathfrak{gl}(3)$ fundamental model, and characterize completely their pseudo-orthogonality in terms of the diagonal coupling $\langle\vec{\mathbf{h}}|\vec{\mathbf{h}}\rangle$ of the separate bases; This allows us to give the form of SoV measure, once again in terms of the diagonal couplings, by computing the decomposition of a generic $|\vec{\mathbf{h}}\rangle_p$ vector as a linear decomposition over the separate basis of the $|\vec{\mathbf{h}}\rangle$.

Comparing with the $\mathfrak{gl}(2)$ case, the picture for non-orthogonal separate bases gets so more complex that it is desirable to construct orthogonal left and right separate bases for the $\mathfrak{gl}(3)$ case as well. In the fourth and last section of this chapter, we remark that orthogonality of the separate bases is achieved in the limit of non-invertible twist (with simple spectrum), and compute scalar product of separate states in this case. We then proceed to describe the existence and properties of an operator $\mathbb{T}(u)$, defined by its eigenvalues and eigenvectors and commuting with the transfer matrices, that can be substituted to the transfer matrix to construct left and right separate bases that are indeed orthogonal.

7.1 Generalities on scalar products of separate states and SoV measure

7.1.1 Dual states of SoV left and right bases

Consider the $\mathscr{Y}(\mathfrak{gl}(\mathfrak{n}))$ twisted inhomogeneous fundamental model. We note $(\cdot,\cdot)_{\mathscr{H}}$ the scalar product of the Hilbert space $\mathscr{H} = \mathbb{C}^{\mathfrak{n}N}$ of the model. We may take it to be the dot product for the canonical basis: noting e_i the vectors of the canonical basis of $\mathscr{H}$, it simply gives

$$\forall i,j \in [\![1,\mathfrak{n}N]\!], \quad (e_i,e_j)_{\mathscr{H}} = \delta_{ij}. \tag{7.1}$$

The induced norm is $\|a\|_{\mathscr{H}} := (a,a)_{\mathscr{H}}$. A similar notation is used for the scalar product on the dual space with $(\cdot,\cdot)_{\mathscr{H}^*}$ and $\|\cdot\|_{\mathscr{H}^*}$.

Consider now the left and right SoV bases made from the powers of the transfer matrices

$$\langle \vec{\mathbf{h}}| := \langle L| \prod_{j=1}^{N} T(\xi_j)^{h_j}, \tag{7.2}$$

$$|\vec{\mathbf{h}}\rangle := \prod_{j=1}^{N} T(\xi_j)^{h_j} |R\rangle . \tag{7.3}$$

A very important point is that a priori $\langle \vec{\mathbf{h}}|$ *is not the dual of* $|\vec{\mathbf{h}}\rangle$ *by the canonical conjugation* $\mathscr{H} \simeq \mathscr{H}^*$. Indeed, the adjoint of a vector of the right SoV basis decomposes as a non-trivial linear combination on the left SoV basis yet to characterize

$$|\vec{\mathbf{h}}\rangle^\dagger = \sum_{\vec{\mathbf{p}}} c_{\vec{\mathbf{p}}} \langle \vec{\mathbf{p}}| , \tag{7.4}$$

and similarly for $\langle \vec{\mathbf{h}}|^\dagger$. The notation could therefore be misleading, so we will clarify it. It is misleading in the sense that for $|\psi\rangle \in \mathscr{H}$, the corresponding bra is *usually* by definition the unique linear form on $\mathscr{H}$ obtained by taking the adjoint, as in $\langle\psi| := |\psi\rangle^\dagger$, and then the braket is used to denote the scalar product involving the corresponding ket: $\langle\psi|\varphi\rangle = (|\psi\rangle, |\varphi\rangle)_{\mathscr{H}}$. Here, for the notations (7.2) and (7.3), this is *not* the case. The left and right states $\langle \vec{\mathbf{h}}|$ and $|\vec{\mathbf{h}}\rangle$ are just notations attached to the covectors and vectors defined in (7.2) and (7.3). In this sense, the braket

$$\langle \vec{\mathbf{h}}|\vec{\mathbf{k}}\rangle \tag{7.5}$$

is the action of the linear form $\langle \vec{\mathbf{h}}|$ in $\mathscr{H}^*$ on a vector $|\vec{\mathbf{k}}\rangle$ of $\mathscr{H}$, with no relation a priori induced by the natural isomorphism $\mathscr{H}^* \simeq \mathscr{H}$. The only common point is that they are both labelled by N-tuple in $\mathcal{I} := [\![0, \mathfrak{n}-1]\!]^N$. We can thus refer to the above brakets as a mere coupling between covectors and vectors. Let us stress again that in this notation,

$$\forall \vec{\mathbf{h}}, \vec{\mathbf{k}} \in \mathcal{I}, \quad \langle \vec{\mathbf{h}}|\vec{\mathbf{k}}\rangle \neq \big(|\vec{\mathbf{h}}\rangle^\dagger , |\vec{\mathbf{k}}\rangle \big)_{\mathscr{H}}. \tag{7.6}$$

Orthogonal states We introduce the $2 \times \mathfrak{n}^N$ orthogonal states

$$|\vec{\mathbf{h}}\rangle_p \in \mathscr{H} \quad \text{and} \quad {}_p\langle \vec{\mathbf{h}}| \in \mathscr{H}^* \tag{7.7}$$

defined as the unique states such that[1]

$$\forall \vec{\ell} \in \mathcal{I}, \quad {}_p\langle \vec{\mathbf{h}}|\vec{\mathbf{k}}\rangle = \delta_{\vec{\mathbf{h}},\vec{\mathbf{k}}} \langle \vec{\mathbf{h}}|\vec{\mathbf{h}}\rangle = \langle \vec{\mathbf{k}}|\vec{\mathbf{h}}\rangle_p . \tag{7.8}$$

We note

$$\mathsf{N}_{\vec{\mathbf{h}}} := \langle \vec{\mathbf{h}}|\vec{\mathbf{h}}\rangle . \tag{7.9}$$

Since $\{ |\vec{\mathbf{h}}\rangle \}$ and $\{ |\vec{\mathbf{h}}\rangle_p \}$ are Hilbert space bases, we have the closure relations

$$\mathrm{Id}_{\mathscr{H}} = \sum_{\vec{\mathbf{h}} \in \mathcal{I}} \frac{|\vec{\mathbf{h}}\rangle\, {}_p\langle \vec{\mathbf{h}}|}{\mathsf{N}_{\vec{\mathbf{h}}}} = \sum_{\vec{\mathbf{h}} \in \mathcal{I}} \frac{|\vec{\mathbf{h}}\rangle_p \langle \vec{\mathbf{h}}|}{\mathsf{N}_{\vec{\mathbf{h}}}} . \tag{7.10}$$

The closure relation gives the linear decomposition of the orthogonal states on the left and right SoV bases

$$|\vec{\mathbf{h}}\rangle_p = \sum_{\vec{\mathbf{k}} \in \mathcal{I}} \frac{{}_p\langle \vec{\mathbf{k}}|\vec{\mathbf{h}}\rangle_p}{\mathsf{N}_{\vec{\mathbf{k}}}} |\vec{\mathbf{k}}\rangle \quad \text{and} \quad {}_p\langle \vec{\mathbf{h}}| = \sum_{\vec{\mathbf{k}} \in \mathcal{I}} \frac{{}_p\langle \vec{\mathbf{h}}|\vec{\mathbf{k}}\rangle_p}{\mathsf{N}_{\vec{\mathbf{k}}}} \langle \vec{\mathbf{k}}| , \tag{7.11}$$

[1] This is the notation introduced in article [LV2] at section 3.3. The index p is a shorthand for *perpendicular*.

which are a more explicit rewriting of (7.4), and the SoV left and right states decompose similarly on the states (7.7)

$$|\vec{\mathbf{h}}\rangle = \sum_{\vec{\mathbf{k}}\in\mathcal{I}} \frac{\langle\vec{\mathbf{k}}|\vec{\mathbf{h}}\rangle}{\mathsf{N}_{\vec{\mathbf{k}}}} |\vec{\mathbf{k}}\rangle_p \quad \text{and} \quad \langle\vec{\mathbf{h}}| = \sum_{\vec{\mathbf{k}}\in\mathcal{I}} \frac{\langle\vec{\mathbf{h}}|\vec{\mathbf{k}}\rangle}{\mathsf{N}_{\vec{\mathbf{k}}}} {}_p\langle\vec{\mathbf{k}}|. \tag{7.12}$$

7.1.2 Scalar products of separate states

Solving the spectral problem of these models in the previous chapter, we saw that the eigenstates $|t\rangle$ of $T(u)$ had factorized wavefunctions of the form

$$\langle\vec{\mathbf{h}}|t\rangle = \prod_{j=1}^{N} t(\xi_j)^{h_j}. \tag{7.13}$$

The same computation can be done for eigencovector $\langle t|$ at the left of the transfer matrix $T(u)$ using the right SoV basis, so that

$$\langle t|\vec{\mathbf{h}}\rangle = \prod_{j=1}^{N} t(\xi_j)^{h_j} \tag{7.14}$$

as well. Therefore, the left and right eigenstate are rewritten as

$$|t\rangle = \sum_{\vec{\mathbf{h}}\in\mathcal{I}} \left(\prod_{j=1}^{N} t(\xi_j)^{h_j} \right) \frac{|\vec{\mathbf{h}}\rangle_p}{\mathsf{N}_{\vec{\mathbf{h}}}}, \tag{7.15}$$

$$\langle t| = \sum_{\vec{\mathbf{h}}\in\mathcal{I}} \left(\prod_{j=1}^{N} t(\xi_j)^{h_j} \right) \frac{{}_p\langle\vec{\mathbf{h}}|}{\mathsf{N}_{\vec{\mathbf{h}}}}. \tag{7.16}$$

This gives a natural definition of separate states as

$$|\alpha\rangle = \sum_{\vec{\mathbf{h}}\in\mathcal{I}} \alpha_{\vec{\mathbf{h}}} \frac{|\vec{\mathbf{h}}\rangle_p}{\mathsf{N}_{\vec{\mathbf{h}}}}, \quad \alpha_{\vec{\mathbf{h}}} = \prod_{j=1}^{N} \alpha_j(h_j) \tag{7.17}$$

$$\langle\beta| = \sum_{\vec{\mathbf{h}}\in\mathcal{I}} \beta_{\vec{\mathbf{h}}} \frac{{}_p\langle\vec{\mathbf{h}}|}{\mathsf{N}_{\vec{\mathbf{h}}}}, \quad \beta_{\vec{\mathbf{h}}} = \prod_{j=1}^{N} \beta_j(h_j), \tag{7.18}$$

which have factorized coordinates $\alpha_{\vec{\mathbf{h}}}$ and $\beta_{\vec{\mathbf{h}}}$. Scalar products of separate states simply reads as

$$\langle\beta|\alpha\rangle = \sum_{\vec{\mathbf{h}},\vec{\mathbf{k}}} \beta_{\vec{\mathbf{h}}}\, \mathcal{M}_{\vec{\mathbf{h}},\vec{\mathbf{k}}}\, \alpha_{\vec{\mathbf{k}}}, \tag{7.19}$$

where the SoV *measure* $\mathcal{M}$ is defined by its matrix elements

$$\mathcal{M}_{\vec{\mathbf{h}},\vec{\mathbf{k}}} = \frac{{}_p\langle\vec{\mathbf{h}}|\vec{\mathbf{k}}\rangle_p}{\mathsf{N}_{\vec{\mathbf{h}}}\mathsf{N}_{\vec{\mathbf{k}}}}. \tag{7.20}$$

We have the closure relation

$$\mathrm{Id} = \sum_{\vec{\mathbf{h}},\vec{\mathbf{k}}\in\mathcal{I}} \mathcal{M}_{\vec{\mathbf{h}},\vec{\mathbf{k}}}\ |\vec{\mathbf{h}}\rangle\langle\vec{\mathbf{k}}|\,. \tag{7.21}$$

Consider now the matrix $\mathcal{N}$ whose elements are defined by the coupling between the left and right SoV bases

$$\mathcal{N}_{\vec{\mathbf{h}},\vec{\mathbf{k}}} = \langle\vec{\mathbf{h}}|\vec{\mathbf{k}}\rangle\,. \tag{7.22}$$

We have $\mathsf{N}_{\vec{\mathbf{h}}} = \mathcal{N}_{\vec{\mathbf{h}},\vec{\mathbf{h}}}$. Then, for all $\vec{\mathbf{h}}, \vec{\mathbf{p}} \in \mathcal{I}$,

$$\sum_{\vec{\mathbf{k}}\in\mathcal{I}} \mathcal{M}_{\vec{\mathbf{h}},\vec{\mathbf{k}}}\, \mathcal{N}_{\vec{\mathbf{k}},\vec{\mathbf{p}}} = \sum_{\vec{\mathbf{k}}\in\mathcal{I}} \frac{{}_p\langle\vec{\mathbf{h}}|\vec{\mathbf{k}}\rangle_p}{\mathsf{N}_{\vec{\mathbf{h}}}\mathsf{N}_{\vec{\mathbf{k}}}} \langle\vec{\mathbf{k}}|\vec{\mathbf{p}}\rangle = \frac{{}_p\langle\vec{\mathbf{h}}|\vec{\mathbf{p}}\rangle}{\mathsf{N}_{\vec{\mathbf{h}}}} = \delta_{\vec{\mathbf{h}},\vec{\mathbf{p}}}, \tag{7.23}$$

thanks to the closure relation (7.10) and the definition of ${}_p\langle\vec{\mathbf{h}}|$, and similarly

$$\sum_{\vec{\mathbf{k}}\in\mathcal{I}} \mathcal{N}_{\vec{\mathbf{h}},\vec{\mathbf{k}}}\, \mathcal{M}_{\vec{\mathbf{k}},\vec{\mathbf{p}}} = \delta_{\vec{\mathbf{h}},\vec{\mathbf{p}}}. \tag{7.24}$$

Hence, the matrices $\mathcal{M}$ and $\mathcal{N}$ are mutual inverse, so the SoV measure is obtained by inverting the matrix of the coupling between left and right SoV bases. They are also the matrices of the change of basis from the SoV basis to the adjoint of the opposite side basis (up to some normalization), namely

$$\langle\vec{\mathbf{h}}| = \sum_{\vec{\mathbf{k}}\in\mathcal{I}} \mathcal{N}_{\vec{\mathbf{h}},\vec{\mathbf{k}}}\, \frac{{}_p\langle\vec{\mathbf{k}}|}{\mathsf{N}_{\vec{\mathbf{k}}}}, \tag{7.25}$$

$$\frac{{}_p\langle\vec{\mathbf{h}}|}{\mathsf{N}_{\vec{\mathbf{k}}}} = \sum_{\vec{\mathbf{k}}\in\mathcal{I}} \mathcal{M}_{\vec{\mathbf{h}},\vec{\mathbf{k}}}\, \langle\vec{\mathbf{k}}|, \tag{7.26}$$

and similar relations hold for vectors $|\vec{\mathbf{h}}\rangle$ and $|\vec{\mathbf{h}}\rangle_p$ of $\mathscr{H}$.

7.2 The $\mathfrak{gl}(2)$ case: orthogonal SoV bases

Let us show how SoV bases constructed from conserved charges can be shown to be orthogonal in the $\mathfrak{gl}(2)$ case.

7.2.1 Left and right separate bases

Let $\mathcal{I} = \{0,1\}^N$. We already know from Theorem 5 that the family of covectors

$$\langle\vec{\mathbf{h}}| := \langle L| \prod_{j=1}^{N} T\big(\xi_j\big)^{h_j} \quad \text{for} \quad \vec{\mathbf{h}} \in \mathcal{I} \tag{7.27}$$

is a separate left basis for almost any choice of $\langle L|$.

Let us note

$$V(x_1,\dots,x_N) := \prod_{1\le a<b\le N} (x_b - x_a) \tag{7.28}$$

the Vandermonde determinant. We define a slightly different family of vectors in $\mathscr{H}$

$$|\vec{\mathbf{h}}\rangle := \prod_{j=1}^{N} T\big(\xi_j - \eta\big)^{1-h_j} |R\rangle \quad \text{for} \quad \vec{\mathbf{h}} \in \mathcal{I}, \tag{7.29}$$

where $|R\rangle \in \mathscr{H}$ is chosen to be the unique vector satisfying

$$\forall \vec{\mathbf{h}} \in \mathcal{I}, \quad \langle\vec{\mathbf{h}}|R\rangle = \frac{\delta_{\vec{\mathbf{h}},\vec{\mathbf{1}}}}{V(\xi_1,\dots,\xi_N)} = \frac{\prod_{j=1}^{N} \delta_{h_j,1}}{V(\xi_1,\dots,\xi_N)}. \tag{7.30}$$

In fact, the basis (7.29) is also a separate basis, as it is a rewriting of the basis constructed from the

power of the transfer matrix acting on the state

$$|\vec{\mathbf{0}}\rangle = \prod_{j=1}^{N} T\big(\xi_j - \eta\big)\,|R\rangle\,, \tag{7.31}$$

which actually is proportional to $\langle L|^{\dagger}$. Using the fusion relations, for all $\vec{\mathbf{h}} \in \mathcal{I}$, one has

$$\begin{aligned}\prod_{j=1}^{N}\left(\frac{T\big(\xi_j\big)}{\text{q-det}\,M(\xi_j)}\right)^{h_j}|\vec{\mathbf{0}}\rangle &= \prod_{j=1}^{N}\left(\frac{T\big(\xi_j\big)}{\text{q-det}\,M(\xi_j)}\right)^{h_j} T\big(\xi_j-\eta\big)\,|R\rangle \\ &= \prod_{j=1}^{N}\left(\frac{\text{q-det}\,M\big(\xi_j\big)}{\text{q-det}\,M\big(\xi_j\big)}\right)^{h_j}\big(T\big(\xi_j-\eta\big)\big)^{1-h_j}\,|R\rangle \\ &\equiv |\vec{\mathbf{h}}\rangle\,.\end{aligned} \tag{7.32}$$

This holds for $\text{q-det}\,M(\xi_j) \neq 0$, i.e. $\det K \neq 0$. If $\det K \neq 0$, it is still possible to prove by direct computation that (7.29) is a separate basis, as it has been done earlier for (7.27). The use of the form (7.29) in the above is just a matter of convenience for the practical computations of the couplings $\langle\vec{\mathbf{h}}|\vec{\mathbf{k}}\rangle$.

7.2.2 Orthogonality of the separate bases

We claim the following proposition.

Proposition 11. *The family of vectors* (7.29) *is dual to the left separate basis* (7.27)*, for it holds*

$$\forall \vec{\mathbf{h}}, \vec{\mathbf{k}} \in \mathcal{I}, \quad \langle\vec{\mathbf{h}}|\vec{\mathbf{k}}\rangle = \frac{\delta_{\vec{\mathbf{h}},\vec{\mathbf{k}}}}{V\Big(\xi_1^{(h_1)},\ldots,\xi_N^{(h_N)}\Big)}. \tag{7.33}$$

Proof. Let us first prove that all non-diagonal terms of (7.33) are zero. The proof is done by induction over the (decreasing) number of coefficients equal to 1 in the right vector $|\vec{\mathbf{k}}\rangle$ of formula (7.33).

Let $\vec{\mathbf{k}} = \vec{\mathbf{1}}$, so that $|\vec{\mathbf{k}}\rangle = |\vec{\mathbf{1}}\rangle = |R\rangle$. By definition,

$$\forall \vec{\mathbf{h}} \in \mathcal{I}, \quad \vec{\mathbf{h}} \neq \vec{\mathbf{k}}, \quad \langle\vec{\mathbf{h}}|\vec{\mathbf{k}}\rangle = \langle\vec{\mathbf{h}}|R\rangle = 0. \tag{7.34}$$

This proves the base case.

Now for the induction step. Fix $m \in [\![1, N]\!]$. Suppose that for all tuples $\vec{\mathbf{k}} \in \mathcal{I}$ such that $\sum_{j=1}^{N} k_j \geq m$, formula (7.33) holds for $\vec{\mathbf{h}} \neq \vec{\mathbf{k}}$, namely that

$$\forall \vec{\mathbf{k}} \in \mathcal{I}, \quad \sum_{j=1}^{N} k_j \geq m, \quad \forall \vec{\mathbf{h}} \in \mathcal{I}, \quad \vec{\mathbf{h}} \neq \vec{\mathbf{k}}, \quad \langle\vec{\mathbf{h}}|\vec{\mathbf{k}}\rangle = 0. \tag{7.35}$$

This is our induction hypothesis. Considering a fixed $\vec{\mathbf{h}} \in \mathcal{I}$, we now pick $\vec{\mathbf{p}} \in \mathcal{I}$ such that $\vec{\mathbf{p}} \neq \vec{\mathbf{h}}$ and $\sum_{j=1}^{N} p_j = m-1$. We will prove that $\langle\vec{\mathbf{h}}|\vec{\mathbf{p}}\rangle = 0$. There exists at least one index $a \in [\![1, N]\!]$ such that $p_a = 0$. Let $\vec{\mathbf{p}} + \hat{\mathbf{e}}_a$ be the N-tuple of $\mathcal{I}$ defined in coefficients by

$$(\vec{\mathbf{p}} + \hat{\mathbf{e}}_a)_j = p_j + \delta_{ja}. \tag{7.36}$$

We have

$$\langle\vec{\mathbf{h}}|\vec{\mathbf{p}}\rangle = \big\langle\vec{\mathbf{h}}\big|T(\xi_a - \eta)\big|\vec{\mathbf{p}} + \hat{\mathbf{e}}_a\big\rangle. \tag{7.37}$$

We now distinguish two cases depending on the value of h_a.

If $h_a = 1$, then we can extract a $T(\xi_a)$ from $\langle\vec{\mathbf{h}}|$, and by the virtue of the fusion relation (6.82) we

have

$$\langle\vec{\mathbf{h}}|\vec{\mathbf{p}}\rangle = \left\langle\vec{\mathbf{h}}-\hat{\mathbf{e}}_a\middle|T(\xi_a)T(\xi_a-\eta)\middle|\vec{\mathbf{p}}+\hat{\mathbf{e}}_a\right\rangle = \text{q-det}\,M(\xi_a)\left\langle\vec{\mathbf{h}}-\hat{\mathbf{e}}_a\middle|\vec{\mathbf{p}}+\hat{\mathbf{e}}_a\right\rangle. \tag{7.38}$$

Because $\vec{\mathbf{p}}\neq\vec{\mathbf{h}}$, it implies $\vec{\mathbf{p}}+\hat{\mathbf{e}}_a\neq\vec{\mathbf{h}}-\hat{\mathbf{e}}_a$. Since $\sum_{j=1}^N(\vec{\mathbf{p}}+\hat{\mathbf{e}}_a)_j = m$, $\left\langle\vec{\mathbf{h}}-\hat{\mathbf{e}}_a\middle|\vec{\mathbf{p}}+\hat{\mathbf{e}}_a\right\rangle = 0$ by the induction hypothesis, so that $\langle\vec{\mathbf{h}}|\vec{\mathbf{p}}\rangle = 0$.

If $h_a = 0$, we cannot extract $T(\xi_a)$ to absorb the $T(\xi_a-\eta)$ in a quantum determinant, so we have to make $T(\xi_a-\eta)$ act on $\langle\vec{\mathbf{h}}|$. Using Lagrange interpolation in the points $\xi_j^{(h_j)}$, this gives

$$\langle\vec{\mathbf{h}}|\,T(\xi_a-\eta) = \operatorname{tr}K\prod_{j=1}^N\left(\xi_a-\eta-\xi_j^{(h_j)}\right)\langle\vec{\mathbf{h}}| + \sum_{r=1}^N\prod_{\substack{s=1\\s\neq r}}^N\frac{\xi_a-\eta-\xi_s^{(h_s)}}{\xi_r^{(h_r)}-\xi_s^{(h_s)}}\left\langle\vec{\mathbf{h}}+(-1)^{\delta_{h_r,1}}\hat{\mathbf{e}}_r\right|. \tag{7.39}$$

For $r=a$, $h_a=0$ so the corresponding N-tuple in the right-hand side is $\vec{\mathbf{h}}+\hat{\mathbf{e}}_a$, which cannot be equal to $\vec{\mathbf{p}}+\hat{\mathbf{e}}_a$ since $\vec{\mathbf{p}}\neq\vec{\mathbf{h}}$. For $r\neq a$, the a-th coordinate of $\vec{\mathbf{h}}$ of the corresponding term is left untouched i.e. $(\vec{\mathbf{h}}+(-1)^{\delta_{h_r,1}}\hat{\mathbf{e}}_r)_a = 1$. Because $(\vec{\mathbf{p}}+\hat{\mathbf{e}}_a)_a = 1$, we have $\vec{\mathbf{h}}+(-1)^{\delta_{h_r,1}}\hat{\mathbf{e}}_r\neq\vec{\mathbf{p}}+\hat{\mathbf{e}}_a$. Therefore, putting $|\vec{\mathbf{p}}+\hat{\mathbf{e}}_a\rangle$ at the right of the above equation, all the terms cancel by the induction hypothesis.

This proves (7.37) for all $\vec{\mathbf{p}}\in\mathcal{I}$ different from $\vec{\mathbf{h}}$ and such that $\sum_{j=1}^N p_j = m-1$. By induction, this proves (7.33) for all, $\vec{\mathbf{h}},\vec{\mathbf{k}}\in\mathcal{I}$, $\vec{\mathbf{h}}\neq\vec{\mathbf{k}}$, so that the left and right separate bases are indeed orthogonal.

Let us now compute $\langle\vec{\mathbf{h}}|\vec{\mathbf{h}}\rangle$. Any $\vec{\mathbf{h}}\in\mathcal{I}$ can be constructed from $\vec{\mathbf{1}}$ by flipping the required coefficients from 1 to 0, so it is sufficient to characterize the ratios

$$\frac{\langle\vec{\mathbf{p}}|\vec{\mathbf{p}}\rangle}{\langle\vec{\mathbf{p}}-\hat{\mathbf{e}}_a|\vec{\mathbf{p}}-\hat{\mathbf{e}}_a\rangle}, \tag{7.40}$$

for $\vec{\mathbf{p}}\in\mathcal{I}$ with $p_a = 1$. We have

$$\frac{\langle\vec{\mathbf{p}}|\vec{\mathbf{p}}\rangle}{\langle\vec{\mathbf{p}}-\hat{\mathbf{e}}_a|\vec{\mathbf{p}}-\hat{\mathbf{e}}_a\rangle} = \frac{\langle\vec{\mathbf{p}}|\vec{\mathbf{p}}\rangle}{\langle\vec{\mathbf{p}}-\hat{\mathbf{e}}_a|T(\xi_a-\eta)|\vec{\mathbf{p}}\rangle}, \tag{7.41}$$

and interpolating $T(\xi_a-\eta)$ in the $\xi_j^{(q_j)}$, $\vec{\mathbf{q}} = \vec{\mathbf{p}}-\hat{\mathbf{e}}_a$, gives

$$\frac{\langle\vec{\mathbf{p}}|\vec{\mathbf{p}}\rangle}{\langle\vec{\mathbf{p}}-\hat{\mathbf{e}}_a|\vec{\mathbf{p}}-\hat{\mathbf{e}}_a\rangle} = \prod_{\substack{b=1\\b\neq a}}^N\frac{\xi_a-\xi_b^{(p_b)}}{\xi_a-\eta-\xi_b^{(p_b)}}. \tag{7.42}$$

A quick induction starting from $\langle\vec{\mathbf{1}}|\vec{\mathbf{1}}\rangle = V(\xi_1,\dots,\xi_N)^{-1}$ proves that

$$\forall\vec{\mathbf{h}}\in\mathcal{I},\quad \langle\vec{\mathbf{h}}|\vec{\mathbf{h}}\rangle = V\left(\xi_1^{(h_1)},\dots,\xi_N^{(h_N)}\right)^{-1}. \tag{7.43}$$

This complete the proof of (7.33). □

A very similar proof can be found in the appendix B of [292], with some minor differences in the notations and the induction reasoning.

7.2.3 SoV measure

The above result states that the dual vector of the left separate basis (7.27) are, up to a normalization, the vectors of the right separate basis (7.29), where the clever choice of indexation in the latter allows for a satisfying $\delta_{\vec{\mathbf{h}},\vec{\mathbf{k}}}$. Indeed, from (7.12) we have

$$|\vec{\mathbf{h}}\rangle_p = |\vec{\mathbf{h}}\rangle, \tag{7.44}$$

and by (7.10) the closure relation is written in terms of the left and right separate bases (7.27) and (7.29) as

$$\mathrm{Id}_{\mathscr{H}} = \sum_{\vec{\mathbf{h}}\in\mathcal{I}} V\big(\xi_1^{(h_1)},\dots,\xi_N^{(h_N)}\big)\,\big|\vec{\mathbf{h}}\big\rangle\big\langle\vec{\mathbf{h}}\big|\,. \tag{7.45}$$

We recognize the SoV measure (7.21) in the above formula: the matrix $\mathcal{N}$ of the coupling $\langle\vec{\mathbf{h}}|\vec{\mathbf{k}}\rangle$ is diagonal and is inverted trivially to give

$$\mathcal{M}_{\vec{\mathbf{h}},\vec{\mathbf{k}}} = \delta_{\vec{\mathbf{h}},\vec{\mathbf{k}}}\, V\big(\xi_1^{(h_1)},\dots,\xi_N^{(h_N)}\big). \tag{7.46}$$

Hence, the left and right SoV bases (7.27) and (7.29) are orthogonal, though not orthonormal in the current choice of normalization. Scalar products of separate states, such as the eigenstates of the transfer matrix, can be computed by (7.19) with the diagonal measure $\mathcal{M}$ calculated above.

7.3 The $\mathfrak{gl}(3)$ case: pseudo-orthogonal SoV bases

7.3.1 Left and right separate bases

Let $\mathcal{I} = \{0,1,2\}$. The family of covectors

$$\big\langle\vec{\mathbf{h}}\big| := \langle\mathscr{L}|\prod_{j=1}^{N} T_2\big(\xi_j-\eta\big)^{\delta_{h_j,0}} T\big(\xi_j\big)^{\delta_{h_j,2}} \quad \text{for} \quad \vec{\mathbf{h}}\in\mathcal{I} \tag{7.47}$$

is a separate basis of $\mathscr{H}^*$.

For $\det K \neq 0$, $\text{q-det}\, M(u) \neq 0$ and we can rewrite the vectors (7.47) as vectors known to form a basis. Indeed, choosing the vector $\langle\mathscr{L}| \equiv \big\langle\vec{\mathbf{1}}\big|$ of the form

$$\langle\mathscr{L}| = \langle L|\prod_{j=1}^{N} T\big(\xi_j\big), \tag{7.48}$$

for some $\langle L|$ as specified in Theorem 5, the family (7.47) coincide with the basis (6.54) up to a non-zero normalization factor involving the quantum determinant. Precisely,

$$\big\langle\vec{\mathbf{h}}\big| = \left(\prod_{j=1}^{N} \text{q-det}\, M(\xi_j)^{\delta_{h_j,0}}\right) \times \langle L|\prod_{j=1}^{N} T\big(\xi_j\big)^{\delta_{h_j,2}-\delta_{h_j,0}+1}, \tag{7.49}$$

and $\delta_{h_j,2}-\delta_{h_j,0}+1 = h_j$ for all $h_j\in\mathcal{I}$.

We now define the 3^N vectors

$$\big|\vec{\mathbf{h}}\big\rangle := \prod_{j=1}^{N} T_2\big(\xi_j\big)^{\delta_{h_j,1}} T\big(\xi_j\big)^{\delta_{h_j,2}}\,|\mathscr{R}\rangle\,, \tag{7.50}$$

where $|\mathscr{R}\rangle \equiv |\vec{\mathbf{0}}\rangle$ is the unique vector such that

$$\forall\vec{\mathbf{h}}\in\mathcal{I}, \quad \langle\vec{\mathbf{h}}|\mathscr{R}\rangle = \delta_{\vec{\mathbf{h}},\vec{\mathbf{0}}}. \tag{7.51}$$

The family (7.50) can be proven to be a separate basis by a proof along the way of the one of Theorem 5. If we happen to take $\big\langle\vec{\mathbf{1}}\big|$ in a tensor product form, the vector $|\mathscr{R}\rangle$ is proven to be of tensor product form as well, whose explicit expression can be computed in terms of the one of $\langle\mathscr{L}|$. This is done in proposition 3.1 of [LV2], see equations (3.8)–(3.10) there.

All of this holds for $\det K \neq 0$. Still, even if $\det K = 0$, it is possible to prove by direct computation that (7.47) is a separate basis, following the usual proof given earlier.

7.3.2 Pseudo-orthogonality

In this higher rank case, the left and right separate bases defined above are not orthogonal, but they are pseudo-orthogonal. By "pseudo" we mean that the coupling matrix $\mathcal{N}_{\vec{\mathbf{h}},\vec{\mathbf{k}}} = \langle \vec{\mathbf{h}} | \vec{\mathbf{k}} \rangle$ is mainly diagonal, meaning that $\langle \vec{\mathbf{h}} | \vec{\mathbf{k}} \rangle$ is zero for "many" indexes $\vec{\mathbf{h}} \neq \vec{\mathbf{k}} \in \mathcal{I}$, but has a few non-zero off-diagonal entries. More precisely, the coupling is characterized by the following theorem

Theorem 7 (Theorem 3.1 of [LV2]). *For any $\vec{\mathbf{k}} = (k_1, \dots, k_N) \in \mathcal{I}$, and α, β, two disjoint subsets of $[\![1, N]\!]$, we define the corresponding modified N-tuple*

$$\vec{\mathbf{k}}^{(\vec{\mathbf{0}},\vec{\mathbf{2}})}_{\alpha,\beta} = (k_1(\alpha,\beta), \dots, k_N(\alpha,\beta)) \quad \textit{where} \tag{7.52}$$

$$\forall j \in [\![1,N]\!], \quad k_j(\alpha,\beta) = \begin{cases} 0 & \textit{for } j \in \alpha, \\ 2 & \textit{for } j \in \beta, \\ k_j & \textit{for } j \notin \{\alpha \cup \beta\}. \end{cases} \tag{7.53}$$

Let $\vec{\mathbf{1}}_{\vec{\mathbf{k}}} := \{ j \in [\![1,N]\!] \mid k_j = 1 \}$ be the indexes of the ones of $\vec{\mathbf{k}}$, and

$$\Gamma_2(\vec{\mathbf{1}}_{\vec{\mathbf{k}}}; r) := \left\{ (\alpha,\beta) \mid \alpha,\beta \subset \vec{\mathbf{1}}_{\vec{\mathbf{k}}}, \quad \alpha \cap \beta = \emptyset \quad \textit{and} \quad \#\alpha = r = \#\beta \right\}, \tag{7.54}$$

the set of couples of non-intersecting subsets of cardinal r of $\vec{\mathbf{1}}_{\vec{\mathbf{k}}}$. With the above notations, the coupling matrix $\mathcal{N}$ between the left and right SoV *basis* (7.47)–(7.50) *has coefficients*

$$\mathcal{N}_{\vec{\mathbf{h}},\vec{\mathbf{k}}} = \langle \vec{\mathbf{h}} | \vec{\mathbf{k}} \rangle = \langle \vec{\mathbf{k}} | \vec{\mathbf{k}} \rangle \left(\delta_{\vec{\mathbf{h}},\vec{\mathbf{k}}} + C^{\vec{\mathbf{k}}}_{\vec{\mathbf{h}}} \sum_{r=1}^{n_{\vec{\mathbf{k}}}} (\det K)^r \sum_{(\alpha,\beta) \in \Gamma_2(\vec{\mathbf{1}}_{\vec{\mathbf{k}}}; r)} \delta_{\vec{\mathbf{h}},\vec{\mathbf{k}}^{(\vec{\mathbf{0}},\vec{\mathbf{2}})}_{\alpha,\beta}} \right), \tag{7.55}$$

where the coefficients $C^{\vec{\mathbf{k}}}_{\vec{\mathbf{h}}}$ are non-zero coefficients independent of $\det K$*, and*

$$n_{\vec{\mathbf{k}}} = \left\lfloor \frac{1}{2} \sum_{j=1}^{N} \delta_{k_j,1} \right\rfloor, \tag{7.56}$$

where $\lfloor x \rfloor$ denotes the integer part of a real number x. The diagonal coefficients explicitly read as quotient of Vandermonde determinants

$$\langle \vec{\mathbf{h}} | \vec{\mathbf{h}} \rangle = \left(\prod_{j=1}^{N} \frac{d(\xi_j - \eta)}{d\left(\xi_j^{\left(1+\delta_{h_j,1}+\delta_{h_j,2}\right)} \right)} \right) \frac{V(\xi_1, \dots, \xi_N)^2}{V\left(\xi_1^{\left(\delta_{h_1,1}+\delta_{h_1,2}\right)}, \dots, \xi_N^{\left(\delta_{h_N,1}+\delta_{h_N,2}\right)} \right) V\left(\xi_1^{\left(\delta_{h_1,1}\right)}, \dots, \xi_N^{\left(\delta_{h_N,1}\right)} \right)}. \tag{7.57}$$

The proof of this theorem is rather convoluted. It is given explicitly in great details in appendix C of [LV2]. We first prove the pseudo-orthogonality and obtain the Kronecker deltas involved in (7.55). Then we characterize the form of the non-diagonal elements in terms of the diagonal ones. The coefficients $C^{\vec{\mathbf{k}}}_{\vec{\mathbf{h}}}$ are characterized completely though implicitly by a heavy recursion. Finally, we derive the explicit expression (7.57) of the diagonal elements.

Let us stress here that all the off-diagonal elements are proportional to a strictly positive power of $\det K$. There is even a grading showing that the power $(\det K)^r$ increases as one goes away from the diagonal. Anticipating on the next section, this indicates that we should get a diagonal SoV measure for

non-invertible simple spectrum twist matrix K.

In the following, we outline the proof and recall the notations necessary to understand it. The general idea to prove (7.55) is to distinguish between cases for both $\vec{\mathbf{k}}$ and $\vec{\mathbf{h}}$, and proceed by induction over the number of indexes in $\vec{\mathbf{k}}$ equal to a certain value. The usual procedure in the induction is to extract a transfer matrix from the vector $\left|\vec{\mathbf{k}}\right\rangle$ and make it act on the other side over $\left\langle\vec{\mathbf{h}}\right|$, lowering the number of indexes equal to a certain value in $\vec{\mathbf{k}}$, allowing us to apply the induction hypothesis. Usually this requires to rewrite the extracted transfer matrix as a sum of transfer matrices in points where the action over $\left\langle\vec{\mathbf{h}}\right|$ is known, and this is done by Lagrange interpolation. While this proof is in the same way as the one of proposition 11 for the $\mathfrak{gl}(2)$ case, the overall procedure is now much heavier; first because there are many cases to distinguish, and secondly because the coupling (7.55) is only pseudo-orthogonal.

We use several notations to make formulas more compact throughout appendix C of [LV2].

- Many Lagrange interpolations are required in the proof. To ease the notation, appendix C make heavy usage of the symbol
$$\underset{\text{UpC}}{=} \tag{7.58}$$
which is interpreted as equality of two objects up to the non-zero numerical coefficients before each term. Hence, we have the handy notation
$$T(u) \underset{\text{UpC}}{=} \operatorname{tr} K + \sum_{j=1}^{N} \mathsf{T}\left(\xi_j^{(h_j)}\right), \tag{7.59}$$
where we use the symbol T to put emphasis on missing coefficients. In particular the dependence in u is not explicit in the right-hand side of the above equations.
- We perform lots of manipulations which shift from one vector $\left|\vec{\mathbf{k}}\right\rangle$ to another $|\vec{\mathbf{q}}\rangle$, where $\vec{\mathbf{q}}$ is a N-tuple constructed primarily from $\vec{\mathbf{k}}$. We thus make great benefit of the compact notations similar to (7.52)–(7.53). In general, for a vector $\vec{\mathbf{k}} = (k_1, \dots, k_N) \in \mathcal{I}$, a subset of indices $\alpha = (\alpha_1, \dots, \alpha_m) \subset [\![1, N]\!]$ with $\#\alpha = m$ and a vector of values $\vec{\mathbf{p}} = (p_1, \dots, p_m) \in \{0, 1, 2\}^m$, we note
$$\vec{\mathbf{k}}_\alpha^{(\vec{\mathbf{p}})} = \left(k_1(\alpha, \vec{\mathbf{p}}), \dots, k_N(\alpha, \vec{\mathbf{p}})\right) \tag{7.60}$$
the N-tuple whose coefficients marked by α are replaced by those specified by $\vec{\mathbf{p}}$, which more precisely are
$$k_j(\alpha, \vec{\mathbf{p}}) = \begin{cases} p_\ell & \text{for } j \in \alpha \text{ with } \ell \in [\![1, m]\!] \text{ such that } \alpha_\ell = j, \\ k_j & \text{for } j \notin \alpha. \end{cases} \tag{7.61}$$
Sometimes the notation is used to recall explicitly the value of some coefficients in the N-tuple at hand, as in (C.67) of [LV2].
- Equation (C.45) defines the notation
$$\vec{\mathbf{h}} \underset{\text{(C.45)}}{\neq} \vec{\mathbf{k}}. \tag{7.62}$$
It means that $\vec{\mathbf{k}}$ cannot be brought into $\vec{\mathbf{h}}$ by substitutions of the form $(1, 1) \to (0, 2)$, i.e. for any couples of disjoint subsets (α, β) of $\vec{\mathbf{1}}_{\vec{\mathbf{k}}}$ with the same cardinality $\#\alpha = r = \#\beta \leq n_{\vec{\mathbf{k}}}$, it must hold
$$\vec{\mathbf{h}} \neq \vec{\mathbf{k}}_{\alpha,\beta}^{(\vec{\mathbf{0}},\vec{\mathbf{2}})}. \tag{7.63}$$
This defines the non-zero coefficients in (7.55): $\mathcal{N}_{\vec{\mathbf{h}},\vec{\mathbf{k}}}$ is zero if and only if $\vec{\mathbf{h}} \neq_{\text{(C.45)}} \vec{\mathbf{k}}$. This is the condition of pseudo-orthogonality for the left and right separate bases.

Another way to state this is to define an equivalence class over $\mathcal{I}$ as follows

$$\forall \vec{\mathbf{k}} \in \mathcal{I}, \quad \left[\vec{\mathbf{k}}\right] := \left\{ \vec{\mathbf{k}}^{(\vec{0},\vec{2})}_{\alpha,\beta} \;\middle|\; \alpha,\beta \in \Gamma_2(\vec{\mathbf{1}}_{\vec{\mathbf{k}}}; r) \quad \text{for} \quad 1 \leq r \leq n_{\vec{\mathbf{k}}} \right\}. \tag{7.64}$$

Then $\vec{\mathbf{h}}$ being different from $\vec{\mathbf{k}}$ in the sense of (C.45) simply means it does not belong to the equivalence class of $\vec{\mathbf{k}}$

$$\vec{\mathbf{h}} \underset{\text{(C.45)}}{\neq} \vec{\mathbf{k}} \quad \Leftrightarrow \quad \vec{\mathbf{h}} \notin \left[\vec{\mathbf{k}}\right]. \tag{7.65}$$

As an example, if $\vec{\mathbf{k}} = (1,1,\vec{\mathbf{p}}) \in \mathcal{I}$ with $\vec{\mathbf{p}} \in \{0,2\}^{N-2}$, the three N-tuples

$$(1,1,\vec{\mathbf{p}}), \quad (0,2,\vec{\mathbf{p}}), \quad (2,0,\vec{\mathbf{p}}), \tag{7.66}$$

are in the equivalence class of $\vec{\mathbf{k}}$ and are *not* different from $\vec{\mathbf{k}}$ in the (C.45) sense: the corresponding $\langle\vec{\mathbf{h}}|$ bras have non-zero coupling with $|\vec{\mathbf{k}}\rangle$, i.e. $\langle\vec{\mathbf{h}}|\vec{\mathbf{k}}\rangle \neq 0$ as we will demonstrate. The six other N-tuples of the form $(a,b,\vec{\mathbf{p}})$ are different from $\vec{\mathbf{k}}$ in the (C.45) sense, and have a vanishing coupling. For $\vec{\mathbf{k}} = (1,1,1,\vec{\mathbf{q}}) \in \mathcal{I}$ with $\vec{\mathbf{q}} \in \{0,2\}^{N-3}$, the 7 bras $\langle\vec{\mathbf{h}}|$ corresponding to the N-tuples

$$\begin{gathered}(1,1,1,\vec{\mathbf{q}}), \quad (0,2,1,\vec{\mathbf{q}}), \quad (2,0,1,\vec{\mathbf{q}}), \quad (0,1,2,\vec{\mathbf{q}}), \quad (2,1,0,\vec{\mathbf{q}}), \\ (1,0,2,\vec{\mathbf{q}}), \quad (1,2,0,\vec{\mathbf{q}}),\end{gathered} \tag{7.67}$$

have non-zero coupling, while the 20 others N-tuples of the form $(a,b,c,\vec{\mathbf{q}})$ have a vanishing coupling with $|\vec{\mathbf{k}}\rangle$.

Let us prove the orthogonality. This is done in details in section C.1 of [LV2], and amounts to prove that

$$\forall \vec{\mathbf{k}} \in \mathcal{I}, \quad \forall \vec{\mathbf{h}} \underset{\text{(C.45)}}{\neq} \vec{\mathbf{k}}, \quad \langle\vec{\mathbf{h}}|\vec{\mathbf{k}}\rangle = 0, \tag{7.68}$$

which is obtained by retaining only the Kronecker deltas dependence of (7.55). First, note that (7.68) is verified with $|\vec{\mathbf{k}}\rangle = |\vec{\mathbf{0}}\rangle$ for all $\langle\vec{\mathbf{h}}|$ by the definition of $|\vec{\mathbf{0}}\rangle \equiv |\mathscr{R}\rangle$. Next, we want to prove that

$$\forall \vec{\mathbf{k}} \in \{0,2\}^N, \quad \forall \vec{\mathbf{h}} \in \mathcal{I}, \quad \vec{\mathbf{h}} \neq \vec{\mathbf{k}}, \quad \langle\vec{\mathbf{h}}|\vec{\mathbf{k}}\rangle = 0. \tag{7.69}$$

This is done by induction over the number of 2's in $\vec{\mathbf{k}}$ in section C.1.1. The base case with one 2 is treated in paragraph i), and the induction step is carried in paragraph ii). Suppose (7.69) holds for all $\vec{\mathbf{k}}$ having at most m 2's. Let $\vec{\mathbf{k}}$ have its first m coefficients equal to 2, the others being 0 (up to a reordering of the indices). We note $\vec{\mathbf{k}}^{(2)}_{m+1}$ the modified vector where $k_{m+1} = 2$, according to notation (7.60)–(7.61). We have $\left\langle\vec{\mathbf{h}}\middle|\vec{\mathbf{k}}^{(2)}_{m+1}\right\rangle = \left\langle\vec{\mathbf{h}}\middle|T(\xi_{m+1})\middle|\vec{\mathbf{k}}\right\rangle$. The $\vec{\mathbf{k}}$ at the right has m 2's so is within the scope of the recursion hypothesis, and we can use the matrix element to show that the scalar product is zero. Depending on the $\left\langle\vec{\mathbf{h}}\right|$ at the right, some cases are immediate, see equations (C.32)–(C.36). The only non-trivial one is the one for a $\vec{\mathbf{h}}$ with its $m+1$ first coefficients equal to 2 as well. The lemma C.1 computes $\left\langle\vec{\mathbf{h}}\middle|T(\xi_{m+1})\middle|\vec{\mathbf{k}}\right\rangle$ in terms of similar matrix elements with a modified $\vec{\mathbf{h}}^{(1,1)}_{r,s}$, where a couple of coefficients at indexes r and s equal to $(0,2)$ has been set to $(1,1)$ (this is done up to some condition on $\vec{\mathbf{h}}$ regarding $\vec{\mathbf{k}}$ stated in the lemma). By using this recursive formula several times until there are no couples of coefficients $(0,2)$ in the covector at the left, we show in corollary C.1 the matrix element is zero. It allows to perform the induction over the number of 2's and prove (7.69).

Next step is to incorporate some 1's in $\vec{\mathbf{k}}$. We want to prove that for a fixed $\vec{\mathbf{h}}$, for any $\vec{\mathbf{k}}$ different from $\vec{\mathbf{h}}$ up to substitutions of couples $(1,1)$ in couples $(0,2)$, $\langle\vec{\mathbf{h}}|\vec{\mathbf{k}}\rangle = 0$.

In section C.1.2, we first consider a $\vec{\mathbf{k}}$ with only one coefficient $k_a = 1$, the others being 0 or 2. No substitution $(1,1) \to (0,2)$ can be done here, so we only have to consider the $\vec{\mathbf{k}}$ at hand. Direct action (by Lagrange interpolation) of the transfer matrix extracted from such a $|\vec{\mathbf{k}}\rangle$ on any $\langle\vec{\mathbf{h}}|$, $\vec{\mathbf{h}} \neq \vec{\mathbf{k}}$, gives 0, see

equations (C.38)–(C.43).

The induction step over the increasing number of 1's in $\vec{\mathbf{k}}$ is carried in section C.1.3, and aims to fully prove (7.68) by induction. Multiple substitutions $(1,1)\to(0,2)$ are now possible in $\vec{\mathbf{k}}$. Suppose (7.68) holds for any $\vec{\mathbf{k}}$ with m coefficients equal to 1. We now fix $\vec{\mathbf{k}}$ such that its $m+1$ first coefficients are equal to 1 as in (C.44) (up to a reordering of the indexes), and let $\vec{\mathbf{h}}\in\mathcal{I}$ be outside $\left[\vec{\mathbf{k}}\right]$. Depending on the values of the coefficients of $\vec{\mathbf{h}}$, some cases are simpler than others such as $h_1=0$, see (C.62). For $h_1\neq 0$, we can rearrange the scalar product $\langle\vec{\mathbf{h}}|\vec{\mathbf{k}}\rangle$ in a matrix element with a $|\vec{\mathbf{k}}_1^{(0)}\rangle$ at the right as stated in (C.64), which has m 1's now and is within the scope of the recursion hypothesis. Lemma C.2 takes a matrix element of this form and computes it as a sum of similar matrix elements, but with a transformation $(0,2)\to(1,1)$ in the $\langle\vec{\mathbf{h}}|$ at the left. Applying this recursive formula several times until there are no remaining $(0,2)$ couples allows proving matrix elements of this form vanish. This is done in corollary C.2. Finally, performing the induction over the number of 1's completes the proof of (7.68). This gives the dependence in the Kronecker deltas of formula (7.55), and completes the proof of the orthogonality.

Now we can compute the actual form (7.55) of the coupling, as well as the explicit form (7.57) of the diagonal elements. This is done in sections C.2.1 and C.2.2 of [LV2] respectively.

Let $\vec{\mathbf{k}}\in\mathcal{I}$ have, up to a reordering of the indices, its m first coefficients equal to 1 and the $N-m$ last ones be in $\{0,2\}$. We want to prove equation (C.67) of [LV2], namely that for any $\vec{\mathbf{h}}\in\mathcal{I}$ with $r\in[\![0,n_{\vec{\mathbf{k}}}-1]\!]$ such that

$$\begin{gathered}\forall a\in[\![1,r+1]\!],\quad (h_{2a-1},h_{2a})=(0,2)\quad\text{and}\quad \forall s\in[\![2r+3,m]\!],\quad h_s=1\\ \text{and}\quad \forall j\in[\![m+1,N]\!],\quad h_j=k_j,\end{gathered}\tag{7.70}$$

it holds

$$\left\langle\vec{\mathbf{h}}\middle|\vec{\mathbf{k}}\right\rangle=\mathsf{c}^{r+1}C_{\vec{\mathbf{h}}}^{\vec{\mathbf{k}}}\left\langle\vec{\mathbf{k}}\middle|\vec{\mathbf{k}}\right\rangle,\tag{7.71}$$

with the notation $\mathsf{c}:=\det K$ for the non-zero determinant of the twist matrix.

Alternatively, $\vec{\mathbf{k}}$ can be written in terms of $\vec{\mathbf{h}}$ as

$$\vec{\mathbf{k}}=\vec{\mathbf{h}}^{(1,1,\vec{\mathbf{q}})}_{1,2,3,\dots,2r+2}\quad\text{with}\quad\vec{\mathbf{q}}=(q_1,\dots,q_{2r})=(1,\dots,1).\tag{7.72}$$

The m first coefficients of $\vec{\mathbf{h}}$ are explicitly recalled with the notation $\vec{\mathbf{h}}^{(0,2,\vec{\mathbf{p}})}_{1,2,3,\dots,2r+2}$ where $\vec{\mathbf{p}}$ is a $2r$-tuple of alternating 0 and 2 as defined in equation (C.69) of [LV2]. Therefore,

$$C_{\vec{\mathbf{h}}}^{\vec{\mathbf{h}}^{(1,1,\vec{\mathbf{q}})}_{1,2,3,\dots,2r+2}}=\frac{\left\langle\vec{\mathbf{h}}\middle|\vec{\mathbf{h}}^{(1,1,\vec{\mathbf{q}})}_{1,2,3,\dots,2r+2}\right\rangle}{\mathsf{c}^{r+1}\left\langle\vec{\mathbf{h}}^{(1,1,\vec{\mathbf{q}})}_{1,2,3,\dots,2r+2}\middle|\vec{\mathbf{h}}^{(1,1,\vec{\mathbf{q}})}_{1,2,3,\dots,2r+2}\right\rangle},\tag{7.73}$$

by its definition in (7.71), which is equation (C.74) of [LV2]. The numerator can be rewritten as

$$\left\langle\vec{\mathbf{h}}\middle|\vec{\mathbf{h}}^{(1,1,\vec{\mathbf{q}})}_{1,2,3,\dots,2r+2}\right\rangle=\mathsf{c}_1\left\langle\vec{\mathbf{h}}^{(1,1,\vec{\mathbf{q}})}_{1,2}\middle|\mathsf{T}_1(x_1^{(1)})\mathsf{T}_1(\xi_2)\middle|\vec{\mathbf{h}}^{(0,1,\vec{\mathbf{q}})}_{1,2,3,\dots,2r+2}\right\rangle,\tag{7.74}$$

with $\mathsf{c}_1=\text{q-det}\,M(\xi_1)$. Lemma C.3 of [LV2] performs the explicit computation of the above quantity as a linear combination of matrix elements of the form

$$\left\langle\vec{\mathbf{h}}^{(1,1,\vec{\mathbf{p}}^{(2)}_{2j})}_{1,2,3,\dots,2r+2}\middle|\mathsf{T}_2(\xi_{2j+2})\middle|\vec{\mathbf{h}}^{(0,1,\vec{\mathbf{q}})}_{1,2,3,\dots,2r+2}\right\rangle.\tag{7.75}$$

These matrix elements are in turn computed in lemma C.4 of [LV2] as linear combinations of matrix elements of the same form, but with a substitution $(0,2)\to(1,1)$ in the covector at the left, effectively

decreasing the number of $(0,2)$ couples in the $\vec{\mathbf{h}}$-like tuples at the left. The formula (C.88) obtained there is recursive and acts only on $(0,2)$ couples to transform it in $(1,1)$ couples. In principles, it allows the exact computation of the $C^{\vec{\mathbf{k}}}_{\vec{\mathbf{h}}}$ coefficients by the repeated use of it on (7.75), though the results is very involved.

The base case of the recursion, corresponding to $r=0$, is tractable. Indeed, it corresponds to only one matrix elements of the form (7.75), which is computable by direct calculations. From equation (C.73) of lemma C.3 in [LV2],

$$C^{\vec{\mathbf{k}}^{(1,1)}_{1,2}}_{\vec{\mathbf{h}}^{(0,2)}_{1,2}} = \frac{d(\xi_2-\eta)}{d(\xi_1-\eta)}\frac{\text{q-det}\, M(\xi_1)}{\mathfrak{c}\eta^{-2}(\xi_1-\xi_2+\eta)}\prod_{a\geq 3}^{N}\frac{\left(\xi_1^{(1)}-\xi_a^{\delta_{h_a,2}}\right)\left(\xi_2-\xi_a^{(1-\delta_{h_a,0})}\right)}{\left(\xi_2^{(1)}-\xi_a^{\delta_{h_a,2}}\right)\left(\xi_1-\xi_a^{(1-\delta_{h_a,0})}\right)}. \tag{7.76}$$

This ends the proof of form (7.71) of off-diagonal couplings. The coefficients there are determined implicitly through explicit recursive formulas summed up in lemmas C.3 and C.4. It remains to compute the diagonal coupling $\langle\vec{\mathbf{h}}|\vec{\mathbf{h}}\rangle$ to completely determined the SoV measure of the fundamental $\mathfrak{gl}(3)$ model.

This calculation is usual and follows the lines of the one done for the proof of proposition 11 for the $\mathfrak{gl}(2)$ case. It is detailed in section C.2.2 of [LV2]. Any N-tuple of $\mathcal{I}$ can be constructed from $\vec{\mathbf{0}}$ by flipping the desired coefficients with a $0\to 1$ transformation, and then a $1\to 2$ one if needed. One only needs to compute the ratios

$$\frac{\langle\vec{\mathbf{h}}^{(0)}_a|\vec{\mathbf{h}}^{(0)}_a\rangle}{\langle\vec{\mathbf{h}}^{(1)}_a|\vec{\mathbf{h}}^{(1)}_a\rangle} \quad\text{and}\quad \frac{\langle\vec{\mathbf{h}}^{(1)}_a|\vec{\mathbf{h}}^{(1)}_a\rangle}{\langle\vec{\mathbf{h}}^{(2)}_a|\vec{\mathbf{h}}^{(2)}_a\rangle} \tag{7.77}$$

for a generic $\vec{\mathbf{h}}\in\mathcal{I}$ and $a\in[\![1,N]\!]$, and one gets the relative normalization of $\langle\vec{\mathbf{h}}|\vec{\mathbf{h}}\rangle$ with respect to $\langle\vec{\mathbf{0}}|\vec{\mathbf{0}}\rangle=1$ as a successive product of the above ratios. One has

$$\langle\vec{\mathbf{h}}^{(0)}_a|\vec{\mathbf{h}}^{(0)}_a\rangle = \langle\vec{\mathbf{h}}^{(1)}_a|T_2(\xi^{(1)}_a)|\vec{\mathbf{h}}^{(0)}_a\rangle, \tag{7.78}$$

and the action of $T_2(\xi^{(1)}_a)$ on $|\vec{\mathbf{h}}_a\rangle^{(0)}$ can be computed by Lagrange interpolation in the right points. The same goes for the other ratios, as

$$\langle\vec{\mathbf{h}}^{(2)}_a|\vec{\mathbf{h}}^{(2)}_a\rangle = \langle\vec{\mathbf{h}}^{(1)}_a|T(\xi_a)|\vec{\mathbf{h}}^{(2)}_a\rangle. \tag{7.79}$$

All calculations done, we recall equations (C.121) and (C.131) to make the proof of eq. (7.57) self-contained

$$\frac{\langle\vec{\mathbf{h}}^{(1)}_a|\vec{\mathbf{h}}^{(1)}_a\rangle}{\langle\vec{\mathbf{h}}^{(0)}_a|\vec{\mathbf{h}}^{(0)}_a\rangle} = \frac{d\left(\xi^{(1)}_a\right)}{d\left(\xi^{(2)}_a\right)}\prod_{\substack{j=1\\ j\neq a}}^{N}\frac{\xi_a-\xi_j^{\left(\delta_{h_j,1}+\delta_{h_j,2}\right)}}{\xi^{(1)}_a-\xi_j^{\left(\delta_{h_j,1}+\delta_{h_j,2}\right)}}, \tag{7.80}$$

$$\frac{\langle\vec{\mathbf{h}}^{(2)}_a|\vec{\mathbf{h}}^{(2)}_a\rangle}{\langle\vec{\mathbf{h}}^{(1)}_a|\vec{\mathbf{h}}^{(1)}_a\rangle} = \prod_{\substack{j=1\\ j\neq a}}^{N}\frac{\xi_a-\xi_j^{\left(\delta_{h_j,2}\right)}}{\xi^{(1)}_a-\xi_j^{\left(\delta_{h_j,2}\right)}}. \tag{7.81}$$

7.3.3 SoV measure

We can now compute the SoV measure associated to the left and right separate bases (7.47)–(7.50). Recall that for separate states of the form

$$|\alpha\rangle = \sum_{\vec{\mathbf{k}}} \alpha_{\vec{\mathbf{k}}} \frac{|\vec{\mathbf{k}}\rangle_p}{\mathsf{N}_{\vec{\mathbf{k}}}}, \quad \alpha_{\vec{\mathbf{k}}} := \prod_{j=1}^{N} \alpha_j^{(h_j)}, \tag{7.82}$$

$$\langle\beta| = \sum_{\vec{\mathbf{h}}} \beta_{\vec{\mathbf{k}}} \frac{{}_p\langle\vec{\mathbf{h}}|}{\mathsf{N}_{\vec{\mathbf{h}}}}, \quad \beta_{\vec{\mathbf{k}}} := \prod_{j=1}^{N} \beta_j^{(h_j)}, \tag{7.83}$$

their scalar products is

$$\langle\beta|\alpha\rangle = \sum_{\vec{\mathbf{h}},\vec{\mathbf{k}}\in\mathcal{I}} \langle\beta_{\vec{\mathbf{h}}}|\mathcal{M}_{\vec{\mathbf{h}},\vec{\mathbf{k}}}|\alpha_{\vec{\mathbf{k}}}\rangle \tag{7.84}$$

where the SoV measure $\mathcal{M}$ is defined by coefficients as

$$\mathcal{M}_{\vec{\mathbf{h}},\vec{\mathbf{k}}} = \frac{{}_p\langle\vec{\mathbf{h}}|\vec{\mathbf{k}}\rangle_p}{\mathsf{N}_{\vec{\mathbf{h}}}\mathsf{N}_{\vec{\mathbf{k}}}}. \tag{7.85}$$

To get the SoV measure, it is sufficient to compute ${}_p\langle\vec{\mathbf{h}}|\vec{\mathbf{h}}\rangle_p$. Somehow, this amounts to invert $\mathcal{N}$.

Could one have $|\vec{\mathbf{h}}\rangle_p = |\vec{\mathbf{h}}\rangle$ here? For the diagonal term, this indeed gives ${}_p\langle\vec{\mathbf{h}}|\vec{\mathbf{k}}\rangle_p = {}_p\langle\vec{\mathbf{h}}|\vec{\mathbf{h}}\rangle = \langle\vec{\mathbf{h}}|\vec{\mathbf{h}}\rangle$, by definition of ${}_p\langle\vec{\mathbf{h}}|$. But then, $\langle\vec{\mathbf{k}}|\vec{\mathbf{h}}\rangle_p = \langle\vec{\mathbf{k}}|\vec{\mathbf{h}}\rangle$, which cannot be equal to $\delta_{\vec{\mathbf{k}},\vec{\mathbf{h}}}\langle\vec{\mathbf{h}}|\vec{\mathbf{h}}\rangle$ as required by the definition (7.8) of $|\vec{\mathbf{h}}\rangle_p$, since we know from Theorem 7 that there are in general several non-zero off-diagonal couplings $\langle\vec{\mathbf{k}}|\vec{\mathbf{h}}\rangle$—namely the ones of the form $\vec{\mathbf{k}} = \vec{\mathbf{h}}^{(\vec{0},\vec{2})}_{\alpha,\beta}$.

$|\vec{\mathbf{h}}\rangle_p$ is therefore decomposed as sum over multiple $|\vec{\mathbf{k}}\rangle$ vectors. The idea to construct it is to start from the corresponding $|\vec{\mathbf{h}}\rangle$ vector and add a "correction" as a linear combination of the $|\vec{\mathbf{h}}^{(\vec{0},\vec{2})}_{\alpha,\beta}\rangle$. The constraints $\langle\vec{\mathbf{k}}|\vec{\mathbf{h}}\rangle_p = 0$ for $\vec{\mathbf{k}} \neq \vec{\mathbf{h}}$ would fix the values of the coefficients in the linear combination. This is done explicitly in lemma 3.1 of [LV2]. There, we obtained the linear decomposition of $|\vec{\mathbf{h}}\rangle_p$ in the right separate basis as

$$|\vec{\mathbf{h}}\rangle_p = |\vec{\mathbf{h}}\rangle + \sum_{r=1}^{n_{\vec{\mathbf{h}}}} \mathsf{c}^r \sum_{\alpha,\beta\in\Gamma_2(\vec{\mathbf{i}}_{\vec{\mathbf{h}}};r)} B_{\alpha,\beta,\vec{\mathbf{h}}} \left|\vec{\mathbf{h}}^{(\vec{0},\vec{2})}_{\alpha,\beta}\right\rangle, \tag{7.86}$$

where the $B_{\alpha,\beta,\vec{\mathbf{h}}}$ coefficients are defined recursively by

$$B_{\alpha,\beta,\vec{\mathbf{h}}} = -\bar{C}^{\vec{\mathbf{h}}}_{\vec{\mathbf{h}}^{(\vec{0},\vec{2})}_{\alpha,\beta}} - \sum_{\substack{\alpha'\subset\alpha,\beta'\subset\beta \\ 1\leq\#\alpha'=\#\beta'<\alpha}} B_{\alpha',\beta',\vec{\mathbf{h}}}\, \bar{C}^{\vec{\mathbf{h}}^{(\vec{0},\vec{2})}_{\alpha',\beta'}}_{\vec{\mathbf{h}}^{(\vec{0},\vec{2})}_{\alpha,\beta}}, \tag{7.87}$$

where the $\bar{C}$ coefficients are proportional to the C coefficients found in (7.55), with

$$\bar{C}^{\vec{\mathbf{r}}}_{\vec{\mathbf{s}}} = \frac{\langle\vec{\mathbf{r}}|\vec{\mathbf{r}}\rangle}{\langle\vec{\mathbf{s}}|\vec{\mathbf{s}}\rangle} C^{\vec{\mathbf{r}}}_{\vec{\mathbf{s}}}. \tag{7.88}$$

Finally, the SoV measure is written as

$${}_p\langle\vec{\mathbf{h}}|\vec{\mathbf{k}}\rangle_p = \langle\vec{\mathbf{h}}|\vec{\mathbf{h}}\rangle \left(\delta_{\vec{\mathbf{h}},\vec{\mathbf{k}}} + \sum_{r=1}^{n_{\vec{\mathbf{k}}}} \mathsf{c}^r \sum_{\alpha,\beta\in\Gamma_2(\vec{\mathbf{i}}_{\vec{\mathbf{h}}};r)} B_{\alpha,\beta,\vec{\mathbf{h}}}\, \delta_{\vec{\mathbf{h}},\vec{\mathbf{k}}^{(\vec{0},\vec{2})}_{\alpha,\beta}}\right) \tag{7.89}$$

This is corollary 3.1 of [LV2].

As we can see, the complexity of the SoV measure increases strongly from the $\mathfrak{gl}(2)$ to the higher-rank $\mathfrak{gl}(3)$ case. Remember that while all the formulas derived here are exact and give the general form of the couplings and the measure, the coefficients $C^{\vec{\mathbf{k}}}_{\vec{\mathbf{h}}}$ and $B_{\alpha,\beta,\vec{\mathbf{h}}}$ are all defined implicitly by recursive formulas. This makes the upcoming calculations of form factors and correlations functions more complicated. Maybe there are local operators for which the computations turn out easy enough in a certain limit, for example $\mathsf{c} \to 0$. Still, it would be much more convenient to have orthogonal separate bases for the $\mathfrak{gl}(3)$ case as well. The next section is devoted to the construction of a such bases.

7.4 Orthogonal bases and the $\mathbb{T}$ operator

7.4.1 The non-invertible twist matrix case

Comparing of the $\mathfrak{gl}(2)$ and $\mathfrak{gl}(3)$ cases, most of the complexity in the computations is already accounted by the fusion relations. Having one more level in the fusion relations greatly increases the number of necessary Lagrange interpolations to compute, say, the action of the transfer matrix over the left and right separate bases.

It is thus tempting to study $\mathfrak{gl}(3)$ models for which the hierarchy of fusion relations simplifies. This is done easily by choosing a non-invertible twist, which forces the quantum determinant to be zero. For a twist matrix $\hat{K}$ with simple spectrum but with one zero eigenvalue, the fusion relations for the associated transfer matrices $T_1^{(\hat{K})}(u)$ and $T_2^{(\hat{K})}(u)$ are

$$T_1^{(\hat{K})}(\xi_j)T_1^{(\hat{K})}(\xi_j-\eta) = T_2^{(\hat{K})}(\xi_j), \tag{7.90}$$

$$T_1^{(\hat{K})}(\xi_j)T_2^{(\hat{K})}(\xi_j-\eta) = 0, \tag{7.91}$$

$$T_2^{(\hat{K})}(\xi_j)T_2^{(\hat{K})}(\xi_j-\eta) = 0. \tag{7.92}$$

The last relations is a consequence of the first two ones. The covectors and vectors families defined in (7.47), (7.50) still form a left and right separate bases, respectively, since it is only required for the twist matrix to be with simple spectrum. The statement of theorem 4.1 of [LV2] elaborates a bit on this.

The picture is strikingly simpler now: the SoV coupling between the left and right separate bases (7.47), (7.50) is now diagonal. This follows immediately from Theorem 7, since $\mathsf{c} = \det \hat{K} = 0$ in the present case.

It is still useful to obtain this results independently of the calculation done in the general case: since the fusion relations are much simpler now, the computations should be less intricate, and it should be possible to derive the orthogonality and the diagonal coupling (7.57) formula directly from the simplified fusion relations.

This is done by theorem 4.1 of [LV2]—whose proof lies from page 23 to 26. Before that, the action of $T_1^{(\hat{K})}(u)$ and $T_2^{(\hat{K})}(u)$ over the left and right separate bases are explicitly computed as linear combinations over the separate bases by proposition 4.1. The actual computations are as usual Lagrange interpolation in the points where the fusion relations (7.90)–(7.92) can be applied. Because they are now simpler, it is quicker and easier to compute the actions exactly.

The description of the eigenstates as separates states is now very direct, since the two bases are dual to each other, up to some N-tuple-dependent normalization. This is the content of theorem 4.2, which states that the vectors and covectors

$$|t_a\rangle = \sum_{\vec{\mathbf{h}}\in\mathcal{I}} \prod_{j=1}^{N} t_{2,a}(\xi_j^{(1)})^{\delta_{h_j,0}} t_{1,a}(\xi_j)^{\delta_{h_j,2}} \frac{|\vec{\mathbf{h}}\rangle}{\mathsf{N}_{\vec{\mathbf{h}}}}, \quad \langle t_a| = \sum_{\vec{\mathbf{h}}\in\mathcal{I}} \prod_{j=1}^{N} t_{2,a}(\xi_j)^{\delta_{h_j,1}} t_{1,a}(\xi_j)^{\delta_{h_j,2}} \frac{\langle\vec{\mathbf{h}}|}{\mathsf{N}_{\vec{\mathbf{h}}}}, \tag{7.93}$$

are eigenstates if and only if the functions $t_{1,a}(u)$, $t_{2,a}(u)$ are eigenvalues of the corresponding transfer

matrices, i.e.

$$T_1^{(\hat{K})}(u)\,|t_a\rangle = t_{1,a}(u)\,|t_a\rangle\,, \quad T_2^{(\hat{K})}(u)\,|t_a\rangle = t_{2,a}(u)\,|t_a\rangle\,, \tag{7.94}$$

$$T_1^{(\hat{K})}(u)\,\langle t_a| = t_{1,a}(u)\,\langle t_a|\,, \quad T_2^{(\hat{K})}(u)\,\langle t_a| = t_{2,a}(u)\,\langle t_a|\,. \tag{7.95}$$

The eigenvalues and eigenvectors in this $\det\hat{K} = 0$ case can be characterized by SoV as described in chapter 6. This is done explicitly in [LV2]. It also ensures the simplicity of the transfer matrix family $(T(u))_{u\in\mathbb{C}}$ when $\hat{K}$ has simple spectrum.

7.4.2 SoV measure and scalar products of separate states

The SoV measure is the inverse of the left and right separate bases coupling matrix. The latter is diagonal, so it is especially simple to invert. Therefore, the scalar product of left and right separate states of the form

$$\langle\alpha| = \sum_{\vec{\mathbf{h}}\in\mathcal{I}} \alpha_{\vec{\mathbf{h}}} \frac{\langle\vec{\mathbf{h}}|}{\mathsf{N}_{\vec{\mathbf{h}}}}, \quad \alpha_{\vec{\mathbf{h}}} = \prod_{j=1}^{N} \alpha_j^{(h_j)}, \tag{7.96}$$

$$|\gamma\rangle = \sum_{\vec{\mathbf{h}}\in\mathcal{I}} \gamma_{\vec{\mathbf{h}}} \frac{|\vec{\mathbf{h}}\rangle}{\mathsf{N}_{\vec{\mathbf{h}}}}, \quad \gamma_{\vec{\mathbf{h}}} = \prod_{j=1}^{N} \gamma_j^{(h_j)}, \tag{7.97}$$

is written as the following sum of products of ratios of Vandermonde determinants

$$\langle\alpha|\gamma\rangle = \sum_{\vec{\mathbf{h}}\in\mathcal{I}} \left(\prod_{j=1}^{N} \frac{d\left(\xi_j^{\left(1+\delta_{h_j,1}+\delta_{h_j,2}\right)}\right)}{d\left(\xi_j-\eta\right)} \alpha_j^{(h_j)} \gamma_j^{(h_j)} \right) \times \frac{V\left(\xi_1^{\left(\delta_{h_1,1}+\delta_{h_1,2}\right)},\dots,\xi_N^{\left(\delta_{h_N,1}+\delta_{h_N,2}\right)}\right) V\left(\xi_1^{\left(\delta_{h_1,1}\right)},\dots,\xi_N^{\left(\delta_{h_N,1}\right)}\right)}{V(\xi_1,\dots,\xi_N)^2}. \tag{7.98}$$

For mixed scalar product of separate states, where the vector at the right is an eigenstate of the transfer matrix, it is possible to repack efficiently this big sum into products of determinants. This is achieved by theorem 4.3 of [LV2], and more precisely in equation (4.77).

For $\hat{K}$ a simple spectrum twist matrix with one zero eigenvalue and $|\gamma\rangle = |t_n\rangle$ for some $n \in [\![1, 3^N]\!]$ an eigenstate of the transfer matrix, the right-hand side of formula (7.98) can be simplified by accounting for the known root pattern of the eigenvalues functions, which we recall is

$$\begin{aligned} \forall (a,b) \in \pi_n(A)\times\pi_n(B), \quad & t_{1,n}(\xi_b) = 0 = t_{2,n}(\xi_a-\eta), \\ & t_{1,n}(\xi_a) \neq 0, \quad t_{2,n}(\xi_b-\eta) \neq 0, \end{aligned} \tag{7.99}$$

where π_n is some permutation of the set $[\![1,N]\!]$ and

$$A = [\![1, \mathsf{M}_n]\!], \quad B = [\![1,N]\!] \setminus A = [\![\mathsf{M}_n+1, N]\!] \tag{7.100}$$

for some integer M_n specific to $|t_n\rangle$. Keeping only the non-zero terms in the right-hand side of (7.98), one can rearrange it as a product of two independent sums, specified in equations (4.85)–(4.88) of [LV2]. These sums are identified to coincide with known determinants, similar to the ones encountered in references [211, 213], for example. Eventually, the mixed scalar product is written as the following

product of determinants

$$\langle \alpha | t_n \rangle = \prod_{a=1}^{N} \frac{d(\xi_a^{(2)})}{d(\xi_a^{(1)})} \frac{V\left(\xi^{(1)}_{\pi_n(1)}, \dots, \xi^{(1)}_{\pi_n(\mathtt{M}_n)}\right)}{V\left(\xi_{\pi_n(1)}, \dots, \xi_{\pi_n(\mathtt{M}_n)}\right)} \times \frac{\det_{N-\mathtt{M}_n} \mathcal{M}^{(\alpha|x_A t_{2,n})}_{+,N-\mathtt{M}_n}}{V\left(\xi_{\pi_n(\mathtt{M}_n+1)}, \dots, \xi_{\pi_n(\mathtt{M}_n)}\right)} \frac{\det_{\mathtt{M}_n} \mathcal{M}^{(\alpha|x_B t_{1,n})}_{-,\mathtt{M}_n}}{V(\xi_{\pi_n(1)}, \dots, \xi_{\pi_n(\mathtt{M}_n)})}, \quad (7.101)$$

where x_A and x_B are scalar coefficients defined in equation (4.80) of [IV2],

$$x_A(u) = \prod_{a \in \pi_n(A)} \frac{u - \xi_a + \eta}{u - \xi_a}, \quad x_B(u) = \prod_{b \in \pi_n(B)} \frac{u - \xi_b - \eta}{u - \xi_b}, \quad (7.102)$$

and the explicit forms of the matrices $\mathcal{M}_\pm$ are

$$\left(\mathcal{M}^{\alpha|x_A t_{2,n}}_{+,N-\mathtt{M}_n}\right)_{ij} = \sum_{h=0}^{1} \alpha^{(h)}_{\pi_n(\mathtt{M}_n+i)} \; x_A^{1-h}\left(\xi_{\pi_n(\mathtt{M}_n+i)}\right) \underline{t}^{\,h}_{2,n}\left(\xi^{(1)}_{\pi_n(\mathtt{M}_n+i)}\right) \left(\xi_{\pi_n(\mathtt{M}_n+i)}\right)^{j-1}, \quad (7.103)$$

$$\left(\mathcal{M}^{\alpha|x_B t_{1,n}}_{-,\mathtt{M}_n}\right)_{ij} = \sum_{h=0}^{1} \alpha^{(h+1)}_{\pi_n(i)} \; x_B^{h}\left(\xi_{\pi_n(i)}\right) t^{h}_{1,n}\left(\xi_{\pi(i)}\right) \left(\xi^{(h)}_{\pi_n(i)}\right)^{j-1}, \quad (7.104)$$

with

$$\underline{t}_{2,n}(u) = t_{2,n}(u) \prod_{j=1}^{N} \frac{u - \xi_j}{u - \xi_j - \eta}. \quad (7.105)$$

They happen to be of the form of determinants similar to the $\mathfrak{gl}(2)$ case; the fusion relation in this degenerate case are indeed close to the ones of $\mathfrak{gl}(2)$. The norm of an eigenstate $|t_n\rangle$ is then obtained by simplification from (7.101), and is written

$$\langle t_n | t_n \rangle = \prod_{a=1}^{N} \frac{V\left(\xi^{(1)}_{\pi_n(1)}, \dots, \xi^{(1)}_{\pi_n(\mathtt{M}_n)}\right)}{V\left(\xi^{(1)}_{\pi_n(1)}, \dots, \xi^{(1)}_{\pi_n(\mathtt{M}_n)}\right)} \cdot \prod_{b \in \pi_n(B)} t_{2,n}(\xi_b^{(1)}) x_A(\xi_b) \cdot \prod_{a \in \pi_n(A)} t_{1,n}(\xi_a) \cdot \det \mathcal{T}_{\mathtt{M}_n}, \quad (7.106)$$

where $\mathcal{T}_{\mathtt{M}_n}$ is the $\mathtt{M}_n \times \mathtt{M}_n$ square matrix with coefficients

$$\left(\mathcal{T}_{\mathtt{M}_n}\right)_{(i,j) \in \pi_n(A) \times A} = \sum_{h=0}^{1} t_{1,n}(\xi_i)^{(1-h)} \; x_B(\xi_i)^h \left(\xi_i^{(h)}\right)^{j-1}. \quad (7.107)$$

Despite working on a $\mathfrak{gl}(3)$ model, these scalar products are similar to product of $\mathfrak{gl}(2)$ type scalar products.

7.4.3 Extension to the invertible twist matrix case

As we see in the above paragraph, it would be very desirable to construct the separate bases for the $\mathfrak{gl}(3)$ chain with a general twist K from conserved quantities that obey a set of simpler fusion relations. Luckily, such an operator exists.

Assume K is a diagonalizable and invertible twist matrix with simple spectrum. From the results of the previous chapter, we know that the associated transfer matrix $T^{(K)}(u)$ is diagonalizable with simple spectrum for almost all values of the inhomogeneities, and have a complete characterization of the spectrum by SoV. Let

$$|t_a^{(K)}\rangle \quad \text{and} \quad \langle t_a^{(K)}| \quad (7.108)$$

be respectively the 3^N eigenvectors and 3^N eigencovectors of $T^{(K)}(u)$. Now let $\hat{K}$ denote the matrix obtained from K by putting one of its eigenvalues to zero, while preserving its diagonalizability. It also has simple spectrum. Using $\hat{K}$ as a twist matrix, we can construct the $T^{(\hat{K})}(u)$ transfer matrix, which is

associated to a $\mathfrak{gl}(3)$ spin chain of same length but different boundary conditions than the one associated to $T^{(K)}(u)$. The transfer matrix $T^{(\hat{K})}(u)$ is diagonalizable with simple spectrum. We note

$$|t_a^{(\hat{K})}\rangle \quad \text{and} \quad \langle t_a^{(\hat{K})}| \tag{7.109}$$

its 3^N eigenvectors and 3^N eigencovectors, respectively, and $t_{1,a}^{(\hat{K})}(u)$, $t_{2,a}^{(\hat{K})}(u)$ the corresponding eigenvalues defined by

$$T_1^{(\hat{K})}(u)\,|t_a^{(\hat{K})}\rangle = t_{1,a}^{(\hat{K})}(u)\,|t_a^{(\hat{K})}\rangle\,, \quad T_2^{(\hat{K})}(u)\,|t_a^{(\hat{K})}\rangle = t_{2,a}^{(\hat{K})}(u)\,|t_a^{(\hat{K})}\rangle\,, \tag{7.110}$$

$$T_1^{(\hat{K})}(u)\,\langle t_a^{(\hat{K})}| = t_{1,a}^{(\hat{K})}(u)\,\langle t_a^{(\hat{K})}|\,, \quad T_2^{(\hat{K})}(u)\,\langle t_a^{(\hat{K})}| = t_{2,a}^{(\hat{K})}(u)\,\langle t_a^{(\hat{K})}|\,. \tag{7.111}$$

We define the new two operators

$$\mathbb{T}_j^{(K)}(u) := \sum_{a=1}^{3N} t_{j,a}^{(\hat{K})}(u) \frac{|t_a^{(K)}\rangle\langle t_a^{(K)}|}{\langle t_a^{(K)}|t_a^{(K)}\rangle}, \quad \text{with} \quad j \in \{1,2\}. \tag{7.112}$$

It is immediate that these operators have been defined specifically to have the eigen(co)vectors of $T^{(K)}(u)$, but with the corresponding eigenvalue of the $T^{(\hat{K})}(u)$ matrix. Because they are diagonalizable in the same basis, the $\mathbb{T}_j^{(K)}(u)$ and the $T_k^{(K)}(u)$ commute

$$\forall j,k \in \{1,2\}, \quad \left[\mathbb{T}_j^{(K)}(u), \mathbb{T}_k^{(K)}(u)\right] = 0 = \left[\mathbb{T}_j^{(K)}(u), T_k^{(K)}(u)\right]. \tag{7.113}$$

But the $\mathbb{T}_j^{(K)}(u)$ shares the same spectrum as the $T_j^{(\hat{K})}(u)$ matrices, so they satisfy the same fusion relations:

$$\mathbb{T}_1^{(K)}(\xi_j)\mathbb{T}_1^{(K)}(\xi_j - \eta) = \mathbb{T}_2^{(K)}(\xi_j), \tag{7.114}$$

$$\mathbb{T}_1^{(K)}(\xi_j)\mathbb{T}_2^{(K)}(\xi_j - \eta) = 0, \tag{7.115}$$

$$\mathbb{T}_2^{(K)}(\xi_j)\mathbb{T}_2^{(K)}(\xi_j - \eta) = 0. \tag{7.116}$$

The above equations can be checked on the eigenvalues. This makes the $\mathbb{T}^{(K)}(u)$ transfer matrix a very interesting operator: it is a conserved quantity of the $\mathfrak{gl}(3)$ model with the boundary conditions associated to the K matrix, but has a much simpler structure in terms of fusion relations than the usual transfer matrix $T^{(K)}(u)$. Following the general philosophy introduced in [225], we can construct separate bases starting from $\mathbb{T}^{(K)}(u)$ instead of $T^{(K)}(u)$. Let

$$\langle\widehat{\vec{\mathbf{h}}}| := \langle\vec{\mathbf{1}}| \prod_{j=1}^{N} \mathbb{T}_2^{(K)}(\xi_j^{(1)})^{\delta_{h_j,0}} \mathbb{T}_1^{(K)}(\xi_j)^{\delta_{h_j,2}}, \tag{7.117}$$

$$|\widehat{\vec{\mathbf{h}}}\rangle := \prod_{j=1}^{N} \mathbb{T}_2^{(K)}(\xi_j)^{\delta_{h_j,1}} \mathbb{T}_1^{(K)}(\xi_j)^{\delta_{h_j,2}} \left|\vec{\mathbf{0}}\right\rangle, \tag{7.118}$$

be the family of 3^N vectors and 3^N covectors indexed by $\vec{\mathbf{h}} \in \mathcal{I}$. By the proposition 3.1 of [LV2], they are respectively left and right separate bases. But besides, the matrix of the couplings is diagonal, i.e. these two bases are orthogonal. This is ensured by theorem 4.1 of [LV2], since it relies only on the fusion

relations (7.90)–(7.92) to compute the couplings. Then,

$$\begin{aligned}\langle\widehat{\vec{\mathbf{h}}}|\widehat{\vec{\mathbf{k}}}\rangle &= \mathsf{N}_{\vec{\mathbf{h}}}\delta_{\vec{\mathbf{h}},\vec{\mathbf{k}}}\\ &= \prod_{j=1}^{N}\delta_{h_j,k_j}\frac{d(\xi_j-\eta)}{d\left(\xi_j^{(1+\delta_{h_j,1}+\delta_{h_j,2})}\right)}\frac{V\left(\xi_1^{(\delta_{h_1,1}+\delta_{h_1,2})},\dots,\xi_N^{(\delta_{h_N,1}+\delta_{h_N,2})}\right)V\left(\xi_1^{(\delta_{h_1,2})},\dots,\xi_N^{(\delta_{h_N,2})}\right)}{V(\xi_1,\dots,\xi_N)^2}.\end{aligned} \tag{7.119}$$

The SoV measure associated to the left and right separate bases (7.117)–(7.118) is therefore diagonal as well. The scalar products of separate states of the form

$$\langle\alpha| = \sum_{\vec{\mathbf{h}}\in\mathcal{I}}\alpha_{\vec{\mathbf{h}}}\frac{\langle\widehat{\vec{\mathbf{h}}}|}{\mathsf{N}_{\vec{\mathbf{h}}}},\quad \alpha_{\vec{\mathbf{h}}} = \prod_{j=1}^{N}\alpha_j^{(h_j)}, \tag{7.120}$$

$$|\beta\rangle = \sum_{\vec{\mathbf{h}}\in\mathcal{I}}\beta_{\vec{\mathbf{h}}}\frac{|\widehat{\vec{\mathbf{h}}}\rangle}{\mathsf{N}_{\vec{\mathbf{h}}}},\quad \beta_{\vec{\mathbf{h}}} = \prod_{j=1}^{N}\beta_j^{(h_j)}, \tag{7.121}$$

therefore write just like the ones for the (co)vectors (7.96)and (7.97). In fact, the formulas obtained in theorem 4.3 of [LV2] applies also in this case, but with the substitution $|t_n\rangle \to |t_n^{(K)}\rangle$ the eigenvector of the transfer matrix $T^{(K)}(u)$.

We have been able to determine an operator $\mathbb{T}^{(K)}(u)$ commuting with $T^{(K)}(u)$ but with simpler fusion rules. The separate bases constructed from it are orthogonal, so their SoV measure is diagonal. This enables the explicit computation of scalar products of separate states in $\mathfrak{gl}(3)$ fundamental models with generic diagonalizable with simple spectrum invertible twist, paving the way for the computation of form factors and correlation functions by separation of variables. This first require to reconstruct the local operators from quantities whose actions on the separate left and right bases is computable, i.e. solving the quantum inverse scattering problem in a way adapted to the above description of the eigenstates.

However, the $\mathbb{T}^{(K)}(u)$ has been defined in (7.112). As defined, the bases (7.117) and (7.118) lack of a direct algebraic construction, for example as a trace of products of some Lax matrices or other objects constructed from the represented generators of the underlying Yangian algebra. Maybe it is possible to express it in terms of the original transfer matrices $T_1^{(K)}(u)$, $T_2^{(K)}(u)$. Another way to characterize $\mathbb{T}^{(K)}(u)$ would be to compute the mixed couplings

$$\left\langle t_a^{(K)}\middle|t_b^{(\hat{K})}\right\rangle \tag{7.122}$$

for all $a,b\in[\![1,3^N]\!]$. Such a computation is made possible by the characterizations of the overlaps computed previously, and should open the way to the practical use of the $\mathbb{T}^{(K)}(u)$ bases.

Chapter

8 Separate bases and spectrum of $\mathfrak{gl}(m|n)$ models

Some quantum integrable models are associated with a $\mathbb{Z}_2$-graded algebra, or superalgebra [107, 125, 128, 229, 293, 294]. This is usually the mark of the presence of fermionic objects in the physical systems. Most notable are the t-J model [295] and the Hubbard model [126, 255, 296].

The structure of the Lie superalgebras $\mathfrak{gl}(\mathfrak{m}|\mathfrak{n})$ differs a bit from the classical Lie algebra $\mathfrak{gl}(\mathfrak{n})$, as the defining relations has to take the grading into account. In particular, the representation theory of Lie superalgebras contains additional subtleties [297–299], and so do the one of the corresponding Yangian algebras $\mathscr{Y}(\mathfrak{gl}(\mathfrak{m}|\mathfrak{n}))$ [117, 300]. The description of quantum integrable models associated with superalgebras can still be done in the QISM language of R-matrix, monodromy and transfer matrices. An important point of divergence is the structure of the fusion relations, as we will highlight in the forthcoming material. The ABA program towards the computation of correlation functions has been developed for the $\mathscr{Y}(\mathfrak{gl}(\mathfrak{m}|\mathfrak{n}))$ models with great success [124, 125, 127, 128, 157, 158, 160–162, 229, 255, 294, 295, 301].

In this chapter, we extend the construction of separate bases from transfer matrices to the supersymmetric case by recalling the results of article [LV1]. The construction is very similar to the non-graded case, though some extra care is required in the definition of the objects at hand to properly account of their grading. We begin with a clear introduction of the graded objects of quantum integrability. Then, we prove the existence of separate bases made out from conserved quantities by showing bases of the form (6.54) are candidate separate bases for $\mathscr{Y}(\mathfrak{gl}(\mathfrak{m}|\mathfrak{n}))$ models. The use of such a basis in the resolution of the spectral problem of the transfer matrix is then discussed, in light of the particular fusion hierarchy of the $\mathscr{Y}(\mathfrak{gl}(\mathfrak{m}|\mathfrak{n}))$ transfer matrices. An *inner-boundary condition* (IBC) is then identified to serve as the necessary functional constraint on eigenvalue, and closure relation for the action of the transfer matrix on the basis. It is used explicitly to solve the spectral problem of the $Y(\mathfrak{gl}(1|2))$ model with some non-invertible quasi-periodic boundary conditions. We close this chapter with the construction of separate bases for the Hubbard model.

8.1 Graded objects

8.1.1 Super vectors spaces

We introduce the notion of $\mathbb{Z}_2$-grading on vector spaces [44].

Definition. *A $\mathbb{Z}_2$-graded vector space, or super vector space, is a complex vector space with $V_0, V_1 \subset V$ such that*

$$V = V_0 \oplus V_1, \tag{8.1}$$

and V_0 and V_1 are stable under the addition:

$$\forall a \in \{0,1\}, \quad \forall v, w \in V_a, \quad v + w \in V_a. \tag{8.2}$$

Vectors that are elements of either one of the two subspaces V_0 or V_1 are said *homogeneous*: they have a well-defined grading, or, because it is a $\mathbb{Z}_2$-grading, a well-defined parity. Vectors of V_0 are *even* and

vectors of V_1 are *odd*. The grading map, or parity map, is an application that associate 0 or one regarding the parity of homogeneous vectors

$$\begin{aligned} p : V_0 \cup V_1 &\longrightarrow \{0,1\} \\ v &\longmapsto p(v) = \begin{cases} 0 & \text{if } v \in V_0, \\ 1 & \text{if } v \in V_1. \end{cases} \end{aligned} \tag{8.3}$$

Suppose V is of finite dimension $\mathfrak{m}+\mathfrak{n}$, with

$$\dim V_0 = \mathfrak{m}, \quad \dim V_1 = n. \tag{8.4}$$

It is handy to define the adapted basis $(e_1, \dots, e_{\mathfrak{m}+\mathfrak{n}})$ with

$$e_1, \dots, e_m \in V_0, \quad e_{\mathfrak{m}+1}, \dots, e_{\mathfrak{m}+\mathfrak{n}} \in V_1. \tag{8.5}$$

A common shorthand notation for these vectors is to define

$$\forall i \in [\![1, \mathfrak{m}+\mathfrak{n}]\!], \quad \bar{i} := p(e_i). \tag{8.6}$$

One also finds the similar notations $[i]$ or $|i|$ in the literature.

Consider the dual space V^* of linear form over V. It is graded as well. Using the braket notation, the grading of the covector $\langle i| = |i\rangle^\dagger \equiv (e_i)^\dagger$ is the same as the one of $|i\rangle$, namely $p(\langle i|) = \bar{i}$.

8.1.2 Superalgebras

Useful references on superalgebras and Lie superalgebras are [44, 298, 299, 302]

Definition. *A complex $\mathbb{Z}_2$-graded algebra, or superalgebra, is a complex algebra A over the complex field admitting the direct sum decomposition*

$$A = A_0 \oplus A_1 \tag{8.7}$$

such that its multiplication operation $A \times A \to A$ verifies

$$\forall i, j \in \{0,1\}, \quad A_i A_j \subseteq A_{i+j \bmod 2}. \tag{8.8}$$

Note that the set of even elements form the subalgebra A_0, while A_1 is not a subalgebra. The grading map $p : A_0 \cup A_1 \to \{0,1\}$ is equivalently defined here, and we note $\bar{x} := p(x)$.

Definition. *A Lie superalgebra $\mathfrak{g}$ is a superalgebra whose product $[\cdot,\cdot]$, called the Lie superbracket, supercommutator or graded commutator, is skew-symmetric and satisfies the graded Jacobi identity:*

$$\forall x, y \in \mathfrak{g}, \quad [x,y] = -(-1)^{\bar{x}\bar{y}}[y,x], \tag{8.9}$$

and

$$\forall x, y, z \in \mathfrak{g}, \quad (-1)^{\bar{x}\bar{z}}[x,[y,z]] + (-1)^{\bar{y}\bar{x}}[y,[z,x]] + (-1)^{\bar{z}\bar{y}}[z,[x,y]] = 0. \tag{8.10}$$

Any associative superalgebra A may be turned into a Lie superalgebra by equipping it with the graded commutator

$$\forall x, y \in A_0 \cup A_1, \quad [x,y] = xy - (-1)^{\bar{x}\bar{y}} yx, \tag{8.11}$$

and extend this definition linearly to the whole superalgebra.

The space $\mathrm{End}(V)$ of endomorphisms over a super vector space V is a super vector space as well. It is also a superalgebra with the multiplication given by the composition, and is a Lie superalgebra with the

graded commutator (8.11); it is noted $\mathfrak{gl}(V)$ as a Lie superalgebra.

We will focus a bit more on the superalgebra $\mathfrak{gl}\big(\mathbb{C}^{\mathfrak{m}|\mathfrak{n}}\big)$, where $\mathbb{C}^{\mathfrak{m}|\mathfrak{n}}$ is the complex super vector space of dimension $\mathfrak{m}+\mathfrak{n}$ whose canonical basis $(e_1,\dots,e_{\mathfrak{m}+\mathfrak{n}})$ splits in two to generate the even and odd subspace:

$$(\mathbb{C}^{\mathfrak{m}|\mathfrak{n}})_0 = \mathrm{Vect}(e_1,\dots,e_{\mathfrak{m}}), \quad (\mathbb{C}^{\mathfrak{m}|\mathfrak{n}})_1 = \mathrm{Vect}(e_{\mathfrak{m}+1,\dots,e_{\mathfrak{m}+\mathfrak{n}}}). \tag{8.12}$$

It is the Lie superalgebra of endomorphisms over $\mathbb{C}^{\mathfrak{m}|\mathfrak{n}}$. We note e_i^j, $i,j \in [\![1,\mathfrak{m}+\mathfrak{n}]\!]$ the elementary operators (or matrices) defined by

$$e_i^j e_k = \delta_k^j e_i. \tag{8.13}$$

Their grading is

$$p(e_i^j) = \bar{i}+\bar{j} \bmod 2. \tag{8.14}$$

Because they multiply as $e_i^j e_k^\ell = \delta_k^j e_i^\ell$, the graded commutator is

$$\left[e_i^j, e_k^\ell\right] = \delta_j^k e_i^\ell - (-1)^{(\bar{i}+\bar{j})(\bar{k}+\bar{\ell})}\delta_i^\ell e_k^j. \tag{8.15}$$

Any element is decomposed as a linear combination on the elementary operators

$$a = \sum_{i,j=1}^{\mathfrak{m}+\mathfrak{n}} a_j^i e_i^j. \tag{8.16}$$

From now on we will often omit sums over repeated indices. It is useful to represent these endomorphisms as block matrices of the form

$$a = \begin{pmatrix} a_{(\mathfrak{m},\mathfrak{m})} & a_{(\mathfrak{m},\mathfrak{n})} \\ a_{(\mathfrak{n},\mathfrak{m})} & a_{(\mathfrak{n},\mathfrak{n})} \end{pmatrix}, \tag{8.17}$$

where the definition of the matrices $a_{(\mathfrak{m},\mathfrak{m})}$, $a_{(\mathfrak{m},\mathfrak{n})}$,... should be self-explanatory. The above matrices form the matrix associative superalgebra $M(\mathfrak{m}|\mathfrak{n},\mathbb{C})$. The even and odd elements are respectively of the form

$$\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}, \quad \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}. \tag{8.18}$$

From the above block form, it is clear that even elements are maps that preserve the parity of the vector, while odd elements flips it. The supertrace is defined on the elementary operators by $\mathrm{str}\, e_i^j = (-1)^{\bar{i}}\delta_i^j$. Using the block form, this gives

$$\mathrm{str}\, a = \mathrm{tr}\, a_{(\mathfrak{m},\mathfrak{m})} - \mathrm{tr}\, a_{(\mathfrak{n},\mathfrak{n})}. \tag{8.19}$$

Similarly, the superdeterminant, or Berezinian, is defined to be

$$\mathrm{Ber}\begin{pmatrix} A & B \\ C & D \end{pmatrix} := \det\big(A - BD^{-1}C\big)\det\big(D^{-1}\big). \tag{8.20}$$

It has properties similar to the ones of the determinant in the non-graded case, see [44]

8.1.3 Tensor products of graded objects

The tensor product $V \otimes W$ of two super vector spaces V and W is itself a super vector space. Indeed, we have the following decomposition in direct sum

$$V \otimes W = \bigoplus_{i,j\in\{0,1\}} V_i \otimes W_j, \tag{8.21}$$

and we can group terms in an even and an odd subspace

$$(V\otimes W)_k := \bigoplus_{i+j=k \bmod 2} V_i \otimes W_j \quad \text{for} \quad k=0 \text{ or } 1. \tag{8.22}$$

The tensor product $A\otimes B$ of two associative superalgebras A, B is defined the same way.

The basic rule of signs in supersymmetric space is that a factor $(-1)^{\bar{x}\bar{y}}$ appears whenever two elements x, y are flipped. We can already see this from (8.9) and (8.11) for instance. The multiplication in $A\otimes B$ of homogeneous elements also benefit from this additional sign factor

$$\forall a,c\in A_0\cup A_1,\quad \forall b,d\in B_0\cup B_1,\quad (a\otimes b)(c\otimes d)=(-1)^{\bar{b}\bar{c}}ac\otimes bd, \tag{8.23}$$

and the above definition is extended linearly to $A\otimes B$ in its whole. The rule extends naturally to N-fold tensor product with $N\geq 2$. For example,

$$(a\otimes b\otimes c)(d\otimes e\otimes f)=(-1)^{(\bar{b}+\bar{c})\bar{d}+\bar{c}\bar{e}}ad\otimes bd\otimes cf. \tag{8.24}$$

Similarly, the action of matrices of $\mathfrak{gl}\left(\mathbb{C}^{m|n}\right)\otimes\mathfrak{gl}\left(\mathbb{C}^{m|n}\right)$ over $\mathbb{C}^{m|n}\otimes\mathbb{C}^{m|n}$ has signs

$$\forall a,b\in\mathfrak{gl}\left(\mathbb{C}^{m|n}\right),\quad \forall v,w\in\mathbb{C}^{m|n},\quad (a\otimes b)(v\otimes w)=(-1)^{\bar{b}\bar{v}}av\otimes bw. \tag{8.25}$$

A consequence of this is that the matrix representation of an operator $a\otimes b\in\mathrm{End}(V)\otimes\mathrm{End}(V)$ is *not* the tensor product of the matrix representations of operator a and operator b. There are additional grading signs, more on this in the following paragraph.

Let $A\in\mathfrak{gl}\left(\mathbb{C}^{m|n}\right)^{\otimes N}$. It can be decomposed on the canonical basis constructed by tensorization from the local ones as

$$A=A^{i_1,\dots,i_N}_{j_1,\dots,j_N}e^{j_1}_{i_1}\otimes\cdots\otimes e^{i_N}_{i_N}. \tag{8.26}$$

It is important to not confuse the coefficients $A^{i_1,\dots,i_N}_{j_1,\dots,j_N}$ in the above linear combination with the components of the image of the vector $e_{j_1}\otimes\cdots\otimes e_{j_N}\in\mathbb{C}^{m|n}$, which are

$$A(e_{j_1}\otimes\cdots\otimes e_{j_N})=(-1)^{\sum_{k=1}^N \bar{i}_k(\bar{i}_{k+1}+\cdots+\bar{i}_N)}A^{i_1,\dots,i_N}_{j_1,\dots,j_N}e_{i_1}\otimes\cdots\otimes e_{i_N}. \tag{8.27}$$

In the non-graded case, these two coefficients would be identical. The coefficients of the matrix representation of operators A are the ones found in (8.27).

Some authors feel necessary to stress the grading dependent behavior of the tensor product and denote it explicitly by referring to the "super tensor product" or "graded tensor product", accompanied by a notation $\otimes_s$ or $\otimes_g$. We will not make use of this notation.

Signs also appear in the definition of dual states in tensor product spaces. Let $V\simeq\mathbb{C}^{m|n}$. We use the braket notation for the canonical basis defined in (8.5):

$$\forall i,j\in[\![1,m+n]\!],\quad \langle i|j\rangle=\delta_{ij}, \tag{8.28}$$

and $p(\langle i|)=p(|i\rangle)=\bar{i}$. Let $|i_1\rangle\otimes\cdots\otimes|i_N\rangle$, $i_1,\dots,i_N\in[\![1,m+n]\!]^N$, be a vector of the basis of $V^{\otimes N}$ obtained by the simple tensorization of the one-site basis. What is its dual state. We would like to recover the usual behavior

$$\forall j_1,\dots,j_N\in[\![1,m+n]\!]^N,\quad \left(|i_1\rangle\otimes\cdots\otimes|i_N\rangle\right)^\dagger|j_1\rangle\otimes\cdots\otimes|j_N\rangle=\delta_{i_1j_1}\dots\delta_{i_Nj_N}. \tag{8.29}$$

It cannot be $\langle i_1|\otimes\cdots\otimes\langle i_N|$, since many additional signs emerge from covectors and vectors passing

through each others. Adding this precise sign factor in the definition of the adjoint does the trick. With

$$\big(|i_1\rangle\otimes\cdots\otimes|i_N\rangle\big)^\dagger := \langle i_1|\otimes\cdots\otimes\langle i_N|(-1)^{\sigma(i_1,\dots,i_N)} \quad \text{with} \quad \sigma(i_1,\dots,i_N)=\sum_{k=2}^{N}\bar{i}_k\big(\bar{i}_1+\cdots+\bar{i}_{k-1}\big), \tag{8.30}$$

the sign compensate for the ones arising from the covectors and vectors passing through each others

$$\begin{aligned}\big(|i_1\rangle\otimes\cdots\otimes|i_N\rangle\big)^\dagger|j_1\rangle\otimes\cdots\otimes|j_N\rangle &= (-1)^{\sigma(i_1,\dots,i_N)}(-1)^{\sum_{k=2}^{N}\bar{i}_k(\bar{j}_1+\cdots+\bar{j}_{k-1})}\langle i_1|j_1\rangle\dots\langle i_N|j_N\rangle\\ &= \delta_{i_1j_1}\dots\delta_{i_Nj_N}.\end{aligned} \tag{8.31}$$

Similarly, for N even operators $A_1,\dots,A_N\in\mathfrak{gl}\big(\mathbb{C}^{m|n}\big)$, where A_j acts non-trivially only on the j^{th} space of V^N, the matrix elements have a factorized form over the spaces

$$\big(|i_1\rangle\otimes\cdots\otimes|i_N\rangle\big)^\dagger A_1\otimes\cdots\otimes A_N\,|j_1\rangle\otimes\cdots\otimes|j_N\rangle = \langle i_1|A_1|j_1\rangle\dots\langle i_N|A_N|j_N\rangle\,. \tag{8.32}$$

Indeed, the A_k are even i.e. $p(A_k)=0$ and produce no sign when passing through, say, the covectors, and then their evenness forces $\bar{i}_k=\bar{j}_k$, so that the signs compensate.

Consider the permutation operator over the tensor product $V\otimes V$ of two copies of the same super vector space V. It accounts for the grading when flipping the vectors, and is defined by

$$\mathcal{P}(v\otimes w)=(-1)^{\bar{v}\bar{w}}w\otimes w. \tag{8.33}$$

Thus, it writes

$$\mathcal{P}=(-1)^{\bar{j}}e^j_i\otimes e^i_j. \tag{8.34}$$

For two homogeneous operators A and B in $\mathrm{End}(V)$,

$$\mathcal{P}(A\otimes B)\mathcal{P}=(-1)^{p(A)p(B)}B\otimes A. \tag{8.35}$$

The permutation operator is globally even, as an operator of the superalgebra $\mathrm{End}(V\otimes V)$.

8.1.4 Yangian of $\mathfrak{gl}(m|n)$

Yangians of classical Lie superalgebras are introduced in details [300]. See also [256, 303] and Molev's monograph [117].

Consider the operator of $\mathrm{End}(\mathbb{C}^{m|n}\otimes\mathbb{C}^{m|n})$

$$R(u)=u+\eta\mathcal{P}. \tag{8.36}$$

It is the R-matrix of the Yangian of the superalgebra $\mathfrak{gl}(m|n)$. It is decomposed on the canonical basis of elementary operators as

$$R(u)=R^{ik}_{j\ell}(u)\;e^j_i\otimes e^\ell_k \quad \text{with} \quad R^{ik}_{j\ell}(u)=u\delta^i_j\delta^k_\ell+\eta(-1)^{\bar{j}}\delta^i_\ell\delta^k_j. \tag{8.37}$$

As in the non-graded case, we write

$$R_{ab}(u):=R^{ik}_{j\ell}(u)\;\mathrm{id}\otimes\cdots\otimes\mathrm{id}\otimes\underbrace{e^j_i}_{\text{space }a}\otimes\mathrm{id}\otimes\cdots\otimes\mathrm{id}\otimes\underbrace{e^\ell_k}_{\text{space }b}\otimes\mathrm{id}\otimes\cdots\otimes\mathrm{id}, \tag{8.38}$$

the R-matrix $R_{ab}(u)=u\mathrm{id}+\eta\mathcal{P}_{ab}$ acting non-trivially the a^{th} and b^{th} spaces. The R-matrix is globally

even. It satisfies the Yang–Baxter equation

$$R_{12}(u-v)R_{13}(u)R_{23}(v) = R_{23}(v)R_{13}(u)R_{12}(u-v). \tag{8.39}$$

This equation can be written in coordinates using the matrix representations $\mathsf{R}(u)$ of the operator $R(u)$

$$R(u)\, e_j \otimes e_\ell = \mathsf{R}(u)^{ik}_{j\ell}(u)\, e_i \otimes e_k \quad \text{with} \quad \mathsf{R}(u)^{ik}_{j\ell}(u) = u\delta^i_j\delta^k_\ell + \eta(-1)^{\bar{i}\bar{k}}\delta^i_\ell\delta^k_j. \tag{8.40}$$

Then (8.39) implies

$$\mathsf{R}^{\alpha\beta}_{\alpha\beta'}(u-v)\mathsf{R}^{\alpha'\gamma}_{\alpha''\gamma'}(u)\mathsf{R}^{\beta'\gamma'}_{\beta''\gamma''}(v)\,(-1)^{\bar{\beta}'(\bar{\alpha}'+\bar{\alpha}'')} = \mathsf{R}^{\beta\gamma}_{\beta'\gamma'}(v)\mathsf{R}^{\alpha\gamma'}_{\alpha'\gamma''}(u)\mathsf{R}^{\alpha'\beta'}_{\alpha''\beta''}(u-v)\,(-1)^{\bar{\beta}'(\bar{\alpha}+\bar{\alpha}')}. \tag{8.41}$$

This is referred to as the *graded* Yang–Baxter equation, since there is an explicit modification by grading signs. The R-matrix (8.36) is invariant by the $\mathfrak{gl}\big(\mathbb{C}^{m|n}\big)$ algebra

$$\forall x \in \mathfrak{gl}\big(\mathbb{C}^{m|n}\big), \quad [R(u), x\otimes \mathrm{id} + \mathrm{id}\otimes x] = 0. \tag{8.42}$$

Also, if K is an even matrix of $M(\mathfrak{m}|\mathfrak{n},\mathbb{C}) \simeq \mathfrak{gl}\big(\mathbb{C}^{m|n}\big)$, we have

$$R(u)(K\otimes \mathrm{id})(\mathrm{id}\otimes K) = (\mathrm{id}\otimes K)(K\otimes \mathrm{id})R(u), \tag{8.43}$$

which is the scalar version of the Yang–Baxter equation, where there is a trivial representation of the $\mathfrak{gl}(\mathfrak{m}|\mathfrak{n})$ algebra on the third space. One sees that any even matrix K can serve as a twist for $\mathfrak{gl}(\mathfrak{m}|\mathfrak{n})$ integrable model.

8.1.5 The $\mathscr{Y}(\mathfrak{gl}(\mathfrak{m}|\mathfrak{n}))$ fundamental chain

Let $\mathbb{V}_0, \mathbb{V}_1, \dots, \mathbb{V}_N$ be N copies of the super vector space $\mathbb{C}^{m|n}$. The space $\mathbb{V}_0$ will serve as the auxiliary space, while the others will be the quantum, physical ones. We note $\mathscr{H} := \mathbb{V}_1 \otimes \cdots \otimes \mathbb{V}_N$. Let K be an even matrix of $\mathfrak{gl}\big(\mathbb{C}^{m|n}\big)$. The $\mathscr{Y}(\mathfrak{gl}(\mathfrak{m}|\mathfrak{n}))$ twisted inhomogeneous fundamental model is defined by the monodromy

$$M(u) = K_0 R_{0N}(u-\xi_N)\dots R_{01}(u-\xi_1). \tag{8.44}$$

It is a globally even, as an operator of the superalgebras $\mathrm{End}(\mathbb{V}_0\otimes\mathscr{H})$. In coordinates, it is written

$$\begin{aligned} M(u) &= M^{i\alpha_1\dots\alpha_N}_{j\beta_1\dots\beta_N}(u) e^j_i \otimes e^{\beta_1}_{\alpha_1}\otimes\cdots\otimes e^{\beta_N}_{\alpha_N} \\ &= K^i_{j_N} R^{j_N\alpha_N}_{j_{N-1}\beta_N}(u-\xi_N)\dots R^{j_1\alpha_1}_{j\beta_1}(u-\xi_1)\, e^j_i\otimes e^{\beta_1}_{\alpha_1}\otimes\cdots\otimes e^{\beta_N}_{\alpha_N}. \end{aligned} \tag{8.45}$$

There are no additional sign because of the evenness of the R-matrices. Noting

$$M(u) = e^j_i \otimes M^i_j(u), \tag{8.46}$$

the above expression shows that the monodromy elements $M^i_j(u)$ are homogeneous elements of the superalgebra $\mathrm{End}(\mathscr{H})$, with their grading depending solely on the coordinates i, j

$$p\Big(M^i_j(u)\Big) = \bar{i} + \bar{j} \bmod 2. \tag{8.47}$$

The monodromy matrix verifies the Yang–Baxter equation

$$R_{ab}(u-v)M_a(u)M_b(v) = M_b(v)M_a(u)R_{ab}(u-v). \tag{8.48}$$

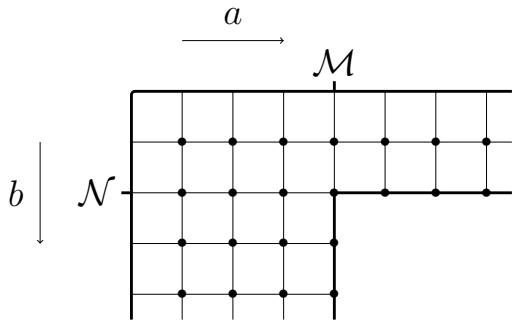

Fig. 8.1 The $H_{\mathfrak{m}|\mathfrak{n}}$ domain for $\mathscr{Y}(\mathfrak{gl}(\mathfrak{m}|\mathfrak{n}))$ fusion

This expands in the $(\mathfrak{m}+\mathfrak{n})^2$ relations

$$\left[M_i^j(u), M_k^\ell(v)\right] = (-1)^{\bar{i}\bar{k}+\bar{i}\bar{\ell}+\bar{k}\bar{\ell}}\Big(M_k^i(v)M_i^\ell(u) - M_k^j(u)M_i^\ell(v)\Big), \tag{8.49}$$

where $[\cdot,\cdot]$ is the graded commutator (8.9). The transfer matrix is obtained by taking the supertrace

$$T(u) := \mathrm{str}_0\, M(u). \tag{8.50}$$

As a sum (and difference) of diagonal elements of the monodromy, the transfer matrix is an even operator of $\mathrm{End}(\mathscr{H})$. Because the super trace vanishes on the graded commutator, the family of the transfer matrices forms is a commuting family

$$\forall u, v \in \mathbb{C}, \quad [T(u), T(v)] = 0. \tag{8.51}$$

8.1.6 The fused transfer matrices

Like in the non-graded case, the family of transfer matrices can be enlarged with the higher order transfer matrices, which are obtained by taking arbitrary representations of the superalgebra $\mathfrak{gl}(\mathfrak{m}|\mathfrak{n})$ on the auxiliary space. And as in the non-graded case, they can be constructed via fusion.

Tensor product of fundamental representations of $\mathfrak{gl}(\mathfrak{m}|\mathfrak{n})$ are decomposable in direct sum of irreducible subrepresentations. Young diagrams can be used to carry out calculations, with mechanics proper to superalgebra though very similar to the non-graded case [128, 297, 298, 304]. Finite dimensional irreducible representations are labelled uniquely by Kac–Dynkin labels, but the correspondence between Kac–Dynkin labels and Young diagrams is not one-to-one [304][1].

Admissible Young diagrams lie inside a *fat hook* domain pictured in fig. 8.1, defined over the bidimensional lattice

$$H_{\mathfrak{m}|\mathfrak{n}} := \mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 1} \setminus \mathbb{Z}_{>\mathfrak{m}} \times \mathbb{Z}_{>\mathfrak{n}}, \tag{8.52}$$

with coordinates (a, b). Young diagrams expand infinitely in both directions a and b, but the points $a > \mathfrak{n}, b > \mathfrak{m}$ are forbidden, leading to the hook shape.

Let λ be some representation of $\mathfrak{gl}(\mathfrak{m}|\mathfrak{n})$. The degeneracy points of the fundamental R-matrix allow to construct the projectors $P_\lambda : (\mathbb{C}^{\mathfrak{m}|\mathfrak{n}})^{\otimes \mathfrak{n}} \to V_\lambda$ which extracts the wanted subrepresentation λ as a product of R-matrices. Fusing on the auxiliary space from the fundamental monodromy $M(u)$, we obtain new monodromy operators associated to λ, and then new transfer matrices one the supertrace is taken. All of these transfer matrices commute together, since an analog Yang–Baxter equation is verified by the fused monodromies, for the projectors are constructed from R-matrices at specific points.

[1] Highest weight irreducible representations are characterized uniquely by their highest weight, which can be noted on a Kac–Dynkin diagram. When using Young diagrams, the corresponding highest weight is obtained from the length of the $\mathfrak{m}$ first rows and height the $\mathfrak{n}$ first columns, see [304] for details. In the case of a Young diagram with at least $\mathfrak{m}+1$ rows of length $\mathfrak{n}$ and $\mathfrak{n}$ columns of height $\mathfrak{m}$, one can delete a length $\mathfrak{n}$ row and add a height $\mathfrak{m}$ row, add compute the same highest weight.

0	$-\eta$	-2η	-3η	-4η
η	0	$-\eta$	-2η	-3η
2η	η	0		
3η	2η	η		

Fig. 8.2 The domain $H_{\mathfrak{m}|\mathfrak{n}}$ domain is filled with multiples of the deformation parameter η. Starting from 0 in box $(1,1)$, one adds η when moving down and $-\eta$ when going right.

For rectangular Young diagram corresponding to the point $(a,b) \in H_{\mathfrak{m}|\mathfrak{n}}$ with b rows and a columns, the corresponding monodromy matrix reads

$$M_b^{(a)}(u) := P_b^{(a)} \left[\overleftarrow{\bigotimes_{\substack{1\le s\le a\\ 1\le r\le b}}} M(u+\eta(r-s)) \right] P_b^{(a)} \tag{8.53}$$

The shifts in (8.53) are obtained by filling the fat hook $H_{\mathfrak{m}|\mathfrak{n}}$ as in fig. 8.2. Then, reading the rectangular domain corresponding to the Young diagram associated to λ column by column, top to bottom from left to right, and tensoring a fundamental shifted monodromy corresponding to the current box at the left of the previous ones, one obtain (8.53). The projectors are obtained as a product of R-matrices [227, 229]. For rectangular diagrams (a,b), they are of the form

$$P_b^{(a)} \propto \prod_{i<j} R_{ij}(s_j - s_i), \tag{8.54}$$

where i, j run over the boxes of the diagram of fig. 8.2 in the same way as described earlier, and the s_i, s_j are the shifts in the boxes. The projectors associated to row and columns Young diagrams are the symmetrizer and antisymmetrizer, respectively, over the $\mathbb{V}^{\otimes a}$ space:

$$P_a^{(1)} = P^+_{1\dots a} = \sum_{\sigma\in \mathsf{S}_a} P_\sigma, \tag{8.55}$$

$$P_1^{(a)} = P^-_{1\dots a} = \sum_{\sigma\in \mathsf{S}_a} \mathrm{sign}(\sigma)\, P_\sigma, \tag{8.56}$$

where S_a is the symmetric group of rank a. We have the following recursion relations for the symmetrizers and antisymmetrizers making explicit use of the R-matrix

$$P^+_{1\dots a} = \frac{1}{a\eta} P^+_{1\dots a-1} R((a-1)\eta) P^+_{2\dots a}, \tag{8.57}$$

$$P^-_{1\dots a} = -\frac{1}{a\eta} P^-_{1\dots a-1} R(-(a-1)\eta) P^-_{2\dots a}, \tag{8.58}$$

in agreement with (8.54).

The fused transfer matrix is obtained by taking the super trace on the representation λ on the auxiliary space. For the rectangular representation, one has

$$T_b^{(a)}(u) := \mathrm{str}_{\mathbb{V}_b^a} M_b^{(a)}(u), \tag{8.59}$$

where $V_b^a \simeq V^{\otimes ab}$. All these transfer matrices commute together, as a consequence of generalized form of the Yang–Baxter equation (8.48) being true for any irreps taken on the auxiliary spaces a and b. Hence,

$$\forall (a,b),(c,d) \in H_{\mathfrak{m}|\mathfrak{n}}, \quad \forall u,v\in\mathbb{C}, \quad \left[T_b^{(a)}(u), T_d^{(c)}(v)\right] = 0. \tag{8.60}$$

There are other ways to construct the fused transfer matrices (8.59). The nice coderivative formalism obtain these matrices and all their characteristics and relations from the characters of the super algebra $\mathfrak{gl}(\mathfrak{m}|\mathfrak{n})$ [229, 231].

Let us recall the most relevant properties of the transfer matrices $T_b^{(a)}(u)$ for our purposes.

Polynomial structure The transfer matrix $T_b^{(a)}(u)$ is a polynomial in u of degree abN, with $(ab-1)N$ central zeroes given by

$$Z_b^{(a)}(u) = \prod_{j=1}^{N}\left[(u-\xi_j)^{-1}\prod_{\ell=1}^{b}\prod_{m=1}^{a}\big(u-\xi_j+\eta(\ell-m)\big)\right], \tag{8.61}$$

which therefore factorizes as

$$T_b^{(a)}(u) = \widetilde{T}_b^{(a)}(u)\, Z_b^{(a)}(u), \tag{8.62}$$

where $\widetilde{T}_b^{(a)}(u)$ is a polynomial of degree ab.

Fusion equations There are bilinear relations between transfer matrices associated to adjacent rectangular diagrams, often called the Hirota relations

$$T_b^{(a)}(u-\eta)T_b^{(a)}(u) = T_{b+1}^{(a)}(u-\eta)T_{b-1}^{(a)}(u) + T_b^{(a-1)}(u-\eta)T_b^{(a+1)}(u), \tag{8.63}$$

where in our normalization, the following boundary conditions are imposed

$$T_{b\geq 1}^{(0)}(u) = 1 = T_0^{(a\geq 1)}(u). \tag{8.64}$$

All the transfer matrices outside the *extended fat hook* $\bar{H}_{\mathfrak{m}|\mathfrak{n}} := (\mathbb{Z}_{\geq 0}\times\mathbb{Z}_{\geq 0})\setminus(\mathbb{Z}_{>\mathfrak{m}}\times\mathbb{Z}_{>\mathfrak{n}})$ are identically zero, i.e.

$$\forall (a,b)\notin \bar{H}_{\mathfrak{m}|\mathfrak{n}},\quad T_b^{(a)}(u) = 0. \tag{8.65}$$

Inner-boundary condition Because the correspondence between Young diagram and the irreducible representation of the superalgebra $\mathfrak{gl}(\mathfrak{m}|\mathfrak{n})$ is not bijective, there exists non-trivial relations linking transfer matrices associated to distinct Young diagrams. This concerns rectangular Young diagrams saturating one arm of the fat hook $H_{\mathfrak{m}|\mathfrak{n}}$ [128, 304, 305]. The first relation is called the *inner-boundary condition* (IBC) and is written

$$(-1)^{\mathfrak{n}}\,\text{q-Ber}(u)T_{\mathfrak{n}}^{(\mathfrak{m}+1)}(u+\eta) = T_{\mathfrak{n}+1}^{(\mathfrak{m})}(u), \tag{8.66}$$

where

$$\text{q-Ber}\,M(u) = \text{sdet}(K)\frac{a(u)\prod_{k=1}^{\mathfrak{m}-1}d(u-k\eta)}{\prod_{\ell=1}^{\mathfrak{n}-\mathfrak{m}}d(u+\ell\eta)}\text{id}_{\mathscr{H}}, \tag{8.67}$$

with

$$a(u-\eta) = d(u) = \prod_{j=1}^{N}(u-\xi_j). \tag{8.68}$$

The operator q-Ber(u) is called the quantum Berezinian, for $\mathfrak{m}\neq\mathfrak{n}$, and plays a role similar to the quantum determinant in the non-graded case [127, 306]. This relation is explicitly verified in appendix C of [LV1] by the use of the coderivative formalism.

Reconstruction of fused matrices from the fundamental one The Bazhanov–Reshetikhin determinant formulas allow to write all the transfer matrices $T_b^{(a)}(u)$ in terms of the row and column type ones

$$T_b^{(a)}(u) = \det_{1\leq i,j\leq a} T_{b+i-j}^{(1)}\big(u-(i-1)\eta\big) \tag{8.69}$$

$$= \det_{1\leq i,j\leq b} T_1^{(a+i-j)}\big(u+(i-1)\eta\big). \tag{8.70}$$

We will use the simpler notations

$$\begin{aligned} T_a(u) &:= T_a^{(1)}(u) \quad \text{and} \quad T_{(a)}(u) := T_1^{(a)}(u), \\ \widetilde{T}_a(u) &:= \widetilde{T}_a^{(1)}(u) \quad \text{and} \quad \widetilde{T}_{(a)}(u) := \widetilde{T}_1^{(a)}(u). \end{aligned} \tag{8.71}$$

The asymptotics of these matrices are given solely by the corresponding symmetrization of the twist matrix:

$$T_{\infty,a} := \lim_{u\to\infty} u^{-aN} T_a(u) = \mathrm{str}_{1\dots a} P^+_{1\dots a} K_1 \dots K_a P^+_{1\dots a}, \tag{8.72}$$

$$T_{\infty,(a)} := \lim_{u\to\infty} u^{-aN} T_{(a)}(u) = \mathrm{str}_{1\dots a} P^-_{1\dots a} K_1 \dots K_a P^-_{1\dots a}. \tag{8.73}$$

One recognizes that $T_{\infty,a}$ and $T_{\infty,(a)}$ are the characters of the completely symmetric and antisymmetric representations, respectively, applied to the twist matrix. It is seen explicitly with a diagonalizable twist matrix with spectrum $\mathrm{Sp}(K) = \{k_1, \dots k_{m+n}\}$, for example.

In the inhomogeneities, the matrices enjoys simpler fusion relations thanks to the structure of their central roots. We have, for all $j \in [\![1, N]\!]$ and for all positive integer n,

$$T_1(\xi_j) T_n(\xi_j + \eta) = T_{n+1}(\xi_j), \tag{8.74}$$

$$T_1(\xi_j) T_{(n)}(\xi_j - \eta) = T_{(n+1)}(\xi_j). \tag{8.75}$$

The proof is done easily by induction, using the general bilinear Hirota fusion relations (8.63), and the central zeroes (8.61). This is done in lemma 2.1 of [LV1]. These row and columns transfer matrices are reconstructed from the fundamental one by successive Lagrange interpolation in the inhomogeneities.

Remark 1 (Comparaison with other conventions). In a great part of literature, such as articles [128, 305], the conventions used for the fused transfer matrices differ from ours. We make the link explicit in the following.

1. *Rectangular representations and orientation of the fat-hook*
   Rectangular diagrams s^a with a rows and s columns in [128, 305] correspond to diagrams (a, s) in our convention. This is because the conventions chosen for the fat-hook differ: figure 1 of [128] matches with fig. 8.1 when rotated by $-\pi/2$.
2. *R-matrix*
   The value of the η parameter, free in our discussion, is fixed at 2 in [128, 305]. This is inconsequential, as it amounts to a rescaling of the spectral parameter, but change the definition of all the related objects.
3. *Definition of transfer matrices*
   Transfer matrices associated to rectangular representations in [128, 305] are defined up to non-constant shifts compared to definition (8.59). We note them $U(a, s, u)$, and they are defined as [see 305, equation (6)]
   $$U(a, s, u) := \mathrm{str}_{\mathbb{V}_b^a}\big(\pi_s^a(g)\mathcal{T}(u - s + a)\big), \tag{8.76}$$
   where $\mathcal{T}(u)$ is the untwisted monodromy constructed from the R-matrices with $\eta = 2$, $g \in \mathtt{GL}(()\mathfrak{m}|\mathfrak{n})$ a twist and π_s^a the representation associated to the rectangular Young diagram s^a. In our notation,

reintroducing the η parameter, the link is done by

$$T_b^{(a)}(u) \overset{\eta=2}{=} U(a,b,u+b-a). \tag{8.77}$$

4. *Hirota equation*
   Together with the specific choice $\eta = 2$, the latter convention leads to a different form of the Hirota equation. In [128, 305], it is written

$$U(a,s,u+1)U(a,s,u-1) = U(a,s+1,u)U(a,s-1,u) + U(a+1,s,u)U(a-1,s,u). \tag{8.78}$$

   Compared to (8.63), each of the three terms is shifted in only one of its variables, by ± 1, making it easy to remember. Introducing the constant shift $u \to u+s-a-1$, one obtains with $\eta = 2$ the following form

$$T_s^{(a)}(u-\eta)T_s^{(a)}(u) = T_{s+1}^{(a)}(u-\eta)T_{s-1}^{(a)}(u) + T_s^{(a+1)}(u)T_s^{(a-1)}(u-\eta), \tag{8.79}$$

   which is exactly equation (8.63) for $s \to b$.
5. *Boundary conditions*
   The boundary conditions chosen here (see equations (8.64) and (8.65)) also differs from the ones in [128, 305]. There, it is imposed that

$$T(0,s,u) = \phi(u-s), \quad T(a,0,u) = \phi(u+a), \tag{8.80}$$

   where $\phi(u)$ is some function fixing the input data of the problem—in this precise case it would be $\phi(u) = \prod_{j=1}^{N}(u-\xi_j)$ for the inhomogeneous chain of length N. In our setting, the input data are fixed by the giving of the transfer matrix $T(u) \equiv T_1^{(1)}(u)$. Both choices are valid, but produce slightly different hierarchies of fused transfer matrix: the $U(a,s,u)$ matrices are polynomial in u of degree N for all values of a and s, while in our case the degree in u of the $T_b^{(a)}(u)$ transfer matrices grows as abN, with $(ab(N-1))$ central zeroes (see equation (8.61)). Consequently, the inner-boundary conditions also differ, up to a scalar factor that happen to be the total quantum Berezinian, and by shifts in the spectral parameter because of the different definitions of the transfer matrices (compare equation (8.66) with equation (28) of [305]).

We have the following lemma, which is corollary 2.1 of [LV1]

Lemma 2. *Under the condition*

$$\forall j,k \in [\![1,N]\!],\ j \neq k, \quad \xi_a \neq \xi_b \bmod \eta \tag{8.81}$$

on the inhomogeneities, the transfer matrices $T_{n+1}(u)$ and $T_{(n+1)}(u)$ are constructed in terms of $T_1(u)$ by the following recursive Lagrange interpolation formulas

$$T_{n+1}(u) = \prod_{r=1}^{n} d(u+r\eta)\left[T_{\infty,n+1}(u) + \sum_{a=1}^{N} f_a^{(n+1)(u)}\ T_1(\xi_a)T_n(\xi_a+\eta)\right], \tag{8.82}$$

$$T_{(n+1)}(u) = \prod_{r=1}^{n} d(u-r\eta)\left[T_{\infty,(n+1)}(u) + \sum_{a=1}^{N} g_a^{(n+1)(u)}\ T_1(\xi_a)T_{(n)}(\xi_a-\eta)\right], \tag{8.83}$$

where the asymptotics coefficients are

$$T_{\infty,a}(u) := T_{\infty,a}\prod_{j=1}^{N}(u-\xi_j) \quad \textit{and} \quad T_{\infty,(a)}(u) := T_{\infty,(a)}\prod_{j=1}^{N}(u-\xi_j), \tag{8.84}$$

and the interpolation coefficients are

$$f_a^{(m)}(u) := \left(\prod_{\substack{b=1 \\ b\neq a}}^{N} \frac{u-\xi_b}{\xi_a-\xi_b} \right) \prod_{b=1}^{N} \prod_{r=1}^{m-1} \frac{1}{\xi_a - \xi_b^{(r)}}, \tag{8.85}$$

$$g_a^{(m)}(u) := \left(\prod_{\substack{b=1 \\ b\neq a}}^{N} \frac{u-\xi_b}{\xi_a-\xi_b} \right) \prod_{b=1}^{N} \prod_{r=1}^{m-1} \frac{1}{\xi_a - \xi_b^{(-r)}}, \tag{8.86}$$

with the usual shorthand notation $\xi_j^{(r)} := \xi_j - r\eta$.

Proof. The proof of this lemma is very direct. The $T_a(u)$ and $T_{(a)}(u)$ are polynomials of degree aN. We already know $(a-1)N$ central roots of them, so they can be reconstructed by Lagrange interpolation from their asymptotic behavior and their values at N different points. The asymptotics are computable and written above, and the fusion relations provide values in the inhomogeneities in terms of the previous transfer matrices in the fusion hierarchy. □

One see easily that the general form of $T_b^{(a)}(u)$ in terms of the fundamental transfer matrix $T_1^{(1)}(u) \equiv T(u)$ is a linear combination of the form

$$T_b^{(a)}(u) = \sum_{s_1,\dots,s_{ab}} c(u)\ T(u+s_1\eta)\dots T(u+s_k\eta), \tag{8.87}$$

where the $s_1,\dots,s_k$ are some shifts governed by the Lagrange interpolations and the shifts in the Bazhanov–Reshetikhin formulas, with $1 \leq k \leq ab$. Hence, the *degree in* $T(u)$ of $T_b^{(a)}(u)$ is ab.

8.2 Separates bases

As in the non-graded case, it is possible to construct bases from conserved quantity that are separate for the spectral problem of the transfer matrix. The basis made from powers of the transfer matrix in Theorem 5 is constructed very similarly for the $\mathscr{Y}(\mathfrak{gl}(\mathfrak{m}|\mathfrak{n}))$ fundamental model. The proof relies mainly on the reduction of the R-matrix to the permutation in a particular point. We reproduce here the theorem and its full proof.

Consider a $(\mathfrak{m}+\mathfrak{n})\times(\mathfrak{m}+\mathfrak{n})$ square twist matrix K of $\mathfrak{gl}\left(\mathbb{C}^{\mathfrak{m}|\mathfrak{n}}\right)$ which is even and non-derogatory. It is block diagonal of the form

$$K = \begin{pmatrix} K_{\mathfrak{m}} & 0 \\ 0 & K_{\mathfrak{n}} \end{pmatrix}. \tag{8.88}$$

For $\mathcal{X} = \mathfrak{m}, \mathfrak{n}$, we note

$$\{k_{1,\mathcal{X}},\dots,k_{m_{\mathcal{X}},\mathcal{X}}\} = \mathrm{Sp}(K_{\mathcal{X}}) \tag{8.89}$$

the $m_{\mathcal{X}}$ eigenvalues of $K_{\mathcal{X}}$ of algebraic multiplicity $d_{1,\mathcal{X}},\dots,d_{m_{\mathcal{X}},\mathcal{X}}$, with $\sum_{a=1}^{m_{\mathcal{X}}} = \mathcal{X}$. The spectrum of the twist matrix is (geometrically) simple

$$\forall i,j \in [\![1, m_{\mathcal{X}}]\!], \quad k_{i,\mathcal{X}} \neq k_{j,\mathcal{X}}. \tag{8.90}$$

There exists a change of basis invertible matrix $W_K \in \mathfrak{gl}\left(\mathbb{C}^{\mathfrak{m}|\mathfrak{n}}\right)$

$$W_K = \begin{pmatrix} W_{K,\mathfrak{m}} & 0 \\ 0 & W_{K,\mathfrak{n}} \end{pmatrix}, \tag{8.91}$$

even as well, that puts K in its Jordan normal form

$$K_J = \begin{pmatrix} K_{J,\mathfrak{m}} & 0 \\ 0 & K_{J,\mathfrak{n}} \end{pmatrix} = W_K^{-1} K W_K, \tag{8.92}$$

where for $\mathcal{X} = \mathfrak{m}$ or $\mathfrak{n}$, the $K_{J,\mathcal{X}}$ matrices write in Jordan normal form with $m_{\mathcal{X}}$ Jordan blocks of the form (6.10)

$$K_{J,\mathcal{X}} = \begin{pmatrix} J(k_{1,\mathcal{X}},d_{1,\mathcal{X}}) & & \\ & \ddots & \\ & & J(k_{m_{\mathcal{X}},\mathcal{X}},d_{m_{\mathcal{X}},\mathcal{X}}) \end{pmatrix}. \tag{8.93}$$

Let $\mathcal{I} := [\![0, \mathfrak{m}+\mathfrak{n}-1]\!]$, and consider the twisted inhomogeneous fundamental model of length N of the $\mathscr{Y}(\mathfrak{gl}(\mathfrak{m}|\mathfrak{n}))$ algebra. We note $T(u)$ its transfer matrix as defined in (8.50).

Theorem 8. *Let the twist matrix K be an even non-derogatory matrix of $\mathfrak{gl}\left(\mathbb{C}^{\mathfrak{m}|\mathfrak{n}}\right)$ as above. For almost any choice of a covector $\langle S| \in \mathscr{H}^*$ and of the inhomogeneities ξ_j under the condition (8.81), the family of covectors*

$$\forall \vec{\mathbf{h}} \in \mathcal{I}, \quad \langle \vec{\mathbf{h}}| := \langle S| \prod_{j=1}^{N} T(\xi_j)^{h_j} \tag{8.94}$$

forms a covector basis of $\mathscr{H}^*$.

In particular, the state $\langle S|$ can be taken as follows. For a one-site $|S,a\rangle := S_i^{(a)}|i\rangle$ in the a-th space $\mathbb{V}_a$. Its dual covector in $\mathbb{V}_a^*$ is $\langle S,a| = |S,a\rangle^\dagger = S_i^{(a)*}\langle i|$. For the vector

$$|S\rangle := |S,1\rangle \otimes \cdots \otimes |S,N\rangle\,, \tag{8.95}$$

constructed by tensoring N copies of the one site state $|S,a\rangle$, the dual state $\langle S| = |S\rangle^\dagger$ is

$$\langle S| = \sum_{p_1,\dots,p_N}^{\mathfrak{m}+\mathfrak{n}} S_{p_1}^{(1)*} \dots S_{p_N}^{(N)*} \left(|p_1\rangle \otimes \cdots \otimes |p_N\rangle\right)^\dagger, \tag{8.96}$$

where $\left(|p_1\rangle \otimes \cdots \otimes |p_N\rangle\right)^\dagger$ has signs in its definition as in (8.30). Noting

$$\langle S,a|\, W_{K,a}^{-1} = \left(x_{1,\mathfrak{m}}^{(1)}, \dots, x_{d_1,\mathfrak{m}}^{(1)}, \dots, x_{1,\mathfrak{n}}^{(m_\mathfrak{n})}, \dots, x_{d_{m_\mathfrak{n}},\mathfrak{n}}^{(m_\mathfrak{n})}\right), \tag{8.97}$$

it is sufficient to take $|S,a\rangle$ such that

$$\left(\prod_{k=1}^{m_\mathfrak{m}} x_{1,\mathfrak{m}}^{(k)}\right)\left(\prod_{k=1}^{m_\mathfrak{n}} x_{1,\mathfrak{n}}^{(k)}\right) \neq 0 \tag{8.98}$$

for the family of covectors (8.94) to form a basis.

Proof. As in the non-graded case, we have

$$R(0) = \eta\mathcal{P} \quad \text{and} \quad \mathrm{str}_a\, \mathcal{P}_{ab} = \mathrm{id}_b, \tag{8.99}$$

so the transfer matrix in the inhomogeneities is

$$T_(\xi_j) = R_{jj-1}(\xi_j - \xi_{j-1}) \dots R_{j1}(\xi_j - \xi_1) K_j R_{jN}(\xi_j - \xi_N) \dots R_{jj+1}(\xi_j - \xi_{j+1}). \tag{8.100}$$

It is now very similar to the proof of Theorem 5: it is sufficient to prove that the determinant of the matrix

M of size $(\mathfrak{m}+\mathfrak{n})^N$, of elements

$$\mathsf{M}_{ij} = \big\langle \vec{\mathbf{h}}(i) \big| e_j \big\rangle \quad \text{for} \quad i,j \in [\![1,(\mathfrak{m}+\mathfrak{n})^N]\!]\,, \tag{8.101}$$

is non-zero. Here $\vec{\mathbf{h}}(i)$ is the unique N-tuple labeled by i defined by

$$i = 1 + \sum_{a=1}^{N} h_a(i) a^{i-1}, \tag{8.102}$$

while the e_j are the vectors of the canonical basis $\mathscr{H}$ —constructed by tensoring the local canonical bases—enumerated by j. For $j \in [\![1,(\mathfrak{m}+\mathfrak{n})^N]\!]$, one has

$$|e_j\rangle = \big|e_{1+h_1(j)}(1)\big\rangle \otimes \cdots \otimes \big|e_{1+h_N(j)}(N)\big\rangle \tag{8.103}$$

from (8.102), where the $|e_a(j)\rangle$, $a \in [\![1,\mathfrak{m}+\mathfrak{n}]\!]$ are the vectors of the canonical basis at site j. The determinant $\det \mathsf{M}$ has polynomial dependence in the inhomogeneities and the twist matrix coefficients, so it is sufficient to prove that $\det \mathsf{M} \neq 0$ in some particular points of these parameters to have it almost everywhere. As in the non-graded case, a sufficient criterion for $\det \mathsf{M} \neq 0$ is therefore obtained in the large inhomogeneities limit and reads

$$\det \hat{\mathsf{M}} \neq 0, \tag{8.104}$$

with $\hat{\mathsf{M}}$ the $(\mathfrak{m}+\mathfrak{n})^N$ square matrix of coefficients

$$\forall i,j \in [\![1,(\mathfrak{m}+\mathfrak{n})^N]\!]\,, \quad \hat{\mathsf{M}}_{ij} = \Big\langle S \Big| \prod_{a=1}^{N} K_a^{h_a(i)} \Big| e_j \Big\rangle. \tag{8.105}$$

Let us compute this last matrix element precisely. From (8.96), it writes the sum

$$\Big\langle S \Big| \prod_{a=1}^{N} K_a^{h_a(i)} \Big| e_j \Big\rangle = \sum_{p_1,\dots,p_N}^{\mathfrak{m}+\mathfrak{n}} S^{(1)*}_{p_1} \dots S^{(N)*}_{p_N} \left(|p_1\rangle \otimes \cdots \otimes |p_N\rangle\right)^\dagger \times \left(\prod_{a=1}^{N} K_a^{h_a(i)} \right) \big|e_{1+h_1(j)}(1)\big\rangle \otimes \cdots \otimes \big|e_{h_N(j)}(N)\big\rangle. \tag{8.106}$$

Because the $K_a^{h_a(i)}$ are even, the matrix elements factorizes by (8.32) as a product over N one-site matrix elements:

$$\forall i,j \in [\![1,(\mathfrak{m}+\mathfrak{n})^N]\!]\,, \quad \left(|p_1\rangle \otimes \cdots \otimes |p_N\rangle\right)^\dagger \left(\prod_{a=1}^{N} K_a^{h_a(i)} \right) \big|e_{1+h_1(j)}(1)\big\rangle \otimes \cdots \otimes \big|e_{h_N(j)}(N)\big\rangle = \prod_{a=1}^{N} \Big\langle p_a \Big| K_a^{h_a(i)} \Big| e_{1+h_a(j)}(a) \Big\rangle. \tag{8.107}$$

The sum over the $p_1,\dots,p_N$ is therefore decoupled as a product of N independent sums

$$\hat{\mathsf{M}}_{ij} = \prod_{a=1}^{N} \Big\langle S, a \Big| K_a^{h_a(i)} \Big| e_{1+h_a(j)}(a) \Big\rangle, \tag{8.108}$$

and the same goes for the determinant:

$$\det \hat{\mathsf{M}} = \prod_{a=1}^{N} \det \hat{\mathfrak{m}}_a, \tag{8.109}$$

where $\hat{\mathfrak{m}}_a$ the $\mathfrak{m}+\mathfrak{n}$ square matrix of coefficients

$$(\hat{\mathfrak{m}})_{ij} = \left\langle S_a \middle| K_a^{i-1} \middle| e_j(a) \right\rangle. \tag{8.110}$$

Consequently, the family (8.94) is a basis if

$$\forall a \in [\![1, N]\!], \quad \det \hat{\mathfrak{m}}_a \neq 0. \tag{8.111}$$

These determinants are explicitly computed in terms of the eigenvalues of K and the coefficients of the covector (8.97) to be

$$\begin{aligned}\det \hat{\mathfrak{m}} = \prod_{a=1}^{m_{\mathfrak{m}}} \left(x_{1,\mathfrak{m}}^{(a)}\right)^{d_{a,\mathfrak{m}}} \prod_{a=1}^{m_{\mathfrak{n}}} \left(x_{1,\mathfrak{n}}^{(a)}\right)^{d_{a,\mathfrak{n}}} \prod_{a=1}^{m_{\mathfrak{m}}} \prod_{b=1}^{m_{\mathfrak{n}}} \left(k_{a,\mathfrak{m}} - k_{b,\mathfrak{n}}\right)^{d_{a,\mathfrak{m}} d_{b,\mathfrak{n}}} \\ \times \prod_{1 \leq a < b \leq m_{\mathfrak{m}}} \left(k_{a,\mathfrak{m}} - k_{b,\mathfrak{m}}\right)^{d_{a,\mathfrak{m}} d_{b,\mathfrak{m}}} \prod_{1 \leq a < b \leq m_{\mathfrak{n}}} \left(k_{a,\mathfrak{n}} - k_{b,\mathfrak{n}}\right)^{d_{a,\mathfrak{n}} d_{b,\mathfrak{n}}}.\end{aligned} \tag{8.112}$$

Since K has simple spectrum, all the coefficients depending on its eigenvalues in the above formula are non-zero. Therefore, $\det \hat{\mathfrak{m}}$ is non-zero under the condition (8.98). This proves the desired result. □

Proposition 12. *If the twist matrix K has simple spectrum, then the transfer matrix $T(u)$ has simple spectrum. If moreover it is diagonalizable, then $T(u)$ is diagonalizable as well.*

Proof. Let the twist K has simple spectrum, then (8.94) is a basis by the above theorem. For an eigenvalue $t(u)$ of $T(u)$, and $|t\rangle$ an associated eigenvector, it is immediate that

$$\forall \vec{\mathbf{h}} \in \mathcal{I}, \quad \left\langle \vec{\mathbf{h}} \middle| t \right\rangle = \langle S | t \rangle \prod_{j=1}^{N} t(\xi_j)^{h_j}. \tag{8.113}$$

This determines completely the unique eigenvector associated to $t(u)$ — up to a normalization. If moreover K is diagonalizable, then one can prove that $T(u)$ is diagonalizable as well with computations similar to the proof of lemma 1, that we do not repeat here. The main idea is to show that there are no non-trivial Jordan block, i.e. of size greater than one, by using the asymptotic form of the eigenvectors. □

8.3 About the characterization of the spectrum

Just like in the non-graded case, we would like to use this SoV basis to solve the spectral problem of the transfer matrix. We will heavily rely on what we did on the non-graded case. As we already know, the unique eigenvector of an eigenvalue of the transfer matrix is fully fixed by the knowledge of the eigenvalue, thanks to the immediate computations of its coefficients in the basis (8.94):

$$\forall \vec{\mathbf{h}} \in \mathcal{I}, \quad \langle \vec{\mathbf{h}} | t \rangle = \prod_{j=1}^{N} t(\xi_j)^{h_j}. \tag{8.114}$$

We thus restrict to the characterization of the eigenvalues, and already have some constraints on them from the characteristics of the transfer matrix itself. Namely, they are polynomials of degree N, with dominant coefficient $\mathrm{str}(K)$ and have to satisfy the fusion relations.

For a polynomial $P \in \mathbb{C}_N[X]$ of degree N, a diagonalizable matrix with simple spectrum K such that the dominant coefficient of P is $\mathrm{str}\, K$, and inhomogeneities $\xi_1, \ldots, \xi_N$ verifying (8.81), one can construct successively the polynomials $P_n^{(1)}$ and $\mathcal{P}_1^{(n)}$ by the interpolation formulas (8.69)–(8.70), and

the polynomial $P_b^{(a)}$ from the determinant formulas (8.82)–(8.83). We also add the additional constraint that all the functions for points outside the extended fat hook are identically zero.

For any vectors of $(x_1,\dots,x_N)\in\mathbb{C}^N$, we define the following polynomial

$$t_1(u|\{x_j\}) := T_{\infty,1}(u) + \sum_{a=1}^{N} f_a^{(1)}(u)\; x_a. \tag{8.115}$$

We can then define recursively the infinite family of higher polynomials

$$t_{n+1}(u|\{x_j\}) := \left(\prod_{r=1}^{n} d(u+r\eta)\right)\left[T_{\infty,n+1}(u) + \sum_{a=1}^{N} f_a^{(n+1)}(u)\; t_n(\xi_a+\eta|\{x_j\})\; x_a\right], \tag{8.116}$$

$$t_{(n+1)}(u|\{x_j\}) := \left(\prod_{r=1}^{n} d(u-\eta)\right)\left[T_{\infty,(n+1)}(u) + \sum_{a=1}^{N} g_a^{(n+1)}(u)\; t_{(n)}(\xi_a-\eta|\{x_j\})\; x_a\right], \tag{8.117}$$

and

$$\begin{aligned}\forall (a,b)\in H_{\mathrm{m|n}},\quad t_b^{(a)}(u|\{x_j\}) &:= \det_{1\le i,j\le a} t_{b+i-j}\big(u-(i-1)\eta|\{x_j\}\big)\\ &= \det_{1\le i,j\le b} t_{(a+i-j)}\big(u+(i-1)\eta|\{x_j\}\big).\end{aligned} \tag{8.118}$$

We set $t_0^{a\ge 1}(u|\{x_j\}) = 1 = t_{b\ge 1}^{(0)}(u|\{x_j\})$ for full consistency with the fusion relations (8.63). Let us note

$$\mathrm{FH}(\{x_j\},\{\xi_j\},K) := \left\{ f_b^{(a)}(u)\in\mathbb{C}[u] \;\middle|\; f_b^{(a)}(u) = \begin{cases} t_b^{(a)}(u|\{x_j\}) & \text{for } (a,b)\in \bar{H}_{\mathrm{m|n}},\\ 0 & \text{for } (a,b)\notin \bar{H}_{\mathrm{m|n}},\end{cases} \right\} \tag{8.119}$$

the infinite family of polynomials obtained from a vector $(x_1,\dots,x_N)\in\mathbb{C}^N$ by the fusion hierarchy formulas (8.115)–(8.118), where the asymptotics are prescribed by the matrix $K\in\mathrm{End}(\mathbb{C}^{\mathrm{m|n}})$ and the Lagrange interpolations are done in the inhomogeneities $\{\xi_j\}$.

For f, g and h functions of one variable $u\in\mathbb{C}$, we note

$$\mathrm{IBC}_{\mathrm{m|n},\eta}(f,g,h)[u] := (-1)^{\mathrm{n}} h(u) f(u+\eta) - g(u). \tag{8.120}$$

For $h(u) = \text{q-Ber}\, M(u)$, and $|t\rangle$ such that

$$T_{\mathrm{n}}^{(\mathrm{m}+1)}(u)\,|t\rangle = t_{\mathrm{n}}^{(\mathrm{m}+1)}(u)\,|t\rangle\,,\quad T_{\mathrm{n}+1}^{(\mathrm{m})}(u)\,|t\rangle = t_{\mathrm{n}+1}^{(\mathrm{m})}(u)\,|t\rangle\,, \tag{8.121}$$

we have the *inner-boundary condition* at the function level

$$\mathrm{IBC}_{\mathrm{m|n},\eta}\left(t_{\mathrm{n}}^{(\mathrm{m}+1)}, t_{\mathrm{n}+1}^{(\mathrm{m})}, \text{q-Ber}\, M\right) = 0. \tag{8.122}$$

We now define the following family of polynomials

$$\begin{aligned}\mathrm{Fam}_{\mathrm{m|n}}^{N}\big(K,\{\xi_j\}\big) := \Big\{ f(u)\in\mathbb{C}_N[u] \;\Big|\; &\exists (x_1,\dots,x_N)\in\mathbb{C}^N,\quad \{x_j\}\neq 0,\quad f(u)\equiv t_1(u|\{x_j\}),\\ &\mathrm{IBC}_{\mathrm{m|n},\eta}\left(t_{\mathrm{n}}^{(\mathrm{m}+1)}(u|\{x_j\}),\, t_{\mathrm{n}+1}^{(\mathrm{m})}(u|\{x_j\}),\, \text{q-Ber}\, M\right) = 0,\\ &\text{for}\quad t_{\mathrm{n}}^{(\mathrm{m}+1)}(u|\{x_j\}),\, t_{\mathrm{n}+1}^{(\mathrm{m})}(u|\{x_j\}) \in \mathrm{FH}\big(\{x_j\},\{\xi_j\},K\big)\Big\}.\end{aligned} \tag{8.123}$$

In words, this is the set of all the polynomials of degree N that can be written in the form (8.115) from some non-zero vectors $(x_1,\dots,x_N)\in\mathbb{C}^N$ and twist matrix K, whose fused higher polynomials obtained by the fusion hierarchy (8.119) satisfy the inner-boundary condition at the function level.

From all we said above, it is clear that

$$\mathrm{Sp}(T(u)) \subseteq \mathrm{Fam}^N_{\mathfrak{m}|\mathfrak{n}}\big(K, \{\xi_j\}\big). \tag{8.124}$$

This is another wording for the statement made by lemma 2.2 of [LV1], that we recall here

Lemma 3. *Any eigenvalue $t(u)$ of the transfer matrix $T(u)$ admits the representation $t_1(u|\{x_j\})$, where the $(x_1,\dots,X_N) \neq (0,\dots,0)$ are N complex numbers such that the fused polynomials $t_b^{(a)}(u|\{x_j\})$ verify the inner-boundary condition*

$$(-1)^{\mathfrak{n}}\,\text{q-Ber}\,M(u)\,t_{\mathfrak{n}}^{(\mathfrak{m}+1)}(u+\eta|\{x_j\}) = t_{\mathfrak{n}+1}^{(\mathfrak{m})}(u|\{x_j\}), \tag{8.125}$$

and the out-boundary conditions

$$\forall n,m \geq 1, \quad \forall u \in \mathbb{C}, \quad t_{\mathfrak{n}+n}^{(\mathfrak{m}+m)}(u) = 0. \tag{8.126}$$

Are these two set equal? If no, what are the additional requirements for a polynomial of $\mathrm{Fam}(K,\{\xi_j\})$ to be an eigenvalue of $T(u)$? This is where the basis (8.94) enters the stage.

Let $t(u) \in \mathrm{Fam}(K,\{\xi_j\})$, and $|t\rangle$ the associated vector of $\mathscr{H}$ defined by

$$\forall \vec{\mathbf{h}} \in \mathcal{I}, \quad \langle \vec{\mathbf{h}}|t\rangle = \prod_{j=1}^{N} t(\xi_j)^{h_j}. \tag{8.127}$$

We want to prove that upon some possible additional constraints on $t(u)$,

$$\forall \vec{\mathbf{h}} \in \mathcal{I}, \quad \forall u \in \mathbb{C}, \quad \langle \vec{\mathbf{h}}|T(u)|t\rangle = t(u)\,\langle \vec{\mathbf{h}}|t\rangle\,. \tag{8.128}$$

To do so, one has to compute $\langle \vec{\mathbf{h}}|T(u)|t\rangle$ and rearrange it in the left-hand side of (8.128), requiring some assumptions on $t(u)$ along the way if necessary. The immediate concerns is therefore, as in the non-graded case, to know how to act on the basis of $\langle \vec{\mathbf{h}}|$ with $T(u)$. As we know from section 6.2.1 for the non-graded case, this is given by some Lagrange interpolation and the use of the algebra of the conserved quantities, whose relations, in practice, are derived from the fusion relations. This is no different here.

Action of $T(u)$ on $\langle \vec{\mathbf{h}}|$ Action of $T(\xi_j)$ over $\langle \vec{\mathbf{h}}|$ is merely a shift if there is some room to increase the power of the corresponding factor in $\langle \vec{\mathbf{h}}|$

$$\forall j \in [\![1,N]\!], \quad \forall \vec{\mathbf{h}} = (h_1,\dots,h_N) \in \mathcal{I},\ h_j < \mathfrak{m}+\mathfrak{n}-1, \quad \langle \vec{\mathbf{h}}|\,T(\xi_j) = \langle \vec{\mathbf{h}} + \hat{\mathbf{e}}_j|\,. \tag{8.129}$$

The non-trivial case is when h_j is at its maximal value $\mathfrak{m}+\mathfrak{n}-1$. In the non-graded $\mathfrak{gl}(\mathfrak{n})$ case, we were able to characterize the linear decomposition of $\langle \vec{\mathbf{h}}|\,T(\xi_j^{(r)})$ over the basis of the $\langle \vec{\mathbf{h}}|$ by the repeated use of the fusion relations. Indeed, the last fusion relation eventually gives a central element, the quantum determinant, so that any product of $\mathfrak{n}$ transfer matrices can be recast after many Lagrange interpolations in a linear combination of products with at most $\mathfrak{n}-1$ transfer matrices.

In the graded $\mathfrak{gl}(\mathfrak{m}|\mathfrak{n})$ case, the picture is not as simple. Indeed, there are no representation of dimension 1 that would yield a central fused transfer matrix, unlike in the non-graded case where the central quantum determinant can be obtained by fusion. The closest we have from such a tool is the inner-boundary condition (8.66). The degree in $T(u)$ of the left-hand side of (8.66) is $(\mathfrak{m}+1)\mathfrak{n}$, and the degree of the right-hand side is $\mathfrak{m}(\mathfrak{n}+1)$. With $\mathfrak{n} \neq \mathfrak{m}$, the degree in $T(u)$ is lower on one side or the other. This was somewhat expected with the presence of the quantum Berezinian, that gather fundamental matrices $T(u)$ in a rational function to produce a central element. Nonetheless, supposing $\mathfrak{m} < \mathfrak{n}$, it means

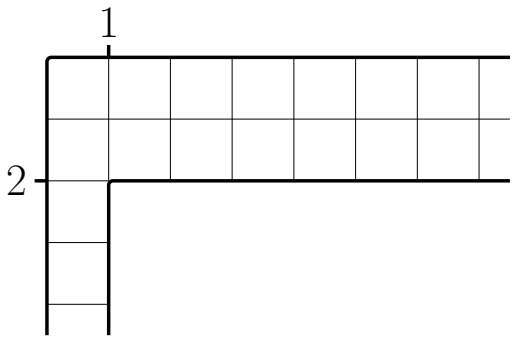

Fig. 8.3 The $H_{\mathfrak{m}|\mathfrak{n}}$ domain for the $\mathscr{Y}(\mathfrak{gl}(1|2))$ model

that a non-trivial linear combination of terms of degree $\mathfrak{mn}+\mathfrak{n}$ in $T(u)$ is actually recombinable in a linear combination of terms of degree $\mathfrak{mn}+\mathfrak{m} < \mathfrak{mn}+\mathfrak{n}$. This gives a very indirect way to possibly rewrite terms of degree $\mathfrak{m}+\mathfrak{n}-1$ in $T(u)$ into terms of smaller degree, effectively computing the missing relations in the abelian algebra of conserved quantities that gives the linear decomposition

$$\langle\vec{\mathbf{h}}|\, T(u) = \sum_{\vec{\mathbf{k}}\in\mathcal{I}} C^{\vec{\mathbf{k}}}_{\vec{\mathbf{h}}}(u)\, \langle\vec{\mathbf{k}}| \tag{8.130}$$

for $\vec{\mathbf{h}} \in \mathcal{I}$ with at least one $h_j = \mathfrak{m}+\mathfrak{n}-1$ for some $j \in [\![1, N]\!]$.

8.4 Specializing for $\mathfrak{gl}(1|2)$ model

We now specialize the discussion to the $\mathfrak{gl}(1|2)$ model. From now on, $\mathcal{I} = \{0,1,2\}$. The fat-hook domain is for $\mathfrak{m}=1$, $\mathfrak{n}=2$ is pictured in fig. 8.3.

Let $K \in \mathfrak{gl}\big(\mathbb{C}^{1|2}\big)$ be diagonalizable. We note k_1, k_2 and k_3 its eigenvalues, and have $\det K_1 = k_1$, $\det K_2 = k_2 k_3$. The associated asymptotics of the column transfer matrices are explicitly

$$\forall n \geq 2, \quad T_{\infty,n} = k_1^{n-2}(k_1-k_2)(k_1-k_3). \tag{8.131}$$

The inner-boundary condition (8.66) reads

$$k_1 T_2^{(1)}(u+\eta) = k_2 k_3 d(u) T_3(u), \tag{8.132}$$

for the fused transfer matrices, and

$$k_1 t_2^{(1)}(u+\eta) = k_2 k_3 d(u) t_3(u) \tag{8.133}$$

at the level of the corresponding eigenvalues. The out-boundary condition (8.126) reads out

$$\forall m,n \geq 0, \quad \forall u \in \mathbb{C}, \quad t^{2+n}_{3+m}(u) = 0. \tag{8.134}$$

We make the following conjecture

Conjecture 1. *For the general $\mathfrak{gl}(1|2)$-graded Yang–Baxter twisted inhomogeneous fundamental model of transfer matrix $T(u)$,*

$$\mathrm{Sp}(T(u)) = \mathrm{Fam}^N_{\mathfrak{m}|\mathfrak{n}}(K, \{\xi_j\}). \tag{8.135}$$

In others words, polynomials $t_1(u|\{x_j\})$ defined in (8.115) are eigenvalues of $T(u)$ if and only if the higher fused polynomials associated to it satisfy the inner-boundary condition (8.125) and the null out boundary condition (8.126).

Indeed, the inner-boundary condition is rewritten by fusion relation as

$$k_2 k_3 d(u) T_3(u) = k_1 (T_2(u) T_2(u+\eta) - T_3(u) T_1(u+\eta)), \tag{8.136}$$

which is of order 3 in the $T_1(\xi_j^{(r)})$ in its left-hand side and of order 4 in its right hand-side. Let K have simple spectrum, so that the family of covectors (8.94) is a basis, and is invertible. The highest power of $T(\xi_j)$ appearing in the basis (8.94) is 2, so acting on it with $T(u)$ produces a power of order 3, which we do not know how to reduce into smaller ones. However, the transfer matrices $T(\xi_j)$ are invertible[2], so the original vectors constructing the basis (8.94) can be rewritten conveniently

$$\exists \langle \bar{S}| \in \mathscr{H}^*, \quad \langle S| = \langle \bar{S}| \prod_{j=1}^{N} T(\xi_j). \tag{8.137}$$

Therefore,

$$\forall \vec{\mathbf{h}} \in \mathcal{I}, \quad \langle \vec{\mathbf{h}}| = \langle \hat{S}| \prod_{j=1}^{N} T(\xi_j)^{1+h_j}. \tag{8.138}$$

The powers in $T(\xi_j)$ now run from 1 to 3 in this expression of the $\langle \vec{\mathbf{h}}|$; acting on it with $T(u)$, this produces terms of order 4, which can be in principles rewritten in terms of order 3 by (8.136).

In the article [LV1], this idea proved fruitful and allowed us to verify the above conjecture in the simpler case where $k_1 = 0$. There, we give a characterization of the spectrum by discrete finite difference equation, as well as by a functional quantum spectral curve. We then compare the results with the ones of the nested Bethe ansatz, and prove its completeness as a corollary of the completeness of the quantum spectral curve. We will only recall the results here, since the proofs are made along the lines of the non-graded case, and are made in details in [LV1].

8.4.1 Non-invertible twist case

Let the twist matrix be

$$\hat{K} = \begin{pmatrix} 0 & 0 \\ 0 & K_2 \end{pmatrix}_{3\times 3} \tag{8.139}$$

with K_2 invertible and diagonalizable with simple spectrum, so that

$$k_1 = 0 \quad \text{and} \quad k_2 \neq k_3, \quad k_2 \neq 0 \neq k_3. \tag{8.140}$$

While one eigenvalue is zero, the vectors (8.94) still form a basis of $\mathscr{H}^*$. The main identity needed here is that, thanks to the inner-boundary condition (8.136),

$$T_3^{(\hat{K})}(u) = 0, \tag{8.141}$$

since $k_1 = 0$. The fusion equations are thus simpler:

$$\forall j \in [\![1, N]\!], \quad T_1^{(\hat{K})}(\xi_j) T_1^{(\hat{K})}(\xi_j + \eta) = T_2^{(\hat{K})}(\xi_j), \tag{8.142}$$

$$T_1^{(\hat{K})}(\xi_j) T_2^{(\hat{K})}(\xi_j + \eta) = 0. \tag{8.143}$$

The two following characterizations of the spectrum of the transfer matrix $T_1^{(\hat{K})}(u)$ hold.

Theorem 9. *For almost any values of the inhomogeneities ξ_j satisfying the condition* (8.81) *and the twist*

[2] This is shown by the reconstruction of the local operators. The transfer matrices $T(\xi_j)$ are shown to coincide with the twist matrix K_j acting locally on the j-th space, dressed by products of shift operators along the chain. See [104, 106].

matrix $\hat{K}$ of the form (8.140), the spectrum of $T_1^{(\hat{K})}(u)$ coincides with the set of polynomials $\mathrm{Fam}_{1|2}^{N}(K,\{\xi_j\})$, namely

$$\left\{ t_1(u)\in\mathbb{C}[u] \;\middle|\; \exists(x_1,\dots,x_N)\in S_{T^{(\hat{K})}},\quad t_1(u)=-(k_2+k_3)\prod_{j=1}^{N}(u-\xi_j)+\sum_{a=1}^{N}f_a^{(1)}(u)\,x_a \right\}, \tag{8.144}$$

with $S_{T^{(\hat{K})}}$ the set of solutions of the system of N cubic equations in the N unknowns $(x_1,\dots,x_N)$

$$\forall a\in[\![1,N]\!],\quad x_a\left[k_2k_3\,d(\xi_a+\eta)+\sum_{r=1}^{N}f_r^{(2)}(\xi_a+\eta)\,t_1(\xi_r+\eta)\,x_r\right]=0. \tag{8.145}$$

Moreover, $T_1^{(\hat{K})}(u)$ is diagonalizable with simple spectrum, and for any eigenvalue $t_1(u)$ of $T_1^{(\hat{K})}(u)$, the associated unique eigenvector $|t\rangle$ is defined by its wavefunctions in a separate form as

$$\forall\vec{\mathbf{h}}\in\mathcal{I},\quad \langle\vec{\mathbf{h}}|t\rangle=\prod_{j=1}^{N}t_1(\xi_j)^{h_j}. \tag{8.146}$$

Theorem 10. *Under the same conditions as the previous theorem, a polynomial $t_1(u)$ is an eigenvalue of $T_1^{(\hat{K})}(u)$ if and only if there exist a unique polynomial of degree $\mathtt{M}\le N$*

$$\varphi_t(u)=\prod_{j=1}^{N}(u-\nu_j), \tag{8.147}$$

with

$$\nu_a\neq\xi_j\quad \textit{for all}\quad (a,j)\in[\![1,\mathtt{M}]\!]\times[\![1,N]\!], \tag{8.148}$$

such that the triplet $\big(t_1(u),\,t_2(u|\{t_1(\xi_j)\}),\,\varphi_t(u)\big)$ lies on the quantum spectral curve of equation

$$\varphi_t(u-\eta)t_2(u-\eta)+\alpha(u)\varphi_t(u)t_1(u)+\beta(u)\varphi_t(u+\eta)=0. \tag{8.149}$$

We have defined

$$\alpha(u)=\bar{\alpha}\prod_{j=1}^{N}\big(u-2\eta-\xi_j\big)\quad\textit{and}\quad\beta(u)=\alpha(u)\alpha(u+\eta), \tag{8.150}$$

with $\bar{\alpha}=k_2$ or k_3, a non-zero eigenvalue of the twist matrix $\hat{K}$.

Moreover, the unique associated eigenvector $|t\rangle$ is defined by its wavefunctions in a separate form as

$$\forall\vec{\mathbf{h}}\in\mathcal{I},\quad \langle\vec{\mathbf{h}}|t\rangle=\prod_{j=1}^{N}\alpha(\xi_j+\eta)^{h_j}\varphi_t(\xi_j+\eta)^{h_j}\varphi_t(\xi_j)^{2-h_j}. \tag{8.151}$$

8.4.2 Comparaison with the NABA results

The spectrum of the $\mathfrak{gl}(1|2)$ fundamental twisted model has already been characterized by NABA techniques [124, 125, 293, 295]. Let

$$Q_1(u)=\prod_{\ell=1}^{L}(u-\lambda_\ell)\quad\text{and}\quad Q_2(u)=\prod_{m=1}^{M}(u-\mu_m) \tag{8.152}$$

be two polynomials of degree $L, M \leq N$, respectively. Eigenvalues of the transfer matrix were shown to be of the form

$$t_1\big(u|\{\lambda_{1\leq\ell\leq L}\},\{\mu_{1\leq m\leq M}\}\big) := \Lambda_1(u)-\Lambda_2(u)-\Lambda_3(u), \tag{8.153}$$

with

$$\Lambda_1(u) = a_1(u)\frac{Q_1(u-\eta)}{Q_1(u)}, \tag{8.154}$$

$$\Lambda_2(u) = a_2(u)\frac{Q_1(u-\eta)Q_2(u+\eta)}{Q_1(u)Q_2(u)}, \tag{8.155}$$

$$\Lambda_3(u) = a_3(u)\frac{Q_2(u-\eta)}{Q_2(u)}, \tag{8.156}$$

and

$$a_1(u) = k_1 a(u) = k_1\prod_{j=1}^{N}(u-\xi_j+\eta) \quad \text{and} \quad \frac{a_2(u)}{k_2} = \frac{a_3(u)}{k_3} = d(u) = \prod_{j=1}^{N}(u-\xi_j), \tag{8.157}$$

if the Bethe roots $\{\lambda_\ell\}$ and $\{\mu_m\}$ satisfy the $L+M$ Bethe equations

$$\frac{a_1(\lambda_\ell)}{a_2(\lambda_\ell)} = \frac{Q_2(\lambda_\ell+\eta)}{Q_2(\lambda_\ell)}, \tag{8.158}$$

$$-\frac{k_2}{k_3} = \frac{Q_1(\mu_m)Q_2(\mu_m-\eta)}{Q_1(\mu_m-\eta)Q_2(\mu_m+\eta)}. \tag{8.159}$$

When this is the case, the apparent pole in the expression (8.153) are regularized, retrieving a polynomial form for the eigenvalue of $T_1(u)$.

The spectral curve characterization stated in Theorem 10 allows to prove the following corollary on the completeness of the NABA description.

Corollary 1. *For inhomogeneities satisfying the condition* (8.81)*, a polynomial $t_1(u)$ is an eigenvalue of $T_1(u)$ if and only if there are Bethe roots $\{\lambda_\ell\}$, $\{\mu_m\}$ solutions to the system of Bethe equations* (8.158)–(8.159) *such that*

$$t_1(u) \equiv t_1\big(u|\{\lambda_\ell\},\{\mu_m\}\big). \tag{8.160}$$

Moreover, the associated solution is unique, and satisfy the following conditions:

$$\{\lambda_\ell\}\subseteq\{\xi_j\} \quad \textit{and} \quad \{\mu_m\}\cap\big\{\{\xi_j\}\cup\{\xi_j+\eta\}\big\} = \emptyset. \tag{8.161}$$

8.5 The Hubbard model case

The Hubbard model is a one-dimensional lattice model where fermionic particle of spin $1/2$ hops from site to site. One can think of it as a chain of hydrogen nuclei, which can accommodate up to two electrons of opposite spin per site. It is of tremendous importance in condensed matter physics, as the simplest generalization beyond the band theory description of solids which still contains enough complexity to capture non-trivial physical behaviors, such as Mott insulators of high temperature superconductivity for example [126, 255, 296].

8.5.1 Preliminary notations

With periodic boundary conditions, its Hamiltonian is written

$$H = -\sum_{j=1}^{N}\sum_{\sigma=\uparrow,\downarrow}\left(c^{\dagger}_{j,\sigma}c_{j+1,\sigma} + c^{\dagger}_{j+1,\sigma}c_{j,\sigma}\right) + U\sum_{j=1}^{N} n_{i,\uparrow}n_{i,\downarrow}, \tag{8.162}$$

where $c^{(\dagger)}_{j,\sigma}$ are the Fermi operators of spin σ at site j, and $n_{j,\sigma} = c^{\dagger}_{j,\sigma}c_{j,\sigma}$ are the number operators. The constant U is some coupling constant. The two terms are seen to correspond to a kinetic terms, allowing the hopping of particles along the chain, and a potential term of an on-site interaction favoring or disfavoring the presence of opposite spin particles regarding the sign of U.

It has been shown to be linked with the Yangian extension of the centrally extended superalgebra $\mathfrak{psu}(2|2)$ [255, 307, 308].

Its description by the quantum inverse scattering method relies on the Shastry R-matrix [see chapter 12 and 13 in 255, for details on its construction]. With $\eta = -2iU$, let

$$R_{ab}(u) := \begin{pmatrix} \cos u & 0 & 0 & 0 \\ 0 & \sin u & 1 & 0 \\ 0 & 1 & \sin u & 0 \\ 0 & 0 & 0 & \cos u \end{pmatrix} \in \mathrm{End}(V_a \otimes V_b), \tag{8.163}$$

where $V_a \simeq \mathbb{C}^2 \simeq V_b$, and

$$\begin{aligned}\hat{R}_{12,34}(u|v) := {} & R_{13}(u-v)R_{24}(u-v) \\ & - \frac{\sin(u-v)}{\sin(u+v)}\tanh\big(h(u)+h(v)\big)\, R_{13}(u+v)\,\sigma^y_1\, R_{24}(u+v)\,\sigma^y_2,\end{aligned} \tag{8.164}$$

where σ^y are the usual Pauli matrices, and

$$h(u) := \frac{1}{2}\,\mathrm{arcsinh}\left(\frac{i\eta}{2}\sin 2u\right). \tag{8.165}$$

Then the Shastry R-matrix reads

$$R_{12,34}(u|v) := I_{12}(h(u))\, I_{34}(h(v))\, \hat{R}_{12,34}(u|v)\, I_{12}(-h(u)) I_{34}(-h(v)), \tag{8.166}$$

where

$$I_{12}(h) := \cosh(h/2) + \sigma^y_1\sigma^y_2\, \sinh(h/2) = e^{\sigma^y_1\sigma^y_2\, h/2}. \tag{8.167}$$

The Shastry R-matrix (8.166) satisfy the Yang–Baxter relation in $\mathrm{End}\,(V_A \otimes V_B \otimes V_C)$

$$R_{A,B}(u|v)R_{A,C}(u|\xi)R_{B,C}(v|\xi) = R_{B,C}(v|\xi)R_{A,C}(u|\xi)R_{A,B}(u|v), \tag{8.168}$$

where the uppercase roman letters A, B, C represent couples of integers. For example, $A = (1,2)$, $B = (3,4)$ and $C = (5,6)$, and then

$$V_A = V_1 \otimes V_2 \simeq \mathbb{C}^4, \quad \text{etc.} \tag{8.169}$$

Properties of the Shastry matrix are summed up in section 4.1 of [LV1]. Let us recall specifically the

following invariance: for any matrix $K \in \mathrm{End}\,\mathbb{C}^4$ of the form

$$K(a,\alpha,\beta,\gamma) = \delta_{a,1}\begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & \gamma & 0 \\ 0 & 0 & 0 & \beta\gamma/\alpha \end{pmatrix} + \delta_{a,2}\begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & 0 & \beta & 0 \\ 0 & \gamma & 0 & 0 \\ 0 & 0 & 0 & \beta\gamma/\alpha \end{pmatrix} + \delta_{a,3}\begin{pmatrix} 0 & 0 & 0 & \alpha \\ 0 & \beta & 0 & 0 \\ 0 & 0 & \gamma & 0 \\ \beta\gamma/\alpha & 0 & 0 & 0 \end{pmatrix} + \delta_{a,4}\begin{pmatrix} 0 & 0 & 0 & \alpha \\ 0 & 0 & \beta & 0 \\ 0 & \gamma & 0 & 0 \\ \beta\gamma/\alpha & 0 & 0 & 0 \end{pmatrix}, \tag{8.170}$$

where α, β, γ are generic complex number, then

$$R_{A,B}(u|v)\,K_A\,K_B = K_B\,K_A\,R_{A,B}(u|v). \tag{8.171}$$

This is the scalar version of the Yang–Baxter equation for the R-matrix (8.166), and matrices of this form can be used to twist the model and change is boundary conditions. Some comments about the simplicity of (8.170) regarding the value of a are made in [LV1].

The monodromy matrix of the twisted Hubbard model of length N of Hilbert space $\mathscr{H} = \bigotimes_{j=1}^{N} V_{A_j}$ is

$$M_A^{(K)}(u) := K_A R_{AA_N}(u|\xi_N)\ldots R_{AA_1}(u|\xi_1) \ \in \mathrm{End}(V_A \otimes \mathscr{H}). \tag{8.172}$$

The transfer matrix is the trace over the auxiliary space $V_A \simeq \mathbb{C}^4$

$$T^{(K)}(u) := \mathrm{tr}_A\, M_A^{(K)}(u), \tag{8.173}$$

and defines a family of commuting operators thanks to (8.168).

8.5.2 Separate basis

The family of covectors (8.94) can be constructed and proved to be a basis for the Hubbard model as well.

Theorem 11. *Suppose K is diagonalizable with simple spectrum. We note W_K the invertible matrix diagonalizing K by*

$$K = W_K\,\mathrm{Diag}(\mathrm{k}_1,\mathrm{k}_2,\mathrm{k}_3,\mathrm{k}_4)\,W_K^{-1}. \tag{8.174}$$

For almost any choice of the twist matrix $K(a,\alpha,\beta,\gamma)$, the transfer matrix $T^{(K)}(u)$ is diagonalizable with simple spectrum, and the family of covectors

$$\forall \vec{\mathbf{h}} \in [\![0,3]\!]^N\,, \quad \langle \vec{\mathbf{h}}| := \langle S| \prod_{j=1}^{N} T^{(K)}(\xi_j)^{h_j} \tag{8.175}$$

forms a basis of $\mathscr{H}^*$ for almost any choice of $\langle S| \in \mathscr{H}^*$. In particular, $\langle S|$ can have the following tensor product form

$$\langle S| = \bigotimes_{j=1}^{N} (x,y,z,w)_a\,\Gamma_W^{-1} \quad \textit{with} \quad \Gamma_W := \bigotimes_{j=1}^{N} W_{K,j}, \tag{8.176}$$

simply asking that $xyzw \neq 0$.

Proof. As always, the proof relies on the identity

$$T^{(K)}(\xi_j) = R_{A_jA_{j-1}}(\xi_j|\xi_{j-1})\dots R_{A_jA_1}(\xi_j|\xi_1)\, K_{A_j}\, R_{A_jA_N}(\xi_j|\xi_N)\dots R_{A_jA_{j+1}}(\xi_j|\xi_{j+1}). \tag{8.177}$$

The other key points is algebraic dependency of the determinant of the matrix whose rows are elements of the family (8.175) expressed in the canonical basis of $\mathscr{H}^*$ in the parameters η and e^{ξ_j}. Hence, it is sufficient to show it is non-zero in some limits of these parameters to show it is non-zero almost everywhere. Here, it is possible to do so in the $\eta = 0$ point. Then,

$$h(u,0) = 0 \quad \text{or} \quad \frac{i\pi}{2} \mod i\pi. \tag{8.178}$$

This implies

$$\tanh(h(u,0) + h(v,0)) = 0, \tag{8.179}$$

so that the Shastry R-matrix (8.166) reduces to the product of two XX R-matrices:

$$R_{A,B}(u-v) = R_{1,3}(u-v)R_{2,4}(u-v), \tag{8.180}$$

where $A = (1,2)$ and $B = (2,4)$. Having said all of this, it is possible to make the same reasoning as in the proof of Theorem 8 to factorize the determinant of the aforementioned matrix into product of N determinant of 4×4 matrices, which are all non-zero due to K having simple spectrum.

Then, in the case $\eta = 0$ as well, one can prove that $\langle t|t\rangle \neq 0$ for any eigenvector $|t\rangle$ of the transfer matrix, using the same argument as in the proof of lemma 1. This implies the diagonalizability and simplicity of the transfer matrix for almost all values of the parameters. □

Currently, we have not yet tried to use this basis to tackle the spectral problem of the Hubbard model. Because the underlying symmetry algebra is bigger and has more structure than, say, the $\mathfrak{gl}(1|2)$ one, there are some subtleties in the fusion procedure, and in particular have an intricate dependency on the spectral parameters [309]. Henceforth, we do not have the full knowledge of the action of the transfer matrix over the vectors (8.175)—a necessity for the full characterization of the spectral problem.

Still, with such a basis, one can construct the unique eigenvector associated to an eigenvalue obtained by other means, say the coordinate Bethe Ansatz for example. Indeed, the construction of the eigenvectors of the Hubbard model from the NABA procedure is intricate. First, the monodromy is a 4×4 matrix, meaning the Yang–Baxter algebra of the model is generated by 16 generators, linked by 256 commutation relations, which is not convenient. Besides, the parametrization of the R-matrix by the spectral parameters u and v is inconvenient; in particular (8.166) is not of difference type. Moreover, hints of hidden 6-vertex structures or link with the t-J model suggests (8.166) is not the most adapted object to perform an algebraic description of the Hubbard model. An ABA for the Hubbard model has still been obtained by Ramos and Martin [307, 310–312], with "an unusual recursive construction of the eigenvectors" [255]. See section 12.6 of the monograph [255] for a detailed account of the reasons for the difficulty to perform ABA for the Hubbard model.

This SoV method provides an alternative way to construct them, which could be of great help in selecting the solution of the NABA equations which are indeed in the spectrum of the transfer matrix. Besides, this could already be useful for numerical methods dealing with finite chains with quite a few sites, where direct diagonalization is out of reach.

Conclusion

The work of this thesis belongs to the studies of quantum integrable models on the one-dimensional lattice. It focuses on the development of a new quantum separation of variables method for these particular models.

Chapters 2 and 3 have been devoted to a detailed introduction to classical and quantum integrable models, and the common techniques used in their studies. The Bethe Ansatz was then described in its algebraic formulation, and the results obtained from ABA techniques, ranging from the spectrum to the correlation functions, were reviewed in chapter 4. With chapter 5, the SoV techniques were introduced in the classical and quantum cases, from first principles to Sklyanin contributions in the context of integrable models. The recent take on SoV proposed in [225] was described in details in chapter 6, being the base techniques of the contributions of the manuscript. Chapters 7 and 8 form the core of the thesis and gather my personal contributions to quantum SoV techniques applied to respectively higher-rank $\mathfrak{gl}(3)$ fundamental chains and supersymmetric $\mathfrak{gl}(m|n)$ fundamental chains.

*
* *

Chapter 7 sums up results of [LV2]. It introduces the $\mathfrak{gl}(3)$ fundamental spin chains, and details how canonical left and right SoV bases obtained following methods of chapter 6 are not orthogonal from each others, but rather pseudo-orthogonal. More precisely, while many ket states of the right SoV basis find their orthogonal state among the bras of the left SoV basis, many others states see their orthogonal states being constructed as a linear combination over a well-defined subset of the left SoV basis. This creates an additional complexity in the computation of scalar products and form factors. The main result of this chapter is two-fold. First, non-diagonal elements of the pseudo-orthogonal SoV measure are characterized exactly in terms of the diagonal ones through recursive formulas. The diagonal elements are computable, and we computed them exactly. Secondly, a novel operator $\mathbb{T}(u)$ is introduced, defined by the giving of its eigenvalues and eigenvectors. By construction, it is shown that the left and right SoV bases constructed from it are orthogonal, greatly reducing the computational cost of scalar products and form factors in the SoV context.

An obvious improvement would be the algebraic construction of the $\mathbb{T}(u)$ operator from the basic blocks of the model, for example from the R matrix or the monodromy. Besides, one can hope for an exact derivation of the non-diagonal elements of the SoV measure by solving for them their defining recursive formulas.

As already stated in the main text, a parallel line of research has been conducted by Cavaglia, Gromov, Levkovich–Maslyuk, Ryan and Volin since 2018 in a series of publications [236–238, 240, 291]. The most recent one [240] is a pedagogical and almost self-contained exposition of their approach on left and right separate bases and their SoV measure towards scalar products, form factors and correlation functions for $\mathfrak{gl}(n)$ models. They explicit the block diagonal form of the SoV measure and exhibit a lexicographic ordering that makes it triangular.

Relying on an orthogonality relation in the form of an integral, they propose an interesting indirect approach to get integral representations of scalar products of on-shell/off-shell Bethe states, as well as some matrix elements, bypassing the explicit computation of the SoV measure from the overlaps of the separate bases vectors.

Still, these publications do not develop the idea of separate bases whose measure is diagonal. One

could argue that most of the complexity of these methods may be drastically reduced with the algebraic construction of an adapted $\mathbb{T}(u)$ operator.

*
* *

Chapter 8 focuses on the $\mathfrak{gl}(\mathfrak{m}|\mathfrak{n})$ fundamental chains and sums up results of [LV1]. Having introduced the graded formalism and models, it demonstrates how separate bases can be constructed following the same ideas as in the non-graded cases, so that separation of variables can be envisioned in the same way for $\mathfrak{gl}(\mathfrak{m}|\mathfrak{n})$ models. In this SoV setting, a closure relation is necessary to enforce constraints on a family of polynomials so that it forms the spectrum of the transfer matrix. In the non-graded case, this relation is identified as one of the fusion relations linking altogether the hierarchy of fused transfer matrices. The transfer matrix obtained by total antisymmetrization over the auxiliary space is actually a central element of the operator algebra, namely the quantum determinant of the monodromy. These fusion relations thus play the role of a functional characterization of the eigenvalues of the transfer matrix, just as the characteristic polynomial does for a numerical matrix. In the graded case however, there is no truncation of fusion hierarchy in either direction of fusion. However, there exists non-trivial relations between different transfer matrices, constructed by different symmetrization procedures. The first of these relations, the Inner Boundary Condition (IBC), is conjectured to play the role of a closure relation for the transfer matrices in fundamental $\mathfrak{gl}(\mathfrak{m}|\mathfrak{n})$ models, namely it allows a functional characterization of the spectrum of the transfer matrix. It is shown explicitly how the IBC succeeds to do so for the $\mathfrak{gl}(1|2)$ fundamental chain with specific quasi-periodic boundary conditions, and the results of the SoV approach are compared with the known NABA ones.

The natural direction of research here is to pursue the likely development of a proof of conjecture 1, and the extension of such a result to general $\mathfrak{gl}(\mathfrak{m}|\mathfrak{n})$ spin chains. This requires either very heavy algebraic computations involving multiple Lagrange interpolation in the spirit of [234], or a better comprehension of the structure of Bethe algebra so that privileged objects for a simple closure relation may be identified. The latter is very reminiscent of the $\mathbb{T}(u)$ idea for the $\mathfrak{gl}(3)$ case. Indeed, the integrability property is, after all, all about conserved quantities. It is expected some choices of conserved quantities are much more adapted to certain computation, like the $\mathbb{T}(u)$ for a diagonal SoV measure in the $\mathfrak{gl}(3)$ case, or a certain choice of operator for the closure relation of $\mathfrak{gl}(\mathfrak{m}|\mathfrak{n})$ models.

Another interesting direction of research would be to extend results from Ryan and Volin [236, 237] to the supersymmetric $\mathfrak{gl}(\mathfrak{m}|\mathfrak{n})$ case. The authors have developed the construction of SoV bases, in the vein of Maillet and Niccoli method [225], for $\mathfrak{gl}(\mathfrak{n})$ spin chains with different representations at each site. The vectors of the separate basis are constructed by a recursive embedding of $\mathfrak{gl}(k)$ to $\mathfrak{gl}(k+1)$ spin chains, controlled by Gelfan'd–Tsetlin patterns, which reveals itself to have the form (6.29). They also propose a construction of the momenta conjugated to the separate variables—namely the operator of shift in the spectrum of SoV coordinates—in Wronksian forms. It would be nice to investigate these operators, say for the fundamental $\mathfrak{gl}(3)$ model, and compare them to the ones found by Sklyanin [206] to find whether they do perform the shift on the whole spectrum [225]. More generally, the extension of results of [236, 237] to $\mathfrak{gl}(\mathfrak{m}|\mathfrak{n})$ models, as well as the developments of the algebraic machinery necessary to prove them, would be beneficial.

*
* *

These developments pave the way towards the computation of correlation function in the new SoV setting for higher rank and supersymmetric spin chains. With the growing interests in quantum separation of variables and its surroundings [175, 221, 222, 313–315], we hope the theory of quantum integrable models will benefit from the rise of more sophisticated SoV techniques. A possible and very desirable

outcome may be a precise definition of quantum integrability founded on the existence and structure of quantum separate variables.

A Appendix
Yangians and quantum groups

The $\mathscr{Y}(\mathfrak{gl}(n))$ Yangian; its RTT presentation The Yangian $\mathscr{Y}(\mathfrak{gl}(n))$ of the Lie algebra $\mathfrak{gl}(n)$ is an associative algebra over $\mathbb{C}$ of infinite dimension [117, 256], generated by the countable generators

$$t_{ij}^{(r)} \quad \text{for} \quad i,j \in [\![1,n]\!] \quad \text{and} \quad r \in \mathbb{N}, \tag{A.1}$$

and relations

$$[t_{ij}^{(r+1)}, t_{kl}^{(r)}] - [t_{ij}^{(r)}, t_{kl}^{(s+1)}] = t_{kj}^{(s)} t_{il}^{(r)} - t_{kj}^{(r)} t_{il}^{(s)}, \tag{A.2}$$

with $t_{ij}^{(0)} = \delta_{ij}$. Using the formal series

$$t_{ij}(u) := \delta_{ij} + t_{ij}^{(1)} u^{-1} + t_{ij}^{(2)} u^{-2} + \cdots = \sum_{r=1}^{+\infty} t_{ij}^{(r)} u^{-r} \in \mathscr{Y}(\mathfrak{gl}(n))[[u^{-1}]], \tag{A.3}$$

the above relations are rewritten as the n^2 equalities

$$(u-v)\big[t_{ij}(u), t_{kl}(v)\big] = t_{kj}(v) t_{il}(u) - t_{kj}(u) t_{il}(v). \tag{A.4}$$

Defining the $n \times n$ matrix $\Gamma(u) \in \mathrm{End}(\mathbb{C}^n) \otimes \mathscr{Y}(\mathfrak{gl}(n))[[u^{-1}]]$[1], whose ij-entry is $t_{ij}(u)$, i.e.

$$\Gamma(u) := \sum_{i,j=1}^{n} e_{ij} \otimes t_{ij}(u) \tag{A.5}$$

the relations (A.4) are packed nicely in the Yang-Baxter equation

$$R(u-v)\Gamma_1(u)\Gamma_2(v) = \Gamma_2(v)\Gamma_1(u)R(u-v), \tag{A.6}$$

where

$$R(u) = 1 + \mathcal{P} u^{-1}, \tag{A.7}$$

$\mathcal{P}$ the permutation operator, and $\Gamma_1(u) = \Gamma(u) \otimes 1$, $\Gamma_2(u) = 1 \otimes \Gamma(u)$. This is the so-called RTT realization of the $\mathscr{Y}(\mathfrak{gl}(n))$ Yangian algebra ($T(u)$ being the usual notation in place of $\Gamma(u)$). Note that there exist two other realizations [117, 256], both introduced by Drinfel'd: the Drinfel'd or first realization [316] and the Chevalley–Serre or second realization [317].

Hopf algebra structure The Yangian $\mathscr{Y}(\mathfrak{gl}(n))$ is a Hopf algebra [318] with comultiplication

$$\Delta : t_{ij}(u) \mapsto \sum_{k=1}^{n} t_{ik}(u) \otimes t_{kj}(u), \tag{A.8}$$

antipode

$$S : \Gamma(u) \mapsto \Gamma^{-1}(u) \tag{A.9}$$

[1] We choose the unconventional notation $\Gamma(u)$, because $T(u)$ is already used for the transfer matrix throughout the whole manuscript.

and counit $\varepsilon : \Gamma(u) \mapsto 1$. The coproduct Δ is also given by the relation

$$\Delta : \Gamma(u) \mapsto \Gamma(u)_{1[1]}(u)\Gamma_{1[2]}(u), \tag{A.10}$$

where the left-hand side is in $\mathrm{End}(\mathbb{C}^n) \otimes \mathscr{Y}(\mathfrak{gl}(n))[[u^{-1}]] \otimes \mathscr{Y}(\mathfrak{gl}(n))[[u^{-1}]]$, and the indexed label the copies of $\mathrm{End}(\mathbb{C}^n)$ and $\mathscr{Y}(\mathfrak{gl}(n))[[u^{-1}]]$. More precisely, when working in $\mathrm{End}(\mathbb{C}^n)^{\otimes m} \otimes \mathscr{Y}(\mathfrak{gl}(n))[[u^{-1}]]^{\otimes n}$, we note

$$\Gamma_{a[b]}(u) := \sum_{i,j=1}^{n} 1^{\otimes(a-1)} \otimes e_{ij} \otimes 1^{\otimes(m-a)} \otimes 1^{\otimes(b-1)} \otimes t_{ij}(u) \otimes 1^{\otimes(n-b)}, \tag{A.11}$$

for $a \in [\![1, m]\!]$, $b \in [\![1, n]\!]$ and $m, n \geq 1$. The Yangian $\mathscr{Y}(\mathfrak{gl}(n))$ is also a Hopf algebra for the *opposite coproduct*

$$\Delta' : \Gamma(u) \mapsto \Gamma(u)_{1[2]}(u)\Gamma(u)_{1[1]}(u), \tag{A.12}$$

for the same antipode and counit as above.

Because $\mathscr{Y}(\mathfrak{gl}(n))$ is a Hopf algebra, the RTT relation also holds for $\Delta\Gamma(u)$, namely

$$R(u-v)\ \Gamma_{1[1]}(u)\Gamma_{1[2]}(u)\ \Gamma_{2[1]}(v)\Gamma_{2[2]}(v) = \Gamma_{2[1]}(v)\Gamma_{2[2]}(v)\ \Gamma_{1[1]}(u)\Gamma_{1[2]}(u)\ R(u-v). \tag{A.13}$$

Relationship with $U(\mathfrak{gl}(n))$ The mapping

$$\begin{aligned} \pi : \mathscr{Y}(\mathfrak{gl}(n)) &\longrightarrow U(\mathfrak{gl}(n)) \\ t_{ij}(u) &\longmapsto \delta_{ij} + e_{ij}u^{-1}, \end{aligned} \tag{A.14}$$

defines a surjective algebra homomorphism, while the assignment $e_{ij} \mapsto t_{ij}(1)$ defines an embedding $U(\mathfrak{gl}(n)) \hookrightarrow \mathscr{Y}(\mathfrak{gl}(n))$. The assignment π is called the *evaluation homomorphism*. Thanks to it, any representation of the Lie algebra $\mathfrak{gl}(n)$ can be regarded as a representation of $\mathscr{Y}(\mathfrak{gl}(n))$, and the irreducibility property is preserved.

Center of $\mathscr{Y}(\mathfrak{gl}(n))$ Viewing $\Gamma(u)$ as a $n \times n$ matrix in the auxiliary space $\mathbb{C}^n$, one defines quantum minors as deformation of the matrix minor, with some shifts integer shifts. The quantum minor of maximal rank is called unsurprisingly the *quantum determinant*. Taking any permutation $q \in \mathfrak{S}_n$, the quantum determinant as the expressions

$$\text{q-det}\,\Gamma(u) = \mathrm{sgn}\, q \sum_{p \in \mathfrak{S}_n} \mathrm{sgn}\, p \cdot t_{p(1),q(1)}(u) \dots t_{p(n),q(n)}(u-n+1) \tag{A.15}$$

$$= \mathrm{sgn}\, q \sum_{p \in \mathfrak{S}_n} \mathrm{sgn}\, p \cdot t_{q(1),p(1)}(u-n-1) \dots t_{q(n),p(n)}(u). \tag{A.16}$$

In particular these simplify for $q = \mathrm{id}$. The quantum determinant can also be written as

$$\begin{aligned} \text{q-det}\,\Gamma(u) &= \mathrm{tr}_{1\dots n}\ A_n \Gamma_1(u) \dots \Gamma_n(u-n+1) \\ &= \mathrm{tr}_{1\dots n}\ \Gamma_n(u-n+1) \dots \Gamma_1(u) A_n, \end{aligned} \tag{A.17}$$

where

$$A_n \equiv P^-_{1\dots n} = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \mathrm{sgn}\, \sigma \cdot P_\sigma \tag{A.18}$$

is the antisymmetrizer on the n copies $1, \dots, n$ of the auxiliary space $\mathbb{C}^n$.

For $Y(\mathfrak{gl}(2))$, this gives four equivalent expressions of the quantum determinant

$$\begin{aligned}\text{q-det}\,\Gamma(u) &= t_{11}(u)t_{22}(u-1) - t_{21}(u)t_{12}(u-1)\\ &= t_{22}(u)t_{11}(u-1) - t_{12}(u)t_{21}(u-1)\\ &= t_{11}(u-1)t_{22}(u) - t_{12}(u-1)t_{21}(u)\\ &= t_{22}(u-1)t_{11}(u) - t_{21}(u-1)t_{12}(u).\end{aligned} \tag{A.19}$$

The coefficients of the formal series expansion of q-det $\Gamma(u)$ are shown to be central elements of the Yangian $\mathscr{Y}(\mathfrak{gl}(\mathfrak{n}))$. Moreover, they are algebraically independent and generate the center of $\mathscr{Y}(\mathfrak{gl}(\mathfrak{n}))$.

Bethe subalgebras and maximal abelian subalgebras Let K be a $\mathfrak{n}\times\mathfrak{n}$ complex matrix. For any $a\in[\![1,\mathfrak{n}]\!]$ set

$$\tau_a(u,K) := \mathrm{tr}_{1\dots a}\; A_a K_1 \dots K_a \Gamma_1(u)\dots\Gamma_a(u-a+1), \tag{A.20}$$

with $A_a \equiv P^-_{1\dots a}$ the antisymmetrizer on a copies of the auxiliary space. In particular, $\tau_1(u,\mathrm{id}) = \Gamma(u)$ and $\tau_{\mathfrak{n}}(u,\mathrm{id}) = \text{q-det}\,\Gamma(u)$. All the coefficients of the $\mathfrak{n}$ series $\tau_1(u,K),\dots,\tau_{\mathfrak{n}}(u,K)$ commute and generate a commutative subalgebra of $\mathscr{Y}(\mathfrak{gl}(\mathfrak{n}))$. Similarly, the coefficients of the series

$$\sigma_a(u,K) := \mathrm{tr}_{1\dots a}\; A_a \Gamma_1(u)\dots\Gamma_a(u-a+1)K_{a+1}\dots K_{\mathfrak{n}} \tag{A.21}$$

also form a commutative subalgebra. They are called the *Bethe subalgebras*. Moreover, if K has simple spectrum, then the coefficients in the $\sigma_a(u,K)$ series are algebraically independent and generate a maximal commutative subalgebra of $\mathscr{Y}(\mathfrak{gl}(\mathfrak{n}))$. Its image under the evaluation homomorphism is also a maximal commutative subalgebra of $U(\mathfrak{gl}(\mathfrak{n}))$.

Evaluation representations A theory of highest weight representation can be constructed for the Yangian $\mathscr{Y}(\mathfrak{gl}(\mathfrak{n}))$. Among the finite-dimensional highest weight representations, the *evaluation representations* are the ones obtained from highest weight representation of $\mathfrak{gl}(\mathfrak{n})$ by the evaluation homomorphism. For any complex number ξ, define the generalized evaluation homomorphism as the assignment

$$\pi_\xi : t_{ij}(u) \mapsto \delta_{ij} + \frac{e_{ij}}{u-\xi} \tag{A.22}$$

from $\mathscr{Y}(\mathfrak{gl}(\mathfrak{n}))$ to $U(\mathfrak{gl}(\mathfrak{n}))$ that produces an additional shift in the spectral parameter.

For (ρ,V) a highest weight representation of $\mathfrak{gl}(\mathfrak{n})$ of highest weight $\lambda=(\lambda_1,\dots,\lambda_n)$, the evaluation representation

$$\mathrm{ev}_{\lambda,\xi} = \rho\circ\pi_\xi \tag{A.23}$$

is a highest weight representation of $\mathscr{Y}(\mathfrak{gl}(\mathfrak{n}))$ over the space V. Its highest weight is the $\mathfrak{n}$-tuple of formal series in u^{-1} $(\lambda(u))$ whose components are

$$\lambda_i(u) = 1 + \frac{\lambda}{u-\xi}, \quad i\in[\![1,\mathfrak{n}]\!]. \tag{A.24}$$

The highest weight vector ζ of the representation $(\mathrm{ev}_{\lambda,\xi},V)$ is the same as the one of the representation (ρ,V) of $U(\mathfrak{gl}(\mathfrak{n}))$, and one has

$$\begin{aligned}&\mathrm{ev}_{\lambda,\xi}(t_{ii}(u))\zeta = \lambda_i(u)\zeta \quad \text{for} \quad i\in[\![1,\mathfrak{n}]\!],\\ &\mathrm{ev}_{\lambda,\xi}(t_{ij}(u))\zeta = 0 \qquad\quad \text{for} \quad 1\le i<j\le\mathfrak{n}.\end{aligned} \tag{A.25}$$

The quantum determinant q-det $\Gamma(u)$ is in the center of the Yangian, so it acts as a multiplication by a

scalar on V, determined by

$$\text{q-det}\,\Gamma(u)|_V = \lambda_1(u)\dots\lambda_{\mathfrak{n}}(u-\mathfrak{n}+1). \tag{A.26}$$

It is customary to abuse of the notation q-det $\Gamma(u)$ to denote this scalar quantity as well.

Consider N highest weight evaluation representations

$$\left(\text{ev}_{\lambda^{(k)},\xi_k}, V_k\right), \quad k \in [\![1,N]\!]\,. \tag{A.27}$$

We note

$$\text{ev}_{\lambda,\{\xi\}} := \bigotimes_{k=1}^{N} \text{ev}_{\lambda^{(k)},\xi_k} \tag{A.28}$$

and

$$\zeta := \zeta_1 \otimes \cdots \otimes \zeta_{\mathfrak{n}} \tag{A.29}$$

the vector of $V_1 \otimes \cdots \otimes V_N$ constructed by tensorization of the highest weight vector of each representation. The submodule $\text{ev}_{\lambda,\{\xi\}}(\mathscr{Y}(\mathfrak{gl}(\mathfrak{n})))\zeta$ of $V_1 \otimes \cdots \otimes V_N$ formed by the descendant states of ζ is a highest weight representation of $\mathscr{Y}(\mathfrak{gl}(\mathfrak{n}))$ as well, with highest weight vector ζ and highest weight $(\lambda_1(u),\dots\lambda_N(u))$, with

$$\lambda_i(u) = \prod_{k=1}^{N}\left(1 + \frac{\lambda_i^{(k)}}{u-\xi_k}\right). \tag{A.30}$$

Spin chains as representations of Yangian algebras Highest weight evaluation representations of Yangian $\mathscr{Y}(\mathfrak{gl}(\mathfrak{n}))$ over a Hilbert space define integrable models. Consider N spaces $V_1,\dots,V_N$, $N \geq 2$, each equipped with a finite irreducible highest weight representation $(\rho_{\lambda^{(k)}}, V_k)$, $k \in [\![1,N]\!]$, of $\mathfrak{gl}(2)$, where $\lambda^{(k)}$ is some Young diagram $\lambda^{(k)} = (\lambda_1^{(k)},\dots,\lambda_{\mathfrak{n}}^{(k)})$ labelling the representation.

The evaluation representation at each site gives the local Lax matrix $L_{0k}(u)$ which is in $\text{End}(\mathbb{C}^{\mathfrak{n}}) \otimes V_k$

$$L_{0k}(u) := \text{ev}_{\lambda^{(k)},\xi_k}[\Gamma(u)] = \sum_{i,j=1}^{\mathfrak{n}} e_{ij} \otimes \left(\rho_{\lambda^{(k)}} \circ \pi_{\xi_k}[t_{ij}(u)]\right). \tag{A.31}$$

The representation of the defining RTT relation impose the Yang–Baxter equation for the Lax matrix at site k with the R-matrix (A.7)

$$R_{12}(u-v)L_{1k}(u)L_{2k}(v) = L_{2k}(v)L_{1k}(u)R_{12}(u-v). \tag{A.32}$$

Tensoring these representations, one gets a highest weight evaluation representation of $\mathscr{Y}(\mathfrak{gl}(\mathfrak{n}))$ on the whole Hilbert space $\mathscr{H} = V_1 \otimes \cdots \otimes V_N$ with the monodromy matrix $M(u)$ in $\text{End}(\mathbb{C}^{\mathfrak{n}}) \otimes \mathscr{H}$

$$\begin{aligned} M(u) &:= \text{ev}_{\lambda,\{\xi\}} \circ (\Delta')^{\otimes(N-1)}\left[\Gamma(u)\right] \\ &= \sum_{i,j=1}^{N} e_{ij} \otimes \sum_{p_1,\dots,p_{N-1}=1}^{\mathfrak{n}} \left(\rho_{\lambda^{(1)}} \circ \pi_{\xi_1}[t_{p_1 j}(u)]\right)\dots\left(\rho_{\lambda^{(N)}} \circ \pi_{\xi_N}[t_{i p_{N-1}}(u)]\right), \end{aligned} \tag{A.33}$$

which identifies to the matrix product of Lax matrices in the auxiliary space

$$M(u) = L_{0N}(u)\dots L_{01}(u). \tag{A.34}$$

$M(u)$ verifies the Yang–Baxter equation with the R-matrix (A.7)

$$R_{12}(u-v)M_{1k}(u)M_{2k}(v) = M_{2k}(v)M_{1k}(u)R_{12}(u-v) \tag{A.35}$$

as a direct consequence of the defining RTT relation (A.6) by the representation on $\mathscr{H}$.

The conventions between this mathematical presentation and the spin chains literature differ a bit. First we introduce the deformation parameter η explicitly by a rescaling of the spectral parameter (and inhomogeneities)

$$u \to \frac{u}{\eta} \quad \text{and} \quad \xi_k \to \frac{\xi_k}{\eta}. \tag{A.36}$$

Then we multiply the monodromy matrix by the scalar factor

$$\prod_{k=1}^{N}(u-\xi_k). \tag{A.37}$$

Equivalently, this amounts to a redefinition of the evaluation homomorphism π_ξ. Now the monodromy is polynomial of $\mathrm{End}(\mathscr{H})[u]$, and is written

$$M(u) = (u-\xi_1+\eta\Pi_{01})\dots(u-\xi_N+\eta\Pi_{0N}), \tag{A.38}$$

where

$$\Pi_{0k} = \sum_{i,j=1}^{\mathfrak{n}} e_{ij}\otimes\rho_{\lambda^{(k)}}[e_{ji}]. \tag{A.39}$$

It still satisfies the Yang–Baxter equation, the YBE is homogeneous. It is customary to use the R-matrix

$$R(u) = u+\eta\mathcal{P}. \tag{A.40}$$

The scalar value of the quantum determinant is now

$$\text{q-det}\, M(u) = \prod_{i=1}^{\mathfrak{n}}\prod_{k=1}^{N}\Big(u-\xi_k+\eta(\lambda_i^{(k)}-i+1)\Big). \tag{A.41}$$

The highest weight of the representation is now given by the polynomials

$$\forall i\in[\![1,\mathfrak{n}]\!],\quad M_{ii}(u)\zeta = \lambda_i(u)\zeta \quad \text{with} \tag{A.42}$$

$$\lambda_i(u) = \prod_{k=1}^{N}\Big(u-\xi_k+\lambda_i^{(k)}\eta\Big). \tag{A.43}$$

Taking the trace of the monodromy over the auxiliary space, this gives the transfer matrix

$$T(u) := \mathrm{tr}\, M(u) \tag{A.44}$$

which is a commuting family of operators of $\mathrm{End}(\mathscr{H})$ thanks to the YBE (A.35)

$$\forall u,v\in\mathbb{C},\quad [T(u),T(v)] = 0. \tag{A.45}$$

Note that the highest weight vector ζ is an eigenstate of the transfer matrix

$$T(u)\zeta = \sum_{i=1}^{\mathfrak{n}}\lambda_i(u)\,\zeta. \tag{A.46}$$

$\mathfrak{gl}(\mathfrak{n})$ and $\mathrm{GL}(\mathfrak{n})$ invariances The one-site Lax matrices are $\mathfrak{gl}(\mathfrak{n})$-invariant: for all $k\in[\![1,N]\!]$

$$\forall i,j\in[\![1,\mathfrak{n}]\!],\quad \big[L_{0k}(u), e_{ij}\otimes 1+1\otimes\rho_{\lambda^{(k)}}(e_{ij})\big]_{\mathscr{H}} = 0. \tag{A.47}$$

This also implies a $\mathtt{GL}(\mathfrak{n})$ group symmetry

$$\forall K \in \mathtt{GL}(\mathfrak{n}), \quad [L_{0k}(u), K \otimes \rho^{\tilde{}}_{\lambda^{(k)}}(K)]_{\mathscr{H}} = 0, \tag{A.48}$$

where $\tilde{\rho}_{\lambda^{(k)}}$ is the induced representation of $\mathtt{GL}(\mathfrak{n})$ by the representation $\rho_{\lambda^{(k)}}$ of $\mathfrak{gl}(\mathfrak{n})$. This invariance extends straightforwardly to the monodromy matrix. Applying a $\mathtt{GL}(\mathfrak{n})$ isomorphism to each site of the chain is equivalent to performing the inverse transformation on the auxiliary space $\mathbb{C}^{\mathfrak{n}}$.

Fusion procedure It is possible to construct $\mathfrak{gl}(\mathfrak{n})$-invariant R-matrices corresponding to higher representations of $\mathfrak{gl}(\mathfrak{n})$ by the *fusion procedure*. It was introduced by Kulish, Reshetikhin and Sklyanin in [227], and explained in details for the $\mathfrak{gl}(3)$ case in [111]. See also [225, 229–231, 319, 320] among the modern literature for their used in the context of integrable system. The general idea is to construct a product of R-matrices along the same auxiliary spaces while fusing along the other, quantum spaces. Let $\lambda = (\lambda_1, \ldots, \lambda_{\mathfrak{n}})$ be a Young diagram corresponding to an irreducible representation $(\rho_\lambda, V_\lambda)$ of $\mathfrak{gl}(\mathfrak{n})$. We write

$$P_\lambda : \otimes_{i=1}^{d} V_i \longrightarrow V_\lambda \tag{A.49}$$

the corresponding projector on the irreducible component V_λ. We note

$$R^{\lambda}_{0V_\lambda}(u) := P_\lambda R_{0d}(u + s_d\eta) \ldots R_{01}(u + s_1\eta) P_\Lambda, \tag{A.50}$$

where shifts $s_1, \ldots, s_d$ are determined by filling each boxes (i, j) of the Young diagram by $-i + j$, and read it from left to right, top to bottom. The R-matrix degenerate in two points

$$R(\pm\eta) = \eta(1 \pm \mathcal{P}) = \pm 2\eta P^{\pm}, \tag{A.51}$$

where P^+ and P^- are respectively the symmetrizer and antisymmetrizer over $V \otimes V$. From them, it is possible to obtain explicit expression of the projector P_λ as a product of shifted R-matrices [319, 321]. For instance, the symmetrizer over n copies of the V space is the ordered product

$$P^{+}_{1\ldots d} = \left(\prod_{\ell=1}^{d-1} \frac{1}{(d+1)!} \right) \overset{\curvearrowleft}{\prod_{\ell=1}^{d-1}} \overset{\curvearrowleft}{\prod_{k=\ell+1}^{d}} R_{\ell k}((k-l)\eta). \tag{A.52}$$

From the choice of the shifts s_i and the R-matrices product form of the projectors P_λ, one proves

$$R_{12}(u-v) R^{\lambda}_{1V_\lambda}(u) R^{\lambda}_{2V_\lambda}(v) = R^{\lambda}_{2V_\lambda}(v) R^{\lambda}_{1V_\lambda}(u) R_{12}(u-v) \tag{A.53}$$

by repeated use of the YBE for the fundamental R-matrix (A.40).

Fusing along the auxiliary space, the fusion procedure allows to construct a wide family of monodromy matrices $M^{\lambda}_{V_\lambda}(u)$ indexed by Young diagrams λ of $\mathfrak{gl}(\mathfrak{n})$. They are intertwined by Yang–Baxter equations of the form

$$R^{\lambda,\mu}_{V_\lambda V_\mu}(u-v) M^{\lambda}_{V_\lambda}(u) M^{\mu}_{V_\mu}(v) = M^{\mu}_{V_\mu}(v) M^{\lambda}_{V_\lambda}(u) R^{\lambda,\mu}_{V_\lambda V_\mu}(u-v), \tag{A.54}$$

where λ, μ are two Young diagrams and $R^{\lambda,\mu}_{V_\lambda V_\mu}(u)$ is the R-matrix constructing by fusing on its two space to the corresponding V_λ, V_μ representations.

The fusion procedure thus generates new transfer matrices

$$T^{\lambda}(u) := \mathrm{tr}_{V_\lambda} M^{\lambda}(u), \tag{A.55}$$

which are shown to form a commuting family by taking the trace of (A.54) over $V_\lambda \otimes V_\mu$:

$$\forall \lambda, \mu \in \mathrm{YD}, \quad \forall u, v \in \mathbb{C}, \quad \left[T^\lambda(u), T^\mu(v)\right] = 0, \tag{A.56}$$

where YD is the set of admissible Young diagrams indexing $\mathfrak{gl}(\mathfrak{n})$ highest weight irreducible representations

$$\mathrm{YD} = \{\, \lambda = (\lambda_1, \ldots, \lambda_\mathfrak{n}) \mid \lambda_1 \geq \cdots \geq \lambda_\mathfrak{n} \geq 0 \,\} \setminus (0, \ldots, 0). \tag{A.57}$$

Hierarchy of fused transfer matrices The transfer matrices $T^\lambda(u)$ are not all independent, but tied together by the fusion relations. They arise from the decomposition in irreducible components of tensor products of highest weight representations of $\mathfrak{gl}(\mathfrak{n})$.

As we already know, the elements $\tau_a(u, K)$ defined in (A.20) generates the Bethe algebra. Their representations over $\mathscr{H}$

$$T_a(u) = \mathrm{ev}_{\lambda,\{\xi\}}[\tau_a(u, K)] \tag{A.58}$$

are the fused transfer matrices corresponding to the pure column Young diagram with a boxes $(1, \ldots, 1, 0, \ldots, 0)$.

All transfer matrices $T^\lambda(u)$ can be expressed in terms of the $T_a(u)$ by means or the Bazhanov–Reshetikhin formula

$$T^\lambda(u) = \det_{1 \leq i,j \leq \lambda_1} T_{(\lambda^\mathrm{T})_j + i - j}(u - \eta(i-1)) \tag{A.59}$$

Also, the bilinear Hirota relations holds between transfer matrices associated to rectangular Young diagrams. Noting (a, b) the rectangular Young diagrams with $a \leq \mathfrak{n}$ rows of length b[2], the other being zeros, it holds

$$T_{(a,b)}(u) T_{(a,b)}(u - \eta) = T_{(a,b-1)}(u) T_{(a,b+1)}(u - \eta) + T_{(a+1,b)}(u) T_{(a-1,b)}(u - \eta). \tag{A.60}$$

The fusion hierarchy for the $T_a^{(K)}(u)$ transfer matrices is given by the relations

$$T_{a-1}^{(K)}(\xi_j - \eta) T^{(K)}(\xi_j) = T_a^{(K)}(\xi_j), \tag{A.61}$$

for $j \in [\![1, N]\!]$ and $a \in [\![1, \mathfrak{n}]\!]$, with $T_\mathfrak{n}^{(K)}(u) \equiv \text{q-det}\, M^{(K)}(u)$. From (A.20) and (A.58), one can see the $T_a(u)$ are polynomials in $\mathrm{End}(\mathscr{H})[u]$ of degree aN. Their leading coefficients is easily identified as the antisymmetric characters of the twist matrix K. Moreover, one can exhibit the $(a-1)N$ central zeroes

$$\forall j \in [\![1, N]\!], \quad \forall r \in [\![1, a-1]\!], \quad T_a(\xi_j + r\eta) = 0. \tag{A.62}$$

This can be shown by bringing two projectors whose product is zero in the expression (A.20) [128, 227]. Therefore, the knowledge of N fusion relations is sufficient to reconstruct the transfer matrix $T_a(u)$ form $T_{a-1}(u)$ by Lagrange interpolation in the inhomogeneities.

$$T_a(u) = \prod_{j=1}^{N} \prod_{r=1}^{a-1} (u - \xi_j - r\eta) \left[T_a^\infty \prod_{j=1}^{N} (u - \xi_j) + \sum_{k=1}^{N} g_k^{(a)}(u) T_{a-1}(\xi_k - \eta) T_1(\xi_k) \right], \tag{A.63}$$

where

$$T_a^\infty = \mathrm{tr}_{1\ldots a}(P_{1\ldots a}^- K_1 \ldots K_a) \tag{A.64}$$

[2] In chapter 8, we use similar but different notations when dealing with transfer matrices associated to rectangular Young diagrams, because the convention chosen there for the orientation of the Young diagrams is different.

and

$$g_k^{(a)}(u) = \prod_{\substack{\ell=1 \\ \ell \neq k}}^{N} \frac{u-\xi_\ell}{\xi_k-\xi_\ell} \cdot \prod_{j=1}^{N} \prod_{r=1}^{a-1} \frac{1}{\xi_k-\xi_j-r\eta}. \tag{A.65}$$

The above discussion still holds for other algebras $\mathcal{A}$, such as $\mathcal{A} = U_q(\widehat{\mathfrak{gl}(\mathfrak{n})})$ which gives generalization of the XXZ model, or $\mathbb{Z}_2$-graded algebras $\mathcal{A} = \mathscr{Y}(\mathfrak{gl}(\mathfrak{m}|\mathfrak{n}))$, $U_q(\widehat{\mathfrak{gl}(\mathfrak{m}|\mathfrak{n})})$ giving supersymmetric quantum integrable models. The latter are described in more details in chapter 8.

Bibliography

Publications of Louis Vignoli

[LV1] J. M. Maillet, G. Niccoli, and L. Vignoli, "Separation of variables bases for integrable $gl_{\mathcal{M}|\mathcal{N}}$ and Hubbard models," *SciPost Phys.*, vol. 9, p. 60, 4 2020. DOI: 10.21468/SciPostPhys.9.4.060 (pages 19, 95, 105, 125, 133–135, 141, 143, 146, 147, 150, 181).

[LV2] J. M. Maillet, G. Niccoli, and L. Vignoli, "On Scalar Products in Higher Rank Quantum Separation of Variables," *SciPost Phys.*, vol. 9, p. 86, 6 2020. DOI: 10.21468/SciPostPhys.9.6.086 (pages 19, 105, 107, 108, 113–124, 149, 239).

References

[1] G. Galilei, *Dialogo sopra i due massimi sistemi del mondo*. Edizioni Studio Tesi, 1992, vol. 36 (page 7).

[2] G. Galilei, *Discorsi e dimostrazioni matematiche*. Elsevier, 2013 (page 7).

[3] R. Descartes, *Les principes de la philosophie*. par la Compagnie des libraires, 1723 (page 7).

[4] S. Roux, "Découvrir le principe d'inertie," *Recherches sur la philosophie et le langage*, vol. 24, pp. 453–515, 2006 (page 7).

[5] I. Newton, *Philosophiæ naturalis principia mathematica*. 1687 (page 7).

[6] R. J. Blackwell and C. Huygens, "Christiaan Huygens' *The Motion of Colliding Bodies*," *Isis*, vol. 68, no. 4, pp. 574–597, 1977 (pages 7, 8).

[7] J. Kepler, *Astronomia nova ..., seu physica coelestis, tradita commentariis de motibus stellae martis*. Heidelberg: Voegelin, 1609. DOI: https://doi.org/10.3931/e-rara-558 (page 8).

[8] J. Kepler, *The harmony of the world*. American Philosophical Society, 1997, vol. 209 (page 8).

[9] D. Bernoulli, *Euler's correspondence with daniel bernoulli*, 1744 (page 8).

[10] A. Borrelli, "Angular momentum between physics and mathematics," in *Mathematics meets physics*, V. H. Deutsch, Ed., 2011 (page 8).

[11] L. Poinsot, *Éléments de statique: suivis de quatre mémoires sur la composition des moments et des aires*. Hachette Livre, 2018 (page 8).

[12] G. W. Leibniz and K. Gerhardt, *Leibnizens mathematische schriften*, A. A. Berlin, Ed. 1849 (page 8).

[13] J. L. Lagrange, *Mécanique analytique*. Mallet-Bachelier, 1853, vol. 1 (page 8).

[14] É. du Châtelet, *Principes Mathématiques de la Philosophie Naturelle*, D. et Saillant, Ed. Lambert, 1859 (page 8).

[15] W. J. M. Rankine, "Xviii. on the general law of the transformation of energy," *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*, vol. 5, no. 30, pp. 106–117, 1853. DOI: 10.1080/14786445308647205. eprint: https://doi.org/10.1080/14786445308647205 (page 8).

[16] H. W. Dickinson, *A short history of the steam engine*. Cambridge University Press, 2011 (page 8).

[17] S. Carnot, "Réflexions sur la puissance motrice du feu et sur les machines propres à développer cette puissance," *Annales scientifiques de l'École normale supérieure*, vol. 1, pp. 393–457, 1872. DOI: 10.24033/asens.88 (page 8).

[18] J. P. Joule and M. Faraday, "III. on the mechanical equivalent of heat," *Philosophical Transactions of the Royal Society of London*, vol. 140, pp. 61–82, 1850. DOI: 10.1098/rstl.1850.0004. eprint: https://royalsocietypublishing.org/doi/pdf/10.1098/rstl.1850.0004 (page 8).

[19] J. R. von Mayer, *Remarks on the forces of nature, 1841. as quoted in: Lehninger, a. bioenergetics-the molecular basis of biological energy transformations*, 1971 (page 8).

[20] R. Clausius, "I. on the moving force of heat, and the laws regarding the nature of heat itself which are deducible therefrom," *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*, vol. 2, no. 8, pp. 1–21, 1851. DOI: 10.1080/14786445108646819. eprint: https://doi.org/10.1080/14786445108646819 (page 8).

[21] J. C. Maxwell, "Illustrations of the dynamical theory of gases.—Part I. On the motions and collisions of perfectly elastic spheres," *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*, vol. 19, no. 124, pp. 19–32, 1860. DOI: 10.1080/14786446008642818. eprint: https://doi.org/10.1080/14786446008642818 (page 8).

[22] J. C. Maxwell, "Illustrations of the dynamical theory of gases.—Part II. On the process of diffusion of two or more kinds of moving particles among one another," *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*, vol. 20, no. 130, pp. 21–37, 1860. DOI: 10.1080/14786446008642902. eprint: https://doi.org/10.1080/14786446008642902 (page 8).

[23] L. Boltzmann, "Weitere studien über das wärmegleichgewicht unter gasmolekülen," in *Kinetische Theorie II*, Springer, 1970, pp. 115–225. DOI: https://doi.org/10.1007/978-3-322-84986-1_3 (page 8).

[24] R. Feynman, R. Leighton, and M. Sands, *The Feynman Lectures on Physics, Vol. I: The New Millennium Edition: Mainly Mechanics, Radiation, and Heat*. Basic Books, 2015 (page 9).

[25] J. Tyndall and W. Francis, *Scientific Memoirs, Selected from the Transactions of Foreign Academies of Science, and from Foreign Journals. Natural Philosophy*, v. 1. 1853 (page 9).

[26] A. Einstein, "Does the inertia of a body depend upon its energy-content," *Ann Phys*, vol. 18, pp. 639–641, 1905 (page 9).

[27] A. Einstein, "Elementary derivation of the equivalence of mass and energy," *Bulletin of the American mathematical society*, vol. 41, no. 4, pp. 223–230, 1935 (page 9).

[28] R. Wald, *General Relativity*. University of Chicago Press, 2010 (page 9).

[29] M. L'Annunziata, *Radioactivity: Introduction and History, From the Quantum to Quarks*. Elsevier Science, 2016 (page 9).

[30] R. V. Pound and G. A. Rebka, "Apparent weight of photons," *Phys. Rev. Lett.*, vol. 4, pp. 337–341, 7 Apr. 1960. DOI: 10.1103/PhysRevLett.4.337 (page 9).

[31] S. Rainville, J. K. Thompson, E. G. Myers, J. M. Brown, *et al.*, "A direct test of e=mc2," *Nature*, vol. 438, no. 7071, pp. 1096–1097, Dec. 2005. DOI: 10.1038/4381096a (page 9).

[32] M. Peskin and D. Schroeder, *An introduction to quantum field theory*. CRC press, 2018 (page 9).

[33] M. D. Schwartz, *Quantum field theory and the standard model*. Cambridge University Press, 2014 (page 9).

[34] H. Goldstein, C. Poole, and J. Safko, *Classical Mechanics (3rd Edition)*. Jun. 2001 (pages 10, 17, 21, 22, 65, 66).

[35] J. Hermann, "Extrait d'une lettre de M. Herman à M. Bernoulli datée de Padoüe le 12 juillet 1710," *Histoire de l'académie royale des sciences (Paris)*, vol. 1732, p. 519, 1710 (page 10).

[36] U. Le Verrier, "Lettre de M. Le Verrier à M. Faye sur la théorie de Mercure et sur le mouvement du périhélie de cette planète," *Comptes Rendus de l'Academie de Sciences (Paris)*, pp. 379–383, 1859 (page 10).

[37] J. V. José and E. J. Saletan, *Classical Dynamics: A Contemporary Approach*. Cambridge University Press, 1998. DOI: 10.1017/CBO9780511803772 (pages 10, 11, 21, 22).

[38] E. Noether, "Invariante variationsprobleme," in *Springer Collected Works in Mathematics*, Springer Berlin Heidelberg, 1983, pp. 231–239. DOI: 10.1007/978-3-642-39990-9_13 (page 11).

[39] M. Nakahara, *Geometry, topology and physics*. CRC press, 2003 (page 11).

[40] F. Gieres, "About symmetries in physics," Dec. 1997. arXiv: hep-th/9712154 (page 11).

[41] M.-L. Ge, "Quantum Groups and Quantum Integrable Systems," in *Quantum Group and Quantum Integrable Systems*, World Scientific, Jul. 1992. DOI: 10.1142/9789814538534 (page 11).

[42] J. F. Cornwell, *Group Theory in Physics: Volume 1*, ser. Techniques of Physics. Elsevier Science, 1986 (page 11).

[43] J. Cornwell, *Group Theory in Physics: Volume 2*, ser. Techniques of Physics. Elsevier Science, 1986 (page 11).

[44] J. F. Cornwell, *Group Theory in Physics. Volume III: Supersymmetries and Infinite-Dimensional Algebras*. Academic Press, 1989, vol. 10, pp. 1–628 (pages 11, 125–127).

[45] J. E. Humphreys, *Introduction to Lie algebras and representation theory*. Springer Science & Business Media, 2012, vol. 9 (page 11).

[46] R. Gilmore, *Lie groups, Lie algebras, and some of their applications*. Courier Corporation, 2012 (page 11).

[47] J. Liouville, "Note sur l'intégration des équations différentielles de la dynamique, présentée au bureau des longitudes le 29 juin 1853.," *Journal de Mathématiques Pures et Appliquées*, pp. 137–138, 1855 (pages 11, 21).

[48] V. I. Arnol'd, *Mathematical methods of classical mechanics*. Springer Science & Business Media, 2013, vol. 60. DOI: 10.1007/978-1-4757-1693-1 (pages 11, 21, 23, 24).

[49] O. Babelon, D. Bernard, and M. Talon, *Introduction to Classical Integrable Systems*, ser. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 2003. DOI: 10.1017/CBO9780511535024 (pages 11–14, 21, 23, 27, 28, 31).

[50] P. D. Lax, "Integrals of nonlinear equations of evolution and solitary waves," *Communications on Pure and Applied Mathematics*, vol. 21, no. 5, pp. 467–490, 1968. DOI: https://doi.org/10.1002/cpa.3160210503. eprint: https://onlinelibrary.wiley.com/doi/pdf/10.1002/cpa.3160210503 (pages 11, 12, 21, 26).

[51] J. Boussinesq, "Théorie de l'intumescence liquide appelée onde solitaire ou de translation se propageant dans un canal rectangulaire," *CR Acad. Sci. Paris*, vol. 72, no. 755-759, p. 1871, 1871 (page 11).

[52] J. Boussinesq, *Éssai sur la théorie des eaux courantes*, ser. Mémoires présentés par divers savants à l'Académie des sciences de l'Institut National de France. Imprimerie nationale, 1877 (page 11).

[53] J. S. Russell, *Report on Waves: Made to the Meetings of the British Association in 1842-43*. 1845 (page 11).

[54] C. S. Gardner, J. M. Greene, M. D. Kruskal, and R. M. Miura, "Method for solving the korteweg-devries equation," *Phys. Rev. Lett.*, vol. 19, pp. 1095–1097, 19 Nov. 1967. DOI: 10.1103/PhysRevLett.19.1095 (pages 12, 13, 21, 29).

[55] A. Shabat and V. Zakharov, "Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media," *Soviet physics JETP*, vol. 34, no. 1, p. 62, 1972 (pages 12, 26).

[56] V. E. Zakharov and A. B. Shabat, "Interaction between solitons in a stable medium," *Sov. Phys. JETP*, vol. 37, no. 5, pp. 823–828, 1973 (pages 12, 26).

[57] V. E. Zakharov and L. D. Faddeev, "Korteweg–de vries equation: A completely integrable hamiltonian system," *Funktsional'nyi Analiz i ego Prilozheniya*, vol. 5, no. 4, pp. 18–27, 1971 (pages 12, 26).

[58] I. M. Gel'fand and B. M. Levitan, "On the determination of a differential equation from its spectral function," pp. 253–304, 1955. DOI: 10.1090/trans2/001/11 (pages 12, 31).

[59] V. A. Marchenko, "On reconstruction of the potential energy from phases of the scattered waves," in *Dokl. Akad. Nauk SSSR*, vol. 104, 1955, pp. 695–698 (pages 12, 31).

[60] M. J. Ablowitz, D. J. Kaup, A. C. Newell, and H. Segur, "Method for solving the sine-gordon equation," *Physical Review Letters*, vol. 30, no. 25, p. 1262, 1973 (pages 12, 31).

[61] V. A. Marchenko, *Sturm-Liouville operators and applications*. American Mathematical Soc., 2011, vol. 373 (page 12).

[62] L. Faddeev, "Integrable models in 1+ 1 dimensional quantum field theory," CEA Centre d'Etudes Nucleaires de Saclay, Tech. Rep., 1982 (pages 12, 27, 37).

[63] L. Faddeev and L. Takhtajan, *Hamiltonian methods in the theory of solitons*. Springer Science & Business Media, 1987 (pages 12, 21, 26, 27, 32).

[64] W. Lenz, "Beitrag zum verständnis der magnetischen erscheinungen in festen körpern," *Z. Phys.*, vol. 21, pp. 613–615, 1920 (page 12).

[65] E. Ising, "Contribution to the Theory of Ferromagnetism," *Z. Phys.*, vol. 31, pp. 253–258, 1925. DOI: 10.1007/BF02980577 (page 12).

[66] L. Onsager, "Crystal statistics. I. A two-dimensional model with an order-disorder transition," *Phys. Rev.*, vol. 65, pp. 117–149, 3-4 Feb. 1944. DOI: 10.1103/PhysRev.65.117 (page 12).

[67] B. Kaufman, "Crystal statistics. II. Partition function evaluated by spinor analysis," *Phys. Rev.*, vol. 76, pp. 1232–1243, 8 Oct. 1949. DOI: 10.1103/PhysRev.76.1232 (page 12).

[68] B. Kaufman and L. Onsager, "Crystal statistics. III. Short-range order in a binary Ising lattice," *Phys. Rev.*, vol. 76, pp. 1244–1252, 8 Oct. 1949. DOI: 10.1103/PhysRev.76.1244 (page 12).

[69] W. Heisenberg, "Zur Theorie des Ferromagnetismus," *Zeitschrift für Physik*, vol. 49, no. 9-10, pp. 619–636, Sep. 1928. DOI: 10.1007/bf01328601 (pages 12, 37, 42).

[70] H. Bethe, "Zur Theorie der Metalle," *Zeitschrift für Physik*, vol. 71, no. 3-4, pp. 205–226, Mar. 1931. DOI: 10.1007/bf01341708 (page 12).

[71] R. Orbach, "Linear antiferromagnetic chain with anisotropic coupling," *Physical Review*, vol. 112, no. 2, pp. 309–316, Oct. 1958. DOI: 10.1103/physrev.112.309 (page 12).

[72] L. R. Walker, "Antiferromagnetic linear chain," *Physical Review*, vol. 116, no. 5, pp. 1089–1090, Dec. 1959. DOI: 10.1103/physrev.116.1089 (page 12).

[73] C. N. Yang and C. P. Yang, "One-dimensional chain of anisotropic spin-spin interactions. i. proof of bethe's hypothesis for ground state in a finite system," *Phys. Rev.*, vol. 150, pp. 321–327, 1 Oct. 1966. DOI: 10.1103/PhysRev.150.321 (page 13).

[74] C. N. Yang and C. P. Yang, "One-dimensional chain of anisotropic spin-spin interactions. ii. properties of the ground-state energy per lattice site for an infinite system," *Phys. Rev.*, vol. 150, pp. 327–339, 1 Oct. 1966. DOI: 10.1103/PhysRev.150.327 (page 13).

[75] C. N. Yang and C. P. Yang, "One-dimensional chain of anisotropic spin-spin interactions. iii. applications," *Phys. Rev.*, vol. 151, pp. 258–264, 1 Nov. 1966. DOI: 10.1103/PhysRev.151.258 (page 13).

[76] L. Pauling, "The structure and entropy of ice and of other crystals with some randomness of atomic arrangement," *Journal of the American Chemical Society*, vol. 57, no. 12, pp. 2680–2684, Dec. 1935. DOI: 10.1021/ja01315a102 (page 13).

[77] E. H. Lieb, "Residual entropy of square ice," *Phys. Rev.*, vol. 162, pp. 162–172, 1 Oct. 1967. DOI: 10.1103/PhysRev.162.162 (page 13).

[78] B. Sutherland, "Two-dimensional hydrogen bonded crystals without the ice rule," *Journal of Mathematical Physics*, vol. 11, no. 11, pp. 3183–3186, 1970. DOI: 10.1063/1.1665111. eprint: https://doi.org/10.1063/1.1665111 (page 13).

[79] B. Sutherland, *Beautiful Models*. WORLD SCIENTIFIC, Jun. 2004. DOI: 10.1142/5552 (pages 13, 39).

[80] R. J. Baxter, "Exactly Solved Models in Statistical Mechanics," in *Series on Advances in Statistical Mechanics*, World Scientific, May 1985, pp. 5–63. DOI: 10.1142/9789814415255_0002 (pages 13, 42).

[81] R. Baxter, "Eight-vertex model in lattice statistics and one-dimensional anisotropic Heisenberg chain. i, II, III," *Annals of Physics*, vol. 76, no. 1, pp. 1–24, 25–47, 48–71, Mar. 1973. DOI: 10.1016/0003-4916(73)90440-5 (page 13).

[82] R. J. Baxter, "Corner transfer matrices of the eight-vertex model. I. low-temperature expansions and conjectured properties," *Journal of Statistical Physics*, vol. 15, no. 6, pp. 485–503, Dec. 1976. DOI: 10.1007/bf01020802 (page 13).

[83] R. J. Baxter, "Corner transfer matrices of the eight-vertex model. II. The Ising model case," *Journal of Statistical Physics*, vol. 17, no. 1, pp. 1–14, Jul. 1977. DOI: 10.1007/bf01089373 (page 13).

[84] R. J. Baxter, "Eight-vertex model in lattice statistics," *Physical Review Letters*, vol. 26, no. 14, pp. 832–833, Apr. 1971. DOI: 10.1103/physrevlett.26.832 (page 13).

[85] R. J. Baxter, "One-dimensional anisotropic Heisenberg chain," *Annals of Physics*, vol. 70, no. 2, pp. 323–337, Apr. 1972. DOI: 10.1016/0003-4916(72)90270-9 (pages 13, 42).

[86] C. N. Yang, "Some exact results for the many-body problem in one dimension with repulsive delta-function interaction," *Phys. Rev. Lett.*, vol. 19, pp. 1312–1315, 23 Dec. 1967. DOI: 10.1103/PhysRevLett.19.1312 (page 13).

[87] J. B. McGuire, "Study of exactly soluble one-dimensional n-body problems," *Journal of Mathematical Physics*, vol. 5, no. 5, pp. 622–636, 1964 (page 13).

[88] C. N. Yang, "S matrix for the one-dimensional n-body problem with repulsive or attractive δ-function interaction," *Physical Review*, vol. 168, no. 5, p. 1920, 1968 (page 13).

[89] E. Sklyanin, "On complete integrability of the Landau-Lifshitz equation," LOMI, Tech. Rep., 1979 (pages 13, 14, 21, 27).

[90] E. K. Sklyanin, "Quantum version of the method of inverse scattering problem," *Journal of Soviet Mathematics*, vol. 19, no. 5, pp. 1546–1596, 1982. DOI: 10.1007/bf01091462 (pages 13, 21, 27, 30–32, 37, 41).

[91] M. A. Semenov-Tyan-Shanskii, "What is a classical r-matrix?" *Functional Analysis and Its Applications*, vol. 17, no. 4, pp. 259–272, Oct. 1983. DOI: 10.1007/BF01076717 (pages 13, 21).

[92] E. H. Lieb and W. Liniger, "Exact analysis of an interacting bose gas. i. the general solution and the ground state," *Phys. Rev.*, vol. 130, pp. 1605–1616, 4 May 1963. DOI: 10.1103/PhysRev.130.1605 (page 13).

[93] E. H. Lieb, "Exact analysis of an interacting bose gas. ii. the excitation spectrum," *Phys. Rev.*, vol. 130, pp. 1616–1624, 4 May 1963. DOI: 10.1103/PhysRev.130.1616 (page 13).

[94] E. K. Sklyanin and L. D. Faddeev, "Quantum-mechanical approach to completely integrable field theory models," in *Fifty Years of Mathematical Physics*, WORLD SCIENTIFIC, Feb. 2016, pp. 290–292. DOI: 10.1142/9789814340960_0025 (page 13).

[95] E. K. Sklyanin, L. A. Takhtadzhyan, and L. D. Faddeev, "Quantum inverse problem method. i," *Theoretical and Mathematical Physics*, vol. 40, no. 2, pp. 688–706, Aug. 1979. DOI: 10.1007/bf01018718 (pages 13, 14, 30).

[96] E. Sklyanin, "Method of the inverse scattering problem and the nonlinear quantum Schroedinger equation," *Sov. Phys. Dokl.*, vol. 24, pp. 107–109, 1979 (pages 13, 21).

[97] L. Faddeev, "Instructive history of the quantum inverse scattering method," in *KdV '95: Proceedings of the International Symposium held in Amsterdam, The Netherlands, April 23–26, 1995, to commemorate the centennial of the publication of the equation by and named after Korteweg and de Vries*, M. Hazewinkel, H. W. Capel, and E. M. de Jager, Eds. Dordrecht: Springer Netherlands, 1995, pp. 69–84. DOI: 10.1007/978-94-011-0017-5_3 (pages 13, 37).

[98] H. B. Thacker, "Exact integrability in quantum field theory and statistical systems," *Rev. Mod. Phys.*, vol. 53, pp. 253–285, 2 Apr. 1981. DOI: 10.1103/RevModPhys.53.253 (page 14).

[99] V. E. Korepin, N. M. Bogoliubov, and A. G. Izergin, *Quantum Inverse Scattering Method and Correlation Functions*, ser. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 1993. DOI: 10.1017/CBO9780511628832 (pages 14, 37, 51, 58).

[100] L. Faddeev, "How Algebraic Bethe Ansatz works for integrable models," *arXiv preprint hep-th/9605187*, 1996 (pages 14, 37, 40–42, 51).

[101] H. Bergknoff and H. B. Thacker, "Method for solving the massive thirring model," *Phys. Rev. Lett.*, vol. 42, pp. 135–138, 3 Jan. 1979. DOI: 10.1103/PhysRevLett.42.135 (page 14).

[102] L. A. Takhtadzhan and L. D. Faddeev, "The quantum method of the inverse problem and the Heisenberg XYZ model," *Russian Mathematical Surveys*, vol. 34, no. 5, pp. 11–68, Oct. 1979. DOI: 10.1070/rm1979v034n05abeh003909 (pages 14, 42).

[103] L. D. Faddeev and L. A. Takhtajan, "Quantum inverse scattering method," *Sov. Sci. Rev. Math*, 1981 (pages 14, 37, 42).

[104] N. Kitanine, J. Maillet, and V. Terras, "Form factors of the XXZ Heisenberg finite chain," *Nuclear Physics B*, vol. 554, no. 3, pp. 647–678, Aug. 1999. DOI: 10.1016/s0550-3213(99)00295-3 (pages 14, 37, 49, 55, 57–59, 143).

[105] N. Kitanine, J. Maillet, and V. Terras, "Correlation functions of the XXZ Heisenberg spin- chain in a magnetic field," *Nuclear Physics B*, vol. 567, no. 3, pp. 554–582, Feb. 2000. DOI: 10.1016/s0550-3213(99)00619-7 (pages 14, 15, 37, 57, 60).

[106] J. Maillet and V. Terras, "On the quantum inverse scattering problem," *Nuclear Physics B*, vol. 575, no. 3, pp. 627–644, Jun. 2000. DOI: 10.1016/s0550-3213(00)00097-3 (pages 14, 37, 49, 50, 58, 143).

[107] F. Gohmann and V. E. Korepin, "Solution of the quantum inverse problem," *J. Phys. A*, vol. 33, pp. 1199–1220, 2000. DOI: 10.1088/0305-4470/33/6/308. arXiv: hep-th/9910253 (pages 14, 37, 49, 50, 125).

[108] A. A. Belavin and V. G. Drinfel'd, "Solutions of the classical yang–baxter equation for simple lie algebras," *Functional Analysis and Its Applications*, vol. 16, no. 3, pp. 159–180, Jul. 1982. DOI: 10.1007/BF01081585 (page 14).

[109] A. A. Belavin and V. G. Drinfel'd, "Classical young–baxter equation for simple lie algebras," *Functional Analysis and Its Applications*, vol. 17, no. 3, pp. 220–221, 1984. DOI: 10.1007/bf01078107 (page 14).

[110] A. Belavin and V. Drinfeld, *Triangle Equations and Simple Lie Algebras*, ser. Soviet scientific reviews: Mathematical physics reviews. Harwood Academic Publishers, 1984 (page 14).

[111] P. P. Kulish and N. Y. Reshetikhin, "GL3-invariant solutions of the Yang-Baxter equation and associated quantum systems," *Journal of Soviet Mathematics*, vol. 34, no. 5, pp. 1948–1971, Sep. 1986. DOI: 10.1007/bf01095104 (pages 14, 18, 158).

[112] P. P. Kulish and E. K. Sklyanin, "Solutions of the Yang-Baxter equation," *Journal of Soviet Mathematics*, vol. 19, no. 5, pp. 1596–1620, 1982. DOI: 10.1007/bf01091463 (pages 14, 18, 29).

[113] P. P. Kulish and N. Y. Reshetikhin, "Quantum linear problem for the sine-Gordon equation and higher representations," *Journal of Soviet Mathematics*, vol. 23, no. 4, pp. 2435–2441, 1983. DOI: 10.1007/bf01084171 (page 14).

[114] M. Jimbo, "A q-difference analogue of $U(g)$ and the Yang-Baxter equation," *Letters in Mathematical Physics*, vol. 10, no. 1, pp. 63–69, Jul. 1985. DOI: 10.1007/bf00704588 (page 14).

[115] V. G. Drinfel'd, "Quantum groups," *Journal of Soviet mathematics*, vol. 41, no. 2, pp. 898–915, 1988. DOI: 10.1007/BF01247086 (page 14).

[116] V. Chari and A. Pressley, *A Guide to Quantum Groups*. Cambridge University Press, 1995 (page 14).

[117] A. Molev, *Yangians and classical Lie algebras*, 143. American Mathematical Soc., 2007 (pages 14, 45, 125, 129, 153).

[118] C. Kassel, *Quantum Groups*, ser. Graduate Texts in Mathematics. Springer New York, 2012 (page 14).

[119] N. Reshetikhin and V. G. Turaev, "Invariants of 3-manifolds via link polynomials and quantum groups," *Inventiones mathematicae*, vol. 103, no. 1, pp. 547–597, Dec. 1991. DOI: 10.1007/BF01239527 (page 14).

[120] V. Turaev and O. Viro, "State sum invariants of 3-manifolds and quantum 6j-symbols," *Topology*, vol. 31, no. 4, pp. 865–902, 1992. DOI: https://doi.org/10.1016/0040-9383(92)90015-A (page 14).

[121] V. G. Turaev, *Quantum Invariants of Knots and 3-Manifolds*. De Gruyter, 2016. DOI: doi:10.1515/9783110435221 (page 14).

[122] P. P. Kulish and N. Y. Reshetikhin, "Diagonalisation of $GL(N)$ invariant transfer matrices and quantum N-wave system (Lee model)," *Journal of Physics A: Mathematical and General*, vol. 16, no. 16, pp. L591–L596, Nov. 1983. DOI: 10.1088/0305-4470/16/16/001 (page 15).

[123] N. A. Slavnov, "Introduction to the nested algebraic Bethe ansatz," *SciPost Phys. Lect. Notes*, p. 19, 2020. DOI: 10.21468/SciPostPhysLectNotes.19 (page 15).

[124] M. P. Pfannmüller and H. Frahm, "Algebraic Bethe ansatz for gl(2, 1) invariant 36-vertex models," *Nuclear Physics B*, vol. 479, no. 3, pp. 575–593, 1996. DOI: https://doi.org/10.1016/0550-3213(96)00425-7 (pages 15, 125, 144).

[125] F. Göhmann, "Algebraic Bethe ansatz for the gl(1|2) generalized model and Lieb–Wu equations," *Nuclear Physics B*, vol. 620, no. 3, pp. 501–518, 2002. DOI: https://doi.org/10.1016/S0550-3213(01)00497-7 (pages 15, 125, 144).

[126] D. Baeriswyl, D. Campbell, J. M. P. Carmelo, F. Guinea, and E. Louis, *The Hubbard Model: Its Physics and Mathematical Physics*. Springer, Boston, MA, Jan. 1995. DOI: 10.1007/978-1-4899-1042-4 (pages 15, 125, 145).

[127] E. Ragoucy and G. Satta, "Analytical Bethe Ansatz for closed and open $gl(M|N)$ super-spin chains in arbitrary representations and for any Dynkin diagram," *Journal of High Energy Physics*, vol. 2007, no. 9, pp. 001–001, 2007. DOI: 10.1088/1126-6708/2007/09/001 (pages 15, 125, 133).

[128] V. Kazakov, A. Sorin, and A. Zabrodin, "Supersymmetric Bethe ansatz and Baxter equations from discrete Hirota dynamics," *Nuclear Physics B*, vol. 790, no. 3, pp. 345–413, Feb. 2008. DOI: 10.1016/j.nuclphysb.2007.06.025 (pages 15, 18, 125, 131, 133–135, 159).

[129] S. Pakuliak, E. Ragoucy, and N. Slavnov, "Nested Algebraic Bethe Ansatz in integrable models: recent results," *SciPost Phys. Lect. Notes*, p. 6, 2018. DOI: 10.21468/SciPostPhysLectNotes.6 (pages 15, 16, 51, 60–62).

[130] M. Gaudin, *The Bethe Wavefunction*, J.-S. Caux, Ed. Cambridge University Press, 2014. DOI: 10.1017/CBO9781107053885 (pages 15, 47).

[131] V. E. Korepin, "Calculation of norms of Bethe wave functions," *Communications in Mathematical Physics*, vol. 86, no. 3, pp. 391–418, 1982. DOI: 10.1007/bf01212176 (pages 15, 57, 62).

[132] N. Slavnov, "Calculation of scalar products of wave functions and form factors in the framework of the algebraic bethe ansatz," *Theoretical and Mathematical Physics - THEOR MATH PHYS-ENGL TR*, vol. 79, pp. 502–508, May 1989. DOI: 10.1007/BF01016531 (pages 15, 56).

[133] N. Kitanine, J. Maillet, N. Slavnov, and V. Terras, "Spin-spin correlation functions of the XXZ-Heisenberg chain in a magnetic field," *Nuclear Physics B*, vol. 641, no. 3, pp. 487–518, Oct. 2002. DOI: 10.1016/s0550-3213(02)00583-7 (pages 15, 60).

[134] N. Kitanine, J. Maillet, N. Slavnov, and V. Terras, "Master equation for spin–spin correlation functions of the XXZ chain," *Nuclear Physics B*, vol. 712, no. 3, pp. 600–622, Apr. 2005. DOI: 10.1016/j.nuclphysb.2005.01.050 (pages 15, 60).

[135] N. Kitanine, K. Kozlowski, J. M. Maillet, N. A. Slavnov, and V. Terras, "On correlation functions of integrable models associated with the six-vertex R-matrix," *Journal of Statistical Mechanics: Theory and Experiment*, vol. 2007, no. 01, P01022–P01022, Jan. 2007. DOI: 10.1088/1742-5468/2007/01/p01022 (page 15).

[136] N. Kitanine, K. K. Kozlowski, J. M. Maillet, N. A. Slavnov, and V. Terras, "Algebraic Bethe ansatz approach to the asymptotic behavior of correlation functions," *Journal of Statistical Mechanics: Theory and Experiment*, vol. 2009, no. 04, P04003, Apr. 2009. DOI: 10.1088/1742-5468/2009/04/p04003 (page 15).

[137] N. Kitanine, K. K. Kozlowski, J. M. Maillet, N. A. Slavnov, and V. Terras, "On the thermodynamic limit of form factors in the massless XXZ Heisenberg chain," *Journal of Mathematical Physics*, vol. 50, no. 9, p. 095 209, Sep. 2009. DOI: 10.1063/1.3136683 (page 15).

[138] N. Kitanine, K. K. Kozlowski, J. M. Maillet, G. Niccoli, *et al.*, "Correlation functions of the open XXZ chain: I," *Journal of Statistical Mechanics: Theory and Experiment*, vol. 2007, no. 10, P10009–P10009, Oct. 2007. DOI: 10.1088/1742-5468/2007/10/p10009 (page 15).

[139] N. Kitanine, K. K. Kozlowski, J. M. Maillet, G. Niccoli, *et al.*, "Correlation functions of the open XXZ chain: II," *Journal of Statistical Mechanics: Theory and Experiment*, vol. 2008, no. 07, P07010, Jul. 2008. DOI: 10.1088/1742-5468/2008/07/p07010 (page 15).

[140] J.-S. Caux and J. M. Maillet, "Computation of dynamical correlation functions of Heisenberg chains in a magnetic field," *Physical Review Letters*, vol. 95, no. 7, Aug. 2005. DOI: 10.1103/physrevlett.95.077201 (page 15).

[141] J.-S. Caux, R. Hagemans, and J. M. Maillet, "Computation of dynamical correlation functions of Heisenberg chains: The gapless anisotropic regime," *Journal of Statistical Mechanics: Theory and Experiment*, vol. 2005, no. 09, P09003–P09003, Sep. 2005. DOI: 10.1088/1742-5468/2005/09/p09003 (page 15).

[142] M. Kenzelmann, R. Coldea, D. A. Tennant, D. Visser, *et al.*, "Order-to-disorder transition in the xy-like quantum magnet Cs_2CoCl_4 induced by noncommuting applied fields," *Phys. Rev. B*, vol. 65, p. 144 432, 14 Apr. 2002. DOI: 10.1103/PhysRevB.65.144432 (page 15).

[143] J.-M. Maillet, "Heisenberg spin chains: From quantum groups to neutron scattering experiments," in *Quantum Spaces: Poincaré Seminar 2007*, B. Duplantier, Ed. Basel: Birkhäuser Basel, 2007, pp. 161–201. DOI: 10.1007/978-3-7643-8522-4_4 (pages 15, 37, 45).

[144] F. Göhmann, A. Klümper, and A. Seel, "Integral representations for correlation functions of the XXZ chain at finite temperature," *Journal of Physics A: Mathematical and General*, vol. 37, no. 31, pp. 7625–7651, Jul. 2004. DOI: 10.1088/0305-4470/37/31/001 (page 15).

[145] F. Göhmann, A. Klümper, and A. Seel, "Integral representation of the density matrix of the xxz chain at finite temperatures," *Journal of Physics A: Mathematical and General*, vol. 38, no. 9, pp. 1833–1841, Feb. 2005. DOI: 10.1088/0305-4470/38/9/001 (page 15).

[146] F. Göhmann, A. Klümper, and A. Seel, "Emptiness formation probability at finite temperature for the isotropic heisenberg chain," *Physica B: Condensed Matter*, vol. 359-361, pp. 807–809, Apr. 2005. DOI: 10.1016/j.physb.2005.01.234 (page 15).

[147] F. Göhmann and J. Suzuki, "Quantum spin chains at finite temperatures," *New Trends in Quantum Integrable Systems*, Oct. 2010. DOI: 10.1142/9789814324373_0005 (page 15).

[148] K. K. Kozlowski, J. M. Maillet, and N. A. Slavnov, "Long-distance behavior of temperature correlation functions in the one-dimensional Bose gas," *Journal of Statistical Mechanics: Theory and Experiment*, vol. 2011, no. 03, P03018, Mar. 2011. DOI: 10.1088/1742-5468/2011/03/p03018 (page 15).

[149] K. K. Kozlowski, J. M. Maillet, and N. A. Slavnov, "Correlation functions for one-dimensional bosons at low temperature," *Journal of Statistical Mechanics: Theory and Experiment*, vol. 2011, no. 03, P03019, Mar. 2011. DOI: 10.1088/1742-5468/2011/03/p03019 (page 15).

[150] S. Belliard, S. Pakuliak, E. Ragoucy, and N. A. Slavnov, "Highest coefficient of scalar products in $SU(3)$-invariant integrable models," *Journal of Statistical Mechanics: Theory and Experiment*, vol. 2012, no. 09, P09003, Sep. 2012. DOI: 10.1088/1742-5468/2012/09/p09003 (pages 15, 62).

[151] S. Belliard, S. Pakuliak, E. Ragoucy, and N. A. Slavnov, "The algebraic Bethe ansatz for scalar products in $SU(3)$-invariant integrable models," *Journal of Statistical Mechanics: Theory and Experiment*, vol. 2012, no. 10, P10017, Oct. 2012. DOI: 10.1088/1742-5468/2012/10/p10017 (pages 15, 62).

[152] S. Pakuliak, E. Ragoucy, and N. Slavnov, "Form factors in quantum integrable models with $GL(3)$-invariant R-matrix," *Nuclear Physics B*, vol. 881, pp. 343–368, 2014. DOI: https://doi.org/10.1016/j.nuclphysb.2014.02.014 (page 15).

[153] S. Z. Pakuliak, E. Ragoucy, and N. A. Slavnov, "Determinant representations for form factors in quantum integrable models with the $GL(3)$-invariant R-matrix," *Theoretical and Mathematical Physics*, vol. 181, no. 3, pp. 1566–1584, Dec. 2014. DOI: 10.1007/s11232-014-0236-0 (page 15).

[154] S. Pakuliak, E. Ragoucy, and N. A. Slavnov, "$GL(3)$-based quantum integrable composite models. I. Bethe vectors," *Symmetry, Integrability and Geometry: Methods and Applications*, Jul. 2015. DOI: 10.3842/sigma.2015.063 (page 15).

[155] S. Pakuliak, E. Ragoucy, and N. A. Slavnov, "$GL(3)$ -based quantum integrable composite models. II. form factors of local operators," *Symmetry, Integrability and Geometry: Methods and Applications*, Jul. 2015. DOI: 10.3842/sigma.2015.064 (page 15).

[156] A. Liashyk and N. A. Slavnov, "On Bethe vectors in $gl(3)$-invariant integrable models," *Journal of High Energy Physics*, vol. 2018, no. 6, Jun. 2018. DOI: 10.1007/jhep06(2018)018 (page 15).

[157] A. Hutsalyuk, A. Liashyk, S. Z. Pakuliak, E. Ragoucy, and N. A. Slavnov, "Multiple actions of the monodromy matrix in $gl(2|1)$-invariant integrable models," *Symmetry, Integrability and Geometry: Methods and Applications*, Oct. 2016. DOI: 10.3842/sigma.2016.099 (pages 15, 125).

[158] S. Z. Pakuliak, E. Ragoucy, and N. A. Slavnov, "Bethe vectors for models based on the super-Yangian $Y(\mathfrak{gl}(m|n))$," *Journal of Integrable Systems*, vol. 2, no. 1, Jan. 2017. DOI: 10.1093/integr/xyx001 (pages 15, 60, 125).

[159] A. A. Hutsalyuk, A. N. Liashyk, S. Z. Pakuliak, E. Ragoucy, and N. A. Slavnov, "Current presentation for the super-Yangian double $DY\,\mathfrak{gl}(m|n)$ and Bethe vectors," *Russian Mathematical Surveys*, vol. 72, no. 1, pp. 33–99, Feb. 2017. DOI: 10.1070/rm9754 (page 15).

[160] A. Hutsalyuk, A. Liashyk, S. Z. Pakuliak, E. Ragoucy, and N. A. Slavnov, "Scalar products of Bethe vectors in models with $\mathfrak{gl}(2|1)$ symmetry 2. determinant representation," *Journal of Physics A: Mathematical and Theoretical*, vol. 50, no. 3, p. 034 004, Dec. 2016. DOI: 10.1088/1751-8121/50/3/034004 (pages 15, 62, 125).

[161] A. Hutsalyuk, A. Liashyk, S. Pakuliak, E. Ragoucy, and N. Slavnov, "Scalar products of bethe vectors in the models with gl(m|n)symmetry," *Nuclear Physics B*, vol. 923, pp. 277–311, Oct. 2017. DOI: 10.1016/j.nuclphysb.2017.07.020 (pages 15, 62, 125).

[162] A. Hutsalyuk, A. Liashyk, S. Pakuliak, E. Ragoucy, and N. Slavnov, "Norm of bethe vectors in models with $gl(m|n)$ symmetry," *Nuclear Physics B*, vol. 926, pp. 256–278, Jan. 2018. DOI: 10.1016/j.nuclphysb.2017.11.006 (pages 15, 62, 125).

[163] N. Beisert, C. Ahn, L. F. Alday, Z. Bajnok, *et al.*, "Review of ads/cft integrability: An overview," *Letters in Mathematical Physics*, vol. 99, no. 1, pp. 3–32, Jan. 2012. DOI: 10.1007/s11005-011-0529-2 (page 15).

[164] N. Beisert, "The S-matrix of AdS/CFT and Yangian symmetry," in *Proceedings of Bethe Ansatz: 75 years later — PoS(Solvay)*, Sissa Medialab, Aug. 2007. DOI: 10.22323/1.038.0002 (page 15).

[165] N. Beisert, "The analytic Bethe ansatz for a chain with centrally extended $su(2|2)$ symmetry," *Journal of Statistical Mechanics: Theory and Experiment*, vol. 2007, no. 01, P01017–P01017, Jan. 2007. DOI: 10.1088/1742-5468/2007/01/p01017 (page 15).

[166] I. Kostov, D. Serban, and D. Volin, "Strong coupling limit of bethe ansatz equations," *Nuclear Physics B*, vol. 789, no. 3, pp. 413–451, 2008. DOI: https://doi.org/10.1016/j.nuclphysb.2007.06.017 (page 15).

[167] H. Saleur, "Superspin chains and supersigma models: A short introduction," *Exact Methods in Low-dimensional Statistical Physics and Quantum Computing: Lecture Notes of the Les Houches Summer School: Volume 89, July 2008*, vol. 89, p. 455, 2010 (page 15).

[168] D. Serban, "A note on the eigenvectors of long-range spin chains and their scalar products," *Journal of High Energy Physics*, vol. 2013, no. 1, Jan. 2013. DOI: 10.1007/jhep01(2013)012 (page 15).

[169] D. Serban, "Eigenvectors and scalar products for long range interacting spin chains II: The finite size effects," *Journal of High Energy Physics*, vol. 2013, no. 8, Aug. 2013. DOI: 10.1007/jhep08(2013)128 (page 15).

[170] Y. Jiang, S. Komatsu, I. Kostov, and D. Serban, "The hexagon in the mirror: The three-point function in the SoV representation," *Journal of Physics A: Mathematical and Theoretical*, vol. 49, no. 17, p. 174 007, Mar. 2016. DOI: 10.1088/1751-8113/49/17/174007 (page 15).

[171] F. Levkovich-Maslyuk, "A review of the AdS/CFT Quantum Spectral Curve," *J. Phys. A*, vol. 53, no. 28, p. 283 004, 2020. DOI: 10.1088/1751-8121/ab7137. arXiv: 1911.13065 [hep-th] (page 15).

[172] V. Tarasov and A. Varchenko, "Completeness of Bethe vectors and difference equations with regular singular points," *International Mathematics Research Notices*, vol. 1995, no. 13, p. 637, 1995. DOI: 10.1155/s1073792895000377 (pages 16, 56).

[173] E. Mukhin, V. Tarasov, and A. Varchenko, "On separation of variables and completeness of the Bethe ansatz for quantum $\mathfrak{gl}_N$ Gaudin model," *Glasgow Mathematical Journal*, vol. 51, no. A, p. 137, Feb. 2009. DOI: 10.1017/s0017089508004850 (pages 16, 56).

[174] W. Hao, R. I. Nepomechie, and A. J. Sommese, "Completeness of solutions of bethe's equations," *Physical Review E*, vol. 88, no. 5, Nov. 2013. DOI: 10.1103/physreve.88.052113 (pages 16, 56).

[175] D. Chernyak, S. Leurent, and D. Volin, *Completeness of Wronskian Bethe equations for rational gl(m|n) spin chain*, 2020. arXiv: 2004.02865 [math-ph] (pages 16, 56, 150).

[176] G. Niccoli, "Antiperiodic spin-1/2 XXZ quantum chains by separation of variables: Complete spectrum and form factors," *Nuclear Physics B*, vol. 870, no. 2, pp. 397–420, May 2013. DOI: 10.1016/j.nuclphysb.2013.01.017 (pages 16, 18, 63, 82–84).

[177] E. K. Sklyanin, "The quantum Toda chain," in *Non-Linear Equations in Classical and Quantum Field Theory*, Springer Berlin Heidelberg, 1985, pp. 196–233. DOI: 10.1007/3-540-15213-x_80 (pages 16, 17, 65).

[178] N. Gromov, F. Levkovich-Maslyuk, and G. Sizov, "New construction of eigenstates and separation of variables for $SU(N)$ quantum spin chains," *Journal of High Energy Physics*, vol. 2017, no. 9, Sep. 2017. DOI: 10.1007/jhep09(2017)111 (pages 16, 18, 19, 63, 85, 105).

[179] N. Gromov and F. Levkovich-Maslyuk, "New compact construction of eigenstates for supersymmetric spin chains," *Journal of High Energy Physics*, vol. 2018, no. 9, Sep. 2018. DOI: 10.1007/jhep09(2018)085 (pages 16, 18, 105).

[180] Y. Wang, W.-L. Yang, J. Cao, and K. Shi, *Off-Diagonal Bethe Ansatz for Exactly Solvable Models*. Springer Berlin Heidelberg, 2015. DOI: 10.1007/978-3-662-46756-5 (page 16).

[181] S. Belliard, "Heisenberg XXX model with general boundaries: Eigenvectors from algebraic Bethe Ansatz," *Symmetry, Integrability and Geometry: Methods and Applications*, Nov. 2013. DOI: 10.3842/sigma.2013.072 (pages 16, 63).

[182] J. L. R. d'Alembert, *Traité de dynamique*. David, 1758 (page 16).

[183] S. Engelsman, "D'Alembert et les équations aux dérivées partielles," 1984. DOI: 10.3406/dhs.1984.1480 (page 16).

[184] J. B. J. Fourier, *Théorie analytique de la chaleur*. Firmin Didot, 1822 (page 16).

[185] M. C. Gutzwiller, "The quantum mechanical Toda lattice, II," *Annals of Physics*, vol. 133, no. 2, pp. 304–331, 1981. DOI: https://doi.org/10.1016/0003-4916(81)90253-0 (page 17).

[186] E. K. Sklyanin, "Goryachev-chaplygin top and the inverse scattering method," *Journal of Soviet Mathematics*, vol. 31, no. 6, pp. 3417–3431, Dec. 1985. DOI: 10.1007/bf02107243 (pages 17, 65, 70).

[187] E. K. Sklyanin, "Separation of variables in the classical integrable $SL(3)$ magnetic chain," *Communications in Mathematical Physics*, vol. 150, no. 1, pp. 181–191, Nov. 1992. DOI: 10.1007/bf02096572 (pages 17, 32, 65, 70–73, 83).

[188] E. K. Sklyanin, "Separation of variables : New trends," *Progress of Theoretical Physics Supplement*, vol. 118, pp. 35–60, 1995. DOI: 10.1143/ptps.118.35 (pages 17, 65, 70–73, 84).

[189] D. R. D. Scott, "Classical functional bethe ansatz for SL(n): Separation of variables for the magnetic chain," *Journal of Mathematical Physics*, vol. 35, no. 11, pp. 5831–5843, Nov. 1994. DOI: 10.1063/1.530712 (pages 17, 65, 71).

[190] M. I. Gekhtman, "Separation of variables in the classicalSL(n) magnetic chain," *Communications in Mathematical Physics*, vol. 167, no. 3, pp. 593–605, Feb. 1995. DOI: 10.1007/bf02101537 (pages 17, 65, 71).

[191] E. K. Sklyanin, "Separation of variables in the gaudin model," *Journal of Soviet Mathematics*, vol. 47, no. 2, pp. 2473–2488, Oct. 1989. DOI: 10.1007/bf01840429 (pages 17, 47).

[192] E. Sklyanin, "Functional Bethe Ansatz," in *Integrable and Superintegrable Systems*, World Scientific, Oct. 1990, pp. 8–33. DOI: 10.1142/9789812797179_0002 (pages 17, 65, 71, 78).

[193] E. K. Sklyanin, "Quantum inverse scattering method. Selected topics," Oct. 1992. arXiv: hep-th/9211111 (pages 17, 37, 65, 71, 73, 78, 84).

[194] E. K. Sklyanin and T. Takebe, "Separation of variables in the elliptic gaudin model," *Communications in Mathematical Physics*, vol. 204, no. 1, pp. 17–38, Jul. 1999. DOI: 10.1007/s002200050635 (page 17).

[195] E. K. Sklyanin, "New approach to the quantum nonlinear schrödinger equation," *Journal of Physics A: Mathematical and General*, vol. 22, no. 17, pp. 3551–3560, Sep. 1989. DOI: 10.1088/0305-4470/22/17/020 (page 17).

[196] E. Sklyanin, "Exact quantization of the sinh-gordon model," *Nuclear Physics B*, vol. 326, no. 3, pp. 719–736, 1989. DOI: https://doi.org/10.1016/0550-3213(89)90552-X (page 17).

[197] V. B. Kuznetsov and E. K. Sklyanin, "Separation of variables in the A_2 type jack polynomials," *arXiv preprint solv-int/9508002*, 1995 (page 17).

[198] V. B. Kuznetsov and E. K. Sklyanin, "Separation of variables for the ruijsenaars model and a new integral representation for the macdonald polynomials," *Journal of Physics A: Mathematical and General*, vol. 29, no. 11, pp. 2779–2804, Jun. 1996. DOI: 10.1088/0305-4470/29/11/014 (page 17).

[199] V. B. Kuznetsov, F. W. Nijhoff, and E. K. Sklyanin, "Separation of variables for the ruijsenaars system," *Communications in Mathematical Physics*, vol. 189, no. 3, pp. 855–877, Nov. 1997. DOI: 10.1007/s002200050231 (page 17).

[200] S. Derkachov, G. Korchemsky, and A. Manashov, "Non-compact Heisenberg spin magnets from high-energy QCD," *Nuclear Physics B*, vol. 617, no. 1-3, pp. 375–440, Dec. 2001. DOI: 10.1016/s0550-3213(01)00457-6 (page 17).

[201] S. Derkachov, G. P. Korchemsky, and A. N. Manashov, "Separation of variables for the quantum $SL(2,\mathbb{R})$ spin chain," *Journal of High Energy Physics*, vol. 2003, no. 07, pp. 047–047, Jul. 2003. DOI: 10.1088/1126-6708/2003/07/047 (page 17).

[202] S. Derkachov, G. P. Korchemsky, and A. N. Manashov, "Baxter $\mathbb{Q}$-operator and separation of variables for the open $SL(2,\mathbb{R})$ spin chain," *Journal of High Energy Physics*, vol. 2003, no. 10, pp. 053–053, Oct. 2003. DOI: 10.1088/1126-6708/2003/10/053 (page 17).

[203] A. G. Bytsko and J. Teschner, "Quantization of models with non-compact quantum group symmetry: Modular XXZ magnet and lattice sinh-Gordon model," *Journal of Physics A: Mathematical and General*, vol. 39, no. 41, pp. 12 927–12 981, Sep. 2006. DOI: 10.1088/0305-4470/39/41/s11 (page 17).

[204] H. Frahm, A. Seel, and T. Wirth, "Separation of variables in the open XXX chain," *Nuclear Physics B*, vol. 802, no. 3, pp. 351–367, Oct. 2008. DOI: 10.1016/j.nuclphysb.2008.04.008 (pages 17, 78).

[205] H. Frahm, J. H. Grelik, A. Seel, and T. Wirth, "Functional Bethe ansatz methods for the open XXX chain," *Journal of Physics A: Mathematical and Theoretical*, vol. 44, no. 1, p. 015 001, Nov. 2010. DOI: 10.1088/1751-8113/44/1/015001 (pages 17, 78).

[206] E. K. Sklyanin, "Separation of variables in the quantum integrable models related to the Yangian $Y[sl(3)]$," *Journal of Mathematical Sciences*, vol. 80, no. 3, pp. 1861–1871, Jun. 1996. DOI: 10.1007/bf02362784 (pages 17, 65, 83, 84, 150).

[207] F. A. Smirnov, "Separation of variables for quantum integrable models related to $U_q(\widehat{sl}_N)$," in *MathPhys Odyssey 2001*, Birkhäuser Boston, 2002, pp. 455–465. DOI: 10.1007/978-1-4612-0087-1_17 (pages 17, 83).

[208] G. Niccoli and J. Teschner, "The sine-Gordon model revisited: I," *Journal of Statistical Mechanics: Theory and Experiment*, vol. 2010, no. 09, P09014, Sep. 2010. DOI: 10.1088/1742-5468/2010/09/p09014 (page 18).

[209] G. Niccoli, "Reconstruction of Baxter-operator from Sklyanin SOV for cyclic representations of integrable quantum models," *Nuclear Physics B*, vol. 835, no. 3, pp. 263–283, Aug. 2010. DOI: 10.1016/j.nuclphysb.2010.03.009 (page 18).

[210] G. Niccoli, "Completeness of Bethe ansatz by Sklyanin SOV for cyclic representations of integrable quantum models," *Journal of High Energy Physics*, vol. 2011, no. 3, Mar. 2011. DOI: 10.1007/jhep03(2011)123 (page 18).

[211] G. Niccoli, "Non-diagonal open spin-1/2 XXZ quantum chains by separation of variables: Complete spectrum and matrix elements of some quasi-local operators," *Journal of Statistical Mechanics: Theory and Experiment*, vol. 2012, no. 10, P10025, Oct. 2012. DOI: 10.1088/1742-5468/2012/10/p10025 (pages 18, 84, 121).

[212] N. Grosjean and G. Niccoli, "The τ_2-model and the chiral Potts model revisited: Completeness of Bethe equations from Sklyanin's SoV method," *Journal of Statistical Mechanics: Theory and Experiment*, vol. 2012, no. 11, P11005, Nov. 2012. DOI: 10.1088/1742-5468/2012/11/p11005 (pages 18, 56).

[213] N. Grosjean, J. M. Maillet, and G. Niccoli, "On the form factors of local operators in the lattice sine–Gordon model," *Journal of Statistical Mechanics: Theory and Experiment*, vol. 2012, no. 10, P10006, Oct. 2012. DOI: 10.1088/1742-5468/2012/10/p10006 (pages 18, 121).

[214] G. Niccoli, "An antiperiodic dynamical six-vertex model: I. complete spectrum by SOV, matrix elements of the identity on separate states and connections to the periodic eight-vertex model," *Journal of Physics A: Mathematical and Theoretical*, vol. 46, no. 7, p. 075 003, Feb. 2013. DOI: 10.1088/1751-8113/46/7/075003 (pages 18, 63, 84).

[215] G. Niccoli, "Form factors and complete spectrum of XXX anti-periodic higher spin chains by quantum separation of variables," *Journal of Mathematical Physics*, vol. 54, no. 5, p. 053 516, May 2013. DOI: 10.1063/1.4807078 (pages 18, 63, 84).

[216] N. Grosjean, J.-M. Maillet, and G. Niccoli, "On the form factors of local operators in the Bazhanov-Stroganov and chiral Potts models," *Annales Henri Poincaré*, vol. 16, no. 5, pp. 1103–1153, Oct. 2014. DOI: 10.1007/s00023-014-0358-9 (page 18).

[217] N. Kitanine, J. M. Maillet, and G. Niccoli, "Open spin chains with generic integrable boundaries: Baxter equation and Bethe ansatz completeness from separation of variables," *Journal of Statistical Mechanics: Theory and Experiment*, vol. 2014, no. 5, P05015, May 2014. DOI: 10.1088/1742-5468/2014/05/p05015 (pages 18, 56, 84).

[218] N. Kitanine, J. M. Maillet, G. Niccoli, and V. Terras, "On determinant representations of scalar products and form factors in the SoV approach: The XXX case," *Journal of Physics A: Mathematical and Theoretical*, vol. 49, no. 10, p. 104 002, Jan. 2016. DOI: 10.1088/1751-8113/49/10/104002 (pages 18, 84, 101).

[219] N. Kitanine, J. M. Maillet, G. Niccoli, and V. Terras, "The open XXX spin chain in the SoV framework: Scalar product of separate states," *Journal of Physics A: Mathematical and Theoretical*, vol. 50, no. 22, p. 224 001, May 2017. DOI: 10.1088/1751-8121/aa6cc9 (pages 18, 84).

[220] N. Kitanine, J. M. Maillet, G. Niccoli, and V. Terras, "The open XXZ spin chain in the SoV framework: Scalar product of separate states," *Journal of Physics A: Mathematical and Theoretical*, vol. 51, no. 48, p. 485 201, Oct. 2018. DOI: 10.1088/1751-8121/aae76f (pages 18, 84).

[221] G. Niccoli, H. Pei, and V. Terras, "Correlation functions by separation of variables: The xxx spin chain," *SciPost Physics*, vol. 10, no. 1, Jan. 2021. DOI: 10.21468/scipostphys.10.1.006 (pages 18, 84, 150).

[222] H. Pei and V. Terras, *On scalar products and form factors by separation of variables: The antiperiodic xxz model*, 2021. arXiv: 2011.06109 [math-ph] (pages 18, 84, 150).

[223] J. M. Maillet, G. Niccoli, and B. Pezelier, "Transfer matrix spectrum for cyclic representations of the 6-vertex reflection algebra I," *SciPost Physics*, vol. 2, no. 1, Feb. 2017. DOI: 10.21468/scipostphys.2.1.009 (pages 18, 84).

[224] J. M. Maillet, G. Niccoli, and B. Pezelier, "Transfer matrix spectrum for cyclic representations of the 6-vertex reflection algebra II," *SciPost Phys.*, vol. 5, p. 26, 3 2018. DOI: 10.21468/SciPostPhys.5.3.026 (pages 18, 84).

[225] J. M. Maillet and G. Niccoli, "On quantum separation of variables," *Journal of Mathematical Physics*, vol. 59, no. 9, p. 091 417, Sep. 2018. DOI: 10.1063/1.5050989 (pages 18, 19, 63, 81, 84, 85, 87, 89, 95, 105, 123, 149, 150, 158).

[226] R. A. Horn and C. R. Johnson, *Topics in Matrix Analysis*. Cambridge University Press, Apr. 1991. DOI: 10.1017/cbo9780511840371 (pages 18, 89).

[227] P. P. Kulish, N. Y. Reshetikhin, and E. K. Sklyanin, "Yang-Baxter equation and representation theory: I," *Letters in Mathematical Physics*, vol. 5, no. 5, pp. 393–403, Sep. 1981. DOI: 10.1007/bf02285311 (pages 18, 132, 158, 159).

[228] A. Kuniba, T. Nakanishi, and J. Suzuki, "Functional relations in solvable lattice models I: Functional relations and representation theory," *International Journal of Modern Physics A*, vol. 09, no. 30, pp. 5215–5266, Dec. 1994. DOI: 10.1142/s0217751x94002119 (page 18).

[229] V. Kazakov and P. Vieira, "From characters to quantum (super)spin chains via fusion," *Journal of High Energy Physics*, vol. 2008, no. 10, pp. 050–050, Oct. 2008. DOI: 10.1088/1126-6708/2008/10/050 (pages 18, 125, 132, 133, 158).

[230] A. Kuniba, T. Nakanishi, and J. Suzuki, "T-systems and Y-systems in integrable systems," *Journal of Physics A: Mathematical and Theoretical*, vol. 44, no. 10, p. 103 001, Feb. 2011. DOI: 10.1088/1751-8113/44/10/103001 (pages 18, 158).

[231] V. Kazakov, S. Leurent, and Z. Tsuboi, "Baxter's Q-operators and operatorial Bäcklund flow for quantum (super)-spin chains," *Communications in Mathematical Physics*, vol. 311, no. 3, pp. 787–814, May 2012. DOI: 10.1007/s00220-012-1428-9 (pages 18, 133, 158).

[232] J. M. Maillet and G. Niccoli, "On quantum separation of variables beyond fundamental representations," *SciPost Physics*, vol. 10, no. 2, Feb. 2021. DOI: 10.21468/scipostphys.10.2.026 (pages 18, 87, 95, 105).

[233] J. M. Maillet and G. Niccoli, "On separation of variables for reflection algebras," *Journal of Statistical Mechanics: Theory and Experiment*, vol. 2019, no. 9, p. 094 020, Sep. 2019. DOI: 10.1088/1742-5468/ab357a (pages 19, 87, 105).

[234] J. M. Maillet and G. Niccoli, "Complete spectrum of quantum integrable lattice models associated to $Y(gl(n))$ by separation of variables," *SciPost Phys.*, vol. 6, p. 71, 6 2019. DOI: 10.21468/SciPostPhys.6.6.071 (pages 19, 87, 95, 102–105, 150).

[235] J. M. Maillet and G. Niccoli, "Complete spectrum of quantum integrable lattice models associated to $\mathcal{U}_q(\widehat{gl_n})$ by separation of variables," *Journal of Physics A: Mathematical and Theoretical*, vol. 52, no. 31, p. 315 203, Jul. 2019. DOI: 10.1088/1751-8121/ab2930 (pages 19, 87, 95, 105).

[236] P. Ryan and D. Volin, "Separated variables and wave functions for rational $gl(N)$ spin chains in the companion twist frame," *Journal of Mathematical Physics*, vol. 60, no. 3, p. 032 701, Mar. 2019. DOI: 10.1063/1.5085387 (pages 19, 63, 85, 87, 105, 149, 150).

[237] P. Ryan and D. Volin, *Separation of variables for rational* $\mathfrak{gl}(\mathfrak{n})$ *spin chains in any compact representation, via fusion, embedding morphism and Bäcklund flow*, 2020. arXiv: 2002.12341 [math-ph] (pages 19, 63, 87, 105, 149, 150).

[238] A. Cavaglià, N. Gromov, and F. Levkovich-Maslyuk, "Separation of variables and scalar products at any rank," *Journal of High Energy Physics*, vol. 2019, no. 9, Sep. 2019. DOI: 10.1007/jhep09(2019)052 (pages 19, 105, 149).

[239] N. Gromov, F. Levkovich-Maslyuk, P. Ryan, and D. Volin, "Dual separated variables and scalar products," *Physics Letters B*, vol. 806, p. 135 494, Jul. 2020. DOI: 10.1016/j.physletb.2020.135494 (page 19).

[240] N. Gromov, F. Levkovich-Maslyuk, and P. Ryan, *Determinant form of correlators in high rank integrable spin chains via separation of variables*, 2020. arXiv: 2011.08229 [hep-th] (pages 19, 105, 149).

[241] O. Babelon and C.-M. Viallet, "Hamiltonian structures and lax equations," *Physics Letters B*, vol. 237, no. 3-4, pp. 411–416, Mar. 1990. DOI: 10.1016/0370-2693(90)91198-k (pages 21, 27, 28).

[242] W. Appel and E. Kowalski, *Mathematics for Physics and Physicists*. Princeton University Press, 2007 (page 23).

[243] J. Avan, H. De Vega, and J. Maillet, "Conformally covariant linear system for the four-dimensional self-dual Yang–Mills theory," *Physics Letters B*, vol. 171, no. 2, pp. 255–260, 1986. DOI: https://doi.org/10.1016/0370-2693(86)91544-3 (page 27).

[244] J. Avan, F. Nijhoff, and J. Maillet, "Spinorial linear system for self-dual Yang–Mills equations," *Physics Letters B*, vol. 211, no. 3, pp. 329–334, 1988. DOI: https://doi.org/10.1016/0370-2693(88)90911-2 (page 27).

[245] N. Reshetikhin and P. Weigmann, "Towards the classification of completely integrable quantum field theories (the Bethe-ansatz associated with Dynkin diagrams and their automorphisms)," *Physics Letters B*, vol. 189, no. 1-2, pp. 125–131, Apr. 1987. DOI: 10.1016/0370-2693(87)91282-2 (page 29).

[246] M. J. Ablowitz, M. Ablowitz, P. Clarkson, and P. A. Clarkson, *Solitons, nonlinear evolution equations and inverse scattering*. Cambridge university press, 1991, vol. 149 (page 31).

[247] V. A. Marchenko, *Sturm–Liouville operators and applications*. American Mathematical Soc., 2011, vol. 373 (page 31).

[248] J.-M. Maillet, "New integrable canonical structures in two-dimensional models," *Nuclear Physics B*, vol. 269, no. 1, pp. 54–76, May 1986. DOI: 10.1016/0550-3213(86)90365-2 (pages 32, 34).

[249] L. Freidel and J. Maillet, "Quadratic algebras and integrable systems," *Physics Letters B*, vol. 262, no. 2, pp. 278–284, 1991. DOI: https://doi.org/10.1016/0370-2693(91)91566-E (page 32).

[250] Y. Ishimori, "An integrable classical spin chain," *Journal of the Physical Society of Japan*, vol. 51, no. 11, pp. 3417–3418, 1982. DOI: 10.1143/JPSJ.51.3417. eprint: https://doi.org/10.1143/JPSJ.51.3417 (page 32).

[251] N. Papanicolaou, "Complete integrability for a discrete heisenberg chain," *Journal of Physics A: Mathematical and General*, vol. 20, no. 12, pp. 3637–3652, Aug. 1987. DOI: 10.1088/0305-4470/20/12/018 (page 32).

[252] J. M. Maillet, "New algebraic canonical structures of integrability in 2-d field theories," in *Super Field Theories*, H. C. Lee, V. Elias, G. Kunstatter, R. B. Mann, and K. S. Viswanathan, Eds. Boston, MA: Springer US, 1987, pp. 527–532. DOI: 10.1007/978-1-4613-0913-0_30 (page 34).

[253] S. Weigert, "The problem of quantum integrability," *Physica D: Nonlinear Phenomena*, vol. 56, no. 1, pp. 107–119, 1992. DOI: https://doi.org/10.1016/0167-2789(92)90053-P (page 37).

[254] J.-S. Caux and J. Mossel, "Remarks on the notion of quantum integrability," *Journal of Statistical Mechanics: Theory and Experiment*, vol. 2011, no. 02, P02023, Feb. 2011. DOI: 10.1088/1742-5468/2011/02/p02023 (pages 37, 39).

[255] F. H. Essler, H. Frahm, F. Göhmann, A. Klümper, and V. E. Korepin, *The one-dimensional Hubbard model*. Cambridge University Press, 2005. DOI: 10.1017/CBO9780511534843 (pages 37, 125, 145, 146, 148).

[256] F. Loebbert, "Lectures on yangian symmetry," *Journal of Physics A: Mathematical and Theoretical*, vol. 49, no. 32, p. 323 002, Jul. 2016. DOI: 10.1088/1751-8113/49/32/323002 (pages 37, 129, 153).

[257] V. Terras, "Méthode du problème inverse, groupes quantiques et fonctions de corrélation des modèles intégrables quantiques," Ph.D. dissertation, ED 52 PHAST, 2000 (pages 37, 50, 58).

[258] P. A. M. Dirac, *The principles of quantum mechanics*, 27. Oxford university press, 1981 (page 37).

[259] J. v. Neumann, "Uber funktionen von funktionaloperatoren," *Annals of Mathematics*, vol. 32, no. 2, pp. 191–226, 1931. DOI: https://doi.org/10.2307/1968185 (page 38).

[260] M. Jimbo, "Introduction to the yang–baxter equation," *International Journal of Modern Physics A*, vol. 04, no. 15, pp. 3759–3777, 1989. DOI: 10.1142/S0217751X89001503. eprint: https://doi.org/10.1142/S0217751X89001503 (page 40).

[261] S. K. Satija, J. D. Axe, G. Shirane, H. Yoshizawa, and K. Hirakawa, "Neutron scattering study of spin waves in one-dimensional antiferromagnet $kcuF_3$," *Phys. Rev. B*, vol. 21, pp. 2001–2007, 5 Mar. 1980. DOI: 10.1103/PhysRevB.21.2001 (page 45).

[262] M. Abramowitz and I. A. Stegun, *Handbook of mathematical functions with formulas, graphs, and mathematical tables*. US Government printing office, 1964, vol. 55 (page 47).

[263] O. Foda, K. Iohara, M. Jimbo, R. Kedem, *et al.*, "An elliptic quantum algebra for $\widehat{sl_2}$," *Letters in Mathematical Physics*, vol. 32, no. 3, pp. 259–268, 1994 (page 47).

[264] G. Felder, "Elliptic quantum groups," in *11th International Conference on Mathematical Physics (ICMP-11) (Satellite colloquia: New Problems in the General Theory of Fields and Particles, Paris, France, 25-28 Jul 1994)*, Jul. 1994. arXiv: hep-th/9412207 (page 47).

[265] M. Gaudin, "Modeles exacts en mécanique statistique: La méthode de Bethe et ses généralizations," *Note CEA*, vol. 1559, no. 1, p. 1973, 1972 (page 47).

[266] M. Gaudin, "Diagonalisation d'une classe d'hamiltoniens de spin," *Journal de Physique*, vol. 37, no. 10, pp. 1087–1098, 1976 (page 47).

[267] E. K. Sklyanin, "Boundary conditions for integrable quantum systems," *Journal of Physics A: Mathematical and General*, vol. 21, no. 10, pp. 2375–2389, May 1988. DOI: 10.1088/0305-4470/21/10/015 (page 48).

[268] B. Pezelier, "Separation of variables and new quantum integrable systems with boundaries," Theses, Université de Lyon, Jun. 2018 (page 48).

[269] E. Mukhin, V. Tarasov, and A. Varchenko, "The B. and M. Shapiro conjecture in real algebraic geometry and the Bethe ansatz," *Annals of Mathematics*, vol. 170, no. 2, pp. 863–881, Sep. 2009. DOI: 10.4007/annals.2009.170.863 (page 56).

[270] J. M. Maillet and J. Sanchez de Santos, "Drinfel'd twists and algebraic Bethe ansatz," Dec. 1996. arXiv: q-alg/9612012 (page 57).

[271] S. Belliard and N. Slavnov, "Why scalar products in the algebraic Bethe ansatz have determinant representation," *Journal of High Energy Physics*, vol. 2019, no. 10, pp. 1–17, 2019. DOI: 10.1007/JHEP10(2019)103 (pages 57, 60).

[272] M. Gaudin, B. M. McCoy, and T. T. Wu, "Normalization sum for the Bethe's hypothesis wave functions of the Heisenberg–Ising chain," *Phys. Rev. D*, vol. 23, no. 2, p. 417, 1981. DOI: 10.1103/PhysRevD.23.417 (page 57).

[273] S. Lukyanov and V. Terras, "Long-distance asymptotics of spin-spin correlation functions for the xxz spin chain," *Nuclear Physics B*, vol. 654, no. 3, pp. 323–356, Mar. 2003. DOI: 10.1016/s0550-3213(02)01141-0 (page 60).

[274] N. Kitanine, J. Maillet, N. Slavnov, and V. Terras, "Dynamical correlation functions of the spin-1/2 chain," *Nuclear Physics B*, vol. 729, no. 3, pp. 558–580, Nov. 2005. DOI: 10.1016/j.nuclphysb.2005.08.046 (page 60).

[275] N. Kitanine, J. M. Maillet, N. A. Slavnov, and V. Terras, "On the spin–spin correlation functions of the XXZ spin-1/2 infinite chain," *Journal of Physics A: Mathematical and General*, vol. 38, no. 34, pp. 7441–7460, Aug. 2005. DOI: 10.1088/0305-4470/38/34/001 (page 60).

[276] N. Kitanine, J. M. Maillet, N. A. Slavnov, and V. Terras, "Exact results for the σ_z two-point function of the XXZ chain at $\Delta = 1/2$," *Journal of Statistical Mechanics: Theory and Experiment*, vol. 2005, no. 09, Sep. 2005. DOI: 10.1088/1742-5468/2005/09/l09002 (page 60).

[277] N. A. Slavnov, "Scalar products in GL(3)-based models with trigonometric r-matrix. determinant representation," *Journal of Statistical Mechanics: Theory and Experiment*, vol. 2015, no. 3, P03019, Mar. 2015. DOI: 10.1088/1742-5468/2015/03/p03019 (page 62).

[278] S. Pakuliak, E. Ragoucy, and N. A. Slavnov, "Form factors of local operators in a one-dimensional two-component Bose gas," *Journal of Physics A: Mathematical and Theoretical*, vol. 48, no. 43, p. 435 001, Oct. 2015. DOI: 10.1088/1751-8113/48/43/435001 (page 62).

[279] S. Belliard, N. A. Slavnov, and B. Vallet, "Modified algebraic bethe ansatz: Twisted xxx case," *Symmetry, Integrability and Geometry: Methods and Applications*, Jun. 2018. DOI: 10.3842/sigma.2018.054 (page 63).

[280] N. Gromov and F. Levkovich-Maslyuk, "New compact construction of eigenstates for supersymmetric spin chains," *Journal of High Energy Physics*, vol. 2018, no. 9, p. 85, Sep. 2018. DOI: 10.1007/JHEP09(2018)085 (page 63).

[281] A. Messiah, *Mécanique quantique. Tome 1*. Dunod, 2003 (pages 65, 77).

[282] I. M. Krichever, "Integration of nonlinear equations by the methods of algebraic geometry," *Functional Analysis and Its Applications*, vol. 11, no. 1, pp. 12–26, Jan. 1977. DOI: 10.1007/BF01135528 (page 70).

[283] I. M. Krichever and S. P. Novikov, "Holomorphic bundles over algebraic curves and non-linear equations," *Russian Mathematical Surveys*, vol. 35, no. 6, pp. 53–79, Dec. 1980. DOI: 10.1070/rm1980v035n06abeh001974 (page 70).

[284] A. Messiah, *Mécanique quantique. Tome 2*. Dunod, 2003 (page 77).

[285] G. Niccoli and V. Terras, "Antiperiodic XXZ chains with arbitrary spins: Complete eigenstate construction by functional equations in separation of variables," *Letters in Mathematical Physics*, vol. 105, no. 7, pp. 989–1031, May 2015. DOI: 10.1007/s11005-015-0759-9 (page 84).

[286] D. Levy-Bencheton, G. Niccoli, and V. Terras, "Antiperiodic dynamical 6-vertex model by separation of variables II: Functional equations and form factors," *Journal of Statistical Mechanics: Theory and Experiment*, vol. 2016, no. 3, p. 033 110, Mar. 2016. DOI: 10.1088/1742-5468/2016/03/033110 (page 84).

[287] S. Faldella, N. Kitanine, and G. Niccoli, "The complete spectrum and scalar products for the open spin-1/2 XXZ quantum chains with non-diagonal boundary terms," *Journal of Statistical Mechanics: Theory and Experiment*, vol. 2014, no. 1, P01011, Jan. 2014. DOI: 10.1088/1742-5468/2014/01/p01011 (page 84).

[288] N. Gromov and F. Levkovich-Maslyuk, "New compact construction of eigenstates for supersymmetric spin chains," *Journal of High Energy Physics*, vol. 2018, no. 9, Sep. 2018. DOI: 10.1007/jhep09(2018)085 (page 85).

[289] A. Liashyk and N. A. Slavnov, "On bethe vectors in $\mathfrak{gl}_3$ -invariant integrable models," *Journal of High Energy Physics*, vol. 2018, no. 6, Jun. 2018. DOI: 10.1007/jhep06(2018)018 (page 85).

[290] S. Derkachov and P. Valinevich, "Separation of variables for the quantum $SL(3,\mathbb{C})$ spin magnet: Eigenfunctions of Sklyanin B-operator," *arXiv preprint arXiv:1807.00302*, 2018 (page 85).

[291] N. Gromov, F. Levkovich-Maslyuk, P. Ryan, and D. Volin, *Dual separated variables and scalar products*, 2019. arXiv: 1910.13442 [hep-th] (pages 105, 149).

[292] J. M. Maillet, G. Niccoli, and L. Vignoli, "On Scalar Products in Higher Rank Quantum Separation of Variables," *SciPost Phys.*, vol. 9, p. 86, 6 2020. DOI: 10.21468/SciPostPhys.9.6.086 (page 112).

[293] P. P. Kulish, "Integrable graded magnets," *Journal of Soviet Mathematics*, vol. 35, no. 4, pp. 2648–2662, Nov. 1986. DOI: 10.1007/bf01083770 (pages 125, 144).

[294] J. Gruneberg, "On the construction and solution of $U_q(gl(2,1;\mathbb{C}))$-symmetric models," *Nuclear Physics. B*, vol. 568, Mar. 2000. DOI: 10.1016/S0550-3213(99)00675-6 (page 125).

[295] F. H. L. Essler and V. E. Korepin, "Higher conservation laws and algebraic Bethe ansätze for the supersymmetric t-J model," *Phys. Rev. B*, vol. 46, pp. 9147–9162, 14 Oct. 1992. DOI: 10.1103/PhysRevB.46.9147 (pages 125, 144).

[296] "The Hubbard model at half a century," *Nature Physics*, vol. 9, 523 EP, Sep. 2013. DOI: 10.1038/nphys2759 (pages 125, 145).

[297] I. Bars, "Supergroups and their representations," in *Introduction to Supersymmetry in Particle and Nuclear Physics*. Springer US, 1984, pp. 107–184. DOI: 10.1007/978-1-4757-0917-9_5 (pages 125, 131).

[298] I. Bars, "Supergroups and superalgebras in physics," *Physica D: Nonlinear Phenomena*, vol. 15, no. 1-2, pp. 42–64, 1985. DOI: 10.1016/0167-2789(85)90147-2 (pages 125, 126, 131).

[299] L. Frappat, P. Sorba, and A. Sciarrino, *Dictionary on Lie algebras and superalgebras*. Academic Press (London), 2000, p. 410 (pages 125, 126).

[300] L. Gow, "Yangians of Lie superalgebras," Ph.D. dissertation, University of Sydney, 2007 (pages 125, 129).

[301] A. M. Grabinski and H. Frahm, "Non-diagonal boundary conditions for $\mathfrak{gl}(1|1)$ super spin chains," *Journal of Physics A: Mathematical and Theoretical*, vol. 43, no. 4, p. 045 207, Jan. 2010. DOI: 10.1088/1751-8113/43/4/045207 (page 125).

[302] V. Kac, "Lie superalgebras," *Advances in Mathematics*, vol. 26, no. 1, pp. 8–96, 1977. DOI: https://doi.org/10.1016/0001-8708(77)90017-2 (page 126).

[303] R. Zhang, "The $gl(M|N)$ super Yangian and its finite-dimensional representations," *Letters in Mathematical Physics*, vol. 37, no. 4, pp. 419–434, 1996 (page 129).

[304] I. Bars, B. Morel, and H. Ruegg, "Kac–Dynkin diagrams and supertableaux," *Journal of mathematical physics*, vol. 24, no. 9, pp. 2253–2262, 1983. DOI: 10.1063/1.525970 (pages 131, 133).

[305] A. V. Zabrodin, "Bäcklund transformations for the difference Hirota equation and the supersymmetric Bethe ansatz," *Theoretical and Mathematical Physics*, vol. 155, no. 1, pp. 567–584, Apr. 2008. DOI: 10.1007/s11232-008-0047-2 (pages 133–135).

[306] M. Nazarov, "Quantum Berezinian and the classical Capelli identity," *Letters in Mathematical Physics*, vol. 21, no. 2, pp. 123–131, 1991. DOI: 10.1007/bf00401646 (page 133).

[307] P. B. Ramos and M. J. Martins, "Algebraic Bethe ansatz approach for the one-dimensional Hubbard model," *Journal of Physics A: Mathematical and General*, vol. 30, no. 7, pp. L195–L202, Apr. 1997. DOI: 10.1088/0305-4470/30/7/009 (pages 146, 148).

[308] F. Göhmann and V. Inozemtsev, "The Yangian symmetry of the Hubbard models with variable range hopping," *Physics Letters A*, vol. 214, no. 3-4, pp. 161–166, May 1996. DOI: 10.1016/0375-9601(96)00174-0 (page 146).

[309] N. Beisert, M. de Leeuw, and P. Nag, "Fusion for the one-dimensional Hubbard model," *Journal of Physics A: Mathematical and Theoretical*, vol. 48, no. 32, p. 324 002, 2015. DOI: 10.1088/1751-8113/48/32/324002 (page 148).

[310] M. Martins and P. Ramos, "The quantum inverse scattering method for Hubbard-like models," *Nuclear Physics B*, vol. 522, no. 3, pp. 413–470, Jul. 1998. DOI: 10.1016/s0550-3213(98)00199-0 (page 148).

[311] P. B. Ramos and M. J. Martins, "One parameter family of an integrable $spl(2|1)$ vertex model: Algebraic Bethe ansatz and ground state structure," *Nucl. Phys. B*, vol. 474, pp. 678–714, 1996. DOI: 10.1016/0550-3213(96)00342-2. arXiv: hep-th/9604072 (page 148).

[312] M. J. Martins and P. B. Ramos, "The Algebraic Bethe ansatz for rational braid-monoid lattice models," *Nucl. Phys. B*, vol. 500, pp. 579–620, 1997. DOI: 10.1016/S0550-3213(97)00342-8. arXiv: hep-th/9703023 (page 148).

[313] P. Ryan, "Integrable systems, separation of variables and the yang-baxter equation," Ph.D. dissertation, Trinity College Dublin, 2021 (page 150).

[314] S. Ekhammar, H. Shu, and D. Volin, "Extended systems of Baxter Q-functions and fused flags I: simply-laced case," Aug. 2020. arXiv: 2008.10597 [math-ph] (page 150).

[315] S. Ekhammar and D. Volin, "Bethe Algebra using Pure Spinors," Apr. 2021. arXiv: 2104.04539 [math-ph] (page 150).

[316] V. G. Drinfeld, "Hopf algebras and the quantum Yang–Baxter equation," *Sov. Math. Dokl.*, vol. 32, pp. 254–258, 1985 (page 153).

[317] V. G. Drinfeld, "A New realization of Yangians and quantized affine algebras," *Sov. Math. Dokl.*, vol. 36, pp. 212–216, 1988 (page 153).

[318] P. Cartier, "A primer of hopf algebras," in *Frontiers in Number Theory, Physics, and Geometry II: On Conformal Field Theories, Discrete Groups and Renormalization*, P. Cartier, P. Moussa, B. Julia, and P. Vanhove, Eds. Berlin, Heidelberg: Springer Berlin Heidelberg, 2007, pp. 537–615. DOI: 10.1007/978-3-540-30308-4_12 (page 153).

[319] A. V. Zabrodin, "Hirota equation and Bethe ansatz," *Theoretical and Mathematical Physics*, vol. 116, no. 1, pp. 782–819, Jul. 1998. DOI: 10.1007/bf02557123 (page 158).

[320] V. Kazakov, S. Leurent, and D. Volin, "T-system on T-hook: Grassmannian solution and twisted Quantum Spectral Curve," *Journal of High Energy Physics*, vol. 2016, no. 12, Dec. 2016. DOI: 10.1007/jhep12(2016)044 (page 158).

[321] A. Zabrodin, "Discrete Hirota's equation in quantum integrable models," *International Journal of Modern Physics B*, vol. 11, no. 26n27, pp. 3125–3158, 1997. DOI: 10.1142/S0217979297001520 (page 158).

Article [LV1]

J. M. Maillet, G. Niccoli, and L. Vignoli, "Separation of variables bases for integrable $gl_{\mathcal{M}|\mathcal{N}}$ and Hubbard models," *SciPost Phys.*, vol. 9, p. 60, 4 2020. DOI: 10.21468/SciPostPhys.9.4.060

SciPost Phys. 9, 060 (2020)

Separation of variables bases for integrable $gl_{\mathcal{M}|\mathcal{N}}$ and Hubbard models

Jean Michel Maillet[⋆], **Giuliano Niccoli**[†] **and Louis Vignoli**[‡]

Univ Lyon, ENS de Lyon, Univ Claude Bernard, CNRS,
Laboratoire de Physique, UMR 5672, F-69342 Lyon, France

⋆ maillet@ens-lyon.fr, † giuliano.niccoli@ens-lyon.fr, ‡ louis.vignoli@ens-lyon.fr

Abstract

We construct quantum Separation of Variables (SoV) bases for both the fundamental inhomogeneous $gl_{\mathcal{M}|\mathcal{N}}$ supersymmetric integrable models and for the inhomogeneous Hubbard model both defined with quasi-periodic twisted boundary conditions given by twist matrices having simple spectrum. The SoV bases are obtained by using the integrable structure of these quantum models, i.e. the associated commuting transfer matrices, following the general scheme introduced in [1]; namely, they are given by set of states generated by the multiple actions of the transfer matrices on a generic co-vector. The existence of such SoV bases implies that the corresponding transfer matrices have non-degenerate spectrum and that they are diagonalizable with simple spectrum if the twist matrices defining the quasi-periodic boundary conditions have that property. Moreover, in these SoV bases the resolution of the transfer matrix eigenvalue problem leads to the resolution of the full spectral problem, i.e. both eigenvalues and eigenvectors. Indeed, to any eigenvalue is associated the unique (up to a trivial overall normalization) eigenvector whose wave-function in the SoV bases is factorized into products of the corresponding transfer matrix eigenvalue computed on the spectrum of the separated variables. As an application, we characterize completely the transfer matrix spectrum in our SoV framework for the fundamental $gl_{1|2}$ supersymmetric integrable model associated to a special class of twist matrices. From these results we also prove the completeness of the Bethe Ansatz for that case. The complete solution of the spectral problem for fundamental inhomogeneous $gl_{\mathcal{M}|\mathcal{N}}$ supersymmetric integrable models and for the inhomogeneous Hubbard model under the general twisted boundary conditions will be addressed in a future publication.

Received 28-04-2020
Accepted 26-08-2020
Published 27-10-2020
doi:10.21468/SciPostPhys.9.4.060

Check for updates

Contents

1 Introduction

In this paper, we generalize the construction introduced in [1] to generate quantum separation of variables (SoV) bases for the class of integrable quantum lattice models associated to the $gl_{\mathscr{M}|\mathscr{N}}$ Yang-Baxter superalgebras [2–4] and to the Hubbard model [5–10] with quasi-periodic twisted boundary conditions given by twist matrices having simple spectrum. The quantum version of the separation of variables and its development in the integrable framework of the quantum inverse scattering method [11–19] originate in the pioneering works of Sklyanin [20–25]. Since then, the SoV method has been successfully applied to several quantum integrable models [26–60] .

Integrable quantum models define the natural background to look for exact non-perturbative results toward the complete solution of some 1+1 dimensional quantum field theories or some equivalent two-dimensional systems in statistical mechanics. They have found natural applications in the exact description of several important phenomena in condensed matter and have provided exact results to be compared with experiments. A prominent example is the quantum Heisenberg spin chain [61] introduced as a model to study phase transitions and critical points of magnetic systems. First exact results for the Hamiltonian's spectrum (eigenvalues and eigenvectors) have been obtained by Bethe for the spin 1/2 XXX chain, thanks to his famous coordinate ansatz [62]. Then it has been extended to the anisotropic spin 1/2 XXZ chain in [63,64], while Baxter [65,66] has obtained first exact results for the Hamiltonian's spectrum of the fully anisotropic spin 1/2 XYZ chain. In statistical mechanics, these quantum models correspond to the six-vertex and eight-vertex models. The ice-type (six-vertex) models [67], accounting for the residual entropy of water ice for crystal lattices with hydrogen bonds, has

first been described in the Bethe Ansatz framework in [68]. For the eight-vertex model [69,70] first exact results in the two-dimensional square lattice are due to Baxter [66,71,72]. The integrable structure[1] of these spin chains and statistical mechanics models has been revealed in the Baxter's papers [65,72,78]. There, the one-parameter family of eight-vertex transfer matrices have been shown to be commutative and the XYZ Hamiltonian to be proportional to its logarithmic derivative, when computed at a particular value of its spectral parameter. The subsequent development of a systematic description of quantum integrable models has been achieved through the development of the quantum inverse scattering framework [11–19]. In particular, the work of Faddeev, Sklyanin and Takhtajan [12] has set the basis for the classification of the Yang-Baxter algebra representations and the natural framework for the discovering of quantum groups [79–82]. The paper [12] has also introduced the Algebraic Bethe Ansatz (ABA), an algebraic version of the original coordinate Bethe Ansatz.

Exact results are also available for the quantum dynamics, *i.e.* form factors and correlations functions, of some integrable quantum models. This is for example the case for XXZ quantum spin 1/2 chain under special boundary conditions whose correlation functions admit multiple integral representations [83–95].

In this context, the form factor expansion has proven to be a very powerful tool: on the one hand, to compute dynamical structure factors [96,97], quantities directly accessible experimentally through neutron scattering [98]; on the other hand, to have access to the asymptotic behaviour of correlation functions of these XXZ chains in the thermodynamic limit and explicit contact with conformal field theory [90–92,99–105].

Integrable quantum models also led to non-perturbative results in the out-of-equilibrium physics context, ranging from the relaxation behaviour of some classical stochastic processes to quantum transport. The XXZ quantum spin chains, under general integrable boundary conditions, appear for example both in the description of the asymmetric simple exclusion processes [106–112] and the description of transport properties of quantum spin systems [113,114].

The new experiments allowing ultra-cold atoms to be trapped in optical lattices have produced concrete realizations of quantum integrable lattices, like the Heisenberg spin chains but also more sophisticated models like the Hubbard model [115,116]. They provide a further natural context for direct comparison of exact theoretical predictions with experiments.

The Hubbard model is of fundamental importance in physics. It is a celebrated quantum model in condensed matter theory, defining a first generalization beyond the band approach for modelling the solid state physics. It manages to describe interacting electrons in narrow energy bands and allows to account for important physical phenomena of different physical systems. Relevant examples are the high temperature superconductivity, band magnetism and the metal-insulator transitions. We refer to the book [10] for a more detailed description of its physical applications and of the known exact results and relevant literature. Here, let us recall that the Hubbard chain is integrable in the quantum inverse scattering framework and it has been first analysed by Bethe Ansatz techniques in a famous paper by Lieb and Wu [117,118]. Coordinate Bethe Ansatz wave-functions for the Hubbard Hamiltonian eigenvectors have been obtained in [119] and subsequent papers. The quantum inverse scattering formulation has been achieved thanks to the Shastry's derivation of the R-matrix [120–122]. From this, the one-parameter family of commuting transfer matrices, generating the Hubbard Hamiltonian by standard logarithmic derivative, can be introduced. The proof that this R-matrix satisfies the Yang-Baxter equation has been given in [123]. In [124–126] a Nested Algebraic Bethe Ansatz for the Hubbard model has been introduced while the quantum transfer matrix approach to study the thermodynamics of the Hubbard model has been considered in [127]. Interestingly, the Hubbard Hamiltonian is invariant under the direct sum of two $Y(sl_2)$ Yangians, as derived

[1] See also [68,69,73–77] for some previous partial understating.

in [128–131], while the structure of the fusion relations for the Hubbard model have been studied in [132], see also [133] for the finite temperature case. It is also relevant to remark that under a strong coupling limit and some special choice of the remaining parameters [10], the Hubbard Hamiltonian leads to the Hamiltonian of the t-$Jgl_{1|2}$-supersymmetric model, another well known model for the description of the high-temperature superconductivity, see [134–136] and references therein.

While integrable quantum models naturally emerge in the description of 1+1 quantum or 2-dimensional statistical mechanics phenomena, they are not really confined to this realm. For example, they play a fundamental role in deriving exact results also for four-dimensional quantum field theory like the planar $N = 4$ Supersymmetric Yang-Mills (SYM) gauge theory, see the review paper [137] and references therein. In this context, integrability tools have been used to derive exact results, like characterizations of the scaling dimensions of local operators for general values of the coupling constant. Notably, such results can be used also as a test of the AdS/CFT correspondence in this planar limit. Indeed, holding for arbitrary values of the coupling, they allow for a verification of the agreement both at weak and strong couplings with the perturbative results obtained respectively in gauge and string theory contexts. Integrability is also becoming relevant in the exact computation of observables. Interestingly, quantum integrable higher rank spin and super-spin chains have found applications in the computation of correlation functions in $N = 4$ SYM, see *e.g.* [138–143]. The same integrable Hubbard model enters in the description of the planar $N = 4$ SYM gauge theory [137, 144] in the large volume asymptotic regime. Indeed, relevant examples are the connection between its dilatation generator at weak coupling and the Hubbard Hamiltonian derived in [145] and the equivalence shown[2] in [146, 147] between the bound state S-matrix for $AdS_5 \times S^5$ superstring [152–154] and two copies of the Shastry's R-matrix of the Hubbard model multiplied by a nontrivial dressing phase. Moreover, this S-matrix enjoys the Yangian symmetry associated to the centrally extended $su(2|2)$ superalgebra [152–155] and an Analytic Bethe Ansatz description of the spectrum has been introduced on this basis in [146, 147, 156].

The large spectrum of applications of these higher rank quantum integrable models and of the Hubbard model, clearly motivate our interest in their analysis by quantum separation of variables. Let us mention that the first interesting analysis toward the SoV description of higher rank models have been presented in [25, 28], see also [57]. More recently, in [157], by the exact analysis of quantum chains of small sizes, the spectrum of the Sklyanin's B-operator has been conjectured together with its diagonalizability for fundamental representations of gl_3 Yang-Baxter algebra associated to some classes of twisted boundary conditions. While in [158] the SoV basis has been constructed for non-compact representations. In [1, 159–161] we have solved the transfer matrix spectral problem[3] for a large class of higher rank quantum integrable models. That is for integrable quantum models associated to the fundamental representations of the $Y(gl_n)$ and $U_q(\widehat{gl_n})$ Yang-Baxter algebra and of the $Y(gl_n)$ reflection algebra. This has been done by introducing and developing a new SoV approach relying only on the integrable structure of the model, i.e. the commutative algebra of conserved charges. In [163, 164] our construction of SoV bases has been extended to general finite dimensional representations of gl_n Yang-Baxter algebra with twisted boundary conditions. In [1], we have proven for the gl_2 representations and for small size gl_3 representations that our SoV bases can be made coinciding with the Sklyanin's ones, if Sklyanin's construction can be applied. In [163, 164] this statement has been extended to the higher rank cases, in this way providing an SoV proof[4]

[2] One should remark that the spin chain approaches, as those in [146, 147], miss the so-called wrapping corrections of the AdS/CFT spectrum while a full description for this spectral problem has been proposed in [148] and thereafter extensively tested, see e.g. [149] and the reviews [150] and [151] for further developments.

[3] While in [162], we have described in detail how our approach works beyond fundamental representations for $Y(gl_2)$.

[4] Indeed, the first proof of this conjecture has been given in [165] in the nested Bethe Ansatz framework.

of the non-nested Bethe Ansatz representation conjectured in [157] of the transfer matrix eigenstates as Sklyanin's B-operator multiple action in the zeros of a polynomial Q-function on a given reference state.

Here, we construct the quantum Separation of Variables (SoV) bases in the representation spaces of both the fundamental inhomogeneous $gl_{\mathscr{M}|\mathscr{N}}$ Yang-Baxter superalgebras and the inhomogeneous Hubbard model under general quasi-periodic twisted boundary conditions defined by twist matrices having simple spectrum. Let us mention here an interesting proposal for a representation of the eigenvectors using a single B operator in [166] for $gl_{\mathscr{M}|\mathscr{N}}$ models inspired by the SoV related methods [157]. In our approach, the SoV bases are constructed by using the known integrable structure of these quantum models, *i.e.* the associated commuting transfer matrices, following our general ideas introduced in [1]. The SoV bases are generated by the multiple actions of the transfer matrices on a generic co-vector of the Hilbert space. The fact that we are able to prove that such sets of co-vectors indeed form bases of the space of states implies important consequences on the spectrum of the transfer matrices. In fact, it follows that the transfer matrices have non degenerate (simple) spectrum or that they are diagonalizable with simple spectrum if the twist matrix respectively has simple spectrum or is diagonalizable with simple spectrum. Moreover, in our SoV bases the resolution of the transfer matrix eigenvalue problem is equivalent to the resolution of the full transfer matrix spectrum (eigenvalues and eigenvectors). Indeed, our SoV bases allow us to associate uniquely to any eigenvalue an eigenvector whose wave-function has the factorized form in terms of product of the transfer matrix eigenvalues on the spectrum of the separated variables.

It is worth pointing out that for these classes of higher rank quantum integrable models, fewer exact results are available when compared to those described for the best known examples of the XXZ spin 1/2 quantum integrable chains. Exact results are mainly confined to the spectral problem and only recently some breakthrough has been achieved toward the dynamics in the framework of the Nested Algebraic Bethe Ansatz (NABA) [167–170], for some higher rank spin and super-spin chains [165, 171–180].

More in detail, in the supersymmetric case, the associated spectral problem has been analysed by using the transfer matrix functional relations, generated by fusion [181–183] of irreducible representations in the auxiliary space of the representation. The Analytic Bethe Ansatz [183–185] developed in this functional framework has been applied to the spectral problem of these supersymmetric models. An important step in the systematic description and analysis of these functional equations has been done by rewriting them in Bazhanov and Reshetikhin's determinant form in [186], and in a Hirota bilinear difference equation form in [187–189]. These so-called T-systems appear both in classical and quantum integrability. An interesting account for their relevance and different application areas can be found in [190]. The validity of these fusion rules and of Analytic Bethe Ansatz description in the supersymmetric case have been derived in [191–194]. In [195, 196] a method has been developed and applied to the supersymmetric case based on the use of Bäcklund transformations on the Hirota-type functional equations [197–199]. It allows a systematic classification of the different Nested Algebraic Bethe Ansatz equations and TQ-functional equations, which emerge naturally in the supersymmetric case, due to different possible choices of the systems of simple roots. It also allows the identification of QQ-functional equations of Hirota type for the Baxter's Q-functions, see for example [200–202]. Nested Algebraic Bethe Ansatz [136, 170, 203] has been successfully used to get Bethe vectors representations for fundamental representations of $Y(gl_{\mathscr{M}|\mathscr{N}})$ and $U_q(\widehat{gl_{\mathscr{M}|\mathscr{N}}})$, see also the recent result [204], while determinant formulae for Bethe eigenvector norms, scalar products and some computations of form factors have been made accessible in [205–208] for the $Y(gl_{1|2})$ and $Y(gl_{2|1})$ case. The completeness of the Nested Algebraic Bethe Ansatz approach for supersymmetric Yangian representations

has been shown in the following papers[5] [209, 210], respectively for the representations of $Y(gl_{1|1})$ and of the general $Y(gl_{\mathcal{M}|\mathcal{N}})$, in the setup of the so-called QQ-Wronskian equations introduced in [211] for the non-supersymmetric case $Y(gl_{\mathcal{M}})$ and more recently in [204] for the $Y(gl_{\mathcal{M}|\mathcal{N}})$ models.

In this paper we start to develop the quantum separation of variables method for these supersymmetric integrable quantum models. The natural advantage of the SoV method is that it is not an Ansatz method and then the completeness of the spectrum description is mainly a built-in feature of it, as proven for a large class of quantum integrable models [39–41, 43–59]. More in detail, no Ansatz is done on the SoV representation of transfer matrix eigenvectors.[6] Indeed, their factorized wave-functions in terms of the eigenvalues of the transfer matrix, or of the Baxter's Q-operator [66, 212–233] are just a direct consequence of the form of the SoV basis. Moreover, these SoV representations are extremely simple and universal and should lead to determinant formulae for scalar products.[7] The SoV representation of transfer matrix eigenvectors also brings to Algebraic Bethe Ansatz rewriting of non-nested type for the eigenvectors,[8] *i.e.* as the action on a SoV induced "reference vector" of a single monomial of SoV induced "B-operators" over the zeros of an associated Q-operator eigenvalue [1]. It is worth to point out that this represents a strong simplification *w.r.t.* the eigenvector representation in NABA approach, where the holding different representations [170] are equivalent to an "explicit representation" which is written in the form of a sum over partitions. This type of results in the SoV framework is even more important in the case of the Hubbard model. Indeed, there, algebraic approaches like NABA are mainly limited to the two particle case [124] and other exact results are accessible only via coordinate Bethe Ansatz.[9]

The paper is organized as follows. In section 2, we first shortly present the graded formalism for the superalgebras $gl_{\mathcal{M}|\mathcal{N}}$ and their fundamental representations, we sum up the main properties of the hierarchy of the fused transfer matrices and their reconstruction in terms of the fundamental[10] one. The SoV basis is then constructed in subsection 2.4 by using the integrable structure of these models. In subsection 2.5, we make some general statement about the closure and admissibility conditions to fix the transfer matrix spectrum for these quantum integrable models. In section 3, we specialize the discussion on the $gl_{1|2}$ model. We state our conjecture on the corresponding closure conditions and we present some first arguments in favour of it in subsection 3.1. Then, we treat in detail a special twisted case in subsection 3.2, for which we prove that the entire spectrum of the transfer matrix is characterized by our conjecture. Then, we give a reformulation of the spectrum in terms of the solutions to a quantum spectral curve equation. Moreover, for these representations, we show the completeness of

[5] Both the papers [209, 210] appeared after the present paper and they are not directly related to our SoV approach.

[6] In the Bethe Ansatz framework, the fact that the form of the eigenvectors is fixed by the Ansatz implies that to prove the completeness of the spectrum description one has to define first admissibility conditions which generate nonzero vectors and then one has to count the number of these solutions and show that it coincides with the dimension of the representation space, in absence of Jordan blocks. This first step is for example done in the papers [204, 210, 211] through the introduction of the isomorphism to the QQ-Wronskian equations.

[7] Indeed, our recent results on higher rank scalar products [234] show the appearance of simple determinant formulae once the SoV basis are appropriately chosen, see also [235, 236] for some interesting SoV analysis of the higher rank scalar products.

[8] Note that this SoV versus ABA rewriting of transfer matrix eigenvectors was first observed in a rank 1 case in [30, 31] and it can be extended in general for polynomial Q-operators, as e.g. argued in [1]. One has to mention that these non-nested forms were first proposed in [157, 166] together with the form of the B-operator for rank higher than 2.

[9] In fact, to our knowledge, the generic N-particle transfer matrix eigenvalues are well verified guesses [10], no Bethe vectors representation is achieved for the corresponding eigenvectors and the conjectured norm formula [10] has to be proven yet.

[10] That is the transfer matrix associated to Lax operators on isomorphic auxiliary and quantum spaces, *i.e.* in our fundamental representations the transfer matrices with Lax operators coinciding with the R-matrix.

the Nested Algebraic Bethe Ansatz, by proving that any eigenvalue can be rewritten in a NABA form using our Q-functions. In section 4, we derive an SoV basis for the Hubbard model with general integrable twist matrix having simple spectrum. Finally, in appendix A for the $gl_{1|2}$ model with general integrable twist matrices, we verify that the NABA form of eigenvalues satisfies the closure and admissibility conditions, implying its compatibility with our conjecture in the SoV framework. In appendix B, we present the proof that our conjecture indeed completely characterizes the transfer matrix spectrum for any integrable twist matrix having simple spectrum for the model defined on two sites while we verify this property by numerical computations for three sites. In appendix C, we give a derivation of the closure relation for the $gl_{\mathcal{M}|\mathcal{N}}$ case.

2 Separation of variables for integrable $gl_{\mathcal{M}|\mathcal{N}}$ fundamental models

Graded structures and Lie superalgebras are treated in great details in [237, 238]. The quantum inverse scattering construction for graded models was introduced in [2–4], and summarized in many articles, see e.g. [10, 156, 239]. Details on Yangians structures for Lie superalgebras can be found in [240, 241].

For the article to be self contained, we introduce the graded algebra $gl_{\mathcal{M}|\mathcal{N}}$ and its fundamental Yangian model in the following, and make explicit the notations and rules for graded computations.

2.1 Graded formalism and integrable $gl_{\mathcal{M}|\mathcal{N}}$ fundamental models

A super vector space V is a $\mathbb{Z}_2$-graded vector space, *ie.* we have

$$V = V_0 \oplus V_1. \tag{2.1}$$

Vectors of V_0 are even, while vectors of V_1 are odd. Objects that have a well-defined parity, either even or odd, are called homogeneous. The parity map, defined on homogeneous objects, writes

$$p : A \in V \longmapsto p(A) = \bar{A} = \begin{cases} 0 & \text{if } A \in V_0 \\ 1 & \text{if } A \in V_1 \end{cases}. \tag{2.2}$$

Maps between $\mathbb{Z}_2$-graded objects are called even if they preserve the parity of objects, or odd if they flip it. An associative superalgebra is a super vector space with an even multiplication map that is associative and the algebra has a unit element for the multiplication. For a superalgebra V we have $V_i V_j \subseteq V_{i+j(\mathrm{mod}\,2)}$. A Lie superalgebra is a super vector space $\mathfrak{g} = \mathfrak{g}_o \oplus \mathfrak{g}_1$ equipped with an even linear map $[\ ,\ ] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ that is graded antisymmetric and satisfies the graded Jacobi identity.

The set of linear maps from V to itself is noted $\mathrm{End}\, V$, and it is a $\mathbb{Z}_2$-graded vector space as well. It is an associative superalgebra with multiplication given by the composition. It is also a Lie superalgebra with the Lie super-bracket defined as the graded commutator between homogeneous objects for the multiplication

$$[A, B] = AB - (-1)^{\bar{A}\bar{B}} BA, \tag{2.3}$$

which extends linearly to the whole space. As a Lie superalgebra, it is denoted $gl(V)$.

Tensor products The tensor product of two super vector spaces V and W is the tensor product of the underlying vectors spaces, with the $\mathbb{Z}_2$-grading structure given by

$$\text{for } k=0 \text{ or } 1, \quad (V\otimes W)_k = \bigoplus_{i+j=k(\mathrm{mod}\,2)} V_i \otimes W_j. \tag{2.4}$$

This also defines the tensor product of associative superalgebras, being defined on the underlying vector space structure, but then we have to define an associative multiplication compatible with the grading. For A, B two associative superalgebras, the multiplication rule on $A\otimes B$ is given by

$$(a_1\otimes b_1)(a_2\otimes b_2) = (-1)^{\bar{b}_1\bar{a}_2} a_1a_2\otimes b_1b_2, \tag{2.5}$$

for $a_1, a_2 \in A$ and $b_1, b_2 \in B$ homogeneous, and extends linearly to $A\otimes B$.

This rule of sign also appears in the action of $A\otimes B$ on $V\otimes W$, where V is an A-module and W is a B-module. We have

$$(a\otimes b)\cdot(v\otimes w) = (-1)^{\bar{b}\bar{v}} a\cdot v\otimes b\cdot w, \tag{2.6}$$

for $a\in A$, $b\in B$, $v\in V$ and $w\in W$.

This rule extends naturally to N-fold tensor product, $N>2$. For example,

$$(a_1\otimes b_1\otimes c_1)(a_2\otimes b_2\otimes c_2) = (-1)^{\bar{a}_2(\bar{b}_1+\bar{c}_1)+\bar{b}_2\bar{c}_1} a_1a_2\otimes b_1b_2\otimes c_1c_2. \tag{2.7}$$

For some authors, the above construction goes explicitly by the name of *super* or *graded* tensor product. We will stick to the name tensor product.[11]

Lie superalgebra $gl_{\mathcal{M}|\mathcal{N}}$ Let $V=\mathbb{C}^{\mathcal{M}|\mathcal{N}}$ be the complex vector superspace with even part of dimension $\mathcal{M}$ and odd part of dimension $\mathcal{N}$. The general linear Lie algebra $gl_{\mathcal{M}|\mathcal{N}} = gl(\mathbb{C}^{\mathcal{M}|\mathcal{N}})$ is the $\mathbb{Z}_2$-graded vector space $\mathrm{End}\,\mathbb{C}^{\mathcal{M}|\mathcal{N}}$ with the Lie super-bracket defined by the graded commutator (2.3).

We fix a homogeneous basis $\{v_1,\dots,v_{\mathcal{M}},v_{\mathcal{M}+1},\dots,v_{\mathcal{M}+\mathcal{N}}\}$ of $\mathbb{C}^{\mathcal{M}|\mathcal{N}}$, where v_i is even for $i\le\mathcal{M}$ and odd for $i\ge\mathcal{M}+1$ and we assign a parity to the index themselves for convenience: $\bar{i}=0$ for $i\le\mathcal{M}$ and $\bar{i}=1$ for $i\ge\mathcal{M}+1$.

The elementary operators e_i^j of $gl_{\mathcal{M}|\mathcal{N}}$ have parity $p(e_i^j)=\bar{i}+\bar{j}(\mathrm{mod}\,2)$. They are defined by their action on the basis of V by

$$e_i^j\cdot v_k = \delta_k^j v_i. \tag{2.8}$$

Since they multiply as

$$e_i^j e_k^l = \delta_k^j e_i^l, \tag{2.9}$$

it follows that the graded commutator is

$$\left[e_i^j, e_k^l\right] = e_i^j e_k^l - (-1)^{p(e_i^j)p(e_k^l)} e_k^l e_i^j = \delta_k^j e_i^l - (-1)^{(\bar{i}+\bar{j})(\bar{k}+\bar{l})}\delta_i^l e_k^j. \tag{2.10}$$

Elements of $gl_{\mathcal{M}|\mathcal{N}}$ decompose on the elementary operators as

$$a = \sum_{i,j=1}^{\mathcal{M}+\mathcal{N}} a_j^i e_i^j \equiv a_j^i e_i^j, \tag{2.11}$$

[11] Note also that some authors prefer to use a matrix formalism and by "super tensor product" denote a morphism between graded and non-graded structure, see [4], appendix A of [136] or appendix A [195]. We will not make any extensive use of it in the following.

where in the last term the sum over repeated indexes is omitted, as we will do in the following. Elements of $\left(gl_{\mathcal{M}|\mathcal{N}}\right)^{\otimes \mathsf{N}}$ writes

$$A = A^{i_1 \dots i_{\mathsf{N}}}_{j_1 \dots j_{\mathsf{N}}} \; e^{j_1}_{i_1} \otimes \dots \otimes e^{j_{\mathsf{N}}}_{i_{\mathsf{N}}}. \tag{2.12}$$

Note that due to the sign rule (2.5), the coordinates $A^{i_1 \dots i_{\mathsf{N}}}_{j_1 \dots j_{\mathsf{N}}}$ do not coincide with the components of the image of $v_{j_1} \otimes \dots \otimes v_{j_{\mathsf{N}}}$ in the tensored basis of $V^{\otimes \mathsf{N}}$:

$$A \cdot v_{j_1} \otimes \dots \otimes v_{j_{\mathsf{N}}} = (-1)^{\sum_{k=1}^{\mathsf{N}-1} \bar{i}_k (\bar{i}_{k+1} + \dots + \bar{i}_{\mathsf{N}})} A^{i_1 \dots i_{\mathsf{N}}}_{j_1 \dots j_{\mathsf{N}}} \; v_{i_1} \otimes \dots \otimes v_{i_{\mathsf{N}}}. \tag{2.13}$$

In the non-graded case, these two tensors would be identical.

One may use the coordinates expression to check the parity of a given operator of $\left(gl_{\mathcal{M}|\mathcal{N}}\right)^{\otimes \mathsf{N}}$. An operator A is homogeneous of parity $p(A)$ if

$$\forall\, i_1, \dots, i_{\mathsf{N}}, j_1, \dots, j_{\mathsf{N}}, \quad (-1)^{\bar{i}_1 + \bar{j}_1 + \dots + \bar{i}_{\mathsf{N}} + \bar{j}_{\mathsf{N}}} A^{i_1 \dots i_{\mathsf{N}}}_{j_1 \dots j_{\mathsf{N}}} = (-1)^{p(A)} A^{i_1 \dots i_{\mathsf{N}}}_{j_1 \dots j_{\mathsf{N}}}. \tag{2.14}$$

The supertrace is defined on the elementary operators as $\mathrm{str}\, e^j_i = (-1)^{\bar{j}} \delta^j_i$. Elements of $gl_{\mathcal{M}|\mathcal{N}}$ may write as a block matrix

$$A = \begin{pmatrix} A_{(\mathcal{M},\mathcal{M})} & A_{(\mathcal{M},\mathcal{N})} \\ A_{(\mathcal{N},\mathcal{M})} & A_{(\mathcal{N},\mathcal{N})} \end{pmatrix} \in gl_{\mathcal{M}|\mathcal{N}}, \tag{2.15}$$

where $A_{(\mathcal{M},\mathcal{M})}$ is an $\mathcal{M}$ by $\mathcal{M}$ square matrix, $A_{(\mathcal{M},\mathcal{N})}$ an $\mathcal{M}$ by $\mathcal{N}$ square matrix, etc. Hence we have $\mathrm{str} A = \mathrm{tr} A_{(\mathcal{M},\mathcal{M})} - \mathrm{tr} A_{(\mathcal{N},\mathcal{N})}$. Note that the supertrace vanishes on the graded commutator

$$\mathrm{str}([A, B]) = 0. \tag{2.16}$$

Dual space Let us denote $|i\rangle \equiv v_i$. The dual basis $\{\langle j|\}_{j=1,\dots,\mathcal{M}+\mathcal{N}}$ is defined by

$$\forall\, i, \quad \langle j|i\rangle = \delta_{ji}. \tag{2.17}$$

The covectors are graded by $p(\langle j|) = \bar{j}$. The dual of a vector $|\psi\rangle = \psi_i |i\rangle$ is

$$\langle \psi| = (\psi_i |i\rangle)^\dagger = \psi_i^* \langle i|, \tag{2.18}$$

where the star $*$ stands for the complex conjugation. For $V^{\otimes \mathsf{N}}$, the dual basis covectors have an additional sign in their definition

$$(|i_1\rangle \otimes \dots \otimes |i_N\rangle)^\dagger \equiv \langle i_1| \otimes \dots \otimes \langle i_N| (-1)^{\sum_{k=2}^{N} \bar{i}_k (\bar{i}_1 + \dots + \bar{i}_{k-1})}, \tag{2.19}$$

such that it compensates for the permutation of vectors and covectors:

$$\begin{aligned} (|i_1\rangle \otimes \dots \otimes |i_N\rangle)^\dagger (|j_1\rangle \otimes \dots \otimes |j_{\mathsf{N}}\rangle) &= (-1)^{\sum_{k=2}^{\mathsf{N}} \bar{i}_k (\bar{i}_1 + \dots + \bar{i}_{k+1}) + \sum_{k=2}^{\mathsf{N}} \bar{i}_k (\bar{j}_1 + \dots + \bar{j}_{k-1})} \langle i_1|j_1\rangle \dots \langle i_N|j_N\rangle \\ &= \delta_{i_1 j_1} \dots \delta_{i_N j_N}. \end{aligned} \tag{2.20}$$

Similarly, for N even operators $A_1, \dots, A_{\mathsf{N}}$ where each A_j acts non-trivially only in the j^{th} space of the tensor product $V^{\otimes \mathsf{N}}$, the following matrix element factorizes over the tensorands

$$(|i_1\rangle \otimes \dots \otimes |i_{\mathsf{N}}\rangle)^\dagger A_1 \dots A_{\mathsf{N}} (|j_1\rangle \otimes \dots \otimes |j_{\mathsf{N}}\rangle) = \langle i_1| A_1 |j_1\rangle \dots \langle i_{\mathsf{N}}| A_{\mathsf{N}} |j_{\mathsf{N}}\rangle. \tag{2.21}$$

Indeed, through the matrix elements $\langle i_a| K_a |j_a\rangle$ that arise from the calculation, the evenness of K forces the grading $\bar{i}_a$ and $\bar{j}_a$ at site a to be equal, so the signs compensate.

Permutation operator The permutation operator has to take account of the grading when flipping vectors $\mathbb{P}\cdot(v\otimes w)=(-1)^{\bar{v}\bar{w}}w\otimes v$. Thus we have

$$\mathbb{P}=(-1)^{\bar{\beta}}e^{\beta}_{\alpha}\otimes e^{\alpha}_{\beta}, \tag{2.22}$$

$$\mathbb{P}\cdot(v_i\otimes v_j)=(-1)^{\bar{i}\bar{j}}v_j\otimes v_i. \tag{2.23}$$

Remark the additional signs in the action as discussed in (2.13). For two homogeneous operators A and B,

$$\mathbb{P}\,(A\otimes B)\,\mathbb{P}=(-1)^{p(A)p(B)}B\otimes A. \tag{2.24}$$

On an N-fold tensor product $V_1\otimes\ldots\otimes V_{\mathsf{N}}$, with $V_i\simeq\mathbb{C}^{\mathscr{M}|\mathscr{N}}$, the permutation operator P_{ab} between spaces V_a and V_b writes

$$\mathbb{P}_{ab}=(-1)^{\bar{\beta}}\,\mathbb{I}\otimes\ldots\otimes\mathbb{I}\otimes\underbrace{e^{\beta}_{\alpha}}_{\text{site }a}\otimes\mathbb{I}\ldots\otimes\mathbb{I}\otimes\underbrace{e^{\alpha}_{\beta}}_{\text{site }b}\otimes\mathbb{I}\ldots\otimes\mathbb{I}, \tag{2.25}$$

where the number of identity operators is obvious by the context. The permutation operator is globally even. We have $\mathbb{P}^2_{ab}=\mathbb{I}^{\otimes N}$, and the usual identities are verified

$$\mathbb{P}_{12}=\mathbb{P}_{21}, \tag{2.26}$$

$$\mathbb{P}_{12}\mathbb{P}_{13}=\mathbb{P}_{13}\mathbb{P}_{23}=\mathbb{P}_{23}\mathbb{P}_{12}, \tag{2.27}$$

$$\mathbb{P}_{13}\mathbb{P}_{24}=\mathbb{P}_{24}\mathbb{P}_{13},\quad \mathbb{P}_{12}\mathbb{P}_{34}=\mathbb{P}_{34}\mathbb{P}_{12}, \tag{2.28}$$

which extend naturally to a N-fold tensor product.

The $\mathscr{Y}(gl_{\mathscr{M}|\mathscr{N}})$ fundamental model The R matrix for the fundamental model of the Yangian $\mathscr{Y}(gl_{\mathscr{M}|\mathscr{N}})$ writes

$$R(\lambda,\mu)=(\lambda-\mu)\,\mathbb{I}\otimes\mathbb{I}+\eta\,\mathbb{P}\quad\in\mathrm{End}(\mathbb{C}^{\mathscr{M}|\mathscr{N}}\otimes\mathbb{C}^{\mathscr{M}|\mathscr{N}}). \tag{2.29}$$

It is of difference type and decomposes on elementary operators as $R(\lambda)=R^{ik}_{jl}(\lambda)\,e^{j}_{i}\otimes e^{l}_{k}$ with

$$R^{ik}_{jl}(\lambda)\equiv\lambda\,\delta^{i}_{j}\delta^{k}_{l}+\eta\,(-1)^{\bar{j}}\delta^{i}_{l}\delta^{k}_{j}. \tag{2.30}$$

It generalizes to a N-fold tensor product : the matrix $R_{ab}(\lambda)=\lambda\,\mathbb{I}^{\otimes(\mathsf{N}+1)}+\eta\,\mathbb{P}_{ab}$ of $\mathrm{End}(V_1\otimes\ldots\otimes V_{\mathsf{N}})$ who acts non trivially only on V_a and V_b writes

$$R_{ab}(\lambda)=R^{ik}_{jl}(\lambda)\,\mathbb{I}\otimes\ldots\otimes\mathbb{I}\otimes\underbrace{e^{j}_{i}}_{\text{site }a}\otimes\mathbb{I}\ldots\otimes\mathbb{I}\otimes\underbrace{e^{l}_{k}}_{\text{site }b}\otimes\mathbb{I}\ldots\otimes\mathbb{I}, \tag{2.31}$$

using the generic notation (2.30). The R matrix is globally even and satisfies the Yang-Baxter equation

$$R_{12}(\lambda-\mu)R_{13}(\lambda)R_{23}(\mu)=R_{23}(\mu)R_{13}(\lambda)R_{12}(\lambda-\mu). \tag{2.32}$$

Sometimes the Yang-Baxter equation is written in coordinates and is explicitly referred to as a "graded" version [2,3]. By equation (2.13),

$$R(\lambda)v_j\otimes v_l=\mathsf{R}^{ik}_{jl}(\lambda)v_i\otimes v_k, \tag{2.33}$$

with $\mathsf{R}^{ik}_{jl}(\lambda)=\lambda\delta^{i}_{j}\delta^{k}_{l}+\eta(-1)^{\bar{i}\bar{k}}\delta^{i}_{l}\delta^{k}_{j}$. Then we have

$$\mathsf{R}^{\alpha\beta}_{\alpha'\beta'}(\lambda,\mu)\mathsf{R}^{\alpha'\gamma}_{\alpha''\gamma'}(\lambda)\mathsf{R}^{\beta'\gamma'}_{\beta''\gamma''}(\mu)(-1)^{\bar{\beta}'(\bar{\alpha}'+\bar{\alpha}'')}=\mathsf{R}^{\beta\gamma}_{\beta'\gamma'}(\mu)\mathsf{R}^{\alpha\gamma'}_{\alpha'\gamma''}(\lambda)\mathsf{R}^{\alpha'\beta'}_{\alpha''\beta''}(\lambda-\mu)(-1)^{\bar{\beta}'(\bar{\alpha}+\bar{\alpha}')}. \tag{2.34}$$

One may check the $gl_{\mathscr{M}|\mathscr{N}}$ invariance of the R matrix (2.29)

$$\forall x \in gl_{\mathscr{M}|\mathscr{N}}, \quad [R(\lambda), x\otimes\mathbb{I}+\mathbb{I}\otimes x]=0. \tag{2.35}$$

If K is an homogeneous even matrix of $gl_{\mathscr{M}|\mathscr{N}}$ of the form

$$K=\begin{pmatrix} K_{\mathscr{M}} & 0 \\ 0 & K_{\mathscr{N}} \end{pmatrix}, \tag{2.36}$$

we have

$$R(\lambda)(K\otimes\mathbb{I})(\mathbb{I}\otimes K)=(\mathbb{I}\otimes K)(K\otimes\mathbb{I})R(\lambda). \tag{2.37}$$

This is a scalar version of the Yang-Baxter equation (2.32), where we put a trivial representation on the third space.

For a spin chain of length N, we denote the Hilbert space by $\mathscr{H}=V_1\otimes\ldots\otimes V_{\mathsf{N}}$ and the auxiliary space by V_0, all the V_j superspaces being isomorphic to $\mathbb{C}^{\mathscr{M}|\mathscr{N}}$. Taking an even twist (2.36), the monodromy is an element of $\mathrm{End}(V_0\otimes V_1\otimes\ldots\otimes V_{\mathsf{N}})$ and writes

$$M_0^{(K)}(\lambda)=K_0R_{0\mathsf{N}}(\lambda-\xi_{\mathsf{N}})\ldots R_{01}(\lambda-\xi_1), \tag{2.38}$$

where $\xi_1,\ldots,\xi_{\mathsf{N}}$ are the inhomogeneities of the chain. The monodromy is globally even as a product and tensor product of even operators. In coordinates, using the notation R^{ik}_{jl} of (2.30), it writes

$$M^{(K)}(\lambda)=M^{i\,\alpha_1\ldots\alpha_{\mathsf{N}}}_{j\,\beta_1\ldots\beta_{\mathsf{N}}}(\lambda)\,e^j_i\otimes e^{\beta_1}_{\alpha_1}\otimes\ldots\otimes e^{\beta_{\mathsf{N}}}_{\alpha_{\mathsf{N}}} \tag{2.39}$$

$$=K^i_{j_{\mathsf{N}}}R^{j_{\mathsf{N}}\alpha_{\mathsf{N}}}_{j_{\mathsf{N}-1}\beta_{\mathsf{N}}}(\lambda-\xi_{\mathsf{N}})\ldots R^{j_1\alpha_1}_{j\beta_1}(\lambda-\xi_1)\,e^j_i\otimes e^{\beta_1}_{\alpha_1}\otimes\ldots\otimes e^{\beta_{\mathsf{N}}}_{\alpha_{\mathsf{N}}}, \tag{2.40}$$

where all the signs from the multiplication of the operators actually vanish because of the evenness of R. We are dropping the superscript (K) from the coordinates to make the notation less cluttered. Writing $M^{(K)}(\lambda)=e^j_i\otimes M^i_j(\lambda)$, the above expression shows that the monodromy elements $M^i_j(\lambda)\in\mathrm{End}(\mathscr{H})$ are homogeneous of parity $p(M^i_j(\lambda))=\bar{i}+\bar{j}$.

The Yang-Baxter scheme generalizes to the monodromy thanks to the global evenness of the R matrix and the form (2.36) of the twist, and we have

$$R_{ab}(\lambda-\mu)M^{(K)}_a(\lambda)M^{(K)}_b(\mu)=M^{(K)}_b(\mu)M^{(K)}_a(\lambda)R_{ab}(\lambda-\mu). \tag{2.41}$$

One can prove the $\mathscr{Y}(gl_{\mathscr{M}|\mathscr{N}})$ Yang-Baxter relations between the monodromy elements write as

$$\left[M^j_i(\lambda),M^l_k(\mu)\right]=(-1)^{\bar{i}\bar{k}+\bar{i}\bar{l}+\bar{k}\bar{l}}\left(M^j_k(\mu)M^l_i(\lambda)-M^j_k(\lambda)M^l_i(\mu)\right), \tag{2.42}$$

where $\left[M^j_i(\lambda),M^l_k(\mu)\right]$ is the graded commutator (2.3).

The transfer matrix is obtained by taking the supertrace over the auxiliary space V_0

$$T^{(K)}(\lambda)=\mathrm{str}_0\,M^{(K)}_0(\lambda). \tag{2.43}$$

It is an even operator of $\mathrm{End}(\mathscr{H})$ as a sum of diagonal elements of the monodromy. Because the supertrace vanishes on the graded commutator, one proves the commutation of the transfer matrices

$$\forall\,\lambda,\mu\in\mathbb{C}, \quad \left[T^{(K)}(\lambda),T^{(K)}(\mu)\right]=0. \tag{2.44}$$

2.2 The tower of fused transfer matrices

Tensor products of fundamental representations of $gl_{\mathscr{M}|\mathscr{N}}$ decompose in direct sum of irreducible subrepresentations (irreps). Young diagrams are used to carry out calculations with a mechanic proper to superalgebras, though very similar to the non graded case [195,242–244]. Finite dimensional irreducible representations are labelled in a unique way by Kac-Dynkin labels, but the correspondence between Kac-Dynkin labels and Young diagrams is not one-to-one [245].

Admissible Young diagrams lie inside a *fat hook* domain pictured in figure 1, defined in the (a,b) bidimensional lattice as $H_{\mathscr{M}|\mathscr{N}} \equiv (\mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 1}) \setminus (\mathbb{Z}_{>\mathscr{M}} \times \mathbb{Z}_{>\mathscr{N}})$. Young diagrams can expand infinitely in both a and b directions, but the box $(a \geq \mathscr{N}+1, b \geq \mathscr{M}+1)$ is forbidden, leading to the hook shape.

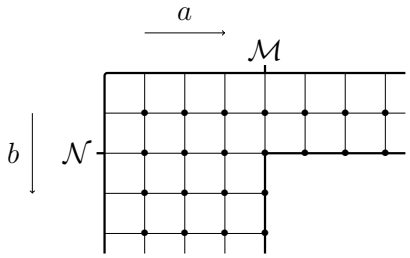

Figure 1: The fat hook domain $H_{\mathscr{M}|\mathscr{N}}$ of admissible Young diagrams for the superalgebra $gl_{\mathscr{M}|\mathscr{N}}$. Bullet points correspond to admissible coordinates (a,b) defining rectangular Young diagrams.

Remark 2.1. For $\mathscr{N} = 0$, the fat hook degenerates to a vertical strip, forcing $a \leq \mathscr{M}$. We recover the usual Young diagram indexation of $gl(\mathscr{M})$ irreducible representations, though the diagrams are here displayed vertically, corresponding to the transposition of the usual $gl(\mathscr{M})$ ones. This is consistent, for example, with the convention of [195], if one rotate the diagrams found there by $-\pi/2$.

The tensoring procedure is called fusion in the context of the quantum inverse scattering method [181,182]. It is used to generate higher dimensional $gl_{\mathscr{M}|\mathscr{N}}$-invariant R-matrices. The Yang-Baxter scheme is preserved, as degeneracy points of the fundamental R-matrix allow to construct the projectors $P_\lambda : (\mathbb{C}^{\mathscr{M}|\mathscr{N}})^{\otimes n} \to V_\lambda$, that extract the wanted subrepresentation λ, as a product of R-matrices. Fusing on the auxiliary space from $M_0^{(K)}$, we obtain new monodromy operators and thus new transfer matrices of $\mathrm{End}(\mathscr{H})$.

For a rectangular Young tableau corresponding to the point $(a,b) \in H_{\mathscr{M}|\mathscr{N}}$, with b rows and a columns, the monodromy matrix reads

$$M_b^{(a),(K)}(\lambda) \equiv P_b^{(a)} \left[\overleftarrow{\bigotimes}_{\substack{1\leq s\leq a \\ 1\leq r\leq b}} M^{(K)}(\lambda + \eta(r-s)) \right] P_b^{(a)}, \tag{2.45}$$

and the transfer matrix

$$T_b^{(a),(K)}(u) \equiv \mathrm{str}_{V_b^{(a)}} M_b^{(a),(K)}(u) \tag{2.46}$$

is obtained by taking the supertrace over the $a \times b$ auxiliary spaces $V \simeq \mathbb{C}^{\mathscr{M}+\mathscr{N}}$, with $V_b^{(a)} \equiv V \otimes \ldots \otimes V$, ab times.

The shifts in (2.45) are given by filling the fat hook as in figure 2. We then read the rectangular Young diagram column by column, top to bottom from left to right, and tensor the shifted monodromy (2.38) corresponding to the current box to the left of the previous ones.

0	$-\eta$	-2η	-3η	-4η
η	0	$-\eta$	-2η	-3η
2η	η	0		
3η	2η	η		

Figure 2: The domain $H_{\mathscr{M}|\mathscr{N}}$ is filled with multiples of the deformation parameter η. Starting from 0 in box $(1,1)$, we add η when moving down and $-\eta$ when going right.

As said earlier, the projectors P_λ can be constructed as a product of R-matrices, like in the non graded case [181, 195, 199]. For rectangular diagrams (a,b), we have[12]

$$P_b^{(a)} \propto \prod_{i<j} R_{ij}(s_j - s_i), \tag{2.47}$$

where i,j run on the boxes of the diagram of figure 2 column by column, top to bottom from left to right, and s_i, s_j are the shift contained in the boxes.

All these transfer matrices commute with each other, as consequence of the Yang-Baxter equation (2.32) being true for any irreps taken in the spaces 1, 2, and 3

$$\forall\,(a,b),(c,d) \in H_{\mathscr{M}|\mathscr{N}},\ \forall \lambda,\mu \in \mathbb{C}, \quad \left[\, T_b^{(a),(K)}(\lambda), T_d^{(c),(K)}(\mu) \,\right] = 0. \tag{2.48}$$

Let us comment that through the nice coderivative formalism [246, 247], in a different but equivalent manner, these fused transfer matrices and the following fusion properties can be also derived. In particular, we make use of this formalism in appendix C to verify (2.54).

Among many others, the fused transfer matrices satisfy the following important properties [191–195]:

Polynomial structure The generic fused transfer matrix $T_b^{(a),(K)}(\lambda)$ is polynomial in λ of degree $ab\mathsf{N}$, with $(ab-1)\mathsf{N}$ central zeros given by

$$Z_b^{(a)}(\lambda) \equiv \prod_{n=1}^{\mathsf{N}} \left[(\lambda - \xi_n)^{-1} \prod_{l=1}^{b} \prod_{m=1}^{a} (\lambda - \xi_n + \eta(l-m)) \right], \tag{2.49}$$

and therefore factorizes as

$$T_b^{(a),(K)}(\lambda) = \widetilde{T}_b^{(a),(K)}(\lambda) Z_b^{(a)}(\lambda), \tag{2.50}$$

where $\widetilde{T}_b^{(a),(K)}(\lambda)$ is polynomial in λ of degree N.

Fusion equations There is a bilinear relation between transfer matrices associated to adjacent rectangular diagrams

$$T_b^{(a),(K)}(\lambda-\eta) T_b^{(a),(K)}(\lambda) = T_{b+1}^{(a),(K)}(\lambda-\eta) T_{b-1}^{(a),(K)}(\lambda) + T_b^{(a-1),(K)}(\lambda-\eta) T_b^{(a+1),(K)}(\lambda), \tag{2.51}$$

where in our normalization, the following boundary conditions are imposed for consistency

$$T_{b\geq 1}^{(0),(K)}(\lambda) = T_0^{(a\geq 1),(K)}(\lambda) = 1. \tag{2.52}$$

[12] Note that in fact one has to use Cherednik regularization to extract the wanted projectors when the diagrams are not purely of row or column type [199].

All the fused transfer matrices outside the *extended fat hook* $\bar{H}_{\mathscr{M}|\mathscr{N}} \equiv \left(\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}\right) / \left(\mathbb{Z}_{>\mathscr{M}} \times \mathbb{Z}_{>\mathscr{N}}\right)$ are identically zero, i.e.

$$\forall (a,b) \notin \bar{H}_{\mathscr{M}|\mathscr{N}}, \quad T_b^{(a),(K)}(\lambda) = 0. \tag{2.53}$$

These relations come from the Jacobi identity applied on the determinant form of the transfer matrices given by the Bazhanov-Reshetikhin formula (2.57), (2.58). We exclude the case $(a,b) = (0,0)$ from the system of the $T_b^{(a)}(\lambda)$, as it cannot be defined uniquely from the boundary conditions.

Inner-boundary condition As the correspondence between Young diagrams and irreps is not bijective, there exist non-trivial relations linking transfer matrices coming from distinct Young diagrams. This is especially the case for rectangular diagrams saturating one of the branch of the fat hook $\bar{H}_{\mathscr{M}|\mathscr{N}}$ [195, 196, 245]. We shall call the first of these relation the *inner-boundary* condition, which writes

$$(-1)^{\mathscr{N}} \mathrm{Ber}(\lambda) T_{\mathscr{N}}^{(\mathscr{M}+1),(K)}(\lambda+\eta) = T_{\mathscr{N}+1}^{(\mathscr{M}),(K)}(\lambda), \tag{2.54}$$

where

$$\mathrm{Ber}(\lambda) = \frac{\det K_{\mathscr{M}}}{\det K_{\mathscr{N}}} \frac{a(\lambda) \prod_{k=1}^{\mathscr{M}-1} d(\lambda - k\eta)}{\prod_{l=1-\mathscr{M}}^{\mathscr{N}-\mathscr{M}} d(\lambda + l\eta)} \, \mathbb{I}_{\mathscr{H}}, \tag{2.55}$$

and

$$a(\lambda-\eta) = d(\lambda) \equiv \prod_{n=1}^{\mathsf{N}} (\lambda - \xi_n). \tag{2.56}$$

$\mathrm{Ber}(\lambda)$ coincides with the central element called the quantum Berezinian, for $\mathscr{N} \neq \mathscr{M}$, and it plays a role similar to the quantum determinant in the non graded case [156, 240]. As anticipated, we verify this relation in appendix C.

2.3 Reconstruction of fused transfer matrix in terms of the fundamental one

Here we want to recall that all these fused transfer matrices are completely determined in terms of the transfer matrix $T_1^{(K)}(\lambda)$ obtained in (2.43). Indeed, the Bazhanov and Reshetikhin's determinant formulae [186] allows us to write all the $T_b^{(a),(K)}(\lambda)$ in terms of those of column type $T_r^{(1),(K)}(\lambda)$ and row type $T_1^{(r),(K)}(\lambda)$ by:

$$T_b^{(a),(K)}(\lambda) = \det_{1\leq i,j\leq a} T_{b+i-j}^{(1),(K)}(\lambda - (i-1)\eta) \tag{2.57}$$

$$= \det_{1\leq i,j\leq b} T_1^{(a+i-j),(K)}(\lambda + (i-1)\eta), \tag{2.58}$$

then our statement is proven once we prove it for the transfer matrix of type $T_r^{(1),(K)}(\lambda)$ and $T_1^{(r),(K)}(\lambda)$. Let us use in the following the simpler notations

$$T_a^{(K)}(\lambda) \equiv T_a^{(1),(K)}(\lambda), \quad T_{(a)}^{(K)}(\lambda) \equiv T_1^{(a),(K)}(\lambda), \tag{2.59}$$

and similarly

$$\widetilde{T}_a^{(K)}(\lambda) \equiv \widetilde{T}_a^{(1),(K)}(\lambda), \quad \widetilde{T}_{(a)}^{(K)}(\lambda) \equiv \widetilde{T}_1^{(a),(K)}(\lambda). \tag{2.60}$$

The fused transfer matrices $T_a^{(K)}(\lambda)$ and $T_{(a)}^{(K)}(\lambda)$ are polynomials of degree $a\mathsf{N}$ in λ, while $\widetilde{T}_a^{(K)}(\lambda)$ and $\widetilde{T}_{(a)}^{(K)}(\lambda)$ are of degree N. We have the following properties for these matrices:

Lemma 2.1. *The following asymptotics hold:*

$$T^{(K)}_{\infty,a} \equiv \lim_{\lambda\to\infty} \lambda^{-a\mathrm{N}} T^{(K)}_a(\lambda) = \mathrm{str}_{1\dots a} P^+_{1\dots a} K_1 \dots K_a P^+_{1\dots a}, \tag{2.61}$$

$$T^{(K)}_{\infty,(a)} \equiv \lim_{\lambda\to\infty} \lambda^{-a\mathrm{N}} T^{(K)}_{(a)}(\lambda) = \mathrm{str}_{1\dots a} P^-_{1\dots a} K_1 \dots K_a P^-_{1\dots a}, \tag{2.62}$$

where, in agreement with (2.47), the projectors admit the following iterative representations in terms of the R-matrix:

$$P^{(1)}_a \equiv P^+_{1\dots a} = \frac{1}{a\eta} P^+_{1\dots a-1} R((a-1)\eta) P^+_{2\dots a}, \tag{2.63}$$

$$P^{(a)}_1 \equiv P^-_{1\dots a} = -\frac{1}{a\eta} P^-_{1\dots a-1} R(-(a-1)\eta) P^-_{2\dots a}. \tag{2.64}$$

In the inhomogeneities the following fusion relations hold:

$$T^{(K)}_{n+1}(\xi_a) = T^{(K)}_1(\xi_a) T^{(K)}_n(\xi_a+\eta), \tag{2.65}$$

$$T^{(K)}_{(n+1)}(\xi_a) = T^{(K)}_1(\xi_a) T^{(K)}_{(n)}(\xi_a-\eta), \tag{2.66}$$

for any positive integer n.

Proof. The asymptotics are an easy corollary of the definition of the fused transfer matrices of type $T^{(K)}_r(\lambda)$ and $T^{(K)}_{(r)}(\lambda)$. Let us now prove the fusion relations in the inhomogeneities, for $n=1$ the identity:

$$T^{(K)}_2(\xi_a) = T^{(K)}_1(\xi_a) T^{(K)}_1(\xi_a+\eta), \tag{2.67}$$

$$T^{(K)}_{(2)}(\xi_a) = T^{(K)}_1(\xi_a) T^{(K)}_1(\xi_a-\eta), \tag{2.68}$$

are obtained by the fusion equations (2.51) just remarking that being:

$$Z^{(2)}_1(\lambda) = d(\lambda-\eta), \quad Z^{(1)}_2(\lambda) = d(\lambda+\eta), \tag{2.69}$$

it holds

$$T^{(K)}_{(2)}(\xi_a+\eta) = 0, \quad T^{(K)}_2(\xi_a-\eta) = 0. \tag{2.70}$$

Then we can proceed by induction to prove the identity, let us assume that it holds for $n \geq 1$ and let us prove it for $n+1$, the relevant fusion identities reads:

$$\begin{aligned} T^{(K)}_n(\xi_a+\eta) T^{(K)}_n(\xi_a) = {} & T^{(K)}_{n+1}(\xi_a) T^{(K)}_{n-1}(\xi_a+\eta) \\ & + T^{(0),(K)}_n(\xi_a) T^{(2),(K)}_n(\xi_a+\eta), \end{aligned} \tag{2.71}$$

and

$$\begin{aligned} T^{(K)}_{(n)}(\xi_a-\eta) T^{(K)}_{(n)}(\xi_a) = {} & T^{(n),(K)}_2(\xi_a-\eta) T^{(n),(K)}_0(\xi_a) \\ & + T^{(K)}_{(n-1)}(\xi_a-\eta) T^{(K)}_{(n+1)}(\xi_a), \end{aligned} \tag{2.72}$$

which being:

$$Z^{(2)}_n(\lambda) \propto d(\lambda-\eta), \quad Z^{(n)}_2(\lambda) \propto d(\lambda+\eta), \tag{2.73}$$

read:

$$T^{(K)}_n(\xi_a+\eta) T^{(K)}_n(\xi_a) = T^{(K)}_{n+1}(\xi_a) T^{(K)}_{n-1}(\xi_a+\eta), \tag{2.74}$$

and

$$T^{(K)}_{(n)}(\xi_a-\eta)T^{(K)}_{(n)}(\xi_a)=T^{(K)}_{(n-1)}(\xi_a-\eta)T^{(K)}_{(n+1)}(\xi_a), \tag{2.75}$$

which lead to our identities for $n+1$ once we use the induction hypothesis for n

$$T^{(K)}_{n}(\xi_a)=T^{(K)}_{1}(\xi_a)T^{(K)}_{n-1}(\xi_a+\eta), \tag{2.76}$$
$$T^{(K)}_{(n)}(\xi_a)=T^{(K)}_{1}(\xi_a)T^{(K)}_{(n-1)}(\xi_a-\eta). \tag{2.77}$$

□

The known central zeros and asymptotic imply that the interpolation formula in the N special points defined by the fusion equations to write $T^{(K)}_{n}(\xi_a)$ and $T^{(K)}_{(n)}(\xi_a)$ completely characterize these transfer matrices. Let us introduce the functions

$$f^{(m)}_a(\lambda)=\prod_{b\neq a,b=1}^{\mathsf{N}}\frac{\lambda-\xi_b}{\xi_a-\xi_b}\prod_{b=1}^{\mathsf{N}}\prod_{r=1}^{m-1}\frac{1}{\xi_a-\xi^{(r)}_b},\quad \xi^{(r)}_b=\xi_b-r\eta, \tag{2.78}$$

$$g^{(m)}_a(\lambda)=\prod_{b\neq a,b=1}^{\mathsf{N}}\frac{\lambda-\xi_b}{\xi_a-\xi_b}\prod_{b=1}^{\mathsf{N}}\prod_{r=1}^{m-1}\frac{1}{\xi_a-\xi^{(-r)}_b}, \tag{2.79}$$

and

$$T^{(K)}_{\infty,a}(\lambda)=T^{(K)}_{\infty,a}\prod_{b=1}^{\mathsf{N}}(\lambda-\xi_b),\qquad T^{(K)}_{\infty,(a)}(\lambda)=T^{(K)}_{\infty,(a)}\prod_{b=1}^{\mathsf{N}}(\lambda-\xi_b), \tag{2.80}$$

then the following corollary holds:

Corollary 2.1. *Under the following conditions on the inhomogeneity parameters* ξ_i

$$\forall a,b\in\{1,\dots\mathsf{N}\},a\neq b,\quad \xi_a\neq\xi_b\mod\eta, \tag{2.81}$$

the transfer matrix $T^{(K)}_{n+1}(\lambda)$ *and* $T^{(K)}_{(n+1)}(\lambda)$ *are completely characterized in terms of* $T^{(K)}_{1}(\lambda)$ *by the fusion equations and the following interpolation formulae:*

$$T^{(K)}_{n+1}(\lambda)=\prod_{r=1}^{n}d(\lambda+r\eta)\left[T^{(K)}_{\infty,n+1}(\lambda)+\sum_{a=1}^{\mathsf{N}}f^{(n+1)}_a(\lambda)T^{(K)}_{n}(\xi_a+\eta)T^{(K)}_{1}(\xi_a)\right], \tag{2.82}$$

$$T^{(K)}_{(n+1)}(\lambda)=\prod_{r=1}^{n}d(\lambda-r\eta)\left[T^{(K)}_{\infty,(n+1)}(\lambda)+\sum_{a=1}^{\mathsf{N}}g^{(n+1)}_a(\lambda)T^{(K)}_{(n)}(\xi_a-\eta)T^{(K)}_{1}(\xi_a)\right]. \tag{2.83}$$

2.4 SoV covector basis for $gl_{\mathscr{M}|\mathscr{N}}$ Yang-Baxter superalgebra

In the next section, we construct a separation of variables basis for the integrable quantum model associated to the fundamental representations of the $gl_{\mathscr{M}|\mathscr{N}}$-graded Yang-Baxter algebra. The construction follows the general ideas presented in the Proposition 2.4 of [1].

As in the non-graded case, the proof relies mainly on the reduction of the R matrix to the permutation at a particular point, and on the centrality of the asymptotics of the transfer matrix.

Let K be a $(\mathscr{M}+\mathscr{N})\times(\mathscr{M}+\mathscr{N})$ square matrix solution of the $gl_{\mathscr{M}|\mathscr{N}}$-graded Yang-Baxter equation of the block form (2.36), then we use the following notation

$$K=W_K K_J W_K^{-1}, \tag{2.84}$$

where K_J is the Jordan form of the matrix K

$$K_J = \begin{pmatrix} K_{J,\mathscr{M}} & 0 \\ 0 & K_{J,\mathscr{N}} \end{pmatrix}, \tag{2.85}$$

where for $\mathscr{X} = \mathscr{M}$ or $\mathscr{N}$

$$K_{J,\mathscr{X}} = \begin{pmatrix} K_{J,\mathscr{X}}^{(1)} & 0 & \cdots & 0 \\ 0 & K_{J,\mathscr{X}}^{(2)} & \ddots & 0 \\ 0 & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & K_{J,\mathscr{X}}^{(m_{\mathscr{X}})} \end{pmatrix}, \tag{2.86}$$

and $K_{J,\mathscr{X}}^{(i)}$ is a $d_{i,\mathscr{X}} \times d_{i,\mathscr{X}}$ upper triangular Jordan block for any i in $\{1,...,m_{\mathscr{X}}\}$ with eigenvalue $k_{i,\mathscr{X}}$, where $\sum_{a=1}^{m_{\mathscr{X}}} d_{a,\mathscr{X}} = \mathscr{X}$. Moreover, it is interesting to point out that the invertible matrix W_K defining the change of basis for the twist matrix is itself a $(\mathscr{M}+\mathscr{N}) \times (\mathscr{M}+\mathscr{N})$ square matrix solution of the $gl_{\mathscr{M}|\mathscr{N}}$-graded Yang-Baxter equation. Indeed, it is of the same block form (2.36):

$$W_K = \begin{pmatrix} W_{K,\mathscr{M}} & 0 \\ 0 & W_{K,\mathscr{N}} \end{pmatrix}. \tag{2.87}$$

Then, the following similarity relation holds for the fundamental transfer matrices:

$$T_1^{(K)}(\lambda) = \mathscr{W}_K T_1^{(K_J)}(\lambda) \mathscr{W}_K^{-1}, \quad \text{with } \mathscr{W}_K = W_{K,\mathsf{N}} \otimes \cdots \otimes W_{K,1}, \tag{2.88}$$

i.e. they are isospectral. We can now state our main result on the form of the SoV basis:

Theorem 2.1. *Let K be a $(\mathscr{M}+\mathscr{N}) \times (\mathscr{M}+\mathscr{N})$ square matrix with simple spectrum of block form (2.36), i.e. we assume that:*

$$k_{i,\mathscr{X}} \neq k_{j,\mathscr{X}'} \text{ for } (i,\mathscr{X}) \neq (j,\mathscr{X}') \ \forall (i,j) \in \{1,...,m_{\mathscr{X}}\} \times \{1,...,m_{\mathscr{X}'}\}, \mathscr{X},\mathscr{X}' \in \{\mathscr{M},\mathscr{N}\}, \tag{2.89}$$

then for almost any choice of the covector $\langle S|$ and of the inhomogeneities under the condition (2.81), the following set of covectors:

$$\langle h_1,...,h_{\mathsf{N}}| \equiv \langle S| \prod_{n=1}^{\mathsf{N}} \left(T_1^{(K)}(\xi_n) \right)^{h_n} \text{ for any } \{h_1,...,h_{\mathsf{N}}\} \in \{0,...,\mathscr{M}+\mathscr{N}-1\}^{\times \mathsf{N}}, \tag{2.90}$$

forms a covector basis of $\mathscr{H}$. In particular, let us take a one-site state $|S,a\rangle = S_i^{(a)}|i\rangle$, $S_i^{(a)} \in \mathbb{C}$. Its dual covector in the single space V_a is $\langle S,a| = |S,a\rangle^\dagger = S_i^{(a)*}\langle i|$. When acting on it by the W_K^{-1} isomorphism, it is noted in coordinates

$$\langle S,a|W_{K,a}^{-1} = \left(S_1^{(a)*}, \ldots, S_{\mathscr{M}+\mathscr{N}}^{(a)*} \right) W_{K,a}^{-1} = \left(x_{1,\mathscr{M}}^{(1)},...,x_{d_1,\mathscr{M}}^{(1)},...,x_{1,\mathscr{N}}^{(m_{\mathscr{N}})},...,x_{d_{m_{\mathscr{N}}},\mathscr{N}}^{(m_{\mathscr{N}})} \right) \in V_a^*. \tag{2.91}$$

Following (2.19), we have $\langle S| = (|S,1\rangle \ldots |S,\mathsf{N}\rangle)^\dagger$ as

$$\langle S| = \sum_{p_1,...,p_{\mathsf{N}}=1}^{\mathscr{M}+\mathscr{N}} S_{p_1}^{(1)*} \ldots S_{p_{\mathsf{N}}}^{(\mathsf{N})*} (|p_1\rangle \ldots |p_{\mathsf{N}}\rangle)^\dagger. \tag{2.92}$$

Then, it is sufficient that

$$\prod_{k=1}^{m_{\mathscr{M}}} x_{1,\mathscr{M}}^{(k)} \prod_{k=1}^{m_{\mathscr{N}}} x_{1,\mathscr{N}}^{(k)} \neq 0, \tag{2.93}$$

for the family of covectors (2.90) to form a basis. Furthermore, the $T_1^{(K)}(\lambda)$ transfer matrix spectrum is simple.

Proof. As in the non graded case, the identity:

$$T_1^{(K)}(\xi_n) = R_{n,n-1}(\xi_n - \xi_{n-1}) \cdots R_{n,1}(\xi_n - \xi_1) K_n R_{n,\mathsf{N}}(\xi_n - \xi_{\mathsf{N}}) \cdots R_{n,n+1}(\xi_n - \xi_{n+1}), \quad (2.94)$$

holds true as a direct consequence of the definition of the transfer matrix $T_1^{(K)}(\lambda)$ and the properties

$$R_{0,n}(0) = \eta \mathbb{P}_{0,n}, \quad \mathrm{str}_{V_0} \mathbb{P}_{0,n} = 1. \quad (2.95)$$

From this point we can essentially follow the proof of Proposition 2.4 of [1]. Indeed, the condition that the set (2.90) form a covector basis of $\mathscr{H}$ is equivalent to

$$\det_{(\mathscr{M}+\mathscr{N})^{\mathsf{N}}} || \mathscr{R}\left(\langle S|, K, \{\xi\}\right) || \neq 0, \quad (2.96)$$

where we have defined:

$$\mathscr{R}\left(\langle S|, K, \{\xi\}\right)_{i,j} \equiv \langle h_1(i), ..., h_N(i) | e_j \rangle, \quad \forall i, j \in \{1, ..., (\mathscr{M}+\mathscr{N})^{\mathsf{N}}\}. \quad (2.97)$$

We are uniquely enumerating the N-tuple $(h_1(i), ..., h_{\mathsf{N}}(i)) \in \{0, ..., \mathscr{M}+\mathscr{N}-1\}^{\times \mathsf{N}}$ by:

$$1 + \sum_{a=1}^{\mathsf{N}} h_a(i) (\mathscr{M}+\mathscr{N})^{a-1} = i \in \{1, ..., (\mathscr{M}+\mathscr{N})^{\mathsf{N}}\}, \quad (2.98)$$

and for any $j \in \{1, ..., (\mathscr{M}+\mathscr{N})^{\mathsf{N}}\}$, $|e_j\rangle = |e_{1+h_1(j)}(1)\rangle \otimes \ldots \otimes |e_{1+h_{\mathsf{N}}(j)}(\mathsf{N})\rangle \in \mathscr{H}$ is the corresponding element of the canonical basis in $\mathscr{H}$, where $|e_r(a)\rangle$ stands for the element $r \in \{1, ..., \mathscr{M}+\mathscr{N}\}$ of the canonical basis in the local quantum space V_a. Now, being the transfer matrix $T_1^{(K)}(\lambda)$ a polynomial in the inhomogeneities $\{\xi_i\}$ and in the parameters of the twist matrix K, the same statement holds true for the determinant on the l.h.s. of (2.96), which is moreover a polynomial in the coefficients $\langle S|e_j\rangle$ of the covector $\langle S|$.

Then it follows that the condition (2.96) holds true for almost any value of these parameters if one can show it under the special limit of large inhomogeneities. Using this argument, the form of the transfer matrix in the inhomogeneities (2.94) and the central asymptotics of the $gl_{\mathscr{M}|\mathscr{N}}$-graded R-matrix one can show that a sufficient criterion is that the following determinant is non-zero:

$$\det_{(\mathscr{M}+\mathscr{N})^{\mathsf{N}}} || \left(\langle S | K_1^{h_1(i)} \cdots K_{\mathsf{N}}^{h_{\mathsf{N}}(i)} | e_j \rangle \right)_{i,j \in \{1,...,(\mathscr{M}+\mathscr{N})^{\mathsf{N}}\}} || \neq 0. \quad (2.99)$$

Let us compute this matrix element precisely: from (2.92), it decomposes as the following sum

$$\langle S | K_1^{h_1(i)} \ldots K_{\mathsf{N}}^{h_{\mathsf{N}}(i)} \left| e_j \right\rangle = \quad (2.100)$$
$$\sum_{p_1,...,p_{\mathsf{N}}=1}^{\mathscr{M}+\mathscr{N}} S_{p_1}^{(1)*} \ldots S_{p_{\mathsf{N}}}^{(\mathsf{N})*} |p_1 \ldots p_{\mathsf{N}}\rangle^\dagger K_1^{h_1(i)} \ldots K_{\mathsf{N}}^{h_{\mathsf{N}}(i)} |e_{1+h_1(j)}(1)\rangle \ldots |e_{1+h_{\mathsf{N}}(j)}(N)\rangle .$$

Now, the $K_a^{h_a(i)}$ being even, the matrix element factorizes by (2.21) as a product over the one site matrix elements

$$|p_1 \ldots p_{\mathsf{N}}\rangle^\dagger K_1^{h_1(i)} \ldots K_{\mathsf{N}}^{h_{\mathsf{N}}(i)} |e_{1+h_1(j)}(1)\rangle \ldots |e_{1+h_{\mathsf{N}}(j)}(N)\rangle$$
$$= \langle p_1 | K_1^{h_1(i)} |e_{1+h_1(j)}(1)\rangle \ldots \langle p_{\mathsf{N}}| K_{\mathsf{N}}^{h_{\mathsf{N}}(i)} |e_{1+h_{\mathsf{N}}(j)}(\mathsf{N})\rangle . \quad (2.101)$$

Therefore the sum over $p_1, \ldots, p_{\mathsf{N}}$ decouples as a product of N sums, and identifying $\langle S, a| = S_{p_a}^{(a)*} \langle p_a|$ in the expression leaves us with

$$\langle S | K_1^{h_1(i)} \ldots K_{\mathsf{N}}^{h_{\mathsf{N}}(i)} \left| e_j \right\rangle = \langle S, 1 | K^{h_1(i)} |e_{1+h_1(j)}(1)\rangle \ldots \langle S, \mathsf{N}| K^{h_{\mathsf{N}}(i)} |e_{1+h_{\mathsf{N}}(j)}(\mathsf{N})\rangle . \quad (2.102)$$

Hence, the determinant factorizes and the criterion amounts to

$$\prod_{a=1}^{\mathsf{N}} \det_{\mathscr{M}+\mathscr{N}} || \left(\langle S, a | K_a^{i-1} | e_j(a) \rangle \right)_{i,j\in\{1,\dots,\mathscr{M}+\mathscr{N}\}} || \neq 0. \tag{2.103}$$

Finally, by Proposition 2.2 of [1], it holds for the factor corresponding to site n in the above product

$$\begin{aligned} \det_{\mathscr{M}+\mathscr{N}} || \langle S, n | K_n^{i-1} | e_j(n) \rangle_{i,j\in\{1,\dots,\mathscr{M}+\mathscr{N}\}} || = \\ \prod_{a=1}^{m_{\mathscr{M}}} \left(x_{1,\mathscr{M}}^{(a)} \right)^{d_{a,\mathscr{M}}} \prod_{a=1}^{m_{\mathscr{N}}} \left(x_{1,\mathscr{N}}^{(a)} \right)^{d_{a,\mathscr{N}}} \prod_{a=1}^{m_{\mathscr{M}}} \prod_{b=1}^{m_{\mathscr{N}}} \left(k_{a,\mathscr{M}} - k_{b,\mathscr{N}} \right)^{d_{a,\mathscr{M}} d_{b,\mathscr{N}}} \\ \times \prod_{1\leq a<b\leq m_{\mathscr{M}}} \left(k_{a,\mathscr{M}} - k_{b,\mathscr{M}} \right)^{d_{a,\mathscr{M}} d_{b,\mathscr{M}}} \prod_{1\leq a<b\leq m_{\mathscr{N}}} \left(k_{a,\mathscr{N}} - k_{b,\mathscr{N}} \right)^{d_{a,\mathscr{N}} d_{b,\mathscr{N}}}, \end{aligned} \tag{2.104}$$

which is clearly nonzero under the condition that the twist K has simple spectrum and that (2.93) holds. The simplicity of the transfer matrix spectrum is then a trivial consequence of the fact that the set of covectors (2.90) is proven to be a basis. Indeed, it implies that given a generic eigenvalue $t(\lambda)$ of $T_1^{(K)}(\lambda)$, the associated eigenvector $|t\rangle$ is unique, being characterized uniquely (up to normalization) by the eigenvalue as

$$\langle h_1, \dots, h_{\mathsf{N}} | t \rangle = \prod_{a=1}^{\mathsf{N}} t^{h_a}(\xi_a), \quad \forall (h_1, \dots, h_{\mathsf{N}}) \in \{0, \dots, \mathscr{M}+\mathscr{N}-1\}^{\times \mathsf{N}}. \tag{2.105}$$

□

Remark 2.2. Note that $\langle S| \neq \langle S, 1| \dots \langle S, \mathsf{N}|$.

The norm of $|S\rangle$ is $\langle S|S\rangle = \prod_{a=1}^{\mathsf{N}} \sum_{i=1}^{\mathscr{M}+\mathscr{N}} |S_i|^2$ and can be set to convenience. In particular, it may be taken to one.

Let us observe that some stronger statement can be done about the transfer matrix diagonalizability and spectrum simplicity according to the following

Proposition 2.1. *Let the twist matrix K be diagonalizable and with simple spectrum on $\mathbb{C}^{\mathscr{M}|\mathscr{N}}$, then $T_1^{(K)}(\lambda)$ is diagonalizable and with simple spectrum, for almost any values of the inhomogeneities satisfying the condition (2.81). Indeed, taken the generic eigenvalue $t(\lambda)$ of $T_1^{(K)}(\lambda)$ it holds:*

$$\langle t | t \rangle \neq 0, \tag{2.106}$$

where $|t\rangle$ and $\langle t|$ are the unique eigenvector and eigencovector associated to it.

Proof. Following the proof of Proposition 2.5 of [1] the non-orthogonality condition (2.106) can be derived. Such condition together with the simplicity of the spectrum implies that we cannot have non-trivial Jordan blocks in the transfer matrix spectrum so that it must be diagonalizable and with simple spectrum. □

2.5 On closure relations and SoV spectrum characterization

In the previous two subsections, we have shown how the transfer matrix $T_1^{(K)}(\lambda)$ associated to general inhomogeneous representations of the $gl_{\mathscr{M}|\mathscr{N}}$-graded Yang-Baxter algebra allows to reconstruct all the fused transfer matrices (mainly by using the known fusion relations (2.65) and (2.66)). Moreover, we have shown that $T_1^{(K)}(\lambda)$ allows to characterize an SoV basis, which also implies its spectrum simplicity or diagonalizability and spectrum simplicity if the twist matrix K is, respectively, with simple spectrum or diagonalizable with simple spectrum.

This analysis shows that the full integrable structure of the $gl_{\mathscr{M}|\mathscr{N}}$-graded Yang-Baxter algebra can be recast in its fundamental transfer matrix as well as the construction of quantum separation of variables. However, there is still one missing information, which is a functional equation, or a discrete system of equations, allowing the complete characterization of the transfer matrix spectrum. As already mentioned, the fusion relations (2.51) alone only give the characterization of higher transfer matrices in terms of the first one. Some further algebra and representation dependent rules are required in order to complete them and extract a closure relation on the transfer matrix.

In the case of the quantum integrable models associated to the fundamental representations of the $gl_{\mathscr{M}}$ and $U_q(\widehat{gl_{\mathscr{M}}})$ Yang-Baxter and reflection algebras, such a closure relation comes from the quantum determinant [1, 45, 46, 159–161]. Indeed, $P^{-}_{1,\ldots,\mathscr{M}}$ is a rank 1 projector in these cases, implying that the corresponding transfer matrix $T^{(K)}_{(\mathscr{M})}(\lambda)$ becomes a computable central element of the Yang-Baxter algebra, namely the quantum determinant. Then, substituting the quantum determinant in the fusion equation (2.65) for $n = \mathscr{M} - 1$ and using the same interpolation formulae for the higher fused transfer matrix eigenvalues, we produce a discrete system of polynomial equations with N equations in N unknowns which was proven [1, 45, 46, 159–161] to completely characterize the transfer matrix spectrum in quantum separation of variables. In the case of non-fundamental representations[13] of the same algebras the closure relation comes instead with the appearing of the first central zeros in the fused transfer matrices of type $T^{(K)}_{n}(\lambda)$. In [162], this analysis has been developed in detail in the case of $\mathscr{M} = 2$. There, it has been shown that imposing the central zeros of the fused transfer matrix $T^{(K)}_{2s+1}(\lambda)$, for a spin $s \geq 1$ representation, a discrete system of polynomial equations with N equations in N unknowns is derived for the transfer matrix eigenvalues. The set of its solutions completely characterizes the transfer matrix spectrum in quantum separation of variables. In the nonfundamental and cyclic representations of the $U_q(\widehat{gl_{\mathscr{M}}})$ Yang-Baxter algebra for q a root of unity such closure relation comes from the so-called truncation identities. For $\mathscr{M} = 2$, it has been shown in [40] how these identities emerge and are proven in the framework of the quantum separation of variables and how they are used to completely characterize the transfer matrix spectrum. In [44] and [59, 60] these results have been extended, respectively, to the most general cyclic representations of the $U_q(\widehat{gl_2})$ Yang-Baxter algebra and reflection algebra.

In the case of integrable quantum lattice models associated to the fundamental representations of the $gl_{\mathscr{M}|\mathscr{N}}$-graded Yang-Baxter algebra, the natural candidate for the closure relation is the *inner-boundary* condition (2.54). Indeed, once we impose it on the eigenvalues $t^{(\mathscr{M}+1),(K)}_{\mathscr{N}}(\lambda)$ and $t^{(\mathscr{M}),(K)}_{\mathscr{N}+1}(\lambda)$ of the transfer matrices $T^{(\mathscr{M}+1),(K)}_{\mathscr{N}}(\lambda)$ and $T^{(\mathscr{M}),(K)}_{\mathscr{N}+1}(\lambda)$, we are left with one nontrivial functional equation containing as unknowns the eigenvalues of the first transfer matrix computed in the inhomogeneities $t^{(K)}_{1}(\xi_{i\leq \mathsf{N}})$. This is the case as the eigenvalues $t^{(\mathscr{M}+1),(K)}_{\mathscr{N}}(\lambda)$ and $t^{(\mathscr{M}),(K)}_{\mathscr{N}+1}(\lambda)$ admit the same expansion in terms of the transfer matrix eigenvalue $t^{(K)}_{1}(\lambda)$ as those derived in subsection 2.3 for the transfer matrices $T^{(\mathscr{M}+1),(K)}_{\mathscr{N}}(\lambda)$ and $T^{(\mathscr{M}),(K)}_{\mathscr{N}+1}(\lambda)$ in terms of the transfer matrix $T^{(K)}_{1}(\lambda)$. Moreover, the *inner-boundary* condition (2.54) involved the quantum Berezinian as a central element hence playing a role similar to the quantum determinant in the bosonic case. More precisely, we can introduce the following polynomials:

$$t_1(\lambda|\{x_a\}) = T^{(K)}_{\infty,1}(\lambda) + \sum_{a=1}^{\mathsf{N}} f^{(1)}_a(\lambda) x_a, \tag{2.107}$$

[13] Here q is not a root of unity for the quantum group case.

and from them recursively the following higher polynomials

$$t_{n+1}(\lambda|\{x_a\}) = \prod_{r=1}^{n} d(\lambda+r\eta)\left[T^{(K)}_{\infty,n+1}(\lambda) + \sum_{a=1}^{\mathsf{N}} f_a^{(n+1)}(\lambda) t_n(\xi_a+\eta|\{x_a\})x_a\right], \tag{2.108}$$

$$t_{(n+1)}(\lambda|\{x_a\}) = \prod_{r=1}^{n} d(\lambda-r\eta)\left[T^{(K)}_{\infty,(n+1)}(\lambda) + \sum_{a=1}^{\mathsf{N}} g_a^{(n+1)}(\lambda) t_{(n)}(\xi_a-\eta|\{x_a\})x_a\right], \tag{2.109}$$

and

$$t_b^{(a)}(\lambda|\{x_a\}) = \det_{1\leq i,j\leq a} t_{b+i-j}(\lambda-(i-1)\eta|\{x_a\}) \tag{2.110}$$

$$= \det_{1\leq i,j\leq b} t_{(a+i-j)}(\lambda+(i-1)\eta|\{x_a\}). \tag{2.111}$$

Then the following lemma holds

Lemma 2.2. *Any transfer matrix $T_1^{(K)}(\lambda)$ eigenvalue*[14] *admits the representation $t_1(\lambda|\{x_a\})$, where the $\{x_a\}$ are solutions of the inner-boundary condition* (2.54)*:*

$$(-1)^{\mathscr{N}} Ber(\lambda) t^{(\mathscr{M}+1)}_{\mathscr{N}}(\lambda+\eta|\{x_a\}) = t^{(\mathscr{M})}_{\mathscr{N}+1}(\lambda|\{x_a\}), \quad \forall \lambda \in \mathbb{C}, \tag{2.112}$$

and of the null out-boundary conditions (2.53)*:*

$$t^{(\mathscr{M}+n)}_{\mathscr{N}+m}(\lambda|\{x_a\}) = 0, \quad \forall \lambda \in \mathbb{C} \text{ and } n,m \geq 1. \tag{2.113}$$

In the next section, we conjecture that the above system of functional equations completely characterizes the transfer matrix spectrum in the case of the $gl_{1|2}$-graded Yang-Baxter algebra. We prove this characterization for some special class of twist matrices while we only give some first motivations of it for general representations. In appendix B we verify it for quantum chains with two and three sites. Let us also mention that this conjecture can be checked explicitly for the simple $gl_{1|1}$ case. It would be interesting in this respect to elucidate the relation of our method with the one developed recently in [204].

3 On SoV spectrum description of $gl_{1|2}$ Yang-Baxter superalgebra

Specialising to the $gl_{1|2}$ case, some results have already been obtained in the context of the NABA, see [136, 205] for instance.

3.1 General statements and conjectured closure relation for general integrable twist

We use this subsection to clarify and justify the following conjecture

Conjecture 3.1. *Taken the general $gl_{1|2}$-graded Yang-Baxter algebra with twisted boundary conditions, the polynomial $t_1(\lambda|\{x_a\})$ defined above is an eigenvalue of the transfer matrix $T_1^{(K)}(\lambda)$ (excluding the trivial solution $x_1 = \ldots = x_{\mathsf{N}} = 0$) iff. the higher polynomials associated to it satisfy, in addition to the fusion relations, the inner-boundary condition* (2.112) *and the null out-boundary conditions* (2.113) *for $\mathscr{M} = 1$ and $\mathscr{N} = 2$.*

The fat hook domain for $gl_{1|2}$ is pictured in figure 3. In the $gl_{1|2}$-graded case under con-

[14] The trivial solution $\{x_1, ..., x_{\mathsf{N}}\} = \{0, ..., 0\}$ has to be excluded for invertible twist matrix.

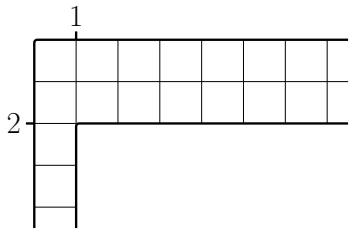

Figure 3: Admissible domain for Young Diagrams of $gl_{1|2}$.

sideration the inner-boundary condition (2.54) reads:

$$T_2^{(2),(K)}(\lambda+\eta)\mathsf{k}_1 = \mathsf{k}_2\mathsf{k}_3 d(\lambda)\, T_3^{(K)}(\lambda), \tag{3.1}$$

where:

$$\det K_{\mathscr{M}=1} = K_{\mathscr{M}=1} = \mathsf{k}_1 \text{ and } \det K_{\mathscr{N}=2} = \mathsf{k}_2\mathsf{k}_3. \tag{3.2}$$

Moreover, we have the following expressions[15] for the asymptotics of the fused transfer matrices

$$T_{\infty,n}^{(K)} = \mathsf{k}_1^{n-2}(\mathsf{k}_1-\mathsf{k}_3)(\mathsf{k}_1-\mathsf{k}_2), \quad \forall n \geq 2. \tag{3.3}$$

Then imposing it for the corresponding eigenvalues we get

$$\mathsf{k}_3\mathsf{k}_2 d(\lambda) t_3(\lambda) = \mathsf{k}_1 t_2^{(2)}(\lambda+\eta), \tag{3.4}$$

which, once we express the eigenvalues $t_2^{(2)}(\lambda+\eta)$ by the use of the fusion relation (2.51):

$$t_2^{(2)}(\lambda+\eta) = t_2(\lambda)t_2(\lambda+\eta) - t_1(\lambda+\eta)t_3(\lambda), \tag{3.5}$$

takes the following closure relation form:

$$\mathsf{k}_3\mathsf{k}_2 d(\lambda) t_3(\lambda) = \mathsf{k}_1(t_2(\lambda)t_2(\lambda+\eta) - t_3(\lambda)t_1(\lambda+\eta)). \tag{3.6}$$

Now, using the interpolation formulae (2.107) and (2.108) for the transfer matrix eigenvalues, we get that the closure relation is indeed a functional equation whose unknowns coincide with the $x_{i\leq \mathsf{N}} \equiv t_1^{(K)}(\xi_{i\leq \mathsf{N}})$. The transfer matrix eigenvalues have to satisfy furthermore the null out-boundary conditions (2.53), which reads:

$$t_{3+m}^{(2+n)}(\lambda) = 0, \quad \forall \lambda \in \mathbb{C} \text{ and } n, m \geq 0, \tag{3.7}$$

in the $gl_{1|2}$-graded case under consideration.

According to our Conjecture the transfer matrix spectrum coincides with the set of solutions to the functional equations (3.6) and (3.7) in the unknowns $t_1^{(K)}(\xi_{i\leq \mathsf{N}})$. This spectrum characterization for general twist matrix will be proven by direct action of the transfer matrices on our SoV basis in our next publication. Here we present some arguments in favour of it while in the next subsections 3.2 we prove it for a special choice of the twist matrix.

Let us consider the case of a twist matrix K invertible with simple spectrum[16], then our SoV basis can be written as follows:

$$\langle h_1, ..., h_{\mathsf{N}}| \equiv \langle \bar{S}| \prod_{n=1}^{\mathsf{N}} (T_1^{(K)}(\xi_n))^{h_n} \text{ for any } \{h_1, ..., h_{\mathsf{N}}\} \in \{1,2,3\}^{\times \mathsf{N}}, \tag{3.8}$$

[15] They can be computed for example by induction starting from the explicit formulae for $T_{\infty,1}^{(K)}$ and $T_{\infty,2}^{(K)}$, by using the fusion equations and the null out-boundary conditions.

[16] Indeed, the case of K non-invertible but having simple spectrum of the form (3.12) will be described in detail in the next subsection.

indeed in this case the transfer matrices $T_1^{(K)}(\xi_n)$ are invertible[17] and so we can write our original vector $\langle S|$ as it follows:

$$\langle S| = \langle \bar{S}| \prod_{n=1}^{\mathsf{N}} T_1^{(K)}(\xi_n). \tag{3.9}$$

If the $t_1(\xi_{i\leq \mathsf{N}})$ solve the closure relation (3.6) and the null out-boundary conditions (3.7), to prove that a vector $|t\rangle$ characterized by

$$\langle h_1, ..., h_{\mathsf{N}}|t\rangle \equiv \prod_{n=1}^{\mathsf{N}} t_1^{h_n}(\xi_n) \quad \forall \{h_1, ..., h_{\mathsf{N}}\} \in \{1,2,3\}^{\times \mathsf{N}}, \tag{3.10}$$

is indeed a transfer matrix eigenvector, the main point is to be able to reduce the covectors containing a fourth order power of $T_1^{(K)}(\xi_n)$ in those of the SoV basis, of maximal order three. Moreover, this reduction must come from relations which are satisfied identically by both the fused transfer matrices and the functions $t_r(\lambda|\{x_a\})$ defined in (2.108), in terms of the given solution $t_1(\xi_{i\leq \mathsf{N}})$. Indeed, this is exactly what it is done by the closure relation for the transfer matrix:

$$\mathsf{k}_3\mathsf{k}_2 d(\lambda) T_3^{(K)}(\lambda) = \mathsf{k}_1 (T_2^{(K)}(\lambda) T_2^{(K)}(\lambda+\eta) - T_3^{(K)}(\lambda) T_1^{(K)}(\lambda+\eta)), \tag{3.11}$$

and by the corresponding one (3.6) for the eigenvalues. As one can easily remark that (3.11) and (3.6) are both of fourth order, respectively, in the $T_1^{(K)}(\xi_n)$ and $t_1(\xi_n)$ on the right hand side while they are both of third order on the left hand side.

In appendix A, we will verify that Nested Algebraic and Analytic Bethe Ansatz are indeed compatible with these requirements, i.e. the functional ansatz for the eigenvalues $t_1(\lambda)$ indeed satisfies the closure relation (3.6) and the null out-boundary conditions (3.7). There, we moreover argue the completeness of the Bethe Ansatz which is compatible with our Conjecture.

It is also worth to mention that we have verified our Conjecture on small lattices, up to three sites. More in detail, we have solved the discrete system of N equations in the N unknowns $t_1^{(K)}(\xi_{i\leq \mathsf{N}})$ obtained particularizing (3.6) in N distinct values of λ. Among these solutions we have selected the solutions verifying the null out-boundary conditions (3.7) for $n = m = 0$. This has produced exactly 3^{N} distinct solutions which are proven to coincide with $T_1^{(K)}(\lambda)$ transfer matrix eigenvalues computed by direct diagonalization, see appendix B.

3.2 SoV spectrum characterization for non-invertible and simple spectrum twist matrix

Let us study here the spectral problem for the transfer matrices associated to the fundamental representations of the $gl_{1|2}$-graded Yang-Baxter algebra in the following class of non-invertible but having simple spectrum $\hat{K}$ twist matrices:

$$\hat{K} = \begin{pmatrix} \mathsf{k}_1 = 0 & 0 \\ 0 & K_2 \end{pmatrix}_{3\times 3}, \tag{3.12}$$

with K_2 any invertible, diagonalizable and simple 2×2 matrix, i.e. it holds:

$$\mathsf{k}_2 \neq \mathsf{k}_3, \ \mathsf{k}_i \neq 0, \ i = 2,3. \tag{3.13}$$

Despite K having a zero eigenvalue, the results of subsection 2.4 imply that the set of covectors (2.90) still forms a covector basis of $\mathscr{H}$. Moreover, these are non-trivial fundamental

[17] The reconstruction of local operators, pioneered in [87,88], implies that the twisted transfer matrix computed in the inhomogeneities coincides with the local matrix K at the site n dressed by the product of shift operators along the chain. So, they are invertible as all these operators are invertible.

representations of the $gl_{1|2}$-graded Yang-Baxter algebra for which our Conjecture is verified, as shown in the following:

Theorem 3.1. *For almost any values of the inhomogeneities* $\{\xi_{a\leq \mathsf{N}}\}$ *satisfying the condition* (2.81) *and the twist matrix eigenvalues satisfying* (3.13)*, the eigenvalue spectrum of* $T_1^{(\hat{K})}(\lambda)$ *coincides with the following set of polynomials*

$$\Sigma_{T^{(\hat{K})}} = \left\{ t_1(\lambda) : t_1(\lambda) = -(\mathsf{k}_2+\mathsf{k}_3)\prod_{a=1}^{\mathsf{N}}(\lambda-\xi_a) + \sum_{a=1}^{\mathsf{N}} f_a^{(1)}(\lambda)x_a, \quad \forall\{x_1,...,x_{\mathsf{N}}\}\in S_{T^{(\hat{K})}} \right\}, \tag{3.14}$$

where $S_{T^{(\hat{K})}}$ *is the set of solutions to the following system of* N *cubic equations:*

$$x_a\left[\mathsf{k}_2\mathsf{k}_3 d(\xi_a+\eta) + \sum_{r=1}^{\mathsf{N}} f_r^{(2)}(\xi_a+\eta)t_1(\xi_r+\eta)x_r\right] = 0, \quad \forall a\in\{1,...,\mathsf{N}\}, \tag{3.15}$$

in N *unknown* $\{x_1,...,x_{\mathsf{N}}\}$*. Moreover,* $T_1^{(\hat{K})}(\lambda)$ *is diagonalizable and with simple spectrum. For any* $t_1(\lambda)\in\Sigma_{T^{(\hat{K})}}$*, the associated and unique eigenvector* $|t\rangle$ *(up-to normalization) has the following wave-functions in our SoV covector basis:*

$$\langle h_1,...,h_{\mathsf{N}}|t\rangle = \prod_{n=1}^{\mathsf{N}} t_1^{h_n}(\xi_n). \tag{3.16}$$

Proof. The main identity to be pointed out here is the following one:

$$T_3^{(\hat{K})}(\lambda)\equiv 0, \tag{3.17}$$

due to the closure relation (3.11) being $\mathsf{k}_1 = 0$. So that the fusion equations (2.66) for $n=1$ and $n=2$ read:

$$T_2^{(\hat{K})}(\xi_a) = T_1^{(\hat{K})}(\xi_a)T_1^{(\hat{K})}(\xi_a+\eta), \tag{3.18}$$

$$0 = T_1^{(\hat{K})}(\xi_a)T_2^{(\hat{K})}(\xi_a+\eta). \tag{3.19}$$

Now it is easy to verify that the system of equations (3.15) just coincides with the above fusion conditions once imposed to functions which have the analytic properties (polynomial form and asymptotics) of eigenvalues. So that it is clear that any eigenvalue has to satisfy them and one is left with the proof of the reverse statement. This proof can be done just showing that the state $|t\rangle$ of the form (3.16) is indeed an eigenvector of the transfer matrix, i.e. that it holds:

$$\langle h_1,...,h_{\mathsf{N}}|T_1^{(\hat{K})}(\lambda)|t\rangle = t_1(\lambda)\langle h_1,...,h_{\mathsf{N}}|t\rangle, \tag{3.20}$$

by direct action of the transfer matrix $T_1^{(\hat{K})}(\lambda)$ on the SoV basis. The steps of the proof are indeed completely similar to those described in the proof of Theorem 5.1 of [1].

Finally, let us point out that Proposition 2.1 implies also that the transfer matrix $T_1^{(\hat{K})}(\lambda)$ is diagonalizable and with simple spectrum for general values of the inhomogeneities parameters. □

Remark 3.1. It is important to point out that the above theorem proves the validity of our Conjecture for the representations considered here, as the system of equations (3.15) is equivalent to the conjectured characterization given by the functional equations (3.6) and (3.7) in the unknowns $t_1^{(K)}(\xi_{i\leq \mathsf{N}})$. Indeed, the system of equations (3.15) is just equivalent to the functional equation

$$t_3(\lambda|\{t_1^{(K)}(\xi_{i\leq \mathsf{N}})\}) = 0, \tag{3.21}$$

which coincides with the closure relation (3.6) for the $\mathsf{k}_1 = 0$ case under consideration. Then it is easy to verify that the null out-boundary conditions (3.7) are also verified. Note for example that the condition (3.21) together with the interpolation formula (2.108) for $\mathsf{k}_1 = 0$ implies:

$$t_n(\lambda|\{t_1^{(K)}(\xi_{i\leq \mathsf{N}})\}) = 0, \quad \forall n \geq 3, \tag{3.22}$$

so that the interpolation formula (2.110) implies:

$$\begin{aligned} t_3^{(2)}(\lambda|\{t_1^{(K)}(\xi_{i\leq \mathsf{N}})\}) &= t_3(\lambda-\eta|\{t_1^{(K)}(\xi_{i\leq \mathsf{N}})\})t_3(\lambda|\{t_1^{(K)}(\xi_{i\leq \mathsf{N}})\}) - t_4(\lambda-\eta|\{t_1^{(K)}(\xi_{i\leq \mathsf{N}})\}) \\ &\quad \times t_2(\lambda|\{t_1^{(K)}(\xi_{i\leq \mathsf{N}})\}) \qquad (3.23) \\ &= 0, \qquad (3.24) \end{aligned}$$

i.e. the null out-boundary conditions (3.7) for $n = m = 0$ and similarly for the others.

Remark 3.2. Let us comment that a different proof of the above theorem can be given by using the fact that the transfer matrix $T_1^{(\hat{K})}(\lambda)$ is diagonalizable with simple spectrum. This in particular means that this transfer matrix admits 3^{N} distinct eigenvalues anyone being a solution of the system (3.15) of N polynomial equations of order three in N unknowns. The Theorem of Bézout[18] states that the above system of polynomial equations admits 3^{N} solutions if the N polynomials, defining the system, have no common components.[19] So, under the condition of no common components, there are indeed exactly 3^{N} distinct solutions to the above system and each one is uniquely associated to a transfer matrix eigenvalue. The proof of the condition of no common components can be done following exactly the same steps presented in appendix B of [160].

Remark 3.3. The fact that $T_3^{(\hat{K})}(\lambda)$ is identically zero in these representations associated to non-invertible simple spectrum twist matrix $\hat{K}$ means, in particular, that it is central so that the algebra shows some strong resemblance to the twisted representations of the gl_3 Yang-Baxter algebra. In fact, taking the gl_3-representation associated to the twist matrix $K' = -\hat{K}$ and $\eta' = -\eta$, then the SoV characterization of the spectrum implies that the transfer matrix $T_1^{(\hat{K})}(\lambda|\eta)$, associated to the $gl_{1|2}$-representation, is isospectral to the transfer matrix $T_1^{(-\hat{K})}(\lambda|-\eta)$, associated to the gl_3-representation. In the same way, we have that $T_2^{(\hat{K})}(\lambda|\eta)$, associated to the $gl_{1|2}$-representation, is isospectral to $T_{(2)}^{(-\hat{K})}(\lambda|-\eta)$, associated to the gl_3-representation.

It is worth remarking that the same type of duality indeed holds between the $gl_{1|\mathcal{N}}$-graded and the $gl_{\mathcal{N}+1}$ non-graded Yang-Baxter algebra when associated to the non-invertible but simple $(\mathcal{N}+1)\times(\mathcal{N}+1)$ twist matrix $\hat{K}$ with first eigenvalue zero. More in detail, we have the isospectrality of the transfer matrices $T_m^{(\hat{K})}(\lambda|\eta)$, associated to the $gl_{1|\mathcal{N}}$-representation, with the transfer matrices $T_{(m)}^{(-\hat{K})}(\lambda|-\eta)$, associated to the $gl_{\mathcal{N}+1}$-representation, for any $1 \leq m \leq \mathcal{N}$. This in particular implies that we can characterize completely as well the spectrum of the transfer matrices of the $gl_{1|\mathcal{N}}$-graded Yang-Baxter algebra for this special class of twist matrices just using the results of [159]. Then the results of the next two subsections can be as well generalized to these special classes of $gl_{1|\mathcal{N}}$-graded Yang-Baxter algebras.

3.2.1 The quantum spectral curve equation for non-invertible twist

The transfer matrix spectrum in our SoV basis is equivalent to the quantum spectral curve[20] functional reformulation as stated in the next theorem.

[18] See for example William Fulton (1974). Algebraic Curves. Mathematics Lecture Note Series. W.A. Benjamin.

[19] Indeed, if there are common components the system admits instead an infinite number of solutions.

[20] To our knowledge, the quantum spectral curve terminology has been introduced by Sklyanin, see for example [25]. It comes natural as the transfer matrices can be seen as the quantum counterpart of the spectral invariants of

Theorem 3.2. *Under the same conditions of the previous theorem, then an entire function* $t_1(\lambda)$ *is a* $T_1^{(\hat{K})}(\lambda)$ *transfer matrix eigenvalue iff. there exists a unique polynomial:*

$$\varphi_t(\lambda) = \prod_{a=1}^{\mathsf{M}} (\lambda - \nu_a) \;\; \textit{with} \;\; \mathsf{M} \leq \mathsf{N} \;\; \textit{and} \;\; \nu_a \neq \xi_n \;\; \forall (a,n) \in \{1,...,\mathsf{M}\} \times \{1,...,\mathsf{N}\}, \tag{3.25}$$

such that $t_1(\lambda)$, $t_2(\lambda|\{t_1(\xi_{a\leq \mathsf{N}})\})$ *and* $\varphi_t(\lambda)$ *are solutions of the following quantum spectral curve functional equation:*

$$\varphi_t(\lambda-\eta)t_2(\lambda-\eta) + \alpha(\lambda)\varphi_t(\lambda)t_1(\lambda) + \beta(\lambda)\varphi_t(\lambda+\eta) = 0\,, \tag{3.26}$$

where we have defined

$$\alpha(\lambda) = -\bar{\alpha}\prod_{a=1}^{\mathsf{N}}(\lambda - 2\eta - \xi_a), \;\; \beta(\lambda) = \alpha(\lambda)\alpha(\lambda+\eta), \tag{3.27}$$

and $\bar{\alpha}$ *is a nonzero solution of the characteristic equation:*

$$\bar{\alpha}\, T_{\infty,2}^{(\hat{K})} - \bar{\alpha}^2\, T_{\infty,1}^{(\hat{K})} + \bar{\alpha}^3 \mathbb{I}_{\mathscr{H}} = 0, \tag{3.28}$$

i.e. $\bar{\alpha} = -\mathsf{k}_2$ *or* $\bar{\alpha} = -\mathsf{k}_3$ *is a nonzero eigenvalue of the twist matrix* $-\hat{K}$*. Moreover, up to a normalization, the common transfer matrix eigenvector* $|t\rangle$ *admits the following separate representation:*[21]

$$\langle h_1,...,h_N|t\rangle = \prod_{a=1}^{\mathsf{N}} \alpha^{h_a}(\xi_a+\eta)\varphi_t^{h_a}(\xi_a+\eta)\varphi_t^{2-h_a}(\xi_a). \tag{3.29}$$

Proof. From the above Remark 3.3, we have that this theorem is a direct consequence of the Theorem 5.2 of [1]. To make the comparison easier, one has just to take the quantum spectral curve characterization of Theorem 4.1 of [159] and use it in the case $n=3$, $K \to -\hat{K}$, $\eta \to -\eta$ to get the quantum spectral curve associated to the non-invertible simple spectrum twist $\hat{K}$. Here, $t_1(\lambda)$ is the eigenvalue associated to the $gl_{1|2}$-transfer matrix $T_1^{(\hat{K})}(\lambda|\eta)$, isospectral to the gl_3-transfer matrix $T_1^{(-\hat{K})}(\lambda|-\eta)$, and $t_2(\lambda|\{t_1(\xi_{a\leq \mathsf{N}})\})$ is the eigenvalue of the $gl_{1|2}$-transfer matrix $T_2^{(\hat{K})}(\lambda|\eta)$, isospectral to the gl_3-transfer matrix $T_{(2)}^{(-\hat{K})}(\lambda|-\eta)$. Then, just removing the common zeros, in the three nonzero terms of the equation and making the common shift $\lambda \to \lambda - 2\eta$, we get our quantum spectral curve equation (3.26). Let us recall that the main elements in the proof of the theorem rely on the fact that the quantum spectral curve

the monodromy matrix. In fact, in [25], these are operatorial functional equations involving just one Q-operator, the canonical operators (i.e. the separate variable operators) and the exponential of their canonical conjugated operators (i.e. the shift operators) and the quantum spectral invariants of the monodromy matrix. In general, we write the quantum spectral curve in its coordinate form, i.e. our quantum spectral curve can be seen as the matrix element of the Sklyanin's one between a transfer matrix eigenstate and an SoV basis element, when Sklyanin's SoV applies, otherwise our results generalize Sklyanin's ones. However, in general, the fact that we can prove that the transfer matrix has simple spectrum and it is diagonalizable allows us to rewrite these quantum spectral curves at the operator level.

[21] Note that (3.29) can be seen as a rewriting of the transfer matrix eigen-wavefunctions in terms of the eigenvalues of a Q-operator. In fact, for $k_1 = 0$, the equation (3.54) and (3.58) imply that the functions $\varphi_t(\lambda)$ are strictly related to the eigenvalues of the operator $Q_2(\lambda)$. Similarly for $k_1 \neq 0$, by using the NABA expression (3.45) and the original SoV representation of the transfer matrix eigen-wavefunctions (3.16), one can argue that (3.29) should be true with $\bar{\alpha} = k_1$ and $\varphi_t(\lambda)$ coinciding with the eigenvalues of the operator $Q_1(\lambda)$.

is equivalent to the following 3N conditions:

$$\alpha(\xi_a+\eta)\frac{\varphi_t(\xi_a+\eta)}{\varphi_t(\xi_a)}=t_1(\xi_a),\quad \text{for } \lambda=\xi_a,\ \forall a\in\{1,...,\mathsf{N}\},\tag{3.30}$$

$$\alpha(\xi_a+\eta)\frac{\varphi_t(\xi_a+\eta)}{\varphi_t(\xi_a)}=\frac{t_2(\xi_a)}{t_1(\xi_a+\eta)},\quad \text{for } \lambda=\xi_a+\eta,\ \forall a\in\{1,...,\mathsf{N}\},\tag{3.31}$$

$$\varphi_t(\xi_a+\eta)t_2(\xi_a+\eta)=0,\quad \text{for } \lambda=\xi_a+2\eta,\ \forall a\in\{1,...,\mathsf{N}\},\tag{3.32}$$

once the asymptotics are fixed as stated above in this theorem. It is then easy to observe that the compatibility of this system of equations is equivalent to impose that $t_1(\lambda)$ and $t_2(\lambda|\{t_1(\xi_{a\leq \mathsf{N}})\})$ satisfy the fusion equations:

$$t_1(\xi_a)t_1(\xi_a+\eta)=t_2(\xi_a),\quad \forall a\in\{1,...,\mathsf{N}\},\tag{3.33}$$

$$t_1(\xi_a)t_2(\xi_a+\eta)=0,\quad \forall a\in\{1,...,\mathsf{N}\}.\tag{3.34}$$

Here, the equation (3.33) is derived as compatibility conditions of (3.30) and (3.31). While, being

$$\alpha(\xi_a+\eta)\neq 0,\quad \varphi_t(\xi_a)\neq 0\ \forall a\in\{1,...,\mathsf{N}\},\tag{3.35}$$

the equation (3.34) is derived from (3.32) multiplying both sides of it for the finite nonzero ratio $\alpha(\xi_a+\eta)/\varphi_t(\xi_a)$ and by using (3.30). □

3.2.2 Completeness of Bethe Ansatz solutions by SoV for non-invertible twist

As detailed in the introduction, Nested Algebraic and Analytic Bethe Ansatz have been used to study the spectrum of the model associated to the fundamental representation of the $gl_{\mathscr{M}|\mathscr{N}}$ Yang-Baxter superalgebra. For the fundamental representation of the $gl_{1|2}$ Yang-Baxter superalgebra associated to a simple and diagonalizable twist matrix K, let us recall here the form of the Bethe Ansatz equations [3, 134–136]:

$$\mathsf{k}_1Q_2(\lambda_j)a(\lambda_j)=\mathsf{k}_2d(\lambda_j)Q_2(\lambda_j+\eta),\tag{3.36}$$

$$\mathsf{k}_2Q_2(\mu_j+\eta)Q_1(\mu_j-\eta)=-\mathsf{k}_3Q_2(\mu_j-\eta)Q_1(\mu_j),\tag{3.37}$$

where

$$Q_1(\lambda)=\prod_{l=1}^{L}(\lambda-\lambda_l),\quad Q_2(\lambda)=\prod_{m=1}^{M}(\lambda-\mu_m),\tag{3.38}$$

and the Bethe Ansatz form of the transfer matrix eigenvalue

$$t_1(\lambda|\{\lambda_{j\leq L}\},\{\mu_{h\leq M}\})=\Lambda_1(\lambda)-\Lambda_2(\lambda)-\Lambda_3(\lambda),\tag{3.39}$$

defined by

$$\Lambda_1(\lambda)=a_1(\lambda)\frac{Q_1(\lambda-\eta)}{Q_1(\lambda)},\tag{3.40}$$

$$\Lambda_2(\lambda)=a_2(\lambda)\frac{Q_1(\lambda-\eta)Q_2(\lambda+\eta)}{Q_1(\lambda)Q_2(\lambda)},\tag{3.41}$$

$$\Lambda_3(\lambda)=a_3(\lambda)\frac{Q_2(\lambda-\eta)}{Q_2(\lambda)},\tag{3.42}$$

where

$$a_1(\lambda)=\mathsf{k}_1a(\lambda),\tag{3.43}$$

$$\frac{a_2(\lambda)}{\mathsf{k}_2}=\frac{a_3(\lambda)}{\mathsf{k}_3}=d(\lambda).\tag{3.44}$$

It is worth to observe that being

$$t_1(\lambda|\{\lambda_{j\leq L}\},\{\mu_{h\leq M}\}) = \mathsf{k}_1 a(\lambda)\frac{Q_1(\lambda-\eta)}{Q_1(\lambda)} - d(\lambda)\left(\mathsf{k}_2\frac{Q_1(\lambda-\eta)Q_2(\lambda+\eta)}{Q_1(\lambda)Q_2(\lambda)} + \mathsf{k}_3\frac{Q_2(\lambda-\eta)}{Q_2(\lambda)}\right), \tag{3.45}$$

under the following *pair-wise distinct* conditions

$$\lambda_l \neq \lambda_m,\ \mu_p \neq \mu_q,\ \mu_q \neq \lambda_m,\ \forall l \neq m \in \{1,...,L\},\ p \neq q \in \{1,...,M\}, \tag{3.46}$$

it follows that the function $t_1(\lambda|\{\lambda_{j\leq L}\},\{\mu_{h\leq M}\})$ has only apparent simple poles in the $\lambda_{j\leq L}$ and $\mu_{h\leq M}$. The regularity of $t_1(\lambda|\{\lambda_{j\leq L}\},\{\mu_{h\leq M}\})$ for $\lambda = \lambda_{j\leq L}$ is implied by the Bethe equation (3.36) while the regularity of $t_1(\lambda|\{\lambda_{j\leq L}\},\{\mu_{h\leq M}\})$ for $\lambda = \mu_{j\leq M}$ is implied by the Bethe equation (3.37). Hence[22] $t_1(\lambda|\{\lambda_{j\leq L}\},\{\mu_{h\leq M}\})$ is a polynomial of degree N with the correct asymptotic for a transfer matrix eigenvalue, i.e. it holds:

$$\lim_{\lambda\to\infty}\lambda^{-\mathsf{N}}t_1(\lambda|\{\lambda_{j\leq L}\},\{\mu_{h\leq M}\}) = \mathsf{k}_1 - (\mathsf{k}_2+\mathsf{k}_3). \tag{3.47}$$

So that the above ansatz is indeed consistent with the analytic properties enjoyed by the transfer matrix eigenvalues. Now, we show that the Bethe ansatz solutions are complete as a corollary of the completeness of the derived quantum spectral curve in the SoV framework, for the class of representations considered in this section. More precisely it holds the next

Corollary 3.1. *Let us consider the class of fundamental representations of the $gl_{1|2}$-graded Yang-Baxter algebra associated to non-invertible but simple spectrum $\hat{K}$ twist matrices, with eigenvalues satisfying (3.13). Then, for almost any values of the inhomogeneities, $t_1(\lambda)$ is an eigenvalue of the transfer matrix $T_1^{(\hat{K})}(\lambda|\eta)$ iff. there exists a solution $\{\{\lambda_{j\leq L}\},\{\mu_{h\leq M}\}\}$ to the Bethe Ansatz equations (3.36) and (3.37) such that $t_1(\lambda) \equiv t_1(\lambda|\{\lambda_{j\leq L}\},\{\mu_{h\leq M}\})$, i.e. $t_1(\lambda)$ has the Bethe Ansatz form (3.45) associated to the solutions $\{\lambda_{j\leq L}\}$, $\{\mu_{h\leq M}\}$. Moreover, for any $t_1(\lambda) \in \Sigma_{T^{(\hat{K})}}$ the associated Bethe Ansatz solution $\{\{\lambda_{j\leq L}\},\ \{\mu_{h\leq M}\}\}$ is unique and satisfies the admissibility conditions:*

$$\{\lambda_{j\leq L}\} \subset \{\xi_{j\leq \mathsf{N}}\},\quad \{\mu_{h\leq M}\} \cap \{\{\xi_{j\leq \mathsf{N}}\} \cup \{\xi_{j\leq \mathsf{N}}+\eta\}\} = \emptyset. \tag{3.48}$$

Proof. This corollary directly follows from our previous theorem. The proof is done pointing out the consequences of the special form of the fusion equations for these representations associated to these non-invertible twists. In particular, from the fusion equations (3.33) and (3.34), which have to be satisfied by all the transfer matrix eigenvalues, we derive the following equation on the second transfer matrix eigenvalues only:

$$t_2^{(\hat{K})}(\xi_a)t_2^{(\hat{K})}(\xi_a+\eta) = 0. \tag{3.49}$$

Being by definition $t_2^{(\hat{K})}(\lambda)$ a degree $2\mathsf{N}$ polynomial in λ, zero in the points $\xi_a - \eta$ for any $a \in \{1,...,\mathsf{N}\}$, it follows that a solution to (3.49) can be obtained iff for any $a \in \{1,...,\mathsf{N}\}$ there exists a unique $h_a \in \{-1,0\}$ such that:

$$t_{2,\mathbf{h}}^{(\hat{K})}(\lambda) = \mathsf{k}_2\mathsf{k}_3\prod_{a=1}^{\mathsf{N}}(\lambda-\xi_a+\eta)(\lambda-\xi_a^{(h_a)}). \tag{3.50}$$

So we have that the system (3.49) has exactly 2^{N} distinct solutions associated to the 2^{N} distinct N-uplet $\mathbf{h} = \{h_{1\leq n\leq \mathsf{N}}\}$ in $\{-1,0\}^{\mathsf{N}}$. Now for any fixed $\mathbf{h} \in \{-1,0\}^{\mathsf{N}}$ we can define a permutation $\pi_{\mathbf{h}} \in S_{\mathsf{N}}$ and a non-negative integer $m_{\mathbf{h}} \leq \mathsf{N}$ such that:

$$h_{\pi_{\mathbf{h}}(a)} = 0,\quad \forall a \in \{1,...,m_{\mathbf{h}}\}\ \text{ and }\ h_{\pi_{\mathbf{h}}(a)} = -1,\quad \forall a \in \{m_{\mathbf{h}}+1,...,\mathsf{N}\}. \tag{3.51}$$

[22] One should also ask for some condition like $\{\mu_{h\leq M}\}\cap\{\xi_{j\leq \mathsf{N}}\} = \emptyset$ from which the Bethe equation (3.36) implies $\{\lambda_{h\leq L}\}\cap\{\xi_{j\leq \mathsf{N}}\} = \emptyset$ unless $\mathsf{k}_1 = 0$.

It is easy to remark now that fixed $\mathbf{h}\in\{-1,0\}^{\mathsf{N}}$ then (3.33), for $a\in\{1,...,m_{\mathbf{h}}\}$, and (3.34) are satisfied iff it holds:

$$t_{1,\mathbf{h}}^{(\hat{K})}(\xi_{\pi_{\mathbf{h}}(a)})=0,\quad \forall a\in\{1,...,m_{\mathbf{h}}\}. \tag{3.52}$$

Indeed, if this is not the case for a given $b\in\{1,...,m_{\mathbf{h}}\}$, then (3.34) implies $t_{2,\mathbf{h}}^{(\hat{K})}(\xi_{h_{\pi_{\mathbf{h}}(b)}}+\eta)=0$ which is not compatible with our choice of $t_{2,\mathbf{h}}^{(\hat{K})}(\lambda)$. So, for any fixed $\mathbf{h}\in\{0,-1\}^{\mathsf{N}}$, we have that the eigenvalues of the transfer matrix have the following form:

$$t_{1,\mathbf{h}}^{(\hat{K})}(\lambda)=\bar{t}_{1,\mathbf{h}}^{(\hat{K})}(\lambda)\prod_{a=1}^{m_{\mathbf{h}}}(\lambda-\xi_{\pi_{\mathbf{h}}(a)}), \tag{3.53}$$

where $\bar{t}_{1,\mathbf{h}}^{(\hat{K})}(\lambda)$ is a degree $\mathsf{N}-m_{\mathbf{h}}$ polynomial in λ, and the function $\varphi_{t,\mathbf{h}}(\lambda)$ associated by the quantum spectral curve to the eigenvalue $t_{1,\mathbf{h}}^{(\hat{K})}(\lambda)$ has the form:

$$\varphi_{t,\mathbf{h}}(\lambda)=\bar{\varphi}_{t,\mathbf{h}}(\lambda)\prod_{a=1}^{m_{\mathbf{h}}}(\lambda-\xi_{\pi_{\mathbf{h}}(a)}-\eta), \tag{3.54}$$

where $\bar{\varphi}_{t,\mathbf{h}}(\lambda)$ is of degree $M-m_{\mathbf{h}}\leq M$ polynomial in λ. Then, simplifying common prefactors, the quantum spectral curve rewrite as it follows:

$$\bar{t}_{1,\mathbf{h}}^{(\hat{K})}(\lambda)\bar{\varphi}_{t,\mathbf{h}}(\lambda)=\bar{\alpha}\,\bar{d}_{\mathbf{h}}(\lambda-\eta)\bar{\varphi}_{t,\mathbf{h}}(\lambda+\eta)+\frac{\mathsf{k}_2\mathsf{k}_3}{\bar{\alpha}}\bar{d}_{\mathbf{h}}(\lambda)\bar{\varphi}_{t,\mathbf{h}}(\lambda-\eta), \tag{3.55}$$

where we have defined:

$$\bar{d}_{\mathbf{h}}(\lambda)=\prod_{a=m_{\mathbf{h}}+1}^{\mathsf{N}}(\lambda-\xi_{\pi_{\mathbf{h}}(a)}). \tag{3.56}$$

So, once we chose $\bar{\alpha}=-\mathsf{k}_2$, we get the following representation of the transfer matrix eigenvalue:

$$t_{1,\mathbf{h}}^{(\hat{K})}(\lambda)=-\prod_{a=1}^{m_{\mathbf{h}}}(\lambda-\xi_{\pi_{\mathbf{h}}(a)})\frac{\mathsf{k}_2\,\bar{d}_{\mathbf{h}}(\lambda-\eta)\bar{\varphi}_{t,\mathbf{h}}(\lambda+\eta)+\mathsf{k}_3\bar{d}_{\mathbf{h}}(\lambda)\bar{\varphi}_{t,\mathbf{h}}(\lambda-\eta)}{\bar{\varphi}_{t,\mathbf{h}}(\lambda)}. \tag{3.57}$$

It is now trivial to verify that this coincides with the Bethe ansatz form (3.45) for $\mathsf{k}_1=0$, once we fix:

$$Q_1^{\left(t_{1,\mathbf{h}}^{(\hat{K})}\right)}(\lambda)\equiv\bar{d}_{\mathbf{h}}(\lambda),\ Q_2^{\left(t_{1,\mathbf{h}}^{(\hat{K})}\right)}(\lambda)\equiv\bar{\varphi}_{t,\mathbf{h}}(\lambda). \tag{3.58}$$

Clearly by definition $Q_1^{\left(t_{1,\mathbf{h}}^{(\hat{K})}\right)}(\lambda)$ and $Q_2^{\left(t_{1,\mathbf{h}}^{(\hat{K})}\right)}(\lambda)$ are solutions of the Bethe Ansatz equations (3.36) and (3.37) and their roots satisfy the conditions (3.48). □

Remark 3.4. It is worth to point out that the above set of Bethe Ansatz solutions indeed satisfies also the pair-wise distinct conditions (3.46). Indeed, from the proof of the previous corollary, we know that for any fixed $\mathbf{h}\in\{-1,0\}^{\mathsf{N}}$, there are $2^{\mathsf{N}-m_{\mathbf{h}}}$ eigenvalues of the transfer matrix of the form (3.57) associated to as many polynomials $\bar{\varphi}_{t,\mathbf{h}}(\lambda)$ of degree $M\leq\mathsf{N}-m_{\mathbf{h}}$ in λ. For any fixed $\mathbf{h}\in\{-1,0\}^{\mathsf{N}}$, these are solutions to (3.37) which coincide with the system of Bethe Ansatz equations associated to an inhomogeneous XXX spin 1/2 quantum chain with $\mathsf{N}-m_{\mathbf{h}}$ quantum sites, with inhomogeneities $\xi_{\pi_{\mathbf{h}}(a)}$ for $a\in\{m_{\mathbf{h}}+1,...,\mathsf{N}\}$ and parameter $-\eta$. Then, to these Bethe Ansatz solutions apply the results of the paper [248] which implies the *pair-wise distinct* conditions

$$\mu_p\neq\mu_q,\ \forall\ p\neq q\in\{1,...,M\}, \tag{3.59}$$

which together with the already proven (3.48) imply in particular (3.46).

4 Separation of variables basis for inhomogeneous Hubbard model

4.1 The inhomogeneous Hubbard model

The 1+1 dimensional Hubbard model is integrable in the quantum inverse scattering approach with respect to the Shastry's R-matrix, which contains as a special case the Lax operator of the Hubbard model [120–122]. In order to introduce them let us start defining the following functions:

$$h(\lambda,\eta):\sinh 2h(\lambda,\eta)=\frac{i\eta}{2}\sin 2\lambda,\quad \Lambda(\lambda)=-i\operatorname{cotg}(2\lambda)\cosh(2h(\lambda,\eta)), \tag{4.1}$$

here, we use the notation $\eta=-2iU$ with the parameter U, the coupling of the Hubbard model, as it plays a similar role as the parameter η in the XXX model from the point of view of the Bethe equations. In the following we omit the η dependence in $h(\lambda,\eta)$ if not required. Then the Shastry's R-matrix reads:

$$R_{12,34}(\lambda|\mu)=I_{12}(h(\lambda))I_{34}(h(\mu))\hat{R}_{12,34}(\lambda|\mu)I_{12}(-h(\lambda))I_{34}(-h(\mu)), \tag{4.2}$$

where

$$\hat{R}_{12,34}(\lambda|\mu)=R_{1,3}(\lambda-\mu)R_{2,4}(\lambda-\mu)-\frac{\sin(\lambda-\mu)}{\sin(\lambda+\mu)}\tanh(h(\lambda)+h(\mu))R_{1,3}(\lambda+\mu)\sigma_1^yR_{2,4}(\lambda+\mu)\sigma_2^y, \tag{4.3}$$

and

$$R_{a,b}(\lambda)=\begin{pmatrix}\cos\lambda & 0 & 0 & 0\\ 0 & \sin\lambda & 1 & 0\\ 0 & 1 & \sin\lambda & 0\\ 0 & 0 & 0 & \cos\lambda\end{pmatrix}\in\mathrm{End}(V_a\otimes V_b), \tag{4.4}$$

where $V_a\cong V_b\cong\mathbb{C}^2$ and we have defined:

$$I_{1,2}(h)=\cosh h/2+\sigma_1^y\otimes\sigma_2^y\sinh h/2=\exp\left(\sigma_1^y\sigma_2^yh/2\right), \tag{4.5}$$

which satisfies the Yang-Baxter equation:

$$R_{A,B}(\lambda|\mu)R_{A,C}(\lambda|\xi)R_{B,C}(\mu|\xi)=R_{B,C}(\mu|\xi)R_{A,C}(\lambda|\xi)R_{A,B}(\lambda|\mu)\in\mathrm{End}(V_A\otimes V_B\otimes V_C), \tag{4.6}$$

where we have used the capital Latin letters to represent a couple of integers, for example $A=(1,2)$, $B=(3,4)$, $C=(5,6)$, meaning that:

$$V_A=V_1\otimes V_2\cong\mathbb{C}^4,\ V_B=V_3\otimes V_4\cong\mathbb{C}^4,\ V_C=V_5\otimes V_6\cong\mathbb{C}^4. \tag{4.7}$$

This R-matrix satisfies the following properties:

$$R_{A,B}(\lambda|\lambda)=P_{1,3}P_{2,4}\,, \tag{4.8}$$

where $P_{i,j}$ are the permutation operators on the two-dimensional spaces $V_i\cong V_j\cong\mathbb{C}^2$, moreover, it holds:

$$R_{A,B}(\lambda|0)=\frac{L_{A,B}(\lambda)}{\cosh h(\lambda)},\quad R_{A,B}(0|\lambda)=\frac{L_{A,B}(-\lambda)}{\cosh h(\lambda)}, \tag{4.9}$$

where $L_{A,B}(\lambda)$ is the Lax operator for the homogeneous Hubbard model:

$$L_{A,B}(\lambda)=I_{12}(h(\lambda))R_{1,3}(\lambda)R_{2,4}(\lambda)I_{12}(h(\lambda)). \tag{4.10}$$

We have the following unitarity property:

$$R_{A,B}(\lambda|\mu)R_{B,A}(\mu|\lambda) = \cos^2(\lambda-\mu)(\cos^2(\lambda-\mu)-\cos^2(\lambda+\mu)\tanh(h(\lambda)-h(\mu))), \tag{4.11}$$

and crossing unitarity relations:

$$R^{-1}_{A,B}(\lambda|\mu) \propto \sigma^y_1 \otimes \sigma^y_2 R^{t_A}_{A,B}(\lambda-\eta|\mu)\sigma^y_1 \otimes \sigma^y_2, \tag{4.12}$$

$$R^{-1}_{A,B}(\lambda|\mu) \propto \sigma^y_3 \otimes \sigma^y_4 R^{t_B}_{A,B}(\lambda|\mu+\eta)\sigma^y_3 \otimes \sigma^y_4. \tag{4.13}$$

This R-matrix satisfies the following symmetry properties, i.e. scalar Yang-Baxter equation:

$$R_{A,B}(\lambda|\mu)K_AK_B = K_BK_AR_{A,B}(\lambda|\mu) \in \mathrm{End}(V_A \otimes V_B), \tag{4.14}$$

where $K \in \mathrm{End}\left(V \cong \mathbb{C}^4\right)$ is any 4×4 matrix of the form:

$$\begin{aligned} K(a,\alpha,\beta,\gamma) = \delta_{a,1}\begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & \gamma & 0 \\ 0 & 0 & 0 & \beta\gamma/\alpha \end{pmatrix} + \delta_{a,2}\begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & 0 & \beta & 0 \\ 0 & \gamma & 0 & 0 \\ 0 & 0 & 0 & \beta\gamma/\alpha \end{pmatrix} \\ + \delta_{a,3}\begin{pmatrix} 0 & 0 & 0 & \alpha \\ 0 & \beta & 0 & 0 \\ 0 & 0 & \gamma & 0 \\ \beta\gamma/\alpha & 0 & 0 & 0 \end{pmatrix} + \delta_{a,4}\begin{pmatrix} 0 & 0 & 0 & \alpha \\ 0 & 0 & \beta & 0 \\ 0 & \gamma & 0 & 0 \\ \beta\gamma/\alpha & 0 & 0 & 0 \end{pmatrix}, \end{aligned} \tag{4.15}$$

where α,β and γ are generic complex values. Note that $K(1,\alpha,\beta,\gamma)$ is simple for generic different values of α,β,γ satisfying $\beta\gamma/\alpha \neq \alpha,\beta,\gamma$. Being $\left\{\alpha,\beta\gamma/\alpha,\sqrt{\beta\gamma},-\sqrt{\beta\gamma}\right\}$ the eigenvalues of $K(2,\alpha,\beta,\gamma)$, then $K(2,\alpha,\beta,\gamma)$ is simple for generic nonzero values of α,β,γ satisfying $\beta\gamma \neq \alpha^2$. Being $\left\{\beta,\gamma,\sqrt{\beta\gamma},-\sqrt{\beta\gamma}\right\}$ the eigenvalues of $K(3,\alpha,\beta,\gamma)$, then $K(3,\alpha,\beta,\gamma)$ is simple for generic different and nonzero values of β,γ. The matrix $K(4,\alpha,\beta,\gamma)$ is instead degenerate being $\left\{\sqrt{\beta\gamma},-\sqrt{\beta\gamma}\right\}$ its eigenvalues.

We can define the following monodromy matrix:

$$M^{(K)}_A(\lambda) \equiv K_AR_{A,A_\mathsf{N}}(\lambda|\xi_\mathsf{N})\cdots R_{A,A_1}(\lambda|\xi_1) \in \mathrm{End}(V_A \otimes \mathscr{H}), \tag{4.16}$$

where $\mathscr{H} = \bigotimes_{n=1}^{\mathsf{N}} V_{A_n}$, $V_{A_n} \cong \mathbb{C}^4$. Then the transfer matrix:

$$T^{(K)}(\lambda) \equiv tr_A M^{(K)}_A(\lambda), \tag{4.17}$$

defines a one-parameter family of commuting operators.

4.2 Our SoV covector basis

The general Proposition 2.4 and 2.5 of [1] for the construction of the SoV covector basis and the diagonalizability and simplicity of the transfer matrix spectrum can be adapted to the inhomogeneous Hubbard model. Let us denote with $K_J(a,\alpha,\beta,\gamma)$ the diagonal form of the matrix $K(a,\alpha,\beta,\gamma)$ and W_K the invertible matrix defining the change of basis to it:

$$K = W_KK_JW_K^{-1}, \tag{4.18}$$

clearly W_K is the identity for $a=1$, then the following theorem holds:

Theorem 4.1. *For almost any choice of the inhomogeneities under the condition (2.81) and of the twist matrix $K(a,\alpha,\beta,\gamma)$, for $a=1,2,3$, the Hubbard transfer matrix $T^{(K)}(\lambda)$ is diagonalizable and with simple spectrum and the following set of covectors:*

$$\langle h_1,...,h_{\mathsf{N}}| \equiv \langle S|\prod_{n=1}^{\mathsf{N}}(T^{(K)}(\xi_n))^{h_n} \quad \text{for any } \{h_1,...,h_{\mathsf{N}}\}\in\{0,1,2,3\}^{\otimes \mathsf{N}}, \tag{4.19}$$

forms a covector basis of $\mathscr{H}$, for almost any choice of $\langle S|$. In particular, we can take the state $\langle S|$ of the following tensor product form:

$$\langle S| = \bigotimes_{a=1}^{\mathsf{N}}(x,y,z,w)_a \Gamma_W^{-1}, \quad \Gamma_W = \bigotimes_{a=1}^{\mathsf{N}} W_{K,a}, \tag{4.20}$$

simply asking $x\,y\,z\,w\neq 0$.

Proof. We have just to remark that also in this case the following identity holds:

$$T^{(K)}(\xi_n) = R_{A_n,A_{n-1}}(\xi_n|\xi_{n-1})\cdots R_{A_n,A_1}(\xi_n|\xi_1)K_{A_n}R_{A_n,A_{\mathsf{N}}}(\xi_n|\xi_{\mathsf{N}})\cdots R_{A_n,A_{n+1}}(\xi_n|\xi_{n+1}). \tag{4.21}$$

Let us now point out that $e^{h(\lambda,\eta)}$ is an algebraic function of order two in η and e^{λ}. Then the determinant of the matrix whose rows are the elements of these covectors in the elementary basis is also an algebraic function of $\{e^{\xi_m}\}_{m\in\{1,...,\mathsf{N}\}}$ and η. So that showing that this determinant is nonzero for a specific value of η one can prove that it is nonzero for almost any value of η and of the others parameters, i.e. the inhomogeneities satisfying (2.81) and the parameters α,β,γ of the twist matrix, for $a=1,2,3$. We can study for example the case $\eta=0$. In this case $h(\lambda,\eta)$ has the following two different determinations:

$$h(\lambda,\eta=0) = 0, i\pi/2 \ \mathrm{mod} i\pi. \tag{4.22}$$

Note that in both the cases, we have that it holds:

$$\tanh(h(\lambda,0)+h(\mu,0)) = 0, \tag{4.23}$$

so that the Shastry's R-matrix reduces to the tensor product of two XX R-matrix, i.e. it holds:

$$R_{A\equiv(1,2),B\equiv(3,4)}(\lambda|\mu)_{\eta=0} = R_{1,3}(\lambda-\mu)R_{2,4}(\lambda-\mu). \tag{4.24}$$

In turn this implies that:

$$\lim_{\lambda\to -i\infty} e^{-i\lambda}R_{A\equiv(1,2),B\equiv(3,4)}(\lambda|\mu)_{\eta=0} = (e^{-i\mu}/4)\,\mathbb{I}_{V_A\otimes V_B}, \tag{4.25}$$

so that we can repeat the same type of proof of the general Proposition 2.4 of [1] to show that for a covector $\langle S|$, of the above tensor product form, the determinant of the full matrix factorizes in the product of the determinants of 4×4 matrices which are nonzero due to the simplicity of the spectrum of the matrix K. This already implies the w-simplicity of the transfer matrix $T^{(K)}(\lambda)$ then in the case $\eta=0$ we can prove the non-orthogonality condition:

$$\langle t|t\rangle \neq 0, \tag{4.26}$$

for any transfer matrix eigenvector by the same argument developed in general Proposition 2.5 of [1], which implies the diagonalizability and simplicity of the transfer matrix spectrum for $\eta=0$ and so for almost any value of η and of the others parameters of the representation. □

Let us briefly comment about the consequences of the existence of such a basis. The first important point to stress is that whenever we have an eigenvalue for the transfer matrix, we can write the corresponding eigenvector in the above basis. It means that if we compute a set of solutions to the Nested Bethe Ansatz equations we can immediately write the transfer matrix eigenvalue and hence the corresponding eigenvector; in particular it will be a true eigenvector as soon as it is non zero. This could be of great use in practice when dealing with finite chains with a number of sites greater than the values accessible by direct diagonalization. In particular, scalar products and form factors could become accessible, at least numerically, from this procedure. For using the above basis on a more fundamental, analytical level, one needs to obtain the complete set of fusion relations that lead to the full closure relations enabling to compute the action of the transfer matrix in the SoV basis (see the discussion on this point given in [1, 159]). This should lead to the full characterization of the spectrum. These fusion relations being rather involved for the Hubbard model, due in particular to the intricate dependence on spectral parameters [132], we will come back to this question in a future publication. Let us nevertheless anticipate that the results obtained for the $gl_{\mathscr{M}|\mathscr{N}}$ case will be of direct importance when dealing with the Hubbard model.

Acknowledgements

J.M.M. and G.N. are supported by CNRS and ENS de Lyon. L.V. is supported by École Polytechnique and ENS de Lyon. The authors would like to thank F. Delduc, M. Magro and H. Samtleben for their availability and for interesting discussions.

A Compatibility of SoV and Bethe Ansatz framework

In this appendix, we verify how the results obtained in the Nested Algebraic and Analytic Bethe Ansatz framework for the $gl_{1|2}$ Yang-Baxter superalgebra are compatible with the conjectured spectrum characterization in the SoV basis. This analysis is done in the fundamental representations of the $gl_{1|2}$ Yang-Baxter superalgebra associated to generic diagonalizable and simple spectrum twist matrices.

A.1 Compatibility conditions for higher transfer matrix eigenvalues

Here we use the Bethe Ansatz form (3.39) of the transfer matrix eigenvalues together with the Bethe ansatz equations (3.36) and (3.37) to describe the eigenvalues of the higher transfer matrices in order to verify that they satisfy both the null out-boundary (3.7) and the inner-boundary (3.4) conditions. Under these hypothesis, we get the following lemma:

Lemma A.1. *The eigenvalues of the higher transfer matrices admit the following representation in terms of the $\Lambda_i(\lambda)$ functions:*

$$t_2(\lambda) = \Lambda_1(\lambda)(\mathsf{k}_1 t_1(\lambda+\eta) + \mathsf{k}_3\mathsf{k}_2 d(\lambda))/\mathsf{k}_1 \tag{A.1}$$

$$= \Lambda_1(\lambda)\left(\Lambda_1(\lambda+\eta) - \Lambda_2(\lambda+\eta) - \Lambda_3(\lambda+\eta) + \mathsf{k}_3\mathsf{k}_2 d(\lambda)/\mathsf{k}_1\right), \tag{A.2}$$

and

$$t_{n+1}(\lambda) = \Lambda_1(\lambda) t_n(\lambda+\eta) \quad \forall n \geq 2. \tag{A.3}$$

Proof. We have already observed that due to the Corollary 2.1 the eigenvalues of the higher transfer matrices admit the interpolation formulae (2.108) in terms of the transfer matrix

eigenvalue $t_1(\lambda)$. Equivalently, given $t_1(\lambda)$ an eigenvalue of the transfer matrix then those of the higher transfer matrices are of the form

$$t_n(\lambda) = \prod_{r=1}^{n-1} d(\lambda + r\eta)\tilde{t}_n(\lambda) \quad \forall n \geq 2, \tag{A.4}$$

where $\tilde{t}_n(\lambda)$ are degree N polynomials in λ, fixed uniquely by the recursive equations:

$$t_2(\xi_a) = t_1(\xi_a)t_1(\xi_a + \eta), \tag{A.5}$$
$$t_{n+1}(\xi_a) = t_1(\xi_a)t_n(\xi_a + \eta), \tag{A.6}$$

and the known asymptotics:

$$\lim_{\lambda\to\infty} \lambda^{-\mathsf{N}} t_n(\lambda) = T^{(K)}_{\infty,n} = \mathsf{k}_1^{n-2}(\mathsf{k}_1 - \mathsf{k}_3)(\mathsf{k}_1 - \mathsf{k}_2), \quad \forall n \geq 2. \tag{A.7}$$

So to prove the above Bethe Ansatz form for the higher transfer matrix eigenvalues we have just to verify these conditions. Concerning the asymptotic behaviour, from the r.h.s. of formula (A.1) and (A.3) we get:

$$\mathsf{k}_1(\mathsf{k}_1 - \mathsf{k}_3 - \mathsf{k}_2) + \mathsf{k}_3\mathsf{k}_2 = (\mathsf{k}_1 - \mathsf{k}_3)(\mathsf{k}_1 - \mathsf{k}_2) \tag{A.8}$$
$$= \left((\mathrm{str}K)^2 + \left(\mathrm{str}K^2\right)\right)/2 = T^{(K)}_{\infty,2}, \tag{A.9}$$

and

$$\lim_{\lambda\to\infty} \lambda^{-\mathsf{N}} t_n(\lambda) = \mathsf{k}_1^{n-2} T^{(K)}_{\infty,2} = T^{(K)}_{\infty,n}, \tag{A.10}$$

so they are satisfied. So we are left with the proof of the fusion properties. It is easy to remark that by the definition of the $\Lambda_i(\lambda)$ it follows that the $t_n(\lambda)$ indeed factorize the coefficients $\prod_{r=1}^{n-1} d(\lambda + r\eta)$, for $n \geq 2$. Let us now show that

$$\widetilde{t}_2(\lambda) = \frac{Q_1(\lambda - \eta)}{Q_1(\lambda)}(\mathsf{k}_1 t_1(\lambda + \eta) + \mathsf{k}_3\mathsf{k}_2 d(\lambda)), \tag{A.11}$$

is indeed a degree N polynomial in λ. This is the case iff the residues of this expression in the zeroes of $Q_1(\lambda)$ are vanishing, namely iff the following identities hold:

$$t_1(\lambda_j + \eta) = -\frac{\mathsf{k}_3\mathsf{k}_2}{\mathsf{k}_1} d(\lambda_j) \text{ for any } \; j \in \{1, ..., L\}, \tag{A.12}$$

and this is the case thanks to the Bethe equation (3.36) in λ_j being:

$$t_1(\lambda_j + \eta) = -\Lambda_3(\lambda_j + \eta) = -\mathsf{k}_3 a(\lambda_j)\frac{Q_2(\lambda_j)}{Q_2(\lambda_j + \eta)}. \tag{A.13}$$

Similarly, we have that

$$\widetilde{t}_n(\lambda) = \mathsf{k}_1 \frac{Q_1(\lambda - \eta)}{Q_1(\lambda)}\widetilde{t}_{n-1}(\lambda + \eta) \tag{A.14}$$
$$= \mathsf{k}_1^{n-2} \frac{Q_1(\lambda - \eta)}{Q_1(\lambda + (n-3)\eta)}\widetilde{t}_2(\lambda + (n-2)\eta) \tag{A.15}$$
$$= \mathsf{k}_1^{n-2} \frac{Q_1(\lambda - \eta)}{Q_1(\lambda + (n-2)\eta)}(\mathsf{k}_1 t_1(\lambda + \lambda + (n-1)\eta) + \mathsf{k}_3\mathsf{k}_2 d(\lambda + (n-2)\eta)), \tag{A.16}$$

which is a degree N polynomial in λ due to the identity (3.36).

So to show that the $t_n(\lambda)$ satisfy the characterization of the higher eigenvalues, we have just to verify that their values in the inhomogeneities agree with (A.5) and (A.6). Indeed, it holds:

$$\begin{aligned} t_2(\xi_a) &= \Lambda_1(\xi_a)(\mathsf{k}_1 t_1(\xi_a+\eta)+\mathsf{k}_3\mathsf{k}_2 d(\xi_a))/\mathsf{k}_1 \\ &= \Lambda_1(\xi_a) t_1(\xi_a+\eta) \\ &= t_1(\xi_a) t_1(\xi_a+\eta), \end{aligned} \tag{A.17}$$

where in the last line we have used the Bethe Ansatz form of $t_1(\lambda)$ and similarly:

$$t_{n+1}(\xi_a) = \Lambda_1(\xi_a) t_n(\xi_a+\eta) = t_1(\xi_a) t_n(\xi_a+\eta). \tag{A.18}$$

□

Here we have explicitly rewritten the eigenvalues form in Bethe Ansatz approach for the higher transfer matrix $T_n^{(\hat{K})}(\lambda)$, by using the fusion we can easily derive those of the others. Now we are interested in showing that these expressions for the higher eigenvalues indeed imply the null out-boundary (3.7) and the inner-boundary (3.4) conditions. Indeed, we have the following lemma:

Lemma A.2. *Let us take a Bethe equation solution and associate to it the $t_1(\lambda)$ of the form (3.45), then the higher functions $t_n^{(m)}(\lambda)$ generated from $t_1(\lambda)$ by the fusion equations, i.e. by using (2.57), (2.58), (2.82) and (2.83), satisfy the null out-boundary condition (3.7) and the inner-boundary (3.4).*

Proof. By using the result of the previous lemma it is easy to show the following null conditions are satisfied:

$$t_{3+n}^{(2)}(\lambda) = 0, \quad \forall n \geq 0, \tag{A.19}$$

indeed, the condition (A.3) implies:

$$\Lambda_1(\lambda) = t_{3+n}(\lambda)/t_{2+n}(\lambda+\eta), \quad \forall n \geq 0, \tag{A.20}$$

so that, in particular, it holds:

$$t_{3+n}(\lambda) = (t_{2+n}(\lambda)/t_{1+n}(\lambda+\eta))(t_{2+n}(\lambda+\eta)), \quad \forall n \geq 1, \tag{A.21}$$

or equivalently:

$$t_{2+n}(\lambda) t_{2+n}(\lambda+\eta) = t_{3+n}(\lambda) t_{1+n}(\lambda+\eta), \quad \forall n \geq 1, \tag{A.22}$$

which by the fusion equations implies the above null conditions. Similarly, we can derive all the other null out-boundary conditions (3.7).

Let us now show the *inner-boundary* condition, from the formula (A.1) we can write:

$$\Lambda_1(\lambda) = \frac{t_2(\lambda)}{t_1(\lambda+\eta)+\mathsf{k}_3\mathsf{k}_2 d(\lambda)/\mathsf{k}_1}, \tag{A.23}$$

and so

$$t_3(\lambda) = \frac{\mathsf{k}_1 t_2(\lambda) t_2(\lambda+\eta)}{\mathsf{k}_1 t_1(\lambda+\eta)+\mathsf{k}_3\mathsf{k}_2 d(\lambda)}, \tag{A.24}$$

which is equivalent to our closure relation:

$$(\mathsf{k}_1 t_1(\lambda+\eta)+\mathsf{k}_3\mathsf{k}_2 d(\lambda)) t_3(\lambda) = \mathsf{k}_1 t_2(\lambda) t_2(\lambda+\eta), \tag{A.25}$$

and taking into account the fusion equation:

$$t_2^{(2)}(\lambda+\eta) = t_2^{(1)}(\lambda) t_2^{(1)}(\lambda+\eta) - t_1(\lambda+\eta) t_3^{(1)}(\lambda), \tag{A.26}$$

we are led to the required identity:

$$\mathsf{k}_3\mathsf{k}_2 d(\lambda)t_3^{(1)}(\lambda) = \mathsf{k}_1 t_2^{(2)}(\lambda+\eta). \tag{A.27}$$

□

It should be noted that all these relations can be proven in a pure algebraic way using the general constructions of T and Q operators and the various relations they satisfy, as given in [246, 247, 249, 250]. Hence the computations presented here, although quite instructive could be considered merely as consistency checks.

A.2 On the relation between SoV and Nested Algebraic Bethe Ansatz

Let us consider the fundamental representation of the $gl_{1|2}$ Yang-Baxter superalgebra associated to generic values of the inhomogeneities $\{\xi_{a\leq \mathsf{N}}\}$, satisfying the condition (2.81), and of the eigenvalues k_1, k_2 and k_3 of a simple and diagonalizable twist matrix K. Note that in this case the transfer matrix is similar to the transfer matrix associated to a diagonal twist with entries the eigenvalues k_1, k_2 and k_3 of K to which Nested Algebraic Bethe Ansatz (NABA) directly applies. Therefore, the following discussion on the connection between the SoV description and the NABA can be directly addressed in this diagonal case.

Let us recall that in the NABA framework, given a solution $\{\{\lambda_{j\leq L}\},\{\mu_{h\leq M}\}\}$ of the Bethe Ansatz equations (3.36) and (3.37) satisfying the pair-wise distinct conditions (3.46), then the associated Bethe Ansatz vector $|t_{\{\lambda_{j\leq L}\},\{\mu_{h\leq M}\}}\rangle$ is proven to satisfy the identity

$$T_1^{(K)}(\lambda)|t^{(NABA)}_{\{\lambda_{j\leq L}\},\{\mu_{h\leq M}\}}\rangle = |t^{(NABA)}_{\{\lambda_{j\leq L}\},\{\mu_{h\leq M}\}}\rangle t_1(\lambda|\{\lambda_{j\leq L}\},\{\mu_{h\leq M}\}), \tag{A.28}$$

with $t_1(\lambda|\{\lambda_{j\leq L}\},\{\mu_{h\leq M}\})$ defined in (3.39), so that it is a transfer matrix eigenvector as soon as it is proven to be nonzero. Then, such a Bethe Ansatz vector has in our SoV basis the following characterization:

$$\langle h_1,...,h_{\mathsf{N}}|t^{(NABA)}_{\{\lambda_{j\leq L}\},\{\mu_{h\leq M}\}}\rangle = \prod_{n=1}^{\mathsf{N}} t_1^{h_n}(\xi_n|\{\lambda_{j\leq L}\},\{\mu_{h\leq M}\})\langle S|t^{(NABA)}_{\{\lambda_{j\leq L}\},\{\mu_{h\leq M}\}}\rangle, \tag{A.29}$$

for any $h_n \in \{0,1,2\}$ and $n \in \{0,...,\mathsf{N}\}$. Note that also in the SoV basis the condition that this Bethe vector is nonzero still remains to be verified. This is the case even for the special representations considered in subsection 3.2. Indeed, we have shown that the specific set of solutions to the Bethe Ansatz equations (3.36) and (3.37) introduced in subsection 3.2.2 is complete and the associated eigenvalues $t_1(\lambda|\{\lambda_{j\leq L}\},\{\mu_{h\leq M}\})$ and eigenvectors $|t^{(SoV)}_{\{\lambda_{j\leq L}\},\{\mu_{h\leq M}\}}\rangle$ have the form (3.45) and (3.29), i.e.

$$\langle h_1,...,h_{\mathsf{N}}|t^{(SoV)}_{\{\lambda_{j\leq L}\},\{\mu_{h\leq M}\}}\rangle = \prod_{n=1}^{\mathsf{N}} t_1^{h_n}(\xi_n|\{\lambda_{j\leq L}\},\{\mu_{h\leq M}\}), \tag{A.30}$$

eigenvectors known to be nonzero by the characterization of the transfer matrix eigenvalues for which there exists at least one N-uplet $h_1,...,h_{\mathsf{N}}$ leading to a nonzero value of the above SoV wave-function. Nevertheless, this a priori does not allow us to rule out the possibility that:

$$|t^{(NABA)}_{\{\lambda_{j\leq L}\},\{\mu_{h\leq M}\}}\rangle = 0, \tag{A.31}$$

as we have still to verify that $\langle S|t^{(NABA)}_{\{\lambda_{j\leq L}\},\{\mu_{h\leq M}\}}\rangle$ is nonzero.

Relying on some already existing results in the literature, we want to present a reasoning that allows to argue that the completeness of the Bethe Ansatz in the SoV framework, for the

special representations of subsection 3.2, indeed implies the completeness for the NABA spectrum description. The reasoning goes as follows. In [1,159] we have shown in general that the SoV characterization of the transfer matrix eigenvectors allows for an Algebraic Bethe Ansatz rewriting on a well defined reference state, see for example section 5 of [159]. Adapting to the current fundamental gl_3-representation associated the twist matrix $K' = -\hat{K}$ and $\eta' = -\eta$ the analysis of [163], it can be argued[23] that the $\mathbb{B}$-operator defined in our SoV basis indeed coincides with the one defined by Sklyanin [25]. Then by adapting the results presented in [165], one can deduce that these SoV eigenvectors rewritten in an Algebraic Bethe Ansatz form, in terms of the Sklyanin B-operator, in turn coincide (up to nonzero normalization) with Nested Algebraic Bethe Ansatz vectors, associated to the same Bethe Ansatz solutions. If implemented with all details this reasoning shows the completeness of the Nested Algebraic Bethe Ansatz as a consequence of the completeness of the SoV characterization derived in subsection 3.2.2 for the fundamental representations associated to non-invertible but simple spectrum $\hat{K}$ twist matrices, with eigenvalues satisfying (3.13).

It is also worth to comment that once the NABA completeness is derived for these special representations, it can be derived for general $gl_{1|2}$-representations by adapting to them the proof given in [248] for the gl_2 fundamental representations associated to general diagonalizable twist. Indeed, one of the main ideas of the proof in [248] is that for a special value of the twist parameter, one can characterize the set of isolated Bethe Ansatz solutions that produce nonzero Bethe vectors, and which is proven to be complete. Then, the results on the completeness of Bethe Ansatz solutions by the SoV approach, derived in subsection 3.2.2, and the above argument on the NABA completeness for these $gl_{1|2}$-representations associated to the twist matrices $\hat{K}$ can be as well the starting point for the proof of completeness by deformation w.r.t. the twist parameters like in [248]. Finally, let us add that relations with [204] would be interesting to explore.

B Verification of the Conjecture for the general twists up to 3 sites

Here, we make a verification of our conjecture on the form of the closure relations for the general twisted representation of the $gl_{\mathscr{M}|\mathscr{N}}$ Yang-Baxter superalgebra, in the case $\mathscr{M} = 1$ and $\mathscr{N} = 2$ for small chain representations, i.e. for a chain having up to $\mathsf{N} = 3$ sites. The verification is done in the following way, we impose the closure relation (3.6) in N pairwise different values[24] of λ to the polynomials (2.107) and (2.108) for $n = 1, 2$. This determines a system of N polynomial equations of order 4 in the N unknowns which are the values of the polynomial (2.107) in the inhomogeneities. We solve this system of equations by Mathematica and we select the solutions which generate polynomials (2.108) which satisfy the null out-boundary conditions 3.7. Our analysis shows that it is enough to impose 3.7 for $n = m = 0$ to select the correct solutions which generate exactly the N^3 different eigenvalues of the diagonalizable and simple spectrum transfer matrix $T^{(K)}(\lambda)$, obtained by diagonalizing it exactly with Mathematica. For $\mathsf{N} = 1, 2$ the results of both the approaches are analytic and we present them here for the interesting $\mathsf{N} = 2$ case. While for $\mathsf{N} = 3$ we have verified our statements for different values of the parameters, i.e. the inhomogeneity parameters and the three eigenvalues of the twist matrix.

We put $\xi_1 = 0$ without loss of generality to shorten the expressions while leaving free all the others parameters ξ_2, k_1, k_2, k_3 and η. Then the solution of the system of equations obtained by (3.6) plus the null out-boundary conditions (3.7) for $n = m = 0$ leads to the following 2^3 distinct solutions for the values of the polynomial (2.107) respectively in $\lambda = \xi_2$

[23] Note that we have proven this statement for a chain with a small number of quantum sites in [1].

[24] Note that any value can be taken if different from the transfer matrix common zeros.

and $\lambda=\xi_1=0$:

$$\{k_1\eta(\eta+\xi_2),\ k_1\eta(\eta-\xi_2)\}\,, \tag{B.1}$$
$$\{k_2\eta(\eta-\xi_2),\ -k_2\eta(\eta+\xi_2)\}\,, \tag{B.2}$$
$$\{k_3\eta(\eta-\xi_2),\ -k_3\eta(\eta+\xi_2)\}\,, \tag{B.3}$$
$$\left\{\frac{\eta}{2}\left((k_1+k_2)\xi_2-\sqrt{4k_1k_2\eta^2+(k_1-k_2)^2\xi_2^2}\right),\ \frac{-\eta}{2}\left((k_1+k_2)\xi_2+\sqrt{4k_1k_2\eta^2+(k_1-k_2)^2\xi_2^2}\right)\right\}, \tag{B.4}$$
$$\left\{\frac{\eta}{2}\left((k_1+k_2)\xi_2+\sqrt{4k_1k_2\eta^2+(k_1-k_2)^2\xi_2^2}\right),\ \frac{-\eta}{2}\left((k_1+k_2)\xi_2-\sqrt{4k_1k_2\eta^2+(k_1-k_2)^2\xi_2^2}\right)\right\}, \tag{B.5}$$
$$\left\{\frac{\eta}{2}\left((k_1+k_3)\xi_2-\sqrt{4k_1k_3\eta^2+(k_1-k_3)^2\xi_2^2}\right),\ \frac{-\eta}{2}\left((k_1+k_3)\xi_2+\sqrt{4k_1k_3\eta^2+(k_1-k_3)^2\xi_2^2}\right)\right\}, \tag{B.6}$$
$$\left\{\frac{\eta}{2}\left((k_1+k_3)\xi_2+\sqrt{4k_1k_3\eta^2+(k_1-k_3)^2\xi_2^2}\right),\ \frac{-\eta}{2}\left((k_1+k_3)\xi_2-\sqrt{4k_1k_3\eta^2+(k_1-k_3)^2\xi_2^2}\right)\right\}, \tag{B.7}$$
$$\left\{\frac{\eta}{2}\left((k_2+k_3)\xi_2-\sqrt{4k_2k_3\eta^2+(k_2-k_3)^2\xi_2^2}\right),\ \frac{-\eta}{2}\left((k_2+k_3)\xi_2+\sqrt{4k_2k_3\eta^2+(k_2-k_3)^2\xi_2^2}\right)\right\}, \tag{B.8}$$
$$\left\{\frac{\eta}{2}\left((k_2+k_3)\xi_2-\sqrt{4k_2k_3\eta^2+(k_2-k_3)^2\xi_2^2}\right),\ \frac{-\eta}{2}\left((k_2+k_3)\xi_2+\sqrt{4k_2k_3\eta^2+(k_2-k_3)^2\xi_2^2}\right)\right\}. \tag{B.9}$$

The values at the points ξ_2 and $\xi_1=0$ and the asymptotic limit allows to reconstruct the polynomials (2.107). The polynomials constructed in this way can be directly verified to coincide with the eigenvalues of $T^{(K)}(\lambda)$, whose expressions are obtained by diagonalizing $T(\lambda)$ exactly with Mathematica:

$$(\mathrm{str}K)\lambda^2+\big(2\eta k_1-(\mathrm{str}K)\xi_2\big)\lambda+k_1\eta(\eta-\xi_2), \tag{B.10}$$
$$(\mathrm{str}K)\lambda^2+\big(2\eta k_2-(\mathrm{str}K)\xi_2\big)\lambda-k_2\eta(\eta+\xi_2), \tag{B.11}$$
$$(\mathrm{str}K)\lambda^2+\big(2\eta k_3-(\mathrm{str}K)\xi_2\big)\lambda-k_3\eta(\eta+\xi_2), \tag{B.12}$$
$$(\mathrm{str}K)\lambda^2+\big((k_1+k_2)\eta-(\mathrm{str}K)\xi_2\big)\lambda+\frac{\eta}{2}\left(-(k_1+k_2)\xi_2-\sqrt{4k_1k_2\eta^2+(k_1-k_2)^2\xi_2^2}\right), \tag{B.13}$$
$$(\mathrm{str}K)\lambda^2+\big((k_1+k_2)\eta-(\mathrm{str}K)\xi_2\big)\lambda+\frac{\eta}{2}\left(-(k_1+k_2)\xi_2+\sqrt{4k_1k_2\eta^2+(k_1-k_2)^2\xi_2^2}\right), \tag{B.14}$$
$$(\mathrm{str}K)\lambda^2+\big((k_1+k_3)\eta-(\mathrm{str}K)\xi_2\big)\lambda+\frac{\eta}{2}\left(-(k_1+k_3)\xi_2-\sqrt{4k_1k_3\eta^2+(k_1-k_3)^2\xi_2^2}\right), \tag{B.15}$$
$$(\mathrm{str}K)\lambda^2+\big((k_1+k_3)\eta-(\mathrm{str}K)\xi_2\big)\lambda+\frac{\eta}{2}\left(-(k_1+k_3)\xi_2+\sqrt{4k_1k_3\eta^2+(k_1-k_3)^2\xi_2^2}\right), \tag{B.16}$$
$$(\mathrm{str}K)\lambda^2+\big((k_2+k_3)\eta-(\mathrm{str}K)\xi_2\big)\lambda+\frac{\eta}{2}\left(-(k_2+k_3)\xi_2-\sqrt{4k_2k_3\eta^2+(k_2-k_3)^2\xi_2^2}\right), \tag{B.17}$$
$$(\mathrm{str}K)\lambda^2+\big((k_2+k_3)\eta-(\mathrm{str}K)\xi_2\big)\lambda+\frac{\eta}{2}\left(-(k_2+k_3)\xi_2+\sqrt{4k_2k_3\eta^2+(k_2-k_3)^2\xi_2^2}\right). \tag{B.18}$$

C Derivation of the inner-boundary condition

One may use the coderivative formalism introduced in [246] and developed in [247] to derive the inner-boundary condition. The coderivative formalism allows to construct the transfer

matrices associated to a given irreducible representation on the auxiliary space by acting on the associated character evaluated at the twist matrix. For a rectangular Young tableau (a,b), we have in our notation

$$\widetilde{T}_b^{(a),K}(\lambda) = \left(\lambda - \xi_1 + \eta\hat{D}\right) \otimes \ldots \otimes \left(\lambda - \xi_{\mathsf{N}} + \eta\hat{D}\right) \chi_b^{(a)}(K). \tag{C.1}$$

Let us take $g = \mathrm{diag}(x_1, \ldots, x_{\mathscr{M}}, y_1, \ldots, y_{\mathscr{N}})$ a diagonal twist. For $k \geq 1$, the characters of the rectangular representations (a,b) which saturate an arm of the fat hook write [196]

$$\chi_{\mathscr{N}+k}^{(\mathscr{M})}(g) = \left(\prod_{i=1}^{\mathscr{M}} x_i^k\right) \prod_{i=1}^{\mathscr{M}} \prod_{j=1}^{\mathscr{N}} (x_i - y_j), \tag{C.2}$$

$$\chi_{\mathscr{N}}^{(\mathscr{M}+k)}(g) = \left(\prod_{j=1}^{\mathscr{N}} (-y_j)^k\right) \prod_{i=1}^{\mathscr{M}} \prod_{j=1}^{\mathscr{N}} (x_i - y_j), \tag{C.3}$$

thus the following relation holds for all $k \geq 1$

$$\chi_{\mathscr{N}+k}^{(\mathscr{M})}(g) = (-1)^{k\mathscr{N}} \mathrm{sdet}(g)^k \chi_{\mathscr{N}}^{(\mathscr{M}+k)}(g), \tag{C.4}$$

where $\mathrm{sdet}(g)$ is the superdeterminant of g defined by

$$\mathrm{sdet}(g) = \frac{\prod_{i=1}^{\mathscr{M}} x_i}{\prod_{j=1}^{\mathscr{N}} y_j}. \tag{C.5}$$

Acting on it with the coderivative $\hat{D}$, we have

$$\hat{D}\, \chi_{\mathscr{N}+k}^{(\mathscr{M})}(g) = (-1)^{k\mathscr{N}} e_{ij} \frac{\partial}{\partial \phi_i^j} \otimes \left(\mathrm{sdet}(e^{\phi\cdot e} g)^k\, \chi_{\mathscr{N}}^{(\mathscr{M}+k)}(e^{\phi\cdot e} g)\right)\Big|_{\phi=0} \tag{C.6}$$

$$= (-1)^{k\mathscr{N}} \mathrm{sdet}(g)^k e_{ij} \frac{\partial}{\partial \phi_i^j} \otimes \left((1 + k\,\mathrm{str}(\phi\cdot e))\, \chi_{\mathscr{N}}^{(\mathscr{M}+k)}(e^{\phi\cdot e} g)\right)\Big|_{\phi=0} \tag{C.7}$$

$$= (-1)^{k\mathscr{N}} \mathrm{sdet}(g)^k \left(\hat{D}\chi_{\mathscr{N}}^{(\mathscr{M}+k)} + k\, e_{ij} \frac{\partial}{\partial \phi_i^j} \otimes \left(\mathrm{str}(\phi\cdot e)\, \chi_{\mathscr{N}}^{(\mathscr{M}+k)}(g)\right)\Big|_{\phi=0}\right) \tag{C.8}$$

$$= (-1)^{k\mathscr{N}} \mathrm{sdet}(g)^k (k + \hat{D}) \chi_{\mathscr{N}}^{(\mathscr{M}+k)}(g). \tag{C.9}$$

Now, acting with $\left(\lambda - \xi_1 + \eta\hat{D}\right) \otimes \ldots \otimes \left(\lambda - \xi_{\mathsf{N}} + \eta\hat{D}\right)$ on (C.4), and putting $g = K$, we thus have

$$\widetilde{T}_{\mathscr{N}+k}^{(M),(K)}(\lambda) = (-1)^{k\mathscr{N}} \mathrm{sdet}(K)^k\, \widetilde{T}_{\mathscr{N}}^{(\mathscr{M}+k),(K)}(\lambda + k\eta). \tag{C.10}$$

Putting $k = 1$, and reintroducing the trivial zeros to recover the $T_b^{(a),(K)}(\lambda)$ matrices, we obtain (2.54).

References

[1] J. M. Maillet and G. Niccoli, *On quantum separation of variables*, J. Math. Phys. **59**, 091417 (2018), doi:10.1063/1.5050989.

[2] P. P. Kulish and E. K. Sklyanin, *Solutions of the Yang-Baxter equation*, J. Math. Sci. **19**, 1596 (1982), doi:10.1007/bf01091463.

[3] P. P. Kulish, *Integrable graded magnets*, J. Math. Sci. **35**, 2648 (1986), doi:10.1007/bf01083770.

[4] P. P. Kulish, *Yang-Baxter equation and reflection equations in integrable models*, in *Low-dimensional models in statistical physics and quantum field theory*, Springer (1996).

[5] J. Hubbard, *Electron correlations in narrow energy bands*, Proc. R. Soc. Lond. A **276**, 238 (1963), doi:10.1098/rspa.1963.0204.

[6] J. Hubbard, *Electron correlations in narrow energy bands III. An improved solution*, Proc. R. Soc. Lond. A **281**, 401 (1964), doi:10.1098/rspa.1964.0190.

[7] M. C. Gutzwiller, *Effect of correlation on the ferromagnetism of transition metals*, Phys. Rev. Lett. **10**, 159 (1963), doi:10.1103/physrevlett.10.159.

[8] J. Kanamori, *Electron correlation and ferromagnetism of transition metals*, Prog. Theor. Phys. **30**, 275 (1963), doi:10.1143/ptp.30.275.

[9] D. Baeriswyl, D. K. Campbell, J. M. P. Carmelo, F. Guinea and E. Louis, *The Hubbard model: Its physics and mathematical physics*, Springer, ISBN 9781489910448 (1995), doi:10.1007/978-1-4899-1042-4.

[10] F. H. L. Essler, H. Frahm, F. Göhmann, A. Klümper and V. E. Korepin, *The one-dimensional Hubbard model*, Cambridge University Press, ISBN 9780521802628 (2005), doi:10.1017/CBO9780511534843.

[11] E. K. Sklyanin and L. D. Faddeev, *Quantum-mechanical approach to completely integrable field theory models*, in Fifty years of mathematical physics, World Scientific, ISBN 9789814340953 (2016), doi:10.1142/9789814340960_0025.

[12] E. K. Sklyanin, L. A. Takhtadzhyan and L. D. Faddeev, *Quantum inverse problem method. I*, Theor. Math. Phys. **40**, 688 (1979), doi:10.1007/bf01018718.

[13] L. A. Takhtadzhan and L. D. Faddeev, *The quantum method of the inverse problem and the Heisenberg XYZ model*, Russ. Math. Surv. **34**, 11 (1979), doi:10.1070/rm1979v034n05abeh003909.

[14] E. K. Sklyanin, *Method of the inverse scattering problem and the nonlinear quantum Schrödinger equation*, Sov. Phys. Dokl. **24**, 107 (1979).

[15] E. K. Sklyanin, *On complete integrability of the Landau-Lifshitz equation*, LOMI-79-3, LOMI-E-79-3 (1979).

[16] L. D. Faddeev and L. A. Takhtajan, *Quantum inverse scattering method*, Sov. Sci. Rev. Math. (1981).

[17] E. K. Sklyanin, *Quantum version of the method of inverse scattering problem*, J. Math. Sci. **19**, 1546 (1982), doi:10.1007/bf01091462.

[18] L. D. Faddeev, *Integrable models in 1+ 1 dimensional quantum field theory*, CEA-CONF–6565, Centre d'Études Nucleaires de Saclay (1982).

[19] L. D. Faddeev, *How Algebraic Bethe Ansatz works for integrable models* (1996), arXiv:hep-th/9605187.

[20] E. K. Sklyanin, *The quantum Toda chain*, in Non-linear equations in classical and quantum field theory, Springer (1985), ISBN 9783540152132, doi:10.1007/3-540-15213-x_80.

[21] E. K. Sklyanin, *Functional Bethe Ansatz*, in Integrable and superintegrable systems, World Scientific (1990), doi:10.1142/9789812797179_0002.

[22] M.-L. Ge, *Quantum group and quantum integrable systems*, World Scientific (1992), doi:10.1142/9789814538534.

[23] E. K. Sklyanin, *Separation of variables in the classical integrable $SL(3)$ magnetic chain*, Commun. Math. Phys. **150**, 181 (1992), doi:10.1007/bf02096572.

[24] E. K. Sklyanin, *Separation of variables: New trends*, Prog. Theor. Phys. Suppl. **118**, 35 (1995), doi:10.1143/ptps.118.35.

[25] E. K. Sklyanin, *Separation of variables in the quantum integrable models related to the Yangiany $Y[sl(3)]$*, J. Math. Sci. **80**, 1861 (1996), doi:10.1007/bf02362784.

[26] O. Babelon, D. Bernard and F. A. Smirnov, *Quantization of solitons and the restricted sine-Gordon model*, Commun. Math. Phys. **182**, 319 (1996), doi:10.1007/bf02517893.

[27] F. A. Smirnov, *Structure of matrix elements in the quantum Toda chain*, J. Phys. A: Math. Gen. **31**, 8953 (1998), doi:10.1088/0305-4470/31/44/019.

[28] F. A. Smirnov, *Separation of variables for quantum integrable models related to $U_q(\widehat{sl}_N)$*, in MathPhys Odyssey 2001, Birkhäuser (2002), ISBN 9781461266051, doi:10.1007/978-1-4612-0087-1_17.

[29] S. É. Derkachov, G. P. Korchemsky and A. N. Manashov, *Noncompact Heisenberg spin magnets from high-energy QCD: I. Baxter Q-operator and separation of variables*, Nucl. Phys. B **617**, 375 (2001), doi:10.1016/s0550-3213(01)00457-6.

[30] S. É. Derkachov, G. P. Korchemsky and A. N. Manashov, *Separation of variables for the quantum $SL(2,\mathbb{R})$ spin chain*, J. High Energy Phys. **07**, 047 (2003), doi:10.1088/1126-6708/2003/07/047.

[31] S. É. Derkachov, G. P. Korchemsky and A. N. Manashov, *Baxter $\mathbb{Q}$-operator and separation of variables for the open $SL(2,\mathbb{R})$ spin chain*, J. High Energy Phys. **10**, 053 (2003), doi:10.1088/1126-6708/2003/10/053.

[32] A. G. Bytsko and J. Teschner, *Quantization of models with non-compact quantum group symmetry: modular XXZ magnet and lattice sinh-Gordon model*, J. Phys. A: Math. Gen. **39**, 12927 (2006), doi:10.1088/0305-4470/39/41/s11.

[33] G. von Gehlen, N. Iorgov, S. Pakuliak and V. Shadura, *The Baxter–Bazhanov–Stroganov model: separation of variables and the Baxter equation*, J. Phys. A: Math. Gen. **39**, 7257 (2006), doi:10.1088/0305-4470/39/23/006.

[34] H. Frahm, A. Seel and T. Wirth, *Separation of variables in the open XXX chain*, Nucl. Phys. B **802**, 351 (2008), doi:10.1016/j.nuclphysb.2008.04.008.

[35] E. Mukhin, V. Tarasov and A. Varchenko, *On separation of variables and completeness of the Bethe ansatz for quantum $\mathfrak{gl}_N$ Gaudin model*, Glasg. Math. J. **51**, 137 (2009), doi:10.1017/s0017089508004850.

[36] E. Mukhin, V. Tarasov and A. Varchenko, *Schubert calculus and representations of the general linear group*, J. Amer. Math. Soc. **22**, 909 (2009), doi:10.1090/s0894-0347-09-00640-7.

[37] E. Mukhin, V. Tarasov and A. Varchenko, *The B. and M. Shapiro conjecture in real algebraic geometry and the Bethe ansatz*, Ann. Math. **170**, 863 (2009), doi:10.4007/annals.2009.170.863.

[38] L. Amico, H. Frahm, A. Osterloh and T. Wirth, *Separation of variables for integrable spin–boson models*, Nucl. Phys. B **839**, 604 (2010), doi:10.1016/j.nuclphysb.2010.07.005.

[39] G. Niccoli and J. Teschner, *The sine–Gordon model revisited: I*, J. Stat. Mech. P09014 (2010), doi:10.1088/1742-5468/2010/09/p09014.

[40] G. Niccoli, *Reconstruction of Baxter-operator from Sklyanin SOV for cyclic representations of integrable quantum models*, Nucl. Phys. B **835**, 263 (2010), doi:10.1016/j.nuclphysb.2010.03.009.

[41] G. Niccoli, *Completeness of Bethe ansatz by Sklyanin SOV for cyclic representations of integrable quantum models*, J. High Energy Phys. **03**, 123 (2011), doi:10.1007/jhep03(2011)123.

[42] H. Frahm, J. H. Grelik, A. Seel and T. Wirth, *Functional Bethe ansatz methods for the open XXX chain*, J. Phys. A: Math. Theor. **44**, 015001 (2010), doi:10.1088/1751-8113/44/1/015001.

[43] N. Grosjean, J. M. Maillet and G. Niccoli, *On the form factors of local operators in the lattice sine–Gordon model*, J. Stat. Mech. P10006 (2012), doi:10.1088/1742-5468/2012/10/p10006.

[44] N. Grosjean and G. Niccoli, *The τ_2-model and the chiral Potts model revisited: completeness of Bethe equations from Sklyanin's SOV method*, J. Stat. Mech. P11005 (2012), doi:10.1088/1742-5468/2012/11/p11005.

[45] G. Niccoli, *Non-diagonal open spin-1/2 XXZ quantum chains by separation of variables: complete spectrum and matrix elements of some quasi-local operators*, J. Stat. Mech. P10025 (2012), doi:10.1088/1742-5468/2012/10/p10025.

[46] G. Niccoli, *Antiperiodic spin-1/2 XXZ quantum chains by separation of variables: Complete spectrum and form factors*, Nucl. Phys. B **870**, 397 (2013), doi:10.1016/j.nuclphysb.2013.01.017.

[47] G. Niccoli, *An antiperiodic dynamical six-vertex model: I. Complete spectrum by SOV, matrix elements of the identity on separate states and connections to the periodic eight-vertex model*, J. Phys. A: Math. Theor. **46**, 075003 (2013), doi:10.1088/1751-8113/46/7/075003.

[48] G. Niccoli, *Form factors and complete spectrum of XXX antiperiodic higher spin chains by quantum separation of variables*, J. Math. Phys. **54**, 053516 (2013), doi:10.1063/1.4807078.

[49] N. Grosjean, J.-M. Maillet and G. Niccoli, *On the form factors of local operators in the Bazhanov–Stroganov and chiral Potts models*, Ann. Henri Poincaré **16**, 1103 (2015), doi:10.1007/s00023-014-0358-9.

[50] S. Faldella and G. Niccoli, *SOV approach for integrable quantum models associated with general representations on spin-1/2 chains of the 8-vertex reflection algebra*, J. Phys. A: Math. Theor. **47**, 115202 (2014), doi:10.1088/1751-8113/47/11/115202.

[51] S. Faldella, N. Kitanine and G. Niccoli, *The complete spectrum and scalar products for the open spin-1/2 XXZ quantum chains with non-diagonal boundary terms*, J. Stat. Mech. P01011 (2014), doi:10.1088/1742-5468/2014/01/p01011.

[52] N. Kitanine, J. M. Maillet and G. Niccoli, *Open spin chains with generic integrable boundaries: Baxter equation and Bethe ansatz completeness from separation of variables*, J. Stat. Mech. P05015 (2014), doi:10.1088/1742-5468/2014/05/p05015.

[53] G. Niccoli and V. Terras, *Antiperiodic XXZ chains with arbitrary spins: complete eigenstate construction by functional equations in separation of variables*, Lett. Math. Phys. **105**, 989 (2015), doi:10.1007/s11005-015-0759-9.

[54] D. Levy-Bencheton, G. Niccoli and V. Terras, *Antiperiodic dynamical 6-vertex model by separation of variables II: functional equations and form factors*, J. Stat. Mech. 033110 (2016), doi:10.1088/1742-5468/2016/03/033110.

[55] G. Niccoli and V. Terras, *The eight-vertex model with quasi-periodic boundary conditions*, J. Phys. A: Math. Theor. **49**, 044001 (2015), doi:10.1088/1751-8113/49/4/044001.

[56] N. Kitanine, J. M. Maillet, G. Niccoli and V. Terras, *On determinant representations of scalar products and form factors in the SoV approach: the XXX case*, J. Phys. A: Math. Theor. **49**, 104002 (2016), doi:10.1088/1751-8113/49/10/104002.

[57] D. Martin and F. Smirnov, *Problems with using separated variables for computing expectation values for higher ranks*, Lett. Math. Phys. **106**, 469 (2016), doi:10.1007/s11005-016-0823-0.

[58] N. Kitanine, J. M. Maillet, G. Niccoli and V. Terras, *The open XXX spin chain in the SoV framework: scalar product of separate states*, J. Phys. A: Math. Theor. **50**, 224001 (2017), doi:10.1088/1751-8121/aa6cc9.

[59] J. M. Maillet, G. Niccoli and B. Pezelier, *Transfer matrix spectrum for cyclic representations of the 6-vertex reflection algebra I*, SciPost Phys. **2**, 009 (2017), doi:10.21468/scipostphys.2.1.009.

[60] J. M. Maillet, G. Niccoli and B. Pezelier, *Transfer matrix spectrum for cyclic representations of the 6-vertex reflection algebra II*, SciPost Phys. **5**, 026 (2018), doi:10.21468/SciPostPhys.5.3.026.

[61] W. Heisenberg, *Zur Theorie des Ferromagnetismus*, Z. Physik **49**, 619 (1928), doi:10.1007/bf01328601.

[62] H. Bethe, *Zur Theorie der Metalle*, Z. Physik **71**, 205 (1931), doi:10.1007/bf01341708.

[63] R. Orbach, *Linear antiferromagnetic chain with anisotropic coupling*, Phys. Rev. **112**, 309 (1958), doi:10.1103/physrev.112.309.

[64] L. R. Walker, *Antiferromagnetic linear chain*, Phys. Rev. **116**, 1089 (1959), doi:10.1103/physrev.116.1089.

[65] R. J. Baxter, *Partition function of the eight-vertex lattice model*, Ann. Phys. **70**, 193 (1972), doi:10.1016/0003-4916(72)90335-1.

[66] R. Baxter, *Eight-vertex model in lattice statistics and one-dimensional anisotropic heisenberg chain. II. Equivalence to a generalized ice-type lattice model*, Ann. Phys. **76**, 25 (1973), doi:10.1016/0003-4916(73)90440-5.

[67] L. Pauling, *The structure and entropy of ice and of other crystals with some randomness of atomic arrangement*, J. Am. Chem. Soc. **57**, 2680 (1935), doi:10.1021/ja01315a102.

[68] E. H. Lieb, *Residual entropy of square ice*, Phys. Rev. **162**, 162 (1967), doi:10.1103/physrev.162.162.

[69] B. Sutherland, *Two-dimensional hydrogen bonded crystals without the ice rule*, J. Math. Phys. **11**, 3183 (1970), doi:10.1063/1.1665111.

[70] C. Fan and F. Y. Wu, *General lattice model of phase transitions*, Phys. Rev. B **2**, 723 (1970), doi:10.1103/physrevb.2.723.

[71] R. J. Baxter, *Eight-vertex model in lattice statistics*, Phys. Rev. Lett. **26**, 832 (1971), doi:10.1103/physrevlett.26.832.

[72] R. J. Baxter, *Exactly solved models in statistical mechanics*, in Series on advances in Statistical Mechanics, World Scientific (1985), ISBN 9789971978112, doi:10.1142/9789814415255_0002.

[73] E. H. Lieb, *Exact solution of the problem of the entropy of two-dimensional ice*, Phys. Rev. Lett. **18**, 692 (1967), doi:10.1103/physrevlett.18.692.

[74] E. H. Lieb, *Exact solution of the F model of an antiferroelectric*, Phys. Rev. Lett. **18**, 1046 (1967), doi:10.1103/physrevlett.18.1046.

[75] E. H. Lieb, *Exact solution of the two-dimensional Slater KDP model of a ferroelectric*, Phys. Rev. Lett. **19**, 108 (1967), doi:10.1103/physrevlett.19.108.

[76] B. Sutherland, *Exact solution of a two-dimensional model for hydrogen-bonded crystals*, Phys. Rev. Lett. **19**, 103 (1967), doi:10.1103/physrevlett.19.103.

[77] B. M. McCoy and T. Tsun Wu, *Hydrogen-bonded crystals and the anisotropic Heisenberg chain*, Nuov. Cim. B **56**, 311 (1968), doi:10.1007/bf02710156.

[78] R. J. Baxter, *One-dimensional anisotropic Heisenberg chain*, Ann. Phys. **70**, 323 (1972), doi:10.1016/0003-4916(72)90270-9.

[79] P. P. Kulish and N. Yu. Reshetikhin, *Quantum linear problem for the sine-Gordon equation and higher representations*, J. Math. Sci. **23**, 2435 (1983), doi:10.1007/bf01084171.

[80] P. P. Kulish and N. Yu. Reshetikhin, *GL_3-invariant solutions of the Yang-Baxter equation and associated quantum systems*, J. Math. Sci. **34**, 1948 (1986), doi:10.1007/bf01095104.

[81] M. Jimbo, *A q-difference analogue of $U(g)$ and the Yang-Baxter equation*, Lett. Math. Phys. **10**, 63 (1985), doi:10.1007/bf00704588.

[82] V. G. Drinfel'd, *Quantum groups*, J. Math. Sci. **41**, 898 (1988), doi:10.1007/BF01247086.

[83] M. Jimbo, K. Miki, T. Miwa and A. Nakayashiki, *Correlation functions of the XXZ model for $\Delta < -1$*, Phys. Lett. A **168**, 256 (1992), doi:10.1016/0375-9601(92)91128-e.

[84] M. Jimbo and T. Miwa, *Algebraic analysis of solvable lattice models*, American Mathematical Society (1994), ISBN 9780821803202, doi:10.1090/cbms/085.

[85] M. Jimbo, R. Kedem, T. Kojima, H. Konno and T. Miwa, *XXZ chain with a boundary*, Nucl. Phys. B **441**, 437 (1995), doi:10.1016/0550-3213(95)00062-w.

[86] M. Jimbo, R. Kedem, H. Konno, T. Miwa and R. Weston, *Difference equations in spin chains with a boundary*, Nucl. Phys. B **448**, 429 (1995), doi:10.1016/0550-3213(95)00218-h.

[87] N. Kitanine, J. M. Maillet and V. Terras, *Form factors of the XXZ Heisenberg spin-12 finite chain*, Nucl. Phys. B **554**, 647 (1999), doi:10.1016/s0550-3213(99)00295-3.

[88] J. M. Maillet and V. Terras, *On the quantum inverse scattering problem*, Nucl. Phys. B **575**, 627 (2000), doi:10.1016/s0550-3213(00)00097-3.

[89] N. Kitanine, J. M. Maillet and V. Terras, *Correlation functions of the XXZ Heisenberg spin-chain in a magnetic field*, Nucl. Phys. B **567**, 554 (2000), doi:10.1016/s0550-3213(99)00619-7.

[90] N. Kitanine, J. M. Maillet, N. A. Slavnov and V. Terras, *Spin–spin correlation functions of the XXZ-12 Heisenberg chain in a magnetic field*, Nucl. Phys. B **641**, 487 (2002), doi:10.1016/s0550-3213(02)00583-7.

[91] N. Kitanine, J. M. Maillet, N. A. Slavnov and V. Terras, *Dynamical correlation functions of the XXZ spin-1/2 chain*, Nucl. Phys. B **729**, 558 (2005), doi:10.1016/j.nuclphysb.2005.08.046.

[92] N. Kitanine, J. M. Maillet, N. A. Slavnov and V. Terras, *Master equation for spin–spin correlation functions of the XXZ chain*, Nucl. Phys. B **712**, 600 (2005), doi:10.1016/j.nuclphysb.2005.01.050.

[93] N. Kitanine, K. K. Kozlowski, J. M. Maillet, G. Niccoli, N. A. Slavnov and V. Terras, *Correlation functions of the open XXZ chain: I*, J. Stat. Mech. P10009 (2007), doi:10.1088/1742-5468/2007/10/p10009.

[94] N. Kitanine, K. K. Kozlowski, J. M. Maillet, G. Niccoli, N. A. Slavnov and V. Terras, *Correlation functions of the open XXZ chain: II*, J. Stat. Mech. P07010 (2008), doi:10.1088/1742-5468/2008/07/p07010.

[95] P. Baseilhac and T. Kojima, *Correlation functions of the half-infinite XXZ spin chain with a triangular boundary*, Nucl. Phys. B **880**, 378 (2014), doi:10.1016/j.nuclphysb.2014.01.011.

[96] J.-S. Caux, R. Hagemans and J. M. Maillet, *Computation of dynamical correlation functions of Heisenberg chains: the gapless anisotropic regime*, J. Stat. Mech. P09003 (2005), doi:10.1088/1742-5468/2005/09/p09003.

[97] J.-S. Caux and J. M. Maillet, *Computation of dynamical correlation functions of Heisenberg chains in a magnetic field*, Phys. Rev. Lett. **95**, 077201 (2005), doi:10.1103/physrevlett.95.077201.

[98] M. Kenzelmann, R. Coldea, D. A. Tennant, D. Visser, M. Hofmann, P. Smeibidl and Z. Tylczynski, *Order-to-disorder transition in the XY-like quantum magnet Cs_2CoCl_4 induced by noncommuting applied fields*, Phys. Rev. B **65**, 144432 (2002), doi:10.1103/physrevb.65.144432.

[99] N. Kitanine, J. M. Maillet, N. A. Slavnov and V. Terras, *Large distance asymptotic behaviour of the emptiness formation probability of the XXZ spin-1/2 Heisenberg chain*, J. Phys. A: Math. Gen. **35**, L753 (2002), doi:10.1088/0305-4470/35/49/102.

[100] N. Kitanine, K. K. Kozlowski, J. M. Maillet, N. A. Slavnov and V. Terras, *Algebraic Bethe ansatz approach to the asymptotic behavior of correlation functions*, J. Stat. Mech. P04003 (2009), doi:10.1088/1742-5468/2009/04/p04003.

[101] N. Kitanine, K. K. Kozlowski, J. M. Maillet, N. A. Slavnov and V. Terras, *On the thermodynamic limit of form factors in the massless XXZ Heisenberg chain*, J. Math. Phys. **50**, 095209 (2009), doi:10.1063/1.3136683.

[102] N. Kitanine, K. K. Kozlowski, J. M. Maillet, N. A. Slavnov and V. Terras, *A form factor approach to the asymptotic behavior of correlation functions in critical models*, J. Stat. Mech. P12010 (2011), doi:10.1088/1742-5468/2011/12/p12010.

[103] K. K. Kozlowski, J. M. Maillet and N. A. Slavnov, *Long-distance behavior of temperature correlation functions in the one-dimensional Bose gas*, J. Stat. Mech. P03018 (2011), doi:10.1088/1742-5468/2011/03/p03018.

[104] N. Kitanine, K. K. Kozlowski, J. M. Maillet, N. A. Slavnov and V. Terras, *Form factor approach to dynamical correlation functions in critical models*, J. Stat. Mech. P09001 (2012), doi:10.1088/1742-5468/2012/09/p09001.

[105] K. K. Kozlowski and J. M. Maillet, *Microscopic approach to a class of 1D quantum critical models*, J. Phys. A: Math. Theor. **48**, 484004 (2015), doi:10.1088/1751-8113/48/48/484004.

[106] B. Derrida, *An exactly soluble non-equilibrium system: The asymmetric simple exclusion process*, Phys. Rep. **301**, 65 (1998), doi:10.1016/s0370-1573(98)00006-4.

[107] G. M. Schütz, *Exactly solvable models for many-body systems far from equilibrium*, in Phase transitions and critical phenomena, Elsevier (2001), ISBN 9780122203190, doi:10.1016/s1062-7901(01)80015-x.

[108] M. Shiroishi and M. Wadati, *Integrable boundary conditions for the one-dimensional Hubbard model*, J. Phys. Soc. Jpn. **66**, 2288 (1997), doi:10.1143/jpsj.66.2288.

[109] F. C. Alcaraz, M. Droz, M. Henkel and V. Rittenberg, *Reaction-diffusion processes, critical dynamics, and quantum chains*, Ann. Phys. **230**, 250 (1994), doi:10.1006/aphy.1994.1026.

[110] Z. Bajnok, *Equivalences between spin models induced by defects*, J. Stat. Mech. P06010 (2006), doi:10.1088/1742-5468/2006/06/p06010.

[111] J. de Gier and F. H. L. Essler, *Bethe Ansatz solution of the asymmetric exclusion process with open boundaries*, Phys. Rev. Lett. **95**, 240601 (2005), doi:10.1103/physrevlett.95.240601.

[112] J. de Gier and F. H. L. Essler, *Exact spectral gaps of the asymmetric exclusion process with open boundaries*, J. Stat. Mech. P12011 (2006), doi:10.1088/1742-5468/2006/12/p12011.

[113] J. Sirker, R. G. Pereira and I. Affleck, *Diffusion and ballistic transport in one-dimensional quantum systems*, Phys. Rev. Lett. **103**, 216602 (2009), doi:10.1103/physrevlett.103.216602.

[114] T. Prosen, *Open XXZ spin chain: Nonequilibrium steady state and a strict bound on ballistic transport*, Phys. Rev. Lett. **106**, 217206 (2011), doi:10.1103/physrevlett.106.217206.

[115] R. Jördens, N. Strohmaier, K. Günter, H. Moritz and T. Esslinger, *A Mott insulator of fermionic atoms in an optical lattice*, Nature **455**, 204 (2008), doi:10.1038/nature07244.

[116] *The Hubbard model at half a century*, Nat. Phys. **9**, 523 (2013), doi:10.1038/nphys2759.

[117] E. H. Lieb and F. Y. Wu, *Absence of Mott transition in an exact solution of the short-range, one-band model in one dimension*, Phys. Rev. Lett. **20**, 1445 (1968), doi:10.1103/physrevlett.20.1445.

[118] E. H. Lieb and F. Y. Wu, *The one-dimensional Hubbard model: a reminiscence*, Phys. A: Stat. Mech. Appl. **321**, 1 (2003), doi:10.1016/S0378-4371(02)01785-5.

[119] F. Woynarovich, *Excitations with complex wavenumbers in a Hubbard chain. I. States with one pair of complex wavenumbers*, J. Phys. C: Solid State Phys. **15**, 85 (1982), doi:10.1088/0022-3719/15/1/007.

[120] B. S. Shastry, *Exact integrability of the one-dimensional Hubbard model*, Phys. Rev. Lett. **56**, 2453 (1986), doi:10.1103/physrevlett.56.2453.

[121] B. S. Shastry, *Infinite conservation laws in the one-dimensional Hubbard model*, Phys. Rev. Lett. **56**, 1529 (1986), doi:10.1103/physrevlett.56.1529.

[122] B. S. Shastry, *Decorated star-triangle relations and exact integrability of the one-dimensional Hubbard model*, J. Stat. Phys. **50**, 57 (1988), doi:10.1007/bf01022987.

[123] M. Shiroishi and M. Wadati, *Yang-Baxter equation for the R-matrix of the one-dimensional Hubbard model*, J. Phys. Soc. Jpn. **64**, 57 (1995), doi:10.1143/jpsj.64.57.

[124] P. B. Ramos and M. J. Martins, *Algebraic Bethe ansatz approach for the one-dimensional Hubbard model*, J. Phys. A: Math. Gen. **30**, L195 (1997), doi:10.1088/0305-4470/30/7/009.

[125] M. J. Martins and P. B. Ramos, *The quantum inverse scattering method for Hubbard-like models*, Nucl. Phys. B **522**, 413 (1998), doi:10.1016/s0550-3213(98)00199-0.

[126] R. Yue and T. Deguchi, *Analytic Bethe ansatz for 1D Hubbard model and twisted coupled XY model*, J. Phys. A: Math. Gen. **30**, 849 (1997), doi:10.1088/0305-4470/30/3/011.

[127] G. Jüttner, A. Klümper and J. Suzuki, *The Hubbard chain at finite temperatures: ab initio calculations of Tomonaga-Luttinger liquid properties*, Nucl. Phys. B **522**, 471 (1998), doi:10.1016/s0550-3213(98)00256-9.

[128] D. B. Uglov and V. E. Korepin, *The Yangian symmetry of the Hubbard model*, Phys. Lett. A **190**, 238 (1994), doi:10.1016/0375-9601(94)90748-x.

[129] F. Göhmann and V. Inozemtsev, *The Yangian symmetry of the Hubbard models with variable range hopping*, Phys. Lett. A **214**, 161 (1996), doi:10.1016/0375-9601(96)00174-0.

[130] S. Murakami and F. Göhmann, *Yangian symmetry and quantum inverse scattering method for the one-dimensional Hubbard model*, Phys. Lett. A **227**, 216 (1997), doi:10.1016/s0375-9601(96)00953-x.

[131] S. Murakami and F. Göhmann, *Algebraic solution of the Hubbard model on the infinite interval*, Nucl. Phys. B **512**, 637 (1998), doi:10.1016/s0550-3213(97)00793-1.

[132] N. Beisert, M. de Leeuw and P. Nag, *Fusion for the one-dimensional Hubbard model*, J. Phys. A: Math. Theor. **48**, 324002 (2015), doi:10.1088/1751-8113/48/32/324002.

[133] A. Cavaglià, M. Cornagliotto, M. Mattelliano and R. Tateo, *A Riemann-Hilbert formulation for the finite temperature Hubbard model*, J. High Energy Phys. 015 (2015), doi:10.1007/jhep06(2015)015.

[134] F. H. L. Essler and V. E. Korepin, *Higher conservation laws and algebraic Bethe Ansätze for the supersymmetric t-J model*, Phys. Rev. B **46**, 9147 (1992), doi:10.1103/PhysRevB.46.9147.

[135] M. P. Pfannmüller and H. Frahm, *Algebraic Bethe ansatz for $gl(2,1)$ invariant 36-vertex models*, Nucl. Phys. B **479**, 575 (1996), doi:10.1016/0550-3213(96)00425-7.

[136] F. Göhmann, *Algebraic Bethe ansatz for the $gl(1|2)$ generalized model and Lieb–Wu equations*, Nucl. Phys. B **620**, 501 (2002), doi:10.1016/S0550-3213(01)00497-7.

[137] N. Beisert et al., *Review of AdS/CFT integrability: An overview*, Lett. Math. Phys. **99**, 3 (2012), doi:10.1007/s11005-011-0529-2.

[138] J. Escobedo, N. Gromov, A. Sever and P. Vieira, *Tailoring three-point functions and integrability*, J. High Energy Phys. **09**, 028 (2011), doi:10.1007/jhep09(2011)028.

[139] N. Gromov and P. Vieira, *Quantum integrability for three-point functions of maximally supersymmetric Yang-Mills theory*, Phys. Rev. Lett. **111**, 211601 (2013), doi:10.1103/physrevlett.111.211601.

[140] P. Vieira and T. Wang, *Tailoring non-compact spin chains*, J. High Energy Phys. **10**, 035 (2014), doi:10.1007/jhep10(2014)035.

[141] J. Caetano and T. Fleury, *Three-point functions and $\mathfrak{su}(1|1)$ spin chains*, J. High Energy Phys. **09**, 173 (2014), doi:10.1007/jhep09(2014)173.

[142] B. Basso, S. Komatsu and P. Vieira, *Structure constants and integrable bootstrap in planar $N = 4$ SYM theory* (2015), arXiv:1505.06745.

[143] Y. Jiang, S. Komatsu, I. Kostov and D. Serban, *The hexagon in the mirror: the three-point function in the SoV representation*, J. Phys. A: Math. Theor. **49**, 174007 (2016), doi:10.1088/1751-8113/49/17/174007.

[144] A. Rej, *Review of AdS/CFT integrability, Chapter I.3: Long-range spin chains*, Lett. Math. Phys. **99**, 85 (2012), doi:10.1007/s11005-011-0509-6.

[145] A. Rej, D. Serban and M. Staudacher, *Planar script $\mathcal{N} = 4$ gauge theory and the Hubbard model*, J. High Energy Phys. **03**, 018 (2006), doi:10.1088/1126-6708/2006/03/018.

[146] N. Beisert, *The analytic Bethe ansatz for a chain with centrally extended $su(2|2)$ symmetry*, J. Stat. Mech. P01017 (2007), doi:10.1088/1742-5468/2007/01/p01017.

[147] M. J. Martins and C. S. Melo, *The Bethe ansatz approach for factorizable centrally extended $su(2|2)$ S-matrices*, Nucl. Phys. B **785**, 246 (2007), doi:10.1016/j.nuclphysb.2007.05.021.

[148] N. Gromov, V. Kazakov, S. Leurent and D. Volin, *Quantum spectral curve for planar $\mathcal{N} = 4$ super-Yang-Mills theory*, Phys. Rev. Lett. **112**, 011602 (2014), doi:10.1103/physrevlett.112.011602.

[149] N. Gromov, V. Kazakov, S. Leurent and D. Volin, *Quantum spectral curve for arbitrary state/operator in AdS_5/CFT_4*, J. High Energy Phys. **09**, 187 (2015), doi:10.1007/jhep09(2015)187.

[150] N. Gromov, *Introduction to the spectrum of $\mathcal{N} = 4$ SYM and the Quantum Spectral Curve* (2017), arXiv:1708.03648.

[151] F. Levkovich-Maslyuk, *A review of the AdS/CFT Quantum Spectral Curve*, J. Phys. A: Math. Theor. **53**, 283004 (2020), doi:10.1088/1751-8121/ab7137.

[152] N. Beisert, *The $su(2|2)$ dynamic S-matrix*, Adv. Theor. Math. Phys. **12**, 948 (2008), doi:10.4310/atmp.2008.v12.n5.a1.

[153] G. Arutyunov, S. Frolov, J. Plefka and M. Zamaklar, *The off-shell symmetry algebra of the light-cone $AdS_5 \times S^5$ superstring*, J. Phys. A: Math. Theor. **40**, 3583 (2007), doi:10.1088/1751-8113/40/13/018.

[154] N. Beisert, *The S-matrix of AdS/CFT and Yangian symmetry*, Proc. Sci. **038**, 0002 (2007), doi:10.22323/1.038.0002.

[155] N. Beisert and M. de Leeuw, *The RTT realization for the deformed $\mathfrak{gl}(2|2)$ Yangian*, J. Phys. A: Math. Theor. **47**, 305201 (2014), doi:10.1088/1751-8113/47/30/305201.

[156] E. Ragoucy and G. Satta, *Analytical Bethe ansatz for closed and open $gl(\mathcal{M}|\mathcal{N})$ super-spin chains in arbitrary representations and for any Dynkin diagram*, J. High Energy Phys. **09**, 001 (2007), doi:10.1088/1126-6708/2007/09/001.

[157] N. Gromov, F. Levkovich-Maslyuk and G. Sizov, *New construction of eigenstates and separation of variables for $SU(N)$ quantum spin chains*, J. High Energy Phys. **09**, 111 (2017), doi:10.1007/jhep09(2017)111.

[158] S. Derkachov and P. Valinevich, *Separation of variables for the quantum $SL(3,\mathbb{C})$ spin magnet: eigenfunctions of Sklyanin B-operator* (2018), arXiv:1807.00302.

[159] J. M. Maillet and G. Niccoli, *Complete spectrum of quantum integrable lattice models associated to Y(gl(n)) by separation of variables*, SciPost Phys. **6**, 071 (2019), doi:10.21468/SciPostPhys.6.6.071.

[160] J. M. Maillet and G. Niccoli, *Complete spectrum of quantum integrable lattice models associated to $\mathcal{U}_q(\widehat{gl_n})$ by separation of variables*, J. Phys. A: Math. Theor. **52**, 315203 (2019), doi:10.1088/1751-8121/ab2930.

[161] J. M. Maillet and G. Niccoli, *On quantum separation of variables: beyond fundamental representations* (2019), arXiv:1903.06618.

[162] J. M. Maillet and G. Niccoli, *On separation of variables for reflection algebras*, J. Stat. Mech. 094020 (2019), doi:10.1088/1742-5468/ab357a.

[163] P. Ryan and D. Volin, *Separated variables and wave functions for rational $gl(N)$ spin chains in the companion twist frame*, J. Math. Phys. **60**, 032701 (2019), doi:10.1063/1.5085387.

[164] P. Ryan and D. Volin, *Separation of variables for rational $\mathfrak{gl}(n)$ spin chains in any compact representation, via fusion, embedding morphism and Bäcklund flow* (2020), arXiv:2002.12341.

[165] A. Liashyk and N. A. Slavnov, *On Bethe vectors in $\mathfrak{gl}(3)$-invariant integrable models*, J. High Energy Phys. **06**, 018 (2018), doi:10.1007/jhep06(2018)018.

[166] N. Gromov and F. Levkovich-Maslyuk, *New compact construction of eigenstates for supersymmetric spin chains*, J. High Energy Phys. **09**, 085 (2018), doi:10.1007/JHEP09(2018)085.

[167] P. Kulish and N. Y. Reshetikhin, *Generalized Heisenberg ferromagnet and the Gross-Neveu model*, Sov. Phys. JETP **53**, 108 (1981).

[168] P. P. Kulish and N. Y. Reshetikhin, *Diagonalisation of $GL(N)$ invariant transfer matrices and quantum N-wave system (Lee model)*, J. Phys. A: Math. Gen. **16**, L591 (1983), doi:10.1088/0305-4470/16/16/001.

[169] S. Belliard and E. Ragoucy, *The nested Bethe ansatz for 'all' closed spin chains*, J. Phys. A: Math. Theor. **41**, 295202 (2008), doi:10.1088/1751-8113/41/29/295202.

[170] S. Pakuliak, E. Ragoucy and N. Slavnov, *Nested Algebraic Bethe Ansatz in integrable models: recent results*, SciPost Phys. Lect. Notes 6 (2018), doi:10.21468/scipostphyslectnotes.6.

[171] G. P. Pronko and Y. G. Stroganov, *Families of solutions of the nested Bethe ansatz for the A_2 spin chain*, J. Phys. A: Math. Gen. **33**, 8267 (2000), doi:10.1088/0305-4470/33/46/309.

[172] S. Belliard, S. Pakuliak, E. Ragoucy and N. A. Slavnov, *Highest coefficient of scalar products in $SU(3)$-invariant integrable models*, J. Stat. Mech. P09003 (2012), doi:10.1088/1742-5468/2012/09/p09003.

[173] S. Belliard, S. Pakuliak, E. Ragoucy and N. A. Slavnov, *The algebraic Bethe ansatz for scalar products in $SU(3)$-invariant integrable models*, J. Stat. Mech. P10017 (2012), doi:10.1088/1742-5468/2012/10/p10017.

[174] S. Belliard, S. Pakuliak, E. Ragoucy and N. A. Slavnov, *Bethe vectors of $GL(3)$-invariant integrable models*, J. Stat. Mech. P02020 (2013), doi:10.1088/1742-5468/2013/02/p02020.

[175] S. Belliard, S. Pakuliak, E. Ragoucy and N. A. Slavnov, *Form factors in $SU(3)$-invariant integrable models*, J. Stat. Mech. P04033 (2013), doi:10.1088/1742-5468/2013/04/p04033.

[176] S. Pakuliak, E. Ragoucy and N. Slavnov, *Form factors in quantum integrable models with $GL(3)$-invariant R-matrix*, Nucl. Phys. B **881**, 343 (2014), doi:https://doi.org/10.1016/j.nuclphysb.2014.02.014.

[177] S. Z. Pakuliak, E. Ragoucy and N. A. Slavnov, *Determinant representations for form factors in quantum integrable models with the $GL(3)$-invariant R-matrix*, Theor. Math. Phys. **181**, 1566 (2014), doi:10.1007/s11232-014-0236-0.

[178] S. Pakuliak, E. Ragoucy and N. A. Slavnov, *$GL(3)$-based quantum integrable composite models. I. Bethe vectors*, SIGMA **11**, 063 (2015), doi:10.3842/sigma.2015.063.

[179] S. Pakuliak, E. Ragoucy and N. A. Slavnov, *GL(3)-based quantum integrable composite models. II. Form factors of local operators*, SIGMA **11**, 064 (2015), doi:10.3842/sigma.2015.064.

[180] S. Pakuliak, E. Ragoucy and N. A. Slavnov, *Form factors of local operators in a one-dimensional two-component Bose gas*, J. Phys. A: Math. Theor. **48**, 435001 (2015), doi:10.1088/1751-8113/48/43/435001.

[181] P. P. Kulish, N. Y. Reshetikhin and E. K. Sklyanin, *Yang-Baxter equation and representation theory: I*, Lett. Math. Phys. **5**, 393 (1981), doi:10.1007/bf02285311.

[182] P. P. Kulish and N. Y. Reshetikhin, *GL_3-invariant solutions of the Yang-Baxter equation and associated quantum systems*, J. Math. Sci. **34**, 1948 (1986), doi:10.1007/bf01095104.

[183] N. Y. Reshetikhin, *A method of functional equations in the theory of exactly solvable quantum systems*, Lett. in Math. Phy. **7**, 205 (1983), doi:10.1007/bf00400435.

[184] E. Ogievetsky and P. Wiegmann, *Factorized S-matrix and the Bethe ansatz for simple Lie groups*, Phys. Lett. B **168**, 360 (1986), doi:10.1016/0370-2693(86)91644-8.

[185] N. Yu. Reshetikhin and P. B. Weigmann, *Towards the classification of completely integrable quantum field theories (the Bethe-Ansatz associated with Dynkin diagrams and their automorphisms)*, Phys. Lett. B **189**, 125 (1987), doi:10.1016/0370-2693(87)91282-2.

[186] V. V. Bazhanov and N. Reshetikhin, *Restricted solid-on-solid models connected with simply laced algebras and conformal field theory*, J. Phys. A: Math. Gen. **23**, 1477 (1990), doi:10.1088/0305-4470/23/9/012.

[187] A. Klümper and P. A. Pearce, *Conformal weights of RSOS lattice models and their fusion hierarchies*, Phys. A: Stat. Mech. Appl. **183**, 304 (1992), doi:10.1016/0378-4371(92)90149-k.

[188] A. Kuniba and T. Nakanishi, *Rogers dilogarithm in integrable systems*, in Differential geometric methods in theoretical physics-Proceedings of the XXI international conference, World Scientific (1993), doi:10.1142/9789814536448.

[189] A. Kuniba, T. Nakanishi and J. Suzuki, *Functional relations in solvable lattice models I: Functional relations and representation theory*, Int. J. Mod. Phys. A **09**, 5215 (1994), doi:10.1142/s0217751x94002119.

[190] A. Kuniba, T. Nakanishi and J. Suzuki, *T-systems and Y-systems in integrable systems*, J. Phys. A: Math. Theor. **44**, 103001 (2011), doi:10.1088/1751-8113/44/10/103001.

[191] Z. Tsuboi, *Analytic Bethe ansatz and functional equations for Lie superalgebra $sl(r+1|s+1)$*, J. Phys. A: Math. Gen. **30**, 7975 (1997), doi:10.1088/0305-4470/30/22/031.

[192] Z. Tsuboi, *Analytic Bethe ansatz related to a one-parameter family of finite-dimensional representations of the Lie superalgebra $sl(r+1|s+1)$*, J. Phys. A: Math. Gen. **31**, 5485 (1998), doi:10.1088/0305-4470/31/24/010.

[193] Z. Tsuboi, *Analytic Bethe ansatz and functional equations associated with any simple root systems of the Lie superalgebra $sl(r+1|s+1)$*, Phys. A: Stat. Mech. Appl. **252**, 565 (1998), doi:10.1016/s0378-4371(97)00625-0.

[194] Z. Tsuboi, *Nonlinear integral equations and high temperature expansion for the $U_q\left(\widehat{sl}(r+1|s+1)\right)$ Perk–Schultz model*, Nucl. Phys. B **737**, 261 (2006), doi:10.1016/j.nuclphysb.2005.12.017.

[195] V. Kazakov, A. Sorin and A. Zabrodin, *Supersymmetric Bethe ansatz and Baxter equations from discrete Hirota dynamics*, Nucl. Phys. B **790**, 345 (2008), doi:10.1016/j.nuclphysb.2007.06.025.

[196] A. V. Zabrodin, *Bäcklund transformations for the difference Hirota equation and the supersymmetric Bethe ansatz*, Theor. Math. Phys. **155**, 567 (2008), doi:10.1007/s11232-008-0047-2.

[197] I. Krichever, O. Lipan, P. Wiegmann and A. Zabrodin, *Quantum integrable systems and elliptic solutions of classical discrete nonlinear equations*, in Low-dimensional applications of quantum field theory, Springer (1997), ISBN 9781489919212, doi:10.1007/978-1-4899-1919-9_16.

[198] A. Zabrodin, *Discrete Hirota's equation in quantum integrable models*, Int. J. Mod. Phys. B **11**, 3125 (1997), doi:10.1142/S0217979297001520.

[199] A. V. Zabrodin, *Hirota equation and Bethe ansatz*, Theor. Math. Phys. **116**, 782 (1998), doi:10.1007/bf02557123.

[200] V. Kazakov, S. Leurent and D. Volin, *T-system on T-hook: Grassmannian solution and twisted Quantum Spectral Curve*, J. High Energy Phys. 044 (2016), doi:10.1007/jhep12(2016)044.

[201] C. Marboe and D. Volin, *Fast analytic solver of rational Bethe equations*, J. Phys. A: Math. Theor. **50**, 204002 (2017), doi:10.1088/1751-8121/aa6b88.

[202] C. Marboe and D. Volin, *The full spectrum of AdS5/CFT4 I: representation theory and one-loop Q-system*, J. Phys. A: Math. Theor. **51**, 165401 (2018), doi:10.1088/1751-8121/aab34a.

[203] A. M. Grabinski and H. Frahm, *Non-diagonal boundary conditions for $\mathfrak{gl}(1|1)$ super spin chains*, J. Phys. A: Math. Theor. **43**, 045207 (2010), doi:10.1088/1751-8113/43/4/045207.

[204] C. Huang, K. Lu and E. Mukhin, *Solutions of $\mathfrak{gl}_{m|n}$ XXX Bethe ansatz equation and rational difference operators*, J. Phys. A: Math. Theor. **52**, 375204 (2019), doi:10.1088/1751-8121/ab1960.

[205] A. Hutsalyuk, A. Liashyk, S. Z. Pakuliak, E. Ragoucy and N. A. Slavnov, *Multiple actions of the monodromy matrix in $gl(2|1)$-invariant integrable models*, SIGMA **12**, 099 (2016), doi:10.3842/sigma.2016.099.

[206] A. A. Hutsalyuk, A. N. Liashyk, S. Z. Pakuliak, E. Ragoucy and N. A. Slavnov, *Current presentation for the super-Yangian double $DY(\mathfrak{gl}(m|n))$ and Bethe vectors*, Russ. Math. Surv. **72**, 33 (2017), doi:10.1070/rm9754.

[207] S. Z. Pakuliak, E. Ragoucy and N. A. Slavnov, *Bethe vectors for models based on the super-Yangian $Y(\mathfrak{gl}(m|n))$*, J. Integrable Sys. **2**, xyx001 (2017), doi:10.1093/integr/xyx001.

[208] A. Hutsalyuk, A. Liashyk, S. Z. Pakuliak, E. Ragoucy and N. A. Slavnov, *Scalar products of Bethe vectors in models with $\mathfrak{gl}(2|1)$ symmetry 2. Determinant representation*, J. Phys. A: Math. Theor. **50**, 034004 (2016), doi:10.1088/1751-8121/50/3/034004.

[209] K. Lu and E. Mukhin, *On the supersymmetric XXX spin chains associated to* $\mathfrak{gl}_{1|1}$ (2019), arXiv:1910.13360.

[210] D. Chernyak, S. Leurent and D. Volin, *Completeness of Wronskian Bethe equations for rational* $\mathfrak{gl}_{m|n}$ *spin chain* (2020), arXiv:2004.02865.

[211] E. Mukhin, V. Tarasov and A. Varchenko, *Spaces of quasi-exponentials and representations of the Yangian* $Y(\mathfrak{gl}_N)$, Transform. Groups **19**, 861 (2014), doi:10.1007/s00031-014-9275-8.

[212] R. J. Baxter, *Spontaneous staggered polarization of the F-model*, J. Stat. Phys. **9**, 145 (1973), doi:10.1007/bf01016845.

[213] R. J. Baxter, *Corner transfer matrices of the eight-vertex model. I. Low-temperature expansions and conjectured properties*, J. Stat. Phys. **15**, 485 (1976), doi:10.1007/bf01020802.

[214] R. J. Baxter, *Corner transfer matrices of the eight-vertex model. II. The Ising model case*, J. Stat. Phys. **17**, 1 (1977), doi:10.1007/bf01089373.

[215] R. J. Baxter, *Solvable eight-vertex model on an arbitrary planar lattice*, Philos. Trans. R. Soc. A: Math. Phys. Eng. Sci. **289**, 315 (1978), doi:10.1098/rsta.1978.0062.

[216] V. Pasquier and H. Saleur, *Common structures between finite systems and conformal field theories through quantum groups*, Nucl. Phys. B **330**, 523 (1990), doi:10.1016/0550-3213(90)90122-t.

[217] M. T. Batchelor, R. J. Baxter, M. J. O'Rourke and C. M. Yung, *Exact solution and interfacial tension of the six-vertex model with anti-periodic boundary conditions*, J. Phys. A: Math. Gen. **28**, 2759 (1995), doi:10.1088/0305-4470/28/10/009.

[218] C. M. Yung and M. T. Batchelor, *Exact solution for the spin-s XXZ quantum chain with non-diagonal twists*, Nucl. Phys. B **446**, 461 (1995), doi:10.1016/0550-3213(95)00168-r.

[219] V. V. Bazhanov, S. L. Lukyanov and A. B. Zamolodchikov, *Integrable structure of conformal field theory II. Q-operator and DDV equation*, Commun. Math. Phys. **190**, 247 (1997), doi:10.1007/s002200050240.

[220] A. Antonov and B. Feigin, *Quantum group representations and the Baxter equation*, Phys. Lett. B **392**, 115 (1997), doi:10.1016/s0370-2693(96)01526-2.

[221] V. V. Bazhanov, S. L. Lukyanov and A. B. Zamolodchikov, *Integrable structure of conformal field theory III. The Yang-Baxter relation*, Commun. Math. Phys. **200**, 297 (1999), doi:10.1007/s002200050531.

[222] S. É. Derkachov, *Baxter's Q-operator for the homogeneous XXX spin chain*, J. Phys. A: Math. Gen. **32**, 5299 (1999), doi:10.1088/0305-4470/32/28/309.

[223] G. P. Pronko, *On Baxter's Q-operator for the XXX spin chain*, Commun. Math. Phys. **212**, 687 (2000), doi:10.1007/s002200000235.

[224] C. Korff, *A Q-operator for the twisted XXX model*, J. Phys. A: Math. Gen. **39**, 3203 (2006), doi:10.1088/0305-4470/39/13/002.

[225] S. E. Derkachov, *Factorization of the R-matrix and Baxter's Q-operator*, J. Math. Sci. **151**, 2880 (2008), doi:10.1007/s10958-008-9010-x.

[226] V. V. Bazhanov, T. Łukowski, C. Meneghelli and M. Staudacher, *A shortcut to the Q-operator*, J. Stat. Mech. P11002 (2010), doi:10.1088/1742-5468/2010/11/p11002.

[227] V. V. Mangazeev, *Q-operators in the six-vertex model*, Nucl. Phys. B **886**, 166 (2014), doi:10.1016/j.nuclphysb.2014.06.024.

[228] H. Boos, F. Göhmann, A. Klümper, K. S. Nirov and A. V. Razumov, *Exercises with the universalR-matrix*, J. Phys. A: Math. Theor. **43**, 415208 (2010), doi:10.1088/1751-8113/43/41/415208.

[229] H. Boos, F. Göhmann, A. Klümper, K. S. Nirov and A. V. Razumov, *Quantum groups and functional relations for higher rank*, J. Phys. A: Math. Theor. **47**, 275201 (2014), doi:10.1088/1751-8113/47/27/275201.

[230] H. Boos, F. Göhmann, A. Klümper, K. S. Nirov and A. V. Razumov, *Oscillator versus prefundamental representations*, J. Math. Phys. **57**, 111702 (2016), doi:10.1063/1.4966925.

[231] H. Boos, F. Göhmann, A. Klümper, K. S. Nirov and A. V. Razumov, *Oscillator versus prefundamental representations. II. Arbitrary higher ranks*, J. Math. Phys. **58**, 093504 (2017), doi:10.1063/1.5001336.

[232] C. Meneghelli and J. Teschner, *Integrable light-cone lattice discretizations from the universal* R*-matrix*, Adv. Theoret. Math. Phys. **21**, 1189 (2017), doi:10.4310/atmp.2017.v21.n5.a3.

[233] R. Frassek, T. Łukowski, C. Meneghelli and M. Staudacher, *Oscillator construction of Q-operators*, Nucl. Phys. B **850**, 175 (2011), doi:10.1016/j.nuclphysb.2011.04.008.

[234] J. M. Maillet, G. Niccoli and L. Vignoli, *On scalar products in higher rank quantum separation of variables* (2020), arXiv:2003.04281.

[235] A. Cavaglià, N. Gromov and F. Levkovich-Maslyuk, *Separation of variables and scalar products at any rank*, J. High Energy Phys. **09**, 052 (2019), doi:10.1007/jhep09(2019)052.

[236] N. Gromov, F. Levkovich-Maslyuk, P. Ryan and D. Volin, *Dual separated variables and scalar products*, Phys. Lett. B **806**, 135494 (2020), doi:10.1016/j.physletb.2020.135494.

[237] V. G. Kac, *Lie superalgebras*, Adv. Math. **26**, 8 (1977), doi:10.1016/0001-8708(77)90017-2.

[238] I. Musson, *Lie superalgebras and enveloping algebras*, American Mathematical Society, Providence, Rhode Island, ISBN 9780821868676 (2012), doi:10.1090/gsm/131.

[239] J. Gruneberg, *On the construction and solution of $U_q(gl(2,1;\mathbb{C}))$-symmetric models*, Nucl. Phys. B **568**, 594 (2000), doi:10.1016/S0550-3213(99)00675-6.

[240] M. L. Nazarov, *Quantum Berezinian and the classical capelli identity*, Lett. Math. Phys. **21**, 123 (1991), doi:10.1007/bf00401646.

[241] L. Gow, *Yangians of Lie superalgebras*, Ph.D. thesis, University of Sydney (2007).

[242] I. Bars, *Supergroups and their representations*, in Introduction to Supersymmetry in Particle and Nuclear Physics, Springer US, Boston, MA, ISBN 9781475709193 (1984), doi:10.1007/978-1-4757-0917-9_5.

[243] I. Bars, *Supergroups and superalgebras in physics*, Phys. D: Nonlinear Phenom. **15**, 42 (1985), doi:10.1016/0167-2789(85)90147-2.

[244] J. Alfaro, R. Medina and L. F. Urrutia, *Orthogonality relations and supercharacter formulas of U(m|n) representations*, J. Math. Phys. **38**, 5319 (1997), doi:10.1063/1.531945.

[245] I. Bars, B. Morel and H. Ruegg, *Kac–Dynkin diagrams and supertableaux*, J. Math. Phys. **24**, 2253 (1983), doi:10.1063/1.525970.

[246] V. Kazakov and P. Vieira, *From characters to quantum (super)spin chains via fusion*, J. High Energy Phys. **10**, 050 (2008), doi:10.1088/1126-6708/2008/10/050.

[247] V. Kazakov, S. Leurent and Z. Tsuboi, *Baxter's Q-operators and operatorial Bäcklund flow for quantum (super)-spin chains*, Commun. Math. Phys. **311**, 787 (2012), doi:10.1007/s00220-012-1428-9.

[248] V. Tarasov and A. Varchenko, *Completeness of Bethe vectors and difference equations with regular singular points*, Internat. Math. Res. Notices 637 (1995), doi:10.1155/s1073792895000377.

[249] V. V. Bazhanov and Z. Tsuboi, *Baxter's Q-operators for supersymmetric spin chains*, Nucl. Phys. B **805**, 451 (2008), doi:10.1016/j.nuclphysb.2008.06.025.

[250] Z. Tsuboi, *Solutions of the T-system and Baxter equations for supersymmetric spin chains*, Nucl. Phys. B **826**, 399 (2010), doi:10.1016/j.nuclphysb.2009.08.009.

Article [LV2]

J. M. Maillet, G. Niccoli, and L. Vignoli, "On Scalar Products in Higher Rank Quantum Separation of Variables," *SciPost Phys.*, vol. 9, p. 86, 6 2020. DOI: 10.21468/SciPostPhys.9.6.086

SciPost Phys. 9, 086 (2020)

On scalar products in higher rank quantum separation of variables

Jean Michel Maillet[⋆], **Giuliano Niccoli**[†] **and Louis Vignoli**[‡]

Univ Lyon, Ens de Lyon, Univ Claude Bernard, CNRS,
Laboratoire de Physique, UMR 5672, F-69342 Lyon, France

⋆ maillet@ens-lyon.fr † giuliano.niccoli@ens-lyon.fr ‡ louis.vignoli@ens-lyon.fr

Abstract

Using the framework of the quantum separation of variables (SoV) for higher rank quantum integrable lattice models [1], we introduce some foundations to go beyond the obtained complete transfer matrix spectrum description, and open the way to the computation of matrix elements of local operators. This first amounts to obtain simple expressions for scalar products of the so-called separate states, that are transfer matrix eigenstates or some simple generalization of them. In the higher rank case, left and right SoV bases are expected to be *pseudo-orthogonal*, that is for a given SoV co-vector $\langle \underline{\mathrm{h}}|$, there could be more than one non-vanishing overlap $\langle \underline{\mathrm{h}}|\underline{\mathrm{k}}\rangle$ with the vectors $|\underline{\mathrm{k}}\rangle$ of the chosen right SoV basis. For simplicity, we describe our method to get these pseudo-orthogonality overlaps in the fundamental representations of the $\mathcal{Y}(gl_3)$ lattice model with N sites, a case of rank 2. The non-zero couplings between the co-vector and vector SoV bases are exactly characterized. While the corresponding *SoV-measure* stays reasonably simple and of possible practical use, we address the problem of constructing left and right SoV bases which do satisfy standard orthogonality (by standard we mean $\langle \underline{\mathrm{h}}|\underline{\mathrm{k}}\rangle \propto \delta_{\underline{\mathrm{h}},\underline{\mathrm{k}}}$). In our approach, the SoV bases are constructed by using families of conserved charges. This gives us a large freedom in the SoV bases construction, and allows us to look for the choice of a family of conserved charges which leads to orthogonal co-vector/vector SoV bases. We first define such a choice in the case of twist matrices having simple spectrum and zero determinant. Then, we generalize the associated family of conserved charges and orthogonal SoV bases to generic simple spectrum and invertible twist matrices. Under this choice of conserved charges, and of the associated orthogonal SoV bases, the scalar products of separate states simplify considerably and take a form similar to the $\mathcal{Y}(gl_2)$ rank one case.

Received 28-08-2020
Accepted 24-11-2020
Published 11-12-2020
doi:10.21468/SciPostPhys.9.6.086

Check for updates

Contents

1 Introduction

The quantum separation of variables (SoV) has been introduced by Sklyanin [2–6] in the framework of the quantum inverse scattering method [7–15]. It enables to analyze the transfer matrix (and Hamiltonian) spectrum using the Yang-Baxter commutation relations. It does not rely on any ansatz, which makes explicit its advantage w.r.t. Bethe Ansatz methods [8,16–22]. This method has been first systematically developed in the class of the rank one integrable quantum models [23–54] proving its wide range of application. The completeness of the transfer matrix spectrum characterization in the SoV approach for compact representations has been clearly addressed and proven in [33–35,37–55]. In this rank one case, the SoV approach has also been shown to lead to simple determinant formulae for scalar products of the so-called separate states [37,39–45,48,53,54]. Those include the transfer matrix eigenstates and their generalizations with factorized but otherwise arbitrary wave functions in the SoV basis. In several important cases, the form factors of local or quasi-local operators have been computed in terms of determinants, while in [50,52,56] a rewriting of the determinants giving the scalar product formulae has been obtained paving the way for the direct analysis of form factors and correlation functions in the homogeneous and thermodynamic limits.

Our aim is to extend these achievements to the higher rank cases. Let us comment that scalar product formulae and matrix elements of local operators have been already computed

in the literature [57–70] for the higher rank case in the nested algebraic Bethe ansatz (NABA) framework [71–74] and that more recently have appeared interesting works analyzing these problems in SoV related frameworks [75, 76].

Sklyanin has also pioneered the SoV approach in the higher rank case[1], in the particular example of rank two [6]. Sklyanin's beautiful SoV construction involves the identification of a B-operator, whose eigenco-vector basis is meant to separate the spectral problem of the transfer matrix. The other fundamental elements of the Sklyanin's construction [6] are the identification of an A-operator, whose role is that of generating the shift operator on the B-spectrum, together with the identification of an operator quantum spectral curve equation involving the transfer matrices, the B-operator and the A-operator. These operator equations should separate the transfer matrix spectrum when computed in the zeroes of the B-operator. However, in [6] the SoV construction has been developed just using the gl_3 Yang-Baxter commutation relations without introducing any specific representations of the algebra. Only more recently, the SoV analysis for higher rank has been revived. For the fundamental representations of gl_3 Yang-Baxter algebra, in [77] the spectrum of the Sklyanin's B-operator has been conjectured together with its diagonalizability for some classes of twisted boundary conditions on the basis of an exact analysis of quantum chains of small sizes. Moreover in [77], the Sklyanin's B-operator has been used to conjecture a formula for the transfer matrix eigenvectors bypassing the traditional nested Bethe Ansatz procedure and consistent with small chains verification[2]. Then, in [79] the separation of variables approach has been initiated for non-compact representations of the gl_3 Yang-Baxter algebra determining the eigenfunctions of the Sklyanin's B-operator. While these findings are quite interesting, the complete implementation of the Sklyanin's SoV program for higher rank seems more involved as, at least for fundamental representations, the proposed A-operator acts as shift only on part of the B-spectrum which leaves unproven the separate relations in this SoV framework. This phenomenon has been already anticipated by Sklyanin in [6] and it occurs when the spectrum of the B-operator zeroes partially coincides with that of the poles of operators appearing in the commutation relations between A-operator and B-operator and/or in the operator quantum spectral curve equation, see [1] for further discussions.

In [1] we have overcome these difficulties by developing a new SoV approach which relies only on the abelian algebra of conserved charges of the given quantum integrable model. In our SoV approach the SoV co-vectors/vectors bases are generated by the action of appropriate sets of conserved charges on some reference co-vector/vector, hence bypassing the construction of the Sklyanin's A and B operators.

In its most general form, our construction uses a family of commuting conserved charges say $T(\lambda)$, $\lambda \in \mathbb{C}$ (typically the transfer matrix, its fused versions or the Baxter Q-operator in most of the cases considered, but in principle more general situations could occur) acting on some Hilbert space $\mathcal{H}$ ($\mathcal{H}^*$ being its dual) of the considered model. Such a family is said to be SoV bases generating if there exist a co-vector $\langle L| \in \mathcal{H}^*$ (resp. a vector $|R\rangle \in \mathcal{H}$) and sets of commuting conserved charges constructed from $T(\lambda)$, $T^{(a)}_{h_a}$ (resp. $\tilde{T}^{(a)}_{k_a}$) where $a = 1, \dots, N$ and $h_a, k_a = 0, \dots, d_a - 1$ with $d = \prod_{a=1}^{N} d_a$ the dimension of the Hilbert spaces $\mathcal{H}$ and $\mathcal{H}^*$, such that the set of co-vectors,

$$\langle h_1, \dots, h_N| = \langle L| \prod_{a=1}^{N} T^{(a)}_{h_a} , \tag{1.1}$$

[1] See also [25, 51] for some interesting analysis toward the SoV description of higher rank cases.

[2] This conjecture has been then proven in the ABA framework in [70]. These observations and conjectures have also been extended to the super-symmetric case in [78].

forms a basis of $\mathcal{H}^*$ and the set of vectors,

$$|k_1,\dots,k_N\rangle = \prod_{a=1}^{N} \tilde{T}^{(a)}_{h_a}|R\rangle \ , \tag{1.2}$$

forms a basis of $\mathcal{H}$. It follows immediately, by construction, that whenever such bases exist, any common eigenvector $|t\rangle$ (resp. eigenco-vector $\langle t|$) of the family $T(\lambda)$ with eigenvalue $t(\lambda)$ is also a common eigenvector (resp. eigenco-vector) of the commuting sets of conserved charges $T^{(a)}_{h_a}$ (resp. $\tilde{T}^{(a)}_{k_a}$) with eigenvalues $t^{(a)}_{h_a}$ (resp. $\tilde{t}^{(a)}_{k_a}$). Hence the corresponding wave functions in the coordinates h_i (resp. k_i) factorize as

$$\Psi_t(h_1,...,h_N) \equiv \langle h_1,...,h_N|t\rangle = \langle L|t\rangle \prod_{i=1}^{N} t^{(i)}_{h_i} \ , \tag{1.3}$$

and similarly,

$$\tilde{\Psi}_t(k_1,...,k_N) \equiv \langle t|k_1,...,k_N\rangle = \langle t|R\rangle \prod_{i=1}^{N} \tilde{t}^{(i)}_{k_i} \ . \tag{1.4}$$

This also means that the eigenvectors coordinates in such SoV bases are completely determined from the eigenvalues of the commuting conserved charges used to construct those bases. Hence, the very existence of such bases implies the simplicity of the spectrum of the family $T(\lambda)$ since the coordinates (wave function) of any eigenvector are completely determined by the corresponding eigenvalue. This in turn implies that the above sets of conserved charges $T^{(a)}_{h_a}$ and $\tilde{T}^{(a)}_{k_a}$ are both basis of the vector space $\mathcal{C}_{T(\lambda)}$ of operators commuting with the family of operators $T(\lambda)$. Hence the *linear* action of the operator $T(\lambda)$ on such bases can be computed in a close form as for any values of $h_1,...,h_N$ (resp. $k_1,...,k_N$), the product $T^{(a)}_{h_a}\cdot T(\lambda)$ (resp. $T(\lambda)\cdot\tilde{T}^{(a)}_{k_a}$) is also a conserved charge commuting with $T(\lambda)$. Hence it is an element of $\mathcal{C}_{T(\lambda)}$ that can be decomposed *linearly* on the basis generated by $T^{(a)}_{h_a}$ (resp. $\tilde{T}^{(a)}_{k_a}$).

To make this more explicitly, let us introduce compact notations we will be using all along this paper, namely, $\underline{\mathbf{h}} = (h_1,...,h_N)$ and similarly $\underline{\mathbf{k}} = (k_1,...,k_N)$, and accordingly, $T_{\underline{\mathbf{h}}} = \prod_{a=1}^{N} T^{(a)}_{h_a}$, $\tilde{T}_{\underline{\mathbf{k}}} = \prod_{a=1}^{N} \tilde{T}^{(a)}_{k_a}$, and also $|k_1,...,k_N\rangle = |\underline{\mathbf{k}}\rangle$, $\langle h_1,...,h_N| = \langle\underline{\mathbf{h}}|$ for the two sets defining the right and left SoV bases[3] , then there exist scalar complex coefficients $N^{\underline{\mathbf{l}}}_{\underline{\mathbf{h}}}(\lambda)$ and $N^{\underline{\mathbf{l}}}_{\underline{\mathbf{h}},\underline{\mathbf{k}}}$ such that[4]:

$$T_{\underline{\mathbf{h}}}\cdot T(\lambda) = \sum_{\underline{\mathbf{l}}} N^{\underline{\mathbf{l}}}_{\underline{\mathbf{h}}}(\lambda)\ T_{\underline{\mathbf{l}}} \ , \tag{1.5}$$

and,

$$T_{\underline{\mathbf{h}}}\cdot \tilde{T}_{\underline{\mathbf{k}}} = \sum_{\underline{\mathbf{l}}} N^{\underline{\mathbf{l}}}_{\underline{\mathbf{h}},\underline{\mathbf{k}}}\ T_{\underline{\mathbf{l}}} \ . \tag{1.6}$$

[3] Using such compact notations it should not be forgotten that these vectors $|\underline{\mathbf{k}}\rangle$ and co-vectors $\langle\underline{\mathbf{h}}|$ defining SoV bases are depending respectively on the chosen sets of conserved charges $\tilde{T}^{(a)}_{k_a}$ and $T^{(a)}_{h_a}$ and on the reference vector $|R\rangle$ and co-vector $\langle L|$. Hence in the following such compact notations will be used only after such choices have been defined.

[4] Let us stress here that these complex coefficients which can be interpreted as the structure constants of the associative and commutative algebra of the conserved charges, are depending directly on the choice of the two sets of commuting conserved charges $T^{(a)}_{h_a}$ and $\tilde{T}^{(a)}_{k_a}$. Hence changing those sets, eventually in a non-linear way, as sums of products of commuting conserved charges are still commuting conserved charges, will modify these structure constants accordingly.

Similarly one can define two other sets of complex coefficients, namely $C^{\underline{\mathbf{l}}}_{\underline{\mathbf{h}},\underline{\mathbf{k}}}$ and $\tilde{C}^{\underline{\mathbf{l}}}_{\underline{\mathbf{h}},\underline{\mathbf{k}}}$ such that:

$$T_{\underline{\mathbf{h}}} \cdot T_{\underline{\mathbf{k}}} = \sum_{\underline{\mathbf{l}}} C^{\underline{\mathbf{l}}}_{\underline{\mathbf{h}},\underline{\mathbf{k}}}\, T_{\underline{\mathbf{l}}}\,, \tag{1.7}$$

and,

$$\tilde{T}_{\underline{\mathbf{h}}} \cdot \tilde{T}_{\underline{\mathbf{k}}} = \sum_{\underline{\mathbf{l}}} \tilde{C}^{\underline{\mathbf{l}}}_{\underline{\mathbf{h}},\underline{\mathbf{k}}}\, \tilde{T}_{\underline{\mathbf{l}}}\,. \tag{1.8}$$

The knowledge of these relations together with the action of the complete family of conserved charges $T(\lambda)$ on our SoV bases has been shown to completely characterize the common spectrum of all the above commuting conserved charges. Particular realizations of this situation include the case where the $T^{(a)}_{h_a}$ are powers of the transfer matrix evaluated in the inhomogeneity parameters as $T(\xi_a)^{h_a}$, or are given as the fused transfer matrices $T_{h_a}(\xi_a^{(h_a)})$ in some shifted points $\xi_a^{(h_a)}$, where h_a is the level of fusion. In the higher spin gl_2 case, they are simply obtained from the Q-operator evaluated in shifted inhomogeneities as $Q(\xi_a^{(h_a)})$. In all these cases, the coefficients $N^{\underline{\mathbf{l}}}_{\underline{\mathbf{h}}}(\lambda)$, $N^{\underline{\mathbf{l}}}_{\underline{\mathbf{h}},\underline{\mathbf{k}}}$, $C^{\underline{\mathbf{l}}}_{\underline{\mathbf{h}},\underline{\mathbf{k}}}$ and $\tilde{C}^{\underline{\mathbf{l}}}_{\underline{\mathbf{h}},\underline{\mathbf{k}}}$ are completely determined by the fusion relations or the T-Q relations satisfied by the transfer matrices and the Baxter Q-operator.

The conditions on the above sets of conserved charges to indeed generate SoV bases were identified and proven[5] in [1], together with the factorization of the wave functions in terms of conserved charge eigenvalues and the proof of the completeness of the description of the transfer matrix spectrum. The discrete separate relations were proven to be equivalent to the quantum spectral curve equations, involving the transfer matrices and the Q-operator holding both at the eigenvalue and operator level, due to the proven simplicity of the transfer matrix spectrum [1]. In our approach, the separate variables relations are themselves proven to be originated by the structure constants of the abelian algebra of conserved charges, in particular by the transfer matrix fusion equations for the charges considered in [1]. From this perspective our SoV approach has the potential to be universal in the realm of quantum integrable model. Indeed, we have proven its applicability for a large class of quantum integrable models from the fundamental representations of gl_n, $gl_{n,m}$ and the $U_q(gl_n)$ Yang-Baxter algebras with simple spectrum twist matrices up to the higher rank reflection algebra cases with general boundary conditions, deriving new and complete descriptions of the transfer matrix spectrum [1,80–84] [6]. Moreover, in [85, 86] our construction of SoV bases using conserved charges has been extended to arbitrary finite dimensional rectangular representations of the gl_n Yang-Baxter algebra.

The relation of our SoV approach with the Sklyanin's one has been first analyzed in [1]. There we have observed the coincidence of our SoV co-vector basis with the Sklyanin's B-operator co-vector eigenbasis for chains of arbitrary length in the gl_2 case. This correspondence has been obtained for special choices of the reference co-vector and of the set of conserved charges used to generate the SoV basis. The same result has been derived in [1] for the gl_3 case for chains of small sizes. In [85] this observation has been proven for arbitrary finite dimensional rectangular representations of the gl_n Yang-Baxter algebra and for chains of any size. Moreover the simple spectrum of the Sklyanin's B-operator, and its gl_n extensions proposed in [77], has been obtained in [85]. This result together with the completeness of the description of the spectrum by factorized wave functions in terms of polynomial Q-functions [1] implies the ABA type formula of [77] for all the transfer matrix eigenvectors[7].

[5] They mainly reduce to properties satisfied by the twist matrix and the inhomogeneities parameters.

[6] Note that our reference [83] describe our approach for higher spin representations for the rank one case. While [84] also contains the SoV basis construction for the quasi-periodic Hubbard model.

[7] Note that it was first remarked in [26] for non-compact rank one models that the factorization of the wavefunctions in terms of polynomial Q-functions imply the ABA form of transfer matrix eigenvectors in the SoV basis

An important feature of our new approach to the SoV bases is that it relies only on finding a suitable set of commuting conserved charges and a corresponding reference co-vector/vector $\langle L| \in \mathcal{H}^*$ and $|R\rangle \in \mathcal{H}$, (the number of choices for those being in fact very large as shown in our first paper [1]). However, any other sets build from sums of products of given commuting conserved charges being again sets of commuting conserved charges, it results in a huge freedom in constructing SoV bases which was not available if one would have stick to SoV bases identified as eigenbasis of the Sklyanin's B-operator or its higher rank extensions.

Clearly, this is a very interesting built in aspect of our new approach to SoV that enables us to ask a new key question in this context: *what would be optimal choices of the sets of conserved charges determining the SoV bases for the quantum integrable model at hand?*

A first answer to this question, from the point of view of the determination of the spectrum, is that an optimal SoV basis is such that the action of the transfer matrix (and hence of the Hamiltonian of the model) on the chosen basis is as simple as possible. This could mean for example that the action of the family of $T(\lambda)$ on any element of the set $T_{\underline{\mathbf{h}}}$ decomposes back on that set with only a very few non-zero coefficients, and moreover that it is given only by local shifts of finite and lowest possible order on the coordinates h_a. This amounts to have chosen the basis $T_{\underline{\mathbf{h}}}$ of the space $\mathcal{C}_{T(\lambda)}$ in such a way that the structure constants $N_{\underline{\mathbf{h}}}^{\underline{\mathbf{l}}}(\lambda)$ have such a simple property; namely that the only non zero coefficients are those where $\underline{\mathbf{h}}$ and $\underline{\mathbf{l}}$ differ only by localized shifts in the coordinates. This is exactly what happens for SoV bases in the gl_2 case that are generated directly from the Baxter Q-operator. Indeed, the Baxter T-Q relation determines an action of the transfer matrix $T(\lambda)$ on the basis generated by $Q(\lambda)$ which involves only two terms with a local shift ± 1 for each coordinate h_a, to be compared to the dimension of the Hilbert space $\mathcal{H}$ and of the Bethe algebra $\mathcal{C}_{T(\lambda)}$ which is 2^N for a spin-1/2 chain of length N. This is in some sense the hallmark of integrability that generate a characteristic equation of degree two, hence much smaller than the dimension of the Hilbert space.

Another meaning of simplicity in the choice of our SoV bases could also be related to the coupling between the two chosen left (1.1) and right (1.2) SoV bases. Namely, a criterion of simplicity could be to take such two SoV covector/vector bases such that their scalar products are calculable in terms of manageable expressions. This is certainly an important question and criterion as it determines to what extend the chosen left (1.1) and right (1.2) SoV bases are easy to use when computing scalar products of separate states, form factors and correlation functions, that are our main goals.

The main purpose of the present paper is to study the important question of scalar products from this perspective.

In the class of rank one quantum integrable models, the SoV analysis so far developed [2–6,23–54] leads to the expectation that the transfer matrix construction of the co-vector/vector SoV bases can be defined in such a way that these are orthogonal bases. Similarly, in the Sklyanin's approach, this leads to the expectation that the co-vector/vector Sklyanin's B-operator eigenbases (orthogonal as soon as B is diagonalizable with simple spectrum) both implement the separation of variables for the transfer matrix spectrum. This feature has been proven to be very useful in computing scalar products of the so-called separate states and also in obtaining

once the Sklyanin's B-operator is proven to be diagonalizable. As we have explained in [1], this proof extends also to the higher rank case under the same assumption as it only uses the SoV representation of the transfer matrix eigenvectors.

determinant formulae for the form factors of local operators. As we will see in the next, in the higher rank quantum integrable models, this is not directly the case if the charges used to construct the co-vector/vector SoV basis are simply the transfer matrices or their fused higher versions, for a generic twist K.

On the one hand, the SoV vector basis is univocally fixed in terms of the co-vector one defined in [1] if one requires that it is of SoV type, i.e. that it is generated by a factorized action of conserved charges, and that it satisfies the orthogonality conditions with the co-vector basis on one quantum site (this is obviously a necessary requirement for general orthogonality!). It turns out that in general such SoV vector basis stays only *pseudo-orthogonal* to the co-vector one for quantum chains of arbitrary length N. More precisely, the matrix of scalar products $\mathcal{N}_{\underline{\mathbf{h}},\underline{\mathbf{k}}} = \langle \underline{\mathbf{h}} | \underline{\mathbf{k}} \rangle$ for the natural SoV bases introduced in [1] is in general not a diagonal matrix.

The aim of the present paper is twofold:

- Characterize the matrix of scalar products $\mathcal{N}_{\underline{\mathbf{h}},\underline{\mathbf{k}}} = \langle \underline{\mathbf{h}} | \underline{\mathbf{k}} \rangle$ and the associated SoV measure (related to the inverse of $\mathcal{N}_{\underline{\mathbf{h}},\underline{\mathbf{k}}}$) for the natural SoV bases introduced in [1] in the example of the rank two gl_3 case in the fundamental representations.

- Determine, in the same gl_3 representations, two sets of commuting conserved charges, $T_{\underline{\mathbf{h}}}$ and $\tilde{T}_{\underline{\mathbf{k}}}$ generating a left and right SoV bases that are orthogonal to each other and compute the corresponding SoV measure.

Given our left SoV co-vector basis, we first prove that the defined set of SoV vectors indeed define a basis and we exactly characterize the *pseudo-orthogonality* conditions writing all the non-zero non-diagonal couplings in terms of the diagonal ones, which we explicitly compute. This set of SoV vectors has been introduced recently in [76] as the set of eigenvectors of a C-operator which plays a similar role to the Sklyanin's B-operator and some integral form has been given for the coupling of the SoV co-vectors/vectors in [76]. Due to the quite different representations, a direct comparison of the results of [76] with those that we obtained stays a complicate task which however deserves further analysis.

Let us comment that this pseudo-orthogonality is intrinsically related to the form of fusion relations of the transfer matrices for higher rank case when computed in the special inhomogeneous points. In fact the matrix of scalar products can be directly related to the structure constants of the algebra of commuting conserved charges (1.6) that are in fact determined completely by the fusion relations as shown in [1]. To be more precise let us illustrate this in the following situation. Suppose we have chosen a left SoV basis of the type (1.1). Then let us consider a right SoV basis (1.2) where we have chosen the right reference vector $|R\rangle$ in such a way that it satisfies $\langle \underline{\mathbf{h}} | R \rangle = \delta_{\underline{\mathbf{h}},\underline{\mathbf{h}}_0}$ for some $\underline{\mathbf{h}}_0$. Then the corresponding matrix $\mathcal{N}_{\underline{\mathbf{h}},\underline{\mathbf{k}}}$ of scalar products can be computed in terms of the structure constant $N_{\underline{\mathbf{h}},\underline{\mathbf{k}}}^{\underline{\mathbf{h}}_0}$ to be:

$$\mathcal{N}_{\underline{\mathbf{h}},\underline{\mathbf{k}}} = N_{\underline{\mathbf{h}},\underline{\mathbf{k}}}^{\underline{\mathbf{h}}_0} \,. \tag{1.9}$$

A very interesting question is thus if there exists an optimal choice of the left (1.1) and right (1.2) SoV bases such that for some $\underline{\mathbf{h}}_0$ we have $N_{\underline{\mathbf{h}},\underline{\mathbf{k}}}^{\underline{\mathbf{h}}_0} = \delta_{\underline{\mathbf{h}},\underline{\mathbf{k}}}\, n(\underline{\mathbf{h}})$ with a calculable non-zero coefficient $n(\underline{\mathbf{h}})$ whose inverse determines the SoV measure.

This naturally leads to the observation that if we want to obtain co-vector/vector SoV bases mutually orthogonal we have to chose in general a different family of commuting conserved charges than the simple choice taken in [1] to generate both of them (or at least look for different points where the transfer matrices are computed). These observations in the Sklyanin's SoV framework for rank two mean that while the Sklyanin's B-operator define the co-vector

SoV basis, its vector eigenbasis is actually only a *pseudo-SoV* basis, i.e. not all the wave functions of transfer matrix eigenco-vectors have factorized form in terms of the transfer matrix eigenvalues.

Despite the absence of direct orthogonality the *SoV-measure* that we derive in section 3 stays reasonably simple and can be used as the starting point to compute matrix elements of local operators in this SoV framework. While, this seems a sensible line of research and we will further analyze it in the future, we would like to further investigate the potentiality of our new SoV approach.

In the present paper, for the rank two gl_3 case in the fundamental representation, we define some new family of commuting conserved charges whose spectral problem is separated for both a co-vector and a vector bases, which are moreover orthogonal to each other. Further, we show that the corresponding *SoV measure* takes a form very similar to the rank one case. The consequence is that w.r.t. this family of commuting conserved charges scalar products simplify considerably and take a form very similar to the rank one case for the separate states. Of course, in order to be able to compute matrix elements of local operators we will need to address the problem of the representation of the local operators in these new SoV bases.

The paper is organized as follows:

Section 2 is dedicated to recall some fundamental properties satisfied by the transfer matrices in the fundamental representations of the gl_3 Yang-Baxter algebra. In subsection 2.2, we moreover recall the results of [1] for the construction of the SoV bases for the considered representations, that is equations (2.23) and (2.24).

In section 3, we introduce a standard construction of co-vector/vector SoV bases (3.6)-(3.7) using the choice of the generating charges made in [1], i.e. given by the transfer matrices evaluated in the inhomogeneity parameters. The Theorem 3.1 characterizes completely the co-vector/vector coupling of these two systems of SoV states. The main results of this section are *i)* that the given system of SoV vectors form a basis, *ii)* the computation in (3.10) of the known tensor product form (3.9) of the reference vector associated to a fixed reference co-vector in the SoV basis, *iii)* the exact characterization in Theorem 3.1 of the pseudo-orthogonality relations (3.14), with the description of the non-diagonal couplings in terms of the diagonal ones, and *iv)* the explicit computation of the diagonal couplings in (3.20). Finally, the subsection 3.3 characterizes with Corollary 3.1 the SoV measure in terms of the non-zero SoV co-vector/vector couplings.

In section 4, we use the freedom in the choice of the generating family of conserved charges to construct orthogonal co-vector/vector SoV bases. The subsection 4.1 is dedicated to this construction in the class of quasi-periodic boundary conditions associated to simple spectrum but non-invertible twist matrices. The main theorem there, Theorem 4.1, states the orthogonality properties and the form of the diagonal SoV co-vector/vector couplings. These are similar to the SoV co-vector/vector couplings of the rank one integrable quantum models. In subsection 4.2, these results are used to compute scalar product formulae of separate states (4.77) and (4.81), showing that they take a form similar to the rank one case. Finally, in section 4.3, we introduce a new set of charges (4.93) that extends the results of subsections 4.1 and 4.2 to the general quasi-periodic boundary conditions, associated to simple spectrum and invertible twist matrices.

We give several technical and important proofs in the three appendices. The appendix A details the proof of the tensor product form of SoV starting co-vector/vector in our SoV con-

struction. The appendix B details how our SoV construction holds in the gl_2 representations, the aim being to establish one simple example to which compare our higher rank construction. Finally, the appendix C is dedicated to the detailed proof of our Theorem 3.1. Subsection C.1 handles the orthogonality proof, while subsection C.2 details the description of the non-zero SoV co-vector/vector couplings.

2 SoV bases for the fundamental representation of the gl_3 Yang-Baxter algebra

2.1 Fundamental representation of the gl_3 Yang-Baxter algebra

We consider here the Yang-Baxter algebra associated to the rational gl_3 R-matrix:

$$R_{a,b}(\lambda) = \lambda I_{a,b} + \eta \mathbb{P}_{a,b} = \begin{pmatrix} a_1(\lambda) & b_1 & b_2 \\ c_1 & a_2(\lambda) & b_3 \\ c_2 & c_3 & a_3(\lambda) \end{pmatrix} \in \mathrm{End}(V_a \otimes V_b), \tag{2.1}$$

where $V_a \cong V_b \cong \mathbb{C}^3$ and we have defined:

$$\begin{aligned} a_j(\lambda) &= \begin{pmatrix} \lambda+\eta\delta_{j,1} & 0 & 0 \\ 0 & \lambda+\eta\delta_{j,2} & 0 \\ 0 & 0 & \lambda+\eta\delta_{j,3} \end{pmatrix}, \quad \forall j \in \{1,2,3\}, \\ b_1 &= \begin{pmatrix} 0 & 0 & 0 \\ \eta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad b_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \eta & 0 & 0 \end{pmatrix}, \quad b_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \eta & 0 \end{pmatrix}, \\ c_1 &= \begin{pmatrix} 0 & \eta & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad c_2 = \begin{pmatrix} 0 & 0 & \eta \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad c_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \eta \\ 0 & 0 & 0 \end{pmatrix}, \end{aligned} \tag{2.2}$$

which satisfies the Yang-Baxter equation

$$R_{12}(\lambda-\mu)R_{13}(\lambda)R_{23}(\mu) = R_{23}(\mu)R_{13}(\lambda)R_{12}(\lambda-\mu) \in \mathrm{End}(V_1 \otimes V_2 \otimes V_3) \tag{2.3}$$

and the scalar Yang-Baxter equation:

$$R_{12}(\lambda)K_1K_2 = K_2K_1R_{12}(\lambda) \in \mathrm{End}(V_1 \otimes V_2), \tag{2.4}$$

where $K \in \mathrm{End}(V)$ is any 3×3 matrix. We can define the following monodromy matrix:

$$M_a^{(K)}(\lambda) \equiv K_a R_{a,\mathrm{N}}(\lambda-\xi_\mathrm{N}) \cdots R_{a,1}(\lambda-\xi_1) \in \mathrm{End}(V_a \otimes \mathcal{H}), \tag{2.5}$$

where $\mathcal{H} = \bigotimes_{n=1}^{\mathrm{N}} V_n$. $M_a^{(K)}(\lambda)$ itself satisfies the Yang-Baxter equation and hence it defines an irreducible 3^{N}-dimensional representation of the gl_3 Yang-Baxter algebra for the *inhomogeneity* parameters $\{\xi_1, ..., \xi_\mathrm{N}\}$ in generic complex positions:

$$\xi_i - \xi_j \neq 0, \pm\eta, \quad \forall i,j \in \{1, ..., \mathrm{N}\}. \tag{2.6}$$

Then, in the framework of the quantum inverse scattering [87–89], the following families of commuting charges exist according to the following:

Proposition 2.1 ([87–89]). *Defined the antisymmetric projectors:*

$$P^{-}_{1,\dots,m} = \frac{1}{m!}\sum_{\pi\in S_m}(-1)^{\sigma_\pi}P_\pi, \tag{2.7}$$

where S_m is the symmetric group of rank m, σ_π the signature of the permutation π and

$$P_\pi(v_1\otimes\cdots\otimes v_m) = v_{\pi(1)}\otimes\cdots\otimes v_{\pi(m)}, \tag{2.8}$$

then the following quantum spectral invariants (the fused transfer matrices):

$$T^{(K)}_m(\lambda) \equiv \mathrm{tr}_{1,\dots,m}\left[P^{-}_{1,\dots,m}M^{(K)}_1(\lambda)M^{(K)}_2(\lambda-\eta)\cdots M^{(K)}_m(\lambda-(m-1)\eta)\right], \quad \forall m\in\{1,2,3\}, \tag{2.9}$$

are one parameter families of mutual commuting operators. Furthermore, the quantum determinant $\text{q-det}\,M^{(K)}(\lambda)\equiv T^{(K)}_3(\lambda)$ *is central, i.e.*

$$\left[\text{q-det}\,M^{(K)}(\lambda), M^{(K)}_a(\mu)\right] = 0. \tag{2.10}$$

Moreover, the general fusion identities [87–89] imply the following

Proposition 2.2 ([87–89]). *The quantum determinant has the following explicit form:*

$$\text{q-det}\,M^{(K)}(\lambda) = \det K\prod_{b=1}^{\mathsf{N}}\left[(\lambda-\xi_b+\eta)\prod_{m=1}^{2}(\lambda-\xi_b-m\eta)\right], \tag{2.11}$$

and $T^{(K)}_1(\lambda)$ and $T^{(K)}_2(\lambda)$ are degree N *and* $2\mathsf{N}$ *in λ. Their asymptotics are central and coincides with the corresponding two spectral invariants of the matrix K:*

$$T^{(K,\infty)}_1 \equiv \lim_{\lambda\to\infty}\lambda^{-\mathsf{N}}T^{(K)}_1(\lambda) = \mathrm{tr}\,K, \quad T^{(K,\infty)}_2 \equiv \lim_{\lambda\to\infty}\lambda^{-2\mathsf{N}}T^{(K)}_2(\lambda) = \frac{(\mathrm{tr}\,K)^2-\mathrm{tr}\,K^2}{2}. \tag{2.12}$$

The fusion identities hold:

$$T^{(K)}_1(\xi_a)T^{(K)}_m(\xi_a-\eta) = T^{(K)}_{m+1}(\xi_a), \quad \forall m\in\{1,2\}, \tag{2.13}$$

and $T^{(K)}_2(\lambda)$ has the following N *central zeroes*

$$T^{(K)}_2(\xi_a+\eta) = 0. \tag{2.14}$$

Let us introduce the functions

$$g^{(m)}_{a,\underline{\mathbf{h}}}(\lambda) = \prod_{b\neq a,b=1}^{\mathsf{N}}\frac{\lambda-\xi^{(h_b)}_b}{\xi^{(h_a)}_a-\xi^{(h_b)}_b}\prod_{b=1}^{(m-1)\mathsf{N}}\frac{1}{\xi^{(h_a)}_a-\xi^{(-1)}_b}, \tag{2.15}$$

$$a(\lambda-\eta) = d(\lambda) = \prod_{a=1}^{\mathsf{N}}(\lambda-\xi_a), \quad \xi^{(h)}_b = \xi_b-h\eta, \quad \underline{\mathbf{h}} = \{h_1,\dots,h_{\mathsf{N}}\}, \tag{2.16}$$

and

$$T^{(K,\infty)}_{m,\underline{\mathbf{h}}}(\lambda) = T^{(K,\infty)}_m\prod_{b=1}^{\mathsf{N}}(\lambda-\xi^{(h_n)}_b). \tag{2.17}$$

The known central zeroes and asymptotic behavior imply that the transfer matrix $T^{(K)}_2(\lambda)$ is completely characterized in terms of $T^{(K)}_1(\lambda)$, e.g. by the following interpolation formula

$$T^{(K)}_2(\lambda) = d(\lambda-\eta)\left(T^{(K,\infty)}_{2,\underline{\mathbf{h}}=\underline{\mathbf{0}}}(\lambda)+\sum_{a=1}^{\mathsf{N}}g^{(2)}_{a,\underline{\mathbf{h}}=\underline{\mathbf{0}}}(\lambda)T^{(K)}_1(\xi_a-\eta)T^{(K)}_1(\xi_a)\right), \tag{2.18}$$

where $\underline{\mathbf{h}}=\underline{\mathbf{0}}$ means that for all $k\in\{1,\dots,\mathsf{N}\}$ we have $h_k=0$.

From now on when we have an $\underline{\mathbf{h}}$ with all the elements equal to the integer 0, 1 or 2 we use directly the bold underlined notation $\underline{\mathbf{0}}$, $\underline{\mathbf{1}}$ and $\underline{\mathbf{2}}$.

2.2 On SoV bases construction in our approach

The general Proposition 2.4 of [1] for the construction of the SoV co-vector basis applies in particular to the fundamental representation of the gl_3 rational Yang-Baxter algebra. Note that we have presented the construction for the co-vector SoV basis just to get a factorized form of the wave-functions of the transfer matrix eigenvectors in terms of the transfer matrix eigenvalues. Evidently, the same construction applies as well to define a vector SoV basis in which the wave-functions of the transfer matrix eigenco-vectors have the same factorized form. In order to clarify this, we present in the following a proposition for this gl_3 case. Let K be a 3×3 simple spectrum matrix and let us denote with K_J the Jordan form of the matrix K and W_K the invertible matrix defining the change of basis:

$$K = W_K K_J W_K^{-1} \text{ with } K_J = \begin{pmatrix} \mathsf{k}_0 & \mathsf{y}_1 & 0 \\ 0 & \mathsf{k}_1 & \mathsf{y}_2 \\ 0 & 0 & \mathsf{k}_2 \end{pmatrix}. \tag{2.19}$$

The requirement K simple spectrum implies that we can reduce ourselves to the following three possible cases:

$$i)\ \mathsf{k}_i \neq \mathsf{k}_j,\ \forall i,j\in\{0,1,2\} \text{ and } \mathsf{y}_1=\mathsf{y}_2=0, \tag{2.20}$$
$$ii)\ \mathsf{k}_0=\mathsf{k}_1\neq\mathsf{k}_2,\ \mathsf{y}_1=1,\ \mathsf{y}_2=0, \tag{2.21}$$
$$ii)\ \mathsf{k}_0=\mathsf{k}_1=\mathsf{k}_2,\ \mathsf{y}_1=1,\ \mathsf{y}_2=1. \tag{2.22}$$

Then,

Proposition 2.3. *Let K be a 3×3 simple spectrum matrix, then for almost any choice of $\langle L|$, $|R\rangle$ and of the inhomogeneities under the condition (2.6), the following set of co-vectors and vectors:*

$$\langle L|\prod_{n=1}^{\mathsf{N}}\left(T_1^{(K)}(\xi_n)\right)^{h_n} \text{ for any } \{h_1,\dots,h_{\mathsf{N}}\}\in\{0,1,2\}^{\mathsf{N}}, \tag{2.23}$$
$$\prod_{n=1}^{\mathsf{N}}\left(T_1^{(K)}(\xi_n)\right)^{h_n}|R\rangle \text{ for any } \{h_1,\dots,h_{\mathsf{N}}\}\in\{0,1,2\}^{\mathsf{N}}, \tag{2.24}$$

forms a co-vector and vector basis of $\mathcal{H}$, respectively. In particular, we can take the following tensor product forms:

$$\langle L| = \bigotimes_{a=1}^{\mathsf{N}}(x,y,z)_a\Gamma_W^{-1},\quad |R\rangle = \bigotimes_{a=1}^{\mathsf{N}}\Gamma_W(r,s,t)_a^{t_a},\quad \Gamma_W=\bigotimes_{a=1}^{\mathsf{N}}W_{K,a} \tag{2.25}$$

simply asking in the case i) $x\,y\,z\neq 0$ for the co-vector and $r\,s\,t\neq 0$ for the vector, in the case ii) $x\,z\neq 0$ for the co-vector and $s\,t\neq 0$ for the vector, in the case iii) $x\neq 0$ for the co-vector and $t\neq 0$ for the vector.

Proof. As shown in the general Proposition 2.4 of [1], the fact that the transfer matrix in the inhomogeneity ξ_n reduces to the twist matrix in the local space n dressed by invertible products of R-matrices implies that the set of co-vectors and vectors above defined form bases of $\mathcal{H}^*$ and $\mathcal{H}$, once the following co-vectors and vectors (obtained by taking the asymptotic limit over the ξ_a)

$$(x,y,z)W_K^{-1},(x,y,z)W_K^{-1}K,(x,y,z)W_K^{-1}K^2, \tag{2.26}$$
$$W_K(r,s,t)^t,KW_K(r,s,t)^t,K^2W_K(r,s,t)^t, \tag{2.27}$$

or equivalently:

$$(x,y,z),(x,y,z)K_J,(x,y,z)K_J^2, \tag{2.28}$$
$$(r,s,t)^t,K_J(r,s,t)^t,K_J^2(r,s,t)^t, \tag{2.29}$$

form bases in $\mathbb{C}^3$, that is the next determinants are non-zero[8]:

$$\det\left((x,y,z)K_J^{i-1}e_j\right)_{i,j\in\{1,2,3\}} = \begin{cases} -xyzV(\mathsf{k}_0,\mathsf{k}_1,\mathsf{k}_2) & \text{in the case i)} \\ x^2zV^2(\mathsf{k}_0,\mathsf{k}_2) & \text{in the case ii)} \\ x^3 & \text{in the case iii)} \end{cases}, \tag{2.30}$$

$$\det\left(e_j^tK_J^{i-1}(r,s,t)\right)_{i,j\in\{1,2,3\}} = \begin{cases} rstV(\mathsf{k}_0,\mathsf{k}_1,\mathsf{k}_2) & \text{in the case i)} \\ s^2tV^2(\mathsf{k}_0,\mathsf{k}_2) & \text{in the case ii)} \\ t^3 & \text{in the case iii)} \end{cases}, \tag{2.31}$$

which leads to the given requirements on the components $x,y,z,r,s,t\in\mathbb{C}$ of the three dimensional co-vector and vectors. □

Note that both these choices of co-vector and vector SoV bases are perfectly fine to fix the transfer matrix spectrum, by factorized wave functions in terms of transfer matrix eigenvalues for both eigenvectors and eigenco-vectors. However, if we wish to go beyond the spectrum, and compute matrix elements of local operators starting with scalar products of the so-called separate states, we need an appropriate choice of the co-vector and vector SoV bases. In the rank one quantum integrable models, the SoV analysis so far developed [2–6, 23–54] leads to the expectation that the transfer matrix construction of the co-vector and vector SoV bases can be defined in such a way that these are orthogonal bases or similarly that the co-vector and vector Sklyanin's B-operator eigenbases both implement the separation of variables for the transfer matrix spectrum. As we will see in the next, in the higher rank quantum integrable models, this is not directly the case if the charges used to construct the co-vector and vector SoV basis are simple powers, or even fusion, of the transfer matrices for general twist K.

3 Scalar products for co-vector/vector SoV bases

3.1 Another construction of co-vector/vector SoV bases

Let us first introduce a slight modification of the co-vector SoV basis w.r.t. the standard one introduced in the previous section by changing the set of conserved charges used to construct them. It reads[9]:

$$\langle\underline{\mathbf{h}}|\equiv\langle h_1,...,h_{\mathsf{N}}|=\langle\underline{\mathbf{1}}|\prod_{n=1}^{\mathsf{N}}T_2^{(K)\delta_{h_n,0}}(\xi_n^{(1)})T_1^{(K)\delta_{h_n,2}}(\xi_n),\quad\forall\ h_n\in\{0,1,2\}, \tag{3.1}$$

where $\langle\underline{\mathbf{1}}|$ is some generic co-vector of $\mathcal{H}$. Let us remark that for an invertible twist matrix K using the identification:

$$\langle\underline{\mathbf{1}}|=\langle L|\prod_{n=1}^{\mathsf{N}}T_1^{(K)}(\xi_n)\,, \tag{3.2}$$

the two sets of co-vectors defined in (2.23) and (3.1) are identical up to a non-zero normalization of each co-vector; hence the two sets are related by the action of a diagonal matrix. To

[8] Here and in the following, we denote by $V(x_1,\ldots,x_n)$ the standard Vandermonde determinant $\prod_{i<j}(x_j-x_i)$.

[9] Throughout this section we use compact notations for the left and right SoV bases defined as in (3.1) and (3.7).

be more precise, with such an identification and using the fact that for an invertible K-matrix the operator $T_2^{(K)}(\xi_n^{(1)})$ is proportional to the inverse of $T_1^{(K)}(\xi_n)$ due to the fusion relations, we get:

$$\langle \underline{\mathbf{h}}| = \alpha_{\underline{\mathbf{h}}} \langle L| \prod_{n=1}^{\mathrm{N}} T_1^{(K)\delta_{h_n,2}-\delta_{h_n,0}+1}(\xi_n), \quad \forall h_n \in \{0,1,2\}, \tag{3.3}$$

where $\alpha_{\underline{\mathbf{h}}} = \prod_{n=1}^{\mathrm{N}} (\text{q-det}\, M^{(K)}(\xi_n))^{\delta_{h_n,0}}$ is a non-zero coefficient. Then, being $\delta_{h_n,2}-\delta_{h_n,0}+1 = h_n$ for any $h_n \in \{0,1,2\}$, we get:

$$\langle \underline{\mathbf{h}}| = \alpha_{\underline{\mathbf{h}}} \langle L| \prod_{n=1}^{\mathrm{N}} T_1^{(K)h_n}(\xi_n) \,, \forall\, h_n \in \{0,1,2\}, \tag{3.4}$$

thus proving that the two sets defined in (2.23) and in (3.1) are equivalent bases up to an invertible diagonal matrix made of the non-zero coefficients $\alpha_{\underline{\mathbf{h}}}$. Moreover, even if K has zero determinant, it can be proven that the two sets (3.1) and (2.23) are both SoV bases (see next section), the linear transformation relating them being in that case more involved.

3.2 Pseudo-orthogonality conditions of these co-vector/vector SoV bases

Here, we show that for the SoV co-vector basis chosen as in (3.1), we can define a *pseudo-orthogonal* vector SoV basis which is orthogonal to the left one for a large set of co-vector/vector couples. We exactly characterize these *pseudo-orthogonality* conditions and the non-zero couplings of these co-vector and vector SoV basis. The corresponding *SoV-measure*, related to the inverse of the scalar products matrix, is completely characterized in the next subsection. It is the starting ingredient to compute matrix elements of local operators in this SoV framework. This will be further employed in forthcoming analysis in this gl_3 case as, despite the absence of direct orthogonality, the *SoV-measure* stays reasonably simple to be used in practical computations.

Let us now introduced the vector $|\underline{\mathbf{0}}\rangle$ uniquely characterized by

$$\langle \underline{\mathbf{k}}|\underline{\mathbf{0}}\rangle = \prod_{a=1}^{\mathrm{N}} \delta_{0,k_a}. \tag{3.5}$$

Then we have the following

Proposition 3.1. *Let K be a 3×3 simple spectrum matrix, then for almost any choice of the co-vector $\langle \underline{\mathbf{1}}|$, of the vector $|\underline{\mathbf{0}}\rangle$ and of the inhomogeneities under the condition (2.6), the set of co-vectors (3.1)*

$$\langle \underline{\mathbf{h}}| = \langle \underline{\mathbf{1}}| \prod_{n=1}^{\mathrm{N}} T_2^{(K)\delta_{h_n,0}}(\xi_n^{(1)}) T_1^{(K)\delta_{h_n,2}}(\xi_n), \tag{3.6}$$

and the set of vectors:

$$|\underline{\mathbf{h}}\rangle \equiv \prod_{n=1}^{\mathrm{N}} T_2^{(K)\delta_{h_n,1}}(\xi_n) T_1^{(K)\delta_{h_n,2}}(\xi_n)|\underline{\mathbf{0}}\rangle, \tag{3.7}$$

form co-vector and vector basis of $\mathcal{H}^*$ and $\mathcal{H}$, respectively. In particular, we can take $\langle \underline{\mathbf{1}}|$ of the following tensor product form:

$$\langle \underline{\mathbf{1}}| = \bigotimes_{a=1}^{\mathrm{N}} (x,y,z)_a \Gamma_W^{-1}, \quad \Gamma_W = \bigotimes_{a=1}^{\mathrm{N}} W_{K,a}, \tag{3.8}$$

simply asking $x\,y\,z \neq 0$ in the case i), $x\,z \neq 0$ in the case ii), $x \neq 0$ in the case iii). Then the associated vector $|\underline{\mathbf{0}}\rangle$ having the property (3.5) also has tensor product form:

$$|\underline{\mathbf{0}}\rangle = \Gamma_W \bigotimes_{a=1}^{\mathsf{N}} |0,a\rangle, \tag{3.9}$$

where we have defined

$$|0,a\rangle = \frac{1}{\Delta}\begin{pmatrix} \mathsf{k}_2(y\mathsf{k}_0 - x\mathsf{y}_1)(z\mathsf{k}_2 + y\mathsf{y}_2) - (y\mathsf{k}_1 + x\mathsf{y}_1)(x\mathsf{y}_1\mathsf{y}_2 + \mathsf{k}_0(z\mathsf{k}_1 - y\mathsf{y}_2)) \\ x(x\mathsf{k}_0\mathsf{y}_1\mathsf{y}_2 + \mathsf{k}_0^2(z\mathsf{k}_1 - y\mathsf{y}_2) - \mathsf{k}_1\mathsf{k}_2(z\mathsf{k}_2 + y\mathsf{y}_2)) \\ x(\mathsf{k}_0 + \mathsf{k}_1)\mathsf{k}_2(y\mathsf{k}_1 + x\mathsf{y}_1 - y\mathsf{k}_0) \end{pmatrix}_a, \tag{3.10}$$

with

$$\Delta = x\Big(y\mathsf{k}_0 - y\mathsf{k}_1 - x\mathsf{y}_1\Big)\Big(z(\mathsf{k}_0 - \mathsf{k}_2)(\mathsf{k}_1 - \mathsf{k}_2) + \mathsf{y}_2\Big(y(\mathsf{k}_2 - \mathsf{k}_1) + x\mathsf{y}_1\Big)\Big)\,\mathrm{q\text{-}det}\, M^{(I)}(\xi_a - 2\eta). \tag{3.11}$$

Proof. The proof that these two sets are indeed bases of the Hilbert space and its dual can be performed along the same lines as the one presented already in [1] and in the previous section. Namely, using the polynomial character of all the expressions involved in the inhomogeneity parameters ξ_n it is enough to prove the proposition in some point in the parameter space. This is achieved by scaling the inhomogeneity parameters from a single scalar, as $\xi_n = n\xi$, and sending the parameter ξ to infinity. In turn, this amounts to obtain the asymptotic behavior of the transfer matrices in that limit. The leading term for the operator $T_1^{(K)}(\xi_n)$ is given by $\xi^{\mathsf{N}-1}K_n$ times some constant, while for the operator $T_2^{(K)}(\xi_n^{(1)})$ it is given by $\xi^{2(\mathsf{N}-1)}(2K_n^2 - 2K_n\,\mathrm{tr}(K) + \mathrm{tr}(K)^2 - \mathrm{tr}(K^2))$ times some other constant. Hence, it is enough to exhibit a co-vector $\langle u|$ such that the set $\langle u|$, $\langle u|K$, $\langle u|K^2$ is a basis of $\mathbb{C}^3$, which is the case as soon as K has simple spectrum. Similarly, the asymptotic of the operator $T_2^{(K)}(\xi_n^{(0)})$ is found proportional to the matrix $\xi^{2(\mathsf{N}-1)}(K_n^2 - K_n\,\mathrm{tr}(K))$, leading to the same conclusion. By these arguments, all we need to prove is that the co-vectors

$$(x,y,z)\tilde{K}_J, (x,y,z), (x,y,z)K_J, \tag{3.12}$$

where $\tilde{K}_J$ is the adjoint matrix of K_J, form a tridimensional basis. If we denote by M_{x,y,z,K_J} the 3×3 matrix which lines are given by these three co-vectors, it holds:

$$\det M_{x,y,z,K_J} = \begin{cases} -x\,y\,z\,V(\mathsf{k}_0,\mathsf{k}_1,\mathsf{k}_2) & \text{in the case i)} \\ x^2 z V^2(\mathsf{k}_0,\mathsf{k}_2) & \text{in the case ii)} \\ x^3 & \text{in the case iii),} \end{cases} \tag{3.13}$$

so that in the case i) we take $x\,y\,z \neq 0$, in the case ii) we take $x\,z \neq 0$ and finally in the case iii) the condition is $x \neq 0$. The construction of the orthogonal vector is a standard computation in $\mathbb{C}^3$ and the fact that it defines a vector basis by action of K and K^2 follows from a direct computation. Another proof uses the characteristic equation of K. Finally, the fact that the reference vector for the right SoV basis can be then chosen of tensor product form is proven in the appendix A. □

Let us now compute the scalar products of these two SoV bases as follows:

Theorem 3.1. *Let all the notations be the same as in Proposition 3.1, then the following pseudo-orthogonality relations hold:*

$$\mathcal{N}_{\underline{\mathbf{h}},\underline{\mathbf{k}}} = \langle \underline{\mathbf{h}}|\underline{\mathbf{k}}\rangle = \langle \underline{\mathbf{k}}|\underline{\mathbf{k}}\rangle \left(\delta_{\underline{\mathbf{h}},\underline{\mathbf{k}}} + C_{\underline{\mathbf{h}}}^{\underline{\mathbf{k}}} \sum_{r=1}^{n_{\underline{\mathbf{k}}}} (\det K)^r \sum_{\substack{\alpha\cup\beta\cup\gamma = \mathbf{1}_{\underline{\mathbf{k}}}, \\ \alpha,\beta,\gamma \text{ disjoint}, \#\alpha = \#\beta = r}} \delta_{\underline{\mathbf{h}},\underline{\mathbf{k}}_{\alpha,\beta}^{(0,2)}} \right), \tag{3.14}$$

where the $C^{\underline{\mathbf{k}}}_{\underline{\mathbf{h}}}$ *are non-zero and independent w.r.t.* $\det K$, $n_{\underline{\mathbf{k}}}$ *is the integer part of* $(\sum_{a=1}^{\mathsf{N}} \delta_{k_a,1})/2$. *We have used the further notations*

$$\underline{\mathbf{k}}^{(\underline{0},\underline{2})}_{\alpha,\beta} \equiv (k_1(\alpha,\beta),...,k_{\mathsf{N}}(\alpha,\beta)) \in \{0,1,2\}^{\mathsf{N}}, \tag{3.15}$$

$$\mathbf{1}_{\underline{\mathbf{k}}} \equiv \{a \in \{1,...,\mathsf{N}\} : k_a = 1\}, \tag{3.16}$$

with

$$k_a(\alpha,\beta) = 0, \quad k_b(\alpha,\beta) = 2, \quad \forall a \in \alpha, b \in \beta \tag{3.17}$$

$$k_c(\alpha,\beta) = k_c, \quad \forall c \in \{1,...,\mathsf{N}\}\backslash\{\alpha \cup \beta\}. \tag{3.18}$$

Moreover, we prove that it holds:

$$\mathsf{N}_{\underline{\mathbf{h}}} = \langle \underline{\mathbf{h}} | \underline{\mathbf{h}} \rangle \tag{3.19}$$

$$= \left(\prod_{a=1}^{\mathsf{N}} \frac{d\left(\xi_a^{(1)}\right)}{d\left(\xi_a^{(1+\delta_{h_a,1}+\delta_{h_a,2})}\right)} \right) \frac{V^2(\xi_1,...,\xi_{\mathsf{N}})}{V\left(\xi_1^{(\delta_{h_1,2}+\delta_{h_1,1})},...,\xi_{\mathsf{N}}^{(\delta_{h_{\mathsf{N}},1}+\delta_{h_{\mathsf{N}},2})}\right) V\left(\xi_1^{(\delta_{h_1,2})},...,\xi_{\mathsf{N}}^{(\delta_{h_{\mathsf{N}},2})}\right)}. \tag{3.20}$$

Proof. The heavy proofs of the pseudo-orthogonality and of the expressions of non-zero SoV co-vector/vector couplings are given in Appendix C. There, the coefficients $C^{\underline{\mathbf{k}}}_{\underline{\mathbf{h}}}$ are characterized completely, but implicitly, by an unwieldy recursion that we do not solve for the generic case. We compute them in the simplest case, see (C.73). □

It is worth to make some remarks on the above theorem. Let us first comment that the sum in (3.14), for any fixed $\underline{\mathbf{k}}$ and $\underline{\mathbf{h}}$, always reduces to at most one single non-zero term. Indeed, fixing $\underline{\mathbf{k}} \neq \underline{\mathbf{h}}$, we can have a non-zero coupling between the vector and co-vector associated if and only if there exists a couple of sets $(\alpha,\beta) \subset \mathbf{1}_{\underline{\mathbf{k}}}$ with the same cardinality $r \leq n_{\underline{\mathbf{k}}}$ such that $\underline{\mathbf{h}} = \underline{\mathbf{k}}^{(\underline{0},\underline{2})}_{\alpha,\beta}$, and of course if the couple (α,β) exists it is unique. The above condition means that if $\sum_{a=1}^{\mathsf{N}} \delta_{k_a,1}$ is smaller or equal to one, then the standard orthogonality works, i.e. only $\underline{\mathbf{h}} = \underline{\mathbf{k}}$ produces a non-zero co-vector/vector coupling. While if $\sum_{a=1}^{\mathsf{N}} \delta_{k_a,1}$ is bigger or equal to two, we have non-zero couplings also for all the co-vectors of (3.1) with[10] $\underline{\mathbf{h}} = \underline{\mathbf{k}}^{(\underline{0},\underline{2})}_{\alpha,\beta}$. Let us remark that if one looks to this pseudo-orthogonality condition in one quantum site, then the basis (3.7) naturally emerges as the candidate to get the orthogonal basis to (3.1). Indeed, for one site, orthogonality is satisfied by them while the fact that the orthogonality is not satisfied for higher number of quantum sites is intrinsically related to the form of fusion relations of the transfer matrices for higher rank. From these considerations follows our statement that if we want to obtain mutually orthogonal co-vector/vector SoV bases, we have to use different[11] families of commuting conserved charges to generate the co-vector and the vector SoV bases.

It is also useful to make some link with the preexisting work [76] in the SoV framework. In fact, the set of vectors (3.7) has been introduced recently in [76] as the set of eigenvectors of a C-operator, which plays a similar role to the Sklyanin's B-operator. There, the starting vector, analogous to our $|\underline{\mathbf{0}}\rangle$, is taken as some not better defined eigenvector of this C-operator, and the proofs that C is diagonalizable and that so (3.7) form a basis are not addressed, while the co-vector/vector coupling of these SoV bases is represented with some integral form.

In our paper, we prove that (3.7) is a basis, we fix the tensor product form of the starting vector $|\underline{\mathbf{0}}\rangle$in terms of the starting co-vector $\langle \underline{\mathbf{1}}|$ and the general twist matrix K, we characterize

[10] That is for the $\underline{\mathbf{h}}$ obtained from $\underline{\mathbf{k}}$ removing one or more couples of $(k_a = 1, k_b = 1)$ and substituting them with $(k_a(\alpha,\beta) = 0, k_b(\alpha,\beta) = 2)$.

[11] w.r.t. those used above.

completely the form of the co-vector/vector couplings of the two SoV bases and from them the *SoV-measure*.

Let us also remark that in [76] is given a selection rule which selects sectors of the quantum space which are orthogonal, which translates in our setting as

$$\langle \underline{\mathbf{h}}|\underline{\mathbf{k}}\rangle = 0 \ \text{ if } \sum_{a=1}^{\mathsf{N}} \delta_{h_a,1} \neq \sum_{a=1}^{\mathsf{N}} \delta_{k_a,1}. \tag{3.21}$$

This is compatible with our result (3.14), but much less restrictive as one can easily understand by looking, for example, to our formula for $r = 1$. In this case, the $\underline{\mathbf{h}}$ fixing the co-vector in (3.7) and $\underline{\mathbf{k}}$ fixing the vector in (3.1) differ only on one couple of index $(h_a, h_b) \neq (k_a, k_b)$. The above selection rule only imposes that $\langle \underline{\mathbf{h}}|\underline{\mathbf{k}}\rangle = 0$ if $h_a + h_b \neq k_a + k_b$, while our formula instead specifies that $\langle \underline{\mathbf{h}}|\underline{\mathbf{k}}\rangle = 0$ unless $k_a = k_b = 1$ and $h_a + h_b = 2$.

3.3 On higher rank SoV measure

In the Theorem 3.1, we have shown that the original higher rank SoV co-vector and vector bases as defined in (3.7) and (3.1) are not mutual orthogonal basis if the twist matrix is invertible. Here, we want to show that from the Theorem 3.1, we can also characterize the SoV measure associated to these bases, i.e. the measure to be used in the computation of scalar products of separate states in these co-vector and vector bases.

Let us start introducing the following sets of co-vectors and vectors that are bases of the Hilbert space orthogonal to our left and right SoV bases:

$$ {}_p\langle \underline{\mathbf{h}}| \text{ and } |\underline{\mathbf{h}}\rangle_p, \quad \forall \underline{\mathbf{h}} \in \{0,1,2\}^{\mathsf{N}}, \tag{3.22}$$

uniquely characterized by the following orthogonality conditions[12]:

$$ {}_p\langle \underline{\mathbf{k}}|\underline{\mathbf{h}}\rangle = \delta_{\underline{\mathbf{k}},\underline{\mathbf{h}}} \langle \underline{\mathbf{h}}|\underline{\mathbf{h}}\rangle, \quad \langle \underline{\mathbf{k}}|\underline{\mathbf{h}}\rangle_p = \delta_{\underline{\mathbf{k}},\underline{\mathbf{h}}} \langle \underline{\mathbf{h}}|\underline{\mathbf{h}}\rangle, \quad \forall \underline{\mathbf{h}}, \underline{\mathbf{k}} \in \{0,1,2\}^{\mathsf{N}}, \tag{3.23}$$

where $|\underline{\mathbf{h}}\rangle$ and $\langle \underline{\mathbf{h}}|$ are the vectors and co-vectors of the SoV basis (3.7) and (3.1), respectively. Clearly, the set generated by the ${}_p\langle \underline{\mathbf{h}}|$ and $|\underline{\mathbf{h}}\rangle_p$ are bases of the Hilbert space, and moreover we have the following decompositions of the identity:

$$\mathbb{I} = \sum_{\underline{\mathbf{h}}} \frac{|\underline{\mathbf{h}}\rangle_p \ \langle \underline{\mathbf{h}}|}{\mathsf{N}_{\underline{\mathbf{h}}}} = \sum_{\underline{\mathbf{h}}} \frac{|\underline{\mathbf{h}}\rangle \ {}_p\langle \underline{\mathbf{h}}|}{\mathsf{N}_{\underline{\mathbf{h}}}}, \tag{3.24}$$

where the sums run over all the possible values of the multiple index $(\underline{\mathbf{h}})$. As a consequence, the transfer matrix eigenco-vectors and eigenvectors admit the following SoV representations in terms of their eigenvalues:

$$|t_a\rangle = \sum_{\underline{\mathbf{h}}} \prod_{n=1}^{\mathsf{N}} t_{2,a}^{\delta_{h_n,0}}(\xi_n^{(1)}) t_{1,a}^{\delta_{h_n,2}}(\xi_n) \ \frac{|\underline{\mathbf{h}}\rangle_p}{\mathsf{N}_{\underline{\mathbf{h}}}}, \tag{3.25}$$

$$\langle t_a| = \sum_{\underline{\mathbf{h}}} \prod_{n=1}^{\mathsf{N}} t_{2,a}^{\delta_{h_n,1}}(\xi_n) t_{1,a}^{\delta_{h_n,2}}(\xi_n) \ \frac{{}_p\langle \underline{\mathbf{h}}|}{\mathsf{N}_{\underline{\mathbf{h}}}}. \tag{3.26}$$

Thus, we can naturally give the following definitions of separate vectors :

$$|\alpha\rangle = \sum_{\underline{\mathbf{k}}} \alpha_{\underline{\mathbf{k}}} \ \frac{|\underline{\mathbf{k}}\rangle_p}{\mathsf{N}_{\underline{\mathbf{k}}}}, \quad \alpha_{\underline{\mathbf{h}}} \equiv \prod_{a=1}^{\mathsf{N}} \alpha_a^{(h_a)} \tag{3.27}$$

[12] Note here that we have included, for convenience, in the orthogonality relations the normalisation factor $\langle \underline{\mathbf{h}}|\underline{\mathbf{h}}\rangle$ as it leads to more natural identifications in the particular case where the right and left SoV bases are directly orthogonal to each other.

with factorized coordinates $\alpha_{\underline{\mathbf{h}}}$ and separate co-vectors,

$$\langle\beta| = \sum_{\underline{\mathbf{h}}} \beta_{\underline{\mathbf{h}}} \frac{{}_p\langle\underline{\mathbf{h}}|}{\mathsf{N}_{\underline{\mathbf{h}}}}, \quad \beta_{\underline{\mathbf{h}}} \equiv \prod_{a=1}^{\mathsf{N}} \beta_a^{(h_a)} \tag{3.28}$$

with factorized coordinates $\beta_{\underline{\mathbf{h}}}$ on the respective bases. The scalar product of such two separate vector and co-vector reads:

$$\langle\beta|\alpha\rangle = \sum_{\underline{\mathbf{h}},\underline{\mathbf{k}}} \beta_{\underline{\mathbf{h}}} \mathcal{M}_{\underline{\mathbf{h}},\underline{\mathbf{k}}} \alpha_{\underline{\mathbf{k}}} \tag{3.29}$$

with the *SoV measure* $\mathcal{M}_{\underline{\mathbf{h}},\underline{\mathbf{k}}}$ defined as:

$$\mathcal{M}_{\underline{\mathbf{h}},\underline{\mathbf{k}}} = \frac{{}_p\langle\underline{\mathbf{h}}|\underline{\mathbf{k}}\rangle_p}{\mathsf{N}_{\underline{\mathbf{h}}}.\mathsf{N}_{\underline{\mathbf{k}}}}. \tag{3.30}$$

It can be obtained from the knowledge of the scalar products between the vectors and co-vectors of the two bases orthogonal to our chosen SoV bases:

$${}_p\langle\underline{\mathbf{h}}|\underline{\mathbf{k}}\rangle_p, \quad \forall \underline{\mathbf{h}}, \underline{\mathbf{k}} \in \{0,1,2\}^{\mathsf{N}}. \tag{3.31}$$

Let us note here that these two matrices $\mathcal{N}_{\underline{\mathbf{h}},\underline{\mathbf{k}}}$ and $\mathcal{M}_{\underline{\mathbf{h}},\underline{\mathbf{k}}}$, modulo some normalisation, have direct interpretation as change of bases matrices between the bases $\langle\underline{\mathbf{h}}|$ and ${}_p\langle\underline{\mathbf{k}}|$, namely we have:

$$\langle\underline{\mathbf{h}}| = \sum_{\underline{\mathbf{k}}} \mathcal{N}_{\underline{\mathbf{h}},\underline{\mathbf{k}}} \frac{{}_p\langle\underline{\mathbf{k}}|}{\mathsf{N}_{\underline{\mathbf{k}}}}, \tag{3.32}$$

and conversely,

$$\frac{{}_p\langle\underline{\mathbf{k}}|}{\mathsf{N}_{\underline{\mathbf{k}}}} = \sum_{\underline{\mathbf{h}}} \mathcal{M}_{\underline{\mathbf{k}},\underline{\mathbf{h}}} \langle\underline{\mathbf{h}}|. \tag{3.33}$$

We also have similar relations (with transposition) for the bases $|\underline{\mathbf{k}}\rangle$ and $|\underline{\mathbf{h}}\rangle_p$. Moreover, it easy to verify that these two matrices are inverse to each other:

$$\sum_{\underline{\mathbf{h}}} \mathcal{M}_{\underline{\mathbf{k}},\underline{\mathbf{h}}} \cdot \mathcal{N}_{\underline{\mathbf{h}},\underline{\mathbf{l}}} = \delta_{\underline{\mathbf{k}},\underline{\mathbf{l}}}. \tag{3.34}$$

Hence to compute the SoV measure $\mathcal{M}_{\underline{\mathbf{k}},\underline{\mathbf{h}}}$, we just need to get the inverse of the matrix of scalar products $\mathcal{N}_{\underline{\mathbf{k}},\underline{\mathbf{h}}}$; in the following we show how to characterize it in terms of $\mathcal{N}_{\underline{\mathbf{k}},\underline{\mathbf{h}}}$, proving in particular that it has in fact the same form as $\mathcal{N}_{\underline{\mathbf{k}},\underline{\mathbf{h}}}$[13].

Let us start proving the following:

Lemma 3.1. *The vectors $|\underline{\mathbf{h}}\rangle_p$ of the basis orthogonal to the SoV co-vector basis (3.1) admit the following decompositions in the SoV vector basis (3.7):*

$$|\underline{\mathbf{h}}\rangle_p = |\underline{\mathbf{h}}\rangle + \sum_{r=1}^{n_{\underline{\mathbf{h}}}} \mathsf{c}^r \sum_{\substack{\alpha\cup\beta\cup\gamma=\mathbf{1}_{\underline{\mathbf{h}}},\\ \alpha,\beta,\gamma \text{ disjoint},\\ \#\alpha=\#\beta=r}} B_{\alpha,\beta,\underline{\mathbf{h}}} |\underline{\mathbf{h}}^{(\underline{\mathbf{0}},\underline{\mathbf{2}})}_{\alpha,\beta}\rangle, \tag{3.35}$$

[13] Somehow this is not surprising as in the appropriate labelling of the SoV bases, the matrix $\mathcal{N}_{\underline{\mathbf{k}},\underline{\mathbf{h}}}$ is lower-triangular with finite depth out of the diagonal and hence its inverse should have a similar form.

where the coefficients $B_{\alpha,\beta,\underline{\mathbf{h}}}$ are completely characterized by the following recursion formula:

$$B_{\alpha,\beta,\underline{\mathbf{h}}} = -\left(\bar{C}^{\underline{\mathbf{h}}}_{\underline{\mathbf{h}}^{(\underline{0},\underline{2})}_{\alpha,\beta}} + \sum_{\substack{\alpha'\subset\alpha,\ \beta'\subset\beta,\\ 1\leq\#\alpha'=\#\beta'\leq\#\alpha-1}} B_{\alpha',\beta',\underline{\mathbf{h}}}\, \bar{C}^{\underline{\mathbf{h}}^{(\underline{0},\underline{2})}_{\alpha',\beta'}}_{\underline{\mathbf{h}}^{(\underline{0},\underline{2})}_{\alpha,\beta}} \right), \tag{3.36}$$

and

$$\bar{C}^{\underline{\mathbf{r}}}_{\underline{\mathbf{s}}} = \frac{\langle \underline{\mathbf{r}}|\underline{\mathbf{r}}\rangle}{\langle \underline{\mathbf{s}}|\underline{\mathbf{s}}\rangle} C^{\underline{\mathbf{r}}}_{\underline{\mathbf{s}}}, \tag{3.37}$$

where $C^{\underline{\mathbf{r}}}_{\underline{\mathbf{s}}}$ are the coefficients of the measure (3.14).

Proof. The fact that we can write each vector $|\underline{\mathbf{h}}\rangle_p$, satisfying (3.23), in terms of the SoV vectors $|\underline{\mathbf{k}}\rangle$ follows from the fact that these last ones form a basis. Here we have to prove that the above expression for $|\underline{\mathbf{h}}\rangle_p$ and for its coefficients indeed imply the orthogonality condition (3.23).

Let us start observing that this is the case for the diagonal term. Indeed, the following identity follows:

$$\langle \underline{\mathbf{h}}|\underline{\mathbf{h}}\rangle_p = \langle \underline{\mathbf{h}}|\underline{\mathbf{h}}\rangle, \tag{3.38}$$

by the measure (3.14) being

$$\underline{\mathbf{h}} \underset{(C.45)}{\neq} \underline{\mathbf{h}}^{(\underline{0},\underline{2})}_{\alpha,\beta}, \quad \forall \alpha,\beta \subset \mathbf{1}_{\underline{\mathbf{h}}},\ \text{disjoint with } 1 \leq \#\alpha = \#\beta \leq n_{\underline{\mathbf{h}}}. \tag{3.39}$$

So we are left with the proof of the orthogonality of

$$\langle \underline{\mathbf{k}}|\underline{\mathbf{h}}\rangle_p = 0, \quad \forall \underline{\mathbf{k}} \neq \underline{\mathbf{h}}, \underline{\mathbf{k}} \in \{0,1,2\}^{\mathsf{N}}. \tag{3.40}$$

Let us start observing that for any $\underline{\mathbf{k}}$ such that

$$\underline{\mathbf{k}} \underset{(C.45)}{\neq} \underline{\mathbf{h}}, \tag{3.41}$$

then it also follows

$$\underline{\mathbf{k}} \underset{(C.45)}{\neq} \underline{\mathbf{h}}^{(\underline{0},\underline{2})}_{\alpha,\beta}, \quad \forall \alpha,\beta \subset \mathbf{1}_{\underline{\mathbf{h}}},\ \text{disjoint with } 1 \leq \#\alpha = \#\beta \leq n_{\underline{\mathbf{h}}}, \tag{3.42}$$

and so by the measure (3.14) the orthogonality holds.

So, we are left with the proof of the orthogonality for the case $\underline{\mathbf{k}} = \underline{\mathbf{h}}^{(\underline{0},\underline{2})}_{\mu,\delta}$ for any fixed disjoint sets $\mu \subset \mathbf{1}_{\underline{\mathbf{h}}}$ and $\delta \subset \mathbf{1}_{\underline{\mathbf{h}}}$ such that $1 \leq \#\mu = \#\delta \leq n_{\underline{\mathbf{h}}}$. Let us observe that the following inequalities holds:

$$\underline{\mathbf{h}}^{(\underline{0},\underline{2})}_{\alpha,\beta} \underset{(C.45)}{\neq} \underline{\mathbf{h}}^{(\underline{0},\underline{2})}_{\mu,\delta}, \tag{3.43}$$

for any disjoint sets α and β contained in $\mathbf{1}_{\underline{\mathbf{h}}}$ with $\#\alpha = \#\beta$ such that

$$\alpha \not\subseteq \mu \text{ and } \beta \not\subseteq \delta. \tag{3.44}$$

Then, by the measure (3.14), we get the following co-vector/vector coupling:

$$\langle \underline{\mathbf{h}}^{(\underline{0},\underline{2})}_{\mu,\delta}|\underline{\mathbf{h}}\rangle_p = \langle \underline{\mathbf{h}}^{(\underline{0},\underline{2})}_{\mu,\delta}|\underline{\mathbf{h}}\rangle + \sum_{r=1}^{n_{\underline{\mathbf{h}}}} \mathsf{c}^r \sum_{\substack{\alpha\cup\beta\cup\gamma=\mathbf{1}_{\underline{\mathbf{h}}},\\ \alpha,\beta,\gamma \text{ disjoint}, \#\alpha=\#\beta=r}} B_{\alpha,\beta,\underline{\mathbf{h}}} \langle \underline{\mathbf{h}}^{(\underline{0},\underline{2})}_{\mu,\delta}|\underline{\mathbf{h}}^{(\underline{0},\underline{2})}_{\alpha,\beta}\rangle \tag{3.45}$$

$$= \langle \underline{\mathbf{h}}^{(\underline{0},\underline{2})}_{\mu,\delta}|\underline{\mathbf{h}}\rangle + \mathsf{c}^{\#\delta} B_{\mu,\delta,\underline{\mathbf{h}}} \langle \underline{\mathbf{h}}^{(\underline{0},\underline{2})}_{\mu,\delta}|\underline{\mathbf{h}}^{(\underline{0},\underline{2})}_{\mu,\delta}\rangle + \sum_{r=1}^{\#\delta-1} \mathsf{c}^r \sum_{\substack{\alpha\subset\mu,\beta\subset\delta,\\ \#\alpha=\#\beta=r}} B_{\alpha,\beta,\underline{\mathbf{h}}} \langle \underline{\mathbf{h}}^{(\underline{0},\underline{2})}_{\mu,\delta}|\underline{\mathbf{h}}^{(\underline{0},\underline{2})}_{\alpha,\beta}\rangle, \tag{3.46}$$

that we impose to be zero to satisfy the orthogonality condition:

$$\langle \underline{\mathbf{h}}^{(\underline{0},\underline{2})}_{\mu,\delta}|\underline{\mathbf{h}}\rangle_p = 0, \quad \forall \mu \subset \mathbf{1}_{\underline{\mathbf{h}}}, \delta \subset \mathbf{1}_{\underline{\mathbf{h}}} \text{ disjoint with } 1 \leq \#\mu = \#\delta \leq n_{\underline{\mathbf{h}}}. \tag{3.47}$$

Here, the main observation is that this can be seen as one equation in one unknown $B_{\mu,\delta,\underline{\mathbf{h}}}$, and solved as it follows:

$$B_{\mu,\delta,\underline{\mathbf{h}}} = -\frac{\langle \underline{\mathbf{h}}^{(\underline{0},\underline{2})}_{\mu,\delta}|\underline{\mathbf{h}}\rangle \mathsf{c}^{-\#\delta}}{\langle \underline{\mathbf{h}}^{(\underline{0},\underline{2})}_{\mu,\delta}|\underline{\mathbf{h}}^{(\underline{0},\underline{2})}_{\mu,\delta}\rangle} - \sum_{r=1}^{\#\delta-1} \mathsf{c}^{r-\#\delta} \sum_{\substack{\alpha\subset\mu,\beta\subset\delta,\\ \#\alpha=\#\beta=r}} B_{\alpha,\beta,\underline{\mathbf{h}}} \frac{\langle \underline{\mathbf{h}}^{(\underline{0},\underline{2})}_{\mu,\delta}|\underline{\mathbf{h}}^{(\underline{0},\underline{2})}_{\alpha,\beta}\rangle}{\langle \underline{\mathbf{h}}^{(\underline{0},\underline{2})}_{\mu,\delta}|\underline{\mathbf{h}}^{(\underline{0},\underline{2})}_{\mu,\delta}\rangle}, \tag{3.48}$$

in terms of the known SoV co-vector/vector couplings and of the coefficients $B_{\alpha,\beta,\underline{\mathbf{h}}}$ for any $\alpha \subset \mu, \beta \subset \delta$, with $1 \leq \#\alpha = \#\beta \leq \#\delta - 1$. Then, by using the formulae (3.14) we get:

$$\frac{\langle \underline{\mathbf{h}}^{(\underline{0},\underline{2})}_{\mu,\delta}|\underline{\mathbf{h}}\rangle \mathsf{c}^{-\#\delta}}{\langle \underline{\mathbf{h}}^{(\underline{0},\underline{2})}_{\mu,\delta}|\underline{\mathbf{h}}^{(\underline{0},\underline{2})}_{\mu,\delta}\rangle} = \bar{C}^{\underline{\mathbf{h}}}_{\underline{\mathbf{h}}^{(\underline{0},\underline{2})}_{\mu,\delta}}, \quad \left.\frac{\langle \underline{\mathbf{h}}^{(\underline{0},\underline{2})}_{\mu,\delta}|\underline{\mathbf{h}}^{(\underline{0},\underline{2})}_{\alpha,\beta}\rangle \mathsf{c}^{r-\#\delta}}{\langle \underline{\mathbf{h}}^{(\underline{0},\underline{2})}_{\mu,\delta}|\underline{\mathbf{h}}^{(\underline{0},\underline{2})}_{\mu,\delta}\rangle}\right|_{\substack{\alpha\subset\mu,\beta\subset\delta,\\ \#\alpha=\#\beta=r}} = \bar{C}^{\underline{\mathbf{h}}^{(\underline{0},\underline{2})}_{\alpha,\beta}}_{\underline{\mathbf{h}}^{(\underline{0},\underline{2})}_{\mu,\delta}}, \tag{3.49}$$

from which our formula (3.36) easily follows.

Now, it is simple to argue that (3.36) gives us recursively all the coefficient $B_{\mu,\delta,\underline{\mathbf{h}}}$ for any $\mu \subset \mathbf{1}_{\underline{\mathbf{h}}}, \delta \subset \mathbf{1}_{\underline{\mathbf{h}}}$ disjoint, with $1 \leq \#\mu = \#\delta \leq n_{\underline{\mathbf{h}}}$.

In the case $\#\mu = \#\delta = 1$, the formula (3.36) reads:

$$B_{a,b,\underline{\mathbf{h}}} = -\bar{C}^{\underline{\mathbf{h}}}_{\underline{\mathbf{h}}^{(\underline{0},\underline{2})}_{a,b}}, \quad \forall a \neq b \in \mathbf{1}_{\underline{\mathbf{h}}}, \tag{3.50}$$

which fixes completely these coefficients. Then, we can consider the case of the generic couple of disjoint sets $\mu \subset \mathbf{1}_{\underline{\mathbf{h}}}, \delta \subset \mathbf{1}_{\underline{\mathbf{h}}}$, with $\#\mu = \#\delta = 2$. In these cases, we have that the formula (3.36) reads:

$$B_{\mu,\delta,\underline{\mathbf{h}}} = -\bar{C}^{\underline{\mathbf{h}}}_{\underline{\mathbf{h}}^{(\underline{0},\underline{2})}_{\mu,\delta}} - \sum_{a\in\mu,b\in\delta} B_{a,b,\underline{\mathbf{h}}} \bar{C}^{\underline{\mathbf{h}}^{(\underline{0},\underline{2})}_{a,b}}_{\underline{\mathbf{h}}^{(\underline{0},\underline{2})}_{\mu,\delta}}, \tag{3.51}$$

which fixes completely these coefficients in terms of those computed in the first step of the recursion.

In this way the formula (3.36) fixes the coefficients $B_{\mu',\delta',\underline{\mathbf{h}}}$ for any fixed couple of disjoint sets $\mu' \subset \mathbf{1}_{\underline{\mathbf{h}}}, \delta' \subset \mathbf{1}_{\underline{\mathbf{h}}}$, with $\#\mu' = \#\delta' = m+1 \leq n_{\underline{\mathbf{h}}}$, in terms of those already computed, i.e. the $B_{\mu,\delta,\underline{\mathbf{h}}}$ for any fixed couple of disjoint sets $\mu \subset \mu' \subset \mathbf{1}_{\underline{\mathbf{h}}}, \delta \subset \delta' \subset \mathbf{1}_{\underline{\mathbf{h}}}$, with $\#\mu = \#\delta \leq m$. □

Let us note that the coefficients $B_{\alpha,\beta,\underline{\mathbf{h}}}$ are, as previously with the coefficients $C^{\underline{\mathbf{k}}}_{\underline{\mathbf{h}}}$, also characterized in a recursive manner, and their generic expression is missing from this Lemma.

The previous lemma implies the following corollary, which completely characterizes the SoV measure.

Corollary 3.1. *Under the same condition of Theorem 3.1, the SoV measure is defined by the following pseudo-orthogonality relations:*

$${}_p\langle \underline{\mathbf{h}}|\underline{\mathbf{k}}\rangle_p = \langle \underline{\mathbf{h}}|\underline{\mathbf{h}}\rangle \left(\delta_{\underline{\mathbf{h}},\underline{\mathbf{k}}} + \sum_{r=1}^{n_{\underline{\mathbf{k}}}} \mathsf{c}^r \sum_{\substack{\alpha\cup\beta\cup\gamma=\mathbf{1}_{\underline{\mathbf{k}}},\\ \alpha,\beta,\gamma \text{ disjoint, } \#\alpha=\#\beta=r}} B_{\alpha,\beta,\underline{\mathbf{k}}} \delta_{\underline{\mathbf{h}},\underline{\mathbf{k}}^{(\underline{0},\underline{2})}_{\alpha,\beta}} \right). \tag{3.52}$$

Proof. We have to use just the expression derived in the previous lemma for the generic vector

$$|\underline{\mathbf{k}}\rangle_p = |\underline{\mathbf{k}}\rangle + \sum_{r=1}^{n_{\underline{\mathbf{k}}}} \mathsf{c}^r \sum_{\substack{\alpha\cup\beta\cup\gamma=\mathbf{1}_{\underline{\mathbf{k}}},\\ \alpha,\beta,\gamma \text{ disjoint}, \#\alpha=\#\beta=r}} B_{\alpha,\beta,\underline{\mathbf{k}}} |\underline{\mathbf{k}}^{(\underline{0},\underline{2})}_{\alpha,\beta}\rangle, \tag{3.53}$$

and the definition of the co-vectors ${}_p\langle\underline{\mathbf{h}}|$ for which it holds:

$${}_p\langle\underline{\mathbf{h}}|\underline{\mathbf{k}}\rangle_p = {}_p\langle\underline{\mathbf{h}}|\underline{\mathbf{k}}\rangle + \sum_{r=1}^{n_{\underline{\mathbf{k}}}} \mathsf{c}^r \sum_{\substack{\alpha\cup\beta\cup\gamma=\mathbf{1}_{\underline{\mathbf{k}}},\\ \alpha,\beta,\gamma \text{ disjoint}, \#\alpha=\#\beta=r}} B_{\alpha,\beta,\underline{\mathbf{k}}}\ {}_p\langle\underline{\mathbf{h}}|\underline{\mathbf{k}}^{(\underline{0},\underline{2})}_{\alpha,\beta}\rangle \tag{3.54}$$

$$= {}_p\langle\underline{\mathbf{h}}|\underline{\mathbf{h}}\rangle \left(\delta_{\underline{\mathbf{h}},\underline{\mathbf{k}}} + \sum_{r=1}^{n_{\underline{\mathbf{k}}}} \mathsf{c}^r \sum_{\substack{\alpha\cup\beta\cup\gamma=\mathbf{1}_{\underline{\mathbf{k}}},\\ \alpha,\beta,\gamma \text{ disjoint}, \#\alpha=\#\beta=r}} B_{\alpha,\beta,\underline{\mathbf{k}}}\ \delta_{\underline{\mathbf{h}},\underline{\mathbf{k}}^{(\underline{0},\underline{2})}_{\alpha,\beta}} \right), \tag{3.55}$$

and being ${}_p\langle\underline{\mathbf{h}}|\underline{\mathbf{h}}\rangle = \langle\underline{\mathbf{h}}|\underline{\mathbf{h}}\rangle$, our result follows. □

4 On the construction of orthogonal co-vector/vector SoV bases

We would like now to introduce a new family of commuting conserved charges in order to construct from them orthogonal co-vector/vector SoV bases. We first describe our construction for the class of simple spectrum and non-invertible K-matrices. Then, from this class, we will define a new family of commuting conserved charges $\mathbb{T}(\lambda)$ which allows for the construction of the co-vector/vector orthogonal SoV bases for a generic simple spectrum K-matrix. The scalar product of separate states w.r.t. the charges $\mathbb{T}(\lambda)$, a class of co-vector/vector which contains the transfer matrix eigenstates, are computed and shown to have a form similar to those of the gl_2 case once one of the two states is a $\mathbb{T}(\lambda)$ eigenvector.

4.1 The case of non-invertible $\hat{K}$-matrices with simple spectrum

In the gl_3 case, the construction of a vector SoV basis orthogonal to the left one is not automatic, as it was in the gl_2 case. Here, it seems that the choice of the appropriate family of commuting conserved charges to construct the basis plays a fundamental role. In this subsection, we consider the special case of a simple spectrum $\hat{K}$-matrix with one zero eigenvalue. The orthogonal co-vector and vector SoV bases will be constructed using the transfer matrices as in the previous section.

Theorem 4.1. *Let $\hat{K}$ be a 3×3 simple spectrum matrix with one zero eigenvalue. For almost any choice of the co-vector $\langle\underline{\mathbf{1}}|$ and of the inhomogeneities under the condition (2.6), the set of co-vectors (3.1) and vectors (3.7) form SoV co-vector and SoV vector bases of $\mathcal{H}^*$ and $\mathcal{H}$, respectively. In particular, we can take $\langle\underline{\mathbf{1}}|$ of the tensor product form (3.8), then the associated vector $|\underline{\mathbf{0}}\rangle$ has the tensor product form defined in (3.9)-(3.10) and (3.1) and (3.7) are basis of $\mathcal{H}$ simply asking $x\,y\,z\neq 0$ in the case i), $x\,z\neq 0$ in the case ii), $x\neq 0$ in the case iii).*

Furthermore, (3.1) and (3.7) are mutually orthogonal SoV bases, i.e. they define the following decomposition of the identity:

$$\mathbb{I} \equiv \sum_{\underline{\mathbf{h}}} \frac{|\underline{\mathbf{h}}\rangle\langle\underline{\mathbf{h}}|}{\mathsf{N}_{\underline{\mathbf{h}}}}, \tag{4.1}$$

with

$$\mathsf{N}_{\underline{\mathbf{h}}} = \prod_{a=1}^{\mathsf{N}} \frac{d(\xi_a^{(1)})}{d\left(\xi_a^{(1+\delta_{h_a,1}+\delta_{h_a,2})}\right)} \frac{V^2(\xi_1,...,\xi_{\mathsf{N}})}{V\left(\xi_1^{(\delta_{h_1,2}+\delta_{h_1,1})},...,\xi_{\mathsf{N}}^{(\delta_{h_{\mathsf{N}},1}+\delta_{h_{\mathsf{N}},2})}\right)V\left(\xi_1^{(\delta_{h_1,2})},...,\xi_{\mathsf{N}}^{(\delta_{h_{\mathsf{N}},2})}\right)}. \tag{4.2}$$

Proof. This theorem follows immediately from the results of Theorem 3.1 putting to zero the determinant of the matrix K. □

However, the proof of our Theorem 3.1 is rather involved and takes quite numerous steps that we give in the appendices. It is therefore of interest to have a more elementary proof in the case at hand, namely whenever the simple spectrum twist matrix K has zero determinant or better to say as soon as the fusion relations for the transfer matrices simplify due to the vanishing of its associated quantum determinant. In fact, this case will provide the generic idea to get orthogonal left and right SoV bases in the general situation. So let us explain from now on a direct proof of this theorem.

Idea of the direct proof. The statement that (3.1) is a co-vector basis of $\mathcal{H}$ is proven as in the previous proposition. Indeed the main condition:

$$\det M_{x,y,z,\hat{K}_J} \neq 0 \tag{4.3}$$

can be satisfied as well in the case $\det \hat{K} = 0$. In fact, if the matrix $\hat{K}$ satisfies the case i), we take $\mathsf{k}_2 = 0$ and the condition is still $x\,y\,z \neq 0$; if the matrix $\hat{K}$ satisfies the case ii), we take $\mathsf{k}_0 = 0$ or $\mathsf{k}_2 = 0$ and the condition is still $x\,z \neq 0$. Finally in the case iii) with $\mathsf{k}_0 = \mathsf{k}_1 = \mathsf{k}_2 = 0$ the condition is still $x \neq 0$. So that we are left with the proof of the orthogonality conditions. which can be proven by using the next results. □

The first step in the direct proof of the above theorem is to obtain the SoV representations for the action of the transfer matrices in the case where the fusion relations simplify due to the vanishing of its associated quantum determinant. It is given in the following Proposition.

Proposition 4.1. *Under the same conditions of the above theorem, the following interpolation formulae hold for the transfer matrices:*

i) On the SoV co-vector basis:

$$\langle \underline{\mathbf{h}}|T_2^{(\hat{K})}(\lambda) = d\,(\lambda-\eta)\left(\sum_{a=1}^{\mathsf{N}} \delta_{h_a,1}\, g^{(2)}_{a,\underline{\mathbf{z}}(\underline{\mathbf{h}})}(\lambda)\langle \underline{\mathbf{h}}|\boldsymbol{T}_a^- + T^{(\hat{K},\infty)}_{2,\underline{\mathbf{z}}(\underline{\mathbf{h}})}(\lambda)\langle \underline{\mathbf{h}}|\right), \tag{4.4}$$

and

$$\begin{aligned}\langle \underline{\mathbf{h}}|T_1^{(\hat{K})}(\lambda) = T^{(\hat{K},\infty)}_{1,\underline{\mathbf{y}}(\underline{\mathbf{h}})}(\lambda)\langle \underline{\mathbf{h}}| + \sum_{a=1}^{\mathsf{N}} \delta_{h_a,1}\, g^{(1)}_{a,\underline{\mathbf{y}}(\underline{\mathbf{h}})}(\lambda)\langle \underline{\mathbf{h}}|\boldsymbol{T}_a^+ + \sum_{a=1}^{\mathsf{N}} \delta_{h_a,2}\, g^{(1)}_{a,\underline{\mathbf{y}}(\underline{\mathbf{h}})}(\lambda) d(\xi_a^{(1)}) \\ \times \left(\sum_{b=1}^{\mathsf{N}} \delta_{h_b(a),1}\, g^{(2)}_{b,\underline{\mathbf{z}}(\underline{\mathbf{h}}_a^{(1)})}(\xi_a)\langle \underline{\mathbf{h}}_a^{(1)}|\boldsymbol{T}_b^- + T^{(\hat{K},\infty)}_{2,\underline{\mathbf{z}}(\underline{\mathbf{h}}_a^{(1)})}(\xi_a)\langle \underline{\mathbf{h}}_a^{(1)}|\right),\end{aligned} \tag{4.5}$$

where

$$\underline{\mathbf{z}}(\underline{\mathbf{h}}) = \{\delta_{h_1,1}+\delta_{h_1,2},...,\delta_{h_{\mathsf{N}},1}+\delta_{h_{\mathsf{N}},2}\},\ \ \underline{\mathbf{y}}(\underline{\mathbf{h}}) = \{\delta_{h_1,2},...,\delta_{h_{\mathsf{N}},2}\}, \tag{4.6}$$

$$\underline{\mathbf{h}}_a^{(1)} = \underline{\mathbf{h}} - (h_a - 1)\underline{\mathbf{e}}_a \ \ \textit{with}\ \ \underline{\mathbf{e}}_a = \{\delta_{1,a},...,\delta_{\mathsf{N},a}\}, \tag{4.7}$$

and

$$\langle h_1,...,h_a,...,h_{\mathsf{N}}|\boldsymbol{T}_a^{\pm} = \langle h_1,...,h_a \pm 1,...,h_{\mathsf{N}}|. \tag{4.8}$$

ii) On the SoV vector basis:

$$T_2^{(\hat{K})}(\lambda)|\underline{\mathbf{h}}\rangle = d(\lambda-\eta)\left(\sum_{a=1}^{\mathsf{N}}\delta_{h_a,0}g^{(2)}_{a,\underline{\mathbf{z}}(\underline{\mathbf{h}})}(\lambda)\boldsymbol{T}_a^+|\underline{\mathbf{h}}\rangle + |\underline{\mathbf{h}}\rangle T^{(\hat{K},\infty)}_{2,\underline{\mathbf{z}}(\underline{\mathbf{h}})}(\lambda)\right), \tag{4.9}$$

and

$$T_1^{(\hat{K})}(\lambda)|\underline{\mathbf{h}}\rangle = |\underline{\mathbf{h}}\rangle T^{(\hat{K},\infty)}_{1,\underline{\mathbf{y}}(\underline{\mathbf{h}})}(\lambda) + \sum_{a=1}^{\mathsf{N}}\delta_{h_a,0}g^{(1)}_{a,\underline{\mathbf{y}}(\underline{\mathbf{h}})}(\lambda)\left(\boldsymbol{T}_a^+\right)^2|\underline{\mathbf{h}}\rangle + \sum_{a=1}^{\mathsf{N}}\delta_{h_a,2}g^{(1)}_{a,\underline{\mathbf{y}}(\underline{\mathbf{h}})}(\lambda)\boldsymbol{T}_a^-|\underline{\mathbf{h}}\rangle$$
$$+\sum_{a=1}^{\mathsf{N}}\delta_{h_a,1}g^{(1)}_{a,\underline{\mathbf{y}}(\underline{\mathbf{h}})}(\lambda)d(\xi_a^{(1)})\left(\sum_{b=1}^{\mathsf{N}}\delta_{\bar{h}_b(a),0}g^{(2)}_{b,\underline{\mathbf{z}}(\underline{\mathbf{h}}^{(2)}_a)}(\xi_a)\boldsymbol{T}_b^+|\underline{\mathbf{h}}^{(2)}_a\rangle + |\underline{\mathbf{h}}^{(2)}_a\rangle T^{(\hat{K},\infty)}_{2,\underline{\mathbf{z}}(\underline{\mathbf{h}}^{(2)}_a)}(\xi_a)\right), \tag{4.10}$$

where

$$\underline{\mathbf{h}}^{(2)}_a = \underline{\mathbf{h}} - (h_a-2)\underline{\mathbf{e}}_a\,, \tag{4.11}$$

and

$$\boldsymbol{T}_a^{\pm}|h_1,\ldots,h_a,\ldots,h_{\mathsf{N}}\rangle = |h_1,\ldots,h_a\pm 1,\ldots,h_{\mathsf{N}}\rangle. \tag{4.12}$$

Proof. The fusion identities take the following form in the case $\det\hat{K}=0$:

$$T_2^{(\hat{K})}(\xi_a^{(1)})T_1^{(\hat{K})}(\xi_a) = \text{q-det}\,M^{(\hat{K})}(\xi_a) = 0, \tag{4.13}$$
$$T_1^{(\hat{K})}(\xi_a^{(1)})T_1^{(\hat{K})}(\xi_a) = T_2^{(\hat{K})}(\xi_a), \tag{4.14}$$
$$T_2^{(\hat{K})}(\xi_a^{(1)})T_2^{(\hat{K})}(\xi_a) = T_1^{(\hat{K})}(\xi_a^{(1)})\,\text{q-det}\,M_a^{(\hat{K})}(\xi_a) = 0. \tag{4.15}$$

Let us take the generic co-vector[14] $\langle h_1,\ldots,h_{\mathsf{N}}|$ and then use the interpolation formula:

$$T_2^{(\hat{K})}(\lambda) = d(\lambda-\eta)\left(T^{(\hat{K},\infty)}_{2,\underline{\mathbf{z}}(\underline{\mathbf{h}})}(\lambda) + \sum_{a=1}^{\mathsf{N}}g^{(2)}_{a,\underline{\mathbf{z}}(\underline{\mathbf{h}})}(\lambda)T_2^{(\hat{K})}\left(\xi_a^{(\delta_{h_a,1}+\delta_{h_a,2})}\right)\right), \tag{4.16}$$

to compute the action of $T_2^{(\hat{K})}(\lambda)$ on $\langle h_1,\ldots,h_{\mathsf{N}}|$:

$$\langle\underline{\mathbf{h}}|T_2(\lambda) = d(\lambda-\eta)\left(T^{(\hat{K},\infty)}_{2,\underline{\mathbf{z}}(\underline{\mathbf{h}})}(\lambda)\langle\underline{\mathbf{h}}| + \sum_{a=1}^{\mathsf{N}}g^{(2)}_{a,\underline{\mathbf{z}}(\underline{\mathbf{h}})}(\lambda)\langle\underline{\mathbf{h}}|T_2^{(\hat{K})}\left(\xi_a^{(\delta_{h_a,1}+\delta_{h_a,2})}\right)\right), \tag{4.17}$$

where it holds:

$$\langle h_1,\ldots,h_a,\ldots,h_{\mathsf{N}}|T_2^{(\hat{K})}\left(\xi_a^{(\delta_{h_a,2}+\delta_{h_a,1})}\right) = \delta_{h_a,1}\langle h_1,\ldots,h'_a=0,\ldots,h_{\mathsf{N}}|\,, \tag{4.18}$$

being by the fusion identities:

$$\langle h_1,\ldots,h_a=2,\ldots,h_{\mathsf{N}}|T_2^{(\hat{K})}(\xi_a^{(1)}) = 0, \tag{4.19}$$
$$\langle h_1,\ldots,h_a=0,\ldots,h_{\mathsf{N}}|T_2^{(\hat{K})}(\xi_a) = 0. \tag{4.20}$$

This proves our interpolation formula for the action of $T_2^{(\hat{K})}(\lambda)$ on the generic element of the co-vector basis $\langle h_1,\ldots,h_{\mathsf{N}}|$. Let us now use the following interpolation formula:

$$T_1^{(\hat{K})}(\lambda) = T^{(\hat{K},\infty)}_{1,\underline{\mathbf{y}}(\underline{\mathbf{h}})}(\lambda) + \sum_{a=1}^{\mathsf{N}}g^{(1)}_{a,\underline{\mathbf{y}}(\underline{\mathbf{h}})}(\lambda)T_1^{(\hat{K})}(\xi_a^{(\delta_{h_a,2})}), \tag{4.21}$$

[14] For convenience, in this section, we do not use uniformly compact notations for the SoV basis co-vectors as their explicit form is sometimes more convenient to write the action of the transfer matrices on them.

to compute the action of $T_1^{(\hat{K})}(\lambda)$ on $\langle h_1,...,h_{\mathsf{N}}|$:

$$\langle \underline{\mathbf{h}}|T_1^{(\hat{K})}(\lambda) = T_{1,\underline{\mathbf{y}}(\underline{\mathbf{h}})}^{(\hat{K},\infty)}(\lambda)\langle \underline{\mathbf{h}}| + \sum_{a=1}^{\mathsf{N}} g_{a,\underline{\mathbf{y}}(\underline{\mathbf{h}})}^{(1)}(\lambda)\Big(\delta_{h_a,1}\langle \underline{\mathbf{h}}|T_1^{(\hat{K})}(\xi_a) + \delta_{h_a,2}\langle \underline{\mathbf{h}}|T_1^{(\hat{K})}(\xi_a^{(1)})\Big), \quad (4.22)$$

where we have used that by the fusion identity it holds:

$$\langle h_1,...,h_a=0,...,h_{\mathsf{N}}|T_1^{(\hat{K})}(\xi_a) = 0, \quad (4.23)$$

so that the above formula reduces to:

$$\langle \underline{\mathbf{h}}|T_1^{(\hat{K})}(\lambda) = T_{1,\underline{\mathbf{y}}(\underline{\mathbf{h}})}^{(\hat{K},\infty)}(\lambda)\langle \underline{\mathbf{h}}| + \sum_{a=1}^{\mathsf{N}} g_{a,\underline{\mathbf{y}}(\underline{\mathbf{h}})}^{(1)}(\lambda)\delta_{h_a,1}\langle \underline{\mathbf{h}}|T_1^{(\hat{K})}(\xi_a) + \sum_{a=1}^{\mathsf{N}} g_{a,\underline{\mathbf{y}}(\underline{\mathbf{h}})}^{(1)}(\lambda)\delta_{h_a,2}\langle \underline{\mathbf{h}}_a^{(1)}|T_2^{(\hat{K})}(\xi_a). \quad (4.24)$$

This leads to our result for the action of $T_1^{(\hat{K})}(\lambda)$ on $\langle \underline{\mathbf{h}}|$ once we use the proven formula for the action of $T_2^{(\hat{K})}(\lambda)$ on $\langle \underline{\mathbf{h}}|$.

Let us now prove the interpolation formulae for the action on SoV vectors. The fusion identities for the case $\det \hat{K} = 0$ imply[15]:

$$T_2^{(\hat{K})}(\xi_a^{(1)})|h_1,...,h_a\neq 0,...,h_{\mathsf{N}}\rangle = 0, \quad (4.25)$$

and so the only contributions to the action of $T_2^{(\hat{K})}(\lambda)$ on a vector $|\underline{\mathbf{h}}\rangle$ come from the central asymptotic term and the terms for $h_a = 0$, from which the action of $T_2^{(\hat{K})}(\lambda)$ easily follows. Let us now remark that the fusion identities together with the commutativity of the transfer matrices also imply the following actions:

$$T_1^{(\hat{K})}(\xi_a)|h_1,...,h_a=1,...,h_{\mathsf{N}}\rangle = T_2^{(\hat{K})}(\xi_a)|h_1,...,h_a=2,...,h_{\mathsf{N}}\rangle, \quad (4.26)$$

$$T_1^{(\hat{K})}(\xi_a^{(1)})|h_1,...,h_a=2,...,h_{\mathsf{N}}\rangle = |h_1,...,h_a=1,...,h_{\mathsf{N}}\rangle, \quad (4.27)$$

$$T_1^{(\hat{K})}(\xi_a)|h_1,...,h_a=0,...,h_{\mathsf{N}}\rangle = |h_1,...,h_a=2,...,h_{\mathsf{N}}\rangle, \quad (4.28)$$

from which we get the following action by interpolation formula

$$\begin{aligned} T_1^{(\hat{K})}(\lambda)|\underline{\mathbf{h}}\rangle = |\underline{\mathbf{h}}\rangle T_{1,\underline{\mathbf{y}}(\underline{\mathbf{h}})}^{(\hat{K},\infty)}(\lambda) + \sum_{a=1}^{\mathsf{N}} \delta_{h_a,0} g_{a,\underline{\mathbf{y}}(\underline{\mathbf{h}})}^{(1)}(\lambda)\left(\boldsymbol{T}_a^+\right)^2|\underline{\mathbf{h}}\rangle \\ + \sum_{a=1}^{\mathsf{N}} \delta_{h_a,2} g_{a,\underline{\mathbf{y}}(\underline{\mathbf{h}})}^{(1)}(\lambda)\boldsymbol{T}_a^-|\underline{\mathbf{h}}\rangle + \sum_{a=1}^{\mathsf{N}} \delta_{h_a,1} g_{a,\underline{\mathbf{y}}(\underline{\mathbf{h}})}^{(1)}(\lambda)T_2^{(\hat{K})}(\xi_a)\boldsymbol{T}_a^+|\underline{\mathbf{h}}\rangle, \end{aligned} \quad (4.29)$$

from which our formula for $T_1^{(\hat{K})}(\lambda)$ on $|\underline{\mathbf{h}}\rangle$ follows by using the one proven for $T_2^{(\hat{K})}(\lambda)$ on $|\underline{\mathbf{h}}\rangle$. □

We can complete now the proof of the Theorem 4.1:

Proof of Theorem 4.1. Let us start proving the orthogonality condition:

$$\langle h_1,...,h_{\mathsf{N}}|k_1,...,k_{\mathsf{N}}\rangle = 0, \quad \forall \{k_1,...,k_{\mathsf{N}}\} \neq \{h_1,...,h_{\mathsf{N}}\} \in \{0,1,2\}^{\otimes \mathsf{N}}. \quad (4.30)$$

The proof is done by induction, assuming that it is true for any vector $|k_1,...,k_{\mathsf{N}}\rangle$ such that $\sum_{n=1}^{\mathsf{N}}(\delta_{k_n,1}+\delta_{k_n,2}) = l$, $l \leq \mathsf{N}-1$, and proving it for vectors $|k'_1,...,k'_{\mathsf{N}}\rangle$ with $\sum_{n=1}^{\mathsf{N}}(\delta_{k'_n,1}+\delta_{k'_n,2}) = l+1$. To this aim we fix a vector $|k_1,...,k_{\mathsf{N}}\rangle$ with $\sum_{n=1}^{\mathsf{N}}\left(\delta_{k_n,1}+\delta_{k_n,2}\right) = l$ and we denote by π a permutation on the set $\{1,...,\mathsf{N}\}$ such that:

$$\delta_{k_{\pi(a)},1} + \delta_{k_{\pi(a)},2} = 1 \ \text{ for } a \leq l \ \text{ and } k_{\pi(a)} = 0 \text{ for } l < a. \quad (4.31)$$

[15] It is important to remark that the only ingredient of the proof for this theorem involve only the simplified fusion relations.

a) Let us first compute:

$$\langle h_1,...,h_{\mathsf{N}}|T_2^{(\hat{K})}(\xi_{\pi(l+1)})|k_1,...,k_{\mathsf{N}}\rangle = \langle h_1,...,h_{\mathsf{N}}|k'_1,...,k'_{\mathsf{N}}\rangle\,, \tag{4.32}$$

where we have defined:

$$k'_{\pi(a)} = k_{\pi(a)},\ \forall a\in\{1,...,\mathsf{N}\}\backslash\{l+1\} \quad \text{and} \quad k'_{\pi(l+1)} = 1, \tag{4.33}$$

for any $\{h_1,...,h_{\mathsf{N}}\}\neq\{k'_1,...,k'_{\mathsf{N}}\}\in\{0,1\}^{\otimes\mathsf{N}}$. There are three cases. The first case is $h_{\pi(l+1)}=0$, then the fusion identity implies:

$$\langle h_1,...,h_{\mathsf{N}}|T_2^{(\hat{K})}(\xi_{\pi(l+1)})|k_1,...,k_{\mathsf{N}}\rangle = 0. \tag{4.34}$$

In the remaining two cases $h_{\pi(l+1)}=1$ or $h_{\pi(l+1)}=2$, we can use the interpolation formula to compute the action of $T_2^{(\hat{K})}(\xi_{\pi(l+1)})$ on the co-vector $\langle h_1,...,h_{\mathsf{N}}|$:

$$\langle h_1,...,h_{\mathsf{N}}|T_2^{(\hat{K})}(\xi_{\pi(l+1)})|k_1,...,k_{\mathsf{N}}\rangle = d(\xi^{(1)}_{\pi(l+1)})T^{(\hat{K},\infty)}_{2,\underline{\mathbf{z}}(\underline{\mathbf{h}})}(\xi_{\pi(l+1)})\langle h_1,...,h_{\mathsf{N}}|k_1,...,k_{\mathsf{N}}\rangle \tag{4.35}$$

$$+d(\xi^{(1)}_{\pi(l+1)})\sum_{a=1}^{\mathsf{N}}\delta_{h_a,1}g^{(2)}_{a,\underline{\mathbf{z}}(\underline{\mathbf{h}})}(\xi_{\pi(l+1)})\langle h_1,...,h'_a=0,...,h_{\mathsf{N}}|k_1,...,k_{\mathsf{N}}\rangle. \tag{4.36}$$

Let us remark now that from $\{h_1,...,h_{\mathsf{N}}\}\neq\{k'_1,...,k'_{\mathsf{N}}\}$ and $h_{\pi(l+1)}=1$ or $h_{\pi(l+1)}=2$, it follows also that $\{h_1,...,h_{\mathsf{N}}\}\neq\{k_1,...,k_{\mathsf{N}}\}$, being $k_{\pi(l+1)}=0$, so that:

$$\langle h_1,...,h_{\mathsf{N}}|k_1,...,k_{\mathsf{N}}\rangle = 0. \tag{4.37}$$

Moreover, it holds:

$$\delta_{h_a,1}\langle h_1,...,h'_a=0,...,h_{\mathsf{N}}|k_1,...,k_{\mathsf{N}}\rangle = 0, \quad \forall a\in\{1,...,\mathsf{N}\}. \tag{4.38}$$

Indeed, if $a\in\{1,...,\mathsf{N}\}\backslash\{\pi(l+1)\}$ and $h_a=1$, we have $\{h_1,...,h'_a=0,...,h_{\mathsf{N}}\}\neq\{k_1,...,k_{\mathsf{N}}\}$, being $k_{\pi(l+1)}=0\neq h_{\pi(l+1)}\in\{1,2\}$. While in the case $a=\pi(l+1)$ either $h_{\pi(l+1)}=2$, so that $\delta_{h_{\pi(l+1)},1}=0$, or $h_{\pi(l+1)}=1$ and the condition $\{h_1,...,h_{\mathsf{N}}\}\neq\{k'_1,...,k'_{\mathsf{N}}\}$ implies that there exists at least a $j\neq\pi(l+1)$ such that $h_j\neq k_j$, so that we have still $\{h_1,...,h'_{\pi(l+1)}=0,...,h_{\mathsf{N}}\}\neq\{k_1,...,k_{\mathsf{N}}\}$.

b) Let us compute now:

$$\langle h_1,...,h_{\mathsf{N}}|T_1^{(\hat{K})}(\xi_{\pi(l+1)})|k_1,...,k_{\mathsf{N}}\rangle = \langle h_1,...,h_{\mathsf{N}}|k'_1,...,k'_{\mathsf{N}}\rangle, \tag{4.39}$$

where we have defined:

$$k'_{\pi(a)} = k_{\pi(a)},\ \forall a\in\{1,...,\mathsf{N}\}\backslash\{l+1\} \quad \text{and} \quad k'_{\pi(l+1)} = 2, \tag{4.40}$$

for any $\{h_1,...,h_{\mathsf{N}}\}\neq\{k'_1,...,k'_{\mathsf{N}}\}\in\{0,1\}^{\otimes\mathsf{N}}$. There are three cases as well. The first case is $h_{\pi(l+1)}=0$, then the fusion identity implies:

$$\langle h_1,...,h_{\mathsf{N}}|T_1^{(\hat{K})}(\xi_{\pi(l+1)})|k_1,...,k_{\mathsf{N}}\rangle = 0\,. \tag{4.41}$$

For the second case for $h_{\pi(l+1)}=1$, it holds:

$$\begin{aligned}&\langle h_1,...,h_{\mathsf{N}}|T_1^{(\hat{K})}(\xi_{\pi(l+1)})|k_1,...,k_{\mathsf{N}}\rangle\\ &=\langle h_1,...,h_{\pi(l+1)}=2,...,h_{\mathsf{N}}|k_1,...,k_{\pi(l+1)}=0,...,k_{\mathsf{N}}\rangle = 0.\end{aligned} \tag{4.42}$$

So we are left with the case $h_{\pi(l+1)} = 2$. Note that in this case the condition $\{h_1,...,h_N\} \neq \{k'_1,...,k'_N\}$ implies that there exists a $j \neq \pi(l+1)$ such that $h_j \neq k_j$, being by definition $h_{\pi(l+1)} = k'_{\pi(l+1)} = 2$. We can use the following interpolation formula to compute the action of $T_1^{(\hat{K})}(\xi_{\pi(l+1)})$ on the co-vector $\langle h_1,...,h_N|$:

$$\begin{aligned}\langle h_1,...,h_N|T_1^{(\hat{K})}(\xi_{\pi(l+1)})|\underline{\mathbf{k}}\rangle = T^{(\hat{K},\infty)}_{1,\underline{\mathbf{y}}(\underline{\mathbf{h}})}(\xi_{\pi(l+1)})\langle h_1,...,h_N|\underline{\mathbf{k}}\rangle \\ + \sum_{a=1}^{N} g^{(1)}_{a,\underline{\mathbf{y}}(\underline{\mathbf{h}})}(\xi_{\pi(l+1)})\delta_{h_a,1}\langle h_1,...,h_N|T_1^{(\hat{K})}(\xi_a)|\underline{\mathbf{k}}\rangle \\ + \sum_{a=1}^{N} g^{(1)}_{a,\underline{\mathbf{y}}(\underline{\mathbf{h}})}(\xi_{\pi(l+1)})\delta_{h_a,2}\langle h_1,...,h_N|T_1^{(\hat{K})}(\xi^{(1)}_a)|\underline{\mathbf{k}}\rangle. \end{aligned} \tag{4.43}$$

From $\{h_1,...,h_N\} \neq \{k_1,...,k_N\}$ it follows:

$$\langle h_1,...,h_N|\underline{\mathbf{k}}\rangle = 0, \tag{4.44}$$

and, moreover, it holds:

$$\delta_{h_a,1}\langle h_1,...,h_N|T_1^{(\hat{K})}(\xi_a)|\underline{\mathbf{k}}\rangle = \delta_{h_a,1}\langle h_1,...,h'_a = 2,...,h_N|\underline{\mathbf{k}}\rangle = 0, \tag{4.45}$$

as for $a = \pi(l+1)$ it holds $\delta_{h_a,1} = 0$, because $h_{\pi(l+1)} = 2$, while for $a \neq \pi(l+1)$ we have still $h_{\pi(l+1)} = 2 \neq k_{\pi(l+1)} = 0$ so that:

$$\langle h_1,...,h'_a = 2,...,h_N|\underline{\mathbf{k}}\rangle = 0. \tag{4.46}$$

So, we are left with the last sum in (4.43), for which it holds:

$$\delta_{h_a,2}\langle h_1,...,h_N|T_1^{(\hat{K})}(\xi^{(1)}_a)|\underline{\mathbf{k}}\rangle = \delta_{h_a,2}\langle h_1,...,h'_a = 1,...,h_N|T_2^{(\hat{K})}(\xi_a)|\underline{\mathbf{k}}\rangle. \tag{4.47}$$

i) For $a = \pi(r)$ for $r \geq l+1$ it holds:

$$\begin{aligned}\delta_{h_a,2}\langle h_{\pi(1)},...,h'_{\pi(r)} = 1,...,h_{\pi(N)}|T_2^{(\hat{K})}(\xi_{\pi(r)})|\underline{\mathbf{k}}\rangle \\ = \delta_{h_a,2}\langle h_{\pi(1)},...,h'_{\pi(r)} = 1,...,h_{\pi(N)}|k_{\pi(1)},...,k''_{\pi(r)} = 1,...,k_{\pi(N)}\rangle, \end{aligned} \tag{4.48}$$

with $\{h_{\pi(1)},...,h'_{\pi(r)} = 1,...,h_{\pi(N)}\} \neq \{k_{\pi(1)},...,k''_{\pi(r)} = 1,...,k_{\pi(N)}\}$. Indeed, if $r = l+1$ we have shown that there is a $j \neq \pi(l+1)$ such that $h_j \neq k_j$, while if $r \geq l+2$ we have still $h_{\pi(l+1)} = 2 \neq k_{\pi(l+1)} = 0$.

So that for $a = \pi(r)$ for $r \geq l+1$, we can use the step a) of the proof to get:

$$\langle h_{\pi(1)},...,h'_{\pi(r)} = 1,...,h_{\pi(N)}|k_{\pi(1)},...,k''_{\pi(r)} = 1,...,k_{\pi(N)}\rangle = 0. \tag{4.49}$$

ii) For $a = \pi(r)$ for $r \leq l$. If $k_{\pi(r)} = 2$, then we can write the l.h.s. of (4.47) as it follows:

$$\delta_{h_{\pi(r)},2}\langle h_1,...,h_N|T_1^{(\hat{K})}(\xi^{(1)}_{\pi(r)})|\underline{\mathbf{k}}\rangle = \delta_{h_{\pi(r)},2}\langle h_1,...,h_N|k_{\pi(1)},...,k''_{\pi(r)} = 1,...,k_{\pi(N)}\rangle, \tag{4.50}$$

which is zero being $h_{\pi(r)} = 2 \neq k''_{\pi(r)} = 1$.

If $k_{\pi(r)} = 1$, then we use the interpolation formula:

$$T_2^{(\hat{K})}(\xi_{\pi(r)})|\underline{\mathbf{k}}\rangle = d(\xi^{(1)}_{\pi(r)})\left(|\underline{\mathbf{k}}\rangle T^{(\hat{K},\infty)}_{2,\underline{\mathbf{z}}(\underline{\mathbf{h}})}(\xi_{\pi(r)}) + \sum_{n=l+1}^{N} g^{(2)}_{n,\underline{\mathbf{z}}(\underline{\mathbf{h}})}(\xi_{\pi(r)})T_2^{(\hat{K})}(\xi_{\pi(n)})|\underline{\mathbf{k}}\rangle\right), \tag{4.51}$$

where we have used that, by the fusion identity:

$$T_2^{(\hat{K})}(\xi_{\pi(n)}^{(1)})|\underline{\mathbf{k}}\rangle = 0 \text{ for } n \leq l \text{ as } T_2^{(\hat{K})}(\xi_{\pi(n)}^{(1)})T_{2-\delta_{k_{\pi(n)},2}}^{(\hat{K})}(\xi_{\pi(n)}) = 0. \tag{4.52}$$

Then, for $a = \pi(r)$ for $r \leq l$ and if $k_{\pi(r)} = 1$, (4.47) reads:

$$\begin{aligned}\delta_{h_{\pi(r)},2}\langle h_{\pi(1)},...,h'_{\pi(r)} = 1,...,h_{\pi(\mathsf{N})}|T_2^{(\hat{K})}(\xi_a)|\underline{\mathbf{k}}\rangle \\ = \delta_{h_{\pi(r)},2}d(\xi_{\pi(r)}^{(1)})(T_{2,\underline{\mathbf{z}}(\underline{\mathbf{h}})}^{(\hat{K},\infty)}(\xi_{\pi(r)})\langle h_{\pi(1)},...,h'_{\pi(r)} = 1,...,h_{\pi(\mathsf{N})}|\underline{\mathbf{k}}\rangle \\ + \sum_{n=l+1}^{\mathsf{N}} g_{n,\underline{\mathbf{z}}(\underline{\mathbf{h}})}^{(2)}(\xi_{\pi(r)})\langle h_{\pi(1)},...,h'_{\pi(r)} = 1,...,h_{\pi(\mathsf{N})}|k_{\pi(1)},...,k''_{\pi(n)} = 1,...,k_{\pi(\mathsf{N})}\rangle).\end{aligned} \tag{4.53}$$

Here we have:

$$\langle h_{\pi(1)},...,h'_{\pi(r)} = 1,...,h_{\pi(\mathsf{N})}|\underline{\mathbf{k}}\rangle = 0, \tag{4.54}$$

being $h_{\pi(l+1)} = 2 \neq k_{\pi(l+1)} = 0$. Moreover, the remaining matrix elements

$$\langle h_{\pi(1)},...,h'_{\pi(r)} = 1,...,h_{\pi(\mathsf{N})}|k_{\pi(1)},...,k''_{\pi(n)} = 1,...,k_{\pi(\mathsf{N})}\rangle, \tag{4.55}$$

for $r \leq l$ and $l+1 \leq n$ are such that $\{h_{\pi(1)},...,h'_{\pi(r)} = 1,...,h_{\pi(\mathsf{N})}\} \neq \{k_{\pi(1)},...,k''_{\pi(n)} = 1,...,k_{\pi(\mathsf{N})}\}$. Indeed, for $n = l+1$ it holds $h_{\pi(l+1)} = 2 \neq k''_{\pi(l+1)} = 1$ while for $l+2 \leq n$ it still holds $h_{\pi(l+1)} = 2 \neq k_{\pi(l+1)} = 0$.

Finally, we can apply the step a) of our proof to show that (4.55) is zero for any fixed $l+1 \leq n$, just exchanging the permutation π with the following one

$$\pi_n(a) = \pi(a)(1-\delta_{a,l+1})(1-\delta_{a,n}) + \pi(n)\delta_{a,l+1} + \pi(l+1)\delta_{a,n}. \tag{4.56}$$

The computation of the SoV measure is standard [37,39] once one uses the interpolation formulae of the transfer matrices given above. Let us write the elements of the proof. Let us first define:

$$\underline{\mathbf{h}}_a^{(j)} = \underline{\mathbf{h}} - (h_a - j)\underline{\mathbf{e}}_a, \quad \forall a, j \in \{1,...,\mathsf{N}\} \times \{0,1,2\},$$

and compute the matrix elements:

$$\langle \underline{\mathbf{h}}_a^{(1)}|T_2^{(\hat{K})}(\xi_a^{(1)})|\underline{\mathbf{h}}_a^{(0)}\rangle = \langle \underline{\mathbf{h}}_a^{(0)}|\underline{\mathbf{h}}_a^{(0)}\rangle. \tag{4.57}$$

We compute the action of $T_2^{(\hat{K})}(\xi_a^{(1)})$ on the right by using the corresponding interpolation formula, and from the orthogonality conditions we get that there is only one term with non-zero contribution, which reads:

$$\langle \underline{\mathbf{h}}_a^{(1)}|T_2^{(\hat{K})}(\xi_a^{(1)})|\underline{\mathbf{h}}_a^{(0)}\rangle = \langle \underline{\mathbf{h}}_a^{(1)}|\underline{\mathbf{h}}_a^{(1)}\rangle \frac{d(\xi_a^{(2)})}{d(\xi_a^{(1)})} \prod_{n\neq a,n=1}^{\mathsf{N}} \frac{\xi_a^{(1)} - \xi_n^{(\delta_{h_n,1}+\delta_{h_n,2})}}{\xi_a - \xi_n^{(\delta_{h_n,1}+\delta_{h_n,2})}}. \tag{4.58}$$

Similarly, we want to compute:

$$\langle \underline{\mathbf{h}}_a^{(1)}|T_1^{(\hat{K})}(\xi_a)|\underline{\mathbf{h}}_a^{(2)}\rangle = \langle \underline{\mathbf{h}}_a^{(2)}|\underline{\mathbf{h}}_a^{(2)}\rangle, \tag{4.59}$$

by using the interpolation formula for the right action of $T_1^{(\hat{K})}(\xi_a)$, we obtain that once again there is just one term that give a non-zero contribution due to the orthogonality and it reads:

$$\langle \underline{\mathbf{h}}_a^{(1)}|T_1^{(\hat{K})}(\xi_a)|\underline{\mathbf{h}}_a^{(2)}\rangle = \langle \underline{\mathbf{h}}_a^{(1)}|\underline{\mathbf{h}}_a^{(1)}\rangle \prod_{n\neq a,n=1}^{\mathsf{N}} \frac{\xi_a - \xi_n^{(\delta_{h_n,2})}}{\xi_a^{(1)} - \xi_n^{(\delta_{h_n,2})}}, \tag{4.60}$$

from which our formula for the normalization holds. □

The following corollary holds:

Corollary 4.1. *Let $\hat{K}$ be a 3×3 simple spectrum matrix with one zero eigenvalue. Then for almost any choice of the co-vector $\langle \underline{\mathbf{1}}|$ and of the inhomogeneities under the condition (2.6) the states*

$$\langle \underline{\mathbf{0}}| = \langle h_1 = 0, ..., h_{\mathsf{N}} = 0|, \quad \langle \underline{\mathbf{2}}| = \langle h_1 = 2, ..., h_{\mathsf{N}} = 2| \tag{4.61}$$

are $T_2^{(\hat{K})}(\lambda)$ eigenstates:

$$\langle \underline{\mathbf{0}}|T_2^{(\hat{K})}(\lambda) = t_{2,0}d(\lambda-\eta)d(\lambda)\langle \underline{\mathbf{0}}|, \tag{4.62}$$

$$\langle \underline{\mathbf{2}}|T_2^{(\hat{K})}(\lambda) = t_{2,0}d(\lambda-\eta)d(\lambda+\eta)\langle \underline{\mathbf{2}}|, \tag{4.63}$$

$$T_2^{(\hat{K})}(\lambda)|\underline{\mathbf{h}}\rangle = |\underline{\mathbf{h}}\rangle t_{2,0}d(\lambda-\eta)d(\lambda+\eta), \quad \forall \underline{\mathbf{h}} \in \{1,2\}^{\mathsf{N}} \tag{4.64}$$

while $\langle \underline{\mathbf{0}}|$ is also $T_1^{(\hat{K})}(\lambda)$ eigenstate:

$$\langle \underline{\mathbf{0}}|T_1^{(\hat{K})}(\lambda) = t_{1,0}d(\lambda)\langle \underline{\mathbf{0}}|. \tag{4.65}$$

Proof. It is enough to take the interpolation formulae for the transfer matrices and apply them over these states. □

Theorem 4.2. *Let $\hat{K}$ be a 3×3 simple spectrum matrix with one zero eigenvalue and with the inhomogeneities under the condition (2.6). Then the transfer matrix spectrum is simple and, for almost any choice of the co-vector $\langle \underline{\mathbf{1}}|$, the vector $|t_a\rangle$ and the co-vector $\langle t_a|$ are transfer matrix eigenstates if and only if they admit (up to an overall normalization) the following separate form in the co-vector and vector SoV eigenbasis:*

$$|t_a\rangle = \sum_{\underline{\mathbf{h}}} \prod_{n=1}^{\mathsf{N}} t_{2,a}^{\delta_{h_n,0}}(\xi_n^{(1)}) t_{1,a}^{\delta_{h_n,2}}(\xi_n) \frac{|\underline{\mathbf{h}}\rangle}{\mathsf{N}_{\underline{\mathbf{h}}}}, \tag{4.66}$$

$$\langle t_a| = \sum_{\underline{\mathbf{h}}} \prod_{n=1}^{\mathsf{N}} t_{2,a}^{\delta_{h_n,1}}(\xi_n) t_{1,a}^{\delta_{h_n,2}}(\xi_n) \frac{\langle \underline{\mathbf{h}}|}{\mathsf{N}_{\underline{\mathbf{h}}}}, \tag{4.67}$$

where the index a run in the set of the transfer matrix eigenvalues of $T_1^{(\hat{K})}(\lambda)$ and the coefficients of the states are written in terms of the corresponding eigenvalues:

$$T_1^{(\hat{K})}(\lambda)|t_a\rangle = |t_a\rangle t_{1,a}(\lambda), \quad T_2^{(\hat{K})}(\lambda)|t_a\rangle = |t_a\rangle t_{2,a}(\lambda), \tag{4.68}$$

$$\langle t_a|T_1^{(\hat{K})}(\lambda) = t_{1,a}(\lambda)\langle t_a|, \quad \langle t_a|T_2^{(\hat{K})}(\lambda) = t_{2,a}(\lambda)\langle t_a|. \tag{4.69}$$

Finally, if the matrix $\hat{K}$ has simple spectrum and is diagonalizable, the same is true for the transfer matrix $T_1^{(\hat{K})}(\lambda)$, which therefore admits 3^{N} distinct eigenvalues $t_{1,a}(\lambda)$ with $a \in \{1, ..., 3^{\mathsf{N}}\}$.

Proof. Let us compute the matrix element:

$$\langle \underline{\mathbf{h}}|t\rangle = \langle \underline{\mathbf{1}}| \prod_{a=1}^{\mathsf{N}} T_2^{(\hat{K})\delta_{h_a,0}}(\xi_a^{(1)}) T_1^{(\hat{K})\delta_{h_a,2}}(\xi_a)|t\rangle = \langle \underline{\mathbf{1}}|t\rangle \prod_{a=1}^{\mathsf{N}} t_2^{\delta_{h_a,0}}(\xi_a^{(1)}) t_1^{\delta_{h_a,2}}(\xi_a). \tag{4.70}$$

From our SoV decomposition of the identity, it holds:

$$|t\rangle = \sum_{\underline{\mathbf{h}}} \langle \underline{\mathbf{h}}|t\rangle \frac{|\underline{\mathbf{h}}\rangle}{\mathsf{N}_{\underline{\mathbf{h}}}}, \tag{4.71}$$

and then fixing the normalization of the state $|t\rangle$ by imposing $\langle \underline{\mathbf{1}}|t\rangle = 1$, our statement is proven. □

The functional equation characterization of the transfer matrix eigenvalues and ABA like representations of the states hold also in the case where the 3×3 simple spectrum matrix $\hat{K}$ has one zero eigenvalue.

4.2 Scalar products of separate states in orthogonal SoV basis

Let us introduce the following class of "separate" co-vectors and vectors:

$$|\alpha\rangle = \sum_{\underline{\mathbf{h}}} \prod_{a=1}^{\mathsf{N}} \alpha_a^{(h_a)} \frac{|\underline{\mathbf{h}}\rangle}{\mathsf{N}_{\underline{\mathbf{h}}}}, \quad \langle\alpha| = \sum_{\underline{\mathbf{h}}} \prod_{a=1}^{\mathsf{N}} \alpha_a^{(h_a)} \frac{\langle\underline{\mathbf{h}}|}{\mathsf{N}_{\underline{\mathbf{h}}}}. \tag{4.72}$$

The eigenvectors and co-vectors of the transfer matrix are of this form, with coefficients $\alpha_a^{h_a}$ constrained by their eigenvalue. We have the following scalar product formulae:

Theorem 4.3. *Let $\hat{K}$ be a 3×3 simple spectrum matrix with one zero eigenvalue and let the inhomogeneity condition (2.6) be satisfied. Then, taken the generic transfer matrix eigenvector:*

$$|t_n\rangle = \sum_{\underline{\mathbf{h}}} \prod_{a=1}^{\mathsf{N}} t_{2,n}^{\delta_{h_a,0}}(\xi_a^{(1)}) t_{1,n}^{\delta_{h_a,2}}(\xi_a) \frac{|\underline{\mathbf{h}}\rangle}{\mathsf{N}_{\underline{\mathbf{h}}}}, \tag{4.73}$$

there exists a permutation π_n of the set $\{1,\dots,\mathsf{N}\}$ such that:

$$t_{1,n}(\xi_{\pi_n(b)}) = t_{2,n}(\xi_{\pi_n(a)} - \eta) = 0, \quad \forall (a,b) \in A\times B, \tag{4.74}$$

$$t_{1,n}(\xi_{\pi_n(a)}) \neq 0, \quad t_{2,n}(\xi_{\pi_n(b)} - \eta) \neq 0, \quad \forall (a,b) \in A\times B, \tag{4.75}$$

where we have defined:

$$A \equiv \{1,\dots,\mathsf{M}_n\}, \quad B \equiv \{\mathsf{M}_n+1,\dots,\mathsf{N}\}. \tag{4.76}$$

Moreover, the action of the generic separate co-vector $\langle\alpha|$ on it reads:

$$\begin{aligned}\langle\alpha|t_n\rangle = &\prod_{a=1}^{\mathsf{N}} \frac{d(\xi_a^{(2)})}{d(\xi_a^{(1)})} \frac{V\big(\xi^{(1)}_{\pi_n(1)},\dots,\xi^{(1)}_{\pi_n(\mathsf{M}_n)}\big)}{V\big(\xi_{\pi_n(1)},\dots,\xi_{\pi_n(\mathsf{M}_n)}\big)} \\ &\times \frac{\det_{\mathsf{N}-\mathsf{M}_n} \mathcal{M}^{(\alpha|x_A t_{2,n})}_{+,\mathsf{N}-\mathsf{M}_n}}{V\big(\xi_{\pi_n(\mathsf{M}_n+1)},\dots,\xi_{\pi_n(\mathsf{N})}\big)} \frac{\det_{\mathsf{M}_n} \mathcal{M}^{(\alpha|x_B t_{1,n})}_{-,\mathsf{M}_n}}{V\big(\xi_{\pi_n(1)},\dots,\xi_{\pi_n(\mathsf{M}_n)}\big)},\end{aligned} \tag{4.77}$$

where we have defined:

$$\Big(\mathcal{M}^{(\alpha|x_A t_{2,n})}_{+,\mathsf{N}-\mathsf{M}_n}\Big)_{(i,j)\in\{1,\dots,\mathsf{N}-\mathsf{M}_n\}^2} = \sum_{h=0}^{1} \alpha^{(h)}_{\pi_n(\mathsf{M}_n+i)} x_A^{1-h}(\xi_{\pi_n(\mathsf{M}_n+i)}) \underline{t}^h_{2,n}(\xi^{(1)}_{\pi_n(\mathsf{M}_n+i)}) (\xi^{(h)}_{\pi_n(\mathsf{M}_n+i)})^{j-1}, \tag{4.78}$$

$$\Big(\mathcal{M}^{(\alpha|x_B t_{1,n})}_{-,\mathsf{M}_n}\Big)_{(i,j)\in\{1,\dots,\mathsf{M}_n\}^2} = \sum_{h=0}^{1} \alpha^{(h+1)}_{\pi_n(i)} x_B^{h}(\xi_{\pi_n(i)}) t^h_{1,n}(\xi_{\pi_n(i)}) (\xi^{(h)}_{\pi_n(i)})^{j-1}, \tag{4.79}$$

with

$$\begin{aligned} x_A(\lambda) &= \prod_{a=1}^{\mathsf{M}_n} \frac{\lambda - \xi_{\pi_n(a)} + \eta}{\lambda - \xi_{\pi_n(a)}}, \quad x_B(\lambda) = \prod_{a=1+\mathsf{M}_n}^{\mathsf{N}} \frac{\lambda - \xi_{\pi_n(b)} - \eta}{\lambda - \xi_{\pi_n(b)}}, \\ \underline{t}_{2,n}(\lambda) &= d(\lambda) t_{2,n}(\lambda)/d(\lambda-\eta). \end{aligned} \tag{4.80}$$

We have the following identity for the action of the eigenco-vector $\langle t_n|$ on the eigenvector $|t_n\rangle$:

$$\langle t_n|t_n\rangle = \prod_{a=1}^{\mathsf{N}} \frac{V\big(\xi^{(1)}_{\pi_n(1)},\dots,\xi^{(1)}_{\pi_n(\mathsf{M}_n)}\big)}{V\big(\xi_{\pi_n(1)},\dots,\xi_{\pi_n(\mathsf{M}_n)}\big)} \prod_{b=1+\mathsf{M}_n}^{\mathsf{N}} t_{2,n}(\xi^{(1)}_{\pi_n(b)}) x_A(\xi_{\pi_n(b)}) \prod_{a=1}^{\mathsf{M}_n} t_{1,n}(\xi_{\pi_n(a)}) \det_{\mathsf{M}_n} \mathcal{T}_{\mathsf{M}_n}, \tag{4.81}$$

where we have defined:

$$\big(\mathcal{T}_{\mathsf{M}_n}\big)_{(i,j)\in A^2} = \sum_{h=0}^{1} t_{1,n}(\xi^{(1-h)}_{\pi_n(i)}) x_B^h(\xi_{\pi_n(i)}) (\xi^{(h)}_{\pi_n(i)})^{j-1}. \tag{4.82}$$

Proof. It is worth recalling that the zero and non-zero pattern of (4.74) and(4.75) has been derived in [81]. There, we have moreover observed that the eigenvalue of the transfer matrix $T_2^{(\hat{K})}(\lambda)$ is completely fixed by them, i.e. it holds

$$t_{2,n}(\lambda) = T_2^{(K,\infty)} d(\lambda-\eta) \prod_{a=1}^{\mathsf{M}_n} (\lambda - \xi^{(1)}_{\pi_n(a)}) \prod_{b=1+\mathsf{M}_n}^{\mathsf{N}} (\lambda - \xi_{\pi_n(b)}). \tag{4.83}$$

The proof of this theorem is a direct consequence of the new found SoV measure (4.2) and of the form of the separate states, from which we get

$$\begin{aligned}\langle \alpha | t_n \rangle = & \sum_{h_1,\dots,h_{\mathsf{N}}=0}^{2} \prod_{a=1}^{\mathsf{N}} \frac{d(\xi_a^{(1+\delta_{h_a,1}+\delta_{h_a,2})})}{d(\xi_a^{(1)})} t_{2,n}^{\delta_{h_a,0}}(\xi_a^{(1)}) t_{1,n}^{\delta_{h_a,2}}(\xi_a)\, \alpha_a^{(h_a)} \\ & \times \frac{V\Big(\xi_1^{(\delta_{h_1,2}+\delta_{h_1,1})},\dots,\xi_{\mathsf{N}}^{(\delta_{h_{\mathsf{N}},1}+\delta_{h_{\mathsf{N}},2})}\Big) V\Big(\xi_1^{(\delta_{h_1,2})},\dots,\xi_{\mathsf{N}}^{(\delta_{h_{\mathsf{N}},2})}\Big)}{V^2(\xi_1,\dots,\xi_{\mathsf{N}})}.\end{aligned} \tag{4.84}$$

We now use the existence of the permutation π_n and the characterization of the zero and non-zero pattern for the transfer matrix eigenvalues (4.74) and (4.75) to factorize the above sum into two sum and get our result. Indeed, by using them the r.h.s. of (4.84) reads ($\mathsf{M}_n^+ = \mathsf{M}_n + 1$):

$$\begin{aligned}&\sum_{h_{\pi_n(1)},\dots,h_{\pi_n(\mathsf{M}_n)}=1}^{2} \sum_{h_{\pi_n(\mathsf{M}_n+1)},\dots,h_{\pi_n(\mathsf{N})}=0}^{1} \prod_{a=1}^{\mathsf{M}_n} \frac{d(\xi^{(2)}_{\pi_n(a)})}{d(\xi^{(1)}_{\pi_n(a)})} t_{1,n}^{\delta_{h_{\pi_n(a)},2}}(\xi_{\pi_n(a)})\, \alpha^{(h_{\pi_n(a)})}_{\pi_n(a)} \\ &\prod_{b=\mathsf{M}_n^+}^{\mathsf{N}} \frac{d\Big(\xi^{(1+\delta_{h_{\pi_n(b)},1})}_{\pi_n(b)}\Big)}{d(\xi^{(1)}_{\pi_n(b)})} t_{2,n}^{\delta_{h_{\pi_n(b)},0}}(\xi^{(1)}_{\pi_n(b)}) \alpha^{(h_{\pi_n(b)})}_{\pi_n(b)} \frac{V\Big(\xi^{(1)}_{\pi_n(1)},\dots,\xi^{(1)}_{\pi_n(\mathsf{M}_n)}\Big) V\Big(\xi^{(\delta_{h_{\pi_n(1)},2})}_{\pi_n(1)},\dots,\xi^{(\delta_{h_{\pi_n(\mathsf{M}_n)},2})}_{\pi_n(\mathsf{M}_n)}\Big)}{V(\xi_{\pi_n(1)},\dots,\xi_{\pi_n(\mathsf{M}_n)}) V(\xi_{\pi_n(1)},\dots,\xi_{\pi_n(\mathsf{M}_n)})} \\ &\prod_{a=1}^{\mathsf{M}_n} \prod_{b=\mathsf{M}_n^+}^{\mathsf{N}} \frac{\xi^{(\delta_{h_{\pi_n(b)},1})}_{\pi_n(b)} - \xi^{(1)}_{\pi_n(a)}}{\xi_{\pi_n(b)} - \xi_{\pi_n(a)}} \prod_{a=1}^{\mathsf{M}_n} \prod_{b=\mathsf{M}_n^+}^{\mathsf{N}} \frac{\xi^{(\delta_{h_{\pi_n(a)},2})}_{\pi_n(a)} - \xi_{\pi_n(b)}}{\xi_{\pi_n(a)} - \xi_{\pi_n(b)}} \frac{V\Big(\xi^{(\delta_{h_{\pi_n(\mathsf{M}_n+1)},1})}_{\pi_n(\mathsf{M}_n+1)},\dots,\xi^{(\delta_{h_{\pi_n(\mathsf{N})},1})}_{\pi_n(\mathsf{N})}\Big)}{V(\xi_{\pi_n(\mathsf{M}_n+1)},\dots,\xi_{\pi_n(\mathsf{N})})}.\end{aligned} \tag{4.85}$$

We can then factorize out of the above sum the factors:

$$\prod_{a=1}^{\mathsf{N}} \frac{d(\xi_a^{(2)})}{d(\xi_a^{(1)})} \frac{V(\xi^{(1)}_{\pi_n(1)},\dots,\xi^{(1)}_{\pi_n(\mathsf{M}_n)})}{V(\xi_{\pi_n(1)},\dots,\xi_{\pi_n(\mathsf{M}_n)})}, \tag{4.86}$$

being left with the product of the following two independent sum, i.e.

$$\sum_{h_{\pi_n(1)},\dots,h_{\pi_n(\mathsf{M}_n)}=1}^{2} \prod_{a=1}^{\mathsf{M}_n} t_{1,n}^{\delta_{h_{\pi_n(a)},2}}(\xi_{\pi_n(a)})\, \alpha^{(h_{\pi_n(a)})}_{\pi_n(a)} x_B^{(h_{\pi_n(a)}-1)}(\xi_{\pi_n(a)}) \frac{V\Big(\xi^{(\delta_{h_{\pi_n(1)},2})}_{\pi_n(1)},\dots,\xi^{(\delta_{h_{\pi_n(\mathsf{M}_n)},2})}_{\pi_n(\mathsf{M}_n)}\Big)}{V(\xi_{\pi_n(1)},\dots,\xi_{\pi_n(\mathsf{M}_n)})} \tag{4.87}$$

times

$$\sum_{h_{\pi_n(\mathsf{M}_n^+)},\dots,h_{\pi_n(\mathsf{N})}=0}^{1} \prod_{b=\mathsf{M}_n^+}^{\mathsf{N}} t_{2,n}^{\delta_{h_{\pi_n(b)},0}}(\xi^{(1)}_{\pi_n(b)}) \alpha^{(h_{\pi_n(b)})}_{\pi_n(b)} x_A^{1-h_{\pi_n(b)}}(\xi_{\pi_n(b)}) \frac{V\Big(\xi^{(\delta_{h_{\pi_n(\mathsf{M}_n+1)},1})}_{\pi_n(\mathsf{M}_n^+)},\dots,\xi^{(\delta_{h_{\pi_n(\mathsf{N})},1})}_{\pi_n(\mathsf{N})}\Big)}{V(\xi_{\pi_n(\mathsf{M}_n^+)},\dots,\xi_{\pi_n(\mathsf{N})})}. \tag{4.88}$$

As previously remarked in [37,39], these sums admit a representation in terms of one determinant formulae, thanks to the multi-linearity of the Vandermonde determinant. From this, our result (4.77) follows.

To derive the formula for the "norm" of the transfer matrix eigenvectors, we just have to observe that by the definition of the vector SoV basis, it holds:

$$\langle t_n|\underline{\mathbf{h}}\rangle = \prod_{a=1}^{\mathsf{N}} t_{2,n}^{\delta_{h_a,1}}(\xi_a) t_{1,n}^{\delta_{h_a,2}}(\xi_a) = \prod_{a=1}^{\mathsf{N}} t_{1,n}^{\delta_{h_a,1}}(\xi_a^{(1)}) t_{1,n}^{(\delta_{h_a,1}+\delta_{h_a,2})}(\xi_a), \tag{4.89}$$

and so we have that

$$\langle t_n|\underline{\mathbf{h}}\rangle = 0, \quad \forall (h_{\pi_n(\mathsf{M}_n+1)},...,h_{\pi_n(\mathsf{N})}) \neq (0,...,0). \tag{4.90}$$

Then the sum (4.88) reduces to

$$\prod_{b=1+\mathsf{M}_n}^{\mathsf{N}} \underline{t}_{2,n}(\xi^{(1)}_{\pi_n(b)})\, x_A(\xi_{\pi_n(b)}), \tag{4.91}$$

while the first one reads:

$$\prod_{a=1}^{\mathsf{M}_n} t_{1,n}(\xi_{\pi_n(a)}) \sum_{h_{\pi_n(1)},...,h_{\pi_n(\mathsf{M}_n)}=1}^{2} \prod_{a=1}^{\mathsf{M}_n} t_{1,n}(\xi^{(2-h_{\pi_n(a)})}_{\pi_n(a)}) x_B^{(h_{\pi_n(a)}-1)}(\xi_{\pi_n(a)}) \times \frac{V\left(\xi^{(\delta_{h_{\pi_n(1)},2})}_{\pi_n(1)},...,\xi^{(\delta_{h_{\pi_n(\mathsf{M}_n)},2})}_{\pi_n(\mathsf{M}_n)}\right)}{V(\xi_{\pi_n(1)},...,\xi_{\pi_n(\mathsf{M}_n)})}. \tag{4.92}$$

It is now quite direct to verify the formula (4.81). □

4.3 On the extension to the case of simple spectrum and invertible K-matrices

The results of the previous subsections give us the possibility to define a new family of conserved charges, from which we can introduce the orthogonal left and right SoV bases also in the case of a general simple spectrum K-matrix with non-zero eigenvalues.

Let us assume that K is 3×3 simple spectrum and diagonalizable matrix with non-zero eigenvalues. Then, by our previous results in the SoV approach [1], we know that the associated transfer matrix $T_1^{(K)}(\lambda)$ is diagonalizable with simple spectrum almost for any value of the inhomogeneities under the condition (2.6), and we have the SoV complete characterization of its spectrum.

Let $\{|t_a^{(K)}\rangle,\ a\in\{1,..,3^{\mathsf{N}}\}\}$ be the eigenvector basis and let $\{\langle t_a^{(K)}|,\ a\in\{1,..,3^{\mathsf{N}}\}\}$ be the eigenco-vector basis associated to the transfer matrix $T_1^{(K)}(\lambda)$. We define the two new families of conserved charges:

$$\mathbb{T}_j^{(K)}(\lambda) = \sum_{a=1}^{3^{\mathsf{N}}} t_{j,a}^{(\hat{K})}(\lambda) \frac{|t_a^{(K)}\rangle\langle t_a^{(K)}|}{\langle t_a^{(K)}|t_a^{(K)}\rangle}, \quad \text{with } j\in\{1,2\}. \tag{4.93}$$

Here, we have denoted with $t_{j,a}^{(\hat{K})}(\lambda)$ the spectrum of the transfer matrices $T_j^{(\hat{K})}(\lambda)$ associated to the matrix $\hat{K}$, obtained from K by putting one of its eigenvalue to zero while keeping its spectrum simplicity and its diagonalizable character, i.e.:

$$T_1^{(\hat{K})}(\lambda)|t_a^{(\hat{K})}\rangle = |t_a^{(\hat{K})}\rangle t_{1,a}^{(\hat{K})}(\lambda), \quad T_2^{(\hat{K})}(\lambda)|t_a^{(\hat{K})}\rangle = |t_a^{(\hat{K})}\rangle t_{2,a}^{(\hat{K})}(\lambda), \tag{4.94}$$

$$\langle t_a^{(\hat{K})}|T_1^{(\hat{K})}(\lambda) = t_{1,a}^{(\hat{K})}(\lambda)\langle t_a^{(\hat{K})}|, \quad \langle t_a^{(\hat{K})}|T_2^{(\hat{K})}(\lambda) = t_{2,a}^{(\hat{K})}(\lambda)\langle t_a^{(\hat{K})}|. \tag{4.95}$$

Note that, by construction, the families $\mathbb{T}_j^{(K)}(\lambda)$ are mutually commuting and they commute with the original transfer matrices as they have diagonal form in the eigenbasis of the original transfer matrix $T_1^{(K)}(\lambda)$:

$$\left[\mathbb{T}_l^{(K)}(\lambda),\mathbb{T}_m^{(K)}(\lambda)\right]=\left[T_l^{(K)}(\lambda),\mathbb{T}_m^{(K)}(\lambda)\right]=0,\quad l,m\in\{1,2\},\tag{4.96}$$

and they share the same spectrum as the transfer matrices $T_j^{(\hat{K})}(\lambda)$. Hence, they satisfy the following fusion equations:

$$\mathbb{T}_2^{(K)}(\xi_a^{(1)})\mathbb{T}_1^{(K)}(\xi_a)=\mathbb{T}_2^{(K)}(\xi_a^{(1)})\mathbb{T}_2^{(K)}(\xi_a)=0,\tag{4.97}$$

$$\mathbb{T}_1^{(K)}(\xi_a^{(1)})\mathbb{T}_1^{(K)}(\xi_a)=\mathbb{T}_2^{(K)}(\xi_a).\tag{4.98}$$

We can now use these new family of conserved charges to construct SoV basis according to (3.1) and (3.7) since the twist matrix $\hat{K}$ has simple spectrum:

$$\langle\widehat{\underline{\mathbf{h}}}|\equiv\langle\underline{\mathbf{1}}|\prod_{n=1}^{\mathsf{N}}\mathbb{T}_2^{(K)\delta_{h_n,0}}(\xi_n^{(1)})\mathbb{T}_1^{(K)\delta_{h_n,2}}(\xi_n),\quad\forall\ h_n\in\{0,1,2\},\tag{4.99}$$

$$|\widehat{\underline{\mathbf{h}}}\rangle\equiv\prod_{n=1}^{\mathsf{N}}\mathbb{T}_2^{(K)\delta_{h_n,1}}(\xi_n)\mathbb{T}_1^{(K)\delta_{h_n,2}}(\xi_n)|\underline{\mathbf{0}}\rangle,\quad\forall\ h_n\in\{0,1,2\}.\tag{4.100}$$

They are mutually orthogonal as the direct proof of Theorem 4.1 uses only the fusion relations which are just identical to the above ones (4.97) and (4.98):

$$\langle\widehat{\underline{\mathbf{k}}}|\widehat{\underline{\mathbf{h}}}\rangle=\mathsf{N}_{\underline{\mathbf{h}}}\prod_{a=1}^{\mathsf{N}}\delta_{h_a,k_a}\tag{4.101}$$

$$=\prod_{a=1}^{\mathsf{N}}\delta_{h_a,k_a}\frac{d(\xi_a^{(1)})}{d\left(\xi_a^{\left(1+\delta_{h_a,1}+\delta_{h_a,2}\right)}\right)}\frac{V\left(\xi_1^{(\delta_{h_1,2}+\delta_{h_1,1})},\ldots,,\xi_{\mathsf{N}}^{(\delta_{h_{\mathsf{N}},1}+\delta_{h_{\mathsf{N}},2})}\right)V\left(\xi_1^{(\delta_{h_1,2})},\ldots,,\xi_{\mathsf{N}}^{(\delta_{h_{\mathsf{N}},2})}\right)}{V^2(\xi_1,\ldots,,\xi_{\mathsf{N}})}.\tag{4.102}$$

They are also SoV bases as the spectrum of the $\mathbb{T}_j^{(K)}(\lambda)$ is separate in these bases. We have the following representation of the vector and co-vector of the original transfer matrix $T_1^{(K)}(\lambda)$:

$$|t_a^{(K)}\rangle=\sum_{\underline{\mathbf{h}}}\prod_{n=1}^{\mathsf{N}}t_{2,a}^{(\hat{K})\delta_{h_n,0}}(\xi_n^{(1)})t_{1,a}^{(\hat{K})\delta_{h_n,2}}(\xi_n)\frac{|\widehat{\underline{\mathbf{h}}}\rangle}{\mathsf{N}_{\underline{\mathbf{h}}}},\tag{4.103}$$

$$\langle t_a^{(K)}|=\sum_{\underline{\mathbf{h}}}\prod_{n=1}^{\mathsf{N}}t_{2,a}^{(\hat{K})\delta_{h_n,1}}(\xi_n)t_{1,a}^{(\hat{K})\delta_{h_n,2}}(\xi_n)\frac{\langle\widehat{\underline{\mathbf{h}}}|}{\mathsf{N}_{\underline{\mathbf{h}}}}.\tag{4.104}$$

Moreover, let us comment that separate states of the form

$$|\alpha\rangle=\sum_{\underline{\mathbf{h}}}\prod_{a=1}^{\mathsf{N}}\alpha_a^{(h_a)}\frac{|\widehat{\underline{\mathbf{h}}}\rangle}{\mathsf{N}_{\underline{\mathbf{h}}}},\ \langle\alpha|=\sum_{\underline{\mathbf{h}}}\prod_{a=1}^{\mathsf{N}}\alpha_a^{(h_a)}\frac{\langle\widehat{\underline{\mathbf{h}}}|}{\mathsf{N}_{\underline{\mathbf{h}}}},\tag{4.105}$$

satisfy the same Theorem 4.3 with the transfer matrix $T_1^{(K)}(\lambda)$ eigenvectors. This is easily derived by using the representation of the transfer matrix eigenvector in the SoV bases constructed by the conserved charges $\mathbb{T}_1^{(K)}(\lambda)$, since from them one gets scalar product formulae similar to those of the gl_2 case, even for the simple spectrum invertible K matrix.

5 Conclusions and perspectives

In the present paper we have addressed the problem of computing the scalar products between the left and right SoV bases introduced earlier in [1] (see also [75]) for the fundamental representations of the $\mathcal{Y}(gl_3)$ lattice model with N sites. These SoV bases are determined from chosen sets of conserved charges generated by the transfer matrix. In the model at hand the left and right SoV bases following the construction given in [1] can be written in terms of the transfer matrix $T_1^{(K)}(\lambda)$ and its fused transfer matrix $T_2^{(K)}(\lambda)$. An important feature of these SoV bases is that they are not orthogonal to each other for generic twist matrix K having simple spectrum. The first key result of the present paper is the computation of the matrix of scalar products between these right and left SoV bases as stated in Theorem 3.1.

Theorem 3.1 also shows that whenever the twist matrix K has simple spectrum and zero determinant, the chosen left and right SoV bases are orthogonal to each others since the off-diagonal elements of the matrix of scalar products are all proportional to some strictly positive power of the determinant of K. Moreover, in that case, we have been able to give a direct proof of this result simply using the simplified fusion relations resulting from the vanishing of the corresponding quantum determinant. As a consequence, it leads to very simple formulae for the scalar products of the so-called separate states. In that case they are just given by products of determinants which are similar to the ones of $\mathcal{Y}(gl_2)$ type.

This observation leads us to consider the generalization of these features for the case of a generic twist matrix K having simple spectrum and non zero determinant. This amounts to define new SoV bases constructed from different sets of conserved charges with respect to the one's so far considered. We have shown that such sets of conserved charges indeed exists and we have characterized them using their generating functional $\mathbb{T}_j^{(K)}(\lambda)$ for $j = 1, 2$ defined in (4.93). By using them, we have determined new left and right SoV bases that are indeed orthogonal to each other, leading to simple scalar products formula for their separate states, and in particular for the scalar products of separate states with transfer matrix eigenstates. They are given as products of $\mathcal{Y}(gl_2)$ type determinants. This paves the way for their use in computing form factors and possibly even correlation functions of local operators. For this we will need to be able to write the resolution of the quantum inverse scattering problem in a form suitable to act in a simple manner on separate states. This question is now under study.

Another important question, with regards to the key results obtained in the present paper is how to determine in general sets of charges having properties similar to the one determined in (4.93). One possible route to this could be to construct explicitly the similarity transformation between the operator families $\mathbb{T}_j^{(K)}(\lambda)$ and $T_j^{(\hat{K})}(\lambda)$. In a future publication, we plan to show for example how to compute

$$\langle t_a^{(K)} | t_b^{(\hat{K})} \rangle, \quad \forall a, b \in \{1, .., 3^{\mathsf{N}}\}, \tag{5.1}$$

which just define the matrix elements of the similarity transformation from $\mathbb{T}_1^{(K)}(\lambda)$ to $T_1^{(\hat{K})}(\lambda)$. This seems accessible thanks to the scalar products analyzed in Theorem 3.1. Another important open problem, deserving further analysis, is the possibility to find a direct construction of the new family of conserved charges $\mathbb{T}_j^{(K)}(\lambda)$ in terms of the original transfer matrix $T_1^{(K)}(\lambda)$ or the associated known family of commuting operators like the $T_2^{(K)}(\lambda)$ and the Baxter Q-operators for the general invertible twist K. More generally, the purely algebraic construction of a family satisfying the same simplified form of the fusion equations, like those written in (4.97)-(4.98), is one of our future goals.

Acknowledgements

J.M.M. and G.N. are supported by CNRS and ENS de Lyon. L.V. is supported by Ecole Polytechnique and ENS de Lyon.

A Explicit tensor product form of SoV starting co-vector/vector

Here, we want to prove the statements of the Proposition 3.1 about the fact that given the co-vector $\langle \underline{\mathbf{1}}|$ of tensor product type then we can write explicitly the vector $|\underline{\mathbf{0}}\rangle$ and it has a tensor product form too according to (3.9) and (3.10).

Let us start proving the following general property, that we state for the gl_3 case but that indeed can be extended to the gl_n cases as well for rational R-matrices:

Proposition A.1. *Let K be a 3×3 matrix, then we have the following explicit formula for the product of transfer matrices:*

$$\prod_{j=1}^{\mathsf{M}} T_1^{(K)}(\xi_{a_j}) = n_{a_1,\dots,a_{\mathsf{M}}} R_{a_1;1,\dots,a_1-1}\, \hat{R}^{(a_1)}_{a_2;1,\dots,a_2-1}\cdots \hat{R}^{(a_1,\dots,a_j)}_{a_{j+1},1,\dots,a_{j+1}-1}\cdots \hat{R}^{(a_1,\dots,a_{\mathsf{M}-1})}_{a_{\mathsf{M}};1,\dots,a_{\mathsf{M}}-1}$$
$$\times \bigotimes_{j=1}^{\mathsf{M}} K_{a_j} \hat{R}^{(a_2,\dots,a_{\mathsf{M}})}_{a_1;a_1+1,\dots,\mathsf{N}}\cdots \hat{R}^{(a_{j+1},\dots,a_{\mathsf{M}})}_{a_j;a_j+1,\dots,\mathsf{N}}\cdots \hat{R}^{(a_{\mathsf{M}})}_{a_{\mathsf{M}-1};a_{\mathsf{M}-1}+1,\dots,\mathsf{N}} R_{a_{\mathsf{M}};a_{\mathsf{M}}+1,\dots,\mathsf{N}}, \tag{A.1}$$

where we have taken $a_1<a_2<\cdots<a_{\mathsf{M}-1}<a_{\mathsf{M}}$ and $\mathsf{M}\leq\mathsf{N}$ and we have used the notation:

$$R_{a;b_1,\dots,b_M} = R_{ab_M}(\xi_a-\xi_{b_M})\cdots R_{ab_1}(\xi_a-\xi_{b_1}), \tag{A.2}$$

while $\hat{R}^{(b_{a_1},\dots,b_{a_k})}_{a;b_1,\dots,b_M}$ denotes the same product of R-matrices however with the factors $R_{ab_{a_1}}$ up to $R_{ab_{a_k}}$ omitted and $n_{a_1,\dots,a_{\mathsf{M}}}=\prod_{i<j} n_{a_i,a_j}$, with $n_{a_i,a_j}=\eta^2-(\xi_{a_i}-\xi_{a_j})^2$. Then, for any choice of $1\leq h_{a_j}\leq 2$ we have:

$$\langle\underline{\mathbf{0}}|\prod_{j=1}^{\mathsf{M}} T_1^{(K)}(\xi_{a_j})^{h_{a_j}} = \sum_{r_1\in A_1,\dots,r_{\mathsf{M}}\in A_{\mathsf{M}}}\ \sum_{s_1\in B_1,\dots,s_{\mathsf{M}}\in B_{\mathsf{M}}} C_{r_1,\dots,r_{\mathsf{M}},s_1,\dots,s_{\mathsf{M}}}$$
$$\times V(r_1,\dots,r_{\mathsf{M}})V(s_1,\dots,s_{\mathsf{M}})\langle\underline{\mathbf{0}}|\bigotimes_{j=1}^{\mathsf{M}} K_{r_j}\bigotimes_{j=1}^{\mathsf{M}} K_{s_j}^{h_j-1}, \tag{A.3}$$

where we take the following tensor product form for the co-vector:

$$\langle\underline{\mathbf{0}}| = \bigotimes_{a=1}^{\mathsf{N}}\langle 0,a|. \tag{A.4}$$

$V(x_1,\dots,x_{\mathsf{M}})$ is the Vandermonde determinant, and $C_{r_1,\dots,r_{\mathsf{M}},s_1,\dots,s_{\mathsf{M}}}$ are some finite non-zero coefficients. We have defined:

$$A_j = \{a_j,\dots,\mathsf{N}\}\cup\begin{cases}\{1,\dots,a_j-1\} & \text{if } h_{a_j}=2,\\ \emptyset & \text{if } h_{a_j}=1,\end{cases} \tag{A.5}$$

$$B_j = a_j\cup\begin{cases}\{a_j+1,\dots,\mathsf{N}\} & \text{if } h_{a_j}=2,\\ \emptyset & \text{if } h_{a_j}=1.\end{cases} \tag{A.6}$$

Proof. Let us consider the following product:

$$R_{a;a+1,\dots,\mathsf{N}}R_{a+l;1,\dots,a+l-1} = R_{a;a+2,\dots,\mathsf{N}}R_{a+l;a+2,\dots,a+l-1}R_{a,a+1}R_{a+l,a+1}R_{a+l,a}R_{a+l;1,\dots,a-1} \tag{A.7}$$

$$= R_{a;a+2,\dots,\mathsf{N}}R_{a+l;a+2,\dots,a+l-1}R_{a+l,a}R_{a+l,a+1}R_{a,a+1}R_{a+l;1,\dots,a-1} \tag{A.8}$$

$$= R_{a;a+3,\dots,\mathsf{N}}R_{a+l;a+3,\dots,a+l-1}R_{a,a+2}R_{a+l,a+2}R_{a+l,a} \times R_{a+l,a+1}R_{a+l;1,\dots,a-1}R_{a,a+1} \tag{A.9}$$

$$= R_{a;a+3,\dots,\mathsf{N}}R_{a+l;a+3,\dots,a+l-1}R_{a+l,a}R_{a+l,a+2}R_{a+l,a+1} \times R_{a+l;1,\dots,a-1}R_{a,a+2}R_{a,a+1}, \tag{A.10}$$

where we have used the commutativity of R-matrices on different spaces and the Yang-Baxter equation. So, by iterating it, we get:

$$R_{a;a+1,\dots,\mathsf{N}}R_{a+l;1,\dots,a+l-1} = n_{a,a+l}\hat{R}^{(a)}_{a+l;1,\dots,a+l-1}\hat{R}^{(a+l)}_{a;a+1,\dots,\mathsf{N}}, \tag{A.11}$$

once we use that:

$$R_{a,a+l}R_{a+l,a} = n_{a,a+l}. \tag{A.12}$$

From this identity we get:

$$T_1^{(K)}(\xi_a)T_1^{(K)}(\xi_{a+l}) = n_{a,a+l}R_{a;1,\dots,a-1}\hat{R}^{(a)}_{a+l;1,\dots,a+l-1}K_a \bigotimes K_{a+l}\hat{R}^{(a+l)}_{a;a+1,\dots,\mathsf{N}}R_{a+l;a+l+1,\dots,\mathsf{N}}, \tag{A.13}$$

from which we easily obtain our statement (A.1) in the case $\mathsf{M}=2$. The general case is proven by induction on M. To get the $\mathsf{M}+1$ case knowing that the formula (A.1) is satisfied for M, we need to prove the following equality for $a_{l-1} < a_l < \cdots < a_k < a_{k+1}$:

$$\hat{R}^{(a_l,\dots,a_k)}_{a_{l-1};a_{l-1}+1,\dots,\mathsf{N}}\hat{R}^{(a_l,\dots,a_k)}_{a_{k+1};1,\dots,a_{k+1}-1} = n_{a_{k+1},a_{l-1}}\hat{R}^{(a_{l-1},\dots,a_k)}_{a_{k+1};1,\dots,a_{k+1}-1}\hat{R}^{(a_l,\dots,a_{k+1})}_{a_{l-1};a_{l-1}+1,\dots,\mathsf{N}}. \tag{A.14}$$

We have the following chain of equalities using the Yang-Baxter commutation relations, then the unitarity relation for the R-matrix and in the last step the fact that two R-matrices acting on different spaces commute:

$$\begin{aligned}
&\hat{R}^{(a_l,\dots,a_k)}_{a_{l-1};a_{l-1}+1,\dots,\mathsf{N}}\hat{R}^{(a_l,\dots,a_k)}_{a_{k+1};1,\dots,a_{k+1}-1} = \\
&= \hat{R}^{(a_l,\dots,a_k)}_{a_{l-1};a_{k+1}+1,\dots,\mathsf{N}}R_{a_{l-1}a_{k+1}}\hat{R}^{(a_l,\dots,a_k)}_{a_{l-1};a_{l-1}+1,\dots,a_{k+1}-1}\hat{R}^{(a_l,\dots,a_k)}_{a_{k+1};a_{l-1}+1,\dots,a_{k+1}-1}R_{a_{k+1}a_{l-1}}\hat{R}^{(a_l,\dots,a_k)}_{a_{k+1};1,\dots,a_{l-1}-1} \\
&= \hat{R}^{(a_l,\dots,a_k)}_{a_{l-1};a_{k+1}+1,\dots,\mathsf{N}}\hat{R}^{(a_l,\dots,a_k)}_{a_{k+1};a_{l-1}+1,\dots,a_{k+1}-1}\hat{R}^{(a_l,\dots,a_k)}_{a_{l-1};a_{l-1}+1,\dots,a_{k+1}-1}R_{a_{l-1}a_{k+1}}R_{a_{k+1}a_{l-1}}\hat{R}^{(a_l,\dots,a_k)}_{a_{k+1};1,\dots,a_{l-1}-1} \\
&= n_{a_{k+1},a_{l-1}}\hat{R}^{(a_l,\dots,a_k)}_{a_{k+1};a_{l-1}+1,\dots,a_{k+1}-1}\hat{R}^{(a_l,\dots,a_k)}_{a_{k+1};1,\dots,a_{l-1}-1}\hat{R}^{(a_l,\dots,a_k)}_{a_{l-1};a_{k+1}+1,\dots,\mathsf{N}}\hat{R}^{(a_l,\dots,a_k)}_{a_{l-1};a_{l-1}+1,\dots,a_{k+1}-1} \\
&= n_{a_{k+1},a_{l-1}}\hat{R}^{(a_{l-1},\dots,a_k)}_{a_{k+1};1,\dots,a_{k+1}-1}\hat{R}^{(a_l,\dots,a_{k+1})}_{a_{l-1};a_{l-1}+1,\dots,\mathsf{N}}.
\end{aligned} \tag{A.15}$$

et us prove the induction going from M to $\mathsf{M}+1$. We have:

$$\begin{aligned}
\prod_{j=1}^{\mathsf{M}+1}T_1^{(K)}(\xi_{a_j}) &= \prod_{j=1}^{\mathsf{M}}T_1^{(K)}(\xi_{a_j})\,T_1^{(K)}(\xi_{a_{\mathsf{M}+1}}) \\
&= n_{a_1,\dots,a_{\mathsf{M}}}R_{a_1;1,\dots,a_1-1}\hat{R}^{(a_1)}_{a_2;1,\dots,a_2-1}\cdots\hat{R}^{(a_1,\dots,a_j)}_{a_{j+1};1,\dots,a_{j+1}-1}\cdots\hat{R}^{(a_1,\dots,a_{\mathsf{M}-1})}_{a_{\mathsf{M}};1,\dots,a_{\mathsf{M}}-1} \\
&\times\bigotimes_{j=1}^{\mathsf{M}}K_{a_j}\hat{R}^{(a_2,\dots,a_{\mathsf{M}})}_{a_1;a_1+1,\dots,\mathsf{N}}\cdots\hat{R}^{(a_{j+1},\dots,a_{\mathsf{M}})}_{a_j;a_j+1,\dots,\mathsf{N}}\cdots\hat{R}^{(a_{\mathsf{M}})}_{a_{\mathsf{M}-1};a_{\mathsf{M}-1}+1,\dots,\mathsf{N}}R_{a_{\mathsf{M}};a_{\mathsf{M}}+1,\dots,\mathsf{N}} \\
&\times R_{a_{\mathsf{M}+1};1,\dots,a_{\mathsf{M}+1}-1}K_{a_{\mathsf{M}+1}}R_{a_{\mathsf{M}+1};a_{\mathsf{M}+1}+1,\dots,\mathsf{N}}.
\end{aligned} \tag{A.16}$$

Then, keeping the last factor as it is and moving the term $R_{a_{\mathsf{M}+1};1,\dots,a_{\mathsf{M}+1}-1}$ to the left using the above proven exchange relation (A.14) successively, and then moving $K_{a_{\mathsf{M}+1}}$ freely (there is no

object acting in the same space) to the left until it will join the products of other matrices K, we get the desired result.

We have to use now that $\langle\underline{\mathbf{0}}|$ is an eigenco-vector for a generic product of rational R-matrices acting on the local quantum spaces, hence:

$$\langle\underline{\mathbf{0}}|R_{a_1;1,\dots,a_1-1}\hat{R}^{(a_1)}_{a_2;1,\dots,a_2-1}\cdots\hat{R}^{(a_1,\dots,a_j)}_{a_{j+1};1,\dots,a_{j+1}-1}\cdots\hat{R}^{(a_1,\dots,a_{\mathsf{M}-1})}_{a_{\mathsf{M}};1,\dots,a_{\mathsf{M}}-1}=m_{a_1,\dots,a_{\mathsf{M}}}\langle\underline{\mathbf{0}}|, \tag{A.17}$$

with $m_{a_1,\dots,a_{\mathsf{M}}}$ some calculable non-zero coefficient. Using the explicit formula for the R-matrix, this implies the following identity:

$$\begin{aligned}\langle\underline{\mathbf{0}}|\prod_{j=1}^{\mathsf{M}}T_1^{(K)}(\xi_{a_j})=n_{a_1,\dots,a_{\mathsf{M}}}m_{a_1,\dots,a_{\mathsf{M}}}\langle\underline{\mathbf{0}}|\bigotimes_{j=1}^{\mathsf{M}}K_{a_j}\hat{R}^{(a_2,\dots,a_{\mathsf{M}})}_{a_1;a_1+1,\dots,\mathsf{N}}\cdots\hat{R}^{(a_{j+1},\dots,a_{\mathsf{M}})}_{a_j;a_j+1,\dots,\mathsf{N}}\cdots\\ \cdots\hat{R}^{(a_{\mathsf{M}})}_{a_{\mathsf{M}-1};a_{\mathsf{M}-1}+1,\dots,\mathsf{N}}R_{a_{\mathsf{M}};a_{\mathsf{M}}+1,\dots,\mathsf{N}},\end{aligned} \tag{A.18}$$

and so:

$$\langle\underline{\mathbf{0}}|\prod_{j=1}^{\mathsf{M}}T_1^{(K)}(\xi_{a_j})=\sum_{r_1\in\{a_1+1,\dots,\mathsf{N}\},\dots,r_{\mathsf{M}}\in\{a_{\mathsf{M}}+1,\dots,\mathsf{N}\}}C_{r_1,\dots,r_{\mathsf{M}}}V(r_1,\dots,r_{\mathsf{M}})\langle\underline{\mathbf{0}}|\bigotimes_{j=1}^{\mathsf{M}}K_{r_j}. \tag{A.19}$$

Applying once again this formula, we get our second statement. □

The following lemma holds for a general simple K matrix.

Lemma A.1. *Let K be a 3×3 w-simple matrix, then if we chose the tensor product form:*

$$\langle\underline{\mathbf{1}}|=\left(\bigotimes_{a=1}^{\mathsf{N}}\langle 1,a|\right)\Gamma_W^{-1},\quad \Gamma_W=\bigotimes_{a=1}^{\mathsf{N}}W_{K,a}, \tag{A.20}$$

we have that the vector $|\underline{\mathbf{0}}\rangle$ defined in (3.5) has the tensor product form:

$$|\underline{\mathbf{0}}\rangle=\Gamma_W\bigotimes_{a=1}^{\mathsf{N}}|0,a\rangle, \tag{A.21}$$

where $|0,a\rangle$ has the form (3.10) and it satisfies the following local properties

$$\langle 1,a|\tilde{K}_J^{(a)}|0,a\rangle=1/\,\text{q-det}\,M^{(I)}(\xi_a), \tag{A.22}$$

$$\langle 1,a|(K_J^{(a)})^h|0,a\rangle=0,\quad \textit{for } h=0,1, \tag{A.23}$$

where $\tilde{K}_J$ is the adjoint matrix of K_J:

$$\tilde{K}_JK_J=K_J\tilde{K}_J=\det K. \tag{A.24}$$

Proof. Let us take the following normalization for the SoV co-vector basis:

$$\langle h_1,\dots,h_{\mathsf{N}}|=\langle\underline{\mathbf{0}}|\prod_{n=1}^{\mathsf{N}}\frac{T_1^{(K)}(\xi_n)^{h_n}}{q\text{-det}M^{(K)}(\xi_n)^{(1-\delta_{h_n,0})}}, \tag{A.25}$$

where we have defined:

$$\langle\underline{\mathbf{0}}|=\langle\underline{\mathbf{1}}|\prod_{a=1}^{\mathsf{N}}q\text{-det}M^{(I)}(\xi_a)\bigotimes_{a=1}^{\mathsf{N}}\tilde{K}^{(a)}. \tag{A.26}$$

Now we can use the previous lemma to get the following statement:

$$\langle h_1,...,h_{\mathsf{N}}| = \sum_{k_1,...,k_{\mathsf{N}}=0,1,2} c^{k_1,...,k_{\mathsf{N}}}_{h_1,...,h_{\mathsf{N}}} \langle \underline{\mathbf{0}}| \bigotimes_{a=1}^{\mathsf{N}} K^{(a)k_a}, \tag{A.27}$$

with

$$c^{k_1=0,...,k_{\mathsf{N}}=0}_{h_1,...,h_{\mathsf{N}}} = 0 \ \text{ if } \exists j \in \{1,...,\mathsf{N}\} : h_j \neq 0. \tag{A.28}$$

By definition:

$$\langle \underline{\mathbf{0}}| \bigotimes_{a=1}^{\mathsf{N}} K^{(a)k_a} |\underline{\mathbf{0}}\rangle = \prod_{a=1}^{\mathsf{N}} \langle 0,a|K_J^{(a)k_a}|0,a\rangle = 0 \ \text{ if } \exists a \in \{1,...,\mathsf{N}\} : k_a \neq 0, \tag{A.29}$$

so we get:

$$\langle h_1,...,h_{\mathsf{N}}|\underline{\mathbf{0}}\rangle = 0 \ \text{ if } \exists j \in \{1,...,\mathsf{N}\} : h_j \neq 0. \tag{A.30}$$

The fact that:

$$\langle \underline{\mathbf{0}}|\underline{\mathbf{0}}\rangle = \prod_{a=1}^{\mathsf{N}} \langle 0,a|0,a\rangle = 1 \tag{A.31}$$

is proven by direct computations.

Finally, let us observe that the following identities:

$$\langle \underline{\mathbf{0}}| \prod_{n=1}^{\mathsf{N}} \frac{\left(T_1^{(K)}(\xi_n)\right)^{h_n}}{\left(q\text{-det}M^{(K)}(\xi_n)\right)^{1-\delta_{h_n,0}}} = \langle \underline{\mathbf{1}}| \prod_{n=1}^{\mathsf{N}} T_2^{(K)\delta_{h_n,0}}(\xi_n-\eta) T_1^{(K)\delta_{h_n,2}}(\xi_n), \tag{A.32}$$

holds for any $h_n \in \{0,1,2\}$. Now, in the limit $\det K \to 0$, keeping K a 3×3 w-simple matrix[16], we have that the r.h.s. of the equation (A.32) is well defined and it defines the limit of the l.h.s., so that our co-vector SoV basis goes back to the one defined in the case $\det K = 0$. Moreover, the $|0,a\rangle$ are well defined and so the $|\underline{\mathbf{0}}\rangle$ above defined in this limit still satisfies (3.5). □

B Orthogonal co-vector/vector SoV basis for gl_2 representations

Here, we consider the fundamental representations of the gl_2 Yang-Baxter algebra associated to generic quasi-periodic boundary conditions, with transfer matrix:

$$T^{(K)}(\lambda) \equiv \text{tr}_{V_a} K_a R_{a,\mathsf{N}}(\lambda-\xi_{\mathsf{N}})\cdots R_{a,1}(\lambda-\xi_1) \in \text{End}(\mathcal{H}), \tag{B.1}$$

where $\mathcal{H}$ is the quantum space of the representation, $R_{a,b}(\lambda) \in \text{End}(V_a \otimes V_b)$, $V_a \simeq \mathbb{C}^2$, $V_b \simeq \mathbb{C}^2$ is the rational 6-vertex R-matrix solution of the Yang-Baxter equation and the twist matrix reads

$$K = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{End}(\mathbb{C}^2). \tag{B.2}$$

The construction of the orthogonal co-vector and vector SoV bases for these gl_2 representations is here implemented to define a reference to compare with for the more involved constructions that we have considered in this paper for gl_3 representations. One should mention that up similarity transformations[17] the SoV bases in these gl_2 cases are already available in the literature using the framework of the traditional Sklyanin's SoV construction, see for

[16] That is according to the three cases considered in the Theorem 3.1.

[17] As discussed in section 3.4 of [1]

example [40] for the antiperiodic case and [55] for more general twists. However, here we are interested in implementing these constructions entirely inside our new approach [1].

The following proposition allows to produce the orthogonal basis to the left SoV basis

$$\langle h_1,...,h_{\mathsf{N}}| \equiv \langle \underline{\mathbf{0}}| \prod_{a=1}^{\mathsf{N}} \left(\frac{T^{(K)}(\xi_a)}{a(\xi_a)} \right)^{h_a} \quad \text{for any } \{h_1,...,h_{\mathsf{N}}\} \in \{0,1\}^{\mathsf{N}}, \tag{B.3}$$

and to show that itself is of SoV type just using the polynomial form of the transfer matrix and the fusion equations.

Let us denote with $|\underline{\mathbf{0}}\rangle$ the non-zero vector orthogonal to all the SoV co-vectors with the exception of $\langle \underline{\mathbf{0}}|$, i.e.

$$\langle h_1,...,h_{\mathsf{N}}|\underline{\mathbf{0}}\rangle = \frac{\prod_{n=1}^{\mathsf{N}} \delta_{h_n,0}}{V^2(\xi_1,...,\xi_{\mathsf{N}})}, \quad \forall \{h_1,...,h_{\mathsf{N}}\} \in \{0,1\}^{\mathsf{N}}, \tag{B.4}$$

with $\langle h_1,\dots,h_{\mathsf{N}}|$ the set of SoV co-vectors a basis. $|\underline{\mathbf{0}}\rangle$ is uniquely defined with the above normalization. Similarly, we can introduce the non-zero vector $|\underline{\mathbf{1}}\rangle$ orthogonal to all the SoV co-vectors with the exception of $\langle 1,...,1|$, i.e.

$$\langle h_1,...,h_{\mathsf{N}}|\underline{\mathbf{1}}\rangle = \frac{\prod_{n=1}^{\mathsf{N}} \delta_{h_n,1}}{V(\xi_1,...,\xi_{\mathsf{N}})V(\xi_1^{(1)},...,\xi_{\mathsf{N}}^{(1)})}, \quad \forall \{h_1,...,h_{\mathsf{N}}\} \in \{0,1\}^{\mathsf{N}}, \tag{B.5}$$

which also fixes the normalization of $|\underline{\mathbf{1}}\rangle$.

Proposition B.1. *Under the same conditions assuring that the set of SoV co-vectors is a basis (i.e. almost any choice of $\langle \underline{\mathbf{0}}|$, $K \neq xI$, for any $x \in \mathbb{C}$, and the condition (2.6)), then the following set of vectors:*

$$|h_1,...,h_{\mathsf{N}}\rangle = \prod_{a=1}^{\mathsf{N}} \left(\frac{T^{(K)}(\xi_a - \eta)}{a(\xi_a)} \right)^{1-h_a} |\underline{\mathbf{1}}\rangle, \quad \forall \{h_1,...,h_{\mathsf{N}}\} \in \{0,1\}^{\mathsf{N}} \tag{B.6}$$

forms an orthogonal basis to the left SoV basis:

$$\langle h_1,...,h_{\mathsf{N}}|k_1,...,k_{\mathsf{N}}\rangle = \frac{\prod_{n=1}^{\mathsf{N}} \delta_{h_n,k_n}}{V(\xi_1,...,\xi_{\mathsf{N}})V(\xi_1^{(h_1)},...,\xi_{\mathsf{N}}^{(h_{\mathsf{N}})})}. \tag{B.7}$$

Let $t(\lambda)$ be an element of the spectrum of $T^{(K)}(\lambda)$, then the uniquely defined eigenvector $|t\rangle$ and eigenco-vector $\langle t|$ admit the following SoV representations:

$$|t\rangle = \sum_{h_1,...,h_{\mathsf{N}}=0}^{1} \prod_{a=1}^{\mathsf{N}} \left(\frac{t(\xi_a)}{a(\xi_a)} \right)^{h_a} V(\xi_1^{(h_1)},...,\xi_{\mathsf{N}}^{(h_{\mathsf{N}})})|h_1,...,h_{\mathsf{N}}\rangle, \tag{B.8}$$

$$\langle t| = \sum_{h_1,...,h_{\mathsf{N}}=0}^{1} \prod_{a=1}^{\mathsf{N}} \left(\frac{t(\xi_a - \eta)}{a(\xi_a)} \right)^{1-h_a} V(\xi_1^{(h_1)},...,\xi_{\mathsf{N}}^{(h_{\mathsf{N}})})\langle h_1,...,h_{\mathsf{N}}|, \tag{B.9}$$

where we have fixed their normalization by imposing:

$$\langle \underline{\mathbf{0}}|t\rangle = \langle t|\underline{\mathbf{1}}\rangle = 1/V(\xi_1,...,\xi_{\mathsf{N}}). \tag{B.10}$$

Proof. Let us start proving the orthogonality condition:

$$\langle h_1,...,h_{\mathsf{N}}|k_1,...,k_{\mathsf{N}}\rangle = 0, \quad \forall \{k_1,...,k_{\mathsf{N}}\} \neq \{h_1,...,h_{\mathsf{N}}\} \in \{0,1\}^{\mathsf{N}}. \tag{B.11}$$

The proof is done by induction, assuming that it is true for any vector $|k_1,...,k_{\mathsf{N}}\rangle$ with $\sum_{n=1}^{\mathsf{N}} k_n = \mathsf{N}-l$ for $l \leq \mathsf{N}-1$ and proving it for vectors $|k'_1,...,k'_{\mathsf{N}}\rangle$ with $\sum_{n=1}^{\mathsf{N}} k'_n = \mathsf{N}-(l+1)$. To this aim we fix a vector $|k_1,...,k_{\mathsf{N}}\rangle$ with $\sum_{n=1}^{\mathsf{N}} k_n = \mathsf{N}-l$ and we denote with π a permutation on the set $\{1,...,\mathsf{N}\}$ such that:

$$k_{\pi(a)} = 0 \text{ for } a \leq l \text{ and } k_{\pi(a)} = 1 \text{ for } l < a, \tag{B.12}$$

then we compute:

$$\langle h_1,...,h_{\mathsf{N}}|T^{(K)}(\xi^{(1)}_{\pi(l+1)})|k_1,...,k_{\mathsf{N}}\rangle = a(\xi_{\pi(l+1)})\langle h_1,...,h_{\mathsf{N}}|k'_1,...,k'_{\mathsf{N}}\rangle\,, \tag{B.13}$$

where we have defined:

$$k'_{\pi(a)} = k_{\pi(a)},\ \forall a \in \{1,...,\mathsf{N}\}\backslash\{l+1\} \text{ and } k'_{\pi(l+1)} = 0, \tag{B.14}$$

for any $\{h_1,...,h_{\mathsf{N}}\} \neq \{k'_1,...,k'_{\mathsf{N}}\} \in \{0,1\}^{\mathsf{N}}$. There are two cases. The first case is $h_{\pi(l+1)} = 1$, then it holds:

$$\langle h_1,...,h_{\mathsf{N}}|T^{(K)}(\xi^{(1)}_{\pi(l+1)})|k_1,...,k_{\mathsf{N}}\rangle = \frac{q\text{-det}M^{(K)}(\xi_{\pi(l+1)})}{a(\xi_{\pi(l+1)})}\langle h'_1,...,h'_{\mathsf{N}}|k_1,...,k_{\mathsf{N}}\rangle, \tag{B.15}$$

where we have defined:

$$h'_{\pi(a)} = h_{\pi(a)},\ \forall a \in \{1,...,\mathsf{N}\}\backslash\{l+1\} \text{ and } h'_{\pi(l+1)} = 0. \tag{B.16}$$

Then from $\{h_1,...,h_{\mathsf{N}}\} \neq \{k'_1,...,k'_{\mathsf{N}}\} \in \{0,1\}^{\mathsf{N}}$ it follows also that $\{h'_1,...,h'_{\mathsf{N}}\} \neq \{k_1,...,k_{\mathsf{N}}\} \in \{0,1\}^{\mathsf{N}}$ and so the induction implies that the r.h.s. of (B.15) is zero and so we get:

$$\langle h_1,...,h_{\mathsf{N}}|k'_1,...,k'_{\mathsf{N}}\rangle = 0. \tag{B.17}$$

The second case is $h_{\pi(l+1)} = 0$. We can use the following interpolation formula:

$$T^{(K)}(\xi^{(1)}_{\pi(l+1)}) = (\mathrm{tr}K)\prod_{a=1}^{\mathsf{N}}(\xi^{(1)}_{\pi(l+1)} - \xi^{(h_{\pi(a)})}_{\pi(a)}) + \sum_{a=1}^{\mathsf{N}}\prod_{b\neq a,b=1}^{\mathsf{N}}\frac{\xi^{(1)}_{\pi(l+1)} - \xi^{(h_{\pi(b)})}_{\pi(b)}}{\xi^{(h_{\pi(a)})}_{\pi(a)} - \xi^{(h_{\pi(b)})}_{\pi(b)}}T^{(K)}\Big(\xi^{(h_{\pi(a)})}_{\pi(a)}\Big), \tag{B.18}$$

from which $\langle h_1,...,h_{\mathsf{N}}|T^{(K)}(\xi^{(1)}_{\pi(l+1)},\{\xi\})|k_1,...,k_{\mathsf{N}}\rangle$ reduces to the following sum:

$$\begin{aligned}&(\mathrm{tr}K)\prod_{a=1}^{\mathsf{N}}(\xi^{(1)}_{\pi(l+1)} - \xi^{(h_{\pi(a)})}_{\pi(a)})\ \langle h_1,...,h_{\mathsf{N}}|k_1,...,k_{\mathsf{N}}\rangle + \sum_{a=1}^{\mathsf{N}}\prod_{b\neq a,b=1}^{\mathsf{N}}\frac{\xi^{(1)}_{\pi(l+1)} - \xi^{(h_{\pi(b)})}_{\pi(b)}}{\xi^{(h_{\pi(a)})}_{\pi(a)} - \xi^{(h_{\pi(b)})}_{\pi(b)}}\\ &\times\frac{(\text{q-det}\,M^{(K)}(\xi_{\pi(a)}))^{h_{\pi(a)}}}{\big(a(\xi_{\pi(a)})\big)^{2h_{\pi(a)}-1}}\langle h^{(a)}_1,...,h^{(a)}_{\mathsf{N}}|k_1,...,k_{\mathsf{N}}\rangle,\end{aligned} \tag{B.19}$$

where we have defined:

$$h^{(a)}_{\pi(j)} = h_{\pi(j)},\ \forall j \in \{1,...,\mathsf{N}\}\backslash\{a\} \text{ and } h^{(a)}_{\pi(a)} = 1 - h_{\pi(a)}. \tag{B.20}$$

Let us now note that from $h_{\pi(l+1)} = 0$ it follows that $\{h_1,...,h_{\mathsf{N}}\} \neq \{k_1,...,k_{\mathsf{N}}\}$, as $k_{\pi(l+1)} = 1$ by definition and similarly $\{h^{(a)}_1,...,h^{(a)}_{\mathsf{N}}\} \neq \{k_1,...,k_{\mathsf{N}}\}$ being by definition $h^{(a)}_{\pi(l+1)} = h_{\pi(l+1)} = 0$ for any $a \in \{1,...,\mathsf{N}\}\backslash\{l+1\}$. Finally, from $\{h_1,...,h_{\mathsf{N}}\} \neq \{k'_1,...,k'_{\mathsf{N}}\}$ with $h_{\pi(l+1)} = k'_{\pi(l+1)} = 0$, clearly it follows that $\{h^{(l+1)}_1,...,h^{(l+1)}_{\mathsf{N}}\} \neq \{k_1,...,k_{\mathsf{N}}\}$. So, by using the induction argument,

we get that any term in the above sum is zero. So that also in the case $h_{\pi(l+1)}=0$, we get that (B.17) is satisfied, and so it is satisfied for any $\{h_1,...,h_{\mathsf{N}}\}\neq\{k'_1,...,k'_{\mathsf{N}}\}$ which proves the induction of the orthogonality to $l+1$. Indeed, by changing the permutation π we can both take for $\{\pi(1),...,\pi(l)\}$ any subset of cardinality l in $\{1,...,\mathsf{N}\}$ and with $\pi(l+1)$ any element in its complement $\{1,...,\mathsf{N}\}\backslash\{\pi(1),...,\pi(l)\}$.

We can compute now the left/right normalization, and to do this we just need to compute the following type of ratio:

$$\frac{\langle h_1^{(a)},...,h_{\mathsf{N}}^{(a)}|h_1^{(a)},...,h_{\mathsf{N}}^{(a)}\rangle}{\langle \bar{h}_1^{(a)},...,\bar{h}_{\mathsf{N}}^{(a)}|\bar{h}_1^{(a)},...,\bar{h}_{\mathsf{N}}^{(a)}\rangle}=a(\xi_a)\frac{\langle h_1^{(a)},...,h_{\mathsf{N}}^{(a)}|h_1^{(a)},...,h_{\mathsf{N}}^{(a)}\rangle}{\langle \bar{h}_1^{(a)},...,\bar{h}_{\mathsf{N}}^{(a)}|T^{(K)}(\xi_a^{(1)})|h_1^{(a)},...,h_{\mathsf{N}}^{(a)}\rangle}, \tag{B.21}$$

with $\bar{h}_j^{(a)}=h_j^{(a)}$ for any $j\in\{1,...,\mathsf{N}\}\backslash\{a\}$ while $\bar{h}_j^{(a)}=0$ and $h_j^{(a)}=1$. We can use now once again the interpolation formula (B.18) which by the orthogonality condition produces only one non-zero term, the one associate to $T^{(K)}(\xi_a,\{\xi\})$. It holds:

$$\frac{\langle h_1^{(a)},...,h_{\mathsf{N}}^{(a)}|h_1^{(a)},...,h_{\mathsf{N}}^{(a)}\rangle}{\langle \bar{h}_1^{(a)},...,\bar{h}_{\mathsf{N}}^{(a)}|\bar{h}_1^{(a)},...,\bar{h}_{\mathsf{N}}^{(a)}\rangle}=\prod_{b\neq a,b=1}^{\mathsf{N}}\frac{\xi_a-\xi_b^{(h_b)}}{\xi_a^{(1)}-\xi_b^{(h_b)}}. \tag{B.22}$$

Using the above result, it is now standard to get the proof of the Vandermonde determinant form for the normalization.

Let us note that being the set of SoV co-vectors and vectors basis in $\mathcal{H}$, it follows that for any transfer matrix eigenstates $|t\rangle$ and $\langle t|$ there exists at least a $\{r_1,...,r_{\mathsf{N}}\}\in\{0,1\}^{\mathsf{N}}$ and a $\{s_1,...,s_{\mathsf{N}}\}\in\{0,1\}^{\mathsf{N}}$ such that:

$$\langle r_1,...,r_{\mathsf{N}}|t\rangle\neq 0,\quad \langle t|s_1,...,s_{\mathsf{N}}\rangle\neq 0, \tag{B.23}$$

which together with the identities:

$$\langle h_1,...,h_{\mathsf{N}}|t\rangle\propto\langle\underline{\mathbf{0}}|t\rangle,\quad \langle t|h_1,...,h_{\mathsf{N}}\rangle\propto\langle t|\underline{\mathbf{1}}\rangle,\quad \forall\{h_1,...,h_{\mathsf{N}}\}\in\{0,1\}^{\mathsf{N}}, \tag{B.24}$$

imply that:

$$\langle\underline{\mathbf{0}}|t\rangle\neq 0,\quad \langle t|\underline{\mathbf{1}}\rangle\neq 0. \tag{B.25}$$

So we are free to fix the normalization of $|t\rangle$ and $\langle t|$ by (B.10). Finally, the representations for these eigenco-vectors and eigenvectors follow from the use of the SoV decomposition of the identity:

$$\mathbb{I}=V(\{\xi\})\sum_{h_1,...,h_{\mathsf{N}}=0}^{1}V(\xi_1^{(h_1)},...,\xi_{\mathsf{N}}^{(h_{\mathsf{N}})})|h_1,...,h_{\mathsf{N}}\rangle\langle h_1,...,h_{\mathsf{N}}|. \tag{B.26}$$

□

Corollary B.1. *Let us assume that the condition (2.6) is satisfied and that $K\neq xI$, for any $x\in\mathbb{C}$, and furthermore $\det K\neq 0$, then the vectors of the right SoV basis admit also the following representations:*

$$|h_1,...,h_{\mathsf{N}}\rangle=\prod_{a=1}^{\mathsf{N}}\left(\frac{T^{(K)}(\xi_a)}{\det K\; d(\xi_a^{(1)})}\right)^{h_a}|\underline{\mathbf{0}}\rangle,\quad \forall\{h_1,...,h_{\mathsf{N}}\}\in\{0,1\}^{\mathsf{N}}, \tag{B.27}$$

as well as for any element of the spectrum of $T^{(K)}(\lambda)$ the unique associated eigenco-vector $\langle t|$ admit the following SoV representations:

$$\langle t|=\mathsf{N}_t\sum_{h_1,...,h_{\mathsf{N}}=0}^{1}\prod_{a=1}^{\mathsf{N}}\left(\frac{t(\xi_a)}{\det K\; d(\xi_a^{(1)})}\right)^{h_a}V(\xi_1^{(h_1)},...,\xi_{\mathsf{N}}^{(h_{\mathsf{N}})})\langle h_1,...,h_{\mathsf{N}}|, \tag{B.28}$$

once we fix the normalization by (B.10), where we have defined:

$$\mathsf{N}_t = \langle t|\underline{\mathbf{0}}\rangle = \prod_{a=1}^{\mathsf{N}} \frac{t(\xi_a^{(1)})}{a(\xi_a)} \neq 0. \tag{B.29}$$

Proof. Taking into account the chosen normalizations clearly it holds:

$$|\underline{\mathbf{0}}\rangle = |h_1 = 0, ..., h_{\mathsf{N}} = 0\rangle = \prod_{a=1}^{\mathsf{N}} \frac{T^{(K)}(\xi_a^{(1)})}{a(\xi_a)} |\underline{\mathbf{1}}\rangle, \tag{B.30}$$

so that:

$$\begin{aligned}\prod_{a=1}^{\mathsf{N}} \left(\frac{T^{(K)}(\xi_a)}{\det K d(\xi_a^{(1)})}\right)^{h_a} |\underline{\mathbf{0}}\rangle &= \prod_{a=1}^{\mathsf{N}} \left(\frac{T^{(K)}(\xi_a)}{\det K d(\xi_a^{(1)})}\right)^{h_a} \frac{T^{(K)}(\xi_a^{(1)})}{a(\xi_a)} |\underline{\mathbf{1}}\rangle \\ &= \prod_{a=1}^{\mathsf{N}} \left(\frac{T^{(K)}(\xi_a) T^{(K)}(\xi_a^{(1)})}{\det K d(\xi_a^{(1)}) a(\xi_a)}\right)^{h_a} \left(\frac{T^{(K)}(\xi_a^{(1)})}{a(\xi_a)}\right)^{1-h_a} |\underline{\mathbf{1}}\rangle \\ &= |h_1, ..., h_{\mathsf{N}}\rangle,\end{aligned} \tag{B.31}$$

by the quantum determinant identity. From this representation of the right SoV vectors it follows also that for any fixed left transfer matrix eigenstate $\langle t|$ it holds:

$$\langle t|h_1, ..., h_{\mathsf{N}}\rangle \propto \langle t|\underline{\mathbf{0}}\rangle, \quad \forall \{h_1, ..., h_{\mathsf{N}}\} \in \{0,1\}^{\mathsf{N}}, \tag{B.32}$$

so that it must holds $\langle t|\underline{\mathbf{0}}\rangle \neq 0$. □

As we have already shown in the previous appendix for gl_3 representations, also in gl_2 representations the tensor product forms hold.

Corollary B.2. *Let the inhomogeneity condition (2.6) be satisfied and $K \neq rI$, for any $r \in \mathbb{C}$, and let $(x, y) \in \mathbb{C}^2$ be such that:*

$$n_K(x, y) = bx^2 + (d-a)xy - cy^2 \neq 0. \tag{B.33}$$

Then, once we define:

$$\langle \underline{\mathbf{0}}| = \bigotimes_{a=1}^{\mathsf{N}} (x, y)_a, \tag{B.34}$$

it holds:

$$|\underline{\mathbf{1}}\rangle = \frac{1}{n_1} \bigotimes_{n=1}^{\mathsf{N}} \begin{pmatrix} -y \\ x \end{pmatrix}_n, \quad |\underline{\mathbf{0}}\rangle = \frac{1}{n_0} \bigotimes_{n=1}^{\mathsf{N}} \begin{pmatrix} bx + dy \\ -(ax + cy) \end{pmatrix}_n, \tag{B.35}$$

where:

$$n_1 = n_{1,...,\mathsf{N}}\, n_K^{\mathsf{N}}(x, y)\, V(\xi_1, ..., \xi_{\mathsf{N}}) V(\xi_1^{(1)}, ..., \xi_{\mathsf{N}}^{(1)}) \left(\prod_{n=1}^{\mathsf{N}} a(\xi_n)\right)^{-1}, \tag{B.36}$$

$$n_0 = n_K^{\mathsf{N}}(x, y)\, V^2(\xi_1, ..., \xi_{\mathsf{N}}),\ n_{1,...,\mathsf{N}} = \prod_{1 \leq i < j \leq \mathsf{N}} (\eta^2 - (\xi_i - \xi_j)^2). \tag{B.37}$$

C Proof of Theorem 3.1

This appendix is dedicated to the completion of the proof of the Theorem 3.1: here we prove the orthogonality properties and the non-zero coupling of the SoV co-vectors/vectors. It is worth remarking that the proof of the "pseudo-orthogonality" is quite intricate and we have divided it in several steps to make it more intelligible. The orthogonality conditions are established in section C.1, while section C.2 is dedicated to the proof of the form of the non-zero couplings of co-vectors/vectors.

The form of the orthogonality condition naturally leads to consider in the first instance vectors with $\underline{\mathbf{k}} \in \{0,2\}^{\mathsf{N}}$, this is achieved in subsection C.1.1. In this case, the co-vector/vector coupling is diagonal, i.e. standard orthogonality holds with non-zero coupling only for co-vector/vector associated to the same N-tuple $\underline{\mathbf{h}} = \underline{\mathbf{k}} \in \{0,2\}^{\mathsf{N}}$. This proof requires already different steps. We prove it first for the case with only one $k_a = 2$ while all the others being zero, and then by induction for the generic N-tuple $\underline{\mathbf{k}} \in \{0,2\}^{\mathsf{N}}$. In subsection C.1.2, we then consider the case with just one $k_a = 1$ while all the others $k_{b\neq a}$ being in $\{0,2\}$. Here, we prove that the standard orthogonality still works. In subsection C.1.3, we finally consider the proof for the case with non-diagonal and diagonal couplings, which correspond to SoV vectors associated to $\underline{\mathbf{k}}$ with at least one couple $(k_a = 1,\ k_b = 1)$. First, the case with just one couple $(k_a = 1,\ k_b = 1)$ is developed, and then the case of vectors associated to a general $\underline{\mathbf{k}} \in \{0,1,2\}^{\mathsf{N}}$.

In subsection C.2.1, we write the non-diagonal couplings in terms of the diagonal ones. In particular, we prove the formula (3.14) and its power dependence w.r.t. $\det K$. The coefficients $C^{\mathbf{k}}_{\underline{\mathbf{h}}}$ in (3.14) are shown to be independent w.r.t. $\det K$ and completely characterized by the Lemma C.3 and by the solutions to the recursion equations derived in Lemma C.4. We do not resolve these recursions in general. Rather we argue the dependence of the coefficients in terms of the involved transfer matrix interpolation formulae and explicitly present them in the case of co-vectors having one couple of $(h_a = 0,\ h_b = 2)$ associated to vectors with a couple $(k_a = 1,\ k_b = 1)$. Finally, in subsection C.2.2, we prove the explicit form of the co-vector/vector diagonal coupling. The proof derived there does not use the fact that for $\det K = 0$ we have an independent derivation of the same SoV measure.

C.1 Orthogonality proof

We use the following *incomplete*[18] notation for the interpolation formulae in the shifted inhomogeneities $\{\xi_n^{(h_n)}\}$ of the transfer matrix:

$$T_a^{(K)}(\lambda) \underset{\text{UpC}}{=} \mathsf{t}_a + \sum_{n=1}^{\mathsf{N}} \mathsf{T}_a(\xi_n^{(h_n)}), \tag{C.1}$$

with

$$\mathsf{t}_a = \mathsf{a}^{\delta_{a,1}} \big(\mathsf{b}\, d(\lambda-\eta)\big)^{\delta_{a,2}} \prod_{a=1}^{\mathsf{N}} (\lambda - \xi_a^{(h_a)}), \quad \mathsf{T}_a(\xi_n^{(h_n)}) = T_a^{(K)}(\xi_n^{(h_n)}) d^{\delta_{a,2}}(\lambda-\eta) g^{(a)}_{n,\underline{\mathbf{h}}}(\lambda) \tag{C.2}$$

for $a \in \{1,2\}$, where

$$\mathsf{a} = \operatorname{tr} K, \quad \mathsf{b} = \frac{(\operatorname{tr} K)^2 - \operatorname{tr}(K^2)}{2}, \quad \mathsf{c} = \det K,$$

are the spectral invariants of the matrix K and

$$\mathsf{c}_n = \text{q-det}\, M^{(K)}(\xi_n) = \mathsf{c}\ \text{q-det}\, M^{(I)}(\xi_n). \tag{C.3}$$

[18] We need only to keep partial information on the interpolation formulae for our current aims.

Note that this shorted notation hides the original value in which the transfer matrix was computed before the interpolation, which is λ in (C.1). It also loses the coefficients of the same interpolation formulae. In the following of this appendix, all the equalities written down with symbol $\underset{\text{UpC}}{=}$ have to be interpreted up to these implicit, missing coefficients. This does not represent a problem, as here we are only interested in proofs that some matrix elements are zero or proportional to each other, which is something that remains true independently of the exact coefficients (as long as they do not vanish).

C.1.1 First step: the case $\left|\underline{\mathbf{k}}\right\rangle$ with $\underline{\mathbf{k}} \in \{0,2\}^{\mathsf{N}}$

In the following, we introduce needed notations to implement operations on N-tuple of indices. Let us denote with $\underline{\mathbf{x}} = \{x_1, ..., x_{\mathsf{N}}\} \in \{0,1,2\}^{\mathsf{N}}$ a generic N-tuple from $\{0,1,2\}$, and with $(j_1, ..., j_m) \in \{0,1,2\}^m$ a generic m-tuple from $\{0,1,2\}$. We introduce the following notations:

$$\underline{\mathbf{x}}^{(j_1,...,j_m)}_{a_1,...,a_m} = \underline{\mathbf{x}} - \sum_{\substack{i=1 \\ i\neq r_1,...,r_l, 0\leq l\leq m-1}}^{m} (x_{a_i} - j_i)\underline{\mathbf{e}}_{a_i} \quad \forall a_i \in \{1,...,\mathsf{N}\},\ i \leq m \leq \mathsf{N}, \tag{C.4}$$

where $\underline{\mathbf{e}}_a = \{\delta_{1,a}, ..., \delta_{\mathsf{N},a}\}$ and the $r_1, ..., r_l$ are defined as follows for any fixed choice of $a_1, ..., a_m$:

$$\forall h \leq l,\ r_h \in \{1,...,m\} : \exists s \in \{1,...,m\}\backslash\{r_1,...,r_l\},\ r_h < s \text{ with } a_{r_h} = a_s, \tag{C.5}$$

while it holds:

$$a_p \neq a_q, \quad \forall p \neq q \in \{1,...,m\}\backslash\{r_1,...,r_l\}. \tag{C.6}$$

In simple words, for any fixed $a_1, ..., a_m$, the $r_1, ..., r_l$ are defined as the minimal set of the smallest integers in $\{1,...,m\}$ such that removing them from $\{1,...,m\}$ make the condition (C.6) satisfied. Clearly, we have $l = 0$ if the $a_1, ..., a_m$ are all distinct.

i) Only one $k_n = 2$ Let us first prove:

$$\langle \underline{\mathbf{h}} | \underline{\mathbf{0}}^{(2)}_n \rangle = \langle \underline{\mathbf{h}} | \mathsf{T}_1(\xi_n) | \underline{\mathbf{0}} \rangle = 0, \quad \text{for } \underline{\mathbf{h}} \neq \underline{\mathbf{0}}^{(2)}_n. \tag{C.7}$$

If $h_n = 0, 1$ this statement is evident, since

$$\langle \underline{\mathbf{h}} | \mathsf{T}_1(\xi_n) | \underline{\mathbf{0}} \rangle = \mathsf{c}_n^{\delta_{h_n,0}} \langle \underline{\mathbf{h}}^{(h_n+1)}_n | \underline{\mathbf{0}} \rangle = 0. \tag{C.8}$$

Now, let us fix $h_n = 2$. Here we proceed by induction, first assuming that all the others $h_{j\neq n} = 0, 1$:

$$\begin{aligned} \langle \underline{\mathbf{h}} | \mathsf{T}_1(\xi_n) | \underline{\mathbf{0}} \rangle &\underset{\text{UpC}}{=} \mathsf{t}_1 \langle \underline{\mathbf{h}} | \underline{\mathbf{0}} \rangle + \langle \underline{\mathbf{h}}^{(1)}_n | \mathsf{T}_2(\xi_n) | \underline{\mathbf{0}} \rangle + \langle \underline{\mathbf{h}} | \sum_{l\neq n, l=1}^{\mathsf{N}} \mathsf{T}_1(\xi_l) | \underline{\mathbf{0}} \rangle \\ &\underset{\text{UpC}}{=} \mathsf{t}_1 \langle \underline{\mathbf{h}} | \underline{\mathbf{0}} \rangle + \langle \underline{\mathbf{h}}^{(1)}_n | \mathsf{T}_2(\xi_n) | \underline{\mathbf{0}} \rangle + \sum_{l\neq n, l=1}^{\mathsf{N}} \mathsf{c}_l^{\delta_{h_l,0}} \langle \underline{\mathbf{h}}^{(h_l+1)}_l | \underline{\mathbf{0}} \rangle \\ &\underset{\text{UpC}}{=} \langle \underline{\mathbf{h}}^{(1)}_n | \mathsf{T}_2(\xi_n) | \underline{\mathbf{0}} \rangle. \end{aligned} \tag{C.9}$$

Now, we have to use the interpolation formula for $\mathsf{T}_2(\xi_n)$

$$\begin{aligned}\langle \underline{\mathbf{h}}|\mathsf{T}_1(\xi_n)|\underline{\mathbf{0}}\rangle &= \langle \underline{\mathbf{h}}_n^{(1)}|\mathsf{T}_2(\xi_n)|\underline{\mathbf{0}}\rangle \\ &\underset{\text{UpC}}{=} \mathsf{t}_2\langle \underline{\mathbf{h}}_n^{(1)}|\underline{\mathbf{0}}\rangle + \langle \underline{\mathbf{h}}_n^{(1)}|\sum_{l=1}^{\mathsf{N}}(\delta_{h'_l,0}\mathsf{T}_2(\xi_l)+\delta_{h'_l,1}\mathsf{T}_2(\xi_l^{(1)}))|\underline{\mathbf{0}}\rangle \\ &\underset{\text{UpC}}{=} \mathsf{t}_2\langle \underline{\mathbf{h}}_n^{(1)}|\underline{\mathbf{0}}\rangle + \sum_{l=1}^{\mathsf{N}}\delta_{h'_l,1}\langle \underline{\mathbf{h}}_{n,l}^{(1,0)}|\underline{\mathbf{0}}\rangle + \sum_{l=1}^{\mathsf{N}}\delta_{h'_l,0}\mathsf{c}_l\langle \underline{\mathbf{h}}_{n,l}^{(1,1)}|\mathsf{T}_1(\xi_l^{(1)})|\underline{\mathbf{0}}\rangle \\ &= 0,\end{aligned} \tag{C.10}$$

where we have defined $\underline{\mathbf{h}}' = \{h'_1,...,h'_{\mathsf{N}}\} = \underline{\mathbf{h}}_n^{(1)}$ and used that $\underline{\mathbf{h}}_{n,l}^{(1,0)} \neq \underline{\mathbf{0}}$ holds even for $l = n$, as the condition $\underline{\mathbf{h}}_n^{(2)} \neq \underline{\mathbf{0}}_n^{(2)}$ implies $\underline{\mathbf{h}}_{n,n}^{(1,0)} = \underline{\mathbf{h}}_n^{(0)} \neq \underline{\mathbf{0}}$. Moreover, it holds

$$\langle \underline{\mathbf{h}}_{n,l}^{(1,1)}|\mathsf{T}_1(\xi_l^{(1)})|\underline{\mathbf{0}}\rangle \underset{\text{UpC}}{=} \mathsf{t}_1\langle \underline{\mathbf{h}}_{n,l}^{(1,1)}|\underline{\mathbf{0}}\rangle + \sum_{m=1}^{\mathsf{N}}\langle \underline{\mathbf{h}}_{n,l}^{(1,1)}|\mathsf{T}_1(\xi_m)|\underline{\mathbf{0}}\rangle, \tag{C.11}$$

and defining $\underline{\mathbf{h}}'' = \{h''_1,...,h''_{\mathsf{N}}\} = \underline{\mathbf{h}}_{n,l}^{(1,1)}$, we get

$$\langle \underline{\mathbf{h}}_{n,l}^{(1,1)}|\mathsf{T}_1(\xi_l^{(1)})|\underline{\mathbf{0}}\rangle \underset{\text{UpC}}{=} \sum_{m=1}^{\mathsf{N}}\langle \underline{\mathbf{h}}_{n,l,m}^{(1,1,h''_m+1)}|\underline{\mathbf{0}}\rangle = 0. \tag{C.12}$$

Let us now consider the induction, i.e. we assume that the orthogonality works when there are $m \geq 1$ values of $h_a = 2$ in $\langle \underline{\mathbf{h}}|$ and we want to prove it for the case of $m+1$ values of $h_a = 2$ in $\langle \underline{\mathbf{h}}|$. Up to a reordering of the index of the $\{\xi_a\}$, this is equivalent to prove that

$$\langle h_1 = 2,...,h_{m+1} = 2, h_{l\geq m+2} \in \{0,1\}|\mathsf{T}_1(\xi_{m+1})|\underline{\mathbf{0}}\rangle = 0. \tag{C.13}$$

Setting

$$\underline{\mathbf{h}} = \{h_1 = 2,...,h_{m+1} = 2, h_{l\geq m+2} \in \{0,1\}\}, \tag{C.14}$$

and once again using the development by interpolation formula, we get

$$\langle \underline{\mathbf{h}}|\mathsf{T}_1(\xi_{m+1})|\underline{\mathbf{0}}\rangle \underset{\text{UpC}}{=} \mathsf{t}_1\langle \underline{\mathbf{h}}|\underline{\mathbf{0}}\rangle + \langle \underline{\mathbf{h}}|\sum_{l=m+2}^{\mathsf{N}}\mathsf{T}_1(\xi_l)|\underline{\mathbf{0}}\rangle + \sum_{l=1}^{m+1}\langle \underline{\mathbf{h}}|\mathsf{T}_1(\xi_l^{(1)})|\underline{\mathbf{0}}\rangle \tag{C.15}$$

$$\underset{\text{UpC}}{=} \sum_{l=1}^{m+1}\langle \underline{\mathbf{h}}_l^{(1)}|\mathsf{T}_2(\xi_l)|\underline{\mathbf{0}}\rangle, \tag{C.16}$$

so expanding $\mathsf{T}_2(\xi_l)$:

$$\langle \underline{\mathbf{h}}_l^{(1)}|\mathsf{T}_2(\xi_l)|\underline{\mathbf{0}}\rangle \underset{\text{UpC}}{=} \mathsf{t}_2\langle \underline{\mathbf{h}}_l^{(1)}|\underline{\mathbf{0}}\rangle + \langle \underline{\mathbf{h}}_l^{(1)}|\sum_{r=1}^{\mathsf{N}}\mathsf{T}_2(\xi_r^{(\delta_{h'_r,1}+\delta_{h'_r,2})})|\underline{\mathbf{0}}\rangle, \tag{C.17}$$

where h'_r are the elements of $\underline{\mathbf{h}}' = \{h'_1,...,h'_{\mathsf{N}}\} \equiv \underline{\mathbf{h}}_l^{(1)}$. Then, we can use the rewriting

$$\left(\delta_{h'_r,1}+\delta_{h'_r,2}\right)\langle \underline{\mathbf{h}}'|\mathsf{T}_2(\xi_r^{(1)})|\underline{\mathbf{0}}\rangle = \left(\delta_{h'_r,1}+\delta_{h'_r,2}\right)\mathsf{c}_r^{\delta_{h'_r,2}}\langle \underline{\mathbf{h}}_{l,r}^{(1,h'_r-1)}|\underline{\mathbf{0}}\rangle = 0. \tag{C.18}$$

Indeed, $\underline{\mathbf{h}}_{l,r}^{(1,h'_r-1)} \neq \underline{\mathbf{0}}$ holds even for $r = l$, as $\underline{\mathbf{h}}_{l,l}^{(1,h'_l-1)} = \underline{\mathbf{h}}_l^{(0)}$ has at least one element equal to 2 being by assumption $m \geq 1$. Then, we get

$$\langle \underline{\mathbf{h}}_l^{(1)}|\mathsf{T}_2(\xi_l)|\underline{\mathbf{0}}\rangle \underset{\text{UpC}}{=} \langle \underline{\mathbf{h}}_l^{(1)}|\sum_{r=m+2}^{\mathsf{N}}\delta_{h'_r,0}\mathsf{T}_2(\xi_r)|\underline{\mathbf{0}}\rangle = \sum_{r=m+2}^{\mathsf{N}}\delta_{h'_r,0}\mathsf{c}_r\langle \underline{\mathbf{h}}_{l,r}^{(1,1)}|\mathsf{T}_1(\xi_r^{(1)})|\underline{\mathbf{0}}\rangle, \tag{C.19}$$

and finally:

$$\langle \underline{\mathbf{h}}_{l,r}^{(1,1)} | \mathsf{T}_1(\xi_r^{(1)}) | \underline{\mathbf{0}} \rangle \underset{\text{UpC}}{=} \mathsf{t}_1 \langle \underline{\mathbf{h}}_{l,r}^{(1,1)} | \underline{\mathbf{0}} \rangle + \langle \underline{\mathbf{h}}_{l,r}^{(1,1)} | \sum_{s=1}^{\mathsf{N}} \mathsf{T}_1(\xi_s) | \underline{\mathbf{0}} \rangle, \tag{C.20}$$

which is zero by the induction. So we have proven the orthogonality:

$$\langle \underline{\mathbf{h}} | \underline{\mathbf{0}}_j^{(2)} \rangle = 0, \quad \text{for any } \underline{\mathbf{h}} \neq \underline{\mathbf{0}}_j^{(2)}. \tag{C.21}$$

ii) The general $\underline{\mathbf{k}} \in \{0,2\}^{\mathsf{N}}$ We now perform the induction over the number of $k_a = 2$ in $\underline{\mathbf{k}} \in \{0,2\}^{\mathsf{N}}$. The orthogonality is assumed to work when there are m values of $k_a = 2$ in $\underline{\mathbf{k}}$, while the others being all 0, and we want to prove it for the case of $m+1$ values of $k_a = 2$ in $\underline{\mathbf{k}}$.

Let us start proving the following

Lemma C.1. *Let $\underline{\mathbf{k}} \in \{0,2\}^{\mathsf{N}}$ with*

$$\sum_{a=1}^{\mathsf{N}} \delta_{k_a,2} = m, \tag{C.22}$$

and $\overset{ab}{\underline{\mathbf{h}}}$ a N-tuple in $\{0,1,2\}$ such that $h_a \neq k_a$ and $h_b \neq k_b$ if $a \neq b$, while $h_a = 1 \neq k_a = 0$ if $a = b$. The following recursive formula holds for any fixed $c \in \{1,\dots,\mathsf{N}\}$:

$$\langle \overset{ab}{\underline{\mathbf{h}}} | \mathsf{T}_1\Big(\xi_c^{(\delta_{h_c,0}+\delta_{h_c,1})}\Big) | \underline{\mathbf{k}} \rangle \underset{\text{UpC}}{=} \sum_{r=1}^{\mathsf{N}} \delta_{h_r,2} \sum_{s=1, s\neq r}^{\mathsf{N}} \delta_{h_s,0} \mathsf{c}_s \langle \overset{ab}{\underline{\mathbf{h}}}_{r,s}^{(1,1)} | \mathsf{T}_1(\xi_s^{(1)}) | \underline{\mathbf{k}} \rangle. \tag{C.23}$$

Proof. Let us use this first interpolation formula:

$$\begin{aligned} \langle \overset{ab}{\underline{\mathbf{h}}} | \mathsf{T}_1\Big(\xi_c^{(\delta_{h_c,0}+\delta_{h_c,1})}\Big) | \underline{\mathbf{k}} \rangle &\underset{\text{UpC}}{=} \mathsf{t}_1 \langle \overset{ab}{\underline{\mathbf{h}}} | \underline{\mathbf{k}} \rangle + \langle \overset{ab}{\underline{\mathbf{h}}} | \sum_{r=1}^{\mathsf{N}} \mathsf{T}_1\Big(\xi_r^{(\delta_{h_r,2})}\Big) | \underline{\mathbf{k}} \rangle \\ &\underset{\text{UpC}}{=} \sum_{r=1}^{\mathsf{N}} \delta_{h_r,2} \langle \overset{ab^{(1)}}{\underline{\mathbf{h}}}_r | \mathsf{T}_2(\xi_r) | \underline{\mathbf{k}} \rangle, \end{aligned} \tag{C.24}$$

as by the orthogonality assumption it holds:

$$\langle \overset{ab}{\underline{\mathbf{h}}} | \underline{\mathbf{k}} \rangle = 0, \tag{C.25}$$

as well as

$$\big(\delta_{h_r,0} + \delta_{h_r,1}\big) \langle \overset{ab}{\underline{\mathbf{h}}} | \mathsf{T}_1(\xi_r) | \underline{\mathbf{k}} \rangle = \big(\delta_{h_r,0}\mathsf{c}_r + \delta_{h_r,1}\big) \langle \overset{ab^{(h_r+1)}}{\underline{\mathbf{h}}}_r | \underline{\mathbf{k}} \rangle = 0, \quad \forall r \in \{1,\dots,\mathsf{N}\}, \tag{C.26}$$

being $\overset{ab^{(h_r+1)}}{\underline{\mathbf{h}}}_r \neq \underline{\mathbf{k}}$ under the condition $\big(\delta_{h_r,0} + \delta_{h_r,1}\big) = 1$. This is easily the case for $b \neq a$ as $\overset{ab^{(h_r+1)}}{\underline{\mathbf{h}}}_r$ keeps at least one $h_j \neq k_j$, for $j = a$ or $j = b$, independently from the choice of r. In the case $a = b$ it holds $\overset{ab^{(h_r+1)}}{\underline{\mathbf{h}}}_r = \underline{\mathbf{h}}_{a,r}^{(1,h_r+1)}$ so that for $r \neq a$ it still holds $h_a = 1 \neq k_a = 0$. Finally, in the case $r = b = a$ it holds $\overset{ab^{(h_r+1)}}{\underline{\mathbf{h}}}_r = \underline{\mathbf{h}}_a^{(2)} \neq \underline{\mathbf{k}}$.

Now, let us use the following second interpolation formula to develop the terms on the r.h.s. of (C.24) :

$$\begin{aligned} \langle \overset{ab^{(1)}}{\underline{\mathbf{h}}}_r | \mathsf{T}_2(\xi_r) | \underline{\mathbf{k}} \rangle &\underset{\text{UpC}}{=} \mathsf{t}_2 \langle \overset{ab^{(1)}}{\underline{\mathbf{h}}}_r | \underline{\mathbf{k}} \rangle + \langle \overset{ab^{(1)}}{\underline{\mathbf{h}}}_r | \sum_{s=1}^{\mathsf{N}} \mathsf{T}_2\Big(\xi_s^{(\delta_{h'_s,1}+\delta_{h'_s,2})}\Big) | \underline{\mathbf{k}} \rangle \\ &\underset{\text{UpC}}{=} \sum_{s=1, s\neq r}^{\mathsf{N}} \delta_{h_s,0} \mathsf{c}_s \langle \overset{ab^{(1,1)}}{\underline{\mathbf{h}}}_{r,s} | \mathsf{T}_1(\xi_s^{(1)}) | \underline{\mathbf{k}} \rangle, \end{aligned} \tag{C.27}$$

where we have defined $\{h'_1,...,h'_{\mathsf{N}}\} \equiv \overset{ab^{(1)}}{\underline{\mathbf{h}}}_r$. Indeed, by the orthogonality condition it holds:

$$\langle \overset{ab^{(1)}}{\underline{\mathbf{h}}}_r \, |\underline{\mathbf{k}}\rangle = 0, \tag{C.28}$$

and

$$\left(\delta_{h'_s,1}+\delta_{h'_s,2}\right)\langle \overset{ab^{(1)}}{\underline{\mathbf{h}}}_r \, |\mathsf{T}_2(\xi_s^{(1)})|\underline{\mathbf{k}}\rangle = \left(\delta_{h'_s,1}+\delta_{h'_s,2}\right)\mathsf{c}_s^{\delta_{h'_s,2}}\langle \overset{ab^{(1,h'_s-1)}}{\underline{\mathbf{h}}}_{r,s} \, |\underline{\mathbf{k}}\rangle = 0, \quad \forall s\in\{1,...,\mathsf{N}\}, \tag{C.29}$$

being $\overset{ab^{(1,h'_s-1)}}{\underline{\mathbf{h}}}_{r,s} \neq \underline{\mathbf{k}}$ under the condition $\left(\delta_{h'_s,1}+\delta_{h'_s,2}\right)=1$. Indeed, this is easily the case for $s\neq r$ as $\overset{ab^{(1,h'_s-1)}}{\underline{\mathbf{h}}}_{r,s}$ keeps $h'_r=1\neq k_r\in\{0,2\}$, independently from the choice of s. In the case $s=r$, it holds $\overset{ab^{(1,h'_s-1)}}{\underline{\mathbf{h}}}_{r,s} = \overset{ab^{(0)}}{\underline{\mathbf{h}}}_r$ so that for $b\neq a$ $\overset{ab^{(0)}}{\underline{\mathbf{h}}}_r$ keeps at least one $h_j\neq k_j$, for $j=a$ or $j=b$. Finally, in the case $s=r$ and $a=b$ it holds $r\neq a$ and so $\overset{ab^{(0)}}{\underline{\mathbf{h}}}_r \neq \underline{\mathbf{k}}$ as by our hypothesis on $\overset{aa}{\underline{\mathbf{h}}}$ we have $h_a=1$ while by (C.24) it must hold $h_r=2$.

Putting together the results of these interpolation developments, we get our recursion formula as a consequence of the orthogonality assumed for m values of $k_j=2$ in $\underline{\mathbf{k}}$. □

Note that the above lemma gives a recursive formula. The terms on the r.h.s. of equation (C.23) are of the same type as the starting one on the l.h.s. , and for any r,s such $\delta_{h_r,2}=\delta_{h_s,0}=1$, the $\overset{ab}{\underline{\mathbf{h}}}{}^{(1,1)}_{r,s}$ surely satisfies the condition to have at least two different elements w.r.t. the given $\underline{\mathbf{k}}\in\{0,2\}^{\mathsf{N}}$. Indeed, $\overset{ab}{\underline{\mathbf{h}}}{}^{(1,1)}_{r,s}$ contains a couple of elements equal to 1. Hence it is possible to apply the same recursion formula to the terms on the r.h.s. of equation (C.23).

The previous lemma implies the following:

Corollary C.1. *Let* $\underline{\mathbf{k}}\in\{0,2\}^{\mathsf{N}}$ *with*

$$\sum_{a=1}^{\mathsf{N}}\delta_{k_a,2}=m, \tag{C.30}$$

and $\overset{ab}{\underline{\mathbf{h}}}$ *such that* $h_a\neq k_a$ *and* $h_b\neq k_b$ *if* $a\neq b$, *while* $h_a=1\neq k_a=0$ *if* $a=b$. *The following orthogonality conditions hold for any fixed* $c\in\{1,...,\mathsf{N}\}$ *:*

$$\langle \overset{ab}{\underline{\mathbf{h}}}|\mathsf{T}_1\left(\xi_c^{\left(\delta_{h_c,0}+\delta_{h_c,1}\right)}\right)|\underline{\mathbf{k}}\rangle = 0\,. \tag{C.31}$$

Proof. Note that if $\overset{ab}{\underline{\mathbf{h}}}$ does not contain $h=2$ or $h=0$, the orthogonality is proven just by applying once the recursion formula. Otherwise, the recursion generate the $\overset{ab}{\underline{\mathbf{h}}}{}^{(1,1)}_{r,s}$ where we have reduced by one the number of $h=2$, reduced by one the number of $h=0$ and increased by two the number of $h=1$. This amounts to change $\underline{\mathbf{h}}$ in $\underline{\mathbf{h}}'$, with $h'_{a\neq r,s}=h_a$ but $h_r=2\to h'_r=1$ and $h_s=0\to h'_s=1$. Then, if $\overset{ab}{\underline{\mathbf{h}}}{}^{(1,1)}_{r,s}$ does not contain $h=2$ or $h=0$, the orthogonality is proven just by applying the recursion formula one more time. Otherwise, we continue to apply (C.23) until we arrive to the condition that there are no $h=2$ or $h=0$ in the index of the SoV co-vectors involved. This proves the above corollary. □

We are now in position to perform the induction over the number m of $k_a = 2$ for the orthogonality.

Up to a reordering in the indices of the $\{\xi_a\}$, this is equivalent to prove:

$$\langle \underline{\mathbf{h}}|\mathsf{T}_1(\xi_{m+1})|\underline{\mathbf{k}}\rangle = 0, \quad \text{for any } \underline{\mathbf{h}} \neq \underline{\mathbf{k}}^{(2)}_{m+1}, \tag{C.32}$$

where we have defined:

$$\underline{\mathbf{k}} = \{k_1 = 2, ..., k_m = 2, k_{l\geq m+1} = 0\}. \tag{C.33}$$

The only case that we have to consider is

$$\underline{\mathbf{h}} \neq \underline{\mathbf{k}}^{(2)}_{m+1} \text{ with } h_1 = 2, ..., h_{m+1} = 2. \tag{C.34}$$

Indeed, if this is not the case we can write:

$$\langle \underline{\mathbf{h}}|\mathsf{T}_1(\xi_{m+1})|\underline{\mathbf{k}}\rangle = \langle \underline{\mathbf{h}}|\mathsf{T}_1(\xi_{l<m+1})|\underline{\mathbf{k}}^{(0,2)}_{l,m+1}\rangle, \tag{C.35}$$

and we can directly apply the corresponding $\mathsf{T}_1(\xi_{l\leq m+1})$ on the left vector $\langle \underline{\mathbf{h}}|$, increasing by one the associated $h_{l\leq m+1} \leq 1$. Then, using the orthogonality assumed for m values of $k_j = 2$ in $\underline{\mathbf{k}}$, we get zero, i.e. for $h_{l\leq m+1} \leq 1$ it holds[19]:

$$\langle \underline{\mathbf{h}}|\mathsf{T}_1(\xi_{m+1})|\underline{\mathbf{k}}\rangle = \langle \underline{\mathbf{h}}^{(h_l+1)}_l|\underline{\mathbf{k}}^{(0,2)}_{l,m+1}\rangle = 0. \tag{C.36}$$

So it is sufficient to consider the tuples $\underline{\mathbf{h}}$ of the form (C.34). But then $\underline{\mathbf{h}}$ has at least two elements different from the given $\underline{\mathbf{k}} \in \{0,2\}^{\mathsf{N}}$. Indeed, from $\underline{\mathbf{h}} \neq \underline{\mathbf{k}}^{(2)}_{m+1}$ it follows that there exists at least one $j \in \{m+2, ..., \mathsf{N}\}$ such that $h_j \neq k_j = 0$, and by the definitions (C.34) and (C.33) of $\underline{\mathbf{h}}$ and $\underline{\mathbf{k}}$, it holds $h_{m+1} = 2 \neq k_{m+1} = 0$. So we get our proof of the orthogonality induction being:

$$\langle \underline{\mathbf{h}}|\mathsf{T}_1(\xi_{m+1})|\underline{\mathbf{k}}\rangle = 0, \tag{C.37}$$

as consequence of (C.31). Note that the proven orthogonality also implies that the above lemma and corollary indeed hold for any $m \leq \mathsf{N}$.

C.1.2 Second step: the case $|\underline{\mathbf{k}}\rangle$ with $k_a = 1$, $k_{b\neq a} \in \{0,2\}$

Let us make the orthogonality proof in the case where $\underline{\mathbf{k}}$ contains only one $a \in \{1, ..., \mathsf{N}\}$, such that $k_a = 1$ while $k_b \in \{0,2\}$ for any $b \neq a \in \{1, ..., \mathsf{N}\}$, i.e. let us show that it holds:

$$\langle \underline{\mathbf{h}}|\mathsf{T}_2(\xi_a)|\underline{\mathbf{k}}^{(0)}_a\rangle = 0, \quad \forall \underline{\mathbf{h}} \neq \underline{\mathbf{k}} \text{ with } \underline{\mathbf{h}} \in \{0,1,2\}^{\mathsf{N}}. \tag{C.38}$$

In the case $h_a = 0$, it rewrites

$$\langle \underline{\mathbf{h}}|\mathsf{T}_2(\xi_a)|\underline{\mathbf{k}}^{(0)}_a\rangle = \mathsf{c}_a \langle \underline{\mathbf{h}}^{(1)}_a|\mathsf{T}_1(\xi^{(1)}_a)|\underline{\mathbf{k}}^{(0)}_a\rangle = 0, \tag{C.39}$$

and this follows by (C.31), observing that $\underline{\mathbf{k}}^{(0)}_a \in \{0,2\}^{\mathsf{N}}$.

In the case $h_a = 1$ or $h_a = 2$, we first implement the interpolation development of $\mathsf{T}_2(\xi_a)$:

$$\begin{aligned}\langle \underline{\mathbf{h}}|\mathsf{T}_2(\xi_a)|\underline{\mathbf{k}}^{(0)}_a\rangle &\underset{\text{UpC}}{=} \mathsf{t}_2\langle \underline{\mathbf{h}}|\underline{\mathbf{k}}^{(0)}_a\rangle + \langle \underline{\mathbf{h}}|\sum_{s=1}^{\mathsf{N}}\mathsf{T}_2(\xi_s^{(\delta_{h_s,1}+\delta_{h_s,2})})|\underline{\mathbf{k}}^{(0)}_a\rangle \\ &\underset{\text{UpC}}{=} \sum_{s=1,s\neq a}^{\mathsf{N}} \delta_{h_s,0}\mathsf{c}_s\langle \underline{\mathbf{h}}^{(1)}_s|\mathsf{T}_1(\xi^{(1)}_s)|\underline{\mathbf{k}}^{(0)}_a\rangle.\end{aligned} \tag{C.40}$$

[19] Note that the above discussion also implies the orthogonality $\langle \underline{\mathbf{h}}|\underline{\mathbf{2}}\rangle = 0$ for any $\underline{\mathbf{h}} \neq \underline{\mathbf{2}}$.

Indeed, we have:

$$\langle \underline{\mathbf{h}} | \underline{\mathbf{k}}_a^{(0)} \rangle = 0, \tag{C.41}$$

and

$$\left(\delta_{h_s,1} + \delta_{h_s,2}\right) \langle \underline{\mathbf{h}} | \mathsf{T}_2(\xi_s^{(1)}) | \underline{\mathbf{k}}_a^{(0)} \rangle = (\delta_{h_s,1} + \delta_{h_s,2} \mathsf{c}_s^{\delta_{h_s,2}}) \langle \underline{\mathbf{h}}_s^{(h_s-1)} | \underline{\mathbf{k}}_a^{(0)} \rangle = 0, \tag{C.42}$$

as it holds $\underline{\mathbf{h}}_s^{(h_s-1)} \neq \underline{\mathbf{k}}_a^{(0)}$ for $s \neq a$ being $h_a \in \{1,2\}$, while for $s = a$ it still holds $\underline{\mathbf{h}}_a^{(h_a-1)} \neq \underline{\mathbf{k}}_a^{(0)}$ evidently for $h_a = 2$ but also for $h_a = 1$. Indeed, in this last case, the condition $\underline{\mathbf{h}} \neq \underline{\mathbf{k}}$ is explicitly written as $\underline{\mathbf{h}}_a^{(1)} \neq \underline{\mathbf{k}}_a^{(1)}$, which is equivalent to $\underline{\mathbf{h}}_a^{(0)} \neq \underline{\mathbf{k}}_a^{(0)}$. Now our orthogonality condition follows just remarking that $\underline{\mathbf{h}}_s^{(1)}$ and $\underline{\mathbf{k}}_a^{(0)}$ have different a and $s(\neq a)$ elements, so that it holds

$$\langle \underline{\mathbf{h}}_s^{(1)} | \mathsf{T}_1(\xi_s^{(1)}) | \underline{\mathbf{k}}_a^{(0)} \rangle = 0, \tag{C.43}$$

by applying (C.31).

C.1.3 Third step: orthogonality by induction on the number of $k_i = 1$ in $|\underline{\mathbf{k}}\rangle$

Let us now prove our final orthogonality statement for the general case of $m+1$ indices $k_a = 1$ in $\underline{\mathbf{k}}$ by induction. Up to a reordering of the indices, this is equivalent to ask that given the N-tuple

$$\underline{\mathbf{k}}: \quad k_1 = k_2 = \cdots = k_m = k_{m+1} = 1,\ k_{l \geq m+2} \in \{0,2\}, \tag{C.44}$$

(we are fixing $\mathbf{1}_{\underline{\mathbf{k}}} = \{1,\ldots,m+1\}$ and so $n_{\underline{\mathbf{k}}}$ is the integer part of $(m+1)/2$) then the covector $\langle \underline{\mathbf{h}} |$ is orthogonal to $|\underline{\mathbf{k}}\rangle$, i.e. $\langle \underline{\mathbf{h}} | \underline{\mathbf{k}} \rangle = 0$, if and only if:

$$\underline{\mathbf{h}} \in \{0,1,2\}^{\mathsf{N}} \quad \text{such that} \quad \sum_{r=0}^{n_{\underline{\mathbf{k}}}} \sum_{\substack{\alpha \cup \beta \cup \gamma = \mathbf{1}_{\underline{\mathbf{k}}}, \\ \alpha,\beta,\gamma \text{ disjoint}, \#\alpha = \#\beta = r}} \delta_{\underline{\mathbf{h}}, \underline{\mathbf{k}}_{\alpha,\beta}^{(\underline{0},\underline{2})}} = 0. \tag{C.45}$$

The above condition on $\underline{\mathbf{h}}$ is denoted by

$$\underline{\mathbf{h}} \underset{(C.45)}{\neq} \underline{\mathbf{k}}. \tag{C.46}$$

It simply says that for any choice of the disjoint subsets $\alpha,\ \beta \subset \mathbf{1}_{\underline{\mathbf{k}}}$ with the same cardinality $0 \leq \#\alpha = \#\beta = r \leq n_{\underline{\mathbf{k}}}$, it must holds:

$$\underline{\mathbf{h}} \neq \underline{\mathbf{k}}_{\alpha,\beta}^{(\underline{0},\underline{2})}. \tag{C.47}$$

In the following, we assume that this orthogonality holds in the case where there are only m values of $k_a = 1$, and prove it for $m+1$.

Let us start proving the following Lemma.

Lemma C.2. *Let $\underline{\mathbf{h}}$ be the generic element of $\{0,1,2\}^{\mathsf{N}}$ with $h_1 \neq 0$ and $h_r \neq 0$, satisfying (C.46) with $\underline{\mathbf{k}}$ of the form (C.44). The following recursive formula*

$$\langle \underline{\mathbf{h}}_{1,r}^{(h_1 \neq 0, h_r \neq 0)} | \mathsf{T}_2(\xi_r) | \underline{\mathbf{k}}_1^{(0)} \rangle \underset{UpC}{=} \sum_{p=1}^{\mathsf{N}} \delta_{h_p,0} \mathsf{c}_p \sum_{q=1}^{\mathsf{N}} \delta_{h_q,2} \langle \underline{\mathbf{h}}_{1,r,p,q}^{(h_1 \neq 0, h_r \neq 0, 1, 1)} | \mathsf{T}_2(\xi_q) | \underline{\mathbf{k}}_a^{(0)} \rangle, \tag{C.48}$$

holds for any fixed $r \in \{1,\ldots,\mathsf{N}\}$, indifferently equal or different from 1.

Proof. Let us make a first interpolation

$$\begin{aligned}\langle \underline{\mathbf{h}}_{1,r}^{(h_1\neq 0,h_r\neq 0)}|\mathsf{T}_2(\xi_r)|\underline{\mathbf{k}}_1^{(0)}\rangle &\underset{\text{UpC}}{=} \mathsf{t}_2\langle \underline{\mathbf{h}}_{1,r}^{(h_1\neq 0,h_r\neq 0)}|\underline{\mathbf{k}}_1^{(0)}\rangle + \langle \underline{\mathbf{h}}_{1,r}^{(h_1\neq 0,h_r\neq 0)}|\sum_{s=1}^{\mathsf{N}}\mathsf{T}_2\Big(\xi_s^{\left(\delta_{h_s,1}+\delta_{h_s,2}\right)}\Big)|\underline{\mathbf{k}}_1^{(0)}\rangle \\ &\underset{\text{UpC}}{=} \mathsf{t}_2\langle \underline{\mathbf{h}}_{1,r}^{(h_1\neq 0,h_r\neq 0)}|\underline{\mathbf{k}}_1^{(0)}\rangle + \sum_{s=1}^{\mathsf{N}}\Big[\mathsf{c}_s^{\delta_{h_s,2}}\left(\delta_{h_s,1}+\delta_{h_s,2}\right)\langle \underline{\mathbf{h}}_{1,r,s}^{(h_1\neq 0,h_r\neq 0,h_s-1)}| \\ &\quad + \delta_{h_s,0}\mathsf{c}_s\langle \underline{\mathbf{h}}_{1,r,s}^{h_1\neq 0,h_r\neq 0,1)}|\Big]\mathsf{T}_1(\xi_s^{(1)})|\underline{\mathbf{k}}_1^{(0)}\rangle.\end{aligned} \tag{C.49}$$

Now we have that it holds:

$$\underline{\mathbf{h}}_{1,r}^{(h_1\neq 0,h_r\neq 0)} \underset{\text{(C.45)}}{\neq} \underline{\mathbf{k}}_1^{(0)}, \tag{C.50}$$

$$\underline{\mathbf{h}}_{1,r,s}^{(h_1\neq 0,h_r\neq 0,h_s-1)} \underset{\text{(C.45)}}{\neq} \underline{\mathbf{k}}_1^{(0)} \quad \text{for any } s \text{ such that } \delta_{h_s,1}+\delta_{h_s,2}=1. \tag{C.51}$$

Let us show the validity of (C.50) first. Clearly, we have $\underline{\mathbf{h}}_{1,r}^{(h_1\neq 0,h_r\neq 0)} \neq \underline{\mathbf{k}}_1^{(0)}$ as $h_1\in\{1,2\}\neq k_1=0$. Moreover, taking $\underline{\mathbf{k}}_{1,a,b}^{(0,0,2)}$ for any $a\neq b\in\{2,\ldots,m+1\}$, it holds $\underline{\mathbf{h}}_{1,r}^{(h_1\neq 0,h_r\neq 0)} \neq \underline{\mathbf{k}}_{1,a,b}^{(0,0,2)}$, because $h_1\in\{1,2\}\neq k_1=0$. Similarly, if we take the generic $\alpha,\beta\subset\{2,\ldots,m+1\}$ with $\alpha\cap\beta=\emptyset$ and $0\leq \#\alpha=\#\beta=r\leq n_{\underline{\mathbf{k}}}$, it must holds:

$$\underline{\mathbf{h}}_{1,r}^{(h_1\neq 0,h_1\neq 0)} \neq \underline{\mathbf{k}}_{1,\alpha,\beta}^{(0,2,\ldots,2,0,\ldots,0)}, \tag{C.52}$$

which is (C.50). Note that our proof of (C.50) holds independently from the value of r : it is valid both for $r\neq 1$ and for $r=1$ where $\underline{\mathbf{h}}_{1,r}^{(h_1\neq 0,h_r\neq 0)}=\underline{\mathbf{h}}_1^{(h_1\neq 0)}$.

Let us now show (C.51). We have to distinguish the two cases $s=1$ and $s\neq 1$. If $s=1$ and $h_1=2$, we have $\underline{\mathbf{h}}_{1,r,s}^{(h_1\neq 0,h_r\neq 0,h_s-1)}=\underline{\mathbf{h}}_{r,1}^{(h_r\neq 0,1)}$, so the proof of (C.51) follows the same steps as the one for (C.50), independently from the value of r. If $s=1$ and $h_1=1$, we have $\underline{\mathbf{h}}_{1,r,s}^{(h_1\neq 0,h_r\neq 0,h_s-1)}=\underline{\mathbf{h}}_{r,1}^{(h_r\neq 0,0)}$, and the following implication holds:

$$\underline{\mathbf{h}}_{r,1}^{(h_r\neq 0,1)} \underset{\text{(C.45)}}{\neq} \underline{\mathbf{k}}_1^{(1)} \quad\Longrightarrow\quad \underline{\mathbf{h}}_{r,1}^{(h_r\neq 0,0)} \underset{\text{(C.45)}}{\neq} \underline{\mathbf{k}}_1^{(0)}, \tag{C.53}$$

where the l.h.s. is our starting point assumption once we fix $h_1=1$, holds independently from the value of r. Note that we have used the notations $\underline{\mathbf{h}}_{r,1}^{(h_r\neq 0,1)}$ and $\underline{\mathbf{h}}_{r,1}^{(h_r\neq 0,0)}$, as these correctly reduce to $\underline{\mathbf{h}}_1^{(1)}$ and $\underline{\mathbf{h}}_1^{(0)}$ for $r=1$, respectively.

Now, if $s\neq 1$, we have by definition $h_1\neq k_1=0$ in $\underline{\mathbf{h}}_{1,r,s}^{(h_1\neq 0,h_r\neq 0,h_s-1)}$ so the proof of (C.51) follows once again the same lines as those of (C.50). This proves equation (C.51).

Returning to (C.49) and using the fact that in $\underline{\mathbf{k}}_1^{(0)}$ there are exactly m entries with $k=1$, equations (C.50), (C.51) and the assumed orthogonality give

$$\langle \underline{\mathbf{h}}_{1,r}^{(h_1\neq 0,h_r\neq 0)}|\underline{\mathbf{k}}_1^{(0)}\rangle = 0 \quad\text{and}\quad \left(\delta_{h_s,1}+\delta_{h_s,2}\right)\langle \underline{\mathbf{h}}_{1,r,s}^{(h_1\neq 0,h_r\neq 0,h_s-1)}|\underline{\mathbf{k}}_1^{(0)}\rangle = 0, \tag{C.54}$$

and we are left with

$$\langle \underline{\mathbf{h}}|\mathsf{T}_2(\xi_1)|\underline{\mathbf{k}}_1^{(0)}\rangle \underset{\text{UpC}}{=} \sum_{p=1}^{\mathsf{N}}\delta_{h_p,0}\mathsf{c}_p\langle \underline{\mathbf{h}}_{1,r,p}^{(h_1\neq 0,h_r\neq 0,1)}|\mathsf{T}_1(\xi_p^{(1)})|\underline{\mathbf{k}}_1^{(0)}\rangle. \tag{C.55}$$

Note that it similarly holds:

$$\underline{\mathbf{h}}_{1,r,p}^{(h_1\neq 0,h_r\neq 0,1)} \underset{\text{(C.45)}}{\neq} \underline{\mathbf{k}}_1^{(0)}, \tag{C.56}$$

as $h_1 = 1$ or 2 does not coincide with $k_1 = 0$ and $p \neq 1$ and r, being associated to the condition $\delta_{h_p,0} = 1$.

Defining $\{h'_1, ..., h'_N\} \equiv \underline{\mathbf{h}}_{1,r,p}^{(h_1\neq 0, h_r \neq 0, 1)}$, we can now perform a second interpolation:

$$\begin{aligned}
&\langle \underline{\mathbf{h}}_{1,r,p}^{(h_1\neq 0, h_r \neq 0, 1)} | \mathsf{T}_1(\xi_p^{(1)}) | \underline{\mathbf{k}}_1^{(0)} \rangle \\
&\underset{\text{UpC}}{=} \mathsf{t}_1 \langle \underline{\mathbf{h}}_{1,r,p}^{(h_1\neq 0, h_r \neq 0, 1)} | \underline{\mathbf{k}}_1^{(0)} \rangle + \langle \underline{\mathbf{h}}_{1,r,p}^{(h_1\neq 0, h_r \neq 0, 1)} | \sum_{q=1}^{N} \mathsf{T}_1(\xi_q^{(\delta_{h'_q,2})}) | \underline{\mathbf{k}}_1^{(0)} \rangle \qquad \text{(C.57)} \\
&\underset{\text{UpC}}{=} \sum_{q=1}^{N} \left(\delta_{h'_q,0} + \delta_{h'_q,1} \right) \mathsf{c}_q^{\delta_{h'_q,0}} \langle \underline{\mathbf{h}}_{1,r,p,q}^{(h_1\neq 0, h_r \neq 0, 1, h'_q+1)} | \underline{\mathbf{k}}_1^{(0)} \rangle + \sum_{q=1}^{N} \delta_{h'_q,2} \langle \underline{\mathbf{h}}_{1,r,p,q}^{(h_1\neq 0, h_r \neq 0, 1, 1)} | \mathsf{T}_2(\xi_q) | \underline{\mathbf{k}}_1^{(0)} \rangle \qquad \text{(C.58)} \\
&\underset{\text{UpC}}{=} \sum_{r=1}^{N} \delta_{h_q,2} \langle \underline{\mathbf{h}}_{1,r,p,q}^{(h_1\neq 0, h_r \neq 0, 1, 1)} | \mathsf{T}_2(\xi_q) | \underline{\mathbf{k}}_1^{(0)} \rangle, \qquad \text{(C.59)}
\end{aligned}$$

where (C.59) follows as it holds:

$$\underline{\mathbf{h}}_{1,r,p,q}^{(h_1\neq 0, h_r \neq 0, 1, h'_q+1)} \underset{\text{(C.45)}}{\neq} \underline{\mathbf{k}}_1^{(0)}, \quad \text{for any } q \text{ such that } \left(\delta_{h'_q,0} + \delta_{h'_q,1} \right) = 1, \qquad \text{(C.60)}$$

while we have suppressed the prime notation in the last line of (C.59), as $h'_q = 2$ iff. $h_q = 2$. Indeed, $q = 1$ is possible iff. $h_1 = 1$ and then $\underline{\mathbf{h}}_{1,r,p,q}^{(h_1\neq 0, h_r \neq 0, 1, h'_q+1)} = \underline{\mathbf{h}}_{r,p,1}^{(h_r\neq 0, 1, 2)}$. Then the component 1 of $\underline{\mathbf{h}}_{r,p,1}^{(h_r\neq 0, 1, 2)}$ is $2 \neq k_1 = 0$, as $p \neq 1, r$, so we can argue the proof of (C.60) as done for the proof of (C.50). Instead, if $q \neq 1$, h_1 is not modified in $\underline{\mathbf{h}}_{1,r,p,q}^{(h_1\neq 0, h_r \neq 0, 1, h'_q+1)}$ so it stays $h_1 \neq k_1 = 0$, and once again we can argue the proof of (C.60) as done for the proof of (C.50). Collecting the results of the two interpolation expansions, we get the wanted formula (C.48) of the Lemma.

Let us now remark that from the fact that $\underline{\mathbf{h}}_{1,r}^{(h_1\neq 0, h_r \neq 0)}$ satisfies (C.46) with $\underline{\mathbf{k}}$ of the form (C.44), then $\underline{\mathbf{h}}_{1,r,p,q}^{(h_1\neq 0, h_r \neq 0, 1, 1)}$ satisfies (C.46) with the same $\underline{\mathbf{k}}$. Moreover, we have that $\underline{\mathbf{h}}_{1,r,p,q}^{(h_1\neq 0, h_r \neq 0, 1, 1)}$ satisfies (C.46) with $\underline{\mathbf{k}}_1^{(0)}$, as it stays true that the component one of $\underline{\mathbf{h}}_{1,r,p,q}^{(h_1\neq 0, h_r \neq 0, 1, 1)}$ is non-zero, independently from the value of $p \in \{2, ..., N\} \backslash \{r\}$ and of $q \in \{1, ..., N\} \backslash \{p\}$. Then, all the terms $\langle \underline{\mathbf{h}}_{1,r,p,q}^{(h_1\neq 0, h_r \neq 0, 1, 1)} | \mathsf{T}_2(\xi_q) | \underline{\mathbf{k}}_1^{(0)} \rangle$ on the r.h.s. of (C.48) can be expanded once again according to the same formula (C.48), as $\underline{\mathbf{h}}_{1,r,p,q}^{(h_1\neq 0, h_r \neq 0, 1, 1)}$ behaves exactly like a $\underline{\mathbf{h}}_{1,r'}^{(h_1\neq 0, 1)}$ with $r' = q$ and $h_{r'} = 1 \neq 0$. This ensure that this is a recursive formula. □

The previous lemma implies the following:

Corollary C.2. *Under the same assumptions on* $\underline{\mathbf{h}}_{1,r}^{(h_1\neq 0, h_r \neq 0)}$ *and* $\underline{\mathbf{k}}$ *as in the previous lemma, the following orthogonality condition holds*

$$\langle \underline{\mathbf{h}}_{1,r}^{(h_1\neq 0, h_r \neq 0)} | \mathsf{T}_2(\xi_r) | \underline{\mathbf{k}}_1^{(0)} \rangle = 0, \qquad \text{(C.61)}$$

for any fixed $r \in \{1, ..., N\}$.

Proof. If $\underline{\mathbf{h}}_{1,r}^{(h_1\neq 0, h_r \neq 0)}$ does not contain $h = 2$ or $h = 0$, this is proven by applying once the recursion formula. Otherwise, a first application of the recursion generates the $\underline{\mathbf{h}}_{1,r,p,q}^{(h_1\neq 0, h_r \neq 0, 1, 1)}$ where we have reduced by one unit the number of $h = 2$ and the number of $h = 0$, while we have increased by two unit the number of $h = 1$, transforming $\underline{\mathbf{h}}$ like $h_p = 0 \to h'_p = 1$ and $h_q = 2 \to h'_q = 1$. Then, if $\underline{\mathbf{h}}_{1,r,p,q}^{(h_1\neq 0, h_r \neq 0, 1, 1)}$ does not contain $h = 2$ or $h = 0$, the orthogonality

is proven just by applying once again the recursion formula. Otherwise, we can continue to apply it over and over until there are no $h=2$ or $h=0$ in the index of the SoV co-vectors involved. This proves the above Corollary. □

Let us now perform the induction step over the number m of $k_a=1$ in the vector $|\underline{\mathbf{k}}\rangle$. Let $\underline{\mathbf{h}}$ be the generic element of $\{0,1,2\}^{\mathsf{N}}$ satisfying (C.46) with a fixed $\underline{\mathbf{k}}$ of the form (C.44). If $h_1\neq 0$, then the orthogonality condition reads

$$0=\langle\underline{\mathbf{h}}|\underline{\mathbf{k}}\rangle=\langle\underline{\mathbf{h}}_1^{(h_1\neq 0)}|\mathsf{T}_2(\xi_1)|\underline{\mathbf{k}}_1^{(0)}\rangle, \tag{C.62}$$

which follows by a direct application of the above corollary. If $h_1=0$, it holds

$$\langle\underline{\mathbf{h}}_1^{(0)}|\underline{\mathbf{k}}\rangle=\langle\underline{\mathbf{h}}_1^{(0)}|\mathsf{T}_2(\xi_1)|\underline{\mathbf{k}}_1^{(0)}\rangle=\mathsf{c}_1\langle\underline{\mathbf{h}}_1^{(1)}|\mathsf{T}_1(\xi_1^{(1)})|\underline{\mathbf{k}}_1^{(0)}\rangle,$$

and so we use the following interpolation

$$\begin{aligned}\langle\underline{\mathbf{h}}_1^{(1)}|\mathsf{T}_1(\xi_1^{(1)})|\underline{\mathbf{k}}_1^{(0)}\rangle &\underset{\text{UpC}}{=}\mathsf{t}_1\langle\underline{\mathbf{h}}_1^{(1)}|\underline{\mathbf{k}}_1^{(0)}\rangle+\langle\underline{\mathbf{h}}_1^{(1)}|\sum_{s=1}^{\mathsf{N}}\mathsf{T}_1(\xi_s^{(\delta_{h'_s,2})})|\underline{\mathbf{k}}_1^{(0)}\rangle\\ &\underset{\text{UpC}}{=}\sum_{s=1}^{\mathsf{N}}(\delta_{h'_s,0}+\delta_{h'_s,1})\mathsf{c}_s^{\delta_{h'_s,0}}\langle\underline{\mathbf{h}}_{1,s}^{(1,h'_s+1)}|\underline{\mathbf{k}}_1^{(0)}\rangle+\sum_{s=2}^{\mathsf{N}}\delta_{h'_s,2}\langle\underline{\mathbf{h}}_{1,s}^{(1,1)}|\mathsf{T}_2(\xi_s)|\underline{\mathbf{k}}_1^{(0)}\rangle,\end{aligned} \tag{C.63}$$

where we have defined $\{h'_1,\ldots,h'_{\mathsf{N}}\}=\underline{\mathbf{h}}_1^{(1)}$. From the assumed orthogonality (i.e. the induction hypothesis) we get

$$\langle\underline{\mathbf{h}}_1^{(0)}|\underline{\mathbf{k}}\rangle=\langle\underline{\mathbf{h}}_1^{(0)}|\mathsf{T}_2(\xi_1)|\underline{\mathbf{k}}_1^{(0)}\rangle\underset{\text{UpC}}{=}\mathsf{c}_1\sum_{s=1}^{\mathsf{N}}\delta_{h_s,2}\langle\underline{\mathbf{h}}_{1,s}^{(1,1)}|\mathsf{T}_2(\xi_s)|\underline{\mathbf{k}}_1^{(0)}\rangle, \tag{C.64}$$

being

$$\underline{\mathbf{h}}_{1,s}^{(1,h'_s+1)}\underset{\text{(C.45)}}{\neq}\underline{\mathbf{k}}_1^{(0)},\quad\text{for any } s \text{ such that } \delta_{h'_s,0}+\delta_{h'_s,1}=1. \tag{C.65}$$

Indeed, for $s=1$ it holds $h'_1=1$ and so $h'_1+1=2\neq k_1=0$, so we can argue the proof of (C.65) as done for the proof of (C.50). While for $s\neq 1$ it stays $h'_1=1$ so we have $h'_1\neq k_1=0$, and once again the proof of (C.65) is done as that of (C.50).

It remains to observe that the terms at the r.h.s. of (C.64) satisfy the requirements of the previous corollary. This completes the proof by the induction of the pseudo-orthogonality (3.14).

Note that the proven orthogonality also implies that the above lemma and corollary indeed hold for any $m\leq\mathsf{N}-1$.

C.2 Non-zero SoV co-vector/vector couplings

C.2.1 Nondiagonal elements from diagonal ones

The orthogonality conditions implied in the formula (3.14) of the Theorem 3.1 have been proven in the previous subsection. Here, we complete the proof of this formula for the non-zero matrix elements $\langle\underline{\mathbf{h}}|\underline{\mathbf{k}}\rangle$ with their expressions in terms of the diagonal ones $\langle\underline{\mathbf{k}}|\underline{\mathbf{k}}\rangle$ and the power dependence w.r.t. $\mathsf{c}=\det K$.

More precisely, let us assume that there are m $k=1$ in $|\underline{\mathbf{k}}\rangle$, let us say

$$k_{\pi_1}=k_{\pi_2}=\cdots=k_{\pi_m}=1, \tag{C.66}$$

then we want to show that it holds

$$\langle \underline{\mathbf{h}}|\underline{\mathbf{k}}^{(1,\dots,1)}_{\pi_1,\dots,\pi_m}\rangle = \mathsf{c}^{r+1} C^{\underline{\mathbf{k}}}_{\underline{\mathbf{h}}} \langle \underline{\mathbf{k}}|\underline{\mathbf{k}}\rangle, \tag{C.67}$$

with $C^{\underline{\mathbf{k}}}_{\underline{\mathbf{h}}}$ non-zero and independent w.r.t. c for[20] $\underline{\mathbf{h}}_{\pi_1,\dots,\pi_m} = \underline{\mathbf{k}}_{\pi_1,\dots,\pi_m} \in \{0,2\}^{\mathsf{N}-m}$ and

$$(h_{\pi_{2a-1}}, h_{\pi_{2a}}) = (0,2), \quad \forall a \in \{1,\dots,r+1\} \quad \text{and} \quad h_{\pi_s} = 1, \quad \forall s \in \{2r+3,\dots,m\}. \tag{C.68}$$

Moreover, the next lemmas completely characterize the coefficients $C^{\underline{\mathbf{k}}}_{\underline{\mathbf{h}}}$ in terms of solutions to a derived recursion relations.

Up to a reordering in the indices of the ξ_i, the generic case of $r+1$ (0,2) couples in $\langle \underline{\mathbf{h}}|$, corresponding to $r+1$ (1,1) in $|\underline{\mathbf{k}}\rangle$, is equivalent to compute $\langle \underline{\mathbf{h}}^{(0,2,\underline{\mathbf{p}})}_{1,2,3,\dots,2r}|\underline{\mathbf{h}}^{(1,1,\underline{\mathbf{q}})}_{1,2,3,\dots,2r+2}\rangle$ in terms of $\langle \underline{\mathbf{h}}^{(1,1,\underline{\mathbf{q}})}_{1,2,3,\dots,2r}|\underline{\mathbf{h}}^{(1,1,\underline{\mathbf{q}})}_{1,2,3,\dots,2r+2}\rangle$, where:

$$\underline{\mathbf{p}} = \{p_1,\dots,p_{2r}\} \text{ with } p_{2a-i} = 2(1-i), \quad \forall a \in \{1,\dots,r\}, i \in \{0,1\}, \tag{C.69}$$

$$\underline{\mathbf{q}} = \{q_1,\dots,q_{2r}\} \text{ with } q_{2a-i} = 1, \quad \forall a \in \{1,\dots,r\}, i \in \{0,1\}, \tag{C.70}$$

while $\underline{\mathbf{h}}_{1,2,3,\dots,2r+2} \in \{0,1,2\}^{\mathsf{N}-2(r+1)}$. Then the following lemma holds:

Lemma C.3. *Under the previous definition of the* $\underline{\mathbf{p}}$ *and* $\underline{\mathbf{q}}$*, the following expansion holds:*

$$C^{\underline{\mathbf{h}}^{(1,1,\underline{\mathbf{q}})}_{1,2,3,\dots,2r+2}}_{\underline{\mathbf{h}}^{(0,2,\underline{\mathbf{p}})}_{1,2,3,\dots,2r+2}} =$$

$$= \frac{\mathsf{c}^{-r}\, q-detM^{(I)}(\xi_1)}{\langle \underline{\mathbf{h}}^{(1,1,\underline{\mathbf{q}})}_{1,2,3,\dots,2r+2}|\underline{\mathbf{h}}^{(1,1,\underline{\mathbf{q}})}_{1,2,3,\dots,2r+2}\rangle}\left[\prod_{a\neq 2,a=1}^{\mathsf{N}} \frac{(\xi^{(1)}_1 - \xi^{(\delta_{h'_a,2})}_a)}{(\xi^{(1)}_2 - \xi^{(\delta_{h'_a,2})}_a)} \langle \underline{\mathbf{h}}^{(1,1,\underline{\mathbf{p}})}_{1,2,3,\dots,2r+2}|\mathsf{T}_2(\xi_2)|\underline{\mathbf{h}}^{(0,1,\underline{\mathbf{q}})}_{1,2,3,\dots,2r+2}\rangle \right.$$

$$\left. + \sum_{j=1}^{r}\prod_{a\neq 2j+2,a=1}^{\mathsf{N}} \frac{(\xi^{(1)}_1 - \xi^{(\delta_{h'_a,2})}_a)}{(\xi^{(1)}_{2j+2} - \xi^{(\delta_{h'_a,2})}_a)} \langle \underline{\mathbf{h}}^{(1,2,\underline{\mathbf{p}}^{(1)}_{2j})}_{1,2,3,\dots,2r+2}|\mathsf{T}_2(\xi_{2j+2})|\underline{\mathbf{h}}^{(0,1,\underline{\mathbf{q}})}_{1,2,3,\dots,2r+2}\rangle \right], \tag{C.71}$$

where we have denoted

$$\underline{\mathbf{h}}' \equiv \{h'_1,\dots,h'_N\} = \underline{\mathbf{h}}^{(1,1,\underline{\mathbf{p}})}_{1,2,3,\dots,2r+2}, \tag{C.72}$$

and for $r=0$ *we get:*

$$C^{\underline{\mathbf{h}}^{(1,1)}_{1,2}}_{\underline{\mathbf{h}}^{(0,2)}_{1,2}} = \frac{d(\xi_2-\eta)}{d(\xi_1-\eta)} \frac{q-detM^{(I)}(\xi_1)}{\eta^{-2}(\xi_1-\xi_2+\eta)^2} \prod_{a\geq 3}^{\mathsf{N}} \frac{(\xi^{(1)}_1 - \xi^{(\delta_{h_a,2})}_a)(\xi_2 - \xi^{(1-\delta_{h_a,0})}_a)}{(\xi^{(1)}_2 - \xi^{(\delta_{h_a,2})}_a)(\xi_1 - \xi^{(1-\delta_{h_a,0})}_a)}. \tag{C.73}$$

Proof. By the definition of the coefficients $C^{\underline{\mathbf{k}}}_{\underline{\mathbf{h}}}$ we have:

$$C^{\underline{\mathbf{h}}^{(1,1,\underline{\mathbf{q}})}_{1,2,3,\dots,2r+2}}_{\underline{\mathbf{h}}^{(0,2,\underline{\mathbf{p}})}_{1,2,3,\dots,2r+2}} = \frac{\langle \underline{\mathbf{h}}^{(0,2,\underline{\mathbf{p}})}_{1,2,3,\dots,2r}|\underline{\mathbf{h}}^{(1,1,\underline{\mathbf{q}})}_{1,2,3,\dots,2r+2}\rangle}{\mathsf{c}^{r+1}\langle \underline{\mathbf{h}}^{(1,1,\underline{\mathbf{q}})}_{1,2,3,\dots,2r+2}|\underline{\mathbf{h}}^{(1,1,\underline{\mathbf{q}})}_{1,2,3,\dots,2r+2}\rangle}, \tag{C.74}$$

then formula (C.71) follows by the following identity

$$\langle \underline{\mathbf{h}}^{(0,2,\underline{\mathbf{p}})}_{1,2,3,\dots,2r}|\underline{\mathbf{h}}^{(1,1,\underline{\mathbf{q}})}_{1,2,3,\dots,2r+2}\rangle = \mathsf{c}_1 \langle \underline{\mathbf{h}}^{(1,1,\underline{\mathbf{p}})}_{1,2,3,\dots,2r+2}|\mathsf{T}_1(\xi^{(1)}_1)\mathsf{T}_1(\xi_2)|\underline{\mathbf{h}}^{(0,1,\underline{\mathbf{q}})}_{1,2,3,\dots,2r+2}\rangle, \tag{C.75}$$

[20] Where we have introduced the notation $\underline{\mathbf{x}}_{r_1,\dots,r_m}$ without the upper index values to indicate the $\mathsf{N}-m$-tuple obtained from the generic N-tuple $\underline{\mathbf{x}}$ removing the entries $\{r_1,\dots,r_m\} \subset \{1,\dots,\mathsf{N}\}$.

once we make an interpolation expansion of $\mathsf{T}_1(\xi_1^{(1)})$. More in detail, up to the coefficients, we use the interpolation identity

$$\mathsf{T}_1(\xi_1^{(1)}) \underset{\text{UpC}}{=} \mathsf{t}_1 + \mathsf{T}_1(\xi_1) + \sum_{s\geq 2}^{\mathsf{N}} \mathsf{T}_1(\xi_s^{(\delta_{h_s,2})}), \tag{C.76}$$

from which it follows

$$\begin{aligned}
&\langle \underline{\mathbf{h}}^{(1,1,\underline{\mathbf{p}})}_{1,2,\dots,2r+2} | \mathsf{T}_1(\xi_1^{(1)}) \mathsf{T}_1(\xi_2) | \underline{\mathbf{h}}^{(0,1,\underline{\mathbf{q}})}_{1,2,\dots,2r+2} \rangle = \\
&\underset{\text{UpC}}{=} \mathsf{t}_1 \langle \underline{\mathbf{h}}^{(1,2,\underline{\mathbf{p}})}_{1,2,\dots,2r+2} | \underline{\mathbf{h}}^{(0,1,\underline{\mathbf{q}})}_{1,2,\dots,2r+2} \rangle + \langle \underline{\mathbf{h}}^{(2,2,\underline{\mathbf{p}})}_{1,2,\dots,2r+2} | \underline{\mathbf{h}}^{(0,1,\underline{\mathbf{q}})}_{1,2,\dots,2r+2} \rangle + \langle \underline{\mathbf{h}}^{(1,1,\underline{\mathbf{p}})}_{1,2,\dots,2r+2} | \mathsf{T}_2(\xi_2) | \underline{\mathbf{h}}^{(0,1,\underline{\mathbf{q}})}_{1,2,\dots,2r+2} \rangle \\
&+ \sum_{a=1}^{r} \sum_{i=0}^{1} \langle \underline{\mathbf{h}}^{(1,2,\underline{\mathbf{p}})}_{1,2,\dots,2r+2} | \mathsf{T}_1(\xi_{2a+2-i}^{(1-i)}) | \underline{\mathbf{h}}^{(0,1,\underline{\mathbf{q}})}_{1,2,\dots,2r+2} \rangle + \sum_{s=1+2(r+1)}^{\mathsf{N}} \langle \underline{\mathbf{h}}^{(1,2,\underline{\mathbf{p}})}_{1,2,\dots,2r+2} | \mathsf{T}_1(\xi_s^{(\delta_{h_s,2})}) | \underline{\mathbf{h}}^{(0,1,\underline{\mathbf{q}})}_{1,2,\dots,2r+2} \rangle
\end{aligned} \tag{C.77}$$

$$\underset{\text{UpC}}{=} \langle \underline{\mathbf{h}}^{(1,1,\underline{\mathbf{p}})}_{1,2,\dots,2r+2} | \mathsf{T}_2(\xi_2) | \underline{\mathbf{h}}^{(0,1,\underline{\mathbf{q}})}_{1,2,\dots,2r+2} \rangle + \sum_{j=1}^{r} \langle \underline{\mathbf{h}}^{(1,2,\underline{\mathbf{p}}^{(1)}_{2j})}_{1,2,\dots,2r+2} | \mathsf{T}_2(\xi_{2j+2}) | \underline{\mathbf{h}}^{(0,1,\underline{\mathbf{q}})}_{1,2,\dots,2r+2} \rangle . \tag{C.78}$$

Indeed, from the previous orthogonality conditions, we have:

$$\langle \underline{\mathbf{h}}^{(1,2,\underline{\mathbf{p}})}_{1,2,3,\dots,2r+2} | \underline{\mathbf{h}}^{(0,1,\underline{\mathbf{q}})}_{1,2,3,\dots,2r+2} \rangle = 0, \quad \langle \underline{\mathbf{h}}^{(2,2,\underline{\mathbf{p}})}_{1,2,3,\dots,2r+2} | \underline{\mathbf{h}}^{(0,1,\underline{\mathbf{q}})}_{1,2,3,\dots,2r+2} \rangle = 0, \tag{C.79}$$

and

$$\langle \underline{\mathbf{h}}^{(1,2,\underline{\mathbf{p}})}_{1,2,3,\dots,2r+2} | \mathsf{T}_1(\xi_{2a+1}) | \underline{\mathbf{h}}^{(0,1,\underline{\mathbf{q}})}_{1,2,3,\dots,2r+2} \rangle = \mathsf{c}_{2a+1} \langle \underline{\mathbf{h}}^{(1,2,\underline{\mathbf{p}}^{(1)}_{2a-1})}_{1,2,3,\dots,2r+2} | \underline{\mathbf{h}}^{(0,1,\underline{\mathbf{q}})}_{1,2,3,\dots,2r+2} \rangle = 0, \tag{C.80}$$

for any $1 \leq a \leq r$. Also, for $s \geq 2r+3$ and $h_s = 0, 1$, we have:

$$\langle \underline{\mathbf{h}}^{(1,2,\underline{\mathbf{p}})}_{1,2,3,\dots,2r+2} | \mathsf{T}_1(\xi_s) | \underline{\mathbf{h}}^{(0,1,\underline{\mathbf{q}})}_{1,2,3,\dots,2r+2} \rangle = \mathsf{c}_s^{\delta_{h_s,0}} \langle \underline{\mathbf{h}}^{(1,2,\underline{\mathbf{p}},h_s+1)}_{1,2,3,\dots,2r+2,s} | \underline{\mathbf{h}}^{(0,1,\underline{\mathbf{q}},h_s)}_{1,2,3,\dots,2r+2,s} \rangle = 0, \tag{C.81}$$

as well as for $s \geq 2r+3$ and $h_s = 2$ we have:

$$\langle \underline{\mathbf{h}}^{(1,2,\underline{\mathbf{p}})}_{1,2,3,\dots,2r+2} | \mathsf{T}_1(\xi_s^{(1)}) | \underline{\mathbf{h}}^{(0,1,\underline{\mathbf{q}})}_{1,2,3,\dots,2r+2} \rangle = \langle \underline{\mathbf{h}}^{(1,2,\underline{\mathbf{p}},2)}_{1,2,3,\dots,2r+2,s} | \underline{\mathbf{h}}^{(0,1,\underline{\mathbf{q}},1)}_{1,2,3,\dots,2r+2,s} \rangle = 0. \tag{C.82}$$

So we are left only with the terms written in (C.78) and our formula (C.71) follows once we reintroduce the missing interpolation coefficients of the formula (C.76).

Let us now compute explicitly the case with only one couple of $(0,2)$, i.e. the case $r = 0$. Formula (C.71) reads:

$$C^{\underline{\mathbf{h}}^{(1,1)}_{1,2}}_{\underline{\mathbf{h}}^{(0,2)}_{1,2}} = q - \det M^{(I)}(\xi_1) \prod_{a\neq 2, a=1}^{\mathsf{N}} \frac{(\xi_1^{(1)} - \xi_a^{(\delta_{h_a,2})})}{(\xi_2^{(1)} - \xi_a^{(\delta_{h_a,2})})} \frac{\langle \underline{\mathbf{h}}^{(1,1)}_{1,2} | \mathsf{T}_2(\xi_2) | \underline{\mathbf{h}}^{(0,1)}_{1,2} \rangle}{\langle \underline{\mathbf{h}}^{(1,1)}_{1,2} | \underline{\mathbf{h}}^{(1,1)}_{1,2} \rangle}, \tag{C.83}$$

then by using the following, up to the coefficients, interpolation identity:

$$\mathsf{T}_2(\xi_2) \underset{\text{UpC}}{=} \mathsf{t}_2 + \mathsf{T}_2(\xi_1) + \sum_{s\geq 2}^{\mathsf{N}} \mathsf{T}_2(\xi_s^{(1-\delta_{h_s,0})}), \tag{C.84}$$

we get:

$$\langle \underline{\mathbf{h}}^{(1,1)}_{1,2} | \mathsf{T}_2(\xi_2) | \underline{\mathbf{h}}^{(0,1)}_{1,2} \rangle \underset{\text{UpC}}{=} \langle \underline{\mathbf{h}}^{(1,1)}_{1,2} | \underline{\mathbf{h}}^{(1,1)}_{1,2} \rangle, \tag{C.85}$$

as by the orthogonality conditions, proven in the previous subsection, it holds:

$$\langle \underline{\mathbf{h}}^{(1,1)}_{1,2} | \mathsf{T}_2(\xi_s^{(1-\delta_{h_s,0})}) | \underline{\mathbf{h}}^{(0,1)}_{1,2} \rangle = 0, \text{ for any } s \geq 2. \tag{C.86}$$

Indeed, we have:

$$\langle \underline{\mathbf{h}}^{(1,1)}_{1,2}|\mathsf{T}_2(\xi_s^{(1-\delta_{h_s,0})})|\underline{\mathbf{h}}^{(0,1)}_{1,2}\rangle = \begin{cases} \langle \underline{\mathbf{h}}^{(1,1)}_{1,2}|\underline{\mathbf{h}}^{(0,1,1)}_{1,2,s}\rangle = 0 \text{ if } h_s = 0\,, \\ \mathsf{c}_s^{\delta_{h_s,2}}\langle \underline{\mathbf{h}}^{(1,1,h_s-1)}_{1,2,s}|\underline{\mathbf{h}}^{(0,1)}_{1,2}\rangle = 0 \text{ if } h_s = 1,2\,. \end{cases} \tag{C.87}$$

Then, reintroducing the missing interpolation coefficients in front to $T_2^{(K)}(\xi_1)$ in (C.84) we get our result (C.73). □

Note that any term in the sum in (C.78), associated to a fixed $j \in \{1,\dots,r\}$, is formally identical to the first term of (C.78) up to the exchange of indices 2 and $2j+2$ in ξ_h.

The following lemma gives a recursive formula to compute the matrix elements on the right hand side of (C.71). To simplify the notations, the lemma is formulated explicitly for the first matrix element but it can be used similarly for the others matrix elements $\langle \underline{\mathbf{h}}^{(1,2,\underline{\mathbf{p}}^{(1)}_{2j})}_{1,2,3,\dots,2r+2}|\,\mathsf{T}_2(\xi_{2j+2})\,|\underline{\mathbf{h}}^{(0,1,\underline{\mathbf{q}})}_{1,2,3,\dots,2r+2}\rangle$,by exchanging the indices $2 \leftrightarrow 2j+2$ in the ξ_h, for every term involving the j index.

Lemma C.4. *Under the previous definition of the* $\underline{\mathbf{p}}$ *and* $\underline{\mathbf{q}}$*, then for* $r \geq 1$ *the following recursion formulae hold*

$$\begin{aligned}\langle \underline{\mathbf{h}}^{(1,1,\underline{\mathbf{p}})}_{1,2,3,\dots,2r+2}|\mathsf{T}_2(\xi_2)|\underline{\mathbf{h}}^{(0,1,\underline{\mathbf{q}})}_{1,2,3,\dots,2r+2}\rangle &= \mathsf{c}_3\mathsf{r}_{1,2}\sum_{s=1}^{r}\mathsf{s}_{1,2,3,2s}\langle \underline{\mathbf{h}}^{(1,1,\underline{\mathbf{p}}^{(1,1)}_{1,2s})}_{1,2,3,\dots,2r+2}|\mathsf{T}_2(\xi_{2s+2})|\underline{\mathbf{h}}^{(1,1,\underline{\mathbf{q}}^{(0)}_1)}_{1,2,3,\dots,2r+2}\rangle \\ &+\sum_{a=1}^{r}\mathsf{c}_{2a+1}\mathsf{r}_{2a+1,2}\left(\sum_{b=1}^{r}\mathsf{s}_{1,2,2a+1,2b}\langle \underline{\mathbf{h}}^{(1,1,\underline{\mathbf{p}}^{(1,1)}_{2a-1,2b})}_{1,2,3,\dots,2r+2}|\mathsf{T}_2(\xi_{2b+2})|\underline{\mathbf{h}}^{(0,1,\underline{\mathbf{q}})}_{1,2,3,\dots,2r+2}\rangle\right),\end{aligned} \tag{C.88}$$

where:

$$\mathsf{r}_{2a+1,2} = \frac{d(\xi_2^{(1)})}{d(\xi_{2a+1}^{(1)})}\prod_{n=0}^{r}\frac{\xi_2-\xi_{2n+2}^{(1)}}{\xi_{2a+1}-\xi_{2n+2}^{(1)}}\prod_{\substack{n=0\\n\neq a}}^{r}\frac{\xi_2-\xi_{2n+1}}{\xi_{2a+1}-\xi_{2n+1}}\prod_{2r+3\leq j\leq \mathsf{N}}\frac{\xi_2-\xi_j^{(1-\delta_{h_j,0})}}{\xi_{2a+1}-\xi_j^{(1-\delta_{h_j,0})}}, \tag{C.89}$$

and

$$\mathsf{s}_{1,2,2a+1,2b} = \prod_{i=1}^{2}\frac{\xi_{2a+1}^{(1)}-\xi_i}{\xi_{2b+2}^{(1)}-\xi_i}\prod_{\substack{n=1\\n\neq b}}^{r}\frac{\xi_{2a+1}-\xi_{2n+2}}{\xi_{2b+2}-\xi_{2n+2}}\prod_{n=1}^{r}\frac{\xi_{2a+1}^{(1)}-\xi_{2n+1}}{\xi_{2b+2}^{(1)}-\xi_{2n+1}}\prod_{2r+3\leq j\leq \mathsf{N}}\frac{\xi_{2a+1}^{(1)}-\xi_j^{(\delta_{h_j,2})}}{\xi_{2b+2}^{(1)}-\xi_j^{(\delta_{h_j,2})}}, \tag{C.90}$$

with the following initial condition for $r = 0$*:*

$$\langle \underline{\mathbf{h}}^{(1,1)}_{1,2}|\mathsf{T}_2(\xi_2)|\underline{\mathbf{h}}^{(0,1)}_{1,2}\rangle = \langle \underline{\mathbf{h}}^{(1,1)}_{1,2}|\underline{\mathbf{h}}^{(1,1)}_{1,2}\rangle\frac{d(\xi_2-\eta)}{d(\xi_1-\eta)}\frac{\eta}{(\xi_1-\xi_2+\eta)^2}\prod_{a\geq 3}^{\mathsf{N}}\frac{\xi_2-\xi_a^{(1-\delta_{h_a,0})}}{\xi_1-\xi_a^{(1-\delta_{h_a,0})}}. \tag{C.91}$$

Proof. Using the interpolation formula (C.84), we get

$$\begin{aligned}
&\langle \underline{\mathbf{h}}^{(1,1,\underline{\mathbf{p}})}_{1,2,3,\dots,2r+2} | \mathsf{T}_2(\xi_2) | \underline{\mathbf{h}}^{(0,1,\underline{\mathbf{q}})}_{1,2,3,\dots,2r+2} \rangle \\
&\qquad \underset{\text{UpC}}{=} \mathsf{t}_2 \langle \underline{\mathbf{h}}^{(1,1,\underline{\mathbf{p}})}_{1,2,3,\dots,2r+2} | \underline{\mathbf{h}}^{(0,1,\underline{\mathbf{q}})}_{1,2,3,\dots,2r+2} \rangle \\
&\qquad\quad + \langle \underline{\mathbf{h}}^{(1,1,\underline{\mathbf{p}})}_{1,2,3,\dots,2r+2} | \underline{\mathbf{h}}^{(1,1,\underline{\mathbf{q}})}_{1,2,3,\dots,2r+2} \rangle + \langle \underline{\mathbf{h}}^{(1,0,\underline{\mathbf{p}})}_{1,2,3,\dots,2r+2} | \underline{\mathbf{h}}^{(0,1,\underline{\mathbf{q}})}_{1,2,3,\dots,2r+2} \rangle \\
&\qquad\quad + \sum_{a=1}^{r} \sum_{i=0}^{1} \langle \underline{\mathbf{h}}^{(1,1,\underline{\mathbf{p}})}_{1,2,3,\dots,2r+2} | \mathsf{T}_2(\xi^{(1-i)}_{2a+2-i}) | \underline{\mathbf{h}}^{(0,1,\underline{\mathbf{q}})}_{1,2,3,\dots,2r+2} \rangle \\
&\qquad\quad + \sum_{s=1+2(r+1)}^{\mathsf{N}} \langle \underline{\mathbf{h}}^{(1,1,\underline{\mathbf{p}})}_{1,2,3,\dots,2r+2} | \mathsf{T}_2(\xi_s^{(\delta_{h_s,1}+\delta_{h_s,2})}) | \underline{\mathbf{h}}^{(0,1,\underline{\mathbf{q}})}_{1,2,3,\dots,2r+2} \rangle \qquad \text{(C.92)} \\
&\qquad \underset{\text{UpC}}{=} \langle \underline{\mathbf{h}}^{(1,1,\underline{\mathbf{p}})}_{1,2,3,\dots,2r+2} | \underline{\mathbf{h}}^{(1,1,\underline{\mathbf{q}})}_{1,2,3,\dots,2r+2} \rangle + \sum_{a=1}^{r} \mathsf{c}_{2a+1} \langle \underline{\mathbf{h}}^{(1,1,\underline{\mathbf{p}}^{(1)}_{2a-1})}_{1,2,3,\dots,2r+2} | \mathsf{T}_1(\xi^{(1)}_{2a+1}) | \underline{\mathbf{h}}^{(0,1,\underline{\mathbf{q}})}_{1,2,3,\dots,2r+2} \rangle . \qquad \text{(C.93)}
\end{aligned}$$

Indeed,

$$\langle \underline{\mathbf{h}}^{(1,1,\underline{\mathbf{p}})}_{1,2,3,\dots,2r+2} | \underline{\mathbf{h}}^{(0,1,\underline{\mathbf{q}})}_{1,2,3,\dots,2r+2} \rangle = 0, \quad \langle \underline{\mathbf{h}}^{(1,0,\underline{\mathbf{p}})}_{1,2,3,\dots,2r+2} | \underline{\mathbf{h}}^{(0,1,\underline{\mathbf{q}})}_{1,2,3,\dots,2r+2} \rangle = 0, \qquad \text{(C.94)}$$

and

$$\langle \underline{\mathbf{h}}^{(1,1,\underline{\mathbf{p}})}_{1,2,3,\dots,2r+2} | \mathsf{T}_2(\xi^{(1)}_{2a+2}) | \underline{\mathbf{h}}^{(0,1,\underline{\mathbf{q}})}_{1,2,3,\dots,2r+2} \rangle = \mathsf{c}_{2a+2} \langle \underline{\mathbf{h}}^{(1,1,\underline{\mathbf{p}}^{(1)}_{2a})}_{1,2,3,\dots,2r+2} | \underline{\mathbf{h}}^{(0,1,\underline{\mathbf{q}})}_{1,2,3,\dots,2r+2} \rangle = 0, \qquad \text{(C.95)}$$

for any $1 \leq a \leq r$. Also, for $s \geq 2r+3$ and $h_s = 1, 2$, we have:

$$\langle \underline{\mathbf{h}}^{(1,1,\underline{\mathbf{p}})}_{1,2,3,\dots,2r+2} | \mathsf{T}_2(\xi^{(1)}_{s}) | \underline{\mathbf{h}}^{(0,1,\underline{\mathbf{q}})}_{1,2,3,\dots,2r+2} \rangle = \mathsf{c}_s^{\delta_{h_s,2}} \langle \underline{\mathbf{h}}^{(1,1,\underline{\mathbf{p}},h_s-1)}_{1,2,3,\dots,2r+2,s} | \underline{\mathbf{h}}^{(0,1,\underline{\mathbf{q}},h_s)}_{1,2,3,\dots,2r+2,s} \rangle = 0, \qquad \text{(C.96)}$$

while for $s \geq 2r+3$ and $h_s = 0$, we have:

$$\langle \underline{\mathbf{h}}^{(1,1,\underline{\mathbf{p}})}_{1,2,3,\dots,2r+2} | \mathsf{T}_2(\xi_{s}) | \underline{\mathbf{h}}^{(0,1,\underline{\mathbf{q}})}_{1,2,3,\dots,2r+2} \rangle = \langle \underline{\mathbf{h}}^{(1,1,\underline{\mathbf{p}},0)}_{1,2,3,\dots,2r+2,s} | \underline{\mathbf{h}}^{(0,1,\underline{\mathbf{q}},1)}_{1,2,3,\dots,2r+2,s} \rangle = 0. \qquad \text{(C.97)}$$

So we are left only with the following terms for $a \in \{1, \dots, r\}$ which read:

$$\langle \underline{\mathbf{h}}^{(1,1,\underline{\mathbf{p}})}_{1,2,3,\dots,2r+2} | \mathsf{T}_2(\xi_{2a+1}) | \underline{\mathbf{h}}^{(0,1,\underline{\mathbf{q}})}_{1,2,3,\dots,2r+2} \rangle = \mathsf{c}_{2a+1} \langle \underline{\mathbf{h}}^{(1,1,\underline{\mathbf{p}}^{(1)}_{2a-1})}_{1,2,3,\dots,2r+2} | \mathsf{T}_1(\xi^{(1)}_{2a+1}) | \underline{\mathbf{h}}^{(0,1,\underline{\mathbf{q}})}_{1,2,3,\dots,2r+2} \rangle . \qquad \text{(C.98)}$$

Now we can use the interpolation formula (C.76) and we get

$$\begin{aligned}
\langle \underline{\mathbf{h}}^{(1,1,\underline{\mathbf{p}}^{(1)}_{2a-1})}_{1,2,3,\dots,2r+2} | \mathsf{T}_1(\xi^{(1)}_{2a+1}) | \underline{\mathbf{h}}^{(0,1,\underline{\mathbf{q}})}_{1,2,3,\dots,2r+2} \rangle &\underset{\text{UpC}}{=} \mathsf{t}_1 \langle \underline{\mathbf{h}}^{(1,1,\underline{\mathbf{p}}^{(1)}_{2a-1})}_{1,2,3,\dots,2r+2} | \underline{\mathbf{h}}^{(0,1,\underline{\mathbf{q}})}_{1,2,3,\dots,2r+2} \rangle \\
&+ \langle \underline{\mathbf{h}}^{(2,1,\underline{\mathbf{p}}^{(1)}_{2a-1})}_{1,2,3,\dots,2r+2} | \underline{\mathbf{h}}^{(0,1,\underline{\mathbf{q}})}_{1,2,3,\dots,2r+2} \rangle + \langle \underline{\mathbf{h}}^{(1,2,\underline{\mathbf{p}}^{(1)}_{2a-1})}_{1,2,3,\dots,2r+2} | \underline{\mathbf{h}}^{(0,1,\underline{\mathbf{q}})}_{1,2,3,\dots,2r+2} \rangle \\
&+ \sum_{b=1}^{r} \mathsf{c}_{2b+1}^{1-\delta_{b,a}} \langle \underline{\mathbf{h}}^{(1,1,\underline{\mathbf{p}}^{(1)}_{2a-1})}_{1,2,3,\dots,2r+2} + \underline{\mathbf{e}}_{2b+1} | \underline{\mathbf{h}}^{(0,1,\underline{\mathbf{q}})}_{1,2,3,\dots,2r+2} \rangle \\
&+ \sum_{b=1}^{r} \langle \underline{\mathbf{h}}^{(1,1,\underline{\mathbf{p}}^{(1,0)}_{2a-1,2b})}_{1,2,3,\dots,2r+2} | \mathsf{T}_1(\xi^{(1)}_{2b+2}) | \underline{\mathbf{h}}^{(0,1,\underline{\mathbf{q}})}_{1,2,3,\dots,2r+2} \rangle \\
&+ \sum_{s=1+2(r+1)}^{\mathsf{N}} \langle \underline{\mathbf{h}}^{(1,1,\underline{\mathbf{p}}^{(1)}_{2a-1})}_{1,2,3,\dots,2r+2} | \mathsf{T}_1(\xi_s^{(\delta_{h_s,2})}) | \underline{\mathbf{h}}^{(0,1,\underline{\mathbf{q}})}_{1,2,3,\dots,2r+2} \rangle \qquad \text{(C.99)} \\
&\underset{\text{UpC}}{=} \sum_{b=1}^{r} \langle \underline{\mathbf{h}}^{(1,1,\underline{\mathbf{p}}^{(1,1)}_{2a-1,2b})}_{1,2,3,\dots,2r+2} | \mathsf{T}_2(\xi_{2b+2}) | \underline{\mathbf{h}}^{(0,1,\underline{\mathbf{q}})}_{1,2,3,\dots,2r+2} \rangle . \qquad \text{(C.100)}
\end{aligned}$$

Indeed, by the orthogonality it holds:

$$\langle \underline{\mathbf{h}}^{(1,1,\underline{\mathbf{p}}^{(1)}_{2a-1})}_{1,2,3,\dots,2r+2} | \underline{\mathbf{h}}^{(0,1,\underline{\mathbf{q}})}_{1,2,3,\dots,2r+2} \rangle = 0, \quad \langle \underline{\mathbf{h}}^{(2,1,\underline{\mathbf{p}}^{(1)}_{2a-1})}_{1,2,3,\dots,2r+2} | \underline{\mathbf{h}}^{(0,1,\underline{\mathbf{q}})}_{1,2,3,\dots,2r+2} \rangle = 0 \tag{C.101}$$

$$\langle \underline{\mathbf{h}}^{(1,2,\underline{\mathbf{p}}^{(1)}_{2a-1})}_{1,2,3,\dots,2r+2} | \underline{\mathbf{h}}^{(0,1,\underline{\mathbf{q}})}_{1,2,3,\dots,2r+2} \rangle = 0, \quad \langle \underline{\mathbf{h}}^{(1,1,\underline{\mathbf{p}}^{(1)}_{2a-1})}_{1,2,3,\dots,2r+2} + \underline{\mathbf{e}}_{2b+1} | \underline{\mathbf{h}}^{(0,1,\underline{\mathbf{q}})}_{1,2,3,\dots,2r+2} \rangle = 0, \tag{C.102}$$

while for $h_s = 2$ it holds:

$$\langle \underline{\mathbf{h}}^{(1,1,\underline{\mathbf{p}}^{(1)}_{2a-1},2)}_{1,2,3,\dots,2r+2,s} | \mathsf{T}_1(\xi^{(1)}_s) | \underline{\mathbf{h}}^{(0,1,\underline{\mathbf{q}})}_{1,2,3,\dots,2r+2} \rangle = \langle \underline{\mathbf{h}}^{(1,1,\underline{\mathbf{p}}^{(1)}_{2a-1},2)}_{1,2,3,\dots,2r+2,s} | \underline{\mathbf{h}}^{(0,1,\underline{\mathbf{q}},1)}_{1,2,3,\dots,2r+2,s} \rangle = 0, \tag{C.103}$$

as well as for $h_s = 0, 1$ it holds:

$$\langle \underline{\mathbf{h}}^{(1,1,\underline{\mathbf{p}}^{(1)}_{2a-1})}_{1,2,3,\dots,2r+2} | \mathsf{T}_1(\xi_s) | \underline{\mathbf{h}}^{(0,1,\underline{\mathbf{q}})}_{1,2,3,\dots,2r+2} \rangle = \langle \underline{\mathbf{h}}^{(1,1,\underline{\mathbf{p}}^{(1)}_{2a-1},h_s+1)}_{1,2,3,\dots,2r+2,s} | \underline{\mathbf{h}}^{(0,1,\underline{\mathbf{q}},h_s)}_{1,2,3,\dots,2r+2,s} \rangle = 0. \tag{C.104}$$

Therefore we obtain the following mixed recursion formula

$$\begin{aligned} \langle \underline{\mathbf{h}}^{(1,1,\underline{\mathbf{p}})}_{1,2,3,\dots,2r+2} | \mathsf{T}_2(\xi_2) | \underline{\mathbf{h}}^{(0,1,\underline{\mathbf{q}})}_{1,2,3,\dots,2r+2} \rangle &\underset{\text{UpC}}{=} \langle \underline{\mathbf{h}}^{(1,1,\underline{\mathbf{p}})}_{1,2,3,\dots,2r+2} | \underline{\mathbf{h}}^{(1,1,\underline{\mathbf{q}})}_{1,2,3,\dots,2r+2} \rangle \\ &+ \sum_{a=1}^{r} \sum_{b=1}^{r} \mathsf{c}_{2a+1} \langle \underline{\mathbf{h}}^{(1,1,\underline{\mathbf{p}}^{(1,1)}_{2a-1,2b})}_{1,2,3,\dots,2r+2} | \mathsf{T}_2(\xi_{2b+2}) | \underline{\mathbf{h}}^{(0,1,\underline{\mathbf{q}})}_{1,2,3,\dots,2r+2} \rangle. \end{aligned} \tag{C.105}$$

Indeed, all the matrix elements $\langle \underline{\mathbf{h}}^{(1,1,\underline{\mathbf{p}}^{(1,1)}_{2a-1,2b})}_{1,2,3,\dots,2r+2} | \mathsf{T}_2(\xi_{2b+2}) | \underline{\mathbf{h}}^{(0,1,\underline{\mathbf{q}})}_{1,2,3,\dots,2r+2} \rangle$ on the r.h.s. of (C.105) have $(r-1)$-couples of $(0,2)$, i.e. one less w.r.t. the first matrix element on the r.h.s. of (C.78) $\langle \underline{\mathbf{h}}^{(1,1,\underline{\mathbf{p}})}_{1,2,3,\dots,2r+2} | \mathsf{T}_2(\xi_2) | \underline{\mathbf{h}}^{(0,1,\underline{\mathbf{q}})}_{1,2,3,\dots,2r+2} \rangle$. Moreover, the matrix element $\langle \underline{\mathbf{h}}^{(1,1,\underline{\mathbf{p}})}_{1,2,3,\dots,2r+2} | \underline{\mathbf{h}}^{(1,1,\underline{\mathbf{q}})}_{1,2,3,\dots,2r+2} \rangle$ contains one couple less of $(0,2)$ that the starting matrix element $\langle \underline{\mathbf{h}}^{(0,2,\underline{\mathbf{p}})}_{1,2,3,\dots,2r+2} | \underline{\mathbf{h}}^{(1,1,\underline{\mathbf{q}})}_{1,2,3,\dots,2r+2} \rangle$, i.e. r-couples of $(0,2)$. Up to a reordering in the indices, $\langle \underline{\mathbf{h}}^{(1,1,\underline{\mathbf{q}})}_{1,2,3,\dots,2r+2} | \underline{\mathbf{h}}^{(1,1,\underline{\mathbf{q}})}_{1,2,3,\dots,2r+2} \rangle$ can be developed just as done in (C.71), generating matrix elements with $(r-1)$-couples of $(0,2)$. In total, we have that

$$\langle \underline{\mathbf{h}}^{(1,1,\underline{\mathbf{p}})}_{1,2,3,\dots,2r+2} | \underline{\mathbf{h}}^{(1,1,\underline{\mathbf{q}})}_{1,2,3,\dots,2r+2} \rangle \underset{\text{UpC}}{=} \mathsf{c}_3 \sum_{j=2}^{r+1} \langle \underline{\mathbf{h}}^{(1,1,\underline{\mathbf{p}}^{(1,1)}_{1,2j})}_{1,2,3,\dots,2r+2} | \mathsf{T}_2(\xi_{2j}) | \underline{\mathbf{h}}^{(1,1,\underline{\mathbf{q}}^{(0)}_1)}_{1,2,3,\dots,2r+2} \rangle, \tag{C.106}$$

and by substituting it in (C.105) we get the recursion formula (C.88), up to the coefficients.

Now that we have identified the non-zero contributions in the used interpolation formulae, we can easily compute the missing coefficients presented in (C.88). From (C.92), the non-zero contributions of $\mathsf{T}_2(\xi_2)$ read:

$$\sum_{a=0}^{r} \frac{d(\xi^{(1)}_2)}{d(\xi^{(1)}_{2a+1})} \prod_{b \neq 2a+1} \frac{\xi_2 - \xi_b^{(1-\delta_{\tilde{h}_b,0})}}{\xi_{2a+1} - \xi_b^{(1-\delta_{\tilde{h}_b,0})}} \mathsf{T}_2(\xi_{2a+1}), \tag{C.107}$$

where $\tilde{h}_0 = 1$ and $\tilde{h}_b$ is the b element of $\underline{\mathbf{h}}^{(1,1,\underline{\mathbf{p}})}_{1,2,3,\dots,2r+2}$ for any $b \geq 2$. Similarly, from (C.99), the non-zero contributions of $\mathsf{T}_1(\xi^{(1)}_{2a+1})$ read:

$$\sum_{b=1}^{r} \prod_{c \neq 2b+2} \frac{\xi^{(1)}_{2a+1} - \xi_c^{(\delta_{h_c,2})}}{\xi^{(1)}_{2b+2} - \xi_c^{(\delta_{h_c,2})}} \mathsf{T}_1(\xi^{(1)}_{2b+2}), \tag{C.108}$$

where h_b is the b element of $\underline{\mathbf{h}}^{(1,1,\underline{\mathbf{p}})}_{1,2,3,\dots,2r+2}$ for any $b \geq 1$. Finally, from (C.106), the non-zero contributions of $\mathsf{T}_1(\xi_2^{(1)})$ read:

$$\sum_{b=1}^{r} \prod_{c\neq 2b+2} \frac{\xi_3^{(1)} - \xi_c^{(\delta_{h_c,2})}}{\xi_{2b+2}^{(1)} - \xi_c^{(\delta_{h_c,2})}} \mathsf{T}_1(\xi_{2b+2}^{(1)}), \tag{C.109}$$

where h_b is the b element of $\underline{\mathbf{h}}^{(1,1,\underline{\mathbf{p}})}_{1,2,3,\dots,2r+2}$ for any $b \geq 1$. From these expansions, it is simple to verify that the recursion holds as written in the lemma.

Finally, the initial condition (C.91) for the recursion just coincides with the identity (C.85), proven in the previous lemma, by reintroducing the missing interpolation coefficients in front to $T_2^{(K)}(\xi_1)$ in (C.84). □

It is worth remarking that in the recursion formula (C.88) the common part $\underline{\mathbf{h}}_{1,2,3,\dots,2r+2}$ of the SoV co-vectors and vectors are left unchanged by the recursion, i.e. the recursion acts only on the $(0,2)$ couples.

Moreover, thanks to Lemma C.3 the solution of these recursions formulae lead to the determination of the coefficient $C^{\underline{\mathbf{k}}}_{\underline{\mathbf{h}}}$, as defined in (C.67). Here, we do not solve these recursions but we use the previous lemmas to complete the proof of the Theorem 3.1 by proving the independence of the $C^{\underline{\mathbf{k}}}_{\underline{\mathbf{h}}}$ w.r.t. c. We have just to remark that at the right hand side of (C.88) we have matrix elements of $\mathsf{T}_2(\xi_{2h})$ with $(r-1)$-couples of $(0,2)$ in the co-vector corresponding to $(r-1)$-couples of $(1,1)$ in the vector. The same statement holds true adapting (C.88) for the development of the others matrix elements $\langle \underline{\mathbf{h}}^{(1,2,\underline{\mathbf{p}}^{(1)}_{2j})}_{1,2,3,\dots,2r+2}| \mathsf{T}_2(\xi_{2j+2}) |\underline{\mathbf{h}}^{(0,1,\underline{\mathbf{q}})}_{1,2,3,\dots,2r+2}\rangle$. Hence, applying $(r-1)$-times the same recursion formulae to all the non-zero matrix elements generated in this first step of the recursion, we end up exactly in the same diagonal matrix element $\langle \underline{\mathbf{h}}^{(1,1,\underline{\mathbf{q}})}_{1,2,3,\dots,2r+2}|\underline{\mathbf{h}}^{(1,1,\underline{\mathbf{q}})}_{1,2,3,\dots,2r+2}\rangle$, proving the following proportionality:

$$\langle \underline{\mathbf{h}}^{(0,2,\underline{\mathbf{p}})}_{1,2,3,\dots,2r+2}|\underline{\mathbf{h}}^{(1,1,\underline{\mathbf{q}})}_{1,2,3,\dots,2r+2}\rangle \propto \mathsf{c}^{r+1} \langle \underline{\mathbf{h}}^{(1,1,\underline{\mathbf{q}})}_{1,2,3,\dots,2r+2}|\underline{\mathbf{h}}^{(1,1,\underline{\mathbf{q}})}_{1,2,3,\dots,2r+2}\rangle, \tag{C.110}$$

as any time that we make a recursion we generate exactly a power one of c. The proportionality coefficient $C^{\underline{\mathbf{k}}}_{\underline{\mathbf{h}}}$ must then be independent with respect to c as the full dependence in c is already made explicit in the previous formula.

C.2.2 Computation of diagonal elements

Here we give a proof of the form of the diagonal coupling between SoV co-vectors and vectors. It is independent from the proof of the same result, but in the special case $\det K = 0$, that is given in the main body of the paper, see Theorem 4.1.

We follow the standard procedure used to prove the "Sklyanin measure" [37,39], by using the usual interpolation formulae of the transfer matrices.

i) We have that

$$\langle \underline{\mathbf{h}}_a^{(1)}|T_2^{(K)}(\xi_a^{(1)})|\underline{\mathbf{h}}_a^{(0)}\rangle = \langle \underline{\mathbf{h}}_a^{(0)}|\underline{\mathbf{h}}_a^{(0)}\rangle. \tag{C.111}$$

Computing the action of $T_2^{(K)}(\xi_a^{(1)})$ by interpolating in the right points

$$T_2^{(K)}(\xi_a^{(1)}) = d(\xi_a^{(2)}) \left(T^{(K,\infty)}_{2,\underline{\mathbf{z}}(\underline{\mathbf{h}})}(\xi_a^{(1)}) + \sum_{b=1}^{\mathsf{N}} g^{(2)}_{b,\underline{\mathbf{z}}(\underline{\mathbf{h}})}(\xi_a^{(1)}) T_2^{(K)}(\xi_b^{(\delta_{h_b,1}+\delta_{h_b,2})}) \right), \tag{C.112}$$

where we recall the definitions

$$\underline{\mathbf{z}}(\underline{\mathbf{h}}) = \{\delta_{h_1,1} + \delta_{h_1,2}, ..., \delta_{h_\mathsf{N},1} + \delta_{h_\mathsf{N},2}\}, \tag{C.113}$$

$$g^{(m)}_{a,\underline{\mathbf{h}}}(\lambda) = \prod_{b\neq a,b=1}^{\mathsf{N}} \frac{\lambda - \xi_b^{(h_b)}}{\xi_a^{(h_a)} - \xi_b^{(h_b)}} \prod_{b=1}^{(m-1)\mathsf{N}} \frac{1}{\xi_a^{(h_a)} - \xi_b^{(-1)}}, \tag{C.114}$$

we get

$$\langle \underline{\mathbf{h}}^{(0)}_a | \underline{\mathbf{h}}^{(0)}_a \rangle = d(\xi_a^{(2)})(T^{(K,\infty)}_{2,\underline{\mathbf{z}}(\underline{\mathbf{h}})}(\xi_a^{(1)}) \langle \underline{\mathbf{h}}^{(1)}_a | \underline{\mathbf{h}}^{(0)}_a \rangle + g^{(2)}_{a,\underline{\mathbf{z}}(\underline{\mathbf{h}})}(\xi_a^{(1)}) \langle \underline{\mathbf{h}}^{(1)}_a | \underline{\mathbf{h}}^{(1)}_a \rangle \tag{C.115}$$

$$+ \sum_{b=1,b\neq a}^{\mathsf{N}} g^{(2)}_{b,\underline{\mathbf{z}}(\underline{\mathbf{h}})}(\xi_a^{(1)}) \langle \underline{\mathbf{h}}^{(1)}_a | T_2^{(K)}(\xi_b^{(\delta_{h_b,1}+\delta_{h_b,2})}) | \underline{\mathbf{h}}^{(0)}_a \rangle). \tag{C.116}$$

Now, we can use the following identities:

$$\langle \underline{\mathbf{h}}^{(1)}_a | T_2^{(K)}(\xi_b^{(\delta_{h_b,1}+\delta_{h_b,2})}) | \underline{\mathbf{h}}^{(0)}_a \rangle = \begin{cases} \langle \underline{\mathbf{h}}^{(1,h_b-1)}_{a,b} | \underline{\mathbf{h}}^{(0,h_b)}_{a,b} \rangle & \text{if } h_b \in \{1,2\}, \\ \langle \underline{\mathbf{h}}^{(1,0)}_{a,b} | \underline{\mathbf{h}}^{(0,1)}_{a,b} \rangle & \text{if } h_b = 0, \end{cases} \tag{C.117}$$

and then being

$$\underline{\mathbf{h}}^{(1)}_a \underset{\text{(C.45)}}{\neq} \underline{\mathbf{h}}^{(0)}_a, \quad \underline{\mathbf{h}}^{(1,0)}_{a,b} \underset{\text{(C.45)}}{\neq} \underline{\mathbf{h}}^{(0,1)}_{a,b}, \tag{C.118}$$

$$\underline{\mathbf{h}}^{(1,h_b-1)}_{a,b} \underset{\text{(C.45)}}{\neq} \underline{\mathbf{h}}^{(0,h_b)}_{a,b} \quad \text{if } h_b \in \{1,2\}, \tag{C.119}$$

the orthogonality conditions implies the identity:

$$\langle \underline{\mathbf{h}}^{(0)}_a | \underline{\mathbf{h}}^{(0)}_a \rangle = d(\xi_a^{(2)}) g^{(2)}_{a,\underline{\mathbf{z}}(\underline{\mathbf{h}})}(\xi_a^{(1)}) \langle \underline{\mathbf{h}}^{(1)}_a | \underline{\mathbf{h}}^{(1)}_a \rangle, \tag{C.120}$$

or equivalently:

$$\frac{\langle \underline{\mathbf{h}}^{(0)}_a | \underline{\mathbf{h}}^{(0)}_a \rangle}{\langle \underline{\mathbf{h}}^{(1)}_a | \underline{\mathbf{h}}^{(1)}_a \rangle} = \frac{d(\xi_a^{(2)})}{d(\xi_a^{(1)})} \prod_{n\neq a,n=1}^{\mathsf{N}} \frac{\xi_a^{(1)} - \xi_n^{(\delta_{h_n,1}+\delta_{h_n,2})}}{\xi_a - \xi_n^{(\delta_{h_n,1}+\delta_{h_n,2})}}. \tag{C.121}$$

ii) Similarly, we have

$$\langle \underline{\mathbf{h}}^{(1)}_a | T_1^{(K)}(\xi_a) | \underline{\mathbf{h}}^{(2)}_a \rangle = \langle \underline{\mathbf{h}}^{(2)}_a | \underline{\mathbf{h}}^{(2)}_a \rangle. \tag{C.122}$$

Computing the action of $T_1^{(K)}(\xi_a)$ by interpolating in the right points

$$T_1^{(K)}(\lambda) = T^{(K,\infty)}_{1,\underline{\mathbf{y}}(\underline{\mathbf{h}})}(\xi_a) + \sum_{a=1}^{\mathsf{N}} g^{(1)}_{a,\underline{\mathbf{y}}(\underline{\mathbf{h}})}(\xi_a) T_1^{(K)}(\xi_a^{(\delta_{h_a,2})}), \tag{C.123}$$

where we recall the definitions

$$\underline{\mathbf{y}}(\underline{\mathbf{h}}) = \{\delta_{h_1,2}, ..., \delta_{h_\mathsf{N},2}\}, \tag{C.124}$$

we get

$$\langle \underline{\mathbf{h}}^{(2)}_a | \underline{\mathbf{h}}^{(2)}_a \rangle = T^{(K,\infty)}_{1,\underline{\mathbf{y}}(\underline{\mathbf{h}})}(\xi_a) \langle \underline{\mathbf{h}}^{(1)}_a | \underline{\mathbf{h}}^{(2)}_a \rangle + g^{(1)}_{a,\underline{\mathbf{y}}(\underline{\mathbf{h}})}(\xi_a) \langle \underline{\mathbf{h}}^{(1)}_a | \underline{\mathbf{h}}^{(1)}_a \rangle \tag{C.125}$$

$$+ \sum_{b=1,b\neq a}^{\mathsf{N}} g^{(1)}_{b,\underline{\mathbf{y}}(\underline{\mathbf{h}})}(\xi_a) \langle \underline{\mathbf{h}}^{(1)}_a | T_1^{(K)}(\xi_b^{(\delta_{h_b,2})}) | \underline{\mathbf{h}}^{(2)}_a \rangle. \tag{C.126}$$

Now, using the following identities:

$$\langle \underline{\mathbf{h}}_a^{(1)} | T_1^{(K)}(\xi_b^{(\delta_{h_b,2})}) | \underline{\mathbf{h}}_a^{(2)} \rangle = \begin{cases} \langle \underline{\mathbf{h}}_{a,b}^{(1,h_b+1)} | \underline{\mathbf{h}}_{a,b}^{(2,h_b)} \rangle & \text{if } h_b \in \{0,1\}, \\ \langle \underline{\mathbf{h}}_{a,b}^{(1,2)} | \underline{\mathbf{h}}_{a,b}^{(2,1)} \rangle & \text{if } h_b = 2, \end{cases} \tag{C.127}$$

and then being

$$\underline{\mathbf{h}}_a^{(1)} \underset{\text{(C.45)}}{\neq} \underline{\mathbf{h}}_a^{(2)}, \quad \underline{\mathbf{h}}_{a,b}^{(1,2)} \underset{\text{(C.45)}}{\neq} \underline{\mathbf{h}}_{a,b}^{(2,1)}, \tag{C.128}$$

$$\underline{\mathbf{h}}_{a,b}^{(1,h_b+1)} \underset{\text{(C.45)}}{\neq} \underline{\mathbf{h}}_{a,b}^{(2,h_b)} \quad \text{if } h_b \in \{1,2\}, \tag{C.129}$$

the orthogonality conditions implies the identity:

$$\langle \underline{\mathbf{h}}_a^{(2)} | \underline{\mathbf{h}}_a^{(2)} \rangle = g_{a,\underline{\mathbf{y}}(\underline{\mathbf{h}})}^{(2)}(\xi_a) \langle \underline{\mathbf{h}}_a^{(1)} | \underline{\mathbf{h}}_a^{(1)} \rangle, \tag{C.130}$$

or equivalently:

$$\frac{\langle \underline{\mathbf{h}}_a^{(2)} | \underline{\mathbf{h}}_a^{(2)} \rangle}{\langle \underline{\mathbf{h}}_a^{(1)} | \underline{\mathbf{h}}_a^{(1)} \rangle} = \prod_{n \neq a, n=1}^{\mathsf{N}} \frac{\xi_a - \xi_n^{(\delta_{h_n,2})}}{\xi_a^{(1)} - \xi_n^{(\delta_{h_n,2})}}. \tag{C.131}$$

References

[1] J. M. Maillet and G. Niccoli, *On quantum separation of variables*, J. Math. Phys. **59**, 091417 (2018), doi:10.1063/1.5050989.

[2] E. K. Sklyanin, *The quantum Toda chain*, In Non-Linear Equations in Classical and Quantum Field Theory, pp. 196, Springer Berlin Heidelberg (1985), doi:10.1007/3-540-15213-x_80.

[3] E. Sklyanin, *Functional Bethe ansatz*, In Integrable and Superintegrable Systems, pp. 8, World Scientific (1990), doi:10.1142/9789812797179_0002.

[4] E. K. Sklyanin, *Separation of variables in the classical integrable $SL(3)$ magnetic chain*, Commun. Math. Phys. **150**, 181 (1992), doi:10.1007/bf02096572.

[5] E. K. Sklyanin, *Separation of variables: New trends*, Prog. Theor. Phys. Suppl. **118**, 35 (1995), doi:10.1143/ptps.118.35.

[6] E. K. Sklyanin, *Separation of variables in the quantum integrable models related to the Yangian $Y[sl(3)]$*, J. Math. Sci. **80**, 1861 (1996), doi:10.1007/bf02362784.

[7] E. K. Sklyanin and L. D. Faddeev, *Quantum-mechanical approach to completely integrable field theory models*, In Fifty Years of Mathematical Physics, pp. 290, World Scientific, (2016) doi:10.1142/9789814340960_0025.

[8] E. K. Sklyanin, L. A. Takhtadzhyan and L. D. Faddeev, *Quantum inverse problem method. i*, Theor. Math. Phys. **40**, 688 (1979), doi:10.1007/bf01018718.

[9] L. A. Takhtadzhan and L. D. Faddeev, *The quantum method of the inverse problem and the Heisenberg XYZ model*, Russ. Math. Surv. **34**, 11 (1979), doi:10.1070/rm1979v034n05abeh003909.

[10] E. Sklyanin, *Method of the inverse scattering problem and the nonlinear quantum Schroedinger equation*, Sov. Phys. Dokl. **24**, 107 (1979).

[11] E. Sklyanin, *On complete integrability of the Landau-Lifshitz equation*, Tech. rep., LOMI (1979).

[12] L. D. Faddeev and L. A. Takhtajan, *Quantum inverse scattering method*, Sov. Sci. Rev. Math (1981).

[13] E. K. Sklyanin, *Quantum version of the method of inverse scattering problem*, J. Math. Sci. **19**, 1546 (1982), doi:10.1007/bf01091462.

[14] L. Faddeev, *Integrable models in 1+1 dimensional quantum field theory*, Tech. rep., CEA Centre d'Etudes Nucleaires de Saclay (1982).

[15] L. Faddeev, *How Algebraic Bethe Ansatz works for integrable models* (1996), arXiv:hep-th/9605187.

[16] H. Bethe, *Zur Theorie der Metalle*, Zeitschrift für Physik **71**, 205 (1931), doi:10.1007/bf01341708.

[17] L. R. Walker, *Antiferromagnetic linear chain*, Phys. Rev. **116**, 1089 (1959), doi:10.1103/physrev.116.1089.

[18] E. H. Lieb, *Residual entropy of square ice*, Phys. Rev. **162**, 162 (1967), doi:10.1103/physrev.162.162.

[19] R. J. Baxter, *Partition function of the Eight-Vertex lattice model*, Ann. Phys. **70**, 193 (1972), doi:10.1016/0003-4916(72)90335-1.

[20] R. J. Baxter, *Eight-vertex model in lattice statistics*, Phys. Rev. Lett. **26**, 832 (1971), doi:10.1103/physrevlett.26.832.

[21] R. Baxter, *Eight-vertex model in lattice statistics and one-dimensional anisotropic Heisenberg chain. i, II, III*, Ann. Phys. **76**, 1 (1973), doi:10.1016/0003-4916(73)90440-5.

[22] R. J. Baxter, *Exactly solved models in statistical mechanics*, In Series on Advances in Statistical Mechanics, pp. 5, World Scientific, doi:10.1142/9789814415255_0002 (1985).

[23] O. Babelon, D. Bernard and F. A. Smirnov, *Quantization of solitons and the restricted sine-Gordon model*, Commun. Math. Phys. **182**, 319 (1996), doi:10.1007/bf02517893.

[24] F. A. Smirnov, *Structure of matrix elements in the quantum Toda chain*, J. Phys. A **31**, 8953 (1998), doi:10.1088/0305-4470/31/44/019.

[25] F. A. Smirnov, *Separation of variables for quantum integrable models related to* $U_q(\widehat{sl}_n)$, In MathPhys Odyssey 2001, pp. 455, Birkhäuser Boston (2002), doi:10.1007/978-1-4612-0087-1_17.

[26] S. Derkachov, G. Korchemsky and A. Manashov, *Non-compact Heisenberg spin magnets from high-energy QCD*, Nucl. Phys. B **617**, 375 (2001), doi:10.1016/s0550-3213(01)00457-6.

[27] S. Derkachov, G. P. Korchemsky and A. N. Manashov, *Separation of variables for the quantum* $sl(2,\mathbb{R})$ *spin chain*, J. High Energy Phys. **07**, 047 (2003), doi:10.1088/1126-6708/2003/07/047.

[28] S. Derkachov, G. P. Korchemsky and A. N. Manashov, *Baxter* $\mathbb{Q}$*-operator and separation of variables for the open* $sl(2,\mathbb{R})$ *spin chain*, J. High Energy Phys. **10**, 053 (2003), doi:10.1088/1126-6708/2003/10/053.

[29] A. G. Bytsko and J. Teschner, *Quantization of models with non-compact quantum group symmetry: modular XXZ magnet and lattice sinh-Gordon model*, J. Phys. A **39**, 12927 (2006), doi:10.1088/0305-4470/39/41/s11.

[30] G. von Gehlen, N. Iorgov, S. Pakuliak and V. Shadura, *The Baxter-Bazhanov-Stroganov model: separation of variables and the Baxter equation*, J. Phys. A **39**, 7257 (2006), doi:10.1088/0305-4470/39/23/006.

[31] H. Frahm, A. Seel and T. Wirth, *Separation of variables in the open XXX chain*, Nucl. Phys. B **802**, 351 (2008), doi:10.1016/j.nuclphysb.2008.04.008.

[32] L. Amico, H. Frahm, A. Osterloh and T. Wirth, *Separation of variables for integrable spin-boson models*, Nucl. Phys. B **839**, 604 (2010), doi:10.1016/j.nuclphysb.2010.07.005.

[33] G. Niccoli and J. Teschner, *The sine-Gordon model revisited: I*, J. Stat. Mech. P09014 (2010), doi:10.1088/1742-5468/2010/09/p09014.

[34] G. Niccoli, *Reconstruction of Baxter Q-operator from Sklyanin SOV for cyclic representations of integrable quantum models*, Nucl. Phys. B **835**, 263 (2010), doi:10.1016/j.nuclphysb.2010.03.009.

[35] G. Niccoli, *Completeness of Bethe ansatz by Sklyanin SOV for cyclic representations of integrable quantum models*, J. High Energy Phys. **03**, 123 (2011), doi:10.1007/jhep03(2011)123.

[36] H. Frahm, J. H. Grelik, A. Seel and T. Wirth, *Functional Bethe ansatz methods for the open XXX chain*, J. Phys. A **44**, 015001 (2010), doi:10.1088/1751-8113/44/1/015001.

[37] N. Grosjean, J. M. Maillet and G. Niccoli, *On the form factors of local operators in the lattice sine–Gordon model*, J. Stat. Mech. P10006 (2012), doi:10.1088/1742-5468/2012/10/p10006.

[38] N. Grosjean and G. Niccoli, *The τ_2-model and the chiral Potts model revisited: completeness of Bethe equations from Sklyanin's SoV method*, J. Stat. Mech. P11005 (2012), doi:10.1088/1742-5468/2012/11/p11005.

[39] G. Niccoli, *Non-diagonal open spin-1/2 XXZ quantum chains by separation of variables: complete spectrum and matrix elements of some quasi-local operators*, J. Stat. Mech. P10025 (2012), doi:10.1088/1742-5468/2012/10/p10025.

[40] G. Niccoli, *Antiperiodic spin-1/2 XXZ quantum chains by separation of variables: Complete spectrum and form factors*, Nucl. Phys. B **870**, 397 (2013), doi:10.1016/j.nuclphysb.2013.01.017.

[41] G. Niccoli, *An antiperiodic dynamical six-vertex model: I. complete spectrum by SOV, matrix elements of the identity on separate states and connections to the periodic eight-vertex model*, J. Phys. A **46**, 075003 (2013), doi:10.1088/1751-8113/46/7/075003.

[42] G. Niccoli, *Form factors and complete spectrum of XXX anti-periodic higher spin chains by quantum separation of variables*, J. Math. Phys. **54**, 053516 (2013), doi:10.1063/1.4807078.

[43] N. Grosjean, J.-M. Maillet and G. Niccoli, *On the form factors of local operators in the Bazhanov-Stroganov and chiral Potts models*, Annales Henri Poincaré **16**, 1103 (2015), doi:10.1007/s00023-014-0358-9.

[44] S. Faldella and G. Niccoli, *SoV approach for integrable quantum models associated with general representations on spin-1/2 chains of the 8-vertex reflection algebra*, J. Phys. A **47**, 115202 (2014), doi:10.1088/1751-8113/47/11/115202.

[45] S. Faldella, N. Kitanine and G. Niccoli, *The complete spectrum and scalar products for the open spin-1/2 XXZ quantum chains with non-diagonal boundary terms*, J. Stat. Mech. P01011 (2014), doi:10.1088/1742-5468/2014/01/p01011.

[46] N. Kitanine, J. M. Maillet and G. Niccoli, *Open spin chains with generic integrable boundaries: Baxter equation and Bethe ansatz completeness from separation of variables*, J. Stat. Mech. P05015 (2014), doi:10.1088/1742-5468/2014/05/p05015.

[47] G. Niccoli and V. Terras, *Antiperiodic XXZ chains with arbitrary spins: Complete eigenstate construction by functional equations in separation of variables*, Lett. Math. Phys. **105**, 989 (2015), doi:10.1007/s11005-015-0759-9.

[48] D. Levy-Bencheton, G. Niccoli and V. Terras, *Antiperiodic dynamical 6-vertex model by separation of variables II: functional equations and form factors*, J. Stat. Mech., 033110 (2016), doi:10.1088/1742-5468/2016/03/033110.

[49] G. Niccoli and V. Terras, *The eight-vertex model with quasi-periodic boundary conditions*, J. Phys. A **49**, 044001 (2015), doi:10.1088/1751-8113/49/4/044001.

[50] N. Kitanine, J. M. Maillet, G. Niccoli and V. Terras, *On determinant representations of scalar products and form factors in the SoV approach: the XXX case*, J. Phys. A **49**, 104002 (2016), doi:10.1088/1751-8113/49/10/104002.

[51] D. Martin and F. Smirnov, *Problems with using separated variables for computing expectation values for higher ranks*, Lett. Math. Phys. **106**, 469 (2016), doi:10.1007/s11005-016-0823-0.

[52] N. Kitanine, J. M. Maillet, G. Niccoli and V. Terras, *The open XXX spin chain in the SoV framework: scalar product of separate states*, J. Phys. A **50**, 224001 (2017), doi:10.1088/1751-8121/aa6cc9.

[53] J. M. Maillet, G. Niccoli and B. Pezelier, *Transfer matrix spectrum for cyclic representations of the 6-vertex reflection algebra I*, SciPost Physics **2**, 009 (2017), doi:10.21468/scipostphys.2.1.009.

[54] J. M. Maillet, G. Niccoli and B. Pezelier, *Transfer matrix spectrum for cyclic representations of the 6-vertex reflection algebra II*, SciPost Phys. **5**, 026 (2018), doi:10.21468/SciPostPhys.5.3.026.

[55] Y. Jiang, S. Komatsu, I. Kostov and D. Serban, *The hexagon in the mirror: the three-point function in the SoV representation*, J. Phys. A **49**, 174007 (2016), doi:10.1088/1751-8113/49/17/174007.

[56] N. Kitanine, J. M. Maillet, G. Niccoli and V. Terras, *The open XXZ spin chain in the SoV framework: scalar product of separate states*, J. Phys. A **51**, 485201 (2018), doi:10.1088/1751-8121/aae76f.

[57] S. Belliard, S. Pakuliak, E. Ragoucy and N. A. Slavnov, *Highest coefficient of scalar products in $SU(3)$-invariant integrable models*, J. Stat. Mech. P09003 (2012), doi:10.1088/1742-5468/2012/09/p09003.

[58] S. Belliard, S. Pakuliak, E. Ragoucy and N. A. Slavnov, *The algebraic Bethe ansatz for scalar products in $SU(3)$-invariant integrable models*, J. Stat. Mech. P10017 (2012), doi:10.1088/1742-5468/2012/10/p10017.

[59] S. Belliard, S. Pakuliak, E. Ragoucy and N. A. Slavnov, *Bethe vectors of $GL(3)$-invariant integrable models*, J. Stat. Mech. P02020 (2013), doi:10.1088/1742-5468/2013/02/p02020.

[60] S. Belliard, S. Pakuliak, E. Ragoucy and N. A. Slavnov, *Form factors in $SU(3)$-invariant integrable models*, J. Stat. Mech. P04033 (2013), doi:10.1088/1742-5468/2013/04/p04033.

[61] S. Pakuliak, E. Ragoucy and N. Slavnov, *Form factors in quantum integrable models with $GL(3)$-invariant R-matrix*, Nucl. Phys. B **881**, 343 (2014), doi:https://doi.org/10.1016/j.nuclphysb.2014.02.014.

[62] S. Z. Pakuliak, E. Ragoucy and N. A. Slavnov, *Determinant representations for form factors in quantum integrable models with the $GL(3)$-invariant R-matrix*, Theor. Math. Phys. **181**, 1566 (2014), doi:10.1007/s11232-014-0236-0.

[63] S. Pakuliak, E. Ragoucy and N. A. Slavnov, *$GL(3)$-based quantum integrable composite models. I. Bethe vectors*, Symmetry Integr. Geom.: Methods Appl. 063 (2015), doi:10.3842/sigma.2015.063.

[64] S. Pakuliak, , E. Ragoucy, N. A. Slavnov and and, *$GL(3)$ -based quantum integrable composite models. II. form factors of local operators*, Symmetry Integr. Geom.: Methods Appl. 064 (2015), doi:10.3842/sigma.2015.064.

[65] S. Pakuliak, E. Ragoucy and N. A. Slavnov, *Form factors of local operators in a one-dimensional two-component Bose gas*, J. Phys. A **48**, 435001 (2015), doi:10.1088/1751-8113/48/43/435001.

[66] A. Hutsalyuk, A. Liashyk, S. Z. Pakuliak, E. Ragoucy and N. A. Slavnov, *Multiple actions of the monodromy matrix in gl(2|1)-invariant integrable models*, Symmetry Integr. Geom.: Methods Appl. 099 (2016), doi:10.3842/sigma.2016.099.

[67] A. A. Hutsalyuk, A. N. Liashyk, S. Z. Pakuliak, E. Ragoucy and N. A. Slavnov, *Current presentation for the super-Yangian double DY $\mathfrak{gl}(m|n)$ and Bethe vectors*, Russ. Math. Surv. **72**, 33 (2017), doi:10.1070/rm9754.

[68] S. Z. Pakuliak, E. Ragoucy and N. A. Slavnov, *Bethe vectors for models based on the super-Yangian $Y(\mathfrak{gl}(m|n))$*, J. Integr. Syst. **2**, xyx001 (2017), doi:10.1093/integr/xyx001.

[69] A. Hutsalyuk, A. Liashyk, S. Z. Pakuliak, E. Ragoucy and N. A. Slavnov, *Scalar products of Bethe vectors in models with $\mathfrak{gl}(2|1)$ symmetry 2. Determinant representation*, J. Phys. A: Math. Theor. **50**, 034004 (2016), doi:10.1088/1751-8121/50/3/034004.

[70] A. Liashyk and N. A. Slavnov, *On Bethe vectors in $gl(3)$-invariant integrable models*, J. High Energy Phys. **06**, 018 (2018), doi:10.1007/jhep06(2018)018.

[71] P. Kulish and N. Y. Reshetikhin, *Generalized Heisenberg ferromagnet and the Gross-Neveu model*, Sov. Phys. JETP **53**, 108 (1981).

[72] P. P. Kulish and N. Yu Reshetikhin, *Diagonalisation of GL(N) invariant transfer matrices and quantum N-wave system (Lee model)*, J. Phys. A: Math. Gen. **16**, L591 (1983), doi:10.1088/0305-4470/16/16/001.

[73] S. Belliard and E. Ragoucy, *The nested Bethe ansatz for 'all' closed spin chains*, J. Phys. A **41**, 295202 (2008), doi:10.1088/1751-8113/41/29/295202.

[74] S. Pakuliak, E. Ragoucy and N. Slavnov, *Nested Algebraic Bethe Ansatz in integrable models: recent results*, SciPost Phys. Lect. Notes 06 (2018), doi:10.21468/scipostphyslectnotes.6.

[75] A. Cavaglià, N. Gromov and F. Levkovich-Maslyuk, *Separation of variables and scalar products at any rank*, J. High Energy Phys. **09**, 052 (2019), doi:10.1007/jhep09(2019)052.

[76] N. Gromov, F. Levkovich-Maslyuk, P. Ryan and D. Volin, *Dual separated variables and scalar products*, Phys. Lett. B, **806**, 135494 (2020), doi:10.1016/j.physletb.2020.135494.

[77] N. Gromov, F. Levkovich-Maslyuk and G. Sizov, *New construction of eigenstates and separation of variables for $SU(N)$ quantum spin chains*, J. High Energy Phys. **09**, 111 (2017), doi:10.1007/jhep09(2017)111.

[78] N. Gromov and F. Levkovich-Maslyuk, *New compact construction of eigenstates for supersymmetric spin chains*, J. High Energy Phys. **09**, 085 (2018), doi:10.1007/JHEP09(2018)085.

[79] S. E. Derkachov and P. A. Valinevich, *Separation of variables for the quantum SL(3,C) spin magnet: eigenfunctions of Sklyanin B-operator* (2018), arXiv:1807.00302.

[80] J. M. Maillet and G. Niccoli, *Complete spectrum of quantum integrable lattice models associated to $Y(gl(n))$ by separation of variables*, SciPost Phys. **6**, 071 (2019), doi:10.21468/SciPostPhys.6.6.071.

[81] J. M. Maillet and G. Niccoli, *Complete spectrum of quantum integrable lattice models associated to $\mathcal{U}_q(\widehat{gl_n})$ by separation of variables*, J. Phys. A: Math. Theor. **52**, 315203 (2019), doi:10.1088/1751-8121/ab2930.

[82] J. M. Maillet and G. Niccoli, *On separation of variables for reflection algebras*, J. Stat. Mech. 094020 (2019), doi:10.1088/1742-5468/ab357a.

[83] J. M. Maillet and G. Niccoli, *On quantum separation of variables beyond fundamental representations* (2019), arXiv:1903.06618.

[84] J. Michel Maillet, G. Niccoli and L. Vignoli, *Separation of variables bases for integrable $gl_{\mathcal{M}|\mathcal{N}}$ and Hubbard models*, SciPost Phys. **9**, 060 (2020), doi:10.21468/SciPostPhys.9.4.060.

[85] P. Ryan and D. Volin, *Separated variables and wave functions for rational $gl(N)$ spin chains in the companion twist frame*, J. Math. Phys. **60**, 032701 (2019), doi:10.1063/1.5085387.

[86] P. Ryan and D. Volin, *Separation of variables for rational gl(n) spin chains in any compact representation, via fusion, embedding morphism and Backlund flow* (2020), arXiv:2002.12341.

[87] P. P. Kulish, N. Y. Reshetikhin and E. K. Sklyanin, *Yang-Baxter equation and representation theory: I*, Lett. Math. Phys. **5**, 393 (1981), doi:10.1007/bf02285311.

[88] P. P. Kulish and N. Y. Reshetikhin, *GL_3-invariant solutions of the Yang-Baxter equation and associated quantum systems*, J. Sov. Math. **34**, 1948 (1986), doi:10.1007/bf01095104.

[89] A. Kuniba, T. Nakanishi and J. Suzuki, *Functional relations in solvable lattice models I: Functional relations and representation theory*, Int. J. Mod. Phys. A **09**, 5215 (1994), doi:10.1142/s0217751x94002119.