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Invariant Gibbs dynamics for the dynamical sine-Gordon model

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In this note, we study the hyperbolic stochastic damped sine-Gordon equation (SdSG), with a parameter $\beta^2 > 0$, and its associated Gibbs dynamics on the two-dimensional torus. After introducing a suitable renormalization, we first construct the Gibbs measure in the range $0 < \beta^2 < 4\pi$ via the variational approach due to Barashkov-Gubinelli (2018). We then prove almost sure global well-posedness and invariance of the Gibbs measure under the hyperbolic SdSG dynamics in the range $0 < \beta^2 < 2\pi$. Our construction of the Gibbs measure also yields almost sure global well-posedness and invariance of the Gibbs measure for the parabolic sine-Gordon model in the range $0 < \beta^2 < 4\pi$.

Keywords: Stochastic sine-Gordon equation; dynamical sine-Gordon model; renormalization; white noise; Gibbs measure; Gaussian multiplicative chaos

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1. Introduction

1.1. Dynamical sine-Gordon model

We consider the following stochastic damped sine-Gordon equation (SdSG) on $\mathbb{T}^2 = (\mathbb{R}/2\pi\mathbb{Z})^2$ with an additive space-time white noise forcing:

$$\begin{cases} \partial_t^2 u + \partial_t u + (1-\Delta)u + \gamma \sin(\beta u) = \sqrt{2}\xi \\ (u, \partial_t u)|_{t=0} = (u_0, v_0), \end{cases} \qquad (t,x) \in \mathbb{R}_+ \times \mathbb{T}^2, \tag{1.1}$$

where γ and β are non-zero real numbers and ξ denotes a (Gaussian) space-time white noise on $\mathbb{R}_+ \times \mathbb{T}^2$. Our main goal in this paper is to construct invariant dynamics of SdSG (1.1) associated with the Gibbs measure, which formally reads

$$\text{`}d\vec{\rho}(u,v) = \mathcal{Z}^{-1}\, \mathrm{e}^{-E(u,v)} \mathrm{d}u\, \mathrm{d}v\text{'}. \tag{1.2}$$

Here, $\mathcal{Z} = \mathcal{Z}(\beta)$ denotes a normalization constant and

$$E(u,v) = \frac{1}{2}\int_{\mathbb{T}^2} \left(u(x)^2 + |\nabla u(x)|^2 + v(x)^2\right) \mathrm{d}x - \frac{\gamma}{\beta}\int_{\mathbb{T}^2} \cos\left(\beta u(x)\right) \mathrm{d}x \tag{1.3}$$

denotes the energy (= Hamiltonian) of the (deterministic undamped) sine-Gordon equation:

$$\partial_t^2 u + (1-\Delta)u + \gamma \sin(\beta u) = 0. \tag{1.4}$$

Our first goal is to provide a rigorous construction of the Gibbs measure $\vec{\rho}$ for $0 < \beta^2 < 4\pi$; see theorem 1.1.

The Gibbs measure $\vec{\rho}$ in (1.2) arises in various physical contexts such as two-dimensional Yukawa and Coulomb gases in statistical mechanics and the quantum sine-Gordon model in Euclidean quantum field theory. We refer the readers to [**2**, **6**, **10**, **21**, **24**–**27**, **36**] and the references therein for more physical motivations and interpretations of the measure $\vec{\rho}$. The dynamical model (1.1) then corresponds to the so-called 'canonical' stochastic quantization [**38**] of the quantum sine-Gordon model represented by the measure $\vec{\rho}$ in (1.2).

From the analytical point of view, the hyperbolic SdSG (1.1) is a good model for the study of singular stochastic nonlinear wave equations (SNLW). SNLW has been studied in various settings; see for example [**8**, Chapter 13] and the references therein. In particular, over the past several years, we have witnessed a fast development in the Cauchy theory of singular SNLW on $\mathbb{T}^d$. When $d = 2$, the well-posedness theory for SNLW with a polynomial nonlinearity:

$$\partial_t^2 u + (1-\Delta)u + u^k = \xi \tag{1.5}$$

is now well understood [**15**, **17**, **29**, **30**]. See also [**33**, **39**] for related results on two-dimensional compact Riemannian manifolds [**33**] and on $\mathbb{R}^2$ [**39**]. The situation is more delicate for $d = 3$. In a recent paper [**16**], Gubinelli *et al.* treated the quadratic case ($k = 2$) by adapting the paracontrolled calculus, originally introduced in the parabolic setting [**14**], to the dispersive setting. For the sine-Gordon model, the

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main new difficulty in (1.1) comes from the non-polynomial nature of the nonlinearity, which makes the analysis of the relevant stochastic object particularly non-trivial.

In the aforementioned works, the main source of difficulty comes from the roughness of the space-time white noise ξ. For $d \geqslant 2$, the stochastic convolution Ψ, solving the following linear stochastic wave equation:[1]

$$\partial_t^2 \Psi + (1 - \Delta)\Psi = \xi, \tag{1.6}$$

belongs almost surely to $C(\mathbb{R}_+; W^{-\varepsilon,\infty}(\mathbb{T}^2))$ for any $\varepsilon > 0$ but not for $\varepsilon = 0$. See lemma 2.1. This lack of regularity shows that there is an issue in forming a nonlinearity of the form Ψ^k and $\sin(\beta\Psi)$, thus requiring a proper renormalization. In our previous work [**32**], we studied the undamped case:

$$\partial_t^2 u + (1 - \Delta)u + \gamma \sin(\beta u) = \xi. \tag{1.7}$$

By introducing a *time-dependent* renormalization, we proved local well-posedness of the undamped model (1.7) for *any* value of $\beta^2 > 0$ for small times (depending on β). The main ingredient in [**32**] was to exploit smallness of Ψ in (1.6) for small times (thanks to the time-dependent nature of the renormalization).

The situation for the damped model (1.1) is, however, different from the undamped case. In studying the problem associated with the (formally) invariant measure $\vec{\rho}$ in (1.2), we work with a *time-independent* renormalization and thus the situation is closer to the parabolic model:[2]

$$\partial_t u + \frac{1}{2}(1 - \Delta)u + \gamma \sin(\beta u) = \xi, \tag{1.8}$$

studied in [**6**, **21**], where the value of $\beta^2 > 0$ played an important role in the solution theory. For $0 < \beta^2 < 4\pi$, the Da Prato-Debussche trick [**7**] along with a standard Wick renormalization yields local well-posedness of (1.8); see remark 1.5. It turns out that there is an infinite number of thresholds: $\beta^2 = (j/(j+1))8\pi$, $j \in \mathbb{N}$, where one encounters new divergent stochastic objects, requiring further renormalizations. By using the theory of regularity structures [**19**], Hairer and Shen [**21**] and Chandra *et al.* [**6**] proved local well-posedness of the parabolic model (1.8) to the entire subcritical regime $0 < \beta^2 < 8\pi$. When $\beta^2 = 8\pi$, the equation (1.8) is critical and falls outside the scope of the current theory.

Due to a weaker smoothing property of the relevant linear propagator, our hyperbolic model (1.1) is expected to be much more involved than the parabolic case. Indeed, as we see below, the standard Da Prato-Debussche trick yields the solution theory for (1.1) only for $0 < \beta^2 < 2\pi$ (which is much smaller than the parabolic case: $0 < \beta^2 < 4\pi$). See theorem 1.2 below. In the next subsection, we provide precise statements of our main results. Before proceeding further, we mention the recent works [**11**, **20**, **34**] on the well-posedness theory for the stochastic heat and wave equations with an exponential nonlinearity in the two-dimensional setting.

[1] The equation (1.6) is also referred to as the linear stochastic Klein-Gordon equation. In the following, however, we simply refer to this as to the wave equation.

[2] We point out that while the spatially homogeneous case with $\partial_t - \frac{1}{2}\Delta$ was studied in [**6**, **21**], the model (1.8) is more relevant for our discussion. See remark 1.5.

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1.2. Main results

Our main goal in this paper is twofold; (i) provide a rigorous construction of (a renormalized version of) the Gibbs measure $\vec{\rho}$ in (1.2) and (ii) construct well-defined dynamics for the hyperbolic SdSG (1.1) associated with the Gibbs initial data. For this purpose, we first fix some notations. Given $s \in \mathbb{R}$, let μ_s denote a Gaussian measure, formally defined by

$$\mathrm{d}\mu_s = Z_s^{-1} \mathrm{e}^{-(1/2)\|u\|_{H^s}^2} \mathrm{d}u = Z_s^{-1} \prod_{n \in \mathbb{Z}^2} \mathrm{e}^{-(1/2)\langle n \rangle^{2s} |\widehat{u}_n|^2} \mathrm{d}\widehat{u}_n, \tag{1.9}$$

where $\langle \cdot \rangle = (1 + |\cdot|^2)^{1/2}$ and $\widehat{u}_n$ denotes the Fourier coefficient of u at the frequency $n \in \mathbb{Z}^2$. We set

$$\vec{\mu}_s = \mu_s \otimes \mu_{s-1}. \tag{1.10}$$

In particular, when $s = 1$, the measure $\vec{\mu}_1$ is defined as the induced probability measure under the map:

$$\omega \in \Omega \longmapsto (u^\omega, v^\omega),$$

where u^ω and v^ω are given by

$$u^\omega = \sum_{n \in \mathbb{Z}^2} \frac{g_n(\omega)}{\langle n \rangle} e_n \quad \text{and} \quad v^\omega = \sum_{n \in \mathbb{Z}^2} h_n(\omega) e_n. \tag{1.11}$$

Here, $e_n = (2\pi)^{-1} \mathrm{e}^{in \cdot x}$ and $\{g_n, h_n\}_{n \in \mathbb{Z}^2}$ denotes a family of independent standard complex-valued Gaussian random variables conditioned so that $\overline{g_n} = g_{-n}$ and $\overline{h_n} = h_{-n}$, $n \in \mathbb{Z}^2$. It is easy to see that $\vec{\mu}_1 = \mu_1 \otimes \mu_0$ is supported on

$$\mathcal{H}^s(\mathbb{T}^2) \stackrel{\text{def}}{=} H^s(\mathbb{T}^2) \times H^{s-1}(\mathbb{T}^2)$$

for $s < 0$ but not for $s \geqslant 0$.

With (1.3), (1.9) and (1.10), we can formally write $\vec{\rho}$ in (1.2) as

$$\mathrm{d}\vec{\rho}(u, v) \sim \mathrm{e}^{(\gamma/\beta) \int_{\mathbb{T}^2} \cos(\beta u) \mathrm{d}x} \mathrm{d}\vec{\mu}_1(u, v). \tag{1.12}$$

In view of the roughness of the support of $\vec{\mu}_1$, the nonlinear term in (1.12) is not well-defined and thus a proper renormalization is required to give a meaning to (1.12).

Let $\mathbf{P}_N$ be a smooth frequency projector onto the frequencies $\{n \in \mathbb{Z}^2 : |n| \leqslant N\}$ defined as a Fourier multiplier operator with a symbol:

$$\chi_N(n) = \chi(N^{-1} n) \tag{1.13}$$

for some fixed non-negative function $\chi \in C_0^\infty(\mathbb{R}^2)$ such that $\operatorname{supp} \chi \subset \{\xi \in \mathbb{R}^2 : |\xi| \leqslant 1\}$ and $\chi \equiv 1$ on $\{\xi \in \mathbb{R}^2 : |\xi| \leqslant \frac{1}{2}\}$. Given $u = u^\omega$ as in (1.11), i.e.[3] $\mathcal{L}(u) = \mu_1$,

[3]Given a random variable X, $\mathcal{L}(X)$ denotes the law of X.

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set σ_N, $N \in \mathbb{N}$, by setting

$$\sigma_N = \mathbb{E}\Big[\big(\mathbf{P}_N u(x)\big)^2\Big] = \frac{1}{4\pi^2} \sum_{n\in\mathbb{Z}^2} \frac{\chi_N(n)^2}{\langle n\rangle^2} = \frac{1}{2\pi} \log N + o(1), \tag{1.14}$$

as $N \to \infty$, independent of $x \in \mathbb{T}^2$. Given $N \in \mathbb{N}$, define the truncated renormalized density:

$$R_N(u) = \frac{\gamma_N}{\beta} \int_{\mathbb{T}^2} \cos\left(\beta \mathbf{P}_N u(x)\right) \mathrm{d}x, \tag{1.15}$$

where $\gamma_N = \gamma_N(\beta)$ is defined by

$$\gamma_N(\beta) = \mathrm{e}^{(\beta^2/2)\sigma_N}. \tag{1.16}$$

In particular, we have $\gamma_N \to \infty$ as $N \to \infty$. We then define the truncated renormalized Gibbs measure:

$$\mathrm{d}\vec{\rho}_N(u,v) = \mathcal{Z}_N^{-1}\, \mathrm{e}^{R_N(u)} \mathrm{d}\vec{\mu}_1(u,v) \tag{1.17}$$

for some normalization constant $\mathcal{Z}_N = \mathcal{Z}_N(\beta) \in (0,\infty)$. We now state our first result.

THEOREM 1.1. *Let* $0 < \beta^2 < 4\pi$.

(i) *The truncated renormalized density* $\{R_N\}_{N\in\mathbb{N}}$ *in* (1.15) *is a Cauchy sequence in* $L^p(\mu_1)$ *for any finite* $p \geqslant 1$, *thus converging to some limiting random variable* $R \in L^p(\mu_1)$.

(ii) *Given any finite* $p \geqslant 1$, *there exists* $C_p > 0$ *such that*

$$\sup_{N\in\mathbb{N}} \left\| e^{R_N(u)} \right\|_{L^p(\mu_1)} \leqslant C_p < \infty. \tag{1.18}$$

Moreover, we have

$$\lim_{N\to\infty} e^{R_N(u)} = e^{R(u)} \quad \text{in } L^p(\mu_1). \tag{1.19}$$

As a consequence, the truncated renormalized Gibbs measure $\vec{\rho}_N$ *in* (1.17) *converges, in the sense of* (1.19), *to the renormalized Gibbs measure* $\vec{\rho}$ *given by*

$$d\vec{\rho}(u,v) = \mathcal{Z}^{-1} e^{R(u)} d\vec{\mu}_1(u,v). \tag{1.20}$$

Furthermore, the resulting Gibbs measure $\vec{\rho}$ *is equivalent to the Gaussian measure* $\vec{\mu}_1$.

The proof of theorem 1.1 also allows us to define the renormalized Gibbs measure:

$$\mathrm{d}\rho(u) = \mathcal{Z}^{-1} \mathrm{e}^{R(u)} \mathrm{d}\mu_1(u) \tag{1.21}$$

as a limit of the truncated measure

$$\mathrm{d}\rho_N(u) = \mathcal{Z}_N^{-1} \mathrm{e}^{R_N(u)} \mathrm{d}\mu_1(u)$$

for $0 < \beta^2 < 4\pi$. The Gibbs measure ρ in (1.21) is relevant to the parabolic model (1.8). See remark 1.5.

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In a recent work [**25**], Lacoin *et al.* constructed a measure associated with the sine-Gordon model in the one-dimensional setting, where the based Gaussian measure is log-correlated (as in the massive Gaussian free field on $\mathbb{T}^2$). Their construction applies to the full subcritical range[4]: $0 < \beta^2 < 8\pi$. At this moment, their argument is restricted to the one-dimensional case and does not extend to the two-dimensional case under consideration.

Theorem 1.1 (i) follows from the construction of the imaginary Gaussian multiplicative chaos; see lemma 2.2 below. The main difficulty in proving theorem 1.1 (ii) appears in showing the uniform bound (1.18). We establish the bound (1.18) by applying the variational approach introduced by Barashkov and Gubinelli in [**1**] in the construction of the Φ^4_3-measure. See also [**18**].

Next, we move onto the well-posedness theory of the hyperbolic SdSG (1.1). Let us first introduce the following renormalized truncated SdSG:

$$\partial_t^2 u_N + \partial_t u_N + (1-\Delta)u_N + \gamma_N \mathbf{P}_N\big\{\sin(\beta \mathbf{P}_N u_N)\big\} = \sqrt{2}\xi, \tag{1.22}$$

where γ_N is as in (1.16). We now state our second result.

THEOREM 1.2. *Let* $0 < \beta^2 < 2\pi$. *Then, the stochastic damped sine-Gordon equation* (1.1) *is almost surely globally well-posed with respect to the renormalized Gibbs measure* $\vec{\rho}$ *in* (1.20)*. Furthermore, the renormalized Gibbs measure* $\vec{\rho}$ *is invariant under the dynamics.*

More precisely, there exists a non-trivial stochastic process $(u, \partial_t u) \in C(\mathbb{R}_+; \mathcal{H}^{-\varepsilon}(\mathbb{T}^2))$ *for any* $\varepsilon > 0$ *such that, for any* $T > 0$*, the solution* $(u_N, \partial_t u_N)$ *to the truncated SdSG* (1.22) *with the random initial data* $(u_N, \partial_t u_N)|_{t=0}$ *distributed according to the truncated Gibbs measure* $\vec{\rho}_N$ *in* (1.17)*, converges in probability to* $(u, \partial_t u)$ *in* $C([0,T]; \mathcal{H}^{-\varepsilon}(\mathbb{T}^2))$*. Moreover, the law of* $(u(t), \partial_t u(t))$ *is given by the renormalized Gibbs measure* $\vec{\rho}$ *in* (1.20) *for any* $t \geqslant 0$.

In view of theorem 1.1 and Bourgain's invariant measure argument [**4**, **5**], theorem 1.2 follows once we construct the limiting process $(u, \partial_t u)$ locally in time. Furthermore, in view of the equivalence of $\vec{\rho}$ and $\vec{\mu}_1$, it suffices to study the dynamics with the Gaussian random initial data (u_0, v_0) with $\mathcal{L}(u_0, v_0) = \vec{\mu}_1$. As in [**32**], we proceed with the Da Prato-Debussche trick. For our damped model, we let Ψ be the solution to the linear stochastic damped wave equation:

$$\begin{cases} \partial_t^2 \Psi + \partial_t \Psi + (1-\Delta)\Psi = \sqrt{2}\xi \\ (\Psi, \partial_t \Psi)|_{t=0} = (u_0, v_0), \end{cases} \tag{1.23}$$

where $\mathcal{L}(u_0, v_0) = \vec{\mu}_1$. Define the linear damped wave propagator $\mathcal{D}(t)$ by

$$\mathcal{D}(t) = e^{-(t/2)} \frac{\sin\left(t\sqrt{(3/4)-\Delta}\right)}{\sqrt{(3/4)-\Delta}} \tag{1.24}$$

[4]Due to a different scaling, the threshold $0 < \beta^2 < 2d$ in [**25**] corresponds to $0 < \beta^2 < 8\pi$ in our convention. See remark 1.14 in [**34**].

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as a Fourier multiplier operator. Then, we have

$$\Psi(t) = \partial_t \mathcal{D}(t)u_0 + \mathcal{D}(t)(u_0 + v_0) + \sqrt{2}\int_0^t \mathcal{D}(t-t')dW(t'), \tag{1.25}$$

where W denotes a cylindrical Wiener process on $L^2(\mathbb{T}^2)$:

$$W(t) = \sum_{n\in\mathbb{Z}^2} B_n(t)e_n, \tag{1.26}$$

and $\{B_n\}_{n\in\mathbb{Z}^2}$ is defined by $B_n(0) = 0$ and $B_n(t) = \langle \xi, \mathbf{1}_{[0,t]} \cdot e_n\rangle_{t,x}$. Here, $\langle\cdot,\cdot\rangle_{t,x}$ denotes the duality pairing on $\mathbb{R}\times\mathbb{T}^2$. As a result, we see that $\{B_n\}_{n\in\mathbb{Z}^2}$ is a family of mutually independent complex-valued[5] Brownian motions conditioned so that $B_{-n} = \overline{B_n}$, $n \in \mathbb{Z}^2$. By convention, we normalized B_n such that $\mathrm{Var}(B_n(t)) = t$.

A direct computation shows that $\Psi_N(t,x) = \mathbf{P}_N\Psi(t,x)$ is a mean-zero real-valued Gaussian random variable with variance

$$\mathbb{E}\big[\Psi_N(t,x)^2\big] = \mathbb{E}\big[\big(\mathbf{P}_N\Psi(t,x)\big)^2\big] = \sigma_N$$

for any $t \geqslant 0$, $x\in\mathbb{T}^2$ and $N \geqslant 1$, where σ_N is as in (1.14).

Let u_N be as in theorem 1.2, satisfying (1.22) with $\mathcal{L}((u_N, \partial_t u_N)|_{t=0}) = \vec{\mu}_1$. Then, write u_N as $u_N = w_N + \Psi$. Then, the residual part w_N satisfies the following equation:

$$\begin{cases} \partial_t^2 w_N + \partial_t w_N + (1-\Delta)w_N + \mathrm{Im}\,\mathbf{P}_N\big\{e^{i\beta\mathbf{P}_N w_N}\Theta_N\big\} = 0,\\ (w_N, \partial_t w_N)|_{t=0} = (0,0). \end{cases} \tag{1.27}$$

Here, Θ_N denotes the so-called imaginary Gaussian multiplicative chaos defined by

$$\Theta_N(t,x) = :e^{i\beta\Psi_N(t,x)}: \overset{\text{def}}{=} \gamma_N e^{i\beta\Psi_N(t,x)} = \mathrm{e}^{(\beta^2/2)\sigma_N}\mathrm{e}^{\mathrm{i}\beta\Psi_N(t,x)}, \tag{1.28}$$

where γ_N is as in (1.16). By proceeding as in [**21**, **32**], we establish the regularity property of Θ_N; see lemma 2.2. In particular, given $0 < \beta^2 < 4\pi$, $\{\Theta_N\}_{N\in\mathbb{N}}$ forms a Cauchy sequence in $L^p(\Omega; L^q([0,T]; W^{-\alpha,\infty}(\mathbb{T}^2)))$ for any finite $p, q \geqslant 1$ and $\alpha > \frac{\beta^2}{4\pi}$. Then, local well-posedness of (1.27), uniformly in N, follows from a standard contraction argument, using the Strichartz estimates, certain product estimates and the fractional chain rule. See § 3. The restriction $\beta^2 < 2\pi$ appears due to a weaker smoothing property in the current wave setting. See remark 1.6.

Remark 1.3. Invariant Gibbs measures for nonlinear wave equations have been studied extensively, starting with the work [**9**]. See the survey papers [**3**, **28**] for the references therein. In the context of the (deterministic) sine-Gordon equation (1.4), McKean [**27**] studied the one-dimensional case and constructed an invariant Gibbs measure for (1.4) on $\mathbb{T}$. A small adaptation of our argument for proving theorem 1.2 allows us to prove almost sure global well-posedness and invariance

[5]In particular, B_0 is a standard real-valued Brownian motion.

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of the (renormalized) Gibbs measure $\vec{\rho}$ for the (deterministic, renormalized) sine-Gordon equation (1.4) on $\mathbb{T}^2$ for $0 < \beta^2 < 2\pi$.

Remark 1.4. In this paper, we use a smooth frequency projector $\mathbf{P}_N$ with the multiplier χ_N in (1.13). As in the parabolic case, it is possible to show that the limiting Gibbs measure $\vec{\rho}$ in theorem 1.1 and the limit $(u, \partial_t u)$ of $(u_N, \partial_t u_N)$ in theorem 1.2 are independent of the choice of the smooth cut-off function χ. See [**31**] for such an argument in the wave case (with a polynomial nonlinearity). Moreover, we may also proceed by smoothing via a mollification and obtain analogous results.

Remark 1.5. As mentioned above, the Da Prato-Debussche approach suffices to prove local well-posedness for the parabolic sine-Gordon model (1.8) in the range $0 < \beta^2 < 4\pi$; see the discussion before theorem 2.1 in [**21**]. Indeed, with the Da Prato-Debussche decomposition $u_N = w_N + \Psi$, where Ψ is the stochastic convolution for the heat case, we see that the residual part w_N satisfies

$$\partial_t w_N + \frac{1}{2}(1-\Delta)w_N + \operatorname{Im} \mathbf{P}_N\{\mathrm{e}^{\mathrm{i}\beta \mathbf{P}_N w_N}\Theta_N\} = 0. \tag{1.29}$$

Here, the imaginary Gaussian multiplicative chaos Θ_N in the heat case has exactly the same regularity as in the wave case stated in lemma 2.2. Namely, it has the spatial regularity $-\alpha < -(\beta^2/4\pi)$. Then, in view of the two degrees of smoothing under the heat propagator (the Schauder estimate), local well-posedness of (1.29) for w_N in the class $C([0,T]; W^{2-\alpha,\infty}(\mathbb{T}^2))$ follows easily from the product estimate (lemma 3.2 (iv)), provided that $\alpha < 2 - \alpha$, namely $\beta^2 < 4\pi$.

Therefore, combining this local well-posedness, the construction of the Gibbs measure (theorem 1.1), and Bourgain's invariant measure argument, we conclude almost sure global well-posedness and invariance of the renormalized Gibbs measure ρ in (1.21) for the parabolic sine-Gordon model (1.8).

Remark 1.6. In our wave case, the linear propagator $\mathcal{D}(t)$ provides only one degree of smoothing, thus requiring $\alpha - 1 < -\alpha$ in proving local well-posedness. This gives the restriction of $\beta^2 < 2\pi$ in theorem 1.2. In view of theorem 1.1, it is therefore of very much interest to study further local well-posedness of the hyperbolic SdSG (1.1) for $2\pi \leqslant \beta^2 < 4\pi$.

As pointed out in [**21**, **34**], the difficulty of the sine-Gordon model on $\mathbb{T}^2$ is heuristically comparable to the one for the dynamical Φ^3_3-model when $\beta^2 = 2\pi$. Namely, when $\beta^2 = 2\pi$, the hyperbolic SdSG (1.1) corresponds to the quadratic SNLW (1.5) on $\mathbb{T}^3$ (with $k = 2$). In [**16**], Gubinelli *et al.* proved local well-posedness of the quadratic SNLW on $\mathbb{T}^3$ by combining the paracontrolled approach with multilinear harmonic analysis. Furthermore, in order to replace a commutator argument (which does not provide any smoothing in the dispersive/hyperbolic setting), they also introduce paracontrolled operators. Hence, in order to treat the hyperbolic SdSG (1.1) for $\beta^2 = 2\pi$, we plan to adapt the paracontrolled approach as in [**16**].

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2. Imaginary Gaussian multiplicative chaos and construction of the Gibbs measure

In this section, we briefly go over the regularity and convergence properties of the imaginary Gaussian multiplicative chaos $\Theta_N = :e^{i\beta\Psi_N}:$ defined in (1.28). We then proceed to the construction of the Gibbs measure ρ as stated in theorem 1.1.

2.1. Imaginary Gaussian multiplicative chaos

In the following, we review the regularity and convergence properties of the truncated stochastic convolution $\Psi_N = \mathbf{P}_N\Psi$, where Ψ is as in (1.25), and Θ_N in (1.28). We use the following notation as in [**32**]; given two functions f and g on $\mathbb{T}^2$, we write

$$f \approx g$$

if there exist some constants $c_1, c_2 \in \mathbb{R}$ such that $f(x) + c_1 \leqslant g(x) \leqslant f(x) + c_2$ for any $x \in \mathbb{T}^2 \backslash \{0\} \cong [-\pi,\pi)^2 \setminus \{0\}$. We first state the regularity and convergence properties of Ψ_N. See [**15**, **16**, **33**].

LEMMA 2.1. *Given any $T, \varepsilon > 0$ and finite $p \geqslant 1$, $\{(\Psi_N, \partial_t\Psi_N)\}_{N\in\mathbb{N}}$ is a Cauchy sequence in $L^p(\Omega; C([0,T]; \mathcal{H}^{-\varepsilon}(\mathbb{T}^2)))$, thus converging to some limiting process $(\Psi, \partial_t\Psi) \in L^p(\Omega; C([0,T]; \mathcal{H}^{-\varepsilon}(\mathbb{T}^2)))$. Moreover, $(\Psi_N, \partial_t\Psi_N)$ converges almost surely to $(\Psi, \partial_t\Psi)$ in $C([0,T]; \mathcal{H}^{-\varepsilon}(\mathbb{T}^2))$.*

Let $G = (1-\Delta)^{-1}\delta_0$ denote the Green function for $1-\Delta$. Then, recall from [**32**, lemma 2.3] that for all $N \in \mathbb{N}$ and $x \in \mathbb{T}^2 \setminus \{0\}$, we have

$$\mathbf{P}_N^2 G(x) \approx -\frac{1}{2\pi}\log\big(|x| + N^{-1}\big). \tag{2.1}$$

Using (2.1), we can proceed as in the proof of lemma 2.7 in [**32**] and show that for any $t \geqslant 0$, the covariance function:

$$\Gamma_N(t, x-y) \stackrel{\text{def}}{=} \mathbb{E}\big[\Psi_N(t,x)\Psi_N(t,y)\big]$$

satisfies

$$\Gamma_N(t, x-y) \approx -\frac{1}{2\pi}\log\big(|x-y| + N^{-1}\big). \tag{2.2}$$

For our problem, the stochastic convolution Ψ defined in (1.25) is a stationary process and thus Γ_N is independent of t. Compare this with the time-dependent case in [**32**]; see (2.23) in [**32**].

Next, we state the regularity and convergence properties of Θ_N.

LEMMA 2.2. *Let $0 < \beta^2 < 4\pi$. Then, for any finite $p, q \geqslant 1$, $T > 0$, and $\alpha > (\beta^2/4\pi)$, $\{\Theta_N\}_{N\in\mathbb{N}}$ is a Cauchy sequence in $L^p(\Omega; L^q([0,T]; W^{-\alpha,\infty}(\mathbb{T}^2)))$ and hence converges to a limiting process Θ in $L^p(\Omega; L^q([0,T]; W^{-\alpha,\infty}(\mathbb{T}^2)))$.*

Due to the stationarity of Ψ, we have $\gamma_N = e^{(\beta^2/2)\sigma_N}$ in (1.16) independent of time. This is the reason why, contrary to [**32**, proposition 1.1], the regularity of Θ

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in lemma 2.2 is independent of time. Compare this with proposition 1.1 in [**32**] for the undamped wave case, where the regularity of the relevant imaginary Gaussian multiplicative chaos decreases over time.

Proof. Lemma 2.2 follows from a straightforward modification of the proof of proposition 1.1 in [**32**] on the construction of the imaginary Gaussian multiplicative chaos in the undamped wave case. Namely, using Minkowski's integral inequality, it suffices to establish convergence of $\Theta_N(t,x)$ for any fixed $t \geqslant 0$ and $x \in \mathbb{T}^2$, which follows from the argument in [**32**, proposition 1.1] by replacing [**32**, Lemma 2.7] with (2.2). In particular, $\beta^2 t/4\pi$ in [**32**] is replaced by $\beta^2/2\pi$. In establishing this lemma (and proposition 1.1 in [**32**]), we need to exploit a key cancellation property of charges; see lemma 2.5 in [**32**]. □

2.2. Construction of the Gibbs measure

In this subsection, we present a proof of rheorem 1.1. The main task here is to establish the uniform integrability (1.18) of the densities $\mathrm{e}^{R_N(u)}$ of the weighted Gaussian measures $\vec{\rho}_N$ in (1.17). For this purpose, we use the variational approach due to Barashkov and Gubinelli [**1**] and express the partition function $\mathcal{Z}_N$ in (1.17) in terms of a minimization problem involving a stochastic control problem (lemma 2.4). We then study the minimization problem and establish uniform boundedness of the partition function $\mathcal{Z}_N$. Our argument follows that in § 4 of [**18**].

From (1.17) and integrating over $\mu_0(v)$, we can express the partition function $\mathcal{Z}_N$ as

$$\mathcal{Z}_N = \int \mathrm{e}^{R_N(u)} \mathrm{d}\mu_1(u). \tag{2.3}$$

We first show the following convergence property of R_N.

Lemma 2.3. *Given any finite $p \geqslant 1$, R_N defined in* (1.15) *converges to some limit R in $L^p(\mu_1)$ as $N \to \infty$.*

Proof. Let $\mathcal{L}(u) = \mu_1$. Then, from (1.15), (1.16) and (1.28), we have

$$R_N(u) = \frac{1}{\beta}\int_{\mathbb{T}^2} \mathrm{Re}\left(:\mathrm{e}^{i\beta \mathbf{P}_N u}: \right)\mathrm{d}x = \frac{2\pi}{\beta}\,\mathrm{Re}\,\widehat{\Theta}_N(t,n)\big|_{(t,n)=(0,0)},$$

where $\widehat{\Theta}_N(t,n)$ denotes the spatial Fourier transform at time t and the frequency n. Then, lemma 2.3 is a direct consequence of the proof of lemma 2.2 (see the proof of proposition 1.1 in [**32**]) since the convergence of $\Theta_N(t,x)$ in $L^p(\mu_1)$ is established for any fixed $t \geqslant 0$ and $x \in \mathbb{T}^2$. □

Next, we prove the uniform integrability (1.18). Once we prove (1.18), the desired convergence (1.19) of the density follows from a standard argument, using lemma 2.3 with (1.18). See [**40**, remark 3.8]. See also the proof of proposition 1.2 in [**35**].

In order to prove (1.18), we follow the argument in [**1**, **18**] and derive a variational formula for the partition function $\mathcal{Z}_N$ in (2.3). Let us first introduce some notations. See also § 4 in [**18**]. Let $W(t)$ be the cylindrical Wiener process in (1.26). We define

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a centred Gaussian process $Y(t)$ by

$$Y(t) = \langle \nabla \rangle^{-1} W(t), \tag{2.4}$$

where $\langle \nabla \rangle = \sqrt{1-\Delta}$. Then, we have $\mathcal{L}(Y(1)) = \mu_1$. By setting $Y_N = \mathbf{P}_N Y$, we have $\mathcal{L}(Y_N(1)) = (\mathbf{P}_N)_{\#}\mu_1$. In particular, we have $\mathbb{E}[Y_N(1)^2] = \sigma_N$, where σ_N is as in (1.14).

Next, let $\mathbb{H}_a$ denote the space of drifts, which are the progressively measurable processes that belong to $L^2([0,1]; L^2(\mathbb{T}^2))$, $\mathbb{P}$-almost surely. Given a drift $\eta \in \mathbb{H}_a$, we define the measure $\mathbb{Q}^\eta$ whose Radon-Nikodym derivative with respect to $\mathbb{P}$ is given by the following stochastic exponential:

$$\frac{d\mathbb{Q}^\eta}{d\mathbb{P}} = \mathrm{e}^{\int_0^1 \langle \eta(t), \mathrm{d}W(t)\rangle - (1/2)\int_0^1 \|\eta(t)\|_{L^2_x}^2 \mathrm{d}t},$$

where $\langle \cdot, \cdot \rangle$ stands for the usual inner product on $L^2(\mathbb{T}^2)$. Then, by letting $\mathbb{H}_c$ denote the subspace of $\mathbb{H}_a$ consisting of drifts such that $\mathbb{Q}^\eta(\Omega) = 1$, it follows from Girsanov's theorem ([**8**, theorem 10.14] and [**37**, theorems 1.4 and 1.7 in Chapter VIII]) that W is a semi-martingale under $\mathbb{Q}^\eta$ with the following decomposition:

$$W(t) = W^\eta(t) + \int_0^t \eta(t')\mathrm{d}t', \tag{2.5}$$

where W^η is now a $L^2(\mathbb{T}^2)$-cylindrical Wiener process under the new measure $\mathbb{Q}^\eta$. Substituting (2.5) in (2.4) leads to the decomposition:

$$Y = Y^\eta + \mathcal{I}(\eta),$$

where

$$Y^\eta(t) = \langle \nabla \rangle^{-1} W^\eta(t) \quad \text{and} \quad \mathcal{I}(\eta)(t) = \int_0^t \langle \nabla \rangle^{-1} \eta(t')\mathrm{d}t'.$$

In the following, we use $\mathbb{E}$ to denote an expectation with respect to $\mathbb{P}$, while we use $\mathbb{E}_\mathbb{Q}$ for an expectation with respect to some other probability measure $\mathbb{Q}$.

Proceeding as in [**1**, lemma 1] and [**18**, proposition 4.4], we then have the following variational formula for the partition function $\mathcal{Z}_N$ in (2.3).

LEMMA 2.4. *For any $N \in \mathbb{N}$, we have*

$$-\log \mathcal{Z}_N = \inf_{\eta \in \mathbb{H}_c} \mathbb{E}_{\mathbb{Q}^\eta}\bigg[- R_N(Y^\eta(1) + \mathcal{I}(\eta)(1)) + \frac{1}{2}\int_0^1 \|\eta(t)\|_{L^2_x}^2 \mathrm{d}t \bigg]. \tag{2.6}$$

Lemma 2.4 follows from a straightforward modification of the proof of proposition 4.4 in [**18**] and thus we omit details. In the following, we use lemma 2.4 and show that the infimum in (2.6) is bounded away from $-\infty$, uniformly in $N \in \mathbb{N}$, which establishes the uniform bound (1.18). To this end, we first state the following lemma to estimate $Y^\eta(1)$ and $\mathcal{I}(\eta)(1)$.

LEMMA 2.5. *Let $Y^\eta(1)$ and $\mathcal{I}(\eta)(1)$ be as above.*

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(i) *Let* $0 < \beta^2 < 4\pi$ *and* $\alpha > \frac{\beta^2}{4\pi}$. *Then, for any finite* $p \geqslant 1$, *we have*

$$\sup_{\eta\in\mathbb{H}_c} \mathbb{E}_{\mathbb{Q}^\eta}\| :\mathrm{e}^{\mathrm{i}\beta Y^\eta(1)}: \|^p_{W_x^{-\alpha,\infty}} < \infty. \tag{2.7}$$

(ii) *For any* $\eta \in \mathbb{H}_c$, *we have*

$$\|\mathcal{I}(\eta)(1)\|^2_{H^1_x} \leqslant \int_0^1 \|\eta(t)\|^2_{L^2_x}\mathrm{d}t. \tag{2.8}$$

Proof.

(i) For any $\eta \in \mathbb{H}_c$, W^η is a cylindrical Wiener process in $L^2(\mathbb{T}^2)$ under $\mathbb{Q}^\eta$. Thus, the law of $Y^\eta(1) = \langle\nabla\rangle^{-1}W^\eta(1)$ under $\mathbb{Q}^\eta$ is always given by μ_1, so in particular, it is independent of $\eta \in \mathbb{H}_c$. Then, (2.7) follows from (the proof of) lemma 2.2.

(ii) The proof of (2.8) is straightforward from Minkowski's and Cauchy-Schwarz's inequalities. See the proof of lemma 4.7 in [**18**].

□

We are now ready to establish the uniform integrability estimate (1.18). For simplicity, we only prove (1.18) for $p = 1$. In view of lemma 2.4, we need to bound from below

$$\mathcal{W}_N(\eta) = \mathbb{E}_{\mathbb{Q}^\eta}\bigg[- R_N\big(Y^\eta(1) + \mathcal{I}(\eta)(1)\big) + \frac{1}{2}\int_0^1 \|\eta(t)\|^2_{L^2_x}\mathrm{d}t\bigg], \tag{2.9}$$

uniformly in the drift $\eta \in \mathbb{H}_c$ and in $N \in \mathbb{N}$. To simplify notations, we fix $\eta \in \mathbb{H}_c$ and $N \in \mathbb{N}$ and drop the dependence in η and N in (2.9). Moreover, we set $Y = Y^\eta(1)$ and $H = \mathcal{I}(\eta)(1)$. From the definition of R_N in (1.15), we have

$$\begin{aligned} R_N(Y+H) &= \frac{1}{\beta}\int_{\mathbb{T}^2} :\cos(\beta(Y+H)):\, \mathrm{d}x \\ &= \frac{1}{\beta}\int_{\mathbb{T}^2}\Big(:\cos(\beta Y): \cos(\beta H) - :\sin(\beta Y): \sin(\beta H)\Big)\mathrm{d}x. \end{aligned}$$

By duality between $H^\alpha(\mathbb{T}^2)$ and $H^{-\alpha}(\mathbb{T}^2)$ and Cauchy's inequality, we have

$$\begin{aligned} |R_N(Y+H)| &\lesssim \| :\cos(\beta Y): \|_{H^{-\alpha}}\| \cos(\beta H)\big\|_{H^\alpha} + \| :\sin(\beta Y): \|_{H^{-\alpha}}\| \cos(\beta H)\|_{H^\alpha} \\ &\lesssim \sum_{\kappa\in\{-1,1\}} \Big\{\delta^{-1}\| : \mathrm{e}^{\mathrm{i}\kappa\beta Y} : \|^2_{H^{-\alpha}} + \delta\|\mathrm{e}^{\mathrm{i}\kappa\beta H}\|^2_{H^\alpha}\Big\} \end{aligned} \tag{2.10}$$

for any $\delta > 0$. Using the fractional chain rule (see lemma 3.2 (ii) below) and lemma 2.5 (ii), we have

$$\begin{aligned} \|\mathrm{e}^{\pm\mathrm{i}\beta H}\|_{H^\alpha} &\sim \|\mathrm{e}^{\pm\mathrm{i}\beta H}\|_{L^2} + \big\||\nabla|^\alpha\big(\mathrm{e}^{\pm\mathrm{i}\beta H})\big)\big\|_{L^2} \\ &\lesssim 1 + \|H\|_{H^\alpha} \lesssim 1 + \Big(\int_0^1 \|\eta(t)\|^2_{L^2_x}\mathrm{d}t\Big)^{1/2}, \end{aligned} \tag{2.11}$$

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as long as $\alpha \leqslant 1$. Moreover, in view of lemma 2.5 (i), we have

$$\mathbb{E}_{\mathbb{Q}^\eta}\Big[\| :\mathrm{e}^{\pm \mathrm{i}\beta Y}: \|_{H^{-\alpha}}^2\Big] \lesssim 1, \tag{2.12}$$

provided that $0 < \beta^2 < 4\pi$ and $\alpha > (\beta^2/4\pi)$. Therefore, from (2.9), (2.10), (2.11) and (2.12), we obtain

$$\mathcal{W}_N(\eta) \geqslant \mathbb{E}_{\mathbb{Q}^\eta}\bigg[\frac{1}{2}\int_0^1 \|\eta(t)\|_{L^2_x}^2\,\mathrm{d}t - C_1\delta^{-1} - C_2\delta\Big(1 + \int_0^1 \|\eta(t)\|_{L^2_x}^2\,\mathrm{d}t\Big)\bigg].$$

By taking $\delta > 0$ sufficiently small, we conclude that there exists finite $C(\delta) > 0$ such that

$$\sup_{N\in\mathbb{N}}\sup_{\eta\in\mathbb{H}_c}\mathcal{W}_N(\eta) \geqslant \sup_{N\in\mathbb{N}}\sup_{\eta\in\mathbb{H}_c}\Big\{ - C(\delta) + \frac{1}{4}\mathbb{E}_{\mathbb{Q}^\eta}\int_0^1 \|\eta(t)\|_{L^2}^2\,\mathrm{d}t\Big\} \geqslant -C(\delta) > -\infty.$$

This proves (1.18) when $p = 1$. The general case $p \geqslant 1$ follows from a straightforward modification.

3. Local well-posedness of the hyperbolic SdSG

In this last section, we present a proof of theorem 1.2. As mentioned in the introduction, thanks to Bourgain's invariant measure argument and the uniform (in N) equivalence of the (truncated) Gibbs measures and the base Gaussian measure $\vec{\mu}_1$, it suffices to prove local well-posedness and convergence of the truncated dynamics (1.22) with the Gaussian random initial data whose law is given by $\vec{\mu}_1$. Furthermore, in view of the uniform (in N) boundedness of the frequency projector $\mathbf{P}_N$ on $W^{s,p}(\mathbb{T}^2)$, $s \in \mathbb{R}$, $1 \leqslant p \leqslant \infty$, and the Da Prato-Debussche decomposition:

$$u_N = w_N + \Psi,$$

it suffices to prove local well-posedness of the following model equation:

$$\begin{cases} \partial_t^2 w + \partial_t w + (1-\Delta)w + \operatorname{Im}\big\{\mathrm{e}^{\mathrm{i}\beta w}\Theta\big\} = 0, \\ (w, \partial_t w)|_{t=0} = (0,0), \end{cases} \tag{3.1}$$

for a given (deterministic) source function Θ.

Proposition 3.1. *Given $0 < \alpha < \frac{1}{2}$, let Θ be a distribution in $L^2([0,1]; W^{-\alpha,\infty}(\mathbb{T}^2))$. Then, there exists $T = T(\|\Theta\|_{L^2([0,1];W_x^{-\alpha,\infty})}) \in (0,1]$ and a unique solution w to (3.1) in the class:*

$$X^{1-\alpha}(T) \overset{\mathrm{def}}{=} C([0,T]; H^{1-\alpha}(\mathbb{T}^2)) \cap C^1([0,T]; H^{-\alpha}(\mathbb{T}^2)) \cap L^\infty([0,T]; L^{2/\alpha}(\mathbb{T}^2)).$$

Moreover, the solution map: $\Theta \mapsto w$ is continuous.

Once we prove proposition 3.1, the convergence of the solution $u_N = w_N + \Psi$ to (1.22) follows from lemma 2.2 and arguing as in our previous work [**32**]. Note that the restriction $0 < \alpha < \frac{1}{2}$ in proposition 3.1 gives the range $0 < \beta^2 < 2\pi$ in theorem 1.2 in view of lemma 2.2.

Before proceeding to the proof of proposition 3.1, we state the following deterministic tools from [**32**]; see (3.3) and lemmas 3.1 and 3.2 in [**32**].

LEMMA 3.2. *Let $0 < \alpha < 1$ and $d \geqslant 1$. Then, the following estimates hold:*

(i) (Strichartz estimate). *Let u be a solution to the linear damped wave equation on $\mathbb{R}_+ \times \mathbb{T}^2$:*

$$\begin{cases} \partial_t^2 u + \partial_t u + (1-\Delta)u = f \\ (u, \partial_t u)|_{t=0} = (u_0, u_1). \end{cases}$$

Then, for any $0 < T \leqslant 1$, we have

$$\|u\|_{C_T H_x^{1-\alpha}} + \|\partial_t u\|_{C_T H_x^{-\alpha}} + \|u\|_{L_T^\infty L_x^{\frac{2}{\alpha}}} \lesssim \|(u_0, u_1)\|_{\mathcal{H}^{1-\alpha}} + \|f\|_{L_T^1 H_x^{-\alpha}}.$$

(ii) (fractional chain rule). *Let F be a Lipschitz function on $\mathbb{R}$ such that $\|F'\|_{L^\infty(\mathbb{R})} \leqslant L$. Then, for any $1 < p < \infty$, we have*

$$\big\||\nabla|^\alpha F(f)\big\|_{L^p(\mathbb{T}^d)} \lesssim L\big\||\nabla|^\alpha f\big\|_{L^p(\mathbb{T}^d)}.$$

(iii) (fractional Leibniz rule). *Let $1 < p_j, q_j, r < \infty$ with $(1/p_j) + (1/q_j) = 1/r$, $j = 1, 2$. Then, we have*

$$\big\|\langle\nabla\rangle^\alpha (fg)\big\|_{L^r(\mathbb{T}^d)} \lesssim \big\|\langle\nabla\rangle^\alpha f\big\|_{L^{p_1}(\mathbb{T}^d)}\|g\|_{L^{q_1}(\mathbb{T}^d)} + \|f\|_{L^{p_2}(\mathbb{T}^d)}\big\|\langle\nabla\rangle^\alpha g\big\|_{L^{q_2}(\mathbb{T}^d)}.$$

(iv) (product estimate). *Let $1 < p, q, r < \infty$ such that $(1/p) + (1/q) \leqslant (1/r) + (\alpha/d)$. Then, we have*

$$\big\|\langle\nabla\rangle^{-\alpha} (fg)\big\|_{L^r(\mathbb{T}^d)} \lesssim \big\|\langle\nabla\rangle^{-\alpha} f\big\|_{L^p(\mathbb{T}^d)}\big\|\langle\nabla\rangle^\alpha g\big\|_{L^r(\mathbb{T}^d)}.$$

The Strichartz estimate on $\mathbb{T}^2$ in (i) follows from the corresponding Strichartz estimate for the wave/Klein-Gordon equation on $\mathbb{R}^2$ (see [**13**, **22**, **23**]) for the $(1-\alpha)$-wave admissible pair $(\infty, 2/\alpha)$, the finite speed of propagation, and the fact that the linear damped wave propagator $\mathcal{D}(t)$ in (1.24) satisfies the same Strichartz estimates as that for the Klein-Gordon equation $\partial_t^2 u + ((3/4) - \Delta)u = 0$. For the fractional chain rule on $\mathbb{T}^d$ [**12**]. See [**15**] for (iii) and (iv).

We now present a proof of proposition 3.1.

Proof of proposition 3.1.. By writing (3.1) in the Duhamel formulation, we have

$$w(t) = \Phi(w)(t) := -\int_0^t \mathcal{D}(t-t')\,\mathrm{Im}\left\{\mathrm{e}^{\mathrm{i}\beta w}\Theta\right\}(t')\mathrm{d}t',$$

where $\mathcal{D}(t)$ is as in (1.24). Fix $0 < T \leqslant 1$ and $0 < \alpha < 1/2$. We use B to denote the ball in $X^{1-\alpha}(T)$ of radius 1 centred at the origin.

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From lemma 3.2 (i), (iv) and then (ii) with $\alpha < 1-\alpha$, we have

$$\begin{aligned}\|\Phi(w)\|_{X^{1-\alpha}(T)} &\lesssim \|\mathrm{e}^{\mathrm{i}\beta w}\Theta\|_{L^1_T H^{-\alpha}_x} \lesssim T^{1/2}\|\mathrm{e}^{\mathrm{i}\beta w}\|_{L^\infty_T H^\alpha_x}\|\Theta\|_{L^2_T W^{-\alpha,2/\alpha}_x} \\ &\lesssim T^{1/2}\big(1+\|w\|_{X^{1-\alpha}(T)}\big)\|\Theta\|_{L^2_T W^{-\alpha,\infty}_x} \\ &\lesssim T^{1/2}\|\Theta\|_{L^2_T W^{-\alpha,\infty}_x}\end{aligned} \tag{3.2}$$

for $w \in B$. By the fundamental theorem of calculus, we have

$$\mathrm{e}^{\mathrm{i}\beta w_1} - \mathrm{e}^{\mathrm{i}\beta w_2} = (w_1-w_2)F(w_1,w_2) \overset{\text{def}}{=} (w_1-w_2)(\mathrm{i}\beta)\int_0^1 \mathrm{e}^{\mathrm{i}\beta(\tau w_1+(1-\tau)w_2)}\mathrm{d}\tau.$$

Thus, from lemma 3.2 (i) and (iv), we have

$$\|\Phi(w_1)-\Phi(w_2)\|_{X^{1-\alpha}(T)} \lesssim T^{1/2}\|(w_1-w_2)F(w_1,w_2)\|_{L^\infty_T W^{\alpha,2/(1+\alpha-\varepsilon)}_x}\|\Theta\|_{L^2_T W^{-\alpha,2/\varepsilon}_x} \tag{3.3}$$

for any small $\varepsilon > 0$. Then, by applying lemma 3.2 (iii) and then (ii) to (3.3), we obtain

$$\begin{aligned}&\|(w_1-w_2)F(w_1,w_2)\|_{L^\infty_T W^{\alpha,2/(1+\alpha-\varepsilon)}_x} \\ &\quad\lesssim \|w_1-w_2\|_{L^\infty_T H^\alpha_x}\|F(w_1,w_2)\|_{L^\infty_T L^{2/(\alpha-\varepsilon)}_x} \\ &\qquad + \|w_1-w_2\|_{L^\infty_T L^{2/\alpha}_x}\|F(w_1,w_2)\|_{L^\infty_T W^{\alpha,2/(1-\varepsilon)}_x} \\ &\quad\lesssim \|w_1-w_2\|_{X^{1-\alpha}(T)}\Big(1+\|w_1\|_{L^\infty_T W^{\alpha,2/(1-\varepsilon)}_x}+\|w_2\|_{L^\infty_T W^{\alpha,2/(1-\varepsilon)}_x}\Big).\end{aligned} \tag{3.4}$$

Given $0<\alpha<1/2$, choose $\varepsilon > 0$ small such that $\alpha+\varepsilon < 1-\alpha$. Then, it follows from (3.3), (3.4) and Sobolev's inequality that

$$\|\Phi(w_1)-\Phi(w_2)\|_{X^{1-\alpha}(T)} \lesssim T^{1/2}\|\Theta\|_{L^2_T W^{-\alpha,\infty}_x}\|w_1-w_2\|_{X^{1-\alpha}(T)} \tag{3.5}$$

for any $w_1, w_2 \in B$.

Hence, we conclude from (3.2) and (3.5) that the map $\Phi = \Phi_\Theta$ is a contraction on $B \subset X^{1-\alpha}(T)$, provided that $T = T(\|\Theta\|_{L^2([0,1];W^{-\alpha,\infty}_x)}) > 0$ is sufficiently small. The uniqueness in the whole space $X^{1-\alpha}(T)$ follows from a standard continuity argument, while a small modification of the argument above shows the continuous dependence on Θ. □

Proposition 3.1 thus establishes local well-posedness of the truncated equation (1.27), uniformly in $N \in \mathbb{N}$, and also for the limiting equation

$$\begin{cases}\partial_t^2 w + (1-\Delta)w + \partial_t w + \operatorname{Im}\{\mathrm{e}^{\mathrm{i}\beta w}\Theta\} = 0, \\ (w,\partial_t w)|_{t=0} = (0,0),\end{cases} \tag{3.6}$$

where Θ is the limit of Θ_N constructed in lemma 2.2. We briefly describe an extra ingredient in showing convergence of w_N to w, satisfying (3.6). Since the flow map constructed in proposition 3.1 is continuous in Θ, there is only one extra term

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$(\mathrm{Id} - \mathbf{P}_N)\{e^{i\beta w}\Theta\}$ in estimating the difference $\|w_N - w\|_{C_T H_x^{1-\alpha}}$. By exploiting the fact that this extra term is supported on high frequencies $\{|n| \gtrsim N\}$, we have

$$\big\|(\mathrm{Id} - \mathbf{P}_N)\{e^{i\beta w}\Theta\}\big\|_{L_T^1 H_x^{-\alpha}} \lesssim N^{-\varepsilon}\big\|e^{i\beta w}\Theta\big\|_{L_T^1 H_x^{-\alpha+\varepsilon}} \lesssim T^{1/2} N^{-\varepsilon}\big(1 + \|w\|_{L_T^\infty H_x^{\alpha-\varepsilon}}\big)\|\Theta\|_{L_T^2 W_x^{-\alpha+\varepsilon,\infty}}.$$

Combining with the argument above, we can then prove convergence $w_N \to w$ as $N \to \infty$. Note that given $0 < \beta^2 < 2\pi$ and $0 < \alpha < 1/2$ with $\beta^2/4\pi < \alpha$, we have $\beta^2/4\pi < \alpha - \varepsilon$ for small $\varepsilon > 0$, which guarantees that $\Theta \in L^2([0,T]; W^{\varepsilon-\alpha}(\mathbb{T}^2))$ in view of lemma 2.2.

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